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Multiple soft radiation at one-loop order and the emission of a soft quark–antiquark pair

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Abstract We consider the radiation of two or more soft partons in QCD hard-scattering at one-loop order. The corresponding scattering amplitude is singular, and the singular behaviour is controlled by a process-independent soft current. Using regularization in $d = 4 - 2\epsilon$ space-time dimensions, we explicitly evaluate the ultraviolet and infrared divergent (ϵ-pole) terms of the one-loop soft current for emission of an arbitrary number of soft partons in a generic hard-scattering process. Then we consider the specific case of soft quark–antiquark ($q\bar{q}$) emission and we compute the one-loop current by including the finite terms. We find that the one-loop soft-$q\bar{q}$ current exhibits a new type of transverse-momentum singularity, which has a quantum (absorptive) origin and a purely non-abelian character. At the squared amplitude (cross section) level, this transverse-momentum singularity produces contributions to multijet production processes in hadron collisions. The one-loop squared current also leads to charge asymmetry terms, which are a distinctive features of soft-$q\bar{q}$ radiation. We also extend these results to the cases of QED and mixed QCD×QED radiative corrections for soft fermion–antifermion emission.

1 Introduction

The physics program carried out at the large hadron collider (LHC) has already produced an impressive amount of high-precision data, and similar data will be obtained in the next runs of the LHC. Theoretical predictions are thus demanded to achieve a corresponding high precision.

In the context of the perturbative evaluation of QCD radiative corrections, the present high-precision frontier is represented by computations at the next-to-next-to-next-to-leading order (N³LO) in the QCD coupling α_S. Some N³LO results for LHC processes are already available (see, e.g., related references in Ref. [1]). In the case of observables that are highly sensitive to multiple radiation of soft and collinear partons, the fixed-order QCD predictions have to be supplemented with the all-order resummed calculations of classes of large logarithmic contributions. In few specific cases (see, e.g., Refs. [2–19]) resummed QCD calculations have reached the next-to-next-to-next-to-leading logarithmic (N³LL) accuracy.

An important feature of QCD scattering amplitudes is the presence of singularities in soft and collinear regions of the phase space, and the corresponding presence of infrared (IR) divergences in virtual contributions at the loop level. The soft and collinear singularities have a process-independent structure, and they are controlled by universal factorization formulae and corresponding soft/collinear factors. As briefly recalled below, these factorization properties are relevant for both fixed-order and resummed QCD calculations.

In the computation of physical observables for hard-scattering processes, phase space soft/collinear singularities and virtual IR divergences cancel between themselves. However, much technical effort is required to achieve and implement the cancellation, and the effort highly increases by increasing the perturbative order. Soft/collinear factorization formulae can be used to organize and greatly simplify the cancellation mechanism of the IR divergences in fixed-oder calculations.

In the evaluation of observables close to the exclusive boundary of the phase space, real and virtual radiative corrections in the scattering amplitudes are kinematically strongly unbalanced. As a consequence, the cancellation mechanism of the IR divergences leaves residual effects in the form of large logarithmic contributions. Soft/collinear factorization formulae and the corresponding singular factors are the basic ingredients for the explicit computation and resummation of these large logarithmic contributions.

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The singular factors at $\mathcal{O}(\alpha_S)$ and $\mathcal{O}(\alpha_S^2)$ for soft and collinear factorization of scattering amplitudes are known since long time. The explicit knowledge of soft/collinear factorization at $\mathcal{O}(\alpha_S)$ has been essential to devise fully general (process-independent and observable-independent) methods to carry out next-to-leading order (NLO) QCD calculations (see, e.g., Refs. [20–22]). Similarly, the knowledge of soft/collinear factorization formulae at $\mathcal{O}(\alpha_S^2)$ [23–34] is exploited to develop methods (see, e.g., the review in Ref. [1]) at the next-to-next-to-leading order (NNLO). Soft/collinear factorization up to $\mathcal{O}(\alpha_S^2)$ contributes to resummed calculations up to next-to-next-to-leading logarithmic (NNLL) accuracy (see, e.g., Refs. [35,36]).

Soft and collinear factorization at $\mathcal{O}(\alpha_S^3)$ can be used in the context of N^{3}LO calculations and of resummed calculations at N^{3}LL accuracy. The process-independent singular factors for the various collinear limits at $\mathcal{O}(\alpha_S^3)$ are presented in Refs. [33,37–48]. Soft factorization of scattering amplitudes at $\mathcal{O}(\alpha_S^3)$ requires the study of various tree-level and loop contributions. Triple soft-gluon radiation at the tree level is studied in Ref. [49]. Double soft emission at one loop level has been considered recently in Ref. [50]. Single soft-gluon radiation at two loop order is examined in detail in Refs. [47,51–53].

This paper is devoted to a study of soft-parton emission at $\mathcal{O}(\alpha_S^3)$ and beyond this order. More precisely, we consider the singular behaviour of scattering amplitudes in the limit in which *two* or *more* external partons are soft. The singularity is controlled in factorized form by a current for soft multiparton radiation from hard partons. At one-loop order the soft current contains IR and ultraviolet (UV) divergent contributions that we explicitly evaluate for the emission of an arbitrary number of soft partons. In the particular case of emission of a soft quark-antiquark ($q\bar{q}$) pair, we explicitly compute also the finite contributions to the one-loop current. We comment on the related results of Ref. [50] in the paper.

The outline of the paper is as follows. In Sect. 2 we introduce our notation, and we recall the soft factorization formula for scattering amplitudes and the known results on the tree-level currents for emission of a single soft gluon and of soft-$q\bar{q}$ pair. We use analytic continuation in $d = 4 - 2\epsilon$ space-time dimensions to regularize IR and UV divergences in loop contributions. In Sect. 3 we discuss general features of the current for multiple soft radiation at the loop level. In particular, we present in explicit form the result of the IR and UV divergent (ϵ-pole) terms of the one-loop soft current. In Sect. 4 we consider the emission of a soft-$q\bar{q}$ pair and we compute the corresponding one-loop current by including the finite (i.e., $\mathcal{O}(\epsilon^0)$) terms. We comment on general features of our result that is valid for generic multiparton scattering processes in arbitrary kinematical configurations. Section 5 is devoted to consider soft-$q\bar{q}$ radiation at the squared amplitude level. We first recall the results for the squared current

at the tree level, and then we explicitly compute the one-loop squared current. We discuss the structure of the charge asymmetry contributions, which are a distinctive feature of soft-$q\bar{q}$ radiation at the loop level. In Sect. 5.3 we present simplified expressions for processes with two and three hard partons. In Sect. 6 we generalize our QCD results for soft $q\bar{q}$ emission to the cases of QED and mixed QCD×QED radiative corrections for soft fermion-antifermion emission. A brief summary of our results is presented in Sect. 7.

2 Soft factorization

We consider the amplitude (the S-matrix element) $\mathcal{M}$ of a generic scattering process whose external particles (the external legs of $\mathcal{M}$) are QCD partons (quarks, antiquarks and gluons) and, possibly, additional non-QCD particles (i.e., partons with no colour charge such as leptons, Higgs and electroweak vector bosons and so forth). We use the notation $\mathcal{M}(p_1, p_2, \ldots, p_n)$, where p_i $(i = 1, \ldots, n)$ is the momentum of the QCD parton A_i ($A_i = g, q$ or $\bar{q}$). Unless otherwise specified, the dependence of $\mathcal{M}$ on the momenta (and quantum numbers) of additional colourless particles is not explicitly denoted.

The external QCD partons are on-shell with physical spin polarizations (thus, $\mathcal{M}$ includes the corresponding spin wave functions), and we always define the external momenta p_i's as *outgoing* momenta. Note, however, that we do no restrict our treatment to processes with physical partons in the final state. In particular, the time-component (i.e. the 'energy') p_i^0 of the momentum vector p_i^ν ($\nu = 0, 1, \ldots, d - 1$) in d space-time dimensions is *not* positive definite. Different types of physical processes are described by considering different kinematical regions of the parton momenta and by simply applying crossing symmetry to the wave functions and quantum numbers of the external partons of the same matrix element $\mathcal{M}(p_1, p_2, \ldots, p_n)$. According to our definition of the momenta, if p_i has positive energy, $\mathcal{M}(\ldots, p_i, \ldots)$ describes a physical process that produces the *parton A_i* in the final state; if p_i has negative energy, $\mathcal{M}(\ldots, p_i, \ldots)$ describes a physical process produced by the collision of the *antiparton $\overline{A}_i$* in the initial state.

The scattering amplitude $\mathcal{M}$ also depends on the colour indices $\{c_1, c_2, \ldots\}$ and on the spin (e.g. helicity) indices $\{s_1, s_2, \ldots\}$ of the external QCD partons, and we write

$$\mathcal{M}_{s_1, s_2, \ldots, s_n}^{c_1, c_2, \ldots, c_n}(p_1, p_2, \ldots, p_n). \tag{1}$$

It is convenient to directly work in colour (and spin) space, and to use the notation of Ref. [22] (see also Ref. [54]). We treat the colour and spin structures by formally introducing an orthonormal basis $\{|c_1, c_2, \ldots, c_n\rangle \otimes |s_1, s_2, \ldots, s_n\rangle\}$ in colour + spin space. The scattering amplitude can be written as

$$\mathcal{M}_{s_1,s_2,\ldots}^{c_1,c_2,\ldots}(p_1, p_2, \ldots) \equiv \Big(\langle c_1, c_2, \ldots| \otimes \langle s_1, s_2, \ldots|\Big)$$
$$|\mathcal{M}(p_1, p_2, \ldots)\rangle. \qquad (2)$$

Thus $|\mathcal{M}(p_1, p_2, \ldots, p_n)\rangle$ is a vector in colour + spin (helicity) space.

As previously stated, we define the external momenta p_i's as *outgoing* momenta. The colour indices $\{c_1, c_2, \ldots c_n\}$ are consistently defined as *outgoing* colour indices: c_i is the colour index of the parton A_i with outgoing momentum p_i (if p_i has negative energy, c_i is the colour index of the physical parton $\overline{A}_i$ that collides in the initial state). An analogous comment applies to spin indices.

The amplitude $\mathcal{M}$ can be evaluated in QCD perturbation theory as a power series expansion (i.e., loop expansion) in the QCD coupling g_S (or, equivalently, in the strong coupling $\alpha_S = g_S^2/(4\pi)$). We write

$$\mathcal{M} = \mathcal{M}^{(0)} + \mathcal{M}^{(1)} + \mathcal{M}^{(2)} + \cdots, \qquad (3)$$

where $\mathcal{M}^{(0)}$ is the tree-level scattering amplitude, $\mathcal{M}^{(1)}$ is the one-loop contribution, $\mathcal{M}^{(2)}$ is the two-loop contribution, and so forth. More generally, $\mathcal{M}^{(0)}$ is not necessarily a tree amplitude, but rather the lowest-order amplitude for that given process. Thus, $\mathcal{M}^{(L)}$ ($L = 1, 2 \ldots$) is the corresponding L-loop correction. For instance, in the cases of the diphoton production process $gg \to \gamma\gamma$ or the Higgs boson (H) production process $gg \to H$, the corresponding amplitude $\mathcal{M}^{(0)}$ involves a quark loop (a massive-quark loop in the case of $gg \to H$). Note that in Eq. (3) we have not made explicit the dependence on powers of g_S. Thus, $\mathcal{M}^{(0)}$ includes an integer power of g_S as overall factor, and $\mathcal{M}^{(1)}$ includes an extra factor of g_S^2 (i.e., $\mathcal{M}^{(1)}/\mathcal{M}^{(0)} \propto g_S^2$).

Physical processes take place in four-dimensional space-time. The four-dimensional evaluation of the L-loop amplitude $\mathcal{M}^{(L)}$ leads to UV and IR divergences that have to be properly regularized. We regularize both kind of divergences by performing the analytic continuation of the loop momenta and phase-space in $d = 4 - 2\epsilon$ space-time dimensions. We postpone comments on different variants of dimensional regularization. The dimensional-regularization scale is denoted by μ. After regularization, the UV and IR divergences appears as ϵ-poles of the Laurent series expansion in powers of ϵ around $\epsilon = 0$. Throughout the paper we formally consider expressions for arbitrary values of $d = 4 - 2\epsilon$ (equivalently, in terms of ϵ expansions, the expressions are valid to all orders in ϵ before they are eventually truncated at some order in ϵ). We always consider unrenormalized amplitudes, and g_S denotes the bare (unrenormalized) coupling constant.

We are interested in studying the behaviour of $\mathcal{M}$ in the kinematical configuration where *one* or *more* of the momenta of the external *massless* partons (gluons or massless quark and antiquarks) become soft. In this kinematical configu-

ration, $\mathcal{M}$ becomes singular. To make the notation more explicit, the soft momenta are denoted by q_k^ν, while the other parton momenta are still denoted by p_i^ν. The behaviour of $\mathcal{M}(\ldots, q_k, \ldots, p_i, \ldots)$ in this *multiparton* soft region is formally specified by performing an overall rescaling of all soft momenta as $q_k \to \lambda q_k$ (the rescaling parameter λ is the same for each soft momentum q_k) and by considering the limit $\lambda \to 0$. In this limit, if the set of soft partons has m ($m \geq 1$) momenta q_k's ($k = 1, \ldots, m$), the amplitude $\mathcal{M}$ is singular and it behaves as

$$\mathcal{M}(\lambda q_1, \ldots, \lambda q_m, p_1 \ldots, p_n) \sim \frac{1}{(\lambda)^m} \, \mathrm{mod}\,(\ln^r \lambda)$$
$$+ \cdots, \quad (\lambda \to 0). \quad (4)$$

The power-like behaviour $(\lambda)^{-m}$ that we have specified in the right-hand side of Eq. (4) determines the *dominant* singular terms of $\mathcal{M}$ in the multiple soft region. The logarithmic corrections $\ln^r \lambda$ ($r = 0, 1, 2, \ldots$) arise from scaling violation, since the naïve (power-like) scaling behaviour is violated by the effects of the UV and IR divergences of the scattering amplitude at the loop level (see Sect. 3). The dots on the right-hand side of Eq. (4) denote the *subdominant* singular behaviour of $\mathcal{M}$. The relative suppression factor between subdominant and dominant terms is (at least) of $\mathcal{O}(\sqrt{\lambda})$.

The computation of physical observables eventually requires the phase-space integration of the *squared* amplitude $|\mathcal{M}|^2$. We note that, after phase-space integration over the soft momenta, the dominant singular behaviour of $|\mathcal{M}|^2$ produces logarithmic soft (IR) divergences (i.e., ϵ poles), whereas the subdominant singular behaviour does not lead to soft divergences. In this paper we are interested in the dominant singular behaviour of Eq. (4).

In the soft multiparton limit, the dominant singular behaviour of $\mathcal{M}$ can be expressed by the following process-independent (universal) factorization formula [27–29,55,56]

$$|\mathcal{M}(q_1, \ldots, q_m, p_1, \ldots, p_n)\rangle$$
$$= \boldsymbol{J}(q_1, \ldots, q_m) \, |\mathcal{M}(p_1, \ldots, p_n)\rangle + \cdots, \qquad (5)$$

where, analogously to Eq. (4), the dots on the right-hand side denote subdominant singular terms. The amplitude $\mathcal{M}(p_1, \ldots, p_n)$ on the right-hand side of Eq. (5) is simply obtained by removing the m external legs with soft parton momenta $q_1, \ldots, q_m$ from the amplitude on the left-hand side. The factor $\boldsymbol{J}(q_1, \ldots, q_m)$ is the *soft multiparton current* that embodies the dominant singular behaviour denoted in the right-hand side of Eq. (4).

In the case of tree-level scattering amplitudes [28,57,58], the factorization formula (5) can be simply derived by considering soft-parton radiation from the hard-parton (the partons with momenta $p_1, \ldots, p_n$) external legs of the amplitude and by directly applying the eikonal approximation for emission vertices and propagators. At the one-loop level, the factoriza-

tion structure of Eq. (5) was worked out in Refs. [27,29,55]. In particular, as discussed in detail in Ref. [29], the one-loop soft current can still be computed by using the eikonal approximation for soft-parton radiation from the external hard partons, and this discussion generalizes to two-loop and higher-loop orders. Owing to its origin by eikonal radiation from the hard partons, the soft current can also straightforwardly be expressed in equivalent form as matrix element of Wilson line operators [56].

The soft current $\boldsymbol{J}(q_1, \ldots, q_m)$ depends on the soft partons, specifically on their momenta and their quantum numbers (flavour, spin, colour), and it also depends on the hard partons (on their momenta and their quantum numbers), though we use a customary notation in which the dependence of $\boldsymbol{J}$ on $p_1, \ldots, p_n$ is not explicitly denoted in its argument. The current $\boldsymbol{J}$ is an operator (a 'rectangular' matrix) that acts from the (lower-dimensional) colour + spin space of the hard partons to the (higher-dimensional) colour + spin space of the soft and hard partons. We remark on the fact that the soft current $\boldsymbol{J}$ is simply proportional to the unit operator in the spin subspace of the hard partons, since soft radiation is insensitive to the spin of the hard radiating partons.

In spite of its dependence on hard partons, the soft current $\boldsymbol{J}$ is completely *universal*, namely, it does not depend on the specific scattering amplitude $\mathcal{M}$ and on its corresponding specific physical process. The universality of $\boldsymbol{J}$ also implies that it is directly applicable in contexts that do not directly refer to the soft behaviour of scattering amplitudes. For instance, $\boldsymbol{J}$ (or, more specifically, $\boldsymbol{J}^\dagger \boldsymbol{J}$) is precisely the *integrand* of any specific soft function (see, e.g., Ref. [35] and references therein) that can be introduced through soft-collinear effective theory (SCET) [59–64] methods.

The colour-space factorization formula (5) does not require any specifications about the detailed colour structure of the scattering amplitudes in its left-hand and right-hand sides. Scattering amplitudes can be decomposed in a form that factorizes the QCD colour from colourless kinematical coefficients, which are colour-ordered subamplitudes (see, e.g., Ref. [65]). Colour-ordered subamplitudes fulfil soft factorization formulae that are analogous to Eq. (5) in terms of colour-stripped (though colour-ordered) soft factors (see, e.g., Refs. [27,58]). The factorization properties of colour-order subamplitudes and the corresponding soft factors can be directly and explicitly derived from Eq. (5). To this purpose it is sufficient to insert the colour decomposition of $\mathcal{M}$ and the explicit colour structure of $\boldsymbol{J}$ in Eq. (5). Therefore, the colour-space factorization of Eq. (5) and soft factorization of colour-ordered subamplitudes are equivalent formulations. The advantage of Eq. (5) is that it leads to a more compact formulation, without the necessity of introducing

the explicit colour decomposition of $\mathcal{M}$, whose actual form depends on the specific partonic content of the amplitude (e.g., on the number of gluons and quark-antiquark pairs[1]) and on the loop order. Moreover, the colour space formulation can simplify the direct computation of the soft limit of *squared* amplitudes (see, e.g., Sect. 5).

The soft current $\boldsymbol{J}$ in Eq. (5) can be evaluated in QCD perturbation theory, and it can be expressed in terms of a loop expansion that is completely analogous to that in Eq. (3). We write

$$\boldsymbol{J} = \boldsymbol{J}^{(0)} + \boldsymbol{J}^{(1)} + \boldsymbol{J}^{(2)} + \cdots , \tag{6}$$

where $\boldsymbol{J}^{(0)}$ is the tree-level current, $\boldsymbol{J}^{(1)}$ is the one-loop current, and so forth. Analogously to Eq. (3), the loop label L in $\boldsymbol{J}^{(L)}$ refers to the unrenormalized current. Inserting the expansions (3) and (6) in Eq. (5) we obtain factorization formulae that are valid order-by-order in the number of loops. The soft factorization formula for tree-level (lowest-order) amplitudes is

$$\begin{aligned} &|\mathcal{M}^{(0)}(q_1, \ldots, q_m, p_1, \ldots, p_n)\rangle \\ &\simeq \boldsymbol{J}^{(0)}(q_1, \ldots, q_m) \, |\mathcal{M}^{(0)}(p_1, \ldots, p_n)\rangle, \end{aligned} \tag{7}$$

where the symbol '$\simeq$' means[2] that we are neglecting subdominant terms in the soft limit (i.e., the terms denoted by dots in the right-hand side of Eqs. (4) and (5)). The soft factorization formula for one-loop amplitudes is

$$\begin{aligned} &|\mathcal{M}^{(1)}(q_1, \ldots, q_m, p_1, \ldots, p_n)\rangle \\ &\simeq \boldsymbol{J}^{(1)}(q_1, \ldots, q_m) \, |\mathcal{M}^{(0)}(p_1, \ldots, p_n)\rangle \\ &+ \boldsymbol{J}^{(0)}(q_1, \ldots, q_m) \, |\mathcal{M}^{(1)}(p_1, \ldots, p_n)\rangle. \end{aligned} \tag{8}$$

The tree-level current for the emission of a *single* soft gluon of momentum q^ν is well known [57]:

$$\boldsymbol{J}^{(0)}(q) = g_S \, \mu^\epsilon \sum_{i \in H} \boldsymbol{T}_i \frac{p_i \cdot \varepsilon(q)}{p_i \cdot q} \equiv \boldsymbol{J}_\nu^{(0)}(q) \varepsilon^\nu(q) \, , \tag{9}$$

where the notation $i \in H$ means that the sum extends over all hard partons (with momenta p_i) in $\mathcal{M}$, $\varepsilon^\nu(q)$ is the spin polarization vector of the soft gluon, and $\boldsymbol{T}_i$ is the colour charge of the hard parton i.

The spin index s and the colour index a ($a = 1, \ldots, N_c^2 - 1$, for $SU(N_c)$ QCD with N_c colours) of the soft gluon can be specified by acting onto Eq. (9) in colour+spin space as in Eq. (2). Considering $(\langle a| \otimes \langle s|) \, \boldsymbol{J}^{(0)}(q) \equiv \boldsymbol{J}_s^{(0)\,a}(q)$, we have $(\langle a| \otimes \langle s|) \, \varepsilon^\nu(q) \, \boldsymbol{T}_i = \varepsilon_{(s)}^\nu(q) \, T_i^a$, where T_i^a denotes the generators of $SU(N_c)$ of the representation of the parton

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[1] In particular, if the set of soft partons includes one or more quark-antiquark pairs, the scattering amplitudes in the left-hand and right-hand sides of Eq. (5) have different numbers of quark-antiquark pairs and, therefore, they have different colour decompositions.

[2] The symbol '$\simeq$' is used throughout the paper with the same meaning as in Eq. (7).

i. We have $\langle c_i | T_i^a | c_i' \rangle = (T_i^a)_{c_i\, c_i'}$, where $(T^a)_{cb} \equiv i f_{cab}$ (colour-charge matrix in the adjoint representation) if the parton i is a gluon and $(T^a)_{\alpha\beta} \equiv t_{\alpha\beta}^a$ (colour-charge matrix in the fundamental representation, with $\alpha, \beta = 1, \ldots, N_c$) if the parton i is a quark ($(T^a)_{\alpha\beta} \equiv \bar{t}_{\alpha\beta}^a = -t_{\beta\alpha}^a$ if the parton i is an antiquark). We normalize the colour matrices such as $[t^a, t^b] = i f_{abc}$ and $\mathrm{Tr}(t^a t^b) = T_R \delta^{ab}$ with $T_R = 1/2$. The colour-charge algebra gives $[T_i^a, T_j^b] = i f_{abc} T_i^c \delta_{ij}$ and $\sum_a T_i^a T_j^a \equiv \boldsymbol{T}_i \cdot \boldsymbol{T}_j$ with $\boldsymbol{T}_i^2 = C_i$, where C_i is the Casimir operator of the representation of the parton i, i.e. $C_i = C_A = N_c$ if i is a gluon and $C_i = C_F = (N_c^2 - 1)/(2N_c)$ if i is a quark or antiquark.

We note that the colour charge operators $\boldsymbol{T}_i$ fulfil some relevant properties related to colour conservation. For instance, we have [22]

$$\sum_{i \in \mathcal{M}} \boldsymbol{T}_i | \mathcal{M} \rangle = 0, \tag{10}$$

which follows from the fact that the scattering amplitude $\mathcal{M}$ is a colour-singlet state (the notation $i \in \mathcal{M}$ means that the sum in Eq. (10) extends over all external partons of $\mathcal{M}$). In particular, Eq. (10) implies that the tree-level soft current in Eq. (9) leads to a gauge invariant soft factor since $q^\nu \boldsymbol{J}_\nu^{(0)}(q) | \mathcal{M}(p_1, \ldots, p_n) \rangle \propto \sum_{i \in H} \boldsymbol{T}_i | \mathcal{M}(p_1, \ldots, p_n) \rangle = 0$.

A property analogous to that in Eq. (10) is fulfilled by the soft multiparton current $\boldsymbol{J}$. We have

$$\left(\sum_{k \in S} \boldsymbol{T}_k + \sum_{i \in H} \boldsymbol{T}_i \right) \boldsymbol{J}(q_1, \ldots, q_m)$$
$$= \boldsymbol{J}(q_1, \ldots, q_m) \sum_{i \in H} \boldsymbol{T}_i, \tag{11}$$

or, equivalently,

$$\sum_{k \in S} \boldsymbol{T}_k \ \boldsymbol{J}(q_1, \ldots, q_m) = \left[\boldsymbol{J}(q_1, \ldots, q_m), \sum_{i \in H} \boldsymbol{T}_i \right], \tag{12}$$

where the notation $k \in S$ ($i \in H$) means that the sum extends over all the soft (hard) partons in $\boldsymbol{J}$. The relations in Eqs. (11) and (12) express the property of colour flow conservation and follow from the fact that the total colour charge is conserved in the radiation process of soft partons by hard partons. Note from Eq. (12) that the total charge of the soft partons acts on $\boldsymbol{J}$ as a rotation of its hard-parton charges. We remark that Eqs. (10)–(12) are valid to all orders in the loop expansion or, equivalently, order-by-order in QCD perturbation theory. It is straightforward to explicitly check that the soft single-gluon current in Eq. (9) fulfils the colour flow conservation property in Eq. (11).

Since scattering amplitudes are colour-singlet states, the structure of soft factorization in Eq. (5) implies that the explicit expression of the universal soft current $\boldsymbol{J}$ necessarily involves some degrees of arbitrariness. Different expressions

for $\boldsymbol{J}$ are indeed permitted, provided the difference is proportional to an operator that is proportional to the total colour charge of the hard partons. Owing to Eq. (10), this degree of arbitrariness is physically harmless (it does not affect the soft behaviour of the scattering amplitude) and the ensuing different expressions of $\boldsymbol{J}$ are fully equivalent (although they are not exactly equal at the formal level, before acting onto colour-singlet states). The property of colour-flow conservation in Eq. (11) does not remove this degree of arbitrariness.

For subsequent use (and similarly to Ref. [49]) we introduce the notation

$$\mathbf{O} \underset{\mathrm{cs}}{=} \mathbf{O}', \tag{13}$$

where the subscript CS in the symbol $\underset{\mathrm{cs}}{=}$ means that the equality between the colour operators $\mathbf{O}$ and $\mathbf{O}'$ (e.g., soft currents or their corresponding squared currents) is valid if these operators act (either on the left or on the right) onto colour-singlet states. The notation in Eq. (13) permits to directly relate (and equate) expressions that simply differ by contributions that are due to the physically harmless arbitrariness of the soft current $\boldsymbol{J}$.

The soft factor for radiation of *two* soft gluons from tree-level colour-ordered subamplitudes with external gluons and with external gluons and an additional quark-antiquark pair was computed in Ref. [58]. The tree-level current $\boldsymbol{J}^{(0)}(q_1, q_2)$ for emission of *two* soft gluons in a generic scattering amplitude was given in Ref. [28]. The tree-level current for emission of *three* soft gluons was computed in Ref. [49].

The tree-level current for emission of a *single* soft quark (or antiquark) vanishes. This result is equivalent to say that the dominant singular behaviour in Eq. (4) is absent in the soft single-quark limit (the radiation of a single soft quark only produces a subdominant behaviour of $\mathcal{O}(1/\sqrt{\lambda})$ in the right-hand side of Eq. (4)).

The tree-level current for emission of a soft $q\bar{q}$ pair was computed in Ref. [28], where the result was explicitly reported at the level of squared amplitudes (i.e., the result refers to $\boldsymbol{J}^\dagger \boldsymbol{J}$). The corresponding result for the $q\bar{q}$ current is

$$\boldsymbol{J}^{(0)}(q_1, q_2) = -\left(g_S \mu^\epsilon \right)^2 \sum_{i \in H} \boldsymbol{t}^c \, \boldsymbol{T}_i^c \, \frac{p_i \cdot j(1, 2)}{p_i \cdot q_{12}}, \tag{14}$$

where we have introduced the fermionic current $j^\nu(1, 2)$,

$$j^\nu(1, 2) \equiv \frac{\bar{u}(q_1) \gamma^\nu v(q_2)}{q_{12}^2}, \quad q_{12} = q_1 + q_2. \tag{15}$$

The soft quark and antiquark have momenta q_1^ν and q_2^ν, respectively, and $u(q)$ and $v(q)$ are the customary Dirac spinors. The spin indices (s_1 and s_2) and the colour indices (α_1 and α_2) of the quark and antiquark are embodied in the colour+spin space notation of Eq. (14). Analogously to Eq. (9), we can consider $(\langle \alpha_1, \alpha_2 | \otimes \langle s_1, s_2 |) \boldsymbol{J}(q_1, q_2) \equiv$

$J^{\alpha_1,\alpha_2}_{s_1,s_2}(q_1, q_2)$ and we have $(\langle \alpha_1, \alpha_2| \otimes \langle s_1, s_2|)\, t^c\, \bar{u}(q_1)\, \gamma^\nu$
$v(q_2) = t^c_{\alpha_1\alpha_2}\, \bar{u}_{(s_1)}(q_1)\, \gamma^\nu\, v_{(s_2)}(q_2)$.

3 One-loop current for multiple soft emission: UV and IR divergences

The soft singular behaviour of one-loop amplitudes (see Eq. (8)) is controlled by $J^{(0)}$ and by an additional new ingredient, the one-loop soft current $J^{(1)}$.

The one-loop soft limit for emission of a *single* soft gluon was worked out independently by two groups [27,29], finding results that are in agreement. The analysis of Ref. [27] is based on the study of colour-ordered subamplitudes, while Ref. [29] considers generic scattering amplitudes. The results of Refs. [27,29] are valid for the case of massless hard partons. The generalization of the results of Ref. [29] to include massive hard partons (such as heavy quarks) was carried out in Ref. [31,32]. In the remaining part of this paper we limit ourselves to consider scattering amplitudes with *massless* hard partons.

The result of the one-loop current for *single* gluon emission is [29] (we explicitly write $J^{(1)\,a} \equiv \langle a|\, J^{(1)}$, where a is the colour index of the soft gluon):

$$J^{(1)\,a} = -\left(g_S\,\mu^\epsilon\right)^3 c_\Gamma \frac{1}{\epsilon^2}\, \Gamma(1-\epsilon)\Gamma(1+\epsilon)\; if_{abc}$$

$$\times \sum_{\substack{i,j\,\in H \\ i \neq j}} T^b_i\, T^c_j \left(\frac{p^\nu_i}{p_i \cdot q} - \frac{p^\nu_j}{p_j \cdot q}\right) \varepsilon_\nu(q)$$

$$\times \frac{(-2p_i \cdot q - i0)^{-\epsilon}\, (-2p_j \cdot q - i0)^{-\epsilon}}{\left(-2p_i \cdot p_j - i0\right)^{-\epsilon}}, \qquad (16)$$

where '$x - i0$' denotes the customary Feynman prescription for analytic continuation in different kinematical regions ($x > 0$ and $x < 0$) and c_Γ is the typical volume factor of d-dimensional one-loop integrals:

$$c_\Gamma \equiv \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{(4\pi)^{2-\epsilon}\,\Gamma(1-2\epsilon)}. \qquad (17)$$

We remark that Eq. (16) gives the complete result to all orders in the ϵ expansion around $\epsilon = 0$ (equivalently, the result in arbitrary $d = 4 - 2\epsilon$ space-time dimensions).

We comment on some features of Eq. (16). The one-loop current is proportional to the structure constants f_{abc} of the gauge group and, therefore, it is *purely* non-abelian. This is in agreement with the absence of one-loop corrections to the soft current for single soft-photon emission in massless QED [66,67]. The current in Eq. (16) involves non-abelian colour correlations, $if_{abc}\, T^b_i\, T^c_j$, with two hard partons. Its kinematical structure has a rational dependence on $p_i \cdot \varepsilon(q)/p_i \cdot q$ (which is analogous to that in the tree-level current of Eq. (9)) that is only modified through logarithmic corrections by the

one-loop interactions. The logarithmic corrections are due to the ϵ expansion of the last factor in the right-hand side of Eq. (16), and they are proportional to powers of $\ln q^2_{\perp ij}$ (modulo branch-cut effects), where $q^2_{\perp ij}$,

$$q^2_{\perp ij} = \frac{2(p_i \cdot q)(p_j \cdot q)}{p_i \cdot p_j}, \qquad (18)$$

has a simple kinematical interpretation since it is the transverse component of the gluon momentum q with respect to the longitudinal direction singled out by the momenta p_i and p_j (in a reference frame in which p_i and p_j are back-to-back) of the colour-correlated partons. The overall scaling behaviour of $J^{(1)}(\lambda q)$ (with $\lambda > 0$) in the limit $\lambda \to 0$ is proportional to $(\lambda^2)^{-\epsilon}/\lambda = (1/\lambda)\mathrm{mod}(\epsilon^r \ln^r \lambda)$, and it is in agreement with Eq. (4). In particular, we explicitly see that the $\ln \lambda$-enhancement is produced by the use of dimensional regularization to avoid the IR and UV divergences in the one-loop contribution to J. Performing the ϵ expansion of Eq. (16), this IR and UV behaviour produces double ($1/\epsilon^2$) and single ($1/\epsilon$) poles near $\epsilon = 0$.

The two-loop current for *single* soft-gluon emission was computed in Ref. [47] up to including contributions of $\mathcal{O}(\epsilon^0)$ for the simplest case of scattering amplitudes with *only* two hard partons. Subsequently this result was extended up to $\mathcal{O}(\epsilon^2)$ [51,52] and to all orders in ϵ [52]. The two-loop result of Refs. [47,51,52] has a structure that is very similar to the one-loop current in Eq. (16). More involved structures, in terms of both colour correlations and kinematical dependence, do appear in the general case of scattering amplitudes with three or more hard partons, and the corresponding two-loop current for single soft-gluon emission was considered and explicitly computed in Ref. [53], by including the finite contributions up to $\mathcal{O}(\epsilon^0)$.

We now discuss multiple soft radiation at one-loop order. The structure of the loop-level current J for multiple soft radiation is expected to be definitely more complex (in terms of both colour and kinematical dependence) than the single soft-gluon current in Eq. (16). The presence of two or more soft partons and the ensuing dependence on their momenta increases the number of relevant kinematical invariants, which drive an increased complexity of colour and kinematical correlations (especially at high orders in the ϵ expansion). In the remaining part of this Section we deal with general properties of the soft current J with $m \geq 2$ soft partons. In particular, we consider the UV and IR divergences of J and we discuss their regularization scheme dependence.

The one-loop current for multiple soft emission has (analogously to Eq. (16)) double ($1/\epsilon^2$) and single ($1/\epsilon$) pole contributions due to the presence of IR and UV divergences in the four-dimensional case ($\epsilon = 0$). At L-loop order, the current $J^{(L)}$ has poles of the type $1/\epsilon^k$ with $2L \geq k \geq 1$. These ϵ-pole contributions are directly related to the corresponding contributions to the multiparton scattering ampli-

tudes [54,68–72]. The ϵ-pole contributions to the one-loop current $J^{(1)}$ have a general structure, whose explicit form can be directly derived from the known universal structure of the IR and UV divergences of *one-loop* scattering amplitudes [22,54,73,74]. Starting from the results in Refs. [22,54,73,74], the procedure to derive the ϵ-pole contributions to $J^{(1)}$ is completely analogous to that used in Refs. [33,42] for the study of the multiparton *collinear* limit of scattering amplitudes (see, in particular, Eqs. (104)–(109) in the arXiv version of Ref. [33] and replace the collinear splitting matrix $Sp^{(1)}$ with the soft current $J^{(1)}$). Moreover that procedure can be extended to higher-loop orders and it leads to a compact representation of the ϵ-pole contributions to J at arbitrary perturbative orders (see the analogous procedure in Sect. 6.1 and, in particular, Eq. (137) in the arXiv version of Ref. [33] and replace Sp with J). Owing to the complete analogy with the collinear limit studied in Ref. [33], we limit ourselves to present the final results for the soft limit.

The general all-order representation of the ϵ-pole contributions to J is

$$J(q_1, \ldots, q_m) = \mathbf{V}(q_1, \ldots, q_m, p_1, \ldots, p_n)$$
$$J^{[\text{no }\epsilon-\text{poles}]}(q_1, \ldots, q_m) \, \mathbf{V}^{-1}(p_1, \ldots, p_n), \qquad (19)$$

where $J^{[\text{no }\epsilon-\text{poles}]}$ is obtained from J by properly subtracting its ϵ-pole part order by order in the loop expansion. Therefore at each perturbative order $J^{[\text{no }\epsilon-\text{poles}]}$ is finite in the limit $\epsilon \to 0$ order by order in the ϵ expansion around $\epsilon = 0$. Note that this statement refers to the loop expansion of $J^{[\text{no }\epsilon-\text{poles}]}$ with respect to renormalized QCD coupling (the use of the renormalized QCD coupling removes ϵ poles of UV origin, which cannot be absorbed in the $\mathbf{V}$ factors of Eq. (19)). Obviously, at the tree level J and $J^{[\text{no }\epsilon-\text{poles}]}$ coincides ($J^{(0)} = J^{(0)\,[\text{no }\epsilon-\text{poles}]}$). The colour space operator $\mathbf{V}(q_1, \ldots, q_m, p_1, \ldots, p_n)$ is the process-independent factor [54,68–72] that controls the IR ϵ-pole contributions to the scattering amplitude $|\mathcal{M}(q_1, \ldots, q_m, p_1, \ldots, p_n)\rangle$ in the left-hand side of the factorization formula (5) (at the tree level, $\mathbf{V}^{(0)} = 1$). The operator $\mathbf{V}(p_1, \ldots, p_n)$ to the right of $J^{[\text{no }\epsilon-\text{poles}]}$ in Eq. (19) is the restriction of $\mathbf{V}$ to the scattering amplitude $|\mathcal{M}(p_1, \ldots, p_n)\rangle$ in the left-hand side of Eq. (5) (i.e., the amplitude with the external soft legs removed). Since the $\mathbf{V}$ factors in Eq. (19) only depends on the colour, flavour and momentum of the soft and hard partons, their perturbative knowledge determines in a recursive manner (i.e., order by order in the loop expansion) the ϵ-pole part of J at a given order in terms of J at lower perturbative orders.

At the one-loop level, using Eq. (19) and the known expression [22,54,73,74] of the one-loop term $\mathbf{V}^{(1)}$ of the operator $\mathbf{V}$, we obtain the following expression for the ϵ-pole contributions to the soft multiparton current $J^{(1)}$:

$$J^{(1)}(q_1, \ldots, q_m) = -g_S^2 \, c_\Gamma \left\{ \sum_{k \in S} \left[\frac{1}{\epsilon^2} C_k + \frac{1}{\epsilon} (\gamma_k - b_0) \right] \right.$$

$$\times J^{(0)}(q_1, \ldots, q_m) + \frac{1}{\epsilon} \left[\sum_{\substack{k,l \in S \\ k \neq l}} \ln\left(\frac{-2q_k \cdot q_l - i0}{\mu^2} \right) T_k \cdot T_l \right.$$

$$+ \sum_{\substack{i \in H \\ k \in S}} \ln\left(\frac{-2p_i \cdot q_k - i0}{\mu^2} \right) 2 \, T_i \cdot T_k \left. \right] J^{(0)}(q_1, \ldots, q_m)$$

$$- \frac{1}{\epsilon} \sum_{\substack{i,j \in H \\ i \neq j}} \ln\left(\frac{-2p_i \cdot p_j - i0}{\mu^2} \right)$$

$$\left. \times \left[J^{(0)}(q_1, \ldots, q_m), \, T_i \cdot T_j \right] \right\} + \mathcal{O}(\epsilon^0), \qquad (20)$$

where C_k ($T_k^2 = C_k$) is the Casimir coefficient of the parton k and, analogously, the coefficient γ_k depends on the flavour of the parton k and, explicitly, we have

$$\gamma_q = \gamma_{\bar{q}} = \frac{3}{2} \, C_F, \quad \gamma_g = \frac{1}{6} \, (11 \, C_A - 4T_R \, N_f), \qquad (21)$$

where N_f is the number of flavours of massless quarks. The coefficient b_0 is the first perturbative coefficient of the QCD β function,

$$b_0 = \frac{1}{6} \, (11 \, C_A - 4T_R \, N_f). \qquad (22)$$

Note that, in our normalization, we have $b_0 = \gamma_g$.

The various ϵ-pole terms in Eq. (20) have different origins. The single-pole term that is proportional to b_0 is of UV origin; it can be removed by renormalizing the soft current J (we recall that we are considering unrenormalized scattering amplitudes and, correspondingly, unrenormalized soft currents). The other ϵ-pole terms are of IR origin. The double-pole terms, which are proportional to the Casimir coefficients C_k ($C_k = C_F$ and C_A for quarks and gluons, respectively), originate from one-loop contributions in which the loop momentum is nearly on-shell, very soft and parallel to the momentum of one of the soft partons involved in the current. The single-pole terms with γ_k coefficients are produced by contributions in which the loop momentum is not soft, though it is nearly on-shell and parallel to the momentum of one of the external soft partons of the current. The single-pole terms with logarithmic dependence on soft-parton and hard-partons subenergies ($q_k \cdot q_l$, $p_i \cdot q_k$, $p_i \cdot p_j$) originate from configurations in which the loop momentum is very soft and at wide angle with respect to the direction of the external-leg (soft and hard) partons. Specifically, the radiative part of these terms (i.e., the real part of the logarithms) is due to a nearly on-shell virtual gluon in the loop, while the absorptive part (i.e., the imaginary part of the logarithms) is due to the exchange of an off-shell Coulomb-type gluon.

The expression in the right-hand side of Eq. (20) is valid for an *arbitrary* number m of soft partons in the current (and for an arbitrary number of hard partons in the scattering amplitude). This expression is given in terms of explicit coefficients and of the tree-level current $J^{(0)}$ for the corresponding parton configuration. Once $J^{(0)}$ (and, in particular, its colour structure) is explicitly known, Eq. (20) can be directly applied to determine the explicit ϵ-pole contributions to the one-loop current $J^{(1)}$.

In particular, in the case of a single ($m = 1$) soft parton, using Eq. (9) it is straightforward to check that Eq. (20) gives the ϵ-pole terms of the one-loop result $J^{(1)}$ in Eq. (16). In this respect, we note that the expression in the right-hand side of Eq. (20) does not identically (in its precise algebraic form) correspond to the ϵ-pole terms in Eq. (16): the difference is due to terms of $\mathcal{O}(1/\epsilon^2)$ and $\mathcal{O}(1/\epsilon)$ that are proportional to the total colour charge ($\sum_{i \in H} T_i$) of the hard partons. As previously discussed (see Eq. (13) and related comments) the presence of such terms in J is physically harmless.

The tree-level currents $J^{(0)}$ for emission of two ($m = 2$) soft partons (either two gluons or a $q\bar{q}$ pair) are also explicitly known [28]. Therefore Eq. (20) can also be straightforwardly applied to explicitly obtain the ϵ-pole terms of the one-loop current $J^{(1)}$ for double soft-parton emission. The case of a soft quark and antiquark is discussed in detail in Sect. 4. The case of two soft gluons is studied in Ref. [50]. We have checked that the ϵ-pole terms of the one-loop double-gluon current computed in Ref. [50] agree with the corresponding result that is obtained by using Eq. (20) and the colour conservation relation (10). To be precise about the absolute normalization of the one-loop current, we think that the factor $e^{-\epsilon \gamma_E}$ has to be removed from the expansion parameter in Eq. (3.2) of Ref. [50].

We comment on the behaviour of the one-loop current $J^{(1)}(q_1, \ldots, q_m)$ with respect to the overall rescaling $q_k \to \lambda q_k$ of all the momenta of the soft partons. To avoid the effects of branch-cut contributions from crossing different kinematical regions of soft and hard momenta, with limit ourselves to considering the case with $\lambda > 0$. According to Eq. (4) (see also the discussion below it) and Eq. (5), the tree-level current $J^{(0)}$ behaves as

$$J^{(0)}(\lambda q_1, \ldots, \lambda q_m) = \frac{1}{(\lambda)^m} J^{(0)}(q_1, \ldots, q_m), \qquad (23)$$

and the expected one-loop behaviour is

$$J^{(1)}(\lambda q_1, \ldots, \lambda q_m) = \frac{(\lambda)^{-2\epsilon}}{(\lambda)^m} J^{(1)}(q_1, \ldots, q_m), \quad (\lambda > 0). \qquad (24)$$

The behaviour as in Eqs. (23) and (24) is indeed observed in the tree-level results of Eqs. (9) and (14) and in the one-loop soft single-parton current of Eq. (16). Using Eq. (23) and applying the λ rescaling to the explicit expression

in the right-hand side of Eq. (20), we obtain the result $J^{(1)}(\lambda q_1, \ldots, \lambda q_m) \underset{\overline{\text{CS}}}{=} \frac{1}{(\lambda)^m}(1 - 2\epsilon \ln \lambda) J^{(1)}(q_1, \ldots, q_m) + \mathcal{O}(\epsilon^0)$ (note that we neglect harmless contributions proportional to the total colour charge of the hard partons). This result is perfectly consistent with Eq. (24), since Eq. (20) only embodies the correct ϵ-pole contributions to $J^{(1)}$. In particular, in Eq. (20) these contributions are embodied in a 'minimal' form by systematically neglecting terms of $\mathcal{O}(\epsilon^n)$ ($n \geq 0$), with the sole exception of terms that arise from the ϵ-expansion of the overall factor c_Γ ($(4\pi)^2 c_\Gamma = 1 + \mathcal{O}(\epsilon)$). The ϵ-pole contributions to $J^{(1)}$ can be expressed in alternative forms with respect to Eq. (20). In particular, the right-hand side of Eq. (20) can be supplemented with terms of $\mathcal{O}(\epsilon^n)$ ($n \geq 0$) in a manner that restores the behaviour in Eq. (24) to all orders in the ϵ expansion.

An alternative explicit form of the ϵ-pole contributions to $J^{(1)}$ for the soft multiparton ($m \geq 2$) limit is as follows

$$
\begin{aligned}
J^{(1)}(q_1, \ldots, q_m) \underset{\overline{\text{CS}}}{=} & -g_{\text{S}}^2 \left(\frac{-q_{1\ldots m}^2 - i0}{\mu^2} \right)^{-\epsilon} c_\Gamma \\
& \times \left\{ \sum_{k \in S} \left[\frac{1}{\epsilon^2} C_k + \frac{1}{\epsilon}(\gamma_k - b_0) \right] J^{(0)}(q_1, \ldots, q_m) \right. \\
& + \frac{1}{\epsilon} \left[\sum_{\substack{k,l \in S \\ k \neq l}} \ln\left(\frac{-2q_k \cdot q_l - i0}{-q_{1\ldots m}^2 - i0} \right) T_k \cdot T_l \right. \\
& \left. + \sum_{\substack{i \in H \\ k \in S}} \ell_{ik}(q_{1\ldots m}) \, 2\, T_i \cdot T_k \right] J^{(0)}(q_1, \ldots, q_m) \\
& \left. + \frac{1}{\epsilon} \sum_{\substack{i,j \in H \\ i \neq j}} L_{ij}(q_{1\ldots m}) \left[J^{(0)}(q_1, \ldots, q_m), \, T_i \cdot T_j \right] \right\} \\
& + \mathcal{O}(\epsilon^0), \quad (m \geq 2),
\end{aligned}
\qquad (25)
$$

where the total soft momentum is denoted by $q_{1\ldots m}$,

$$q_{1\ldots m} \equiv \sum_{k \in S} q_k = q_1 + \cdots + q_m, \qquad (26)$$

and we have introduced the logarithmic functions ℓ_{ik} and L_{ij} of hard and soft momenta:

$$\ell_{ik}(q_{1\ldots m}) \equiv \ln\left(\frac{-p_i \cdot q_k - i0}{-p_i \cdot q_{1\ldots m} - i0} \right), \qquad (27)$$

$$
\begin{aligned}
L_{ij}(q_{1\ldots m}) = L_{ji}(q_{1\ldots m}) \equiv & \ln\left(\frac{-p_i \cdot q_{1\ldots m} - i0}{-p_i \cdot p_j - i0} \right) \\
& + \ln\left(\frac{-2p_j \cdot q_{1\ldots m} - i0}{-q_{1\ldots m}^2 - i0} \right).
\end{aligned}
\qquad (28)
$$

It can be explicitly checked that the two expressions in Eqs. (20) and (25) only differ by terms of $\mathcal{O}(\epsilon^0)$ and higher orders in ϵ while acting onto colour singlet quantities. We note that the logarithmic functions ℓ_{ik} and L_{ij} in Eqs. (27) and (28) are invariant under the overall rescaling $q_k \to \lambda q_k$ ($\lambda > 0$) of the soft momenta. Therefore, the explicit expression in the right-hand side of Eq. (25) exactly fulfils the scaling behaviour in Eq. (24). We also note that both expressions of Eqs. (20) and (25) fulfil the colour flow conservation property in Eq. (11).

Throughout the paper we use the dimensional regularization procedure to deal with UV and IR divergences and, therefore, the momenta (and their associated phase space) of the virtual particles inside loops are analytically continued to $d = 4 - 2\epsilon$ space-time dimensions [75–79]. Different variants of dimensional regularization can be used, and each variant defines a specific regularization scheme (RS). The RSs that are mostly used are conventional dimensional regularization (CDR) [76–79], the 't Hooft–Veltman (HV) [75] scheme, dimensional reduction (DR) [80] and the four-dimensional helicity (4DH) scheme [81]. The momenta of the external-leg particles in the scattering amplitude can be either d-dimensional (CDR and DR schemes) or four-dimensional (HV and 4DH schemes). The number of spin polarization (helicity) states of the gluon also depends on the RS: external-leg gluons can have either $d - 2 = 2 - 2\epsilon$ polarizations (CDR) or 2 polarizations (HV, DR, 4DH), and virtual gluons can have either $d - 2 = 2 - 2\epsilon$ polarizations (CDR, HV) or 2 polarizations (DR, 4DH). Scattering amplitudes and, consequently, soft currents (as defined by the soft limit) depend on the RS. As for the RS dependence on external-leg particles, throughout the paper we formally express soft (tree-level and one-loop) currents in terms of external-leg momenta (p_i, q_k) and corresponding polarization wave functions $(\varepsilon(q_k), u(q_k), v(q_k))$: these expressions are formally RS invariant, although momenta and wave functions implicitly embody an RS dependence (which can be regarded as a dependence of $\mathcal{O}(\epsilon)$). At the one-loop level, soft currents (and scattering amplitudes) have a residual RS dependence that can be explicitly parametrized by the number of polarization states h_g of virtual gluons. We write $h_g = 2 - 2\epsilon \delta_R$ and, therefore, we have (this is the same notation as used, e.g., in Refs. [26,27])

$$\delta_R = 1 \,(\text{CDR, HV}), \quad \delta_R = 0 \,(\text{4DH, DR}). \tag{29}$$

To formally express the explicit δ_R dependence of the one-loop soft current $\boldsymbol{J}^{(1)}$ we then define

$$\boldsymbol{J}^{(1)}_{\text{RS}} \equiv \boldsymbol{J}^{(1)} - \left[\boldsymbol{J}^{(1)}\right]_{\delta_R = 1}, \tag{30}$$

where both terms in the right-hand side are expressed through the *same* formal external-leg variables (momenta and wave functions), which embody an implicit dependence on the RS, and $\left[\boldsymbol{J}^{(1)}\right]_{\delta_R = 1}$ is obtained by setting $\delta_R = 1$ in the explicit expression of $\boldsymbol{J}^{(1)}$. Roughly speaking (e.g., modulo the implicit RS dependence due to the number of polarizations of the external partons), $\boldsymbol{J}^{(1)}_{\text{RS}}$ in Eq. (30) represents the difference of $\boldsymbol{J}^{(1)}$ between a given RS and the CDR (or HV) scheme.

One-loop scattering amplitudes have an explicit RS dependence on δ_R. Considering the ϵ expansion up to including terms of $\mathcal{O}(\epsilon^0)$, the dependence on δ_R can be written in *factorized* form through the tree-level scattering amplitude and universal (process independent) coefficients [82,83]. Using these scattering amplitude results, we can obtain the explicit δ_R dependence of the one-loop current up to the same order in the ϵ expansion. In particular, the δ_R dependence of the one-loop scattering amplitude can be controlled through an ensuing δ_R dependence of the one-loop expression $\mathbf{V}^{(1)}$ [54,82–84] of the operator $\mathbf{V}$ in Eq. (19) and, therefore, we can explicitly compute the right-hand side of Eq. (30) up to $\mathcal{O}(\epsilon^0)$. The expression of $\boldsymbol{J}^{(1)}_{\text{RS}}$ for m soft partons is

$$
\begin{aligned}
\boldsymbol{J}^{(1)}_{\text{RS}}(q_1, \ldots, q_m) = {} & -(g_{\text{S}}\mu^\epsilon)^2\, c_\Gamma\, (\delta_R - 1) \\
& \times \sum_{k \in S} (\widetilde{\gamma}_k - \widetilde{b}_0)\, \boldsymbol{J}^{(0)}(q_1, \ldots, q_m) + \mathcal{O}(\epsilon),
\end{aligned}
\tag{31}
$$

where the coefficient $\widetilde{\gamma}_k$ depends on the flavour of the soft parton k and it has an IR origin, while the coefficient $\widetilde{b}_0$ has an UV origin. The explicit IR coefficients [82,83] and the UV coefficient [85] are

$$\widetilde{\gamma}_q = \widetilde{\gamma}_{\bar{q}} = \frac{1}{2}\, C_F \quad \widetilde{\gamma}_g = \widetilde{b}_0 = \frac{1}{6}\, C_A. \tag{32}$$

Note that, analogously to the structure of its ϵ-pole contributions (see Eqs. (20) and (25)), the δ_R dependence dependence of $\boldsymbol{J}^{(1)}$ has a factorized structure in terms of its corresponding tree-level current $\boldsymbol{J}^{(0)}$. Since $\widetilde{\gamma}_g = \widetilde{b}_0$, in the case of a single soft gluon ($m = 1$), Eq. (31) agrees with the explicit result in Eq. (16). Incidentally, we recall [29] that the result in Eq. (16) (to all orders in the ϵ expansion) is valid in any RS, and thus the expression of the single soft gluon current in Eq. (16) is basically RS invariant (it does not depend on δ_R, and the RS dependence is formally encoded in the corresponding RS dependence on $\varepsilon(q)$ and the dimensionality of the external momenta p_i, p_j, q). As shown in Eq. (31), in the soft multiparton ($m \geq 2$) case $\boldsymbol{J}^{(1)}_{\text{RS}}$ is of $\mathcal{O}(\epsilon^0)$. Conceptually, however, the RS dependence of $\boldsymbol{J}^{(1)}$ (and of one-loop scattering amplitudes) starts and $\mathcal{O}(1/\epsilon)$: the effect of $\mathcal{O}(1/\epsilon)$ is formally hidden in Eq. (20) (or Eq. (25)) through the product $\boldsymbol{J}^{(0)} \cdot 1/\epsilon^2$ ($\boldsymbol{J}^{(0)}$ conceptually embodies an RS dependence at $\mathcal{O}(\epsilon)$ through its external-leg momenta and polarization vectors).

Throughout the paper we explicitly consider unrenormalized scattering amplitudes and currents. However, UV renormalization commutes with the soft limit and, therefore, the renormalization procedure can be straightforwardly applied to all the explicit expressions presented herein. In particu-

lar, since we are considering amplitudes and currents with (on-shell) massless hard partons, the renormalization procedure simply amounts to replace the bare coupling g_S (or $\alpha_S = g_S^2/(4\pi)$) with its expression in terms of the renormalized running coupling $g_S(\mu_R)$ (or $\alpha_S(\mu_R)$) at the renormalization scale μ_R. In this respect, we recall that also the coupling renormalization is affected by RS subtleties. For instance, renormalizing the coupling by subtraction of the sole UV ϵ-poles (e.g., the term proportional to b_0 in Eq. (20)) in a given RS does not lead to an RS invariant definition of the renormalized coupling $g_S(\mu_R)$: an additional finite renormalization shift of $g_S(\mu_R)$ (whose size depends on the RS dependent coefficient $\delta_R \widetilde{b}_0$ in Eq. (32)) [85] is necessary to achieve an RS independent definition of $g_S(\mu_R)$.

As discussed and presented in this Section, the ϵ-pole contributions (and also the RS dependent contributions at $\mathcal{O}(\epsilon^0)$) to the one-loop current $\boldsymbol{J}^{(1)}$ for the general case of m ($m \geq 2$) soft partons and an arbitrary number of hard partons are completely determined by Eqs. (20) or (25) (and Eq. (31)), and they are explicitly known as soon as the corresponding tree-level current $\boldsymbol{J}^{(0)}$ is known. The determination of $\boldsymbol{J}^{(1)}$ at $\mathcal{O}(\epsilon^0)$ and, possibly, at higher orders in ϵ requires detailed one-loop computations and they have a high complexity. To have a rough idea of the computational complexity, we can simply observe that $\boldsymbol{J}^{(1)}$ can (in principle) be determined by performing the soft limit of one-loop amplitudes according to Eq. (8). To apply Eq. (8) we have to consider amplitudes with $m+n$ external legs, and the number n of non-soft external legs cannot be 'too small', otherwise the amplitude on the right-hand side of Eq. (8) vanishes. For example, the amplitude should have at least two external hard QCD partons (because of colour conservation) and one additional colourless external leg (because of momentum conservation): the soft limit of such an amplitude with $m+3$ external legs leads to the current $\boldsymbol{J}^{(1)}$ in the simplest case with two hard QCD partons. To get information on the colour-correlation structure of $\boldsymbol{J}^{(1)}$ in the general case of several hard partons, the amplitude should have at least $n = 4$ hard QCD partons in its external legs (owing to colour-conservation relations, the cases with $n = 2$ and 3 hard partons lead to simplified colour structures; see, e.g., Sect. 5.3). In summary, the amplitudes to be considered should have $m+n$ external legs with $n \geq 4$: even in the case of double soft-parton radiation ($m = 2$), this implies (at least) six external legs. As is well known, one-loop computations of these multileg scattering amplitudes are definitely complex to be carried out in analytic form (which is necessary to perform the soft limit). The computation of the one-loop current $\boldsymbol{J}^{(1)}$ can be highly simplified by using general methods (e.g., the method of Ref. [29])) that do not require a full direct computation of scattering amplitudes. However, despite some relevant simplification, even these methods have to deal with multileg one-loop Feynman integrals, whose evaluation is definitely complex, especially at high orders in the ϵ expansion.

4 Soft $q\bar{q}$ emission: the one-loop current

In this section we present and discuss the results of our explicit computation of the QCD one-loop current for soft $q\bar{q}$ radiation. In Sect. 6 we also generalize the results to the cases of QED and mixed QCD×QED radiative corrections.

The tree-level current $\boldsymbol{J}^{(0)}$ for emission of a soft-$q\bar{q}$ pair in a scattering amplitude with an *arbitrary* number of hard partons is given in Eq. (14). To evaluate the one-loop contribution $\boldsymbol{J}^{(1)}$ we use the general (process-independent) method of Ref. [29] (the same method is used in the computations of Refs. [31,32] and [51]). The computational procedure is completely analogous to that in Ref. [29] (though it is extended from the case of a single soft gluon to the case of a soft-$q\bar{q}$ pair) and we do not repeat all the details. We have to evaluate a set of one-loop Feynman diagrams (as example, in Fig. 1 we show two contributing Feynman diagrams) in which the external-leg hard partons are coupled to virtual gluons by using the eikonal approximation (for both vertices and propagators), while the other vertices and propagators are computed by using the customary QCD Feynman rules. We perform the calculation by using both the Feynman gauge and the axial gauge $n \cdot A = 0$, with an auxiliary light-like ($n^2 = 0$) gauge vector n^μ. Combining all the contributing Feynman diagrams, the dependence on the gauge vector cancels at the integrand level (i.e., before performing the integration over the loop momentum) and the total axial-gauge integrand coincides with the Feynman gauge integrand: this provides us with an explicit check of the gauge invariance of the procedure and of the calculation.

As usual in the context of dimensional regularization, scaleless one-loop integrals vanish. Eventually we have to compute several (non-vanishing) tensor, vector and scalar one-loop Feynman integrals. Tensor and vector integrals are expressed in terms of scalar integrals by using customary techniques [86]. One-loop integrals with five external legs (pentagon integrals; see, e.g. the Feynman diagram in Fig. 1a) are expressed [87,88] in terms of one-loop integrals with four external legs (box integrals) plus remaining pentagon integrals in $6-2\epsilon$ space-time dimensions, which only contribute at $\mathcal{O}(\epsilon)$ (and higher orders in ϵ). We do not explicitly evaluate these contributions at $\mathcal{O}(\epsilon)$. We eventually express the complete result in terms of a minimal set of basic one-loop scalar integrals. The set involves customary two-point and three-point (with at least one on-shell leg) Feynman integrals and some soft box integrals (box integrals with eikonal propagators). Part of these soft box integrals was already computed in Ref. [29] and the additional integrals are analogous to those encountered in Ref. [89]. We have performed an independent calculation of these soft box integrals and we find agreement with the results reported in Ref. [89] (see 'soft box 2' and 'soft box 4' in Sect. 4.2 of Ref. [89]). Our final result for the one-loop current $\boldsymbol{J}^{(1)}$ is reported below.
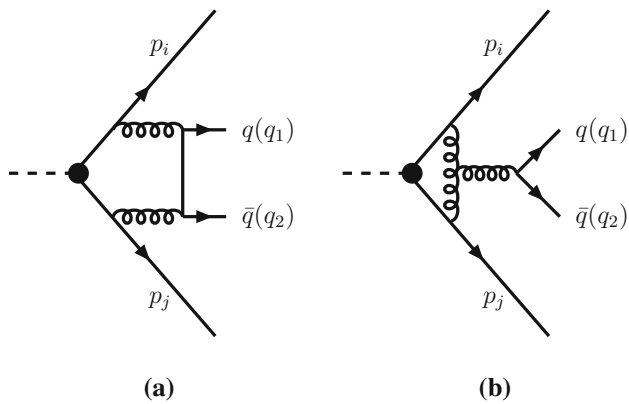
Fig. 1 Example of two one-loop Feynman diagrams that contribute to the one-loop current $\boldsymbol{J}^{(1)}$ for soft $q\bar{q}$ emission. The external-leg hard partons with momenta p_i and p_j are coupled to virtual gluons by using the eikonal approximation. The dashed line symbolically denotes the colour indices and momenta of the additional external legs. The effect of the tree-level scattering amplitude $\mathcal{M}^{(0)}(p_1, \ldots, p_n)$ (see Eq. (8)) is given by an effective pointlike vertex (little black circle) that only depends on the colour structure of $\mathcal{M}^{(0)}$ (on the colour indices of the hard partons)

To present our results, we first define the tree-level and one-loop rescaled currents $\hat{\boldsymbol{J}}^{(0)}$ and $\hat{\boldsymbol{J}}^{(1)}$ as follows

$$\boldsymbol{J}^{(0)}(q_1, q_2) = \left(g_S \mu^\epsilon\right)^2 \ \hat{\boldsymbol{J}}^{(0)}(q_1, q_2), \tag{33}$$

$$\boldsymbol{J}^{(1)}(q_1, q_2) = \left(g_S \mu^\epsilon\right)^4 \left(-q_{12}^2 - i0\right)^{-\epsilon} c_\Gamma \ \hat{\boldsymbol{J}}^{(1)}(q_1, q_2), \tag{34}$$

$$\hat{\boldsymbol{J}}^{(1)}(q_1, q_2) = \hat{\boldsymbol{J}}^{(1, \text{div})}(q_1, q_2) + \hat{\boldsymbol{J}}^{(1, \text{fin})}(q_1, q_2), \tag{35}$$

where $\hat{\boldsymbol{J}}^{(1)}$ is written in terms of two components, $\hat{\boldsymbol{J}}^{(1, \text{div})}$ and $\hat{\boldsymbol{J}}^{(1, \text{fin})}$. The rescaled current $\hat{\boldsymbol{J}}^{(0)}$ can be read from comparing Eqs. (14) and (33). The component $\hat{\boldsymbol{J}}^{(1, \text{div})}$ of Eq. (35) embodies the ϵ-pole contributions to $\boldsymbol{J}^{(1)}$, while $\hat{\boldsymbol{J}}^{(1, \text{fin})}$ includes all the remaining UV/IR finite contributions at $\mathcal{O}(\epsilon^0)$ and higher orders in ϵ. The explicit expressions of $\hat{\boldsymbol{J}}^{(1, \text{div})}$ and $\hat{\boldsymbol{J}}^{(1, \text{fin})}$ are

$$\hat{\boldsymbol{J}}^{(1, \text{div})}(q_1, q_2) = -2 \left[\frac{1}{\epsilon^2} C_F \right.$$
$$+ \frac{1}{\epsilon} \left(\frac{3}{2} C_F - \frac{1}{6} (11 C_A - 4 T_R N_f) \right) \right]$$
$$\times \hat{\boldsymbol{J}}^{(0)}(q_1, q_2) - \frac{2}{\epsilon} j_\nu(1, 2) \, \boldsymbol{t}^a \boldsymbol{t}^b \sum_{\substack{i, j \in H \\ i \neq j}} T_i^a T_j^b$$
$$\times \left(\frac{p_i^\nu}{p_i \cdot q_{12}} - \frac{p_j^\nu}{p_j \cdot q_{12}} \right) \left(L_{ij} + \ell_{i1} + \ell_{j2} \right), \tag{36}$$

$$\hat{\boldsymbol{J}}^{(1, \text{fin})}(q_1, q_2) = \left[\left(-8 - (\delta_R - 1) \right) C_F \right.$$

$$+ \left(\frac{76}{9} - \frac{\pi^2}{3} + \frac{1}{3} (\delta_R - 1) \right) C_A - \frac{20}{9} T_R N_f \right]$$
$$\times \hat{\boldsymbol{J}}^{(0)}(q_1, q_2) + j_\nu(1, 2) \, \boldsymbol{t}^a \boldsymbol{t}^b \sum_{\substack{i, j \in H \\ i \neq j}} T_i^a T_j^b$$
$$\times \left[\left(\frac{p_i^\nu}{p_i \cdot q_{12}} - \frac{p_j^\nu}{p_j \cdot q_{12}} \right) \left(L_{ij}^2 + (\ell_{i1} - \ell_{j2})^2 \right) \right.$$
$$+ \frac{q_{12}^2}{q_{12\perp ij}^2} \left(\frac{p_i^\nu}{p_i \cdot q_{12}} + \frac{p_j^\nu}{p_j \cdot q_{12}} \right) 2 L_{ij} \left(\ell_{i1} - \ell_{j2} \right) \right]$$
$$+ \mathcal{O}(\epsilon), \tag{37}$$

where we have used the logarithmic functions of Eqs. (27) and (28) and we have introduced the shorthand notation $\ell_{ik}(q_{12}) \equiv \ell_{ik}$ (with $k = 1, 2$) and $L_{ij}(q_{12}) \equiv L_{ij}$ (i.e., we omit the explicit dependence on the argument q_{12}). The kinematical variable $q_{12\perp ij}^2$ that is used in Eq. (37) is

$$q_{12\perp ij}^2 = \frac{2(p_i \cdot q_{12})(p_j \cdot q_{12})}{p_i \cdot p_j} - q_{12}^2. \tag{38}$$

We remark that the results in Eqs. (36) and (37) are valid in *arbitrary* kinematical regions, since the time component ('energy') of the outgoing momenta $\{q_1, q_2, p_i, p_j\}$ of the soft and hard partons can have an arbitrary sign. According to the notation in Eq. (14) the colour indices α_1 and α_2 of the soft quark and antiquark are specified by considering $\langle \alpha_1, \alpha_2 | \boldsymbol{J}^{(1)}(q_1, q_2) \equiv \boldsymbol{J}^{(1) \alpha_1, \alpha_2}(q_1, q_2)$, and this leads to $\langle \alpha_1, \alpha_2 | \boldsymbol{t}^a \boldsymbol{t}^b = (t^a t^b)_{\alpha_1 \alpha_2}$ in Eqs. (36) and (37).

The result in Eq. (36), which follows from our direct computation of $\boldsymbol{J}^{(1)}$, agrees with the ϵ-pole contributions that can straightforwardly be obtained by applying the general results in Eqs. (20) or (25) to the specific case of a soft $q\bar{q}$ pair (note that this agreement is valid modulo harmless terms that are proportional to the total colour charge $\sum_{i \in H} \boldsymbol{T}_i$ of the hard partons). The expression in Eq. (36) has a term that is directly proportional to $\boldsymbol{J}^{(0)}$ and additional terms that involve colour (and kinematical) correlations of the soft $q\bar{q}$ pair with *two* hard partons. These colour correlations are produced by the colour matrix factor $\boldsymbol{t}^a \boldsymbol{t}^b T_i^a T_j^b$. We remark that these correlations are not purely non-abelian, but they also include a component that is still present in the *abelian* limit of commuting colour matrices (this feature has to be contrasted with the one-loop single soft-gluon case of Eq. (16), in which correlations are purely non-abelian). In particular, this also implies that the soft current for lepton-antilepton radiation in massless QED has non-vanishing QED radiative corrections at one-loop level (see Sect. 6). The kinematical coefficients of these colour-correlation terms are proportional to the momentum function

$$L_{ij} + \ell_{i1} + \ell_{j2} = \ln\left(\frac{-p_i \cdot q_1 - i0}{-p_i \cdot p_j - i0}\right)$$
$$+ \ln\left(\frac{-p_j \cdot q_2 - i0}{-q_1 \cdot q_2 - i0}\right), \qquad (39)$$

whose real part is the logarithm of a conformally invariant cross ratio, namely,

$$\text{Re}(L_{ij} + \ell_{i1} + \ell_{j2}) = \ln\left(\frac{|p_i \cdot q_1|\,|p_j \cdot q_2|}{|p_i \cdot p_j|\,|q_1 \cdot q_2|}\right). \qquad (40)$$

We note that (analogously to the treatment in Sect. 3) in the computation of $\hat{\boldsymbol{J}}^{(1)}(q_1, q_2)$ we have dressed gluon propagators with one-loop vacuum polarization effects that are due only to massless partons. In particular, the terms proportional to $N_f \hat{\boldsymbol{J}}^{(0)}$ in the right-hand side of Eqs. (36) and (37) are due to the vacuum polarization of N_f massless quarks. Vacuum polarization effects of *massive* quarks can straightforwardly be included in $\hat{\boldsymbol{J}}^{(1)}(q_1, q_2)$, and they produce corresponding (mass-dependent) contributions that are proportional to $\hat{\boldsymbol{J}}^{(0)}(q_1, q_2)$.

We comment on the structure of $\hat{\boldsymbol{J}}^{(1,\,\text{fin})}$. We have explicitly computed it up to $\mathcal{O}(\epsilon^0)$ and the result is presented in Eq. (37). The expression of $\hat{\boldsymbol{J}}^{(1,\,\text{fin})}$ at $\mathcal{O}(\epsilon^0)$ is quite compact and remarkably much simpler than expected. In particular, although it involves momentum functions of trascendentality equal to two, they are only powers of logarithmic functions with no additional dependence on dilogarithms Li_2. Dilogarithms do appear in the computation of individual Feynman diagrams and loop integrals at $\mathcal{O}(\epsilon^0)$, but they cancel in the complete result for $\hat{\boldsymbol{J}}^{(1,\,\text{fin})}$. The finite component $\hat{\boldsymbol{J}}^{(1,\,\text{fin})}$ includes a term that is proportional to $\boldsymbol{J}^{(0)}$ and additional correlation terms with *two* hard partons whose colour structure is exactly analogous to that in Eq. (36) (and it embodies both abelian and non-abelian components). We have explicitly checked that no different colour-correlation structures occur at *any* higher orders in the ϵ expansion. The term that is proportional to $\boldsymbol{J}^{(0)}$ explicitly depends on the RS parameter δ_R: this dependence exactly agrees with that of the general result in Eq. (31).

We also comment on the kinematical dependence of the colour correlation terms. At the tree level the soft-$q\bar{q}$ current $\boldsymbol{J}^{(0)}$ has a kinematical structure with a rational dependence on $j(1,2) \cdot p_i / p_i \cdot q_{12}$ (see Eqs. (14) and (15)). In particular, this dependence leads to a collinear singularity if $q_{12}^2 \to 0$ (i.e., if the momenta of the soft quark and antiquark are parallel). Exactly the same rational dependence (though possibly modified by logarithmic factors) occurs in the one-loop contributions $\hat{\boldsymbol{J}}^{(1,\,\text{div})}$ and $\hat{\boldsymbol{J}}^{(1,\,\text{fin})}$. However, by inspection of Eq. (37) we see that the one-loop interaction at $\mathcal{O}(\epsilon^0)$ also produces a different type of kinematical dependence as given by the factor $j(1,2) \cdot p_i \, q_{12}^2/(p_i \cdot q_{12}\, q_{12\perp ij}^2)$. This rational

factor has no collinear singularity at $q_{12}^2 \to 0$, but it potentially leads to a singularity in the limit $q_{12\perp ij}^2 \to 0$. This is a 'transverse-momentum singularity', since the kinematical variable $\sqrt{q_{12\perp ij}^2}$ in Eq. (38) is the transverse component of the momentum q_{12} of the soft $q\bar{q}$ pair with respect to the momenta p_i and p_j of the colour-correlated hard partons in a reference frame in which p_i and p_j are back-to-back.

The transverse-momentum singularity in the current is partly screened by the logarithmic function L_{ij}, and we have

$$\frac{1}{q_{12\perp ij}^2} L_{ij} \underset{q_{12\perp ij}\to 0}{\simeq} \frac{1}{q_{12\perp ij}^2}\left[2\pi i\,\text{sign}(q_{12}^2)\,\Theta\left(\frac{-p_i \cdot q_{12}}{p_i \cdot p_j}\right)\right.$$
$$\left.\Theta\left(\frac{-p_j \cdot q_{12}}{p_i \cdot p_j}\right) + \mathcal{O}\left(\frac{q_{12\perp ij}^2}{q_{12}^2}\right)\right], \qquad (41)$$

which shows that the current has a one-loop singularity of absorptive origin. Considering the physically most relevant kinematical region in which the soft quark and antiquark are produced in the final state ($q_1^0 > 0, q_2^0 > 0$), Eq. (41) becomes

$$\frac{1}{q_{12\perp ij}^2} L_{ij} \underset{q_{12\perp ij}\to 0}{\simeq} \frac{1}{q_{12\perp ij}^2}\, 2\pi i\,\Theta(-p_i^0)\,\Theta(-p_j^0),$$
$$(q_1^0 > 0,\ q_2^0 > 0), \qquad (42)$$

which shows that the transverse-momentum singularity is present in the scattering amplitude of a physical process in which the final-state soft $q\bar{q}$ pair is produced by the collision of the hard partons i and j in the initial state. We remark that this singularity has a pure quantum mechanics (loop) origin, and it occurs in the limit $q_{12\perp ij}^2 \to 0$ even if the transverse momenta $q_{1\perp ij}$ and $q_{2\perp ij}$ ($q_{k\perp ij}^2 = 2p_i \cdot q_k\, p_j \cdot q_k/p_i \cdot p_j$, $k = 1, 2$) of the soft quark and antiquark are separately large (i.e., they are separately non-vanishing) and q_{12}^2 is large. We also note that, setting $q_{12\perp ij} = 0$ at fixed non-vanishing values of q_{12}^2 and $q_{1\perp ij}$ (or $q_{2\perp ij}$), we have $(\ell_{i1} - \ell_{j2}) = -(\ell_{j1} - \ell_{i2})$. Therefore, in the limit $q_{12\perp ij}^2 \to 0$ the factor $\ell_{i1} - \ell_{j2}$ is (approximately) antisymmetric with respect to the exchange $p_i \leftrightarrow p_j$ and this implies that we can perform the following replacement in Eq. (37):

$$\boldsymbol{t}^a \boldsymbol{t}^b\, T_i^a\, T_j^b\, \frac{L_{ij}}{q_{12\perp ij}^2}\left(\ell_{i1} - \ell_{j2}\right) \underset{q_{12\perp ij}\to 0}{\longrightarrow}$$
$$-f^{abc}\boldsymbol{t}^c\, T_i^a\, T_j^b\, \frac{\pi\,\Theta(-p_i^0)\,\Theta(-p_j^0)}{q_{12\perp ij}^2}\left(\ell_{i1} - \ell_{j2}\right),$$
$$(q_1^0, q_2^0 > 0), \qquad (43)$$

and it follows that, in the kinematical region with $q_1^0 > 0$ and $q_2^0 > 0$, the transverse-momentum singularity has a *purely* non-abelian character (see the factor f^{abc} in the right-hand side of the relation (43)).

As we have just discussed, the singularity of the soft-$q\bar{q}$ current in the limit $q_{12\perp ij} \to 0$ originates from one-loop

interactions of the two soft partons. Therefore, we expect the presence of the transverse-momentum singularity also in the case of double soft-gluon emission at one-loop level. The one-loop double-gluon current computed in Ref. [50] indeed shows singular terms at $q_{12\perp ij} \to 0$.

We have also computed the soft-$q\bar{q}$ one-loop current $\boldsymbol{J}^{(1)}$ by explicitly evaluating its dependence on the RS parameter δ_R to *all* orders in ϵ. Using the notation of Eq. (30) and considering the rescaled currents in Eqs. (33) and (35), we find the result

$$\hat{\boldsymbol{J}}_{\mathrm{RS}}^{(1,\,\mathrm{fin})}(q_1, q_2) = - (\delta_R - 1)\frac{1}{1-\epsilon}\left[C_F - C_A \right.$$
$$\left. \times \frac{1 - 4\epsilon + 2\epsilon^2}{(1-2\epsilon)(3-2\epsilon)} \right] \hat{\boldsymbol{J}}^{(0)}(q_1, q_2). \tag{44}$$

Note that the δ_R dependence at one-loop order is completely factorized with respect to $\boldsymbol{J}^{(0)}$. We also note that this factorized structure and the explicit expression of the ϵ-dependent factor in Eq. (44) are exactly equal to the corresponding RS dependence of the splitting function, $\mathrm{Split}(g \to q(q_1)\bar{q}(q_2))$, of one-loop scattering amplitudes for radiation of a $q\bar{q}$ pair in the collinear limit [26,27,34].

The one-loop current $\boldsymbol{J}^{(1)}$ for soft-$q\bar{q}$ emission has been independently computed in Ref. [50], and the corresponding result is presented in Sect. 3.3 therein. We first note that the one-loop result of Ref. [50] differs from our result already at the level of ϵ-pole contributions. However, we also note that we can remove such difference by adjusting the relative size of the four contributions in the right-hand side of Eq. (3.20) of Ref. [50]. More precisely, we modify the size of $\mathcal{M}^{\mathrm{s.l.}}$ by applying the replacement $\left(-\frac{4}{N_c} + \frac{N_c}{2}\right) \to \left(-C_F + \frac{N_c}{2}\right) = \frac{1}{2N_c}$ to its colour coefficient (see the line 10 of Eq. (3.21)). We have contacted the author of Ref. [50] and he agreed with this correction. Performing such replacement, we have explicitly checked that the expression of $\boldsymbol{J}^{(1)}$ in Eq.(3.20) of Ref. [50] agrees with our result (modulo the overall normalization of the one-loop current, which is not clearly specified in Ref. [50]) for both the ϵ-pole terms and the finite contributions at $\mathcal{O}(\epsilon^0)$. However, we note that this check and comparison involve some 'limitations'. The explicit result of Ref. [50] only refers to the 'time-like' region, namely to the kinematical region in which the soft partons and all the hard partons are physically produced in the final state. Moreover, the result of Ref. [50] is specified for fixed (four dimensional) helicities of the soft quark and antiquark, and the comparison with our result requires the repeated use of the Schouten identity (which, precisely speaking, is valid only in four space-time dimensions) for the product of helicity spinors.

5 Soft $q\bar{q}$ radiation: squared amplitudes and current

Using the colour+spin space notation of Sect. 2, the squared amplitude $|\mathcal{M}|^2$ (summed over the colours and spins of its external legs) is written as follows

$$|\mathcal{M}|^2 = \langle \mathcal{M} | \mathcal{M} \rangle. \tag{45}$$

Accordingly, the square of the soft-emission factorization formula in Eq. (5) gives

$$|\mathcal{M}(q_1, \ldots, q_m, p_1, \ldots, p_n)|^2$$
$$\simeq \langle \mathcal{M}(p_1, \ldots, p_n) | \, |\boldsymbol{J}(q_1, \ldots, q_m)|^2 \, |\mathcal{M}(p_1, \ldots, p_n) \rangle, \tag{46}$$

where, analogously to Eqs. (7) and (8), the symbol $\simeq$ means that we have neglected contributions that are subdominant in the soft multiparton limit (i.e., the contributions that are denoted by the dots on the right-hand side of Eq. (5)). In the right-hand side of Eq. (46), $|\boldsymbol{J}|^2$ denotes the all-loop squared current summed over the colours $\{c_1, \ldots, c_m\}$ and spins $\{s_1, \ldots, s_m\}$ of the soft partons:

$$|\boldsymbol{J}(q_1, \ldots, q_m)|^2 = \left[J_{s_1,\ldots,s_m}^{c_1,\ldots,c_m}(q_1, \ldots, q_m) \right]^\dagger$$
$$J_{s_1,\ldots,s_m}^{c_1,\ldots,c_m}(q_1, \ldots, q_m)$$
$$\equiv [\boldsymbol{J}(q_1, \ldots, q_m)]^\dagger \, \boldsymbol{J}(q_1, \ldots, q_m). \tag{47}$$

The squared current $|\boldsymbol{J}|^2$ is a colour operator that depends on the colour charges (and momenta) of the hard partons in $\mathcal{M}(p_1, \ldots, p_n)$. These colour charges produce colour correlations and, therefore, the right-hand side of Eq. (46) is not proportional to $|\mathcal{M}(p_1, \ldots, p_n)|^2$ in the case of a generic scattering amplitude[3]. As remarked on in Sect. 2, $\boldsymbol{J}$ is simply proportional to the unit operator in the spin subspace of the hard partons. Therefore, we note that the squared current $|\boldsymbol{J}|^2$ of Eq. (47) still applies to *spin-polarized* hard-scattering processes, namely, to processes in which the spin polarizations of the *hard* partons are fixed (rather than summed over). Obviously, Eqs. (45)–(47) can also be properly generalized to the case in which the spin polarizations of one or more soft partons are fixed.

In the following part of this Section, we only consider soft-$q\bar{q}$ radiation and the corresponding soft current $\boldsymbol{J}(q_1, q_2)$ (see Eq. (14) and Sect. 4). We define the loop expansion of the squared current as follows

$$|\boldsymbol{J}(q_1, q_2)|^2 \equiv \left(g_S \mu^\epsilon\right)^4 |\hat{\boldsymbol{J}}(q_1, q_2)|^2_{(0\ell)}$$
$$+ \left(g_S \mu^\epsilon\right)^6 \left(|q_{12}^2|\right)^{-\epsilon} c_\Gamma \, |\hat{\boldsymbol{J}}(q_1, q_2)|^2_{(1\ell)}$$
$$+ \mathcal{O}(g_S^8), \tag{48}$$

---

[3] Colour correlations can be simplified in the case of scattering amplitudes with two and three hard partons (see Sect. 5.3).

where $|\hat{\boldsymbol{J}}|^2_{(0\ell)}$ and $|\hat{\boldsymbol{J}}|^2_{(1\ell)}$ are the tree-level (0 loop) and one-loop rescaled contributions to $|\boldsymbol{J}|^2$, respectively.

5.1 The tree-level squared current

The tree-level squared current in Eq. (48) is

$$|\hat{\boldsymbol{J}}(q_1, q_2)|^2_{(0\ell)} = \left[\hat{\boldsymbol{J}}^{(0)}(q_1, q_2)\right]^\dagger \hat{\boldsymbol{J}}^{(0)}(q_1, q_2), \tag{49}$$

where $\hat{\boldsymbol{J}}^{(0)}$ is the rescaled current in Eqs. (14) and (33). The computation of the right-hand side of Eq. (49) is straightforward and the explicit result was first presented in Sect. 3.2 of Ref. [28]. We have

$$
\begin{aligned}
&|\hat{\boldsymbol{J}}(q_1, q_2)|^2_{(0\ell)} \\
&= T_R \sum_{i,j \in H} \boldsymbol{T}_i \cdot \boldsymbol{T}_j \ \mathcal{I}_{ij}(q_1, q_2),
\end{aligned}
\tag{50}
$$

where the momentum-dependent function $\mathcal{I}_{ij}(q_1, q_2)$ is (see Eq. (96) in Ref. [28])

$$
\begin{aligned}
&\mathcal{I}_{ij}(q_1, q_2) \\
&= \frac{(p_i \cdot q_1)(p_j \cdot q_2) + (p_j \cdot q_1)(p_i \cdot q_2) - (p_i \cdot p_j)(q_1 \cdot q_2)}{(q_1 \cdot q_2)^2 (p_i \cdot q_{12})(p_j \cdot q_{12})}.
\end{aligned}
\tag{51}
$$

Using colour charge conservation (see Eq. (10)), the tree-level squared current $|\hat{\boldsymbol{J}}|^2_{(0\ell)}$ can be recast in the following different form

$$|\hat{\boldsymbol{J}}(q_1, q_2)|^2_{(0\ell)} \underset{\overline{\text{cs}}}{=} -\frac{1}{2} T_R \sum_{\substack{i,j \in H \\ i \neq j}} \boldsymbol{T}_i \cdot \boldsymbol{T}_j \ w_{ij}(q_1, q_2), \tag{52}$$

where the soft function w_{ij} is

$$w_{ij}(q_1, q_2) = \mathcal{I}_{ii}(q_1, q_2) + \mathcal{I}_{jj}(q_1, q_2) - 2\mathcal{I}_{ij}(q_1, q_2). \tag{53}$$

The expressions in the right-hand side of Eqs. (50) and (52) are not identical at the algebraic level, but they are fully equivalent by acting onto scattering amplitudes (or, generically, colour-singlet states). The expression in Eq. (52) has a more straightforward physical interpretation, since the function $w_{ij}(q_1, q_2)$ is directly related (see Sect. 5.3.1) to the intensity of soft-$q\bar{q}$ radiation from two hard partons, i and j, in a colour-singlet configuration.

The tree-level squared current in Eqs. (50) or (52) produces *two-particle* correlations between the hard partons. Their colour structure has the form of *dipole* contributions $\boldsymbol{T}_i \cdot \boldsymbol{T}_j$. We note that the momentum-dependent functions $\mathcal{I}_{ij}(q_1, q_2)$ and $w_{ij}(q_1, q_2)$ are *symmetric* with respect to the exchange $q_1 \leftrightarrow q_2$ (they are also symmetric with respect to $p_i \leftrightarrow p_j$). In contrast, our result for the one-loop squared

current (see Sect. 5.2) produces both two-particle and three-particle correlations and, moreover, it involves also an antisymmetric dependence on the momenta q_1 and q_2.

5.2 The one-loop squared current

The one-loop squared current in Eq. (48) is

$$
\begin{aligned}
\hat{\boldsymbol{J}}(q_1, q_2)|^2_{(1\ell)} &= \left(\frac{-q_{12}^2 - i0}{|q_{12}^2|}\right)^{-\epsilon} \\
&\times \left[\hat{\boldsymbol{J}}^{(0)}(q_1, q_2)\right]^\dagger \hat{\boldsymbol{J}}^{(1)}(q_1, q_2) + \text{h.c.},
\end{aligned}
\tag{54}
$$

where 'h.c' denotes the hermitian-conjugate contribution, and the rescaled currents $\hat{\boldsymbol{J}}^{(0)}$ and $\hat{\boldsymbol{J}}^{(0)}$ are defined in Eqs. (33) and (34).

The explicit computation of Eq. (54) produces some contributions that involve the fully-symmetric colour tensor d^{abc},

$$d^{abc} = \frac{1}{T_R} \text{Tr}\left(\{t^a, t^b\} t^c\right). \tag{55}$$

with indices $\{a, b, c\}$ in the adjoint representation of $SU(N_c)$. The presence of d^{abc} is a distinctive feature of (squared) currents for radiation of soft quarks and antiquarks.

Using d^{abc} we also define the d-conjugated (quadratic) charge operator $\widetilde{\boldsymbol{D}}_i$ of the parton i as follows

$$\widetilde{D}_i^a \equiv d^{abc} T_i^b T_i^c. \tag{56}$$

Performing the $SU(N_c)$ colour algebra, we explicitly find

$$i = q : \widetilde{D}_i^a = \frac{1}{2} d_A T_i^a, \tag{57}$$

$$i = \bar{q} : \widetilde{D}_i^a = -\frac{1}{2} d_A T_i^a, \tag{58}$$

$$i = g : \widetilde{D}_i^a = \frac{1}{2} C_A D_i^a, \quad \langle b|D^a|c\rangle = d^{bac}, \tag{59}$$

where we have used

$$d^{abc} d^{dbc} = d_A \delta^{ad}, \quad d_A = \frac{N_c^2 - 4}{N_c}. \tag{60}$$

Note that the tensor d^{abc} is *odd* under charge conjugation. This fact is responsible for the opposite overall sign between the d-charge $\widetilde{\boldsymbol{D}}_i$ and the colour charge $\boldsymbol{T}_i$ of quarks and antiquarks (see Eqs. (57) and (58)). Analogously, in the gluon case the d-charge $\langle b|D_i^a|c\rangle$ in Eq. (59) is symmetric with respect to $b \leftrightarrow c$, while the colour charge $\langle b|T_i^a|c\rangle$ is antisymmetric with respect to $b \leftrightarrow c$.

The explicit expression of the one-loop squared current $|\hat{\boldsymbol{J}}(q_1, q_2)|^2_{(1\ell)}$ is obtained by inserting $\hat{\boldsymbol{J}}^{(0)}$ (see Eqs. (14) and (33)) and $\hat{\boldsymbol{J}}^{(1)}$ (see Eqs. (35)–(37)) in the right-hand side of Eq. (54), and by performing the sum over the colours and

spins of the soft quark and antiquark. We find the following result:

$$
\begin{aligned}
|\hat{\boldsymbol{J}}(q_1, q_2)|^2_{(1\ell)} = &-\frac{1}{2} T_R \sum_{\substack{i,j \in H \\ i \neq j}} \Big[\boldsymbol{T}_i \cdot \boldsymbol{T}_j \; w_{ij}^{[S]}(q_1, q_2) \\
&+ \widetilde{\boldsymbol{D}}_i \cdot \boldsymbol{T}_j \; w_{ij}^{[A]}(q_1, q_2) \Big] \\
&- T_R \sum_{\substack{i,j,k \in H \\ \text{dist.}\{i,j,k\}}} T_i^a T_j^b T_k^c \Big[f^{abc} F_{ijk}^{[S]}(q_1, q_2) \\
&+ d^{abc} \Big(F_{ijk}^{[A]}(q_1, q_2) - \frac{1}{2} F_{iji}^{[A]}(q_1, q_2) \\
&- \frac{1}{2} F_{ijj}^{[A]}(q_1, q_2) \Big) \Big],
\end{aligned}
\tag{61}
$$

which is valid to arbitrary orders in the ϵ expansion. The ϵ dependence is embodied in the *c*-number functions $w^{[S]}$, $w^{[A]}$, $F^{[S]}$ and $F^{[A]}$. The dependence on the colours of the hard partons is due to the colour charges T_i^a and $\widetilde{D}_i^a$. The structure of Eq. (61) involves contributions with both *two* hard-parton correlations and *three* hard-parton correlations. In the case of three hard-parton correlations, the subscript '$\text{dist.}\{i,j,k\}$' in $\sum_{\substack{i,j,k \in H \\ \text{dist.}\{i,j,k\}}}$ denotes the sum over *distinct* hard-parton indices i, j and k (i.e., $i \neq j$, $j \neq k$, $k \neq i$).

The functions $w_{ij}^{[S]}$, $w_{ij}^{[A]}$, $F_{ijk}^{[S]}$ and $F_{ijk}^{[A]}$ in Eq. (61) depend on the momenta of the hard partons and on the momenta q_1 and q_2 of the soft quark and antiquark. The superscript [S] in $w_{ij}^{[S]}$ and $F_{ijk}^{[S]}$ denotes the fact that these functions are *symmetric* under the exchange $q_1 \leftrightarrow q_2$ of the momenta of the soft quark and antiquark:

$$
\begin{aligned}
w_{ij}^{[S]}(q_1, q_2) &= w_{ij}^{[S]}(q_2, q_1), \\
F_{ijk}^{[S]}(q_1, q_2) &= F_{ijk}^{[S]}(q_2, q_1).
\end{aligned}
\tag{62}
$$

Analogously, the superscript [A] in $w_{ij}^{[A]}$ and $F_{ijk}^{[A]}$ highlights the fact that these functions are *antisymmetric* under the exchange $q_1 \leftrightarrow q_2$:

$$
\begin{aligned}
w_{ij}^{[A]}(q_1, q_2) &= -w_{ij}^{[A]}(q_2, q_1), \\
F_{ijk}^{[A]}(q_1, q_2) &= -F_{ijk}^{[A]}(q_2, q_1).
\end{aligned}
\tag{63}
$$

Therefore, $w^{[A]}$ and $F^{[A]}$ produce a quark–antiquark *charge asymmetry* in the one-loop squared current. We note that the charge-asymmetry contributions appear in Eq. (61) with the associated colour factors $\widetilde{\boldsymbol{D}}_i \cdot \boldsymbol{T}_j = d^{abc} T_i^a T_i^b T_j^c$ and $d^{abc} T_i^a T_j^b T_k^c$ that have a linear dependence on the colour tensor d^{abc} (which is odd under charge conjugation). The charge-asymmetry contributions to $|\boldsymbol{J}(q_1, q_2)|^2$ have a *quantum* origin and are characteristic of the radiation of soft quark–antiquark pairs (the squared current $|\boldsymbol{J}(q_1, \ldots, q_m)|^2$ for radiation of m soft gluons is instead fully symmetric with respect to the soft-gluon momenta $q_1, \ldots, q_m$).

We present the explicit result of the ϵ expansion of the functions $w^{[S]}$, $w^{[A]}$, $F^{[S]}$ and $F^{[A]}$ up to $\mathcal{O}(\epsilon^0)$. More precisely, we limit ourselves to presenting the expressions of these functions in the kinematical region where $q_1^0 > 0$ and $q_2^0 > 0$ (i.e., the soft quark and antiquark are produced in the physical final state), which is the most relevant physical region[4]. In this region, the squared current depends (see Eqs. (36) and (37)) on the logarithms $\ell_{i1} \pm \ell_{j2}$ (which are purely real, independently of whether the momenta p_i and p_j are physically incoming or outgoing) and on the real part L_{ijR} and discontinuity $\Theta_{ij}^{(\text{in})}$ of the logarithm L_{ij}. We have (see Eqs. (27) and (28))

$$
\begin{aligned}
\ell_{i1} + \ell_{j2} &= \ln \frac{(p_i \cdot q_1)(p_j \cdot q_2)}{(p_i \cdot q_{12})(p_j \cdot q_{12})}, \\
\ell_{i1} - \ell_{j2} &= \ln \frac{(p_i \cdot q_1)(p_j \cdot q_{12})}{(p_i \cdot q_{12})(p_j \cdot q_2)}, \\
L_{ij} &= L_{ijR} + 2i\pi \, \Theta_{ij}^{(\text{in})},
\end{aligned}
\tag{64}
$$

where

$$
\begin{aligned}
L_{ijR} &= \ln \frac{(p_i \cdot q_{12})(p_j \cdot q_{12})}{(p_i \cdot p_j)(q_1 \cdot q_2)} = \ln \left(1 + \frac{q_{12 \perp ij}^2}{q_{12}^2} \right), \\
\Theta_{ij}^{(\text{in})} &\equiv \Theta(-p_i^0) \, \Theta(-p_j^0) .
\end{aligned}
\tag{65}
$$

The function $w_{ij}^{[S]}$ has the following expression in the region where $q_1^0 > 0$ and $q_2^0 > 0$:

$$
\begin{aligned}
w_{ij}^{[S]}(q_1, q_2) = \Bigg\{ & w_{ij}(q_1, q_2) \Bigg[-C_F \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} \right. \\
&- \pi^2 + 8 + (\delta_R - 1) \Big) - \frac{4}{3} T_R N_f \left(\frac{1}{\epsilon} + \frac{5}{3} \right) \\
&+ \frac{1}{3} C_A \Big(\frac{11}{\epsilon} + \frac{76}{3} - \pi^2 + (\delta_R - 1) \Big) \\
&+ \frac{1}{2} C_A \Big(\frac{2}{\epsilon} \big(L_{ijR} + \ell_{i1} + \ell_{j2} \big) \\
&- L_{ijR}^2 - (\ell_{i1} - \ell_{j2})^2 \Big) \Bigg] \\
&- C_A \left[\mathcal{I}_{ii}(q_1, q_2) - \mathcal{I}_{jj}(q_1, q_2) \right] \\
&\frac{q_{12}^2}{q_{12 \perp ij}^2} L_{ijR} \big(\ell_{i1} - \ell_{j2} \big) + \mathcal{O}(\epsilon) \Bigg\} \\
&+ (q_1 \leftrightarrow q_2).
\end{aligned}
\tag{66}
$$

This function (which is *symmetric* under the exchange $i \leftrightarrow j$) controls the size of the one-loop radiative corrections to the tree-level colour dipole correlations $\boldsymbol{T}_i \cdot \boldsymbol{T}_j$.

We note that $w_{ij}^{[S]}$ also depends on colour coefficients, while $F_{ijk}^{[S]}$, $w_{ij}^{[A]}$ and $F_{ijk}^{[A]}$ only depends on parton momenta.

---

[4] Expressions in other kinematical regions can be obtained by using the fully general one-loop current in Eqs. (36) and (37).

The function $F_{ijk}^{[S]}$ is associated with non-abelian three-particle correlations with colour charge factor $f^{abc} T_i^a T_j^b T_k^c$. In the region where $q_1^0 > 0$ and $q_2^0 > 0$, we have the explicit result

$$
\begin{aligned}
F_{ijk}^{[S]}(q_2, q_1) = {} & 2\pi \, \mathcal{I}_{ki}(q_1, q_2) \, \Big\{ L_{ijR} + \ell_{i1} + \ell_{j2} \\
& + \Theta_{ij}^{(\mathrm{in})} \Big[2 \Big(\frac{1}{\epsilon} - L_{ijR} \Big) \\
& - 2 \frac{q_{12}^2}{q_{12\perp ij}^2} (\ell_{i1} - \ell_{j2}) \Big] + \mathcal{O}(\epsilon) \Big\} \\
& + (q_1 \leftrightarrow q_2).
\end{aligned}
\tag{67}
$$

The charge-asymmetry contributions to Eq. (61) can be expressed through the function $F_{ijk}^{[A]}$. In the region where $q_1^0 > 0$ and $q_2^0 > 0$ we have

$$
\begin{aligned}
F_{ijk}^{[A]}(q_2, q_1) = {} & \Big\{ \mathcal{I}_{ki}(q_1, q_2) \Big[-\frac{2}{\epsilon} (\ell_{i1} + \ell_{j2}) \\
& + (\ell_{i1} - \ell_{j2})^2 + 2 \frac{q_{12}^2}{q_{12\perp ij}^2} L_{ijR} (\ell_{i1} - \ell_{j2}) \Big] \\
& + \mathcal{O}(\epsilon) \Big\} - (q_1 \leftrightarrow q_2).
\end{aligned}
\tag{68}
$$

At arbitrary orders in the ϵ expansion, the two-particle correlation function $w_{ij}^{[A]}$ is directly related to $F_{ijk}^{[A]}$ as follows

$$
w_{ij}^{[A]}(q_1, q_2) = \Big[F_{iji}^{[A]}(q_1, q_2) + F_{jii}^{[A]}(q_1, q_2) \Big] - (i \leftrightarrow j).
\tag{69}
$$

In contrast to $w_{ij}^{[S]}$, we note that $w_{ij}^{[A]}$ is *antisymmetric* under the exchange $i \leftrightarrow j$ of the hard-parton momenta. In particular, this antisymmetry of $w_{ij}^{[A]}$ implies that in the sum over i and j of Eq. (61) we can replace $\widetilde{\boldsymbol{D}}_i \cdot \boldsymbol{T}_j$ by its antisymmetric component, namely, $\widetilde{\boldsymbol{D}}_i \cdot \boldsymbol{T}_j \to (\widetilde{\boldsymbol{D}}_i \cdot \boldsymbol{T}_j - \widetilde{\boldsymbol{D}}_j \cdot \boldsymbol{T}_i)/2$. Inserting Eq. (68) in Eq. (69), $w_{ij}^{[A]}$ has the following expression:

$$
\begin{aligned}
w_{ij}^{[A]}(q_1, q_2) = {} & \Big\{ w_{ij}(q_1, q_2) \Big[-\frac{2}{\epsilon} (\ell_{i1} + \ell_{j2}) + (\ell_{i1} \\
& - \ell_{j2})^2 \Big] + \Big[\mathcal{I}_{ii}(q_1, q_2) - \mathcal{I}_{jj}(q_1, q_2) \Big] \\
& \times \frac{2 q_{12}^2}{q_{12\perp ij}^2} L_{ijR} (\ell_{i1} - \ell_{j2}) + \mathcal{O}(\epsilon) \Big\} \\
& - (q_1 \leftrightarrow q_2).
\end{aligned}
\tag{70}
$$

By inspection of Eqs. (66)–(70) we note that only the function $F_{ijk}^{[S]}$ exhibits a discontinuity with respect to the momenta of the hard partons (see $\Theta_{ij}^{(\mathrm{in})}$ in Eqs. (65) and (67)). The discontinuity contributes in the kinematical region where two hard-parton momenta i and j have negative time component ($p_i^0 < 0$ and $p_j^0 < 0$), namely, the partons i and j collide in the physical initial state. This discontinuity term of the squared current in Eq. (61) originates as interference between a one-loop absorptive (imaginary) contribution and

the antihermitian colour factor $i f^{abc} T_i^a T_j^b T_k^c$ (we recall that i, j and k refer to three distinct partons). Actually, the entire term proportional to $f^{abc} T_i^a T_j^b T_k^c$ in Eq. (61) has this origin[5] as absorptive/colour interference (the absorptive term being related to the kinematical region where $q_1^0 > 0$ and $q_2^0 > 0$).

As discussed in Sect. 4 (see Eqs. (41)–(43) and accompanying comments) the one-loop current of soft-$q\bar{q}$ emission has a transverse-momentum singularity at $q_{12\perp ij} \to 0$. This singularity has a non-abelian character and an absorptive origin. At the level of the one-loop squared current, this singularity does appear in the function $F_{ijk}^{[S]}$ (see the term $(q_{12\perp ij}^2)^{-1} \Theta_{ij}^{(\mathrm{in})}$ in Eq. (67)), while it is absent in all the other contributions (in Eqs. (66), (68) and (70) we see the term $(q_{12\perp ij}^2)^{-1} L_{ijR} \to (q_{12}^2)^{-1}$, which is not singular at $q_{12\perp ij} \to 0$). Therefore, the transverse-momentum singularity at $q_{12\perp ij} \to 0$ contributes through colour correlation $f^{abc} T_i^a T_j^b T_k^c$ to one-loop squared amplitudes for the class of processes with initial-state colliding partons i and j and *two* or *more* final-state hard partons (as recalled below in Eq. (71), the colour correlation vanishes if there is only one final-state hard parton). This class of processes includes, for instance, dijet (or heavy-quark pair) production in hadron–hadron collisions and the transverse-momentum singularity is directly related to the transverse momentum of the dijet system (heavy-quark pair). Interestingly, we note that this is the same class of processes that is sensitive to effects due to the violation of strict collinear factorization [33]. However, we remark on the fact that the transverse-momentum singularity at $q_{12\perp ij} \to 0$ and violation of strict collinear factorization are independent phenomena (e.g., the singularity at $q_{12\perp ij} \to 0$ is not due to violation of strict factorization in the one-loop collinear limit of three partons, such as the soft quark and antiquark and a hard parton i or j).

Regarding three-particle correlations of the type $f^{abc} T_i^a T_j^b T_k^c$ with three distinct hard partons, we also recall two general features. As first noticed in Ref. [29], such colour correlations vanish by acting onto scattering amplitudes with only three hard partons (plus additional colourless external particles). Indeed, we have [29]

$$
f^{abc} T_i^a T_j^b T_k^c \, |i \, j \, k\rangle = 0, \quad \mathrm{dist.}\{i \, j \, k\},
\tag{71}
$$

where $|i \, j \, k\rangle$ denotes a generic colour singlet state of three distinct hard partons i, j and k (the result in Eq. (71) simply follows from the colour conservation relation $(\boldsymbol{T}_i + \boldsymbol{T}_j + \boldsymbol{T}_k)|i \, j \, k\rangle = 0$). As pointed out in Refs. [90,91], such colour correlations vanish by considering their expectation value

---

[5] The one-loop squared current for single soft-gluon radiation [29] has three-particle correlations of the type $f^{abc} T_i^a T_j^b T_k^c$, which have an analogous origin as absorptive/colour interference.

onto *pure* QCD amplitudes at the *tree* level. Namely, we have [90,92]

$$\langle \mathcal{M}^{(0)}(p_1, \ldots, p_n) | \, f^{abc} \, T_i^a \, T_j^b \, T_k^c \, | \mathcal{M}^{(0)}(p_1, \ldots, p_n) \rangle$$
$$= 0 \quad \text{dist.} \{i \, j \, k\}, \tag{72}$$

where $\mathcal{M}^{(0)}$ is a generic scattering amplitude with *only* quark and gluons external lines (and no additional colourless external particles) as obtained by *tree-level* QCD interactions. Therefore, the three-particle correlations $f^{abc} \, T_i^a \, T_j^b \, T_k^c$ contributes to the one-loop squared current in Eq. (61) only (see Eq. (71)) for processes with four or more hard partons and only (see Eq. (72)) through the introduction of either QCD loop corrections or electroweak interactions (see Refs. [29,91]) in the hard-parton scattering amplitude $\mathcal{M}(p_1, \ldots, p_n)$.

We present some general comments on the charge-asymmetry contributions to the one-loop squared current of Eq. (61). Such contributions produce non-vanishing effects only for specific classes of scattering amplitudes (see the discussion below and in Sect. 5.3) and quantities that are not invariant under charge conjugation. Obviously, due to their antisymmetry under the exchange $q_1 \leftrightarrow q_2$, the charge-asymmetry contributions in Eq. (61) give vanishing effects after phase-space symmetric integration over the momenta q_1 and q_2 of the soft quark and antiquark. At the cross section level, the charge-asymmetry contributions can give non-vanishing effects to quantities in which the soft quark (or antiquark) is triggered, either directly (as it can be done for bottom or charm quark) or indirectly (e.g., through its fragmentation), in the final state. For instance, we recall that the Altarelli–Parisi splitting functions for collinear evolution of parton densities and fragmentation functions have a quark–antiquark charge asymmetry [42,93–95], which starts at $\mathcal{O}(\alpha_S^3)$ (the same perturbative order of the soft-$q\bar{q}$ one-loop squared current) and which does not vanish in the soft limit.

Considering $\langle \mathcal{M}(p_1, \ldots, p_n) \| J(q_1, q_2) |^2 | \mathcal{M}(p_1, \ldots, p_n) \rangle$, the charge-asymmetry contributions vanish if $\mathcal{M}(p_1, \ldots, p_n)$ is a pure multigluon scattering amplitude, namely, if it has only gluon external lines (with no additional external $q\bar{q}$ pairs or colourless particles). This is a general consequence of the fact that the original (i.e., before performing the soft-$q\bar{q}$ limit) squared amplitude $|\mathcal{M}(q_1, q_2, p_1, \ldots, p_n)|^2$ is charge-conjugation invariant, since its external legs are gluons and a single $q\bar{q}$ pair (one cannot distinguish between the quark and the antiquark at the squared amplitude level). At the purely technical level, it turns out (as it can be verified) that the colour charge operators $\widetilde{D}_i \cdot T_j$ and $d^{abc} \, T_i^a \, T_j^b \, T_k^c$ in Eq. (61) have vanishing expectation value onto pure multigluon amplitudes.

We also note that the three-particle correlations of the type $d^{abc} \, T_i^a \, T_j^b \, T_k^c$ in Eq. (61) contribute only for processes with *four* or *more* hard partons. Indeed, in the case of only three

hard partons we have

$$d^{abc} \, T_i^a \, T_j^b \, T_k^c \, |i \, j \, k\rangle = 0,$$
$$(\{i \, j \, k\} = \{g \, g \, g\}, \{g \, q \, \bar{q}\}), \tag{73}$$

where the three distinct hard partons (i, j and k) in the colour singlet state $|i \, j \, k\rangle$ are *either* three gluons *or* a gluon and a $q\bar{q}$ pair. The proof of Eq. (73) is given in Sect. 5.3.2 (see Eqs. (85), (89) and related comments).

5.3 Processes with two and three hard partons

The soft-emission factorization formula (46) for squared amplitudes embodies colour correlations produced by the squared current $|J|^2$. In the case of scattering amplitudes with two or three hard partons (plus, necessarily, additional colourless external particles) the colour correlations have a simplified structure. A related discussion and some general results for multiple soft-gluon radiation can be found in Ref. [49]. The radiation of soft-$q\bar{q}$ pairs produces additional colour correlations from charge-asymmetry contributions: their main features are discussed in this Section.

5.3.1 Processes with two hard partons

We consider a generic scattering amplitude $\mathcal{M}_{BC}(q_1, q_2, p_B, p_C)$ whose external legs are two hard partons (denoted as B and C), a soft $q\bar{q}$ pair and additional colourless particles (which are never explicitly denoted). The two hard partons can be either a $q\bar{q}$ pair (note that we specify $B = q$ and $C = \bar{q}$) or two gluons ($\{BC\} = \{gg\}$). The corresponding scattering amplitude $|\mathcal{M}_{BC}(p_B, p_C)\rangle$ without the soft-$q\bar{q}$ pair is a colour singlet state. There is only *one* colour singlet configuration of the two hard partons, and the corresponding one-dimensional colour space is generated by a single colour state vector that we denote as $|BC\rangle$.

Since the soft-$q\bar{q}$ squared current $|J(q_1, q_2)|^2$ conserves the colour charge of the hard partons, the state $|J|^2 |BC\rangle$ is also proportional to $|BC\rangle$. We write

$$|J(q_1, q_2)|^2 \, |BC\rangle = |BC\rangle \, |J(q_1, q_2)|^2_{BC}, \tag{74}$$

where $|J|^2_{BC}$ is a c-number (it is the eigenvalue of the operator $|J|^2$ onto the colour state $|BC\rangle$). Therefore, the soft-factorization formula (46) has the following factorized c-number form:

$$|\mathcal{M}_{BC}(q_1, q_2, p_B, p_C)|^2$$
$$\simeq |J(q_1, q_2)|^2_{BC} \, |\mathcal{M}_{BC}(p_B, p_C)|^2, \tag{75}$$

with no residual correlation effects in colour space (the dependence on $SU(N_c)$ colour coefficients is embodied in the c-number factors $|J|^2_{BC}$ and $|\mathcal{M}_{BC}|^2$). In this respect, the structure of Eq. (75) is similar to that of soft-photon factorization formulae in QED. We recall that a c-number fac-

torization formula analogous to Eq. (75) is equally valid for multiple soft-gluon radiation from two hard partons [49].

We note that Eqs. (74) and (75) are valid at *arbitrary* loop orders in the perturbative expansion of both the squared amplitude and the squared current. Therefore, by considering Eq. (74) and the loop expansion in Eq. (48), we can limit ourselves to evaluate the eigenvalues (c-numbers) $|\hat{\boldsymbol{J}}|^2_{(0\ell)BC}$ and $|\hat{\boldsymbol{J}}|^2_{(1\ell)BC}$, which are the tree-level and one-loop contributions to $|\boldsymbol{J}|^2_{BC}$.

The tree-level squared current $|\hat{\boldsymbol{J}}|^2_{(0\ell)}$ in Eqs. (48) and (52) depends on the colour dipole factor $\boldsymbol{T}_B \cdot \boldsymbol{T}_C$ and, by simply using charge conservation ($\boldsymbol{T}_C |BC\rangle = -\boldsymbol{T}_B |BC\rangle$), we have $\boldsymbol{T}_B \cdot \boldsymbol{T}_C |BC\rangle = -|BC\rangle \boldsymbol{T}_B^2$ ($\boldsymbol{T}_B^2 = C_B$). This leads to the tree-level result first presented in Ref. [28]:

$$|\hat{\boldsymbol{J}}(q_1, q_2)|^2_{(0\ell)BC} = T_R C_B w_{BC}(q_1, q_2),\qquad (76)$$

where $w_{ij}(q_1, q_2)$ is given in Eq. (53) and C_B is the quadratic Casimir coefficient of the hard parton (either $C_B = C_F$ for $\{BC\} = \{q\bar{q}\}$, or $C_B = C_A$ for $\{BC\} = \{gg\}$).

The one-loop squared current in Eq. (61) depends on the colour dipole $\boldsymbol{T}_B \cdot \boldsymbol{T}_C$ (as at the tree level) and on charge-asymmetry colour correlations.

By using Eqs. (56)–(59), we have the following colour algebra results:

$$i = q : \widetilde{\boldsymbol{D}}_i \cdot \boldsymbol{T}_i = \frac{1}{2} d_A C_F, \qquad (77)$$

$$i = \bar{q} : \widetilde{\boldsymbol{D}}_i \cdot \boldsymbol{T}_i = -\frac{1}{2} d_A C_F, \qquad (78)$$

$$i = g : \widetilde{\boldsymbol{D}}_i \cdot \boldsymbol{T}_i = 0. \qquad (79)$$

We note that the operators $\widetilde{\boldsymbol{D}}_i \cdot \boldsymbol{T}_i = d^{abc} T_i^a T_i^b T_i^c$ in Eqs. (77)–(79) are proportional to the unit operator in colour space. This proportionality is actually valid for $\widetilde{\boldsymbol{D}}_i \cdot \boldsymbol{T}_i$ in any colour (irreducible) representation T_i^a, and the proportionality factor is known as cubic Casimir coefficient of $SU(N_c)$.

The action onto $|BC\rangle$ of the charge-asymmetry colour correlations in Eq. (61) can be explicitly evaluated by using colour conservation (we have $\widetilde{\boldsymbol{D}}_B \cdot \boldsymbol{T}_C |BC\rangle = -\widetilde{\boldsymbol{D}}_B \cdot \boldsymbol{T}_B |BC\rangle$ and $\widetilde{\boldsymbol{D}}_C \cdot \boldsymbol{T}_B |BC\rangle = -\widetilde{\boldsymbol{D}}_C \cdot \boldsymbol{T}_C |BC\rangle$) and the cubic Casimir coefficients in Eqs. (77)–(79).

Combining all the contributions in Eq. (61), we find the following final result:

$$\begin{aligned} |\hat{\boldsymbol{J}}(q_1, q_2)|^2_{(1\ell)BC} &= T_R C_F \left[w_{BC}^{[S]}(q_1, q_2) \right. \\ &\left. + \frac{1}{2} d_A w_{BC}^{[A]}(q_1, q_2) \right], \\ &\quad (\{B = q, C = \bar{q}\}), \qquad (80) \end{aligned}$$

$$\begin{aligned} |\hat{\boldsymbol{J}}(q_1, q_2)|^2_{(1\ell)BC} &= T_R C_A w_{BC}^{[S]}(q_1, q_2), \\ &\quad (\{BC\} = \{gg\}), \qquad (81) \end{aligned}$$

where the functions $w_{ij}^{[S]}$ and $w_{ij}^{[A]}$ are given in Eqs. (66) and (70), respectively

In the case of soft-$q\bar{q}$ radiation from the hard partons $\{BC\} = \{q\bar{q}\}$ (see Eq. (80)), we do find charge-asymmetry contributions in $|\hat{\boldsymbol{J}}(q_1, q_2)|^2_{(1\ell)BC}$. We recall that the function $w_{ij}^{[A]}(q_1, q_2)$ is antisymmetric with respect to the separate exchanges $q_1 \leftrightarrow q_2$ and $i \leftrightarrow j$. Therefore, in Eq. (80) the asymmetry in the momenta of the soft-$q\bar{q}$ pair is correlated with a corresponding asymmetry in the momenta p_B and p_C of the hard q and $\bar{q}$. In particular, $|\hat{\boldsymbol{J}}(q_1, q_2)|^2_{(1\ell)BC}$ is invariant under the overall exchange of fermions and antifermions (i.e. $\{q_1, p_B\} \leftrightarrow \{q_2, p_C\}$), consistently with charge-conjugation invariance.

In the case of soft-$q\bar{q}$ radiation from two hard gluons, the one-loop result in Eq. (81) shows no charge-asymmetry effects. We state that this feature persists at arbitrary orders in the QCD loop expansion. The absence of charge-asymmetry effects follows from the fact that the c-number squared current $|\boldsymbol{J}(q_1, q_2)|^2_{BC}$ for $\{BC\} = \{gg\}$ is entirely controlled by QCD interactions, with absolutely no dependence (both explicitly and implicitly) on the production mechanism of the two hard gluons. Therefore, such squared current is charge-conjugation invariant (similarly to the squared amplitude for the process $gg \to q\bar{q}$) and one cannot distinguish between the soft quark and antiquark.

5.3.2 Processes with three hard partons

Before considering the explicit evaluation of the soft-$q\bar{q}$ squared current $|\boldsymbol{J}(q_1, q_2)|^2$ for processes with three hard partons, we recall and derive some general algebraic relations for the action of colour charge correlations operators onto a generic colour singlet state $|ijk\rangle$ formed by three distinct partons i, j and k ($i \neq j$, $j \neq k$, $k \neq i$) in arbitrary representations of the gauge group $SU(N_c)$. We consider the correlations operators that appear in $|\boldsymbol{J}(q_1, q_2)|^2$ up to one-loop level, namely, $\boldsymbol{T}_i \cdot \boldsymbol{T}_j$, $f^{abc} T_i^a T_j^b T_k^c$, $\widetilde{\boldsymbol{D}}_i \cdot \boldsymbol{T}_j$ and $d^{abc} T_i^a T_j^b T_k^c$.

As is well known, the action of dipole factors onto $|ijk\rangle$ can be evaluated in terms of quadratic Casimir coefficients $\boldsymbol{T}_i^2 = C_i$ (see the Appendix A of Ref. [22]). We have

$$2 \boldsymbol{T}_i \cdot \boldsymbol{T}_j |ijk\rangle = |ijk\rangle (C_k - C_i - C_j), \qquad (82)$$

and related permutations of i, j, k. In particular, any generic colour singlet state $|ijk\rangle$ is an eigenstate of $\boldsymbol{T}_i \cdot \boldsymbol{T}_j$ or, equivalently, the action of $\boldsymbol{T}_i \cdot \boldsymbol{T}_j$ onto $|ijk\rangle$ is always proportional to the unit operator in colour space. The result in Eq. (82) simply follows from the charge conservation relation $(\boldsymbol{T}_i + \boldsymbol{T}_j + \boldsymbol{T}_k)|ijk\rangle = 0$, which also leads to the result in Eq. (71) for the operator $f^{abc} T_i^a T_j^b T_k^c$.

Considering charge-asymmetry correlations and using colour conservation ($T_k^a |ijk\rangle = -(T_i^a + T_j^a)|ijk\rangle$), we

have the following relations

$$\widetilde{\boldsymbol{D}}_k \cdot \boldsymbol{T}_k \, |ijk\rangle = - \left(\widetilde{\boldsymbol{D}}_k \cdot \boldsymbol{T}_i + \widetilde{\boldsymbol{D}}_k \cdot \boldsymbol{T}_j \right) |ijk\rangle, \quad (83)$$

$$d^{abc} T_i^a \, T_j^b \, T_k^c \, |i\,j\,k\rangle = - \left(\widetilde{\boldsymbol{D}}_i \cdot \boldsymbol{T}_j + \widetilde{\boldsymbol{D}}_j \cdot \boldsymbol{T}_i \right) |ijk\rangle, \quad (84)$$

and related permutations of i, j, k. We note that we are deal-ing with seven colour correlations operators (six two-particle correlations of the type $\widetilde{\boldsymbol{D}}_i \cdot \boldsymbol{T}_j$, and the three-particle correla-tion $d^{abc} T_i^a T_j^b T_k^c$) whose action onto $|ijk\rangle$ is 'non-trivial', while the action of the three operators $\widetilde{\boldsymbol{D}}_i \cdot \boldsymbol{T}_i$ is directly worked out in c-number form in terms of cubic Casimir coefficients (see Eqs. (77)–(79)). Since colour conservation leads to the six linear relations (exploiting permutations) in Eqs. (83) and (84), all the non-trivial colour correlations can be expressed in terms of a *single* correlation operators. To explicitly show this, we derive the following relations. The three-particle correlation is directly related to cubic Casimir coefficients as follows

$$d^{abc} T_i^a \, T_j^b \, T_k^c \, |i\,j\,k\rangle = \frac{1}{3} \left(\widetilde{\boldsymbol{D}}_i \cdot \boldsymbol{T}_i \right.$$
$$\left. + \widetilde{\boldsymbol{D}}_j \cdot \boldsymbol{T}_j + \widetilde{\boldsymbol{D}}_k \cdot \boldsymbol{T}_k \right) |ijk\rangle. \quad (85)$$

The three symmetric (with respect to $i \leftrightarrow j$) two-particle correlations are equal and directly related to cubic Casimir coefficients as follows

$$\left(\widetilde{\boldsymbol{D}}_i \cdot \boldsymbol{T}_j + \widetilde{\boldsymbol{D}}_j \cdot \boldsymbol{T}_i \right) |ijk\rangle = \left(\widetilde{\boldsymbol{D}}_j \cdot \boldsymbol{T}_k + \widetilde{\boldsymbol{D}}_k \cdot \boldsymbol{T}_j \right) |ijk\rangle =$$
$$\left(\widetilde{\boldsymbol{D}}_k \cdot \boldsymbol{T}_i + \widetilde{\boldsymbol{D}}_i \cdot \boldsymbol{T}_k \right) |ijk\rangle = -\frac{1}{3} \left(\widetilde{\boldsymbol{D}}_i \cdot \boldsymbol{T}_i + \widetilde{\boldsymbol{D}}_j \cdot \boldsymbol{T}_j \right.$$
$$\left. + \widetilde{\boldsymbol{D}}_k \cdot \boldsymbol{T}_k \right) |ijk\rangle. \quad (86)$$

The three antisymmetric (with respect to $i \leftrightarrow j$) two-particle correlations fulfil two independent linear relations (which are related through $i \leftrightarrow j$) as follows

$$\left[(\widetilde{\boldsymbol{D}}_j \cdot \boldsymbol{T}_k - \widetilde{\boldsymbol{D}}_k \cdot \boldsymbol{T}_j) - (\widetilde{\boldsymbol{D}}_i \cdot \boldsymbol{T}_j - \widetilde{\boldsymbol{D}}_j \cdot \boldsymbol{T}_i) \right.$$
$$\left. + \frac{2}{3} (\widetilde{\boldsymbol{D}}_i \cdot \boldsymbol{T}_i + \widetilde{\boldsymbol{D}}_k \cdot \boldsymbol{T}_k) - \frac{4}{3} \widetilde{\boldsymbol{D}}_j \cdot \boldsymbol{T}_j \right] |ijk\rangle = 0, \quad (87)$$

$$\left[(\widetilde{\boldsymbol{D}}_k \cdot \boldsymbol{T}_i - \widetilde{\boldsymbol{D}}_i \cdot \boldsymbol{T}_k) - (\widetilde{\boldsymbol{D}}_i \cdot \boldsymbol{T}_j - \widetilde{\boldsymbol{D}}_j \cdot \boldsymbol{T}_i) \right.$$
$$\left. - \frac{2}{3} (\widetilde{\boldsymbol{D}}_j \cdot \boldsymbol{T}_j + \widetilde{\boldsymbol{D}}_k \cdot \boldsymbol{T}_k) + \frac{4}{3} \widetilde{\boldsymbol{D}}_i \cdot \boldsymbol{T}_i \right] |ijk\rangle = 0. \quad (88)$$

The derivation of Eqs. (85)–(88) from Eqs. (83) and (84) is relatively straightforward. For instance, Eq. (85) is derived by first summing Eq. (84) and its two independent permutations to obtain $3 d^{abc} T_i^a T_j^b T_k^c |ijk\rangle = -[\widetilde{\boldsymbol{D}}_k \cdot (\boldsymbol{T}_i + \boldsymbol{T}_j) + (k \leftrightarrow i) + (k \leftrightarrow j)] |ijk\rangle$, and then by using Eq. (83). Similar algebraic operations lead to Eqs. (86)–(88).

In the specific cases in which i, j, k are either three glu-ons or a gluon and a $q\bar{q}$ pair, we can use the explicit results for $\widetilde{\boldsymbol{D}}_i \cdot \boldsymbol{T}_i$ in Eqs. (77)–(79) and, consequently, Eq. (85) gives $d^{abc} T_i^a T_j^b T_k^c |ijk\rangle = 0$ (this proves Eq. (73)) and from

Eq. (86) we obtain

$$\left(\widetilde{\boldsymbol{D}}_i \cdot \boldsymbol{T}_j + \widetilde{\boldsymbol{D}}_j \cdot \boldsymbol{T}_i \right) |ijk\rangle = 0,$$
$$(\{i\,j\,k\} = \{g\,q\,\bar{q}\}, \{g\,g\,g\}), \quad (89)$$

and related permutations of i, j, k.

Regarding the vanishing value of the correlations $f^{abc} T_i^a T_j^b T_k^c$ and $d^{abc} T_i^a T_j^b T_k^c$ in Eqs. (71) and (73), a comment is in order. The result in Eq. (71) applies to arbitrary colour representations of $\{i\,j\,k\}$, while Eq. (73) is valid (as we have specified in its derivation from Eq. (85)) for some types of colour representations. For instance, in the case of $SU(N_c)$ with $N_c = 3$, the colour singlet state $|ijk\rangle$ can be formed by three quarks and in such case $d^{abc} T_i^a T_j^b T_k^c |ijk\rangle$ does not vanish.

We summarize our general discussion on colour correla-tions for processes with three hard partons i, j, k in arbitrary colour representations of $SU(N_c)$. The charge-symmetric component of $|\boldsymbol{J}(q_1, q_2)|^2$ up to one-loop order is propor-tional to the unit operator in colour space, and it can be expressed in c-number form, in terms of quadratic Casimir coefficients (see Eqs. (71) and (82)). The charge-asymmetry component of $|\boldsymbol{J}(q_1, q_2)|^2$ at one-loop order can eventually be expressed (see Eqs. (85)–(88)) in terms of cubic Casimir coefficients (c-numbers) and a single operator (e.g., $\widetilde{\boldsymbol{D}}_i \cdot \boldsymbol{T}_j$) whose action onto the colour singlet state $|ijk\rangle$ has to be explicitly computed (the result depends on the specific state $|ijk\rangle$).

We come to explicitly discuss soft-$q\bar{q}$ radiation from scat-tering amplitudes with three hard partons in the specific cases that are relevant within perturbative QCD. We consider a generic scattering amplitude $\mathcal{M}_{ABC}(q_1, q_2, p_A, p_B, p_C)$ whose external legs are colourless particles (which are not explicitly denoted), a soft-$q\bar{q}$ pair and three hard partons (denoted as A, B, C) that can be either a gluon and a $q\bar{q}$ pair ($\{ABC\} = \{gq\bar{q}\}$) or three gluons ($\{ABC\} = \{ggg\}$). The corresponding scattering amplitude $|\mathcal{M}_{ABC}(p_A, p_B, p_C)\rangle$ without the soft-$q\bar{q}$ pair is a colour singlet state formed by the three hard partons A, B and C. We consider the cases $\{ABC\} = \{gq\bar{q}\}$ and $\{ABC\} = \{ggg\}$ in turn.

$gq\bar{q}$ case

We specifically set $A = g$, $B = q$ and $C = \bar{q}$.

There is only *one* colour singlet configuration of the three hard partons, $gq\bar{q}$, and the corresponding one-dimensional colour space is generated by a single colour state vector that we denote as $|ABC\rangle$. Therefore, we are in a situation in which we can apply the same reasoning of Sect. 5.3.1 (see Eqs. (74) and (75) and the accompanying discussion). The state $|ABC\rangle$ is an eigenstate of the soft-$q\bar{q}$ squared current $|\boldsymbol{J}(q_1, q_2)|^2$,

$$|\boldsymbol{J}(q_1, q_2)|^2 \, |ABC\rangle = |ABC\rangle \, |\boldsymbol{J}(q_1, q_2)|^2_{ABC},$$
$$(\{ABC\} = \{gq\bar{q}\}), \quad (90)$$

and the soft-factorization formula (46) has the following factorized form:

$$
\begin{aligned}
&|\mathcal{M}_{ABC}(q_1, q_2, p_A, p_B, p_C)|^2 \\
&\simeq |\boldsymbol{J}(q_1, q_2)|^2_{ABC} \ |\mathcal{M}_{ABC}(p_A, p_B, p_C)|^2, \\
&(\{ABC\} = \{gq\bar{q}\}),
\end{aligned}
\tag{91}
$$

where $|\boldsymbol{J}(q_1, q_2)|^2_{ABC}$ is the c-number eigenvalue in Eq. (90). Analogously to Eq. (75), Eq. (91) has a c-number factorized form with no residual correlation effects in colour space (the dependence on $SU(N_c)$ colour coefficients is embodied in the c-number factors $|\boldsymbol{J}|^2_{ABC}$ and $|\mathcal{M}_{ABC}|^2$). We also recall that a c-number factorized formula analogous to Eq. (91) applies [49] to multiple soft-gluon radiation from the three hard partons $\{ABC\} = \{gq\bar{q}\}$.

Equations (90) and (91) are valid at *arbitrary* loop orders in the perturbative expansion of both the squared amplitude and the squared current. Therefore, considering Eq. (90) and the loop expansion in Eq. (48), we can directly evaluate the eigenvalues $|\hat{\boldsymbol{J}}|^2_{(0\ell)ABC}$ and $|\hat{\boldsymbol{J}}|^2_{(1\ell)ABC}$, which are the tree-level and one-loop contributions to $|\hat{\boldsymbol{J}}|^2_{ABC}$.

The tree-level squared current $|\hat{\boldsymbol{J}}|^2_{(0\ell)}$ in Eqs. (48) and (52) involves colour dipole correlations. Using Eq. (82), dipole correlations can be expressed in terms of quadratic Casimir coefficients and this leads to the tree-level result first presented in Ref. [28]:

$$
\begin{aligned}
|\hat{\boldsymbol{J}}(q_1, q_2)|^2_{(0\ell)ABC} = T_R \Big\{ &C_F \, w_{BC}(q_1, q_2) + \frac{1}{2} C_A \left[w_{AB}(q_1, q_2) \right. \\
&\left. + w_{AC}(q_1, q_2) - w_{BC}(q_1, q_2) \right] \Big\}, \\
&(\{ABC\} = \{gq\bar{q}\}),
\end{aligned}
\tag{92}
$$

where $w_{ij}(q_1, q_2)$ is given in Eq. (53). Note that the result in Eq. (92) is symmetric under the exchange $p_B \leftrightarrow p_C$ of the momenta of the hard quark and antiquark.

The one-loop squared current $|\hat{\boldsymbol{J}}(q_1, q_2)|^2_{(1\ell)}$ in Eq. (61) has contributions with and without charge asymmetry. Owing to Eq. (71), the charge-symmetric contributions only involve colour dipole correlations, as at the tree level. As discussed in Eqs. (85)–(88), the charge-asymmetry contributions require the explicit evaluation of a single correlation operator of the type $\widetilde{\boldsymbol{D}}_i \cdot \boldsymbol{T}_j$. We consider the operator $\widetilde{\boldsymbol{D}}_B \cdot \boldsymbol{T}_C$, whose action onto $|ABC\rangle$ can be related to the action of the dipole operator $\boldsymbol{T}_B \cdot \boldsymbol{T}_C$. Indeed, we have

$$
\begin{aligned}
\widetilde{\boldsymbol{D}}_B \cdot \boldsymbol{T}_C |ABC\rangle &= \frac{1}{2} d_A \, \boldsymbol{T}_B \cdot \boldsymbol{T}_C |ABC\rangle \\
&= |ABC\rangle \frac{1}{4} d_A (C_A - 2C_F), \\
&(\{ABC\} = \{gq\bar{q}\}),
\end{aligned}
\tag{93}
$$

where we have used first Eq. (57) and then Eq. (82). We note that $|ABC\rangle$ is an eigenstate of $\widetilde{\boldsymbol{D}}_B \cdot \boldsymbol{T}_C$, as expected one the basis of the general relation in Eq. (90). Using Eq. (93) and the cubic Casimir coefficients in Eqs. (77)–(79), we can express

all the charge-asymmetry colour correlations in c-number form (see Eqs. (85)–(88)). We find the following result for the eigenvalue $|\hat{\boldsymbol{J}}(q_1, q_2)|^2_{(1\ell)ABC}$ of the one-loop squared current for soft-$q\bar{q}$ radiation:

$$
\begin{aligned}
|\hat{\boldsymbol{J}}(q_1, q_2)|^2_{(1\ell)ABC} = T_R \Big\{ &\Big[C_F \, w_{BC}^{[S]}(q_1, q_2) \\
&+ \frac{1}{2} C_A \big(w_{AB}^{[S]}(q_1, q_2) \\
&+ w_{AC}^{[S]}(q_1, q_2) - w_{BC}^{[S]}(q_1, q_2) \big) \Big] \\
&+ \frac{1}{2} d_A \Big[C_F \, w_{BC}^{[A]}(q_1, q_2) \\
&- \frac{1}{2} C_A \big(w_{AB}^{[A]}(q_1, q_2) + w_{CA}^{[A]}(q_1, q_2) \\
&+ w_{BC}^{[A]}(q_1, q_2) \big) \Big] \Big\}, \\
&(\{ABC\} = \{gq\bar{q}\}),
\end{aligned}
\tag{94}
$$

where the functions $w_{ij}^{[S]}(q_1, q_2)$ and $w_{ij}^{[A]}(q_1, q_2)$ are given in Eqs. (66) and (70), respectively. We note that the charge symmetric contribution in Eq. (94) is symmetric under the exchange $p_B \leftrightarrow p_C$ of the hard quark and antiquark. The charge-asymmetry contribution in Eq. (94) is instead *antisymmetric* under the exchange $p_B \leftrightarrow p_C$, in complete analogy with the corresponding contribution for soft-$q\bar{q}$ radiation from two hard partons (see Eq. (80)).

ggg case

We now consider the case in which the three hard partons A, B and C are gluons. The colour singlet space spanned by the three hard gluons is two-dimensional. It is convenient to choose the basis formed by the orthogonal colour state vectors $|(ABC)_f\rangle$ and $|(ABC)_d\rangle$ that are defined as follows

$$
\begin{aligned}
\langle abc | (ABC)_f \rangle &\equiv i f^{abc}, \quad \langle abc | (ABC)_d \rangle \equiv d^{abc}, \\
&(\{ABC\} = \{ggg\}),
\end{aligned}
\tag{95}
$$

where a, b, c are the colour indices of the three gluons. We note that the two states in Eq. (95) have different charge conjugation. The scattering amplitude $|\mathcal{M}_{ABC}(p_A, p_B, p_C)\rangle$ is, in general, a linear combination of the colour antisymmetric state $|(ABC)_f\rangle$ and the colour symmetric state $|(ABC)_d\rangle$, and we write

$$
\begin{aligned}
|\mathcal{M}_{ABC}(p_A, p_B, p_C)\rangle = &|(ABC)_f\rangle \ \mathcal{M}_f(p_A, p_B, p_C) \\
&+ |(ABC)_d\rangle \ \mathcal{M}_d(p_A, p_B, p_C),
\end{aligned}
\tag{96}
$$

where $\mathcal{M}_f$ and $\mathcal{M}_d$ are colour stripped amplitudes. Owing to the Bose symmetry of $|\mathcal{M}_{ABC}\rangle$ with respect to the three gluons, the amplitude $\mathcal{M}_f(p_A, p_B, p_C)$ is antisymmetric under the exchange of two gluon momenta (e.g., $p_A \leftrightarrow p_B$), while $\mathcal{M}_f(p_A, p_B, p_C)$ has a symmetric dependence on p_A, p_B, p_C.

As examples of the scattering amplitude $|\mathcal{M}_{ABC}(p_A, p_B, p_C)\rangle$, we can mention the three scattering processes $H \to ggg$, $\gamma \to ggg$ and $Z \to ggg$. In the Higgs boson process $H \to ggg$ (see, e.g., Ref. [96]) the amplitude component $\mathcal{M}_d$ of Eq. (96) vanishes, while in the photon process $\gamma \to ggg$ (see, e.g., Ref. [97]) we have $\mathcal{M}_f = 0$. In the case of the Z boson process $Z \to ggg$ both components $\mathcal{M}_f$ and $\mathcal{M}_d$ are not vanishing (see, e.g., Ref. [97]). We also note that all these scattering amplitudes are produced through QCD interactions involving quark loops (within the Standard Model, gluons have tree-level interactions only with quarks and, consequently, $\mathcal{M}_{ABC}$ vanishes at the tree level).

We have previously discussed the case of the three hard partons $\{ABC\} = \{gq\bar{q}\}$, which generate a one-dimensional colour singlet space. The fact that the colour singlet space is two-dimensional for $\{ABC\} = \{ggg\}$ is an essential difference. In particular, in the case $\{ABC\} = \{ggg\}$ the all-order soft-factorization formula (46) for squared amplitudes cannot be recast in the factorized c-number form of Eq. (91). The action of the squared current $|\boldsymbol{J}|^2$ onto $|\mathcal{M}_{ABC}\rangle$ of Eq. (96) is colour conserving, but it can produce colour correlations between the two colour singlet states $|(ABC)_f\rangle$ and $|(ABC)_d\rangle$ of the three hard gluons. In general, the squared soft current $|\boldsymbol{J}|^2$ can be represented as a 2×2 correlation matrix that acts onto the two-dimensional space generated by $|(ABC)_f\rangle$ and $|(ABC)_d\rangle$. The all-order structure of this correlation matrix is discussed in Ref. [49] for the case of multiple soft-gluon radiation. In the following we explicitly consider soft-$q\bar{q}$ radiation at the tree level and one-loop order.

The tree-level squared current $|\hat{\boldsymbol{J}}|^2_{(0\ell)}$ in Eqs. (48) and (52) only involves colour dipole correlations, whose action onto both $|(ABC)_f\rangle$ and $|(ABC)_d\rangle$ is proportional to the unit matrix in colour space (see Eq. (82)). Therefore, the contribution of $|\hat{\boldsymbol{J}}|^2_{(0\ell)}$ to the factorization formula (46) can be expressed in factorized c-number form and, using Eq. (82), we have (see also Ref. [28])

$$\langle \mathcal{M}_{ABC}(p_A, p_B, p_C)| \, |\hat{\boldsymbol{J}}(q_1, q_2)|^2_{(0\ell)} \, |\mathcal{M}_{ABC}(p_A, p_B, p_C)\rangle$$
$$= |\mathcal{M}_{ABC}(p_A, p_B, p_C)|^2 \, \frac{T_R \, C_A}{2} \, w_{\{ABC\}}(q_1, q_2),$$
$$(\{ABC\} = \{ggg\}), \tag{97}$$

where

$$w_{\{ABC\}}(q_1, q_2) = w_{AB}(q_1, q_2) + w_{BC}(q_1, q_2)$$
$$+ w_{CA}(q_1, q_2), \tag{98}$$

and $w_{ij}(q_1, q_2)$ is given in Eq. (53). Since w_{ij} is symmetric under the exchange $p_i \leftrightarrow p_j$, we note that the function $w_{\{ABC\}}$ has a completely symmetric dependence on the gluon momenta p_A, p_B, p_C (as required by Bose symmetry). We also note that Eq. (97) is valid at arbitrary orders in the loop expansion of the amplitude $\mathcal{M}_{ABC}(p_A, p_B, p_C)$.

The action of the one-loop squared current $|\hat{\boldsymbol{J}}(q_1, q_2)|^2_{(1\ell)}$ in Eqs. (48) and (61) onto $|\mathcal{M}_{ABC}\rangle$ involves charge symmet-

ric and charge-asymmetry contributions. As summarized in the discussion below Eq. (89), the charge symmetric contributions are proportional to the unit matrix in colour space, while the charge-asymmetry contributions can be expressed in terms of a single colour correlation operator. Specifically, by using Eq. (82)) and Eqs. (85)–(88), we explicitly find

$$|\hat{\boldsymbol{J}}(q_1, q_2)|^2_{(1\ell)} \, |ABC\rangle = T_R \left\{ \frac{C_A}{2} \, w^{[S]}_{\{ABC\}}(q_1, q_2) \right.$$
$$\left. + \widetilde{\boldsymbol{D}}_B \cdot \boldsymbol{T}_A \, w^{[A]}_{[ABC]}(q_1, q_2) \right\} |ABC\rangle,$$
$$(\{ABC\} = \{ggg\}), \tag{99}$$

where

$$w^{[S]}_{\{ABC\}}(q_1, q_2) = w^{[S]}_{AB}(q_1, q_2) + w^{[S]}_{BC}(q_1, q_2)$$
$$+ w^{[S]}_{CA}(q_1, q_2), \tag{100}$$
$$w^{[A]}_{[ABC]}(q_1, q_2) = w^{[A]}_{AB}(q_1, q_2) + w^{[A]}_{BC}(q_1, q_2)$$
$$+ w^{[A]}_{CA}(q_1, q_2), \tag{101}$$

and the functions $w^{[S]}_{ij}(q_1, q_2)$ and $w^{[A]}_{ij}(q_1, q_2)$ are given in Eqs. (66) and (70), respectively. We note that the charge symmetric contribution to Eq. (99) depends on the function $w^{[S]}_{\{ABC\}}$ that has a fully symmetric dependence on the hard-gluon momenta p_A, p_B, p_C. The charge-asymmetry function $w^{[A]}_{[ABC]}$ is instead antisymmetric under the exchange of two gluon momenta (e.g., $p_A \leftrightarrow p_B$).

The charge-asymmetry operator $\widetilde{\boldsymbol{D}}_B \cdot \boldsymbol{T}_A$ in the right-hand side of Eq. (99) acts differently onto the two colour states $|(ABC)_f\rangle$ and $|(ABC)_d\rangle$ of Eq. (96). By explicitly performing the $SU(N_c)$ colour algebra, we find the following result:

$$\widetilde{\boldsymbol{D}}_B \cdot \boldsymbol{T}_A \, |(ABC)_f\rangle = \frac{C_A^2}{4} \, |(ABC)_d\rangle,$$
$$\widetilde{\boldsymbol{D}}_B \cdot \boldsymbol{T}_A \, |(ABC)_d\rangle = \frac{C_A \, d_A}{4} \, |(ABC)_f\rangle, \tag{102}$$

and we note that the operator $\widetilde{\boldsymbol{D}}_B \cdot \boldsymbol{T}_A$ produces 'pure' transitions between the colour symmetric and colour antisymmetric states $|(ABC)_f\rangle$ and $|(ABC)_d\rangle$, which have different charge conjugation.

Using Eqs. (96), (99) and (102), we obtain the final result for the contribution of the one-loop soft-$q\bar{q}$ squared current to squared amplitudes with three hard gluons. We find

$$\langle \mathcal{M}_{ABC}(p_A, p_B, p_C)| \, |\hat{\boldsymbol{J}}(q_1, q_2)|^2_{(1\ell)} \, |\mathcal{M}_{ABC}(p_A, p_B, p_C)\rangle$$
$$= \frac{T_R \, C_A}{2} \left\{ w^{[S]}_{\{ABC\}}(q_1, q_2) \, |\mathcal{M}_{ABC}(p_A, p_B, p_C)|^2 \right.$$
$$+ w^{[A]}_{[ABC]}(q_1, q_2) \, \frac{1}{2} C_A d_A (N_c^2 - 1)$$
$$\left. \times \left[\mathcal{M}_d^{\dagger}(p_A, p_B, p_C) \mathcal{M}_f(p_A, p_B, p_C) + \text{h.c.} \right] \right\},$$
$$(\{ABC\} = \{ggg\}), \tag{103}$$

which is not simply proportional to $|\mathcal{M}_{ABC}|^2$ (unlike the corresponding result in Eq. (91) for $\{ABC\} = \{gq\bar{q}\}$)). In contrast with the case of scattering amplitudes with two hard gluons (see Eq. (81)), we note that the expression in Eq. (103) involves a charge-asymmetry contribution that is not vanishing, provided the hard-scattering amplitude includes non-vanishing components $\mathcal{M}_f$ and $\mathcal{M}_d$ (i.e., $\mathcal{M}_{ABC}$ has no definite charge conjugation). Such feature of $\mathcal{M}_{ABC}$ depends on the specific production mechanism of the three hard gluons. The functions $w_{[ABC]}^{[A]}$ and $(\mathcal{M}_d^\dagger \mathcal{M}_f + \text{h.c.})$ are separately antisymmetric under the exchange of two gluon momenta and, consequently, their product is symmetric. Therefore, the right-hand side of Eq. (103) (including its charge-asymmetry contribution) is fully symmetric under permutations of the three hard gluons, as expected and required by Bose symmetry.

6 Soft fermion-antifermion radiation in QED and mixed QCD×QED

Our results in Sects. 4 and 5 for soft-$q\bar{q}$ emission can be generalized to consider the emission of a soft fermion–antifermion ($f\bar{f}$) pair through QED (photon) interactions and mixed QCD×QED (gluon and photon) interactions. Before presenting the results, we precisely specify our framework.

The soft fermions can be either massless quarks ($f = q$) or electrically-charged massless leptons ($f = \ell$). We consider generic scattering amplitudes, $\mathcal{M}$, whose external particles are massless quarks and gluons, massless leptons and, additionally, particles that carry no colour charge and no electric charge (i.e., photons, Higgs and Z bosons in the context of Standard Model). The external particles (i.e., their momenta and quantum numbers) of $\mathcal{M}$ are treated as outgoing particles (as already specified in Sect. 2 for the pure QCD case). The internal legs of $\mathcal{M}$ can include massless (photons, gluons) and massive (e.g., heavy quarks and/or $W^\pm$ bosons) particles. If an external $f\bar{f}$ pair becomes soft, the scattering amplitude $\mathcal{M}$ is singular and the singular behaviour is due to the production of the soft-$f\bar{f}$ pair through QCD (gluon) and QED (photon) interactions. We formally treat QCD, QED and mixed QCD×QED interactions on equal footing. Therefore, the scattering amplitude $\mathcal{M}$ has a generalized perturbative (loop) expansion in powers of two unrenormalized couplings: the QCD coupling g_S and the QED coupling g ($g^2/(4\pi) = \alpha$ is the fine structure constant at the unrenormalized level). Regarding the RS of the UV and IR divergences, photons and charged leptons are treated in the same way (see Sect. 3) as gluons and massless quarks, respectively.

6.1 The soft-$f\bar{f}$ current

The dominant singular behaviour of $|\mathcal{M}\rangle$ for emission of a soft-$f\bar{f}$ pair is given by the factorization formula in Eq. (5) through the replacement $\boldsymbol{J}(q_1, \ldots, q_m) \rightarrow \boldsymbol{J}_{f\bar{f}}(q_1, q_2)$. Here $\boldsymbol{J}_{f\bar{f}}(q_1, q_2)$ is the soft current for emission of a fermion f and an antifermion $\bar{f}$ with momenta q_1 and q_2, respectively. Analogously to the scattering amplitude $\mathcal{M}$, the current $\boldsymbol{J}_{f\bar{f}}$ is perturbatively computable by performing a loop expansion, and we write

$$\boldsymbol{J}_{f\bar{f}}(q_1, q_2) = \boldsymbol{J}_{f\bar{f}}^{(0)}(q_1, q_2) + \boldsymbol{J}_{f\bar{f}}^{(1)}(q_1, q_2) \\ + \boldsymbol{J}_{f\bar{f}}^{(2)}(q_1, q_2) + \cdots . \quad (104)$$

Since we formally treat QCD and QED interactions on equal footing, the k-th loop term $\boldsymbol{J}_{f\bar{f}}^{(k)}$ include contributions that are proportional to powers of both coupling constants g_S and g. The pure-QCD and pure-QED cases are recovered by setting $\{g = 0, f = q\}$ and $\{g_S = 0, f = \ell\}$, respectively.

The lowest-order (tree-level) term $\boldsymbol{J}_{f\bar{f}}^{(0)}$ of Eq. (104) is

$$\boldsymbol{J}_{f\bar{f}}^{(0)}(q_1, q_2) = \left(g_S\, \mu^\epsilon\right)^2 \hat{\boldsymbol{J}}^{(0)}(q_1, q_2) \\ + \left(g\, \mu^\epsilon\right)^2 \hat{\boldsymbol{J}}_{(1\gamma)}^{(0)}(q_1, q_2), \quad (105)$$

where $\hat{\boldsymbol{J}}^{(0)}(q_1, q_2)$ is the rescaled current in Eqs. (14) and (33) for soft-$q\bar{q}$ emission in QCD (note that $\hat{\boldsymbol{J}}^{(0)}$ vanishes if $f = \ell$). The term $\hat{\boldsymbol{J}}_{(1\gamma)}^{(0)}$ has the following explicit expression:

$$\hat{\boldsymbol{J}}_{(1\gamma)}^{(0)}(q_1, q_2) = -e_f\, \boldsymbol{\Delta}_f \sum_{i \in H} e_i\, \frac{p_i \cdot j(1, 2)}{p_i \cdot q_{12}}, \quad (106)$$

where $j^\nu(1, 2)$ is the fermionic current in Eq. (15). The current $\hat{\boldsymbol{J}}_{(1\gamma)}^{(0)}$ is due to a single-photon interaction between the soft fermion f (with electric charge e_f) and the other external charged particles (with electric charges e_i), $i \in H$, of $\mathcal{M}$. The charges e_f and e_i are expressed in units of the positron charge (e.g., for the up-quark u we have $e_u = +2/3$). The factor $\boldsymbol{\Delta}_f$ in the right-hand side of Eq. (106) is a colour operator that depends on the type of soft fermion f. If $f = \ell$, we simply have $\boldsymbol{\Delta}_f = 1$. If $f = q$, $\boldsymbol{\Delta}_f$ is the projection operator onto the colour singlet state of the $f\bar{f}$ pair, namely, by using the colour space notation of Sect. 2 we have $\langle \alpha_1, \alpha_2 | \boldsymbol{\Delta}_f = \delta_{\alpha_1 \alpha_2}$.

The soft-$f\bar{f}$ current $\boldsymbol{J}_{f\bar{f}}^{(1)}$ in Eq. (104) is due to the one-loop corrections (with respect to both g_S and g) to the tree-level current $\boldsymbol{J}_{f\bar{f}}^{(0)}$. We can write

$$\boldsymbol{J}_{f\bar{f}}^{(1)}(q_1, q_2) = \left(\mu^\epsilon\right)^4 \left(-q_{12}^2 - i0\right)^{-\epsilon} c_\Gamma \left[g_S^4\, \hat{\boldsymbol{J}}^{(1)}(q_1, q_2) \right. \\ \left. + g_S^2\, g^2\, \hat{\boldsymbol{J}}_{(1\gamma)}^{(1)}(q_1, q_2) + g^4\, \hat{\boldsymbol{J}}_{(2\gamma)}^{(1)}(q_1, q_2)\right], \quad (107)$$

where the rescaled currents $\hat{\boldsymbol{J}}^{(1)}$, $\hat{\boldsymbol{J}}^{(1)}_{(1\gamma)}$ and $\hat{\boldsymbol{J}}^{(1)}_{(2\gamma)}$ are introduced similarly to Eq. (34). The term $\hat{\boldsymbol{J}}^{(1)}(q_1, q_2)$ in the right-hand side of Eq. (107) is exactly the soft-$q\bar{q}$ current of Eqs. (34)–(37) for the QCD case. The results for the terms $\hat{\boldsymbol{J}}^{(1)}_{(1\gamma)}$ and $\hat{\boldsymbol{J}}^{(1)}_{(2\gamma)}$ are obtained by properly modifying the QCD result in Eqs. (36) and (37).

The rescaled current $\hat{\boldsymbol{J}}^{(1)}_{(2\gamma)}$ in Eq. (107) is entirely due to QED interactions, and it has the following expression:

$$
\begin{aligned}
\hat{\boldsymbol{J}}^{(1)}_{(2\gamma)}(q_1, q_2) = & \left[-e_f^2 \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + 8 + (\delta_R - 1) \right) \right. \\
& \left. - \frac{4}{3} N_{\text{ch.}} \left(\frac{1}{\epsilon} + \frac{5}{3} \right) \right] \hat{\boldsymbol{J}}^{(0)}_{(1\gamma)}(q_1, q_2) \\
& + j_\nu(1, 2)\, e_f^2\, \boldsymbol{\Delta}_f \sum_{\substack{i, j \in H \\ i \neq j}} e_i e_j \\
& \times \left[\left(\frac{p_i^\nu}{p_i \cdot q_{12}} - \frac{p_j^\nu}{p_j \cdot q_{12}} \right) \right. \\
& \times \left(-\frac{2}{\epsilon}(\ell_{i1} + \ell_{j2}) + (\ell_{i1} - \ell_{j2})^2 \right) \\
& + \frac{q_{12}^2}{q_{12\perp ij}^2} \left(\frac{p_i^\nu}{p_i \cdot q_{12}} \right. \\
& \left. \left. + \frac{p_j^\nu}{p_j \cdot q_{12}} \right) 2 L_{ij} \left(\ell_{i1} - \ell_{j2} \right) \right] + \mathcal{O}(\epsilon),
\end{aligned}
\tag{108}
$$

where $\hat{\boldsymbol{J}}^{(0)}_{(1\gamma)}$ is given in Eq. (106). In Eq. (108) the soft fermion f can be either a quark or a lepton and, similarly, the charged hard particles $i, j \in H$ can include quarks and leptons. The factor $N_{\text{ch.}}$ in the right-hand side of Eq. (108) is analogous to the factor $T_R N_f$ of the QCD expressions in the right-hand side of Eqs. (36) and (37). The coefficient $N_{\text{ch.}}$ depends on the squared electric charges of the massless[6] quarks and leptons in the theory, and we have

$$
N_{\text{ch.}} = \sum_\ell e_\ell^2 + N_c \sum_q e_q^2.
\tag{109}
$$

The one-loop term $\hat{\boldsymbol{J}}^{(1)}_{(1\gamma)}$ in Eq. (107) is due to mixed QCD×QED interactions. It has the following explicit expression:

$$
\begin{aligned}
\hat{\boldsymbol{J}}^{(1)}_{(1\gamma)}(q_1, q_2) = \delta_{fq} & \left\{ \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + 8 + (\delta_R - 1) \right) \right. \\
& \left. \times \left[-C_F \hat{\boldsymbol{J}}^{(0)}_{(1\gamma)}(q_1, q_2) - e_f^2 \hat{\boldsymbol{J}}^{(0)}(q_1, q_2) \right] \right.
\end{aligned}
$$

---

[6] Analogously to the QCD case, in the one-loop current $\hat{\boldsymbol{J}}^{(1)}_{(2\gamma)}(q_1, q_2)$ we have not considered and included vacuum polarization effects due to *massive* particles (charged leptons, quarks and $W^\pm$).

$$
\begin{aligned}
& + j_\nu(1, 2)\, e_f\, \boldsymbol{t}^c \sum_{\substack{i, j \in H \\ i \neq j}} \left(e_i\, T_j^c + e_j\, T_i^c \right) \\
& \times \left[\left(\frac{p_i^\nu}{p_i \cdot q_{12}} - \frac{p_j^\nu}{p_j \cdot q_{12}} \right) \right. \\
& \times \left(-\frac{2}{\epsilon}(\ell_{i1} + \ell_{j2}) + (\ell_{i1} - \ell_{j2})^2 \right) \\
& + \frac{q_{12}^2}{q_{12\perp ij}^2} \left(\frac{p_i^\nu}{p_i \cdot q_{12}} \right. \\
& \left. \left. + \frac{p_j^\nu}{p_j \cdot q_{12}} \right) 2 L_{ij} \left(\ell_{i1} - \ell_{j2} \right) \right] + \mathcal{O}(\epsilon) \Bigg\},
\end{aligned}
\tag{110}
$$

where $\hat{\boldsymbol{J}}^{(0)}$ and $\hat{\boldsymbol{J}}^{(0)}_{(1\gamma)}$ are the tree-level currents in the right-hand side of Eq. (105). We note that $\hat{\boldsymbol{J}}^{(1)}_{(1\gamma)}$ is entirely proportional to the Kronecker delta symbol δ_{fq} and, consequently, it is not vanishing only if the soft fermion f is a quark. Therefore, if the soft fermion f is a charged lepton, the total one-loop current $\boldsymbol{J}^{(1)}_{f\bar{f}}$ in Eq. (107) receives a non-vanishing contribution only from the QED interaction term $\hat{\boldsymbol{J}}^{(1)}_{(2\gamma)}$.

In Sect. 4 we have discussed the singularity at $q_{12\perp ij} \to 0$ of the current $\hat{\boldsymbol{J}}^{(1)}$ for soft-$q\bar{q}$ QCD radiation at the one-loop level, and we have concluded that it has a purely non-abelian character. The results for $\hat{\boldsymbol{J}}^{(0)}_{(2\gamma)}$ and $\hat{\boldsymbol{J}}^{(0)}_{(1\gamma)}$ are consistent with this conclusion, since the expressions in Eqs. (108) and (110) do not have the transverse-momentum singularity. Although the right-hand side of Eqs. (108) and (110) include the factor $1/q_{12\perp ij}^2$, its singular contribution at $q_{12\perp ij} \to 0$ turns out to be antisymmetric under $i \leftrightarrow j$ (see Eqs. (41)–(43) and accompanying comments) and it cancels by summing over $i, j \in H$.

6.2 The square of the soft-$f\bar{f}$ current

The singular behaviour of squared amplitudes for soft-$f\bar{f}$ radiation is controlled by the square of the current in Eq. (104). We have

$$
\begin{aligned}
|\boldsymbol{J}_{f\bar{f}}(q_1, q_2)|^2 = & \left[\boldsymbol{J}^{(0)}_{f\bar{f}}(q_1, q_2) \right]^\dagger \boldsymbol{J}^{(0)}_{f\bar{f}}(q_1, q_2) \\
& + \left\{ \left[\boldsymbol{J}^{(0)}_{f\bar{f}}(q_1, q_2) \right]^\dagger \boldsymbol{J}^{(1)}_{f\bar{f}}(q_1, q_2) + \text{h.c.} \right\} + \cdots,
\end{aligned}
\tag{111}
$$

where the dots stand for higher-loop contributions (i.e., terms of $\mathcal{O}\left((g_S^2)^{4-n}(g^2)^n\right)$ with $0 \leq n \leq 4$).

Using Eq. (105), the tree-level term in Eq. (111) is

$$
\begin{aligned}
\left[\boldsymbol{J}^{(0)}_{f\bar{f}}(q_1, q_2) \right]^\dagger \boldsymbol{J}^{(0)}_{f\bar{f}}(q_1, q_2) \equiv & \left(g_S\, \mu^\epsilon \right)^4 |\hat{\boldsymbol{J}}(q_1, q_2)|^2_{(0\ell)} \\
& + \left(g\, \mu^\epsilon \right)^4 |\hat{\boldsymbol{J}}(q_1, q_2)|^2_{(0\ell; 2\gamma)},
\end{aligned}
\tag{112}
$$

where $|\hat{\boldsymbol{J}}(q_1, q_2)|^2_{(0\ell)}$ is the pure QCD contribution given in Eq. (52). We note that the right-hand side of Eq. (112) does not include a term proportional to $g_S^2 g^2$ (such QCD×QED interference is proportional to $[\hat{\boldsymbol{J}}^{(0)}]^\dagger \hat{\boldsymbol{J}}^{(0)}_{(1\gamma)}$ and it leads to an overall vanishing colour factor, $\mathrm{Tr}(\boldsymbol{t}^c \boldsymbol{\Delta}_f) = 0$). The term $|\hat{\boldsymbol{J}}|^2_{(0\ell;2\gamma)}$ in Eq. (112) is due to QED interactions, and it is

$$
\begin{aligned}
|\hat{\boldsymbol{J}}(q_1, q_2)|^2_{(0\ell;2\gamma)} &= \left[\hat{\boldsymbol{J}}^{(0)}_{(1\gamma)}(q_1, q_2) \right]^\dagger \hat{\boldsymbol{J}}^{(0)}_{(1\gamma)}(q_1, q_2) \\
&= -\left(\delta_{f\ell} + N_c\, \delta_{fq} \right) e_f^2 \\
&\quad \frac{1}{2} \sum_{\substack{i,j\,\in H \\ i \neq j}} e_i\, e_j\; w_{ij}(q_1, q_2),
\end{aligned} \tag{113}
$$

where the function w_{ij} is given in Eq. (53).

The one-loop term in the squared current of Eq. (111) includes all possible contributions that are proportional to the powers $(g_S^2)^{3-n}(g^2)^n$ with $0 \leq n \leq 3$. We write it in the following form:

$$
\begin{aligned}
&\left[\boldsymbol{J}^{(0)}_{f\bar{f}}(q_1, q_2) \right]^\dagger \boldsymbol{J}^{(1)}_{f\bar{f}}(q_1, q_2) + \text{h.c.} \\
&= (\mu^\epsilon)^6 \left(|q_{12}^2| \right)^{-\epsilon} c_\Gamma \left\{ g_S^6\, |\hat{\boldsymbol{J}}(q_1, q_2)|^2_{(1\ell)} \right. \\
&\quad \left. + \sum_{n=1}^{3} (g_S^2)^{3-n} (g^2)^n\, |\hat{\boldsymbol{J}}(q_1, q_2)|^2_{(1\ell;n\gamma)} \right\},
\end{aligned} \tag{114}
$$

where $|\hat{\boldsymbol{J}}(q_1, q_2)|^2_{(1\ell)}$ is the pure QCD contribution given in Eq. (61). Using Eqs. (105) and (107), the other contributions in the right-hand side of Eq. (114) are given in terms of the rescaled currents $\hat{\boldsymbol{J}}^{(0)}$, $\hat{\boldsymbol{J}}^{(0)}_{(1\gamma)}$, $\hat{\boldsymbol{J}}^{(1)}$, $\hat{\boldsymbol{J}}^{(1)}_{(1\gamma)}$ and $\hat{\boldsymbol{J}}^{(1)}_{(2\gamma)}$.

The one-loop contribution $|\hat{\boldsymbol{J}}(q_1, q_2)|^2_{(1\ell;3\gamma)}$ is entirely due to QED interactions, and we explicitly obtain

$$
\begin{aligned}
&|\hat{\boldsymbol{J}}(q_1, q_2)|^2_{(1\ell;3\gamma)} = \left[\hat{\boldsymbol{J}}^{(0)}_{(1\gamma)}(q_1, q_2) \right]^\dagger \hat{\boldsymbol{J}}^{(1)}_{(2\gamma)}(q_1, q_2) + \text{h.c.} \\
&= \left(\delta_{f\ell} + N_c\, \delta_{fq} \right) e_f^2 \left\{ -\frac{1}{2} \sum_{\substack{i,j\,\in H \\ i\neq j}} e_i\, e_j\; w_{ij}^{[S](f)}(q_1, q_2) \right. \\
&\quad \left. - \sum_{k\in H} \sum_{\substack{i,j\,\in H \\ i\neq j}} e_f\, e_k\, e_i\, e_j\, 2\, F_{ijk}^{[A]}(q_1, q_2) \right\},
\end{aligned} \tag{115}
$$

where $F_{ijk}^{[A]}(q_1, q_2)$ is given in Eq. (68) and the one-loop function $w_{ij}^{[S](f)}(q_1, q_2)$ is

$$
\begin{aligned}
w_{ij}^{[S](f)}(q_1, q_2) &= \left\{ w_{ij}(q_1, q_2) \left[-e_f^2 \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} - \pi^2 \right. \right. \right. \\
&\quad \left. \left. +8 + (\delta_R - 1) \right) - \frac{4}{3} N_{\text{ch.}} \left(\frac{1}{\epsilon} + \frac{5}{3} \right) \right] \\
&\quad \left. + \mathcal{O}(\epsilon) \right\} + (q_1 \leftrightarrow q_2).
\end{aligned} \tag{116}
$$

Note that $w_{ij}^{[S](f)}$ explicitly depends on the squared electric charge e_f^2 of the radiated soft fermion f.

We note that the result in Eq. (115) has a charge symmetric contribution (which is proportional to the two-particle correlation function $w_{ij}^{[S](f)}(q_1, q_2)$) and an abelian charge-asymmetry contribution that is proportional to the momentum function $F_{ijk}^{[A]}(q_1, q_2)$. This structure is consistent with the QCD result in Eq. (61), since the charge symmetric three-particle correlations in Eq. (61) are purely non-abelian. At variance with the expression in Eq. (61), in the right-hand side of Eq. (115) we do not explicitly distinguish between two-particle and three-particle charge-asymmetry correlations (i.e., the summed index k can also be equal to either i or j). In the QCD case, we also noticed that three-particle correlations do not contribute to the squared of the soft-$f\bar{f}$ current for emission from three hard partons (see Eqs. (71) and (73)). A corresponding observation does not apply to the one-loop contribution in Eq. (115). For example, we can consider soft-$f\bar{f}$ emission from the hard-scattering process $\bar{u}d \to W^- \to \bar{\nu}_e e^-$ (the charges of the outgoing hard particles are $\{+2/3, +1/3, -1\}$) and we see that the product $e_k e_i e_j$ of three distinct charges in Eq. (115) does not vanish.

The terms $|\hat{\boldsymbol{J}}|^2_{(1\ell;1\gamma)}$ and $|\hat{\boldsymbol{J}}|^2_{(1\ell;2\gamma)}$ in Eq. (114) are due to mixed QCD×QED interactions.

The contribution $|\hat{\boldsymbol{J}}|^2_{(1\ell;1\gamma)}$ can be regarded as a one-loop QED correction to the QCD radiation of the soft fermion–antifermion pair. We obtain the following result:

$$
\begin{aligned}
&|\hat{\boldsymbol{J}}(q_1, q_2)|^2_{(1\ell;1\gamma)} = \left\{ \left[\hat{\boldsymbol{J}}^{(0)}(q_1, q_2) \right]^\dagger \hat{\boldsymbol{J}}^{(1)}_{(1\gamma)}(q_1, q_2) \right. \\
&\quad \left. + \left[\hat{\boldsymbol{J}}^{(0)}_{(1\gamma)}(q_1, q_2) \right]^\dagger \hat{\boldsymbol{J}}^{(1)}(q_1, q_2) \right\} + \text{h.c.} \\
&= \delta_{fq}\, T_R \left\{ \frac{1}{2} \sum_{\substack{i,j\,\in H \\ i\neq j}} \boldsymbol{T}_i \cdot \boldsymbol{T}_j \right. \\
&\quad \times \left[w_{ij}(q_1, q_2)\, e_f^2 \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} - \pi^2 + 8 \right. \right. \\
&\quad \left. \left. + (\delta_R - 1) + \mathcal{O}(\epsilon) \right) + (q_1 \leftrightarrow q_2) \right] \\
&\quad - e_f \sum_{k\in H} \sum_{\substack{i,j\,\in H \\ i\neq j}} \left(e_i\, \boldsymbol{T}_k \cdot \boldsymbol{T}_j + e_j\, \boldsymbol{T}_i \cdot \boldsymbol{T}_k \right. \\
&\quad \left. \left. + e_k\, \boldsymbol{T}_i \cdot \boldsymbol{T}_j \right) 2\, F_{ijk}^{[A]}(q_1, q_2) \right\},
\end{aligned} \tag{117}
$$

where the charge symmetric function $w_{ij}(q_1, q_2)$ and the charge-asymmetry function $F_{ijk}^{[A]}(q_1, q_2)$ are given in Eqs. (53) and (68), respectively. We note that the one-loop term in Eq. (117) is not vanishing only if the soft fermion is a quark.

Similarly to Eq. (115), the summed index k in Eq. (117) can also be equal to either i or j.

The term $|\hat{\boldsymbol{J}}|^2_{(1\ell;2\gamma)}$ can be regarded as a one-loop QCD correction to the tree-level QED radiation (see Eq. (113)) of the soft fermion–antifermion pair. Its explicit expression is

$$
|\hat{\boldsymbol{J}}(q_1, q_2)|^2_{(1\ell;2\gamma)} = \left[\hat{\boldsymbol{J}}^{(0)}_{(1\gamma)}(q_1, q_2)\right]^\dagger \hat{\boldsymbol{J}}^{(1)}_{(1\gamma)}(q_1, q_2) + \text{h.c.}
$$

$$
= \delta_{fq} N_c e_f^2 \frac{1}{2} \sum_{\substack{i,j \in H \\ i \neq j}} e_i e_j \Bigg\{ w_{ij}(q_1, q_2)
$$

$$
\times \left[C_F \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} - \pi^2 + 8 + (\delta_R - 1) \right) + \mathcal{O}(\epsilon) \right]
$$

$$
+ (q_1 \leftrightarrow q_2) \Bigg\}, \tag{118}
$$

where the function $w_{ij}(q_1, q_2)$ is given in Eq. (53). We note that, analogously to Eq. (117), the term $|\hat{\boldsymbol{J}}|^2_{(1\ell;2\gamma)}$ is not vanishing only if the soft fermion is a quark. Unlike the cases of the one-loop terms in Eqs. (115) and (117), charge-asymmetry contributions do not appear in $|\hat{\boldsymbol{J}}|^2_{(1\ell;2\gamma)}$.

By direct inspection of the tree-level and one-loop results in Eqs. (113), (115), (117) and (118), we see that the charge symmetric (charge-asymmetry) contributions are proportional to even (odd) powers of e_f, as expected from charge conjugation symmetry.

7 Summary

We have considered the radiation of two or more soft partons in QCD hard scattering. In this soft limit the scattering amplitude is singular, and the singular behaviour is controlled in factorized form by a multiparton soft current, which has a process-independent structure. At loop level, the scattering amplitudes and the soft current have UV and IR divergences, which we regularize in the form of ϵ poles by analytic continuation in $d = 4 - 2\epsilon$ space-time dimensions.

We have discussed the general structure of the ϵ-pole divergences of the multiparton soft current. We have considered the soft current at one-loop order and we have presented the explicit form of its ϵ-pole (divergent) contributions. We have also discussed the RS dependence of the one-loop soft current.

In the remaining part of the paper we have considered the specific case of soft $q\bar{q}$ radiation, by presenting a detailed study at one-loop order. Considering arbitrary kinematical regions of the soft-parton and hard-parton momenta, we have explicitly computed the one-loop current by including the finite terms at $\mathcal{O}(\epsilon^0)$. We find a relatively simple expression, which, for instance, includes powers of logarithmic functions but no dilog functions.

We find that the one-loop current produces a new type of singularity if the soft-$q\bar{q}$ pair is radiated with a vanishing transverse momentum with respect to the direction of two colliding hard partons in the initial state. This new transverse-momentum singularity has a quantum (more precisely, absorptive) origin and a purely non-abelian character. Owing to its dynamical origin, the transverse-momentum singularity can appear also in the one-loop current for double soft-gluon emission.

We have computed the one-loop contribution of the squared current for soft-$q\bar{q}$ emission and the ensuing colour correlations for squared amplitudes of generic multiparton hard-scattering processes. We have also explicitly considered the specific cases of processes with two or three hard partons, in which the colour correlation structure can be partly simplified.

We find that, despite its absorptive origin, the new one-loop transverse-momentum singularity contributes to squared amplitudes (and, hence, cross sections) of scattering processes with two initial-state colliding partons (hadrons) and two or more hard partons (jets) in the final state.

At variance with the case of multiple soft-gluon radiation, the emission of soft fermions and antifermions lead to charge asymmetry effects. We have discussed in details the charge asymmetry contributions of the one-loop squared current for soft $q\bar{q}$ radiation.

We have finally generalized our QCD study of soft $q\bar{q}$ emission to the study of QED and mixed QCD×QED radiative corrections in the context of soft fermion–antifermion radiation. We have presented the corresponding one-loop results for the soft current and its square.

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