Massive Amplitudes from Twistors on the Worldsheet

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Abstract

The subject of this thesis are ambitwistor string models that describe massive particles by gauging currents to implement a symmetry reduction. Because the amplitude formulae one obtains as correlators in these models are really reductions of the ones presented in [1, 2], the body of the thesis will open with a discussion of properties and features of the six-dimensional superamplitudes that the massive formulae will inherit. Two different instances of symmetry reduction in the ambitwistor string will be considered. The first is a massive version of the RNS ambitwistor string. This provides a derivation of massive amplitude formulae that have support on massive scattering equations such as the ones predicted by Dolan and Goddard [3] and Naculich [4], together with a solid understanding of mass assignment both to external and propagating particles. The second consists of four dimensional twistorial models that will be shown to have an alternative interpretation as theories of maps into the phase space of complexified massive particles. This representation is more suitable to describe supersymmetric theories, such as the Coulomb branch of $\mathcal{N} = 4$ sYM. An interesting class of theories is presented, which is obtained by symmetry reduction along the R-symmetry generators. For supergravity, this produces CSS gauged supergravities in four dimensions. In these theories a novel instance of 'massive' double copy structure arises.

Chapters 2, 3 and 4 are based on:

- G. Albonico, Y. Geyer and L. Mason, *Recursion and worldsheet formulae for 6d superamplitudes*, *JHEP* **08** (2020) 066, [2001.05928]
- G. Albonico, Y. Geyer and L. Mason, *From Twistor-Particle Models to Massive Amplitudes*, *SIGMA* **18** (2022) 045, [2203.08087]

and they present considerable overlap with the original papers. Chapters 5 and 6 contain unpublished work in collaboration with Yvonne Geyer and Lionel Mason.

Contents

Chapter 1

Introduction

When I first learned about scattering in particle physics, I was given the image of a small child who learns what objects and toys do by knocking them into one another. When toys are elementary particles and his arms have grown up to be high-energy colliders, he can try to understand the fundamental laws of nature by observing the results of scattering. Scattering amplitudes are crucial physical observables in many branches of physics. A meeting point between theory and experiment, they have been at the core of the development of the Standard Model of particle physics. Today, the pursuit of higher precision in standard model prediction and the computation of classical gravitational observables from quantum scattering amplitudes put them on the front line in the search for New Physics and in the booming field of gravitational-wave astronomy. Reassured by the idea that someone might be inspired by their work to make concrete predictions, the mathematical physicist can then safely venture in a world of abstraction.

Recent years have seen incredible progress in the computation of scattering amplitudes, to the point that there is today a whole field of study that goes by this name. Famously, one motivation for these developments were the remarkably simple formulae for MHV gluon scattering amplitudes [6]:

$$A^{\mathrm{MHV}}\left(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+\right)=\frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle (n-1)n\rangle\langle n1\rangle}\,, \tag{1.1}$$

where $\pm$ labels the helicity of the particles and the formula is written in spinor-helicity variables. Deriving amplitudes for the same processes in the traditional formalism of lagrangian and Feynman diagrams would involve pages-long computations, only to simplify under great efforts, indicating that some organising principle is obscured. The amplitude community is a very heterogeneous one, composed of particle phenomenologists, string theorists and a wide range of mathematicians who were attracted by the beautiful structures underlying these physical observables. The guiding principle in all works that fall under this umbrella is a search for simplicity, a desire to explore how symmetries and physical principles constrain amplitudes as mathematical objects, bypassing their description in terms of quantum fields. The variety of backgrounds of the people involved mean that the field has ancestry in many different lines of research across mathematics and physics. As a result, a number of previously undiscovered dualities, symmetries and connections between seemingly unrelated phenomena have come to light. Before I dive into the subject of the thesis, I will take a tour around the amplitude landscape, pointing out the heritage from various fields of research and referencing their applications in the thesis.

Behind the formulae for gluon scattering amplitudes (1.1), is one of the major tools of all modern amplitude methods, the *spinor-helicity formalism*. In order to manifest the need for it, we'll argue that the description of particles via quantum fields is redundant. This story has by now appeared countless times in the amplitudes literature. Wigner's classification defines particles as unitary irreducible representations of the Poincaré group, momentum eigenstates labelled by p and a set of quantum numbers σ such as helicity h. Under a Lorentz transformation, these are shown to transform in representations of the little group and this behaviour is reflected in the transformation of scattering amplitudes. On the other hand, quantum fields and hence Feynman amplitudes computed from Lagrangians are Lorentz tensors. This forces one to introduce polarisation vectors that give amplitudes with the correct behaviour under Lorentz transformations:

$$\mathcal{A}(p_i,\epsilon_i)=\epsilon_{1,\sigma}^{\mu_1}\cdots\epsilon_{n,\sigma}^{\mu_n}\mathcal{A}_{\mu_1\cdots\mu_n}\,,\qquad\qquad \epsilon_{\sigma}^{\mu}(\Lambda p)=D_{\sigma\sigma'}(W)\Lambda^{\mu}_{\nu}\epsilon^{\nu}_{\sigma'}(p)\,,$$

with $D_{\sigma\sigma'}(W)$ a representation of the little group. For massless particles such vectors ϵ as functions of p,h can only be defined up to equivalence $\epsilon^{\mu}\sim\epsilon^{\mu}+\alpha p^{\mu}$, so that the full amplitude described with field theory has to obey the on-shell Ward identity[1]:

$$\mathcal{A}(p_i, \epsilon_i)|_{\epsilon_j \to p_j} = 0 \qquad \forall j\,. \tag{1.2}$$

One then expects that a more natural description exists that avoids this redundancy by working directly with variables that transform in representations of the little group, instead of (p^μ, h). This is achieved by the spinor-helicity formalism that splits momentum into its 'square roots' that transform in little group transformations. We will describe this in detail in §2.3 and make use of it throughout the thesis.

Quantum field theory is a century old science and has lead to the greatest advances in our understanding of particle interactions. Obviously the idea is not to start from scratch but rather to find a new language to express what we know. Quantum fields were defined axiomatically [8] based on the physical principles of locality, unitarity and causality. Looking directly at the amplitude, locality gives us information on the singularity structure of amplitudes. In particular it tells us that any pole of a tree level scattering amplitude corresponds to a propagating particle going on-shell. Unitarity, through the statement of the optical theorem, selects processes that factorize over their poles as the product of two subamplitudes corresponding to the diagrams on either side of the on-shell propagator:

$$\lim_{P^2 \to 0} \mathcal{A}_n = \frac{1}{P^2} \mathcal{A}_L \mathcal{A}_R \tag{1.3}$$

We will refer to this property as factorization and the subamplitudes are on-shell vertices, the basic building blocks of modern on-shell recursions such as BCFW [9–11]. These methods exploit (1.3) to build amplitudes iteratively from processes involving fewer particles [12]. We will use a BCFW argument in §3.4 to prove superamplitude formulae. Off-shell recursion relations by Berends and Giele [13] are ancestors of modern on-shell recursion and remain to this day an algorithm employed for efficient evaluation of scattering amplitudes.

If locality and unitarity already give very powerful constraints on scattering amplitudes

[1] In an 'amplitude' approach [7], gauge invariance in the form (1.2) has been investigated as a more fundamental principle and it was shown to uniquely select Yang Mills amplitudes.

and methods to compute them, the role of causality is at the heart of a beautifully rich subject that is fundamental to the contents of this thesis. In the late 1960s, Roger Penrose first formulated twistor theory [14–17] as a new framework for a quantum understanding of spacetime. The conventional point of view at the time was that quantization should be applied to the metric as a field on spacetime, thus introducing quantum aspects to the notion of null, timelike and spacelike directions - and hence to causality. In Penrose's perspective, causality should be a fundamental principle that is untouched by quantum corrections, indicating that null-rays are more fundamental objects than spacetime *events*. Broadly speaking, twistor space is the space of such null-rays in spacetime and a point in spacetime is only determined as the focus point of a set of null-rays. This space has a natural action of conformal transformations and a non-local correspondence with spacetime, pointing to a quantised picture where spacetime points rather than null cones are smeared out. Consolidating the role of twistor space, Penrose found that massless free fields are encoded in geometric data on twistor space [18]. Lacking a connection with quantum field theory, for decades the framework of twistor theory was mainly oriented towards the study of integrable systems and geometry, see [19] for a review. Twistor diagrams, introduced by Penrose as a twistorial analogue of Feynman diagrams, were developed by Hodges in the ′80s and ′90s, coming very close to a breakthrough in the connection with field theory amplitudes [20].

In the meantime, string theory had lived through two revolutions and made it more and more natural to think about amplitudes having to do with moduli spaces of Riemann surfaces. Inspired by Nair's interpretation [21] of the Parke-Taylor formula (1.1) for MHV amplitudes in Yang-Mills as a current algebra correlator on a Riemann sphere, Witten was the first to combine the power of twistors with that of string theory. In the original twistor-string [22–24] the target space is (supersymmetric) twistor space $\mathbb{CP}^{3|4}$. At tree level and N^{k-2}MHV degree scattering amplitudes in $\mathcal{N} = 4$ super Yang Mills were shown to localise on degree $k - 1$ curves in twistor space. Soon after, the RSV formula [25,26] re-expressed these amplitude formulae as sums over residues. These results showed that, although a string theory, the twistor string was remarkably simpler than all of its known cousins: correlators appeared to be localised on solutions of a set of equations [27], greatly simplifying the problem of performing the moduli integrals.

Hodges' gravitational analogue of (1.1) in terms of reduced determinants [28] inspired Skinner and Cachazo's worldsheet formula [29,30] and twistor string [31] for $\mathcal{N} = 8$ supergravity, leading to the interpretation of Hodges' reduced determinants as fermion correlators on the worldsheet.

Any formula one might hope to obtain from worldsheet theories such as the twistor string must obey the requirements we discussed earlier regarding unitarity, locality and the singularity structure of amplitudes. In quantum field theory, it is natural to describe singularities on the space of kinematic configurations:

$$\mathcal{K} = \{\{k_i^\mu\}\, i = 1, ...n \,|\, \sum_i k_i = 0,\, k_i^2 = 0\}\,. \tag{1.4}$$

In general an n-point tree level scattering amplitude becomes singular as the sum of a subset of momenta becomes null, corresponding to a physical particle propagating in an internal channel:

$$s_{a_1...a_j} = (k_{a_1} + ... + k_{a_j})^2 = 0 \qquad a_i \in 1, ...n\,. \tag{1.5}$$

Different subsets of momenta can go simultaneously on shell but not all channels are compatible (e.g. the channels $s_{12} = 0$ and $s_{13} = 0$ are inconsistent), so that a precise characterisation of the singularities is quite complicated. Worldsheet formulae derived from the twistor string point us toward a picture where we have an auxiliary space that offers a better understanding of singular configurations and a map from this space to the space on kinematic invariants such that a correspondence is established between the singularities of the two spaces.

At tree level this space is the moduli space $\mathfrak{M}_{0,n}$ of the n-punctured Riemann sphere, a space of dimension $n-3$ because of the $SL(2,\mathbb{C})$ symmetry acting on it. The moduli space has boundaries corresponding to the configurations in which the Riemann sphere collapses to two subspheres glued at an extra puncture. These singular configurations can be identified by the subset of punctures lying in one of the subspheres and one can find all compatible singularities by examining iteratively the singularities of the remaining subspheres until all the subspheres contain exactly three punctures.

Suppose we want to derive the map that relates degenerations of the moduli space and singularities of the scattering amplitude in such a way that degenerations of the sphere where punctures $\{\sigma_i\}_{i\in S}$ belong to one subsphere correspond to poles of the form $(\sum_{i\in S} k_i)^2 \to 0$. Wanting to recover the kinematic data from the Riemann sphere, we define a meromorphic one form on the Riemann sphere that has at most simple poles, one at each puncture, where the residues are given by

$$\mathrm{Res}_{\sigma=\sigma_i} P(\sigma) = k_i \,, \tag{1.6}$$

which is equivalent to

$$\bar{\partial} P = 2\pi i \mathrm{d}\sigma \sum_i k_i \bar{\delta}(\sigma - \sigma_i) \,. \tag{1.7}$$

Fixing n punctures on the Riemann sphere, i.e. a point on $\mathfrak{M}_{0,n}$, this is solved by

$$P(\sigma) = \sum_i \frac{k_i}{\sigma - \sigma_i} \mathrm{d}\sigma \,. \tag{1.8}$$

Momentum conservation ensures that the pole at infinity vanishes. In order to establish a map that gives the desired correspondence between singular configurations, we are led to demand that the quadratic differential P^2 vanishes:

$$P^2 = \sum_{1<i<j<n} \frac{2k_i \cdot k_j}{(\sigma - \sigma_i)(\sigma - \sigma_j)} \mathrm{d}\sigma^2 = 0 \,. \tag{1.9}$$

Because the external momenta k_i are null, P^2 is again a meromorphic function with only simple poles, so that it is enough to impose

$$\mathrm{Res}_{\sigma=\sigma_i} P^2(\sigma) = k_i \cdot P(\sigma_i) = \sum_{j\neq i} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} \,\mathrm{d}\sigma = 0 \,. \tag{1.10}$$

This is precisely the statement of the *scattering equations*. By virtue of momentum conservation, the set of equations is invariant under $SL(2,\mathbb{C})$ transformations of the Riemann sphere, so that three punctures can be fixed to arbitrary distinct values. One can also show that there are only $n-3$ independent equations. Given the kinematic data (1.4), the scattering equations have $(n-3)!$ solutions for the set of punctures $\{\sigma_i\}_{i=1}^n$ [32]. This led Cachazo, He and Yuan [33–35] to conjecture that amplitude formulae for massless particles at tree level can be written as an integral over the moduli space pf the n-punctured Riemann sphere:

$$\mathcal{A}_n^{\text{CHY}} = \delta^d(\sum k_i) \int_{\mathfrak{M}_{0,n}} \frac{\prod' \, \mathrm{d}\sigma_i \bar{\delta}\left(k_i \cdot P\left(\sigma_i\right)\right)}{\text{Vol}(SL(2,\mathbb{C}))} I(\{k, e, \sigma\})\,, \tag{1.11}$$

where the measure is quotiented by the action of $SL(2,\mathbb{C})$ that fixes three punctures. The prime on the product indicates that the measure only imposes $n-3$ scattering equations in a permutation invariant way. Since there are $n-3$ delta functions for $n-3$ integration variables, this integral is actually completely localised on the solutions to the scattering equations. Because we have not made any assumptions about the type of massless particles involved nor on the number of dimensions that the momenta live in, we expect this to hold universally for all of massless scattering, indicating that there is a common component to all theories that can be separated from the matter specific contributions, here contained in the worldsheet integrand $I(\{k, e, \sigma\})$.

In order to complete the formula, we need an understanding of the integrands one would need to insert in (1.11) to obtain amplitudes for scalars, gluons and gravitons. Parallel to the development of worldsheet formulae, Bern, Carrasco and Johannsson had discovered that color ordered scattering amplitudes for Yang-Mills theories enjoy a set of relations (known as BCJ relations) by virtue of a duality between color and kinematic factors [36]. The way in which color and kinematic could be disentangled so neatly was suggestive of a picture where one could pair them up as color-color, color-kinematic, kinematic-kinematic: the remarkable result of the BCJ double copy [37] is that gravity can be obtained by replacing the color factor in YM with another kinematic one. This type of duality had an ancestor in the Kawai-Lewellen-Tye relations expressing amplitudes for closed strings as products of pairs of open string partial amplitudes via a specific bilinear form called the (KLT) momentum kernel [38]. In [39], Cachazo showed that individual residues contributing to the RSV formula of [26] satisfied the BCJ relations, not only their sum, which allowed a double copy construction of gravity formulae in four dimensions [40]. This line of research led to the discovery that Parke-Taylor integrands evaluated at different solutions of the scattering equations (1.10) are orthogonal with respect to the KLT kernel [34] and to the completion of the formula of Cachazo, He and Yuan for scattering of massless particles in all dimensions [35]. The integrands I in (1.11) were conjectured to be a product of two factors $I = I_L I_R$ with each of I_L and I_R transforming under Möbius transformations as a 1-form in each σ_i.[2] For scalars, gluons and gravitons, each of these were taken from two possibilities. The first was a Parke-Taylor factor that depends on a permutation p

$$\mathrm{PT}(p) = \prod_{i=1}^{n} \frac{1}{\sigma_{p(i)\,p(i+1)}} \,. \tag{1.12}$$

The second was the CHY reduced Pfaffian $\mathrm{Pf}'(M)$ where M is the skew matrix that depends on polarization vectors $e_{i\mu}$ associated to each null momenta $k_{i\mu}$:

$$M = \begin{pmatrix} A & C \\ -C^T & B \end{pmatrix} , \quad A_{ij} = \frac{k_i \cdot k_j}{\sigma_{ij}} , \quad B_{ij} = \frac{e_i \cdot e_j}{\sigma_{ij}} , \quad C_{ij} = \begin{cases} \frac{k_i \cdot e_j}{\sigma_{ij}} , & i \neq j \\ \sum_l \frac{k_i \cdot e_l}{\sigma_{li}} , & i = j \,. \end{cases} \tag{1.13}$$

On the support of the scattering equations, the matrices M have a two-dimensional kernel, and so to obtain a nontrivial Pfaffian, one must delete two rows and columns, say i and j produce M_{ij} and then define the reduced Pfaffian

$$\mathrm{Pf}'(M) := \frac{1}{\sigma_{ij}} \mathrm{Pf}(M_{ij}). \tag{1.14}$$

For biadjoint scalar amplitudes, the integrand is a product of two Parke-Taylors, for Yang-Mills a Parke-Taylor and a Pfaffian, and for gravity two Pfaffians. Further integrands were conjectured for a variety of theories including Einstein-Yang-Mills and DBI [41, 42].

The form of the amplitude (1.11) was strongly suggestive of a stringy origin and it wasn't long before Mason and Skinner introduced ambitwistor strings [43] and showed they reproduced CHY formulae as correlators. These novel string theories are theories of holomorphic maps into the space of complex null geodesics, ambitwistor space. Similarly to twistors, ambitwistors have a non-local correspondence with spacetime and they admit representations in terms of twistors in various low-dimensional spacetimes, which connect them to Witten's twistor string. Ambitwistor strings will be at the center of this thesis and their original form

[2] It is customary to suppress the form degree when writing expresions for the integrands.

will be reviewed in detail in the next chapter. While the validation of the CHY formulae was an important result, the importance of these worldsheet models goes far beyond it. They provide a framework that can be modified to approach loop integrands, to introduce different matter systems, to include supersymmetry and to find new parametrisations of amplitudes.

Up until this point we have focused on results about massless scattering. Despite the fact that most of the particles we observe have mass, due to technical aspects of their description, massive on-shell methods came later. As we argued earlier, modern amplitude methods often rely on novel parametrisations that shed light onto the symmetries and otherwise hidden features of known quantum field theories. The first formulations of spinor helicity for massive particles appeared decades ago [44, 45] and were exploited in the first on-shell approaches to amplitudes in the Coulomb branch of $\mathcal{N} = 4$ SYM [46–52]. These works explored how state-of-the-art amplitude methods such as on-shell recursion, supersymmetric Ward identities, could be adapted to account for massive particles. The main shortcoming of these primordial forms of massive spinor-helicity was the need to pick a decomposition of massive momenta into a pair of massless ones, thus inserting auxiliary parameters into the amplitude formulae and often breaking Lorentz invariance: this gave the image that massive amplitudes didn't have as much to gain from on-shell methods as massless ones. All of these issues were resolved in [12], where spinorial variables for massive momenta were introduced that are little group covariant and give rise to beautifully compact formulae.

Over the past decade, massive amplitudes have been the object of increasing interest, even more so after the detection of gravitational waves from a black hole merger [53]. A long standing program aimed at deriving classical gravity observables from scattering amplitudes has produced results relevant to black hole scattering [54–69], see [70] for a review. While this community has mainly been interested in applications of the double copy, progress has also been made on the front of massive BCFW relations, both in terms of massive shifts [71] and of the formulae that have been produced [72, 73].

The work contained in this thesis has the purpose of introducing worldsheet models of massive particles that produce amplitude formulae relying on a massive version of the scattering equations. The first attempts at generalizing the scattering equations to massive particles came from Dolan and Goddard [3, 32], who proposed a form of the scattering equations for scalars of all equal masses in a ϕ^3 theory. This form of the scattering equations however breaks permutation invariance. Naculich generalised these equations to allow for the appearance of a mixture of massive and massless particles in the amplitude, possibly with several mass parameters [4,74,75]. Naculich's proposal for the massive scattering equations is:

$$E_i := \sum_{j\neq i} \frac{k_i \cdot k_j + \Delta_{ij}}{\sigma_{ij}} = 0, \qquad \sigma_i \in \mathbb{CP}^1, \qquad i = 1, ...n\,, \tag{1.15}$$

where Δ is such that

$$\Delta_{ij} = \Delta_{ji} \qquad\qquad \sum_{j\neq i} \Delta_{ij} = m_i^2\,, \tag{1.16}$$

The conditions (1.16) are necessary to preserve the invariance of the set of equations (1.15) under $SL(2,\mathbb{C})$ transformations. The Δs are functions involving the masses m_i of the external particles. For a general assignment of masses, such functions might be hard to determine and are not guaranteed to exist.

Another idea that has long been exploited is to consider massive particles in four dimensions as massless in six and five [76–81]. The six-dimensional worldsheet formulae of [1, 82] where dimensionally reduced to four dimensions to write massive particles on the Coulomb branch. Exploiting the independence of the original CHY representation (1.11) on the number of spacetime dimensions, Naculich [4] reformulated the equations (1.15) by taking the external momenta to lie in $(d+M)$ dimensions so that it can be split into a $d-$dimensional physical momentum and an $M-$dimensional *internal momentum* k[3]:

$$K_i = (k_i|\mathsf{k}_i) \qquad k_i^2 = \mathsf{k}_i^2 =: m_i^2\,. \tag{1.17}$$

With this notation, the scattering equations for a set of n external momenta K_i become:

$$E_i = \sum_{j\neq i} \frac{k_i \cdot k_j - \mathsf{k}_i \cdot \mathsf{k}_j}{\sigma_{ij}} = 0\,. \tag{1.18}$$

[3] Notice the use of a special character k here to distinguish internal momentum from spinorial kinematic variables κ.

In his work internal momentum had to be assigned by hand, making it hard to formulate a consistent and complete description of the corresponding amplitude formulae and the theories they could belong to. The models we will present in chapters 4 and 5 describe a variety of massive deformations of known gauge and gravity theories. They assign values of internal momentum systematically and produce amplitude formulae that have support on equations like (1.18) and a spinorial version of these. We will present the models both as standalone theories of maps into a massive version of ambitwistor space and as symmetry reductions of massless models in higher dimensions.

Outline of the thesis The thesis is organised as follows. After a review of background material in chapter 2, we begin by presenting important properties of the six-dimensional super-amplitude of [2] that will be relevant for the formulae derived in the massive models. Chapter 4 presents massive models in four dimensions as theories of holomorphic maps into the phase space of complexified massive particle. In chapter 5 a more general formalism is presented to build models of massive particles obtained by symmetry reduction. The first half of the chapter covers the treatment of massive models in the RNS ambitwistor string and their implications at the level of the amplitude formulae. In the second part twistorial models such as the ones of chapter 4 are presented as symmetry reductions and we explore different ways of introducing masses in maximally supersymmetric gauge and gravity theories. Finally chapter 6 gives a massless model in four dimension that is equivalent to the ones in [83] and allows us write twistorial version of the gluing operator of [84], opening the way to loop integrand formulae.

CHAPTER 2

Review

This chapter contains a review of background material that is relevant to the rest of the thesis. It is intended both as a technical treatment of the motivational arguments discussed in the introduction and as an aide-mémoire to support chapters §3-6. It does not aspire to be complete nor always rigorous: we hope to convey the gist of the subject and will point the reader to the relevant literature and modern reviews whenever possible. While in the previous chapter we have made an argument for CHY formulae from a rather historical perspective, here the main focus will be on the ambitwistor string, the formulae being presented as correlators in this type of theories. We begin with a brief review of the geometry of ambitwistor space. Section 2.2 contains a review of the RNS models of [42, 43]. After a section on spinor-helicity variables for massless particles, the attention will turn to twistorial realisations of ambitwistor space and the models for which they lay the basis [1, 83, 85]. Section 2.5 presents the $6d$ superamplitudes of [1,2]. This is meant as a preview of chapter 3, where properties of this formula are discussed in more detail: here we point the reader to the main results so that the following chapter can be consulted rather than read from beginning to end.

Because there is considerable overlap between the models in [83,85] and the ones presented in this thesis, some technical aspects are left to the main body in order to avoid repetitions.

2.1 Ambitwistor space

Ambitwisor space is the space of complex null geodesics in complexified spacetime $\mathcal{M}$ with holomorphic metric g. The name comes from the fact that in four dimensions it can be represented as a quadric in the product of chiral twistor and dual twistor space, as we will describe in §2.4.1. Detailed descriptions of this space can be found in the original papers [86–90] and for modern reviews see [43, 91].

Complex null directions are defined by cotangent vectors $P \in T^*_x\mathcal{M}$ that span the zero energy surface of the Hamiltonian

$$\mathcal{H} = \frac{1}{2} g^{-1}(P, P)\,. \tag{2.1}$$

We can write the space of such null directions:

$$T^*_N = \left\{(X^\mu, P_\mu) \in T^*\mathcal{M} | g^{-1}(P, P) = 0\right\}\ . \tag{2.2}$$

The space of complex null geodesics can be obtained by quotienting out the action of shifts along null directions, parametrised by $P \cdot \partial_X$:

$$\mathbb{A} = \left\{(X^\mu, P_\mu) \in T^*\mathcal{M} | g^{-1}(P, P) = 0\right\} / \ \{P \cdot \partial_X\}\ . \tag{2.3}$$

Projective ambitwistor space $\mathrm{P}\mathbb{A}$ is the space of unscaled null geodesics, obtained by quotienting by the rescaling generated by $\Upsilon = P \cdot \partial_P$ so that the projective scale can be taken to be the scale of P.

By construction, a point in $P\mathbb{A}$ corresponds to a complex null geodesic in $\mathcal{M}$ and conversely a point in spacetime corresponds to a quadric $Q_x \in P\mathbb{A}$ which can be interpreted as the space of complex null rays through x. The non locality in the correspondence is what is responsible for the simplifications in the ambitwistor string representation.

The cotangent bundle is a symplectic manifold, and a more rigorous description of ambitwistor space uses the language of symplectic geometry, defining a symplectic potential $\theta = P_a \, \mathrm{d}X^a$ on $T^*\mathcal{M}$ and associating a Hamiltonian vector field to (2.1) to generate the flow along null geodesics. Ambitwistor space inherits from this a holomorphic $1-$form θ, homogeneous of weight $+1$ in P. On $\mathrm{P}\mathbb{A}$, θ defines what is called a *contact structure* and it encodes information on the complex structure of the space. Non-trivial deformations of the contact structure are cohomology classes $\delta\theta \in H^{0,1}(\mathrm{P}\mathbb{A}, \mathcal{O}_P(1))$ and the (ambitwistor) Penrose transform relates these to non-trivial deformations of the metric on spacetime, i.e. of the conformal structure. More generally, the Penrose transform relates cohomology classes on ambitwistor space to off-shell fields in spacetime. The field equations only arise as quantum consistency conditions in the ambitwistor string.

2.2 The RNS ambitwistor strings

Ambitwistor strings are theories of holomorphic maps from a Riemann surface Σ to projective ambitwistor space $P\mathbb{A}$, as described above. Their action was first formulated by Mason and Skinner [43] as the complexification of the worldline action for a massless particle in $d-$dimensional spacetime. Under their prescription the worldline becomes a Riemann surface Σ with holomorphic coordinate $\sigma \in \mathbb{C}$ and (X^μ, P_μ) on the target space become holomorphic coordinates for the cotangent bundle $T^*\mathcal{M}$ of complexified spacetime.

Bosonic action The simplest ambitwistor string describes a bosonic system:[1]

$$S^B = S^B[X,P] = \frac{1}{2\pi}\int_\Sigma P\cdot\bar{\partial}X - \frac{\tilde{e}}{2}P^2\,, \tag{2.4}$$

where $\bar{\partial}X = \mathrm{d}\bar{\sigma}\partial_{\bar{\sigma}}X$. Here X and P are fields on Σ and therefore carry a conformal weight: while $X \in \Omega^0(\Sigma)$ has weight $(0,0)$, P takes value in the canonical line bundle $K_\Sigma \simeq \Omega^{(1,0)}(\Sigma)$ of holomorphic $(1,0)$ forms. Then $\tilde{e}$ is a $(0,1)$-form taking values in the holomorphic tangent bundle $T_\Sigma \simeq K_\Sigma^{-1}$. This field is a Lagrange multiplier for the constraint $P^2 = 0$ and it is the gauge field for the transformation:

$$\delta\tilde{e} = \bar{\partial}\alpha, \quad \delta X = \alpha P, \quad \delta P = 0\,, \tag{2.5}$$

where $\alpha \in \Omega^0(\Sigma, T_\Sigma)$ is a bosonic gauge parameter. This gauge symmetry generates translations along null geodesics on the complexified cotangent bundle of Minkowski space, therefore

[1] The usual worldsheet diffeomorphism freedom can be parametrised by the gauge field e via the operator $\bar{\partial}_e = \bar{\partial} + e\partial$, where in the action (2.4) the field has been gauge fixed to $e = 0$.

confirming that the fields (X^μ, P_μ) parametrise ambitwistor space, with the gauge field $\tilde{e}$ constraining P to be null and quotienting by the action of (2.5). As recalled in the previous section, projective ambitwistor space is obtained by quotienting by the scale of the null vector P: this assigns projective weight $+1$ to P on $\mathrm{P}\mathbb{A}$ and we identify the canonical line bundle K_Σ with the pullback to the worldsheet of the line bundle $\mathcal{O}_P(1)$ so that we can recognise the target space of 2.4 with $\mathrm{P}\mathbb{A}$.

Models in the ambitwistor string are built out of just a few matter systems on the worldsheet, taken in pairs as:

$$S = S^B + S^L + S^R \,. \tag{2.6}$$

In [42] a variety of choices were introduced for $S^{L/R}$ but here we will focus on the ones that are needed for the models we will study in the course of the thesis, namely those that give rise to gauge and gravity theories.

Current algebra This system is a worldsheet realization of a current algebra $j \in \Omega^0(\Sigma, K_\Sigma \otimes \mathfrak{g})$ of level k, with $\mathfrak{g}$ some Lie algebra, satisfying the standard current algebra OPE:

$$j^a(\sigma) j^b(0) \sim \frac{k\delta^{ab}}{\sigma^2} + \frac{i f^{abc} j^c(\sigma)}{\sigma} + \dots \,, \tag{2.7}$$

where f^{abc} are the structure constants of the algebra $\mathfrak{g}$ and δ^{ab} is the Killing form. This model can be realized via free fermions, WZW models or other constructions that we will indicate generically by S_C, as it represents a color contribution to the model.

Worldsheet fermions Under a choice of spin structure on the worldsheet, the second type of system we consider is built out of a fermionic field $\Psi^\mu \in \Pi\Omega^0(\Sigma, K_\Sigma^{1/2} \otimes \mathbb{C}^d)$ together with a gauge field $\chi \in \Pi\Omega^{0,1}(\Sigma, T_\Sigma^{1/2})$ imposing the constraint $P \cdot \Psi = 0$:

$$S_\Psi = \int g_{\mu\nu} \Psi^\mu \bar{\partial} \Psi^\nu - \chi P \cdot \Psi \,. \tag{2.8}$$

The gauge transformations associated to χ are:

$$\delta X^{\mu} = \epsilon \Psi^{\mu} \quad \delta \Psi^{\mu} = \epsilon P^{\mu} \quad \delta P_{\mu} = 0 \quad \delta \chi = \bar{\partial} \epsilon \,, \tag{2.9}$$

and they generate a form of worldsheet supersymmetry, with $\epsilon \in \Pi\Omega^0(\Sigma, T_{\Sigma}^{1/2})$. We refer the reader to the original discussion in [43] for a better understanding of these gauge transformations and a description of super ambitwistor space.

Models With this choice of matter systems, the simplest model that one can build is the one made up of two current algebrae:

$$S_{\text{BAS}} = S^B + S_C + S_{\tilde{C}} \,, \tag{2.10}$$

describing bi-adjoint scalars. Much like the ϕ^3 theory in QFT, this is a very popular toy model in the amplitude community. Having no polarisation states, its amplitudes are purely kinematic and they are the backbone of gauge and gravitational scattering. In the double copy literature it is known as the *zeroth* copy of gauge theory and gravity. We will describe it in more detail in §5.2.

The heterotic model has one current algebra and one worldsheet fermion system:

$$S_{\text{het}} = S^B + S_C + S_{\Psi} \,. \tag{2.11}$$

It produces amplitudes for Yang Mills. The type II model contains two fermion systems S_{Ψ_r}

$$S_{II} = S^B + S_{\Psi_1} + S_{\Psi_2} \,, \tag{2.12}$$

and it produces amplitudes for type II supergravity.

BRST gauge fixing Gauge fixing worldsheet diffeomorphisms and translations along null geodesics via the BRST procedure introduces fermionic (b, c) and $(\tilde{b}, \tilde{c})$ ghosts associated to the gauge fields $e, \tilde{e}$ as well as bosonic (β, γ) ghost fields associated to χ, with:

$$b, \tilde{b} \in \Pi\Omega^0\left(\Sigma, K_\Sigma^2\right), \quad c, \tilde{c} \in \Pi\Omega^0\left(\Sigma, T_\Sigma\right),$$

$$\beta_r \in \Omega^0\left(\Sigma, K_\Sigma^{3/2}\right), \quad \gamma_r \in \Omega^0\left(\Sigma, T_\Sigma^{1/2}\right).$$

The BRST operator is constructed via the standard procedure. One can compute the central charge to find that the purely bosonic model is critical in 26 dimensions, the type II model is critical in 10 and for the ones involving S_C the critical dimension depends on the choice of current algebra.

Vertex operators As in string theory, amplitudes are obtained as correlators of vertex operators, one for each external particle, inserted at puctures σ_i on the worldsheet. Fixed vertex operators for all possible left and right matter systems are of the form

$$c_i\tilde{c}_i V_i = c_i\tilde{c}_i w_i e^{ik_i\cdot X(\sigma_i)}\,. \tag{2.13}$$

at a puncture σ_i, for some operator $w_i \in \Omega^0(\Sigma, K_\Sigma^2)$, determined by the choice of worldsheet matter and constrained by quantum consistency, BRST invariance and possibly other symmetries. The matter contribution w_i mirrors the left/right structure of the action and factorises into two independent currents:

$$w = v^l v^r, \quad v^l, v^r \in K_\Sigma\,. \tag{2.14}$$

For all matter systems we are going to consider, the contractions of these operators v_i factorise from the rest of the correlation function.

Gauge fixing revisited In the presence of vertex operators, the gauge fixing procedure presents some subtleties. For the ambitwistor string, these were first treated carefully in [92]. We only cite the result here and keep a more detailed description for the models in chapter 4 and 5, the procedure is analogous. The gauge fixing of e is standard in the literature [93] and introduces integrations over the position of $n-3$ punctures, thus making the distinction between fixed and integrated vertex operators. Similarly gauge fixing $\tilde{e}$ and χ gives rise to different types of vertex operators, according to how much residual gauge freedom they have. As explained in [92], these two gauge fields have moduli ($n-3$ for $\tilde{e}$ and $n-2$ for χ at tree level) that cannot be fixed to zero. Integrating these moduli produces bosonic and fermionic delta functions respectively. The gauge fixing of $\tilde{e}$ produces $n-3$ delta functions that are paired with integrated vertex operators to give:

$$\int_\Sigma \mathcal{V}_i := \int_\Sigma \bar{\delta}(\mathrm{Res}_{\sigma_i}(P(\sigma_i)^2) V_i(\sigma_i)\,, \tag{2.15}$$

where

$$\bar{\delta}(z) = \bar{\partial}\frac{1}{2\pi i z} = \delta(\Re z)\delta(\Im z) d\bar{z}\,. \tag{2.16}$$

The integrand in (2.15) has the correct weight as V is a quadratic differential and the delta function takes value in $\Omega^{(0,1)}(\Sigma, T_\Sigma)$. Gauge fixing χ also produces two sorts of vertex operators v_i in (2.14): we'll refer to these as picture -1 and picture 0 vertex operators. In the twistor string literature the nomenclature of fixed and integrated is commonly extended to this type of gauge fixing.

Penrose transform Spacetime fields are represented on ambitwistor space via the Penrose transform. Spin s plane-waves of the form $\epsilon_{\mu_1}\dots\epsilon_{\mu_s}\mathrm{e}^{ik\cdot X}$ were found to correspond to cohomology classes

$$(\epsilon\cdot P)^s\bar{\delta}(k\cdot P)\mathrm{e}^{ik\cdot x} \in H^1(\mathbb{PA}, \mathcal{O}(s-1))\,, \tag{2.17}$$

with $s=1$ for a Maxwell field and $s=2$ for linear gravitons. Matter models such as (2.11) and (2.12) generate fully integrated vertex operators of this form, with quantum consistency in the form of BRST invariance imposing the linearised equations of motion.

Correlators and CHY formulae Amplitude formulae are obtained as correlation functions of vertex operators:

$$\mathcal{M}_n = \left\langle c_1\tilde{c}_1 V_1 c_2\tilde{c}_2 V_2 c_3\tilde{c}_3 V_3 \prod_{i=4}^n \int_\Sigma \mathcal{V}_i \right\rangle \tag{2.18}$$

With respect to each gauge field, one should include as many fixed vertex operators as the number of ghost zero modes, here we only make the disctinction for worldsheet diffeomorphisms, leaving the rest implicit. The X path integral can be carried out explicitly, and it produces both a momentum conserving delta function and a set of equations for P that are solved, at genus zero, by

$$P_\mu(\sigma) = \sum_{i=1}^{n} \frac{k_{i\mu}}{\sigma - \sigma_i} \, \mathrm{d}\sigma_i \, . \tag{2.19}$$

Integrating out P, the correlator localises on the solutions to the scattering equations:

$$E_i := \mathrm{Res}_{\sigma_i}(P^2) = k_i \cdot P(\sigma_i) = \sum_{j=1}^{n} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} \, \mathrm{d}\sigma_j \, . \tag{2.20}$$

We will discuss in more detail the derivation of these equations when treating the massive model in chapter 5. As expected from the ambitwistor string, (2.20) shows that the geometric interpretation of the scattering equations is that they encode the reduction of the target space to ambitwistor space by enforcing the constraint $P^2 = 0$.

Putting everything together, the correlators reproduce the CHY formulae (1.11) of [35,94]:

$$\mathcal{A}_n = \delta^d \left(\sum_i k_i \right) \int_{\mathfrak{M}_{0,n}} \frac{\prod' \mathrm{d}\sigma_i \bar{\delta}\left(k_i \cdot P\left(\sigma_i\right)\right)}{\mathrm{Vol}(SL(2,\mathbb{C}))} \langle w_1 \cdots w_n \rangle \Bigg|_{P^*(\sigma)} \, , \tag{2.21}$$

where the correlator $\langle w_1 \cdots w_n \rangle$ is determined by the worldsheet matter and the one-form $P^*(\sigma)$ is given by (2.19). The Faddeev-Popov volume factor $1/\mathrm{Vol}(SL(2,\mathbb{C}))$ comes from the contribution of the c-ghost zero modes, whereas the one from the $\tilde{c}$-ghosts combines with the $n-3$ delta functions $\bar{\delta}(k \cdot P)$ to give a permutation invariant measure on the moduli space of the n-punctured Riemann sphere.

Integrands Because the two matter systems don't interact with each other, integrands mirror their 'double copy' structure:

$$\mathcal{I}_n = \langle w_1 \cdots w_n \rangle = \langle v_1^l \cdots v_n^l \rangle \langle v_1^r \cdots v_n^r \rangle = \mathcal{I}^L \mathcal{I}^R \, . \tag{2.22}$$

For the models we have described, each of the left/right contributions can either come from a current algebra or from worldsheet fermions.

The current algebra gives contributions:

$$v(\sigma) = T \cdot j(\sigma) \tag{2.23}$$

where $T \in \mathfrak{g}$ selects the color charge of the external state and j is a current in S_C. Correlators $\langle T_1 \cdot j(\sigma_1) \cdots T_n \cdot j(\sigma_n)\rangle$ of n such states give Parke Taylor factors:[2]

$$\mathcal{C}_n(\alpha) = \frac{\mathrm{tr}\left(T_{\alpha(1)} \ldots T_{\alpha(n)}\right)}{\sigma_{\alpha(1)\alpha(2)} \cdots \sigma_{\alpha(n)\alpha(1)}} = \mathrm{tr}\left(T_{\alpha(1)} \ldots T_{\alpha(n)}\right) \mathrm{PT}(\alpha)\,. \tag{2.24}$$

Fully integrated vertex operators from the S_Ψ matter system have:

$$v(\sigma) = \epsilon \cdot P(\sigma) + k \cdot \Psi(\sigma) \epsilon \cdot \Psi(\sigma)\,, \tag{2.25}$$

with ϵ and k the polarisation vector and momentum of the external state. Correlators involve two fixed vertex operators, that we don't specify here and produce:

$$\left\langle v_1^F v_2^F v_3 \ldots v_n \right\rangle = \mathrm{Pf}'(M) = \frac{1}{\sigma_{12}} \mathrm{Pf} M^{[12]}_{[12]}\,, \tag{2.26}$$

where M is the CHY matrix defined in (1.13). This completes the correlators with:

$$\mathcal{I}_n = \mathcal{I}^L \mathcal{I}^R = \begin{cases} \mathrm{PT}(\alpha)\,\mathrm{PT}(\beta), & \text{Biadjoint scalar} \\ \mathrm{PT}(\alpha)\,\mathrm{Pf}'(M), & \text{Yang-Mills theory} \\ \mathrm{Pf}'(M)\,\mathrm{Pf}'(\tilde{M}), & \text{RNS gravity.} \end{cases} \tag{2.27}$$

The original models can be found in [42, 43] and we refer the reader to [95, 96] for reviews of the subject.

[2]These also produce terms containing non cyclic permutations and multi-trace terms that are not expected to appear in the amplitude for the bi-adjoint scalar.

2.3 Interlude: spinor-helicity formalism

When working in a specific dimension, ambitwistor space has very convenient representations in terms of variables that solve the $P^2 = 0$ constraint explicitly. Ignoring for the time being the fact that we are dealing with fields on the worldsheet, we begin by introducing a decomposition of a massless fixed momentum k that solves the $k^2 = 0$ constraint explicitly. This is the well known *spinor helicity formalism* discussed in the introduction. Because this representation exploits the accidental isomorphisms of the spin group, it is specific to each dimension. Spinor helicity for massless particles is an essential tool of modern amplitude methods and there are now a number of excellent reviews in the literature, e.g. [12, 97].

Massless particles in four dimensions In four dimensions we have the isomorphism $\mathrm{Spin}(4,\mathbb{C}) \simeq \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$. Positive and negative chirality spinors transform under this group in the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representation respectively. Undotted and dotted indices label the left and right handed representations and can be raised and lowered with the Levi-Civita symbols $\varepsilon^{\alpha\beta}$ and $\varepsilon_{\dot\alpha\dot\beta}$ defining inner products:

$$\langle\lambda_1\lambda_2\rangle = \varepsilon^{\alpha\beta}\lambda_{1\,\alpha}\lambda_{2\,\beta} = -\langle\lambda_2\lambda_1\rangle \qquad [\lambda_1\lambda_2] = \varepsilon_{\dot\alpha\dot\beta}\tilde\lambda_1^{\dot\alpha}\tilde\lambda_2^{\dot\beta} = -[\lambda_2\lambda_1]\,. \tag{2.28}$$

Four-momentum k^μ transforms in the $(\frac{1}{2}, \frac{1}{2})$ representation and can thus be mapped to an object carrying two spinor indices, one of each chirality:

$$k_{\alpha\dot\alpha} = \sigma^\mu_{\alpha\dot\alpha} k_\mu\,. \tag{2.29}$$

Then the on shell condition imposes $k^2 = \det(k_{\alpha\dot\alpha}) = m^2$. We'll focus here on the $m = 0$ case and leave the discussion of massive particles to chapter 4 as we present our novel massive models. For a massless particle the momentum $k_{\alpha\dot\alpha}$ is a hermitian matrix of rank 1 and it can be decomposed as the outer product of two complex chiral spinors:

$$k_{\alpha\dot\alpha} = \kappa_\alpha\tilde\kappa_{\dot\alpha}\,, \tag{2.30}$$

From the decomposition one can see that the little group $\mathrm{SL}(1,\mathbb{C})$ acts as:

$$\kappa \to w^{-1}\kappa \qquad \tilde{\kappa} \to w\tilde{\kappa}\,. \tag{2.31}$$

Polarization data corresponds to irreducible representations of the little group. For massless particles of helicity h, these are objects that scale as w^{2h} under a little group transformation (2.31). Weyl spinors have polarization data $\epsilon\kappa$ and $\tilde{\epsilon}\tilde{\kappa}$. The Maxwell field strength $F_{\alpha\dot{\alpha}\beta\dot{\beta}}$ splits into an antiself-dual $F_{\dot{\alpha}\dot{\beta}} = \tilde{e}\tilde{\kappa}_{\dot{\alpha}}\tilde{\kappa}_{\dot{\beta}}$ and a self-dual $F_{\alpha\beta} = e\kappa_\alpha\kappa_\beta$ component corresponding to helicity ± 1 states and scaling accordingly.

Massless particles in six dimensions Spinor helicity variables in six dimensions were first introduced by Cheung and O'Connell in [98]. The spin group of the complexified Lorentz group in six dimensions is $\mathrm{Spin}(6,\mathbb{C}) \simeq \mathrm{SL}(4,\mathbb{C})$. This group has independent fundamental ($\mathbf{4}$) and antifundamental ($\bar{\mathbf{4}}$) representations, giving two independent Weyl spinor representations. The simplest $\mathrm{SL}(4,\mathbb{C})$ invariant is given by the singlet in ($\mathbf{4} \otimes \bar{\mathbf{4}}$):

$$\mathbf{4} \,:\, \nu_A \qquad \bar{\mathbf{4}} \,:\, \pi^A \qquad \mathbf{1} \,:\, \nu_A \pi^A\,. \tag{2.32}$$

The only non-trivial invariant tensor is the four index object ϵ_{ABCD}, which can be used to raise pairs of skew indices and to construct invariants:

$$\langle \kappa_1\kappa_2\kappa_3\kappa_4\rangle = \kappa_{1A}\kappa_{2B}\kappa_{3C}\kappa_{4D}\epsilon^{ABCD} \qquad [\kappa_1\kappa_2\kappa_3\kappa_4] = \kappa_1^A\kappa_2^B\kappa_3^C\kappa_4^D\epsilon_{ABCD}\,. \tag{2.33}$$

The six-vector k_μ is in the fundamental $\mathbf{6}$ of $\mathrm{SO}(6,\mathbb{C})$, which can be expressed as the antisymmetric product of two fundamentals or equivalently of two antifundamentals, the isomorphism being established through the chiral (skew) Pauli matrices σ^μ_{AB}. Because the matrix k_{AB} is skew, it has even rank, and since it doesn't have full rank on account of the on-shell condition, it has rank 2. Then there's a two dimensional space of solutions to the Dirac equation both for chiral and anti-chiral spinors:

$$k^{AB} = \varepsilon^{\dot{a}\dot{b}}\kappa^A_{\dot{a}}\kappa^B_{\dot{b}} \equiv \left[\kappa^A\kappa^B\right]\,, \qquad k_{AB} = \kappa^a_A\kappa^b_B\varepsilon_{ab} \equiv \langle\kappa_A\kappa_B\rangle\,. \tag{2.34}$$

It is clear that both these definitions hold up to two distinct $SL(2,\mathbb{C})$ actions on the undotted and dotted indices of the Weyl spinors, so that the little group is $\mathrm{SO}(4,\mathbb{C}) \cong \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$ and ϵ in (2.34) implements inner products in each of the two copies of $\mathrm{SL}(2,\mathbb{C})$.[3]

Polarization data is given by representations of the little group. A Dirac particle has polarization data $\epsilon_A = \epsilon_a \kappa_A^a$. A Maxwell field strength is represented by F_B^A, with $F_A^A = 0$. For a momentum eigenstate the Maxwell equations require $k_{AB}\epsilon^A = 0 = k^{AB}\epsilon_B$, so that all polarization data is encoded in little group spinors $(\epsilon_a, \epsilon_{\dot{a}})$ with[4]

$$F_B^A = \epsilon_B \epsilon^A \,, \qquad \epsilon^A = \epsilon_{\dot{a}} \kappa^{A\dot{a}} \,, \qquad \epsilon_A = \epsilon_a \kappa_A^a \,. \tag{2.35}$$

Massless particles in five dimensions As it was shown in [2, 82, 85], in order to dimensionally reduce to five dimensions, one picks a fixed non-null six-vector, Ω^{AB} in spinor form, and considers the five-dimensional plane $\mathbb{C}^5$ that is orthogonal to it. The choice of Ω breaks the spin group $\mathrm{SL}(4,\mathbb{C}) \to \mathrm{Sp}(4,\mathbb{C})$, isomorphic to $\mathrm{Spin}(5,\mathbb{C})$, and allows one to raise and lower spinor indices using Ω^{AB} and $\Omega_{AB} = \frac{1}{2}\epsilon_{ABCD}\Omega^{AB}$.

Five-vectors then have the same spinor helicity decomposition as in six dimensions, with the additional constraint:

$$\Omega \cdot k = \Omega^{AB}(\kappa_A \kappa_B) = 0 \,. \tag{2.36}$$

Because the fundamental and antifundamental representations are equivalent, the little group is $\mathrm{SL}(2,\mathbb{C})$ and we denote its contractions as $(\cdot,\cdot)$.

2.4 Twistorial ambitwistor string

While the RNS ambitwistor string produces beautifully compact formulae for bosonic amplitudes in any dimension, it poses some difficulties when it comes to the study of its fermionic sector and target space supersymmetry [99]. On the other hand the RSVW formulae [10, 26] in four dimensions extended to supersymmetric theories, exploiting the spinorial nature of

[3] We denote contractions of pairs of little group indices $[\cdot,\cdot]$ and $\langle\cdot,\cdot\rangle$. This notation is used in four dimensions for contractions of chiral spinors, the distinction should be clear from the context.

[4] Note that ϵ_a and $\epsilon_{\dot{a}}$ cannot be taken to be real in Lorentz signature.

twistors. We mentioned in the introduction that these two frameworks are related by twistorial representations of ambitwistor space. Geyer, Lipstein and Mason [83] wrote a twistorial realisation of the ambitwistor string in four dimensions and proposed new models and amplitude formulae for gauge and gravity theories with any amount of supersymmetry §2.4.2. In this context it becomes apparent how spinor-helicity variables are naturally 'twistorial'. Following the same approach, one can seek analogous representations in six and five dimensions §2.4.3. Although we will see that the six dimensional models present some issues [85], the formulae they produce have undergone numerous checks and they have been proven by BCFW recursion [1, 2] for gauge theory and gravity. In §2.5 we introduce the formulae and summarise the results of [2], referring to chapter 3 for details.

2.4.1 Twistors and ambitwistors in four dimensions

In four dimensions, twistor space $\mathrm{P}\mathbb{T}$ is an open subset of $C\mathbb{P}^3$. We take homogeneous coordinates on $\mathbb{CP}^3$ Z^A carrying a natural action of $\mathrm{SL}(4,\mathbb{C})$. The connection with spacetime is established by a geometric correspondence between the space of lines in $\mathbb{CP}^3$ and the complexified compactified Minkowski spacetime represented as a quadric Q in $\mathbb{CP}^5$. In order to lie on the line defined by a point x in spacetime, a twistor has to obey linear relations, known as *incidence relations*, that are more easily expressed by splitting the twistor coordinates into two Weyl spinors of opposite chirality, carrying the same weight:

$$Z^A = (\lambda_\alpha, \mu^{\dot\alpha})\,. \tag{2.37}$$

Then the incidence relations become:

$$\mu^{\dot\alpha} = i x^{\alpha\dot\alpha}\lambda_\alpha, \tag{2.38}$$

Thus establishing the non-local correspondence anticipated in the introduction. The $\mathrm{SL}(4,\mathbb{C})$ action on twistor is isomorphic to the complexified conformal group. We can find a representation of $SL(4,\mathbb{C})$ for which the generators are linear and holomorphic and we can form conformal invariants, such as the inner product defining dual twistors $\tilde{Z} \in \mathrm{P}\mathbb{T}^*$:

$$Z \cdot \tilde{Z} := \lambda_\alpha \tilde{\mu}^\alpha + \mu^{\dot{\alpha}} \tilde{\lambda}_{\dot{\alpha}} \,. \tag{2.39}$$

Going back to the task of writing an alternative parametrization of ambitwistor space, given a null momentum P, we can seek spinor helicity variables for it in the form $P = \lambda\tilde{\lambda}$. Then by the twistor correspondence, given a null geodesic with momentum $P = \lambda\tilde{\lambda}$ going through a point x, we can introduce:

$$Z = (\lambda_\alpha, -ix^{\alpha\dot{\alpha}}\lambda_\alpha) \in \mathrm{P}\mathbb{T} \qquad \tilde{Z} = (ix^{\alpha\dot{\alpha}}\tilde{\lambda}_{\dot{\alpha}}, \tilde{\lambda}_{\dot{\alpha}}) \in \mathrm{P}\mathbb{T}^* \,. \tag{2.40}$$

One can easily verify that these define a null geodesic iff they satisfy:

$$Z \cdot \tilde{Z} = 0 \,. \tag{2.41}$$

The twistor and dual twistor have two independent scalings, one corresponds to the scale of the null geodesic. The second is generated by $\Upsilon = Z \cdot \partial_Z - \tilde{Z} \cdot \partial_{\tilde{Z}}$ and is redundant. We can then write ambitwistor space as a symplectic reduction from the product of twistor and dual twistor space:

$$\mathbb{A} = \left\{ (Z, \tilde{Z}) \in \mathrm{P}\mathbb{T} \times \mathrm{P}\mathbb{T}^* | Z \cdot \tilde{Z} = 0 \right\} / \, \Upsilon \tag{2.42}$$

Quotienting by the scale of the null geodesic further reduces to $\mathrm{P}\mathbb{A}$. The Penrose transform relates massless on-shell fields on spacetime to cohomology classes on twistor and dual twistor space. On ambitwistor space, representatives are built out of the pullback of cohomology classes on $\mathrm{P}\mathbb{T}$ and $\mathrm{P}\mathbb{T}^*$.

2.4.2 Four dimensional ambitwistor string

We give here a brief review of the four dimensional model by Geyer, Lipstein and Mason [83]. This is both a warm up for the later discussions of six dimensional formulae and massive models as well as a special case that presents some unique features. We will only discuss the model for super Yang-Mills and we will not be very rigorous about the implementation of supersymmetry. What we'd like to emphasise is how the reparametrisation of ambitwistor space $\mathbb{A}$ carries with it an enhancement of the scattering equations to incorporate the polarisation data and the redundancy of the little group action.

The bosonic action for the four-dimensional ambitwistor string of [83] is based on the parametrisation of ambitwistor space as a quadric in the product of twistor and dual twistor space (2.42). When constructing the model, twistors and dual twistors become fields on the worldsheet and they must be taken to have value in some line bundles $\mathcal{L}$ and $K_\Sigma \otimes \mathcal{L}^{-1}$ on Σ. Contrary to the twistor string [22, 100], where the line bundle degree wasn't fixed in the model but rather summed over all possible values, the four-dimensional ambitwistor string takes $\mathcal{L} = K_\Sigma^{1/2}$ so that both the twistor and dual twistors are valued in $K_\Sigma^{1/2}$, see [95, 96] for a more detailed comparison. Beside this choice, both models are built on the action:

$$S = \frac{1}{2\pi} \int_\Sigma \tilde{Z} \cdot \bar{\partial} Z - Z \cdot \bar{\partial} \tilde{Z} + a\, Z \cdot \tilde{Z}\,, \tag{2.43}$$

where the Lagrange multiplier $a \in \Omega^{(0,1)}(\Sigma)$ imposes the constraint $Z \cdot \tilde{Z} = 0$ and gauges the transformations generated by Υ. Identifying $K_\Sigma^{1/2}$ with both the pullback to the worldsheet of the line bundle $\mathcal{O}_Z(1)$ on $\mathbb{PT}$ and $\mathcal{O}_{\tilde{Z}}(1)$ on $\mathbb{PT}^*$ reduces the target space to projective ambitwistor space.

One can write models with $\mathcal{N}$ supersymmetries by extending the twistor and dual twistor to their supersymmetric analogues. Here we employ a notation that will be natural in the context of higher dimensional models and dimensional reduction. We can repackage the degrees of freedom of both the twistor and dual twistor into one *Dirac* supertwistor $\mathcal{Y} = (\lambda_A, \mu^A, \eta^I)$, where λ and μ are Dirac spinors made up of the homonymous chiral and antichiral components of Z and $\tilde{Z}$. The fermionic components η^I carry an $\mathcal{N}$-dimensional R-symmetry index. In this notation, the supersymmetric analogue of (2.43) is given by:

$$S = \frac{1}{2\pi} \int_\Sigma \mathcal{Y} \cdot \bar{\partial} \mathcal{Y} + a\, \mathcal{Y} \cdot \mathcal{Y}\,, \tag{2.44}$$

with the inner product $\mathcal{Y} \cdot \mathcal{Y} = Z \cdot \tilde{Z} + \eta_I \eta^I$. While the liberty to choose the degree of the line bundle $\mathcal{L}$ in (2.43) is controversial in the ambitwistor string community, in (2.44) we are clearly compelled to take $\mathcal{Y} \in \Omega^0(\Sigma, K_\Sigma^{1/2})$.

BRST gauge fixing introduces fermionic ghosts for the field a and for e,[5] gauging worldsheet reparametrisations. Constructing the BRST charge, one can verify that the obstructions to $Q^2 = 0$ vanish for maximal supersymmetry and a choice of worldsheet matter with central charge $\mathfrak{c} = 14$ for super Yang-Mills, which we take to be a current algebra.

We mentioned that gauge fixing in the presence of vertex operators is a subtle procedure that generates different types of vertex operators on account of how much residual gauge freedom they have. Here we will not be rigorous about this and simply discuss integrated vertex operators, we refer the reader to [101] for details. We will insert manually the effects of this gauge fixing in the formula for correlators in the form of a Faddeev-Popov determinant.

The Penrose transform on twistor and dual twistor space relates massless on-shell fields of helicity h to cohomology classes $H^1(\mathrm{PT}, \mathcal{O}(2h-2))$ and $H^1(\mathrm{PT}^*, \mathcal{O}(-2h-2))$. For a Maxwell field of momentum $k = \kappa\tilde{\kappa}$ we have representatives (taking $\mathcal{N} = 0$):

$$\begin{aligned} v &= \int \frac{\mathrm{d}s}{s} \bar{\delta}^2 \left(\kappa_\alpha - s\lambda_\alpha(\sigma)\right) \mathrm{e}^{is[\mu\tilde{\kappa}]} \in H^1(\mathbb{PT}) \\ \tilde{v} &= \int \frac{\mathrm{d}s}{s} \bar{\delta}^2 \left(\tilde{\kappa}_{\dot\alpha} - s\tilde{\lambda}_{\dot\alpha}(\sigma)\right) \mathrm{e}^{is\langle\tilde{\mu}\kappa\rangle} \in H^1\left(\mathbb{PT}^*\right) \end{aligned} \tag{2.45}$$

These cohomology classes pull back to ambitwistor space and combine to give the spin 1 plane wave representative (2.17). In order to understand the equations enforced by the delta functions in the vertex operators, it is insightful to inspect the ordinary scattering equations written in spinor helicity variables:

$$k_i \cdot P(\sigma_i) = 0 = \langle \kappa_i \lambda(\sigma_i)\rangle [\tilde{\kappa}_i \tilde{\lambda}(\sigma_i)] \,. \tag{2.46}$$

As per the vertex operators, on the support of the ordinary scattering equations the polarised data must satisfy either $\langle \kappa_i \lambda(\sigma_i)\rangle = 0$ or $[\tilde{\kappa}_i \tilde{\lambda}(\sigma_i)] = 0$.

Amplitude formulae at N^{k-2}MHV degree[6] were obtained by taking correlators of k vertex operators on dual twistor space and $n-k$ on twistor space, built out of the ones in (2.45) with

[5] This should be included as always by writing $\bar\partial_e = \bar{\partial} + e\partial$ and then gauge fixing e to zero.
[6] Here k is the number of negative helicity gluons.

a standard contribution from the current algebra. The path integral localises on:

$$Z(\sigma) = (\lambda, \mu) = \sum_{i=1}^{k} \frac{s_i\left(\kappa_i, 0\right)}{\sigma - \sigma_i} \qquad \tilde{Z}(\sigma) = (\tilde{\mu}, \tilde{\lambda}) = \sum_{p=k+1}^{n} \frac{s_p\left(0, \tilde{\kappa}_p\right)}{\sigma - \sigma_p} \tag{2.47}$$

The polarisation data s can be absorbed as a component of homogeneous coordinates on the Riemann sphere $\sigma_\alpha = \frac{1}{s}(1, \sigma)$, whose contraction we write $(\sigma_i \sigma_j)$. With this notation we write:

$$\mathcal{A}_n^{(4\mathrm{d})} = \underbrace{\int \frac{\prod_{a=1}^{n} \mathrm{d}^2\sigma_a}{\mathrm{vol\,GL}(2,\mathbb{C})} \prod_{i=1}^{k} \bar{\delta}^2\left(\tilde{\kappa}_i - \tilde{\lambda}\left(\sigma_i\right)\right) \prod_{p=k+1}^{n} \bar{\delta}^2\left(\kappa_p - \lambda\left(\sigma_p\right)\right)}_{\mathrm{d}\mu^{4\mathrm{d}}} \mathrm{PT}(\alpha)\,, \tag{2.48}$$

where the $GL(2,\mathbb{C})$ extends the $SL(2,\mathbb{C})$ Mobius invariance to include the little group $\mathbb{C}^* = GL(1)$ generated by

$$\sum_{i \leq k} s_i \partial_{s_i} - \sum_{i>k} s_i \partial_{s_i}\,. \tag{2.49}$$

A quick counting tells us that we have four more delta functions than integrations, the remaining ones encoding momentum conservation. The Parke-Taylor factor comes from the integration of the current algebra.

The amplitude formulae have support on the solutions of the polarised scattering equations:

$$\tilde{\kappa}_i = \tilde{\lambda}\left(\sigma_i\right) \qquad \kappa_p = \lambda\left(\sigma_p\right)\,, \tag{2.50}$$

for $i \leq k$ and $p > k$. From (2.46) we know that the ordinary scattering equations vanish on their support. An inspection of the $(n-3)!$ solutions $\{\sigma_i\}$ of (2.20) shows that they split into $n-3$ sectors, each corresponding to a different MHV degree: for a given k the polarised measure only has support on $A(n-3, k-2)$[7] solutions, defining the corresponding sector. The evaluation of the formula was studied in [102]. It is important to notice that, despite the fact the new moduli s_i 'polarise' the scattering equations on factorisation channels, the pole structure of the amplitude formula is still described by the degenerations of $\mathfrak{M}_{0,n}$, since the s_i don't introduce any singularities in the scattering equations.

[7] The Eulerian number $A(n,m)$ is the number of permutations of n elements in which m elements are greater than their predecessors after the permutation.

2.4.3 Twistors and ambitwistors in six dimensions

In six dimensions, twistors are pure spinors of the conformal group $\mathrm{SO}(8,\mathbb{C})$. This group has three eight dimensional representations, two spinorial ones with opposite chirality and the vector. Triality permutes these three representations into each other.

Both chirality spinors can be employed to define twistor spaces. They can be represented as pairs of six-dimensional spinors:

$$Z^{\mathcal{A}} = \left(\mu^A, \lambda_A\right) \in \bar{\mathbf{4}} \oplus \mathbf{4}, \quad \tilde{Z}_{\mathcal{A}} = \left(\tilde{\mu}_A, \tilde{\lambda}^A\right) \in \mathbf{4} \oplus \bar{\mathbf{4}} \tag{2.51}$$

where A labels the fundamental and antifundamental representations of $\mathrm{SL}(4,\mathbb{C})$ as in (2.32). Both chiralities have natural inner products with themselves, so that twistor space Q is a quadric in the projectivisation of the chiral spinor representation of $\mathrm{SO}(8,\mathbb{C})$ defined as [103–106]:

$$Q = \left\{[Z] \in \mathbb{CP}^7 \mid Z \cdot Z = 2\mu^A\lambda_A = 0\right\} \,. \tag{2.52}$$

Similarly one can define primed twistor space Q', built on antichiral spinors. Here twistor space and primed twistor space are each dual to themselves through the canonical inner product in (2.52), contrary to the four dimensional case where primed twistor space is isomorphic to the dual of twistor space.

Another feature of six dimensions that follows from this concerns the non-local correspondence between ambitwistor space and complexified compactified Minkowski spacetime, viewed as a quadric $\mathbb{M} \subset \mathbb{CP}^7$. Points of Q are related to totally null self-dual $3-$planes in $\mathbb{M}$ via the incidence relations:

$$\mu^A = x^{AB}\lambda_B \,. \tag{2.53}$$

More insightfully, if two twistors are such that $\mathcal{Z}_1 \cdot \mathcal{Z}_2 = 0$, meaning that the line they define lies entirely in twistor space and not just in $\mathbb{CP}^7$, then the two corresponding $3-$planes, α_1 and α_2, intersect in $\mathbb{M}$ and they do so along a null line $\mathcal{L} = \alpha_1 \cap \alpha_2$. Then the correspondence is between complex null geodesics and null lines in Q and ambitwistor space can be defined by a *pair* of twistors (one twistor and one dual twistor in four dimensions), via:

$$\mathbb{A}_6 = \left\{ [Z^a] \in \mathbb{CP}^7 \mid Z^a \cdot Z^b = 0 \quad a, b = 1, 2 \right\} / \, \mathrm{SL}(2, \mathbb{C}) \,, \tag{2.54}$$

This description can be extended to superambitwistor space by replacing twistors Z with supertwistors $\mathcal{Z}$. Supertwistor space is defined as the quadric Q_N in $\mathbb{CP}^{7|2N}$:

$$Q_N = \left\{ [\mathcal{Z}] \in \mathbb{CP}^7 \mid \mathcal{Z} \cdot \mathcal{Z} = 2\mu^A \lambda_A + \omega_{IJ} \eta^I \eta^J = 0 \right\} \,, \tag{2.55}$$

parametrized by $[\mathcal{Z}] = \left[\mu^A, \lambda_A, \eta^I\right]$. Here ω_{IJ} is a skew $2N \times 2N$ matrix, $N = \mathcal{N}^{(6d)} = \mathcal{N}^{(5d)}$ is the number of supercharges in six and five dimensions. The incidence relations

$$\mu^A = x^{AB} \lambda_B + \omega_{IJ} \theta^{AI} \eta^J \qquad \eta^I = \theta^{AI} \lambda_A \,, \tag{2.56}$$

establish the correspondence with chiral Minkowski superspace $\mathbb{C}^{6|8N}$, parametrized by (x^{AB}, θ^{AI}), $I = 1, \ldots 2N$.

By our initial remark, an alternative description of ambitwistor space exists based on antichiral spinors in Q' and by triality a third one is also on an equal footing. Intuitively, the way we have a correspondence between Q and $\mathbb{M}$ and Q' and $\mathbb{M}$, there is also one between Q and Q' as well as a parametrisation of ambitwistor space based on $\mathbb{M}$ [85].

2.4.4 Models in six dimensions

In light of the geometry presented in the previous section, we can proceed as in four dimensions and write a twistorial model that solves the $P^2 = 0$ constraint explicitly. Contrary to what we argued in four dimensions, here we have no choice but to take the twistors to be spinors on the worldsheet, as is clear from the action (2.57).

In [1] a bosonic action was formulated that was further studied in [2, 85]:

$$S_{6d} = \int_\Sigma \frac{1}{2} \left(Z \cdot \bar{D} Z \right) \,, \tag{2.57}$$

where $\bar{D}Z^a = \bar{\partial}Z^a + A^a_b Z^b$, and $A^a_b \in \Omega^{0,1}(\Sigma, \mathfrak{sl}_2)$ a worldsheet $(0,1)-$form gauging the $\mathfrak{sl}_2$ little group. In (2.57) deformations of the complex structure have been gauged fixed via the Beltrami differential e in $\bar{\partial}_e = (\bar{\partial} + e\partial)$. The twistors Z^a are sections of $K_\Sigma^{1/2}$ and here also we identify this line bundle with the pullback to the worldsheet of $\mathcal{O}(1) \to \mathbb{CP}^7$ so that they define the projective scale on twistor space. One can include $(0, N)$ supersymmetry by replacing the twistors Z^a with supertwistors $\mathcal{Z}^a$. As it should be clear from the discussion above, two other distinct models exist based on the alternative representations of ambitwistor space in six dimensions.

For a discussion of BRST gauge fixing and a model for the biadjoint scalar, including a discussion of the vertex operators that will be at the origin of our massive ones in later chapters, we refer the reader to [85]. A model was proposed in [1] with worldsheet matter that would produce the expected integrands for gauge theory and gravity. This can be realised by introducing worldsheet fermions $(\rho_A, \tilde{\rho}^A) \in \Omega^0(\Sigma, K_\Sigma^{1/2})$ with action:

$$S_\rho = \int_\Sigma \tilde{\rho}^A \bar{\partial} \rho_A + b_a \lambda^{Aa} \rho_A + \tilde{b}_a \lambda^a_A \tilde{\rho}^A \,, \tag{2.58}$$

where $(\rho_A, \tilde{\rho}^A) \in \Omega^0(\Sigma, K_\Sigma^{1/2})$ are worldsheet fermions and $\left(b^a, \tilde{b}^a\right)$ are $(0,1)-$forms on the worldsheet and fermionic gauge fields imposing the constraints $\lambda^{Aa}\rho_A = 0 = \lambda^a_A \tilde{\rho}^A$. The issue with this model that has not been solved yet is that this type of matter requires both chiral and antichiral spinors and the two models are not easily combined. Once the model is reduced to five dimensions, the fundamental and antifundamental representations are equivalent so that this issue vanishes.

Although the six dimensional model is problematic, it has inspired amplitude formulae that were presented in [1] and underwent a number of checks in [2]. We will review their main features in section §2.5 and present some of the checks in chapter §3.

2.4.5 Models in five dimensions

Ambitwistor string models in five dimensions take the form

$$S_{5d} = \int_\Sigma \frac{1}{2} \left(Z \cdot \bar{D}Z\right) + a\Omega_1^{AB}(\lambda_A \lambda_B) + S^m \,, \tag{2.59}$$

Here S^m denotes the action for the matter systems, we will specify their content in later chapters as we further reduce these models. The field a is a Lagrange multiplier for the constraint $(\lambda^A \lambda_A)$ and it acts as a gauge field for the transformations:

$$\delta a = \bar{\partial}\alpha \qquad \delta\mu^{Aa} = \alpha\Omega_1^{AB}\lambda_B^a \qquad \delta\lambda_A^a = 0\,. \tag{2.60}$$

From the incidence relations, we see that these correspond to translations in the Ω_1 direction[8]. These are generated by the Hamiltonian vector field $\Omega_1^{AB}\lambda_A^a\partial/\partial\mu^{Ba}$ associated to the constraint $(\lambda^A\lambda_A) = 0$. The gauging of this constraint then reduces the target space to the space of null geodesics in five dimensions as the symplectic quotient:

$$\mathbb{A}_5 = \left\{ Z^a \in \mathbb{T}\times\mathbb{T} \mid Z^a \cdot Z^b = 0, (\lambda^A\lambda_A) = 0 \right\} / \left\{ \mathrm{SL}(2,\mathbb{C}) \times \mathbb{C} \right\}, \tag{2.61}$$

with the extra quotient by $\mathbb{C}$ accounting for the transformations (2.60). As described in (2.36), this description picks a fixed non-null six-vector, Ω_1^{AB} in spinor form, and considers null geodesics along null tangent vectors in the five-dimensional plane $\mathbb{C}^5$ that is orthogonal to it. The choice of Ω_1 breaks the spin group $\mathrm{SL}(4,\mathbb{C}) \to \mathrm{Sp}(4,\mathbb{C})$, isomorphic to $\mathrm{Spin}(5,\mathbb{C})$, and allows one to raise and lower spinor indices using Ω_1^{AB} and $\Omega_{1\,AB} = \frac{1}{2}\epsilon_{ABCD}\Omega_1^{AB}$. For supersymmetric theories, the model is naturally extended by replacing Z^a with $\mathcal{Z}^a$, thus obtaining the target space $\mathbb{A}_{5|2N}$.

A rigorous discussion of BRST gauge fixing and vertex operators for this model can be found in [85] and we will discuss it in the context of the four dimensional massive models of chapters 4-5.

[8] We label this direction Ω_1 anticipating the further reduction we will perform in chapter 5.

2.5 Six dimensional formula

We present here the integral formula for massless six-dimensional tree level amplitudes, with $(N, \bar{N})$ supersymmetry:

$$\mathcal{A}_n = \int \frac{\prod_{i=1}^n \delta^4\left(\mathcal{E}_{iA}\right) \delta\big(\langle v_i \epsilon_i\rangle - 1\big)\ \mathrm{d}\sigma_i\ \mathrm{d}^2 u_i\ \mathrm{d}^2 v_i}{\text{vol SL}(2,\mathbb{C})_\sigma \times \text{SL}(2,\mathbb{C})_u}\ \mathcal{I}_n^{(N,\bar{N})}\,. \tag{2.62}$$

These formulae have support on the polarised scattering equations:

$$\mathcal{E}_{iA} := u_{ia}\lambda_A^a(\sigma_i) - v_{ia}\kappa_{iA}^a = 0 \qquad \langle \epsilon_i v_i\rangle = 1\,, \tag{2.63}$$

Integration is carried out over the positions of the n punctures σ_i as well as on extra moduli (u_i^a, v_i^a) for each particle, with the Faddeev-Popov volume in the denominator taking care of gauge fixing three σs for Möbius invariance and three u components for the action on the little group index a that we'll explain briefly. By a quick counting, we can see that after integration of the $5n-6$ unconstrained moduli there are 6 residual delta functions. We will show in chapter 3 that these enforce momentum conservation. The integrand $\mathcal{I}_n^{(N,\bar{N})}$ contains all the matter-specific factors.

The aim of this section is to simply introduce all the ingredients of this formula and give a preview of statements that we will develop further in the next chapter. For this we will:

→ Present the intuitive origin of the polarised scattering equations from the original scattering equations, see §2.5.1.

→ Introduce the parametrisation we employ for $(N, \bar{N})$ on shell superspace, see §2.5.2.

→ Describe the integrands $\mathcal{I}_n^{(N,\bar{N})}$ for theories in [2] with different amounts of supersymmetry and show that the supersymmetric part factors out and can be seen as an expansion in supermomenta around the leading term for the top states of the multiplet, see §2.5.3.

2.5.1 Polarized scattering equations framework in 6 dimensions

We proceed as we did in four dimensions, seeking a factorisation into spinor helicity variables for the map $P(\sigma)$ over the Riemann sphere:

$$P_{AB}(\sigma) = \lambda_A^a(\sigma)\lambda_{Ba}(\sigma) = \frac{1}{2}\varepsilon_{ABCD}\lambda_{\dot{a}}^C(\sigma)\lambda^{D\dot{a}}(\sigma)\,. \tag{2.64}$$

The ordinary scattering equations are the statement:

$$k_i \cdot P(\sigma_i) = \det(\kappa_{iA}^a, \lambda_A^b) = 0\,. \tag{2.65}$$

This determinant vanishes iff there exist non zero (u_i^a, v_i^a) defined up to a scale such that:

$$\mathcal{E}_{iA} := u_{ia}\lambda_A^a(\sigma_i) - v_{ia}\kappa_{iA}^a = 0\,. \tag{2.66}$$

We can fix the scale by imposing:

$$\langle \epsilon_i v_i \rangle = 1\,. \tag{2.67}$$

Motivated by the ambitwistor string model (2.57) we write an ansatz for $\lambda_{Aa}(\sigma)$:

$$\lambda_{aA}(\sigma) = \sum_{i=1}^{n} \frac{u_{ia}\epsilon_{iA}}{\sigma - \sigma_i}\,. \tag{2.68}$$

Then (2.66)-(2.68) constitute the polarised scattering equations. One can easily verify that on their support:

$$\lambda_{aA}(\sigma)\lambda_B^a(\sigma) = P_{AB}(\sigma) = \sum_i \frac{k_{iAB}}{\sigma - \sigma_i} \tag{2.69}$$

They are a set of $5n$ equations on $5n-6$ variables $(\sigma_i, u_i^a, v_i^a)/\mathrm{SL}(2,\mathbb{C})_\sigma \times \mathrm{SL}(2,\mathbb{C})_u$. The vs can be eliminated from the equations by exploiting their normalization:

$$\sum_i \frac{u_{ij}\epsilon_{j[A}\epsilon_{iB]}}{\sigma_{ij}} = v_{ia}\kappa_{i[A}^a\epsilon_b\kappa_{iB]}^b = k_{iAB}\,. \tag{2.70}$$

In this form it is straightforward that they imply momentum conservation:

$$K_{AB} = \sum_i k_{iAB} = \sum_{i=1}^{3} \epsilon_{i[A}\mathcal{E}_{iB]} = 0\,. \tag{2.71}$$

where in the first equality we have used (2.70) and the last one is due to the symmetry properties of the object we are summing.

Although these are 6 equations, skewing with ϵ_{iC} vanishes identically by construction and there are only three independent equations per point that serve to determine the u_{ia} and σ_i. Despite their non-linear appearance, later we will see that

→ They are underpinned by linear equations, see §3.1.2,

→ And there exists a unique solution to these equations for each solution σ_i to the unpolarized scattering equation, see §3.1.1.[9] It can also be shown that (2.66)-(2.67) have a unique solution $\{u_i, v_i, \sigma_i\}$

We will also see that

→ The polarised measure is equivalent to the CHY measure in §3.1.3.

2.5.2 Supersymmetry in 6d

Here we review supersymmetry representations in 6d, in particular that in [1]. That representation depends on individual solutions to the scattering equations, so we introduce a variant that maintains the same simple structure, but that is global. This representation was introduced in [2].

Supersymmetry in six dimensions has been studied in the context of scattering amplitudes by a number of authors [79, 98, 107], [82], [108]. The generators of $(N, \tilde{N})$ supersymmetry in six dimensions are $Q_{AI}, Q^{A\dot{I}}$, where $I, \dot{I}$ are indices for the R-symmetry group $\mathrm{Sp}(N) \times \mathrm{Sp}(\tilde{N})$ and range from 1 to $2N/2\tilde{N}$ respectively. Their action on momentum eigenstates is defined by:

$$\{Q_{AI}, Q_{BJ}\} = k_{AB}\,\Omega_{IJ}, \quad \{Q^A_{\dot{I}}, Q^B_{\dot{J}}\} = k^{AB}\,\Omega_{\dot{I}\dot{J}} \tag{2.72}$$

[9]Unique up to an SL(2, $\mathbb{C}$)-transformation on the global a index.

where Ω_{IJ} and $\Omega_{\dot{I}\dot{J}}$ are the R-symmetry symplectic metrics.
For massless particles, the generators reduce to the little group as:

$$Q_{AI} = \kappa_A^a Q_{aI}\,, \qquad Q_{\dot{I}}^A = \kappa_{\dot{a}}^A Q_{\dot{I}}^{\dot{a}}\,, \tag{2.73}$$

with the anti-commutation relations:

$$\{Q_{aI}, Q_{bJ}\} = \epsilon_{ab}\Omega_{IJ}\,, \qquad \{Q_{\dot{a}\dot{I}}, Q_{\dot{b}\dot{J}}\} = \epsilon_{\dot{a}\dot{b}}\Omega_{\dot{I}\dot{J}}\,. \tag{2.74}$$

We now construct the on-shell superspace, i.e. we introduce Grassmann variables that allow us to group all the states of a supermultiplet into a superfield. Different choices for these coordinates are available, depending on the specific theory and on the symmetries we wish to keep manifest. We will focus on $(1,1)$ super Yang-Mills but the description is easily generalised to $(N,\tilde{N})$ supersymmetry.

$(1,1)$ super Yang-Mills has 16 supercharges. It arises as a dimensional reduction of $\mathcal{N}=1$ SYM in ten dimensions and its reduction to four dimensions is $\mathcal{N}=4$ SYM. The linearized 'super-Maxwell' multiplet is

$$\mathscr{F} := (F_A^B,\, \psi_I^A,\, \tilde{\psi}_{A\dot{I}},\, \phi_{I\dot{I}})\,, \tag{2.75}$$

consisting of a 2-form curvature F_A^B as described in section 2.3, spinors of each chirality ψ_I^A and $\tilde{\psi}_{A\dot{I}}$ and four scalars $\phi_{I\dot{I}}$. On momentum eigenstates with null momentum k_{AB}, Q_{CJ} acts on this multiplet by

$$Q_{CJ}\mathscr{F} \quad = \quad (k_{AC}\psi_J^B,\, \Omega_{JI}F_C^A,\, k_{AC}\phi_{J\dot{I}},\, \Omega_{JI}\tilde{\psi}_{\dot{I}C})\,. \tag{2.76}$$

To construct on-shell superspace, we need to choose half of the Q_{aI} as anticommuting supermomenta. The possibilities discussed in the literature [79, 98, 107] focus on halving either the I or the a-indices manifesting only full little-group or only R-symmetry respectively. The former was used successfully in recent work on 6d scattering amplitudes for a variety of theories [82, 108]. However, the latter is more natural from the perspective of the ambitwistor string [109], and will be the formulation we work with here. The two approaches are of course related by appropriate Grassmann Fourier transforms.

The polarised scattering equations provide a natural basis (ϵ_a, v_a) for the little group space. This basis gives a choice of supermomenta that manifests the full R-symmetry because the supercharges $\epsilon^a Q_{aI}$ anti-commute. The normalization condition ensures that the basis is always non degenerate. However, v introduces a dependence on the solutions to the scattering equations, making the basis dynamic. While we could in principle work with this basis by taking special care when extracting component amplitudes in the final supermomentum expansion, we choose to work with a global basis for each particle

$$(\epsilon_{ia}, \xi_{ia}) , \qquad \text{with } (\xi_i \epsilon_i) = 1 . \tag{2.77}$$

Using this basis, $\epsilon^a Q_{aI}$ again anti-commute, and can be represented by Grassmann viariables $q_I = \epsilon^a Q_{aI}$. However, the supersymmetry generators are now globally defined,

$$Q_{aI} = \left(\xi_a q_I + \epsilon_a \Omega_{IJ} \frac{\partial}{\partial q_J} \right) , \qquad \tilde{Q}^{\dot{a}}_{\dot{I}} = \left(\xi^{\dot{a}} \tilde{q}_{\dot{I}} + \epsilon^{\dot{a}} \tilde{\Omega}_{\dot{I}\dot{J}} \frac{\partial}{\partial \tilde{q}_{\dot{J}}} \right) . \tag{2.78}$$

Note that due to the normalization condition $(v\epsilon) = 1$, we know that v_a and ξ_a are related by

$$v_a = \xi_a + (\xi v) \epsilon_a . \tag{2.79}$$

Returning to the example of super Yang-Mills, the multiplet now takes the form

$$\mathscr{F}(q_I, \tilde{q}_{\dot{I}}) = \big((\epsilon_A + q^2 \langle \xi \kappa_A \rangle)(\epsilon^B + \tilde{q}^2 \langle \xi \kappa^B \rangle),\ q_I(\epsilon^A + \tilde{q}^2 \langle \xi \kappa^A \rangle),\ \tilde{q}_{\dot{I}}(\epsilon_A + q^2 \langle \xi \kappa_A \rangle),\ q_I \tilde{q}_{\dot{I}} \big) , \tag{2.80}$$

and the $(1,1)$-super-Yang-Mills superfield becomes

$$\Phi^{\mathrm{R}} = g^{\epsilon\tilde{\epsilon}} + q_I\, \psi^{I\tilde{\epsilon}} + \tilde{q}_{\dot{J}}\, \tilde{\psi}^{\epsilon\dot{J}} + q^2 g^{\xi\tilde{\epsilon}} + \tilde{q}^2 g^{\epsilon\tilde{\xi}} + q_I \tilde{q}_{\dot{J}}\, \phi^{I\dot{J}} + \cdots + q^2 \tilde{q}^2\, g^{\xi\tilde{\xi}} . \tag{2.81}$$

where $g^{\epsilon\tilde{\epsilon}} = \epsilon_a \tilde{\epsilon}_{\dot{a}}\, g^{a\dot{a}}$ denotes the gluon field strength with polarization $\epsilon_a \tilde{\epsilon}_{\dot{a}}$. By construction, this representation is now global and independent of the solution to the polarized scattering equations. Of course, this global definition comes at the expense of having to introduce an additional reference spinor ξ_a.

2.5.3 Integrands

Supersymmetry determines the full super-amplitude from the amplitudes involving only the top of the multiplet. For an $(N, \tilde{N})$ theory, supersymmetry imposes that the integrand in (2.62) breaks down as $\mathcal{I}_n^{(N,\tilde{N})} = \mathcal{I}_n \; e^{F_N + \tilde{F}_{\tilde{N}}}$ so that the total dependence on the supermomenta is encoded in the exponential factor that we define below.

Supersymmetry The dependence on supermomenta is carried by the exponential e^F, with $F = F_N + \tilde{F}_{\tilde{N}}$ where[10]

$$F_N = F_N^{\text{pol}} - \frac{1}{2}\sum_{i=1}^{n}\langle \xi_i v_i\rangle \, q_i^2 \,, \qquad F_N^{\text{pol}} = \sum_{i<j} \frac{\langle u_i u_j\rangle}{\sigma_{ij}} q_{iI} q_j^I \,, \tag{2.82a}$$

$$\tilde{F}_{\tilde{N}} = \tilde{F}_{\tilde{N}}^{\text{pol}} - \frac{1}{2}\sum_{i=1}^{n}[\xi_i v_i] \, \tilde{q}_i^2 \,, \qquad \tilde{F}_{\tilde{N}}^{\text{pol}} = \sum_{i<j} \frac{[\tilde{u}_i \tilde{u}_j]}{\sigma_{ij}} \tilde{q}_{i\dot{I}} \tilde{q}_j^{\dot{I}} \qquad . \tag{2.82b}$$

For example for $\mathcal{N} = (1,1)$ super Yang-Mills we take the exponential factor $\exp F^{\text{YM}} = \exp(F_1 + \tilde{F}_1)$. In the dynamic R-symmetry preserving representation we mentioned in the previous section, as used in [1], only the terms F_N^{pol} remain in the exponential. In the next chapter we will show that the exponential factors above are indeed invariant under supersymmetry transformations §3.2.2.

Matter For the ambidextrous spin one contribution, define an $n \times n$ matrix H by

$$H_{ij} = \begin{cases} \frac{\epsilon_{iA}\epsilon_j^A}{\sigma_{ij}} & i \neq j \\ e_i \cdot P(\sigma_i)\,, & i = j \end{cases} \tag{2.83}$$

where e_i is the null polarization vector and $P(\sigma)$ is as defined in (2.19). We can define H_{ii} equivalently by

$$\lambda_{aA}(\sigma_i)\epsilon_i^A = -u_{ia} H_{ii}\,, \quad \lambda^{\dot{a}A}(\sigma_i)\epsilon_{iA} = -u_i^{\dot{a}} H_{ii}\,. \tag{2.84}$$

[10] Here we decompose our factors for the new fixed SUSY representation in terms of the F_N^{pol} factors used in [1].

See §3.2.1 for details. On the polarized scattering equations, the determinant $\det H$ vanishes because H has co-rank 2 due to

$$\sum_i u_{ia} H_{ij} = \lambda_{aA}(\sigma_j)\epsilon_j^A + u_{ja} H_{jj} = 0\,. \tag{2.85}$$

The first term follows from the definition (2.68) of λ_{aA} and the second equality from (2.84). Similarly, $\sum_j H_{ij} u_{j\dot{a}} = 0$. These identities nevertheless imply that H has a well defined reduced determinant

$$\det{}' H := \frac{\det(H^{[i_1 i_2]}_{[j_1 j_2]})}{\langle u_{i_1} u_{i_2}\rangle [u_{j_1} u_{j_2}]}\,. \tag{2.86}$$

Here $H^{[i_1 j_1]}_{[i_2 j_3]}$ denotes the matrix H with the rows $i_1,\ i_2$ and columns $j_1,\ j_2$ deleted.

→ We will show that $\det{}' H$ is well-defined in the sense that the (2.86) is invariant under permutations of particle labels, and thus independent of the choice of $i_{1,2},\ j_{1,2}$, see §3.2.1 for the proof.

The reduced determinant $\det{}' H$ is manifestly gauge invariant in all particles, carries $\mathrm{SL}(2,\mathbb{C})_\sigma$ weight -2, as expected for a half-integrand $\mathcal{I}^{\text{spin}-1}$ and is equally valid for even and odd numbers of external particles, in contrast to earlier formulae.

→ On the support of the polarized scattering equations, it is verified using factorization in §3.4.2 that $\det{}' H$ is equal to the CHY half-integrand $\mathrm{Pf}{}' M$.

In [2] integrands were given for theories of D5 and M5 branes. In this thesis we will mainly be concerned with gauge theory and gravity. Therefore we refer the interested reader to the original paper for details of the construction and only briefly define the integrands here. For these theories two additional integrands need to be defined: the skew matrix $A:\ A_{ij} = {}^{k_i \cdot k_j}/_{\sigma_{ij}}$, familiar from the CHY formulae [35, 110] and a family of matrices $U^{(a,b)}$ constructed from $(\sigma_i, u_{ia}, \tilde{u}_{i\dot{a}})$, only needed for M5-branes

$$U^{(a,b)}_{ij} = \frac{\langle u_i u_j\rangle^a\, [\tilde{u}_i \tilde{u}_j]^b}{\sigma_{ij}}\,. \tag{2.87}$$

With these ingredients, we have the following integrands of various supersymmetric theories:

$$\text{(1,1)-Super Yang-Mills:} \qquad \mathrm{PT}(\alpha)\ \det{}'H\ e^{F_1+\tilde{F}_1} \tag{2.88a}$$

$$\text{(2,2)-Supergravity:} \qquad \det{}'H\ \det{}'\tilde{H}\ e^{F_2+\tilde{F}_2} \tag{2.88b}$$

$$\text{(1,1)-D5-branes:} \qquad \det{}'A\ \det{}'H\ e^{F_1+\tilde{F}_1} \tag{2.88c}$$

$$\text{(2,0)-M5-branes:} \qquad \det{}'A\ \frac{\mathrm{Pf}\,'A}{\mathrm{Pf}\,U^{(2,0)}}\ e^{F_2} \tag{2.88d}$$

The resulting superamplitudes are $\mathrm{SL}(2,\mathbb{C})_\sigma \times \mathrm{SL}(2,\mathbb{C})_\pm$ invariant, the super Yang-Mills and supergravity amplitudes are gauge invariant, and the supergravity amplitudes are permutation invariant. We also see colour-kinematics duality expressed in the form of the super Yang-Mills and supergravity amplitudes. The M5 amplitudes are manifestly chiral. The most important statement about these formulae is:

→ The amplitude formulae (2.62) with integrands (2.88) all factorize correctly. There exists valid BCFW shifts for the gauge and gravity formulae so that their equivalence with the corresponding tree-level S-matrices is guaranteed by recursion and the three-point examples of §3.3. We will prove this statement in §3.4.

In computing low-multiplicity amplitudes, we will:

→ Verify that we reproduce known formulae §3.3.2.

→ Show the features of the novel supersymmetry representation and illustrate how to extract component amplitudes §3.3.4.

The solutions to the polarised scattering equations (2.63) as well as the integrands depend on the polarisation data of the external particles. This differs from the usual spinor-helicity representation of the amplitude as an object with free little group indices. From the explicit low particle results we obtain it is obvious that one can go from representation to the other via $A^{\epsilon_1\tilde{\epsilon}_1\ldots\epsilon_n\tilde{\epsilon}_n} = \prod_i \epsilon_{ia_i}\tilde{\epsilon}_{i\dot{a}_i}\ldots A_n^{a_1\dot{a}_1\ldots a_n\dot{a}_n}$. When considering the full superamplitude formula, two checks are in order:

→ We verify that the integrands are indeed linear in the polarisation data §3.2.4.

→ The fact that we can pick the polarisation data also means that we can select equivalent gluon states both from the top and from the bottom state of the multiplet. We verify that the two different representations of the same state are compatible in that they give rise to the same amplitude formula §3.2.3.

CHAPTER 3

Properties of the six dimensional superamplitude

A framework was developed in six dimensions [82, 108] that allows the supersymmetric extension of the original CHY formulae and those for brane theories. These models have some features of the original RSVW formulae [25, 26] in that moduli of maps from the worldsheet to chiral spin space in six-dimensions are integrated out against delta functions. These authors were able to obtain amplitude formulae for a variety of supersymmetric theories in this way. However the formulae distinguish between even and odd numbers of particles, and become quite awkward for odd numbers of particles in gauge and gravity theories where such distinctions are not natural. Their possible origins from worldsheet models remain obscure. The amplitude formulae we presented in the review, based on [1, 2], give a different take on six dimensional scattering amplitudes, where some of these issues are resolved.

In this chapter we give a more rigorous analysis of these formulae. In §3.1 the polarized scattering equations and measure are studied in more detail. It is shown that given a solution to the original scattering equations, there exists generically a unique solution to the polarized scattering equations which can be obtained essentially by solving linear equations and then normalizing. The associated measures are also shown to reduce to the CHY measure. Section 3.2 goes on to prove basic properties of the integrands we use, permutation invariance, invariance under supersymmetry and compatibility of the supersymmetry factors with the reduced determinants. In §3.3 the three and four point amplitudes are computed from the new formulae and shown to agree with the standard answers for the corresponding theories.

The full proof of the gauge and gravity formulae by BCFW recursion is given in section

3.4. Along the way we prove factorization for all non-controversial formulae. Somewhat surprisingly, despite their poor power counting at large momenta, our brane formulae have no boundary contribution for large BCFW shifts, as outlined in §7.3.3 of [2].

Finally in §3.5 we discuss further issues and directions. These include a brief discussion of the Grassmannian approach of [82] and its use in [111] to obtain a correspondence between the formulae studied in this chapter and those of [82]. This leads to some brief remarks concerning analogues of the momentum amplituhedron of [112] in 6d.

3.1 Polarized scattering equations and measure

In this section we prove various statements made in the review section 2.5 regarding the measure $\mathrm{d}\mu^{\mathrm{pol}}$ in the six-dimensional superamplitude. In §3.1.1 we prove the existence and uniqueness for solutions to the polarized scattering equations given an initial solution to the scattering equations. Underlying this is a linear formulation of the polarized scattering equations that we make explicit in §3.1.2. The final subsection §3.1.3 proves that the polarized scattering equations measure is equivalent to the standard CHY measure.

With the definitions we presented in §2.5, one can berify that, on the support of the polarised scattering equations:

$$\lambda_{Aa}(\sigma)\lambda_B^a(\sigma) = P_{AB}(\sigma) := \sum_i \frac{k_{iAB}}{\sigma - \sigma_i}\,. \tag{3.1}$$

The LHS has no double poles and taking its residues one finds

$$\mathrm{Res}_{\sigma_i}\lambda_{Aa}(\sigma)\lambda_B^a(\sigma) = \epsilon_{i[A}u_{ia}\lambda(\sigma_i)^a_{B]} = \langle v_i\epsilon_i\rangle\kappa_{i[A|a}\kappa^a_{B]i} =: k_{iAB}\,, \tag{3.2}$$

where the PSE were used in the second equality. When the scattering equations are not imposed, although the residue of $\mathrm{Res}_{\sigma_i}P(\sigma)$ is no longer k_i, there is nevertheless an alpha-plane that contains both $P(\sigma_i)$ and k_i.

3.1.1 Existence and uniqueness of solutions

In this subsection we prove existence and uniqueness using algebreo-geometric arguments. We define the bundle over $\mathbb{CP}^1$ in which λ_{aA}, $a = 0, 1$, takes its values to show that it is a rank-two bundle with canonically defined skew form, and so generically has a pair of sections that can be normalized. We work with bundles on $\mathbb{CP}^1$ which will be direct sums of line bundles $\mathcal{O}(n)$ whose sections can be represented in terms of homogeneous functions of degree n in terms of homogeneous coordinates σ_α, $\alpha = 0, 1$ on $\mathbb{CP}^1$ with skew inner product $(\sigma_i \sigma_j) := \sigma_{i0}\sigma_{j1} - \sigma_{i1}\sigma_{j0}$. We prove:

Proposition 3.1.1 *For each solution $\{\sigma_i\}$ to the scattering equations and compatible polarization data in general position, there exists a unique solution to the polarized scattering equations* (2.63) *and* (2.68) *up to a global action of $SL(2, \mathbb{C})$ on the little-group index.*

Proof: Let $P^{AB}(\sigma)$ arise from the given solution to the scattering equations as the spinor form of (2.19). To remove the poles, define $\Pi(\sigma)^{AB} := P^{AB} \prod(\sigma\sigma_i)$ which is now a holomorphic object of weight $n-2$ on $\mathbb{CP}^1$ and is a null 6-vector so as a skew matrix has rank 2 on $\mathbb{CP}^1$ (for momentum and σ_i in general position it will be vanishing on $\mathbb{CP}^1$).

We require $\lambda_{aA} P^{AB} = 0$ for $a = 0, 1$ so to study solutions to this equation, define the rank-2 bundle $E = \ker P \subset \mathbb{S}_A$ on $\mathbb{CP}^1$ where $\mathbb{S}_A$ is the rank four trivial bundle of spinors over $\mathbb{CP}^1$. To calculate the number of sections we wish to compute the degree of this bundle. To do so consider the short exact sequence

$$0 \longrightarrow E \longrightarrow \mathbb{S}_A \longrightarrow E^0(n-2) \longrightarrow 0\,, \tag{3.3}$$

where the second map is multiplication by $\Pi(\sigma)^{AB}$ and $E^0(n-2) \subset \mathbb{S}^A(n-2)$ is the annihilator of E twisted by $\mathcal{O}(n-2)$, that being the weight of Π^{AB}. In such a short exact sequence the degree of $\mathbb{S}_A$ is the sum of that of E and $E^0(n-2)$ since the degree is the winding number of the determinant of the patching function, and the maps of the exact sequence determine these up to upper triangular terms that don't contribute to the determinant. Since $\mathbb{S}_A$ is trivial, it has degree 0, so we find

$$\deg E + \deg E^0 + 2(n-2) = 0\,. \tag{3.4}$$

Because $E^0 = (\mathbb{S}/E)^*$ and $\mathbb{S}$ is trivial, we have $\deg E^0 = \deg E$ so this gives $\deg E = 2 - n$.

Now $\Lambda_{aA} := \lambda_{aA}\prod(\sigma\sigma_i)$ is a section of $E(n-1)$ which by the above has degree n. Our Λ_{aA} is subject to the n conditions, one at each marked point, as we impose $\Lambda_{aA}|_{\sigma=\sigma_j} \propto \epsilon_{jA}$. This has the effect of defining a subbundle with a reduction of degree by 1 at each marked point, so the total degree is now zero. Thus this subbundle therefore has degree zero. For data in general position, it will therefore be trivial with a two-dimensional family of sections spanned by Λ_{aA}, $a = 0, 1$. These can be normalized because $\Lambda_{0[A}\Lambda_{1B]} = f\Pi_{AB}$ where f is a holomorphic function of the sphere of weight n. The conditions on Λ_{aA} at σ_i imply that f vanishes at each σ_i so $f = c\prod_i(\sigma\,\sigma_i)$ and we can normalize our sections so that $c = 1$ reducing the freedom in the choiced of frame Λ_{aA} to SL(2). On dividing through by $\prod_i(\sigma\,\sigma_i)^2$ we obtain $P_{AB} = \lambda_{aA}\lambda^a_B$. □

For the non-chiral theories that we are considering, we will need both chiralities of spinors satisfying polarized scattering equations i.e, we can also define

$$\lambda^A_{\dot a}(\sigma) := \sum_i \frac{u_{i\dot a}\epsilon^A_i}{\sigma - \sigma_i}\,, \qquad u_{i\dot a}\lambda^{\dot a A}(\sigma_i) = v_{i\dot a}\kappa^{\dot a A}_i\,. \tag{3.5}$$

3.1.2 An explicit linear version of the polarized scattering equations

The above argument is rather abstract and it is helpful to see explicitly at least the underlying linearity of the problem of solving the polarized scattering equations. However we have not been able to give explicit versions of all the algebreo geometric proofs above.

According to the above, we are trying to find a pair of solutions λ_{aA}, $a = 1, 2$ to the equations

$$P(\sigma)^{AB}\lambda_B(\sigma) = 0\,, \tag{3.6}$$

where $\lambda_A(\sigma)$ has projective weight -1 in σ and P weight -2. The argument above gives $\lambda_A\prod(\sigma\sigma_i)$ as a section of $E(n-1)$ which has degree n and rank 2 so generically has $n+2$

global sections. To make this more explicit, make the ansatz[1]

$$\lambda_A = \sum_i \frac{u_{ia_i}\kappa_A^{a_i}}{(\sigma\sigma_i)}\,, \tag{3.7}$$

which removes double poles from (3.6). Given that the total weight of (3.6) is negative, it will be satisfied if the residues at its poles vanish. The vanishing of the residue at σ_i yields

$$k_i^{AB}\sum_j \frac{\kappa_{jB}^{a_j}}{\sigma_{ij}}u_{a_j j} + P(\sigma_i)^{AB}\kappa_{iB}^{a_i}u_{a_i i} = 0\,. \tag{3.8}$$

Now define $p_i^{a\dot{a}}$ after solving the CHY scattering equations (2.20) by

$$P^{AB}(\sigma_i)\kappa_{iA}^{a} = \kappa_{i\dot{a}}^{B}p_i^{a\dot{a}}\,. \tag{3.9}$$

This makes sense at σ_i as κ_{iA}^a annihilates the pole, and a second contraction with κ_{iB}^b leads to zero as it gives $k_i\cdot P$, so it must be a multiple of $\kappa_{i\dot{a}}^B$. We can understand this also by considering the 2-form $P(\sigma_i)\wedge k_i$ which in spinors gives, using the above,

$$P(\sigma_i)_{AC}k_i^{BC} = P(\sigma_i)^{BC}k_{iAC} = p_{iA}^B\,, \qquad p_{iA}^B = \kappa_{iAa}\kappa_{\dot{a}}^B p_i^{a\dot{a}}\,. \tag{3.10}$$

We can now see for example that

$$e_i\cdot P(\sigma_i) = [\epsilon_i|p_i|\epsilon_i\rangle\,, \tag{3.11}$$

using $e_{iAB} = \epsilon_{i[A}\tilde{\epsilon}_{B]i}$ where $\tilde{\epsilon}_A\kappa_i^{AB} = \epsilon_i^B$. Following Cheung and O'Connell [98], we further define

$$\kappa_{ij}^{\dot{a}a} := \kappa_i^{A\dot{a}}\kappa_{jA}^{a}\,, \tag{3.12}$$

that relate the ij-particles little group indices.

[1]We attach the additional i-index to a_i here to distinguish this $u_{a_i i}$ from the u_{ia} in the original ansatz for λ_{Aa}; the a_i is a little group index associated to momentum k_i rather than the global one associated to λ_{Aa}. We will drop these sub-indices when the equations are unambiguous.

With this notation we see that (3.8) can be written as $\kappa^{A}_{i\dot{a}}$ multiplied by

$$\sum_{a,j} H_{ij}^{\dot{a}a_j} u_{a_j j} = 0, \qquad H_{ij}^{\dot{a}a} = \begin{cases} \frac{\kappa_{ij}^{\dot{a}a}}{\sigma_{ij}} & i \neq j \\ p_i^{a\dot{a}} & i = j\,. \end{cases} \tag{3.13}$$

The discussion of the previous subsection implies that generically these equations have $n+2$ solutions. These equations reduce to the original polarized scattering equations if we supplement them with n further equations $\langle \epsilon_j u_j \rangle = 0$, since we will then have $u_{a_j j} = \epsilon_{j a_j} u_j$ as in the original ansatz (2.68). We then expect to find a pair of linearly independent solutions u_{ia}, with $a = 1, 2$ now global little group indices, so that we now have

$$u_{a_i i}^{a} = \epsilon_{i a_i} u_i^{a}. \tag{3.14}$$

In order to normalize these solutions, observe that for a pair of solutions λ_A^1, λ_A^2 to (3.6), we must have that

$$\lambda^1_{[A} \lambda^2_{B]} = f P_{AB} \tag{3.15}$$

for some meromorphic function f on $\mathbb{CP}^1$ with poles at the σ_i. However, when we impose (3.14), the double poles in (3.15) vanish and f must be constant, so we can normalize the pair of solutions u_i^a so that the coefficient is 1. The full $n+2$-dimensional space of solutions also has a volume form determined by (3.15).

In general (3.13) are $2n$-equations on $2n$-unknowns, so we must have $n+2$ relations to agree with the discussion of the previous subsection and to allow us to impose these extra n conditions. The relations follow from the original equation (3.6) and the nilpotency $P^{AB}P_{BC} = 0$ that follows from the original scattering equations. This leads to the nilpotency

$$\sum_{ja} H_{ji}^{a\dot{a}} H_{jk}{}_{a}^{\dot{b}} = 0\,. \tag{3.16}$$

This can be checked explicitly using a Schouten identity. We can use this nilpotency to generate solutions

$$\lambda_A(\sigma) = P(\sigma)_{AB} W^B(\sigma)\,, \qquad W(\sigma)^A = \sum_i \frac{\kappa^A_{i\dot{a}} w^{\dot{a}}_i}{(\sigma\sigma_i)} \tag{3.17}$$

where the W^B has weight 1 in σ so $w_{\dot{a}i}$ has weight 1 in σ_i and 2 in σ. The ansatz guarantees no double poles in λ_A and by taking residues we obtain[2]

$$u^a_i = \sum_{\dot{a},j} H^{a\dot{a}}_{ij} w_{\dot{a}j}\,. \tag{3.19}$$

3.1.3 The equivalence of measures

We first show that

$$\bar{\delta}(k \cdot P) = \int d^2u\, d^2v\, \delta^4(\mathcal{E}_A)\delta(\langle \epsilon v\rangle - 1)\,, \qquad \text{with } \mathcal{E}_A := \langle u\lambda_A\rangle - \langle v\kappa_A\rangle\,. \tag{3.20}$$

After integrating out the four components of (u_a, v_b), we are left with a single delta-function on both sides of the equation. It is easy to see that they have the same support as the latter delta function on the left implies that $v_a \neq 0$, but this can only be true when $(\lambda^a_A, \kappa^b_A)$ have rank less than four, which happens iff $\varepsilon^{ABCD}\lambda^0_A\lambda^1_B\kappa^0_C\kappa^1_D := k \cdot P = 0$. Furthermore the weights in λ^a_A and κ^a_A are -2 on both sides. A systematic proof uses a basis with $\epsilon_a = (0,1)$, $\kappa^0_3 = \kappa^1_4 = 1$ and all other components zero. This allows us to integrate out the v^a directly against the delta functions reducing the right side to

$$\int d^2u\, \delta(u_a\lambda^a_0)\, \delta(u_a\lambda^a_1)\, \delta(u_a\lambda^a_3 - 1) = \delta(\langle\lambda_0\, \lambda_1\rangle)\,, \tag{3.21}$$

where the latter equality follows by direct calculation integrating out the u_a; this gives (3.20) in this basis.

[2] We also have the special solutions when $W(\sigma)^A$ has no poles that leads to the 8 solutions

$$u_{ai} = \kappa_{iaA}(W^A_0 + \sigma_i W^A_1)\,. \tag{3.18}$$

The CHY measure is defined to be

$$d\mu_n^{\text{CHY}} := \delta^6\left(K\right) \frac{\prod_{i=1}^n \bar{\delta}(k_i \cdot P(\sigma_i)) d\sigma_i}{\text{Vol}(\text{SL}(2,\mathbb{C})_\sigma \times \mathbb{C}^3)} = \delta^6\left(K\right) (\sigma_{12}\sigma_{23}\sigma_{31})^2 \prod_{i=4}^n \bar{\delta}(k_i \cdot P(\sigma_i)) d\sigma_i\,, \tag{3.22}$$

where $K = \sum_i k_i$, the volume of $\text{SL}(2,\mathbb{C})_\sigma$ quotients by the Möbius invariance of σ, and the $\mathbb{C}^3$ is a symmetry of the ambitwistor string whose quotient removes the linearly dependent scattering equations delta functions.

Proposition 3.1.2 *We have*

$$d\mu_n^{\text{pol}} := \int \frac{\prod_{i=1}^n d^2u_i\, d^2v_i\, d\sigma_i\, \delta^4(\mathcal{E}_{iA})\delta(\langle\epsilon_i v_i\rangle - 1)}{\text{Vol}(\text{SL}(2,\mathbb{C})_\sigma \times \text{SL}(2,\mathbb{C})_u)} = d\mu_n^{\text{CHY}}\,, \tag{3.23}$$

where $SL(2,\mathbb{C})_\sigma$ *denotes Möbius invariance of* σ *as above in the CHY measure, the* $SL(2,\mathbb{C})_u$ *is acting on the little group index of* u_a*, and the integrals are over the* (u_i, v_i) *variables.*

Proof: We first reduce the $\text{SL}(2,\mathbb{C})_\sigma$ factor fixing $(\sigma_1,\sigma_2,\sigma_3)$ to be constant with the standard

$$\frac{\prod_i d\sigma_i}{\text{Vol SL}(2,\mathbb{C})_\sigma} = \sigma_{12}\sigma_{13}\sigma_{23} \prod_{i\geq 4} d\sigma_i\,. \tag{3.24}$$

Similarly Faddeev-Popov gauge fixing[3] $\text{SL}(2,\mathbb{C})_u$ by

$$u_1^a = (1,0), \qquad u_2^a = (0, u_{12}), \qquad u_3^a = \left(-\frac{u_{23}}{u_{12}}, u_{13}\right)\,, \tag{3.25}$$

so that $u_{ij} = \langle u_i u_j\rangle$ for $i < j \leq 3$ yields

$$\frac{\prod_i d^2u_i}{\text{Vol}\; SL(2)_u} = du_{12}du_{13}du_{23} \prod_{i=3}^n d^2u_i\,, \tag{3.26}$$

On the support of the delta functions $\prod_{i>3} \delta^4(\mathcal{E}_{iA})$ we can write, using (2.71),

$$K_{AB} = \left(\sum_{i=1}^3 \epsilon_{i[A}\mathcal{E}_{iB]}\right)\,. \tag{3.27}$$

[3] This entails contracting a normalized basis of the Lie algebra of $\text{SL}(2,\mathbb{C})_u$ into the form $\prod_i d^2u_i$ and restricting to the given slice.

We can trivially perform one of each of the v_i integrals against the $\delta(\langle v_i \epsilon_i\rangle - 1)$ delta functions by choosing a basis of the little group spin space for each i so that $\epsilon_{ia} = (1,0)$ fixing $v_i^a = (v_i, 1)$.

Choosing a basis of spin space consisting of $\{\epsilon_{iA}, \epsilon_{0A}\}$ with $i = 1,2,3$ and ϵ_{0A} chosen so that $\langle 0123\rangle = 1$, and dual basis $\tilde{\epsilon}_i^A$, $i =, 0, \ldots, 3$ we find via (3.27)

$$K_{0i} = \mathcal{E}_{i0}\,, \qquad K_{ij} = \mathcal{E}_{[ij]}\,, \tag{3.28}$$

so that these polarized scattering equations can be replaced by $\delta^6(K)$. The remaining scattering equations in $\prod_{i=1}^3 \delta^4(\mathcal{E}_{iA})$ are, for $i,j = 1,\ldots,3$,

$$\mathcal{E}_{(ij)} = \begin{cases} \frac{u_{ij}}{\sigma_{ij}} + \ldots & i \neq j \\ v_i + \ldots\,, & i = j \end{cases} \tag{3.29}$$

where the $\ldots$ denotes terms involving $i,j > 3$. Thus we can integrate out du_{ij} and dv_i against these remaining polarized scattering equation delta functions $\delta(\mathcal{E}_{(ij)})$ for $i,j \leq 3$ yielding an extra numerator factor of $\sigma_{12}\sigma_{23}\sigma_{13}$.

Finally we can use (3.20) to replace the remaining polarized scattering equations delta functions by standard ones thus yielding the desired formula. □

3.2 Integrands

In this section, we discuss the integrands $\mathcal{I}_n$ and the supersymmetry representation in more detail. We first show that the spin-one contribution $\det{}'H$ is permutation invariant, and that it is equivalent to the CHY pfaffian $\mathrm{Pf}'M$ in providing the correct dependence on the spin-one polarization data. We move on to giving further details of the supersymmetry factors. Finally, we prove crucial properties such as linearity of the spin-one contribution in the polarization data, and the compatibility of the reduced determinant with the supersymmetry representation.

3.2.1 The kinematic reduced determinant $\det{}'H$.

For our ambidextrous spin one contribution, recall that we defined an $n \times n$ matrix H by

$$H_{ij} = \begin{cases} \frac{\epsilon_{iA}\epsilon_j^A}{\sigma_{ij}} & i \neq j \\ e_i \cdot P(\sigma_i)\,, & i = j \end{cases}, \tag{3.30}$$

where e_i is the null polarization vector above and $P(\sigma)$ is as defined in (2.19). We first prove the equivalence between this definition of H_{ii} and that in (2.84). In order to use the vector representation of the polarization vector, we introduce a spinor $\tilde{\epsilon}_A$ so that $\epsilon^A = k^{AB}\tilde{\epsilon}_B$. Then the polarization vector is $e_{AB} = \epsilon_{[A}\tilde{\epsilon}_{B]}$. The equivalent definition of H_{ii} (2.84) is

$$\lambda_{aA}(\sigma_i)\epsilon_i^A = -u_{ia}H_{ii}\,, \quad \lambda^{\dot{a}A}(\sigma_i)\epsilon_{iA} = -u_i^{\dot{a}}H_{ii}\,. \tag{3.31}$$

The left side is a multiple of u_{ia} (or $u_i^{\dot{a}}$) due to the scattering equation and the identity $k^{AB}\kappa_A^a = 0$. Starting from the second last formula we obtain the first from

$$e_i \cdot P(\sigma_i) = \epsilon^{[A}\tilde{\epsilon}^{B]}\lambda_{aA}(\sigma_i)\lambda_B^a(\sigma_i) = -H_{ii}\tilde{\epsilon}^B u_a \lambda_B^a(\sigma_i) = -H_{ii}\tilde{\epsilon}^B v_a \kappa_B^a = -H_{ii}\,. \tag{3.32}$$

This then, being neither chiral nor antichiral justifies the equivalence.

The matrix H_{ij} is not full rank because

$$\sum_i u_{ia}H_{ij} = \lambda_{aA}(\sigma_j)\epsilon_j^A + u_{ja}H_{jj} = 0\,, \tag{3.33}$$

and so, as above, we define the generalized determinant

$$\det{}'(H) := \frac{\det(H^{[ij]})}{\langle u_i u_j\rangle[u_i u_j]} = \frac{\det(H^{[i_1 i_2]}_{[j_1 j_2]})}{\langle u_{i_1} u_{i_2}\rangle[u_{j_1} u_{j_2}]} \tag{3.34}$$

where $H^{[ij]}$ denotes the matrix H with the ij rows and columns deleted and $H^{[i_1 i_2]}_{[j_1 j_2]}$ the matrix with the with rows i_1, i_2 and columns j_1, j_2 removed. These are well-defined as

Lemma 3.2.1 *The generalized determinant defined above is permutation invariant.*

Proof: We can extend the argument of appendix A of [30] on such generalized determinants as follows.

Consider an $n \times n$ matrix H_i^j with a p-dimensional kernel and cokernel, i.e., that satisfies $\sum_i w_a^i H_i^j = 0$ and $\sum_j H_i^j \tilde{w}_j^b = 0$ where $a, b = 1, \ldots, p$. We must also assume that there are volume p-forms on these kernels, $\langle w_1 \ldots w_p\rangle$ and $[\tilde{w}_1, \ldots \tilde{w}_p]$. Our reduced determinant can be understood as the determinant of the exact sequence

$$0 \to \mathbb{C}^p \overset{\tilde{w}}{\to} \mathbb{C}^n \overset{H}{\to} \mathbb{C}^n \overset{w}{\to} \mathbb{C}^p \to 0\,. \tag{3.35}$$

To make this explicit, note that we have

$$\varepsilon_{j_1 \ldots j_n} \varepsilon^{i_1 \ldots i_n} H_{i_{p+1}}^{j_{p+1}} \ldots H_{i_n}^{j_n} \langle w_1 \ldots w_p\rangle \langle \tilde{w}^1 \ldots \tilde{w}^p\rangle = \det{}'(H) w_1^{[i_1} \ldots w_p^{i_p]} \tilde{w}^1_{[j_1} \ldots \tilde{w}^p_{j_p]} \tag{3.36}$$

for some $\det'(H)$. This formula follows because skew symmetrizing a free index on the left with a w_r or $\tilde{w}_r$ vanishes as it dualizes via the ε to contraction with H_i^j. Thus it must be a multiple of the right hand side as defined. The definitions (3.34), (2.86) then follow by taking components of this definition in the case $p = 2$ on the i_1, i_2, j_1, j_2 indices. In our context the natural volume form on the kernel is defined on the 2-dimensional space of $u_{ia_i} = u_i \epsilon_{a_i}$ by the f on the right hand side of (3.15) but for our polarized scattering equation framework, the normalizations are such that this is 1 so the bracketed terms on the left of (3.36) reduce to unity in (3.34). □

Note that the first term on the left side of (3.36) is simply the p^{th} derivative of $\det H$ where we have to relax the scattering equations and momentum conservation to make the determinant not identically zero. The CHY matrix is also non-degenerate away from the support of the scattering equations and momentum conservation. We have

Proposition 3.2.1 *The determinant is related to the full CHY Paffian by* $\det(H) = \mathrm{Pf}\, M$.

Proof: We use the form of the CHY Pfaffian due to Lam & Yao [113]. They show that the full

Pfaffian of M can be expanded into a sum over the permutations $\rho \in S_n$ of the particle labels,

$$\mathrm{Pf}(M) = \sum_{\rho \in S_n} \mathrm{sgn}(\rho) M_I ... M_J \,, \tag{3.37}$$

where each term has been decomposed into the disjoint cycles $I = (i_1 \dots i_I)$, $J = (j_1 \dots j_J)$ of the permutation ρ. The terms in this cycle expansion are given by

$$M_I = \begin{cases} \frac{\mathrm{tr}(F_{i_1} ... F_{i_I})}{\sigma_I} & \text{if } |I| > 1 \,, \\ C_{ii} & \text{if } I = \{i\} \,, \end{cases} \tag{3.38}$$

and $\sigma_I = \left(\sigma_{i_1 i_2} \dots \sigma_{i_I i_1}\right)^{-1}$ denotes the Parke-Taylor factor associated to the cycle.

Euler's formula for the determinant of H similarly gives

$$\det(H) = \sum_{\rho \in S_n} \mathrm{sgn}(\rho) H_I ... H_J \tag{3.39}$$

where the terms H_I are given by

$$H_I = H_{i_1 i_2} ... H_{i_I i_1} = \begin{cases} \frac{\mathrm{tr}(F_{i_1} ... F_{i_I})}{\sigma_I} & \text{if } |I| > 1 \,, \\ H_{ii} & \text{if } I = \{i\} \,, \end{cases} . \tag{3.40}$$

Here the trace over the Fs is taken in the spin representation and we have $C_{ii} = H_{ii}$ hence the equivalence. □

This result provides some circumstantial evidence that $\mathrm{Pf}' M = \det' H$ on the support of the scattering equations, but we do not have a direct proof. We prove this only indirectly via factorization in §3.4.2. Our $\det' H$ can therefore be used as a half-integrand in place of $\mathrm{Pf}'(M)$

in the theories as described in [110] to give full integrands

$$\text{Yang-Mills:} \qquad \mathrm{PT}(\alpha)\ \det{}'H \tag{3.41a}$$

$$\text{Gravity:} \qquad \det{}'H\ \det{}'\tilde{H} \tag{3.41b}$$

$$\text{D5-branes:} \qquad \det{}'A\ \det{}'H\,. \tag{3.41c}$$

3.2.2 The supersymmetry factors

Here we show that the supersymmetry factors e^{F_N}, with

$$F_N = F_N^{\text{pol}} - \frac{1}{2}\sum_{i=1}^{n}\langle\xi_i v_i\rangle\, q_i^2\,, \qquad F_N^{\text{pol}} = \sum_{i<j}\frac{\langle u_i u_j\rangle}{\sigma_{ij}} q_{iI} q_j^I\,, \tag{3.42a}$$

$$\tilde{F}_{\tilde{N}} = \tilde{F}_{\tilde{N}}^{\text{pol}} - \frac{1}{2}\sum_{i=1}^{n}[\xi_i v_i]\ \tilde{q}_i^2\,, \qquad \tilde{F}_{\tilde{N}}^{\text{pol}} = \sum_{i<j}\frac{[\tilde{u}_i \tilde{u}_j]}{\sigma_{ij}} \tilde{q}_{iI} \tilde{q}_j^I\,, \tag{3.42b}$$

are invariant under supersymmetry. The full supersymmetry generator for n particles is defined by the sum $Q_{AI} = \sum_{i=1}^n Q_{iAI}$ for each particle as defined by (2.78),

$$Q_{iAI} = \langle\xi_i \kappa_{iA}\rangle q_{iI} + \epsilon_{iA}\,\Omega_{IJ}\frac{\partial}{\partial q_{iJ}}\,, \qquad \tilde{Q}_{iI}^A = \left[\xi_i \kappa_i^A\right]\tilde{q}_{iI} + \epsilon_i^A\,\tilde{\Omega}_{IJ}\frac{\partial}{\partial \tilde{q}_{iJ}}\,. \tag{3.43}$$

Superamplitudes must be supersymetrically invariant and so are annihilated by the total Q_{AI} and indeed this determines the amplitude for the whole multiplet from the amplitudes involving only the top of the multiplets.

It is easily verified that the supersymmetry factors give an amplitude that is supersymetrically invariant, since

$$\begin{aligned} Q_{AI}\, e^{F_N} &= \left(\sum_i \Big(\langle\xi_i\kappa_{iA}\rangle + \langle\xi_i v_i\rangle\epsilon_{iA}\Big)\, q_{iI} - \sum_{i,j}\frac{\langle u_i u_j\rangle\,\epsilon_{iA}}{\sigma_{ij}} q_{jI}\right) e^{F_N} \\ &= \left(\sum_i \langle v_i \kappa_{iA}\rangle\, q_{iI} - \sum_{i,j}\frac{\langle u_i u_j\rangle\,\epsilon_{iA}}{\sigma_{ij}} q_{jI}\right) e^{F_N} = 0\,, \end{aligned} \tag{3.44}$$

and similarly $Q_I^A\, e^F = 0$. Here, the second equality follows from $v_i = \xi_i + \langle\xi_i v_i\rangle\epsilon_i$, and the sum vanishes on the support of the polarized scattering equations. Conversely, given an integrand

$\mathcal{I}_n$ for the top states of a multiplet, (2.62) is the unique supersymmetric completion using the supersymmetry representation (2.78), as can be verified using supersymmetric Ward identities.

3.2.3 Consistency of the reduced determinant with the supersymmetry representation

Our gauge (and gravity) formulae in effect give two different representations of bosonic amplitudes with gluons coming from different parts of the multiplets. One comes from simply substituting gluon polarizations from different parts of the multiplet in the kinematic integrand $\det{}'H$ and the other from expanding out the supersymmetry factors. In this subsection we show that these give the same formulae.

When a subset I of the particles are in states at the bottom of the (chiral part of the) supersymmetry multipet, the integrals over the supercharges lead to the integrand

$$\mathcal{I}_n^{\mathrm{h}} = \det U^I \det{}'H \; e^{F^{\bar{I}}+\tilde{F}} \,, \tag{3.45}$$

where $U^I_{ij} = U^{(1,0)}_{ij}$ and the superscripts indicate the restriction to the subsets I and $\bar{I}$ respectively. On the other hand, for *any* choice of polarization data, the integrand for gluons (gravitons) takes the form of a reduced determinant,

$$\mathcal{I}_n^{v_{i_1}\ldots v_{i_{|I|}}} = \det{}'H^I \; e^{F^{\bar{I}}+\tilde{F}} \,, \qquad \text{with} \quad H^I_{ij} = \begin{cases} H_{ij} & i \notin I \\ \frac{\langle \xi_i \kappa_{iA}\rangle \epsilon_j^A}{\sigma_{ij}} & i \in I \,, \end{cases} \tag{3.46}$$

where H^I is defined with polarization spinors $\langle \xi_i \kappa_{iA}\rangle$ instead of ϵ_{iA} for $i \in I$. For the supersymmetry to be compatible with the representation of the integrand, the two prescriptions for the amplitude must agree, $\mathcal{I}_n^{\mathrm{h}} = \mathcal{I}_n^{\xi_{i_1}\ldots\xi_{i_{|I|}}}$.

A lemma on reduced determinants. To prove the equivalence of (3.45) and (3.46), the general strategy will be to first identify the relation between H and H^I. To draw conclusions about the behaviour of their reduced determinants though, we will need a few results discussed in appendix A of [30], which we review here for convenience.

In contrast to regular determinants, it does not make sense to ask how a reduced determinant behaves under the addition of an arbitrary vector to a row or column of H, because this will in general spoil the linearity relations among its rows and columns. On the other hand, we *can* define a new reduced determinant by multiplication with an invertible $n \times n$ matrix U, since this leaves the (full) determinant $\det H = \det \hat{H} = 0$ unaffected,

$$\hat{H}_i^j := U_i^k \, H_k^j \,. \tag{3.47}$$

Since the kernel and co-kernel of H are spanned by w and $\tilde{w}$,[4] the kernel of $\hat{H} = UH$ is $\hat{w} = U^{-1}w$. To be explicit, $\hat{H}$ and $\hat{w}$ satisfy relations analogous to (2.85),

$$\sum_i \hat{w}_a^i \hat{H}_i^j = 0 \,, \qquad \sum_j \tilde{w}_j^b \hat{H}_i^j = 0 \,, \qquad \text{for } \hat{w}_a^i = \left(U^{-1}\right)_k^i \, w_a^k \,. \tag{3.48}$$

We can thus define a reduced determinant $\det{}'\hat{H}$ as in (3.36) by

$$\varepsilon^{i_1 i_2 \ldots i_n} \varepsilon_{j_1 j_2 \ldots j_n} \hat{H}_{i_{p+1}}^{j_{p+1}} \ldots \hat{H}_{i_n}^{j_n} \left\langle \hat{w}_1 \ldots \hat{w}_p \right\rangle \left[\tilde{w}^1 \ldots \tilde{w}^p\right] = \det{}'\hat{H} \;\, \hat{w}_1^{[i_1} \ldots \hat{w}_p^{i_p]} \;\, \tilde{w}^1_{[j_1} \ldots \tilde{w}^p_{j_p]} \,. \tag{3.49}$$

Let us multiply this equation by p facors of U. On the right-hand-side, this cancels the factors of U^{-1} from the kernel $\hat{w}_1^{[i_1} \ldots \hat{w}_p^{i_p]}$, whereas on the left, it combines with the $(n-p)$ factors from $\hat{H} = UH$ to $\det U$. Putting this all together, we arrive at the following lemma [30]:

Lemma 3.2.2 *Under multiplication by an invertible matrix U, the reduced determinant of a matrix $\hat{H} := U\,H$ behaves as*

$$\det{}'\hat{H} = \det U \det{}'H \,, \tag{3.50}$$

with the reduced determinant defined using the kernel $\hat{w} = U^{-1}w$.

This implies in particular that the usual row- and column operations leave the reduced determinant unaffected, $\det{}'\hat{H} = \det{}'H$, due to $\det U = 1$.

[4] As discussed above, for super Yang-Mills and supergravity, we take $w_a^i = u_{ia}$, where a denotes the chiral little group index, and similarly for $\tilde{w}_j^b = \tilde{u}_j^b$.

Equivalence of the reduced determinants. Lemma 3.2.2 now allows us to prove the compatibility of the supersymmetry representation with the reduced determinant. We first note that on the support of the polarized scattering equations, H^I and H are related via

$$\begin{aligned} H^I_{ij} &= \sum_{k\neq i} \frac{\langle u_i u_k\rangle}{\sigma_{ik}} \frac{\epsilon_{kA}\epsilon^A_j}{\sigma_{ij}} - \langle \xi_i v_i\rangle \frac{\epsilon_{iA}\epsilon^A_j}{\sigma_{ij}} \\ &= \sum_{k\neq i} \frac{\langle u_i u_k\rangle}{\sigma_{ik}} H_{kj} - \frac{1}{\sigma_{ij}} \underbrace{\sum_{k\neq i} \langle u_i u_k\rangle H_{kj}}_{=0} - \langle \xi_i v_i\rangle\, H_{ij} =: \sum_k U^I_{ik} H_{kj}\,, \end{aligned} \tag{3.51}$$

for $i \in I$. In the second equality, the middle term vanishes because u spans the kernel of H, and we use the last equality to define U^I. Combining the above result with $H^I_{ij} = H_{ij}$ for $i \notin I$, we thus have

$$H^I = U^I H\,, \qquad\qquad \text{with} \quad U^I_{ij} = \begin{cases} U^{(1,0)}_{ij} & i\neq j\,,\, i\in I \\ -\langle \xi_i v_i\rangle & i = j \in I \\ \delta_{ij} & i\notin I\,. \end{cases} \tag{3.52}$$

Since $\det U^I$ is generically non-zero, and lemma 3.2.2 gives directly that

$$\det{}' H^I = \det U^I \det{}' H\,, \tag{3.53}$$

confirming the equivalence of the two prescriptions.

3.2.4 Linearity in the polarization data

As another important check on the amplitudes (2.88), we verify that they are multilinear in the polarization data. This is of course a mandatatory requirement for amplitudes, but is not manifest in the integrands for gauge and gravity theories because the reduced determinants depend on the u-variables and these can potentially depend in a complicated way on the polarization data via the polarized scattering equations. We first observe that linearity is manifest for amplitudes with two external scalars and $n-2$ gluons. Given the supersymmetry of the formulae this provides strong circumstantial evidence. Then we show explicitly that the reduced determinant is linear on the support of the polarized scattering equations and go on to the full superamplitude.

3.2.4.1 Linearity from supersymmetry

Linearity of the gluon states is most easily seen from the mixed amplitudes with two external scalars, e.g. $j = 1, 2$, and $n-2$ gluons. In this case, we can choose to reduce the determinant $\det{}'H$ on the scalar states, giving

$$\mathcal{A}^{\phi_1\phi_2\epsilon_3\tilde{\epsilon}_3\cdots} = \int d\mu_n^{\text{pol}}\, \frac{1}{\sigma_{12}^2} \det H^{[12]}_{[12]}\, \text{PT}(\alpha)\,. \tag{3.54}$$

The integrand is then manifestly independent of $\{u_i, v_i\}$ as well as $\epsilon_{1,2}$, and only depends on the punctures σ_i and the polarization of the gluons. Due to the invariance of the measure established by proposition 3.1.2, the 'polarization' spinors of the scalars $\epsilon_{1,2}$ are choices of reference spinors. For the gluons on the other hand, the integrand is now manifestly linear in ϵ_i. Supersymmetry then guarantees that linearity extends to the all-gluon amplitude.

The consistency between the supersymmetry representation and the reduced determinant discussed in the last section further guarantees that the argument above holds for gluons both at the top and the bottom of the multiplet; we simply replace H by H^I. For gravity and brane-amplitudes, the argument is completely analogous, and follows again from the multilinearity of the amplitude $\mathcal{M}^{\phi_1\phi_2\epsilon_3\tilde{\epsilon}_3\cdots}$ with two scalars and $n-2$ gravitons.

3.2.4.2 Linearity for non-supersymmetric amplitudes.

We now study the dependence of the reduced determinant on the polarization data directly by expanding the spinors ϵ^a in a basis. This gives the desired linearity for pure Yang-Mills and gravity directly, where the above supersymmetry argument seems excessive, but can equally be applied to supersymmetric theories. We first discuss (chiral) linearity for gluons, but the proof extends straightforwardly to linearity in the anti-chiral polarization data, as well as (bi-)linearity for gravity amplitudes.

Consider the amplitude A^{ϵ_1} or the superamplitude $\mathcal{A}^{\epsilon_1}$, where one of the particles is a gluon with polarization ϵ_1, and all other particles are in arbitrary states. We can expand ϵ_1 in an (arbitrarily chosen) polarization basis ζ_1^a, ζ_2^a via

$$\epsilon_1^a = \alpha_1 \zeta_1^a + \alpha_2 \zeta_2^a\,, \qquad\qquad \text{with } \langle \zeta_2 \zeta_1 \rangle = 1\,. \tag{3.55}$$

It will be helpful to think of this new basis $(\zeta_1, \zeta_2 =: \xi_1^{\zeta_1})$ as playing a similar role to (ϵ_1, ξ_1), both in the polarized scattering equations and in the integrands. To prove linearity of the (super-) amplitudes in the polarization, we then have to show that amplitudes in the two different bases are related via

$$A^{\epsilon_1} = \alpha_1\, A^{\zeta_1} + \alpha_2\, A^{\zeta_2}\,, \tag{3.56}$$

where the amplitudes A^{ϵ_1} and A^{ζ_r} are respectively given by

$$A^{\epsilon_1} = \int d\mu_n^{\text{pol}} \det{}' H \;\text{PT}(\alpha)\,, \quad A^{\zeta_r} = \int d\mu_n^{\text{pol},\zeta_r} \det{}' H^{\zeta_r} \;\text{PT}(\alpha)\,, \tag{3.57}$$

and the superscripts ζ_r indicate that the respective quantities are defined using the polarization ζ_r. For the measure, proposition 3.1.2 guarantees that $d\mu_n^{\text{pol}} = d\mu_n^{\text{pol},\zeta_r}$, but the integration variables $u_i^{\zeta_r} = u_i(\zeta_r)$ defined by $d\mu_n^{\text{pol},\zeta_r}$ enter into the definition of the reduced determinant $\det{}' H^{\zeta_r}$. Since the measure and the Parke-Taylor factors are invariant under changes of polarization, the linearity relation (3.56) for the amplitude is equivalent to linearity of the spin-one contribution;

$$\det{}' H = \alpha_1 \det{}' H^{\zeta_1} + \alpha_2 \det{}' H^{\zeta_2}\,, \tag{3.58}$$

where the (implicit) map between $\{u_i, v_i\}$ on the left-hand side and $\{u_i^{\zeta_r}, v_i^{\zeta_r}\}$ on the right hand side is determined by the polarized scattering equations.

Proposition 3.2.2 *For* $\epsilon_1^a = \alpha_1 \zeta_1^a + \alpha_2 \zeta_2^a$ *expand also* $v_1^a = \beta_1 \zeta_1^a + \beta_2 \zeta_2^a$ *so that* $\langle \epsilon_1 v_1 \rangle = 1$ *gives*

$\alpha_1\beta_2 - \alpha_2\beta_1 = 1$. *Then we have that* $\{u_i, v_i\}$ *and* $\{u_i^{\zeta_r}, v_i^{\zeta_r}\}$ *are related by*

$$v_1^a = \beta_2\, v_1^{\zeta_1\, a} \qquad\qquad u_1^a = \beta_2\, u_1^{\zeta_1\, a} \tag{3.59a}$$

$$v_i^a = v_i^{\zeta_1\, a} + \alpha_2\beta_2\, \frac{\langle u_1^{\zeta_1} u_i^{\zeta_1}\rangle^2}{\sigma_{1i}^2}\epsilon_i^a \qquad\qquad u_i^a = u_i^{\zeta_1\, a} - \alpha_2\beta_2 \frac{\langle u_1^{\zeta_1} u_i^{\zeta_1}\rangle}{\sigma_{1i}} u_1^{\zeta_1\, a}\,, \tag{3.59b}$$

with identical expressions for $\{u_i, v_i\}$ *in terms of* $\{u_i^{\zeta_2}, v_i^{\zeta_2}\}$.

Proof: First note that the punctures σ_i are unaffected so we omit the superscripts here. First write $\epsilon_1^a = (\zeta_1^a + \alpha_2 v_1^a)/\beta_2$. Using this, the polarized scattering equations $\mathcal{E}_i$ can be written in the form

$$\mathcal{E}_{1A} = \sum_{j\neq 1} \frac{\langle u_1 u_j\rangle}{\sigma_{1j}}\epsilon_{jA} - \langle v_1 \kappa_{1A}\rangle \tag{3.60}$$

$$\mathcal{E}_{iA} = \sum_{j\neq 1,i} \underbrace{\left(\frac{\langle u_i u_j\rangle}{\sigma_{ij}} + \frac{\alpha_2}{\beta_2}\frac{\langle u_1 u_i\rangle}{\sigma_{1i}}\frac{\langle u_1 u_j\rangle}{\sigma_{1j}}\right)}_{\overset{!}{=}\frac{\langle u_i^{\zeta_1} u_j^{\zeta_1}\rangle}{\sigma_{ij}}}\epsilon_{jA} + \frac{1}{\beta_2}\frac{\langle u_1 u_i\rangle}{\sigma_{1i}}\langle \zeta_1 \kappa_{1A}\rangle - \underbrace{\left(\langle v_i \kappa_{iA}\rangle - \frac{\alpha_2}{\beta_2}\frac{\langle u_1 u_i\rangle^2}{\sigma_{1i}^2}\epsilon_{iA}\right)}_{\overset{!}{=}\langle v_i^{\zeta_1}\kappa_{iA}\rangle}\,.$$

It is now simple to map this to the polarized scattering equations $\mathcal{E}_i^{\zeta_1}$ via the change of variables (3.59a). ☐

As an aside, although Proposition 3.1.2 implies that the measures are unchanged, it is easily checked directly that $d\mu_n^{\text{pol}} = d\mu_n^{\text{pol},\zeta_1}$: the rescaling (3.59a) gives an overall factor of β_2^{-4} coming from the scattering equation $\delta(\mathcal{E}_1) = \beta_2^{-4}\delta(\mathcal{E}_1^{\zeta_1})$, which exactly compensates the factor from $d^2u_1 d^2v_1 = \beta_2^4\, d^2u_1^{\zeta_1} d^2v_1^{\zeta_1}$. The remaining part of the measure is invariant under the linear shift in $\alpha_2\beta_2$, and thus the polarized measure is invariant under the choice of polarization data.

Theorem 1 *With the above definitions*

$$\det{}'H = \alpha_1 \det{}'H^{\zeta_1} + \alpha_2 \det{}'H^{\zeta_2}\,. \tag{3.61}$$

Proof: For each solution to the scattering equations, the above correspondence (3.59) maps the reduced determinant by

$$\det{}'H = \frac{1}{\langle u_1 u_i\rangle \, [\tilde{u}_1 \tilde{u}_i]} \det H^{[1i]}_{[1i]} = \frac{1}{\beta_2} \det{}'H^{\zeta_1} \,. \tag{3.62}$$

Here, we have reduced on particle 1 for convenience, and used the fact that the diagonal entries H_{ii} for $i \neq 1$ are independent of the polarization ϵ_1 by (3.32). Similarly, the map from $\{u_i, v_i\}$ to $\{u_i^{\zeta_2}, v_i^{\zeta_2}\}$ induced by the polarized scattering equations gives

$$\det{}'H = -\frac{1}{\beta_1} \det{}'H^{\zeta_2} \,. \tag{3.63}$$

Note that $\beta_{1,2}$ depend on the solutions to the polarized scattering equations, so the relations (3.62) and (3.63) between the reduced determinants only hold on individual solutions to the scattering equations, and do not lead to an analogous relation for the amplitudes. However, by combining the two expression we get the following linearity relation

$$\det{}'H = (\alpha_1\beta_2 - \alpha_2\beta_1) \det{}'H = \alpha_1 \det{}'H^{\zeta_1} + \alpha_2 \det{}'H^{\zeta_2} \,, \tag{3.64}$$

as required. This is now independent of the solutions to the scattering equations, and thus lifts to the full amplitudes, confirming (3.56). □

Superamplitudes. The above analysis extends straightforwardly to superamplitudes to give checks on the supersymmetry factors. As before, we take particle 1 to be a gluon, though we do not restrict its position in the multiplet in the supersymmetric case. In the top state, its polarization is $\epsilon_1 = \alpha_1\zeta_1 + \alpha_2\zeta_2$ as above, and in the bottom state we choose the polarization

$$\xi_1 = \alpha_1^{\xi}\zeta_1 + \alpha_2^{\xi}\zeta_2 \,, \tag{3.65}$$

with constant $\alpha_{1,2}^{\xi}$ such that $\alpha_1\alpha_2^{\xi} - \alpha_2\alpha_1^{\xi} = 1$ due to the normalization condition $\langle \epsilon_1 \xi_1 \rangle = 1$.

As indicated above, in the supersymmetric case it will be helpful to treat the basis spinors (ζ_1, ζ_2) as the new basis for the multiplet of particle 1. In the explicit change of variables given in proposition 3.2.2, ζ_1 plays the rôle of the original ϵ_1, and ζ_2 provides the additional polarization spinor to parametrize the full mutiplet, i.e. $\xi_1^{\zeta_1} = \zeta_2$.[5] Using this choice, we can verify by expanding out both sides and using the relation between $\{u_i, v_i\}$ and $\{u_i^{\zeta_1}, v_i^{\zeta_1}\}$ from proposition 3.2.2 that

$$\int d^2q_1\, q_1^2\, e^F = \int d^2q_1^{\zeta_1}\, \beta_2\, \left(\alpha_1 \big(q_1^{\zeta_1}\big)^2 + \alpha_2\right)\, e^{F^{\zeta_1}}\,. \tag{3.66}$$

The superscript ζ_1 again indicates that the supersymmetry factor is defined with the multiplet parametrized by the polarization ζ_1, as well as the variables $u_i^{\zeta_1}$. Similarly, for gluon states at the bottom of the multiplet, we find

$$\int d^2q_1\, e^F = \int d^2q_1^{\zeta_1}\, \beta_2\, \left(\alpha_1^\xi\, \big(q_1^{\zeta_1}\big)^2 + \alpha_2^\xi\right)\, e^{F^{\zeta_1}}\,. \tag{3.67}$$

Combining this with the result (3.62) for the reduced determinant $\det{}'H = \beta_2^{-1} \det{}'H^{\zeta_1}$, we find the expected linearity relations for supersymmetric integrands with one gluon,

$$\det{}'H \int d^2q_1\, q_1^2\, e^F = \det{}'H^{\zeta_1} \int d^2q_1^{\zeta_1}\, \left(\alpha_1 \big(q_1^{\zeta_1}\big)^2 + \alpha_2\right)\, e^{F^{\zeta_1}}\,, \tag{3.68}$$

and similarly for the gluon at the bottom of the multiplet with polarization ξ_1. The simplicity of this relation is due to our choice of $\xi_1^{\zeta_1} = \zeta_2$: using this, as well as the results from §3.2.3, the second term on the right gives indeed the amplitude for a gluon with polarization ζ_2 with a proportionality factor of α_2. As in the bosonic case, the final linearity relation (3.68) is independent of the solution to the polarized scattering equations, and thus lifts to the full superamplitude,

$$\mathcal{A}^{\epsilon_1} = \alpha_1\, \mathcal{A}^{\zeta_1} + \alpha_2\, \mathcal{A}^{\zeta_2}\,, \qquad\qquad \mathcal{A}^{\xi_1} = \alpha_1^\xi\, \mathcal{A}^{\zeta_1} + \alpha_2^\xi\, \mathcal{A}^{\zeta_2}\,. \tag{3.69}$$

3.3 The three and four-point amplitudes

In this section, we discuss the three-particle and four-particle amplitudes in our polarized scattering equations formalism (2.88), and compare them to previous results available in the lit-

[5]Of course, we are free to reverse the roles of ζ_1 and ζ_2 in this discussion, at the expense of a minus sign due to our normalization conventions.

erature, e.g. [98]. We first focus on the three-particle amplitudes that will serve as the seed amplitudes for the BCFW recursion relation of section 3.4. Since the configuration of three momenta is highly degenerate, we include a treatment of the four-particle case for further illustration.

For the calculations below, two general observations will be helpful. First, for low numbers of external particles, the most useful form of the scattering equations arises from (2.70), obtained by skew-symmetrizing the ith polarized scattering equation with ϵ_{iA} to give

$$\sum_j \frac{\langle u_i u_j \rangle \epsilon_{j[A}\epsilon_{B]i}}{\sigma_{ij}} = K_{iAB}\,. \tag{3.70}$$

This can be skewed with further polarization spinors to obtain formulae for $U_{ij} := \langle u_i u_j \rangle / \sigma_{ij}$. We will use this below to construct explicit solutions to the polarized scattering equations, both for three and four particles.

After solving the polarized scattering equations and simplifying the integrands on these solutions, amplitudes are expressed in the form $A^{\epsilon_1 \tilde{\epsilon}_1 \ldots \epsilon_n \tilde{\epsilon}_n}$, with all little group indices contracted linearly into the polarization spinors ϵ_i^a and $\tilde{\epsilon}_i^{\dot{a}}$. To compare our results to the formulae obtained in e.g. [98], we thus have to convert between our polarized formalism and the standard, little-group covariant spinor-helicity formalism, where amplitudes $A_n^{a_1 \dot{a}_1 \ldots a_n \dot{a}_n}$ carry the little group indices of the scattered particles. Using that the amplitudes (2.88) are linear in the polarization spinors ϵ_i^a and $\tilde{\epsilon}_i^{\dot{a}}$ as shown in §3.2.4, the two formalisms are related via

$$A^{\epsilon_1 \tilde{\epsilon}_1 \ldots \epsilon_n \tilde{\epsilon}_n} = \prod_i \epsilon_{i a_i} \tilde{\epsilon}_{i \dot{a}_i} \ldots A_n^{a_1 \dot{a}_1 \ldots a_n \dot{a}_n}\,. \tag{3.71}$$

3.3.1 Three-point amplitudes

We now compute the three particle case to compare to the Yang-Mills result given in [98]. This case is somewhat degenerate as momentum conservation implies that the three null momenta are also mutually orthogonal. In Lorentz signature they would of necessity be proportional, which would be too degenerate to calculate with. We therefore allow complex momenta so that they span a null two-plane. This can be expressed by the non-vanishing $2-$form that is given in spinors by

$$\kappa_B \kappa^A := (k_1 \wedge k_2)^A_B = -(k_1 \wedge k_3)^A_B = (k_2 \wedge k_3)^A_B \,. \tag{3.72}$$

The spinors κ_A and κ^A are defined up to an overall scale and its inverse and are orthogonal to each momentum.

We can represent each momentum k_{iAB} as a line in the projective spin space $\mathbb{CP}^3$ through the two spinors κ_{iaA} for $a = 1, 2$. That each line contains κ_A means that they are concurrent and that they are orthogonal to κ^A means that they are co-planar as in the diagram 3.3.1.

To compare to the results of [98], we introduce little group spinors m_i^a, $\tilde{m}_i^{\dot{a}}$ for each i

$$\kappa_A = m_i^a \kappa_{iAa} \,, \qquad \kappa^A = \tilde{m}_i^{\dot{a}} \kappa^A_{i\dot{a}} \,. \tag{3.73}$$

These are defined in [98] equivalently by

$$\kappa_{iAa} \kappa^A_{j\dot{b}} = m_{ia} \tilde{m}_{j\dot{b}} \,. \tag{3.74}$$

As in [98], we further introduce spinors $w_i, \tilde{w}_i$ normalized against m_i, $\tilde{m}_i$ such that

$$m_{ia} w_i^a = 1 \,, \qquad \tilde{m}_{i\dot{a}} \tilde{w}_i^{\dot{a}} = 1 \,. \tag{3.75}$$

This normalization does not fully fix $w_i, \tilde{w}_i$, since we have the further freedom to add on terms proportional to $m_i, \tilde{m}_i$. We can partially fix this redundancy $w_{ia} \to w_{ia} + c_i m_{ia}$ by the condition

$$w_1^a \kappa_{1Aa} + w_2^a \kappa_{2Aa} + w_3^a \kappa_{3Aa} = 0 \,, \tag{3.76}$$

which imposes co-linearity of the three points $\langle w_i \kappa_{iA} \rangle$ on the lines k_i and reduces the redundancy to shifts satisfying $c_1 + c_2 + c_3 = 0$.

In what follows we will compute the three gluon amplitude from the general formula (2.62) in Yang Mills theory. For three particles the σ_i can be fixed to $(0, 1, \infty)$ and the formula reduces to

$$A_3 = \det{}' H|_* = \frac{\epsilon_{1A}\epsilon_2^A}{U_{23}\tilde{U}_{13}} \,, \tag{3.77}$$

evaluated on the solution to the polarized scattering equations, as indicated by the star. Note that the Jacobian from solving the polarized scattering equations is trivial due to proposition 3.1.2. Having gauge fixed three of the u variables as in §3.1.3, we only need to solve the polarized scattering equations for the three $U_{ij} := U_{ij}^{(1,0)} = \langle u_i u_j\rangle/\sigma_{ij}$, with $U_{ij} = U_{ji}$ for $i \neq j$,

$$U_{12}\epsilon_{2A} + U_{13}\epsilon_{3A} = \langle v_1 \kappa_{1A}\rangle \,, \quad \text{and cyclic,} \tag{3.78}$$

together with the normalization conditions $\langle v_i \epsilon_i\rangle = 1$. These three scattering equations define lines in the plane spanned by the three momenta in the projective spin space as in the diagram 3.3.1.

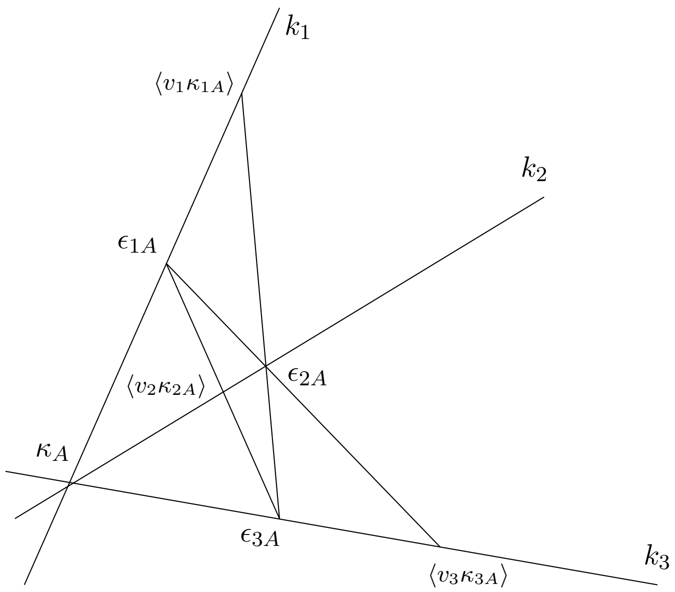

Figure 3.1: Each k_i corresponds to a line in the projective spin space spanned by κ_{iaA}. The lines lie in a common two-plane orthogonal to κ^A and are concurrent meeting at κ_A defined by (3.73). Thus the line k_1 joins ϵ_{1A} and κ_A and so on. The polarized scattering equations give 3 further lines, e.g. with $\mathcal{E}_{1A}$ giving the line joining ϵ_{2A} and ϵ_{3A} and intersecting k_1 at $\langle v_1\kappa_{1A}\rangle$.

In order to solve the polarized scattering equations we use the ϵ_{iA} as a basis of the plane in the projective spin space orthogonal to κ^A to write

$$\kappa_A = \sum_i a_i \epsilon_{iA} \tag{3.79}$$

Using the normalization $\langle v_i \epsilon_i \rangle = 1$, we can further expand v_i in the polarization basis ϵ_i, m_i;

$$v_{ia} = \frac{1}{\langle m_i \epsilon_i \rangle} \left(\langle m_i v_i \rangle \epsilon_{ia} + m_{ia} \right) \,,$$

and solve the system (3.78) to obtain

$$U_{ij} = \frac{a_i}{\langle m_j \epsilon_j \rangle} = \frac{a_j}{\langle m_i \epsilon_i \rangle}\,, \qquad \langle m_i v_i \rangle = a_i \,. \tag{3.80}$$

To compare to [98], we can similarly decompose

$$w_i = -\frac{1}{\langle \epsilon_i m_i \rangle} \epsilon_i + \frac{\langle \epsilon_i w_i \rangle}{\langle \epsilon_i m_i \rangle} m_i, \tag{3.81}$$

and impose the condition (3.76) to obtain:

$$a_i = \frac{\prod_{k \neq i} \langle \epsilon_k m_k \rangle}{\langle \epsilon_1 m_1 \rangle \langle \epsilon_2 m_2 \rangle \langle \epsilon_3 w_3 \rangle + \text{cyc.}} \tag{3.82}$$

The scattering equations for spinors in the antifundamental representation are solved entirely analogously and together we obtain from (3.77) the three point amplitude as

$$A_3 = \Big(\langle \epsilon_1 m_1 \rangle \langle \epsilon_2 m_2 \rangle \langle \epsilon_3 w_3 \rangle + \text{cyc.} \Big) \Big(\langle \tilde{\epsilon}_1 \tilde{m}_1 \rangle \langle \tilde{\epsilon}_2 \tilde{m}_2 \rangle \langle \tilde{\epsilon}_3 \tilde{w}_3 \rangle + \text{cyc.} \Big) \,, \tag{3.83}$$

where we have used that $\epsilon_{1A} \epsilon_2^A = \langle \epsilon_1 m_1 \rangle [\epsilon_2 \tilde{m}_2]$ from (3.74). This is precisely the result in [98], contracted into the polarization spinors as discussed around (3.71).

3.3.2 Four-point Yang-Mills amplitudes

To illustrate these techniques in a slightly more generic setting, consider next the four-gluon amplitude in Yang-Mills theory. As before, we can fix three of the marked points on the sphere, e.g. σ_1, σ_2 and σ_4, so that the solution to the scattering equation in homogeneous coordinates is

$$\sigma_1 = [(1,0)] \;\; \sigma_2 = [(1,1)] \;\; \sigma_3 = [(1, -\frac{s_{13}}{s_{12}})] \;\; \sigma_4 = [(0,1)] \,. \tag{3.84}$$

From the measure, we thus pick up the CHY Jacobian $|\Phi|^{[j_1 j_2 j_3]}_{[i_1 i_2 i_3]} := |\partial \mathcal{E}_i / \partial \sigma_j|^{[j_1 j_2 j_3]}_{[i_1 i_2 i_3]}$ as well as the usual Fadeev-Popov factors $(\sigma_{i_1 i_2}\sigma_{i_2 i_3}\sigma_{i_3 i_1})$ and $(\sigma_{j_1 j_2}\sigma_{j_2 j_3}\sigma_{j_3 j_1})$ due to the equality between the polarized measure and the usual CHY measure established in proposition 3.1.2. Combining this with the four-particle Yang-Mills integrand (2.88a) gives

$$\begin{aligned} A_4^{\epsilon_1 \tilde\epsilon_1 \ldots \epsilon_4 \tilde\epsilon_4} &= \frac{(\sigma_{i_1 i_2}\sigma_{i_2 i_3}\sigma_{i_3 i_1})(\sigma_{j_1 j_2}\sigma_{j_2 j_3}\sigma_{j_3 j_1})}{\det \Phi^{[j_1 j_2 j_3]}_{[i_1 i_2 i_3]}} \, \mathrm{PT}(1234) \det{}' H \Bigg|_* \\ &= \frac{\sigma_{12}^2 (\sigma_{13}\sigma_{34}\sigma_{41})(\sigma_{23}\sigma_{34}\sigma_{42})}{s_{12}} \, \mathrm{PT}(1234) \, \frac{H_{13}H_{24} - H_{14}H_{23}}{\langle u_3 u_4\rangle \, [\tilde u_1 \tilde u_2]} \Bigg|_* \\ &= \frac{1}{\langle u_3 u_4\rangle \, [\tilde u_1 \tilde u_2]} \frac{\sigma_{12}\sigma_{34}}{s_{12}} \left(\epsilon_{1A}\epsilon_3^A \epsilon_{2B}\epsilon_4^B - \frac{\sigma_{31}\sigma_{42}}{\sigma_{41}\sigma_{32}} \epsilon_{1A}\epsilon_4^A \epsilon_{2B}\epsilon_3^B \right) \Bigg|_* , \end{aligned} \tag{3.85}$$

where $*$ again denotes evaluation on the (single) solution to the polarized scattering equations. Using (3.84), the amplitude then becomes

$$A_4^{\epsilon_1 \tilde\epsilon_1 \ldots \epsilon_4 \tilde\epsilon_4} = - \frac{1}{s_{12} U_{34} \tilde U_{12}} \left(\epsilon_{1A}\epsilon_3^A \epsilon_{2B}\epsilon_4^B + \frac{s_{13}}{s_{14}} \epsilon_{1A}\epsilon_3^A \epsilon_{2B}\epsilon_4^B \right) \Bigg|_* , \tag{3.86}$$

evaluated on the solution to the scattering equations. At four points there are $8-3$ independent variables u_i^a and we can take them to be $U_{ij} = \langle u_i u_j\rangle / \sigma_{ij} = U_{ji}$, $i \neq j$, with the extra relation

$$\langle u_i u_j\rangle \langle u_k u_l\rangle + \big(\text{ cyc } jkl \big) = 0, \tag{3.87}$$

given by the Schouten identity. The skewed form (3.70) of the scattering equations give

$$\sum_{j \neq i} U_{ij} \epsilon_{j[A}\epsilon_{iB]} = k_{iAB} \,, \tag{3.88}$$

In order to solve for U_{34} we contract this for $i = 3$ with $\varepsilon^{ABCD}\epsilon_{1C}\epsilon_{2D}$ to obtain

$$U_{34} = - \frac{\langle k_3 12\rangle}{\langle 1234\rangle}, \tag{3.89}$$

where we define

$$\langle 1234\rangle = \varepsilon^{ABCD}\epsilon_{1A}\epsilon_{2B}\epsilon_{3C}\epsilon_{4D}\,, \qquad \langle k_3 12\rangle = \varepsilon^{ABCD}k_{3AB}\epsilon_{1C}\epsilon_{2D}\,. \tag{3.90}$$

Similarly we obtain, using square brackets for 4-brackets of upper-indexed quantities,

$$\tilde{U}_{12} = -\frac{[k_1 34]}{[1234]}. \tag{3.91}$$

Using these we can solve for the v_{ia} to give

$$v_{1a} = \frac{\langle \kappa_{1a} 234\rangle}{\langle 1234\rangle}\,, \tag{3.92}$$

and so on.

The resulting expression for A_4 can be simplified by expanding the product of upper and lower ε tensors as skew product of Kronecker deltas. Consider the quantity

$$\langle k_3 12\rangle[k_1 34] = 4\,\epsilon_{1D}\epsilon_3^D\,k_{3AB}\,k_1^{AC}\,\epsilon_4^B\epsilon_{2C} + 2k_1\cdot k_3(\epsilon_{1A}\epsilon_4^A\,\epsilon_{2B}\epsilon_3^B - \epsilon_{1A}\epsilon_3^A\,\epsilon_{2B}\epsilon_4^B). \tag{3.93}$$

The first term can be rewritten using using momentum conservation as

$$k_{3AB}\,k_1^{AC}\,\kappa^B_{4\dot{a}}\kappa_{2Ca} = -k_{2AB}k_1^{AC}\,\kappa^B_{4\dot{a}}\kappa_{2Ca} = -\frac{1}{2}\,\kappa_{2Aa}\kappa^A_{4\dot{a}}\;k_1\cdot k_2\,, \tag{3.94}$$

such that $\langle k_3 12\rangle[k_1 34]$ is proportional to the numerator of the amplitude,

$$\langle k_3 12\rangle[k_1 34] = s_{14}\left(\epsilon_{1A}\epsilon_3^A\,\epsilon_{2B}\epsilon_4^B + \frac{s_{13}}{s_{14}}\epsilon_{1A}\epsilon_4^A\epsilon_{2B}\epsilon_3^B\right)\,. \tag{3.95}$$

The amplitude then agrees with the result of [98],

$$A_4^{\epsilon_1\tilde{\epsilon}_1\ldots\epsilon_4\tilde{\epsilon}_4} = \frac{\langle 1234\rangle[1234]}{s_{12}s_{14}}\,, \tag{3.96}$$

upon the usual identification (3.71).

As discussed in section 4.1.4, the supersymmetry representation we use breaks little group symmetry so that little group multiplets are spread in different degrees in the superfield expansion (2.81) in terms of supermomenta. All above expressions are for gluons in the top state $g^{\epsilon\tilde{\epsilon}}$, but the calculations extend directly to other amplitudes as well. As we have seen in section 3.2.3, amplitudes for gluons appearing at order q^2 in the multiplet can be calculated either from the supersymmetry representation, or by replacing $\epsilon_i \to \xi_i$ in the integrand. At four points, this can be seen explicitly: consider first the amplitude $A_4(g^{\epsilon_1\tilde{\epsilon}_1}g^{\epsilon_2\tilde{\epsilon}_2}g^{\xi_3\tilde{\epsilon}_3}g^{\xi_4\tilde{\epsilon}_4})$ obtained from the supersymmetry representation,

$$A_4(g^{\epsilon_1\tilde{\epsilon}_1}g^{\epsilon_2\tilde{\epsilon}_2}g^{\xi_3\tilde{\epsilon}_3}g^{\xi_4\tilde{\epsilon}_4}) = A_4^{\epsilon_1\tilde{\epsilon}_1\ldots\epsilon_4\tilde{\epsilon}_4}\,\Omega^{IJ}\Omega^{KL}\frac{\partial}{\partial q_3^I}\frac{\partial}{\partial q_3^J}\frac{\partial}{\partial q_4^K}\frac{\partial}{\partial q_4^K}\; e^{F+\tilde{F}}\Big|_*\Big|_{q_i=0}. \tag{3.97}$$

The only non-vanishing term comes from the F^2 in the expansion of the exponential, and gives an extra factor of $\det U^{\{34\}} = -U_{34}^2 + \langle\xi_3 v_3\rangle\langle\xi_4 v_4\rangle$ in the amplitude. When we evaluate this on the solutions to the polarized scattering equations we obtain, using (3.89) and (3.92),

$$\det U^{\{34\}}\Big|_* = \frac{1}{\langle 1234\rangle^2}\Big(\langle\xi_3\,312\rangle\langle\xi_4\,412\rangle - \langle\xi_3\,124\rangle\langle\xi_4\,123\rangle\Big) = \frac{\langle 12\,\xi_3\xi_4\rangle}{\langle 1234\rangle}. \tag{3.98}$$

Here we have used $k_{i\,AB} = \xi_{i[A}\epsilon_{i|B]}$ in the first equality, as well as the notation $\xi_{iA} := \langle\xi_i\kappa_{iA}\rangle$, and the last equality follows from a Schouten identity in the two-dimensional space defined by $\varepsilon^{ABCD}\epsilon_{1C}\epsilon_{2D}$. Using the result (3.96) for the amplitude where all gluons are in the top state, we thus find

$$A_4(g^{\epsilon_1\tilde{\epsilon}_1}g^{\epsilon_2\tilde{\epsilon}_2}g^{\xi_3\tilde{\epsilon}_3}g^{\xi_4\tilde{\epsilon}_4}) = \frac{\langle 12\,\xi_3\xi_4\rangle[1234]}{s_{12}s_{14}}. \tag{3.99}$$

This clearly agrees with the result from the integrand $\det{}'H_I$ for $I = \{3,4\}$, i.e. by replacing ϵ_{ia} by ξ_{ia} for $i = 3,4$ in (3.96). Similar conjugate formulae apply for amplitudes with a pair of external particles in the $g^{\epsilon\tilde{\xi}}$ states.

3.3.3 Other theories

The Yang-Mills calculations extend directly to the other theories expressed as integrals over the polarized scattering equations. For any theory that admits the representation (2.62), the four point amplitude for the top states of the supersymmetry multiplet has the form:

$$A_4 = \frac{1}{\det'\Phi} \mathcal{I}_L^{\rm h} \mathcal{I}_R^{\rm h} \bigg|_* , \tag{3.100}$$

where the $*$ indicates that the formula is evaluated on the solutions to the polarized scattering equations. Having solved the polarized scattering equations at four point, (3.84), it is now an easy task to evaluate the amplitude for other theories than Yang-Mills (2.88). We have already discussed the Jacobian,

$$\frac{1}{\det'\Phi} = \frac{(\sigma_{i_1 i_2}\sigma_{i_2 i_3}\sigma_{i_3 i_1})(\sigma_{j_1 j_2}\sigma_{j_2 j_3}\sigma_{j_3 j_1})}{\det \Phi^{[j_1 j_2 j_3]}_{[i_1 i_2 i_3]}} = -\frac{s_{12}^4}{s_{12}s_{13}s_{14}} \tag{3.101}$$

The main ingredients that appear in the half integrands evaluated on such solutions are as follows:

$$\mathrm{PT}(1234) = -\frac{s_{12}}{s_{14}} \qquad \det{}'H = \langle 1234\rangle[1234]\frac{s_{12}^2}{s_{12}s_{13}s_{14}} \tag{3.102a}$$

$$\mathrm{Pf}\, U^{(1,1)} = \frac{s_{13}s_{14}}{\langle 1234\rangle\,[1234]} \qquad \mathrm{Pf}\, U^{(2,0)} = \frac{s_{13}s_{14}}{\langle 1234\rangle^2} \tag{3.102b}$$

$$\mathrm{Pf}'A = s_{12}\,. \tag{3.102c}$$

It is then straightforward to calculate all four-particle amplitudes for the theories we have discussed. In $(2,2)$ supergravity, for all particles in the top state, we obtain:

$$M_4^{\rm grav} = \frac{\langle 1234\rangle^2[1234]^2}{s_{12}s_{13}s_{14}}\,, \tag{3.103}$$

which corresponds to the result in [79, 82] and reproduces the KLT relation. For the brane theories we have

$$A_4^{\rm D5} = \langle 1234\rangle[1234]\,, \tag{3.104}$$

$$A_4^{\rm M5} = \langle 1234\rangle^2\,, \tag{3.105}$$

agreeing with [108]. As expected these give the same result on reducing to four or five dimensions where fundamental and anti-fundamental spinors are identified.

The more exotic and controversial formulae in [2], obtained by double-copying the above integrands. When combining the M5 half integrand with a Parke Taylor factor, we get

$$A_4^{(2,0)-\mathrm{PT}} = \frac{\langle 1234\rangle^2}{s_{12}s_{14}}\,. \tag{3.106}$$

As expected, the formula is chiral, and has the same reduction to 5d as the Yang-Mills amplitude. We can also look at the formulae for other 'double copied' theories:

$$A_4^{(3,1)} = \frac{\langle 1234\rangle^3[1234]}{s_{12}s_{13}s_{14}} \tag{3.107}$$

$$A_4^{(4,0)} = \frac{\langle 1234\rangle^4}{s_{12}s_{13}s_{14}}\,. \tag{3.108}$$

We note that (3.107)-(3.108) give the same result as the gravity amplitudes (3.103) upon reduction to four and five dimensions. However, in six dimensions, as remarked in [82, 107], the formulae are more problematic as soft limits (or factorization) to three-point amplitudes are not obviously well-defined. This is because the three-particle kinematics $\kappa_A = m_i^a\kappa_{iaA}$ and $\kappa^A = \tilde{m}_i^{\dot{a}}\kappa^A_{i\dot{a}}$ of (3.73) each have a scaling ambiguity

$$m_i^a \to \alpha\, m_i^a\,, \qquad\qquad \tilde{m}_i^{\dot{a}} \to \alpha^{-1}\tilde{m}_i^{\dot{a}}\,, \tag{3.109}$$

that cancels in $\kappa_A\kappa^B$. In our discussion of the Yang-Mills three-particle amplitudes, this was reflected in the the two factors $\big(\langle\epsilon_1 m_1\rangle\langle\epsilon_2 m_2\rangle\langle\epsilon_3 w_3\rangle + \text{cyc.}\big)\times$ (its tilded version) not being individually invariant under the scaling (3.109), although of course this ambiguity cancels in the full amplitude (3.83). In the chiral double-copied amplitudes (3.106) - (3.108) however, this scaling ambiguity cannot cancel anymore, so there are no invariant three-point amplitudes for gerbe theories. On reduction to 5d, there is an identification between the chiral and anti-chiral spinors so the scaling in (3.109) is fixed up to sign. This is also reflected in the factorization discussion of the related formulae in [82], where it was shown that the resulting three-particle formulae are non-local. As discussed there, the non-locality can be made manifest in two different ways. To factorize the four-particle formula into the product of two three-particle objects summed over internal states, we have to either fix a scale α or fix the shift redundancy $w_{ia} \to w_{ia} + c_i m_{ia}$ of the dual variables. In both cases, the required 'frame choice' depends on the kinematics of all *four* particles, and the three-particle objects are not invariant under the a rescaling of α (in the first case) or a shift in c_i (in the latter case).

Thus it seems unlikely that the formulae (3.106) - (3.108) can be interpreted as tree-level S-matrices in the normal sense.

3.3.4 Fermionic amplitudes

We can also evaluate amplitudes involving the fermionic sector. We will show here how this works for the scattering of two gluons with two gluini in $(1,1)$ super Yang-Mills, but the results can be adapted easily to supergravity and the brane theories.

Consider the four particle amplitude $A_4(g_1^{\epsilon\tilde{\epsilon}}, g_2^{\epsilon\tilde{\epsilon}}, \psi_3^{I\tilde{\epsilon}}, \psi_4^{J\tilde{\epsilon}})$ for two gluons and two gluini, obtained in our supersymmetry representation by extracting the fermionic components as follows,

$$
\begin{aligned}
A_4(g_1^{\epsilon\tilde{\epsilon}}, g_2^{\epsilon\tilde{\epsilon}}, \psi_3^{I\tilde{\epsilon}}, \psi_4^{J\tilde{\epsilon}}) &= \frac{\langle 1234\rangle[1234]}{s_{12}s_{14}} \frac{\partial}{\partial q_3^I}\frac{\partial}{\partial q_4^J}(1 + F_1 + \tilde{F}_1 + ...)\Big|_{q_i=\tilde{q}_i=0} \\
&= \frac{\langle 1234\rangle[1234]}{s_{12}s_{14}} U_{34}\Omega_{IJ}
\end{aligned}
\tag{3.110}
$$

Inserting the solution to the polarized scattering equations (3.89) we obtain,

$$
A_4(g_1^{\epsilon\tilde{\epsilon}}, g_2^{\epsilon\tilde{\epsilon}}, \psi_3^{I\tilde{\epsilon}}, \psi_4^{J\tilde{\epsilon}}) = \frac{\langle 12k_3\rangle[1234]}{s_{12}s_{14}}\Omega_{IJ} \tag{3.111}
$$

We can compare this to the amplitude representation of [79] in the little-group preserving supersymmetry representation;

$$
A_4^{\text{susy}} = \frac{\delta^4(\sum q)\delta^4(\sum \tilde{q})}{s_{12}s_{14}}\,, \tag{3.112}
$$

where the supercharges are $q^{AI} = \varepsilon^{\dot{a}\dot{b}}\kappa^A_{\dot{a}}\tilde{\eta}^I_{\dot{b}}$ and $q^I_A = \varepsilon_{ab}\kappa^a_A\eta^{bI}$. The amplitude $A_4(g_1^{a\dot{a}}, g_2^{b\dot{b}}, \psi_3^{\dot{c}}, \psi_4^{\dot{d}})$

is now the following coefficient of the Grassmann variables η and $\tilde{\eta}$,

$$\begin{aligned} A_4(g_1^{a\dot{a}}, g_2^{b\dot{b}}, \psi_3^{\dot{c}}, \psi_4^{\dot{d}}) &= \frac{\partial}{\partial\eta_1^a}\frac{\partial}{\partial\tilde{\eta}_1^{\dot{a}}}\frac{\partial}{\partial\eta_2^b}\frac{\partial}{\partial\tilde{\eta}_2^{\dot{b}}}\frac{\partial}{\partial\tilde{\eta}_3^{\dot{c}}}\frac{\partial}{\partial\tilde{\eta}_4^{\dot{d}}}\frac{\partial}{\partial\eta_4^e}\frac{\partial}{\partial\eta_4^g}\;\varepsilon^{eg}\;\frac{\delta^4(\sum q)\delta^4(\sum\tilde{q})}{s_{12}s_{14}}\Bigg|_{\eta_i=\tilde{\eta}_i=0} \\ &= \frac{\langle 1_a 2_b k_3\rangle\,[1_{\dot{a}}2_{\dot{b}}3_{\dot{c}}4_{\dot{d}}]}{s_{12}s_{14}} \end{aligned} \tag{3.113}$$

This agrees with our result (3.111) after contraction into the external polarization states.

3.4 Proof of the formula by BCFW recursion

In this section, we give a proof of the gravity and Yang-Mills formulae using BCFW recursion [9, 11]. This is a powerful on-shell tool that has been used to prove a variety of explicit amplitude representations. This technique has two main ingredients. The first is to introduce a deformation of the formula for the amplitude depending on a complex parameter z, and to use complex analysis to reconstruct the amplitude in terms of its residues at poles in z. The second key ingredient in the argument is the factorization property of amplitudes. We know from the Feynman diagram representation of amplitudes that they are multilinear in the polarization vectors and rational in the momenta. The only poles arise from propagators, so that they can only arise along *factorization channels*, where partial sums of the momenta go on shell. At tree-level, factorization is the statement that the residues at such poles are tree amplitudes on each side of the propagator. This then allows us to identify the residues in z in terms of lower point amplitudes, setting up the recursion.

BCFW shifts are generally based on the following one-parameter deformation of the external momenta,

$$\hat{k}_{1\mu} = k_{1\mu} + z\,l_\mu\,, \qquad \hat{k}_{n\mu} = k_{n\mu} - z\,l_\mu\,, \tag{3.114}$$

with $q^2 = q\cdot k_1 = q\cdot k_n = 0$. Cauchy's theorem applied to $\mathcal{A}/z$ then gives an equality between the original undeformed amplitude at $z=0$ and the sum over all other residues at the possible factorisation channels of the amplitude and at ∞. If

$$\lim_{z\to\infty}\mathcal{A}(z) = 0\,, \tag{3.115}$$

we say that there are no boundary terms at $z = \infty$ and the shift is valid. Provided that the amplitude has the factorisation properties we expect from unitarity, the residue theorem then expresses it as a sum over products of lower point amplitudes $\mathcal{A}_{n_L+1}$ and $\mathcal{A}_{n_R+1}$, with $n_L + 1$ and $n_R + 1 = (n - n_L) + 1$ particles respectively, but at shifted values of z

$$\mathcal{A}_n = \sum_{L,R} \mathcal{A}_{n_L+1}\left(z_L\right) \frac{1}{k_L^2} \mathcal{A}_{n_R+1}\left(z_L\right) . \tag{3.116}$$

The sum runs over partitions of the n particles into two sets L and R, with one of the deformed momenta in each subset, $1 \in L$ and $n \in R$. In the propagator, $k_L = \sum_{i\in L} k_i$ denotes the (undeformed, off-shell) momentum, whereas the amplitudes are evaluated on the on-shell deformed momentum $\hat{k}_L = \sum_{i\in L} k_i + z_L\, q$ with $z_L = -k_L^2/2q \cdot k_L$. See also fig. 3.2 for a diagrammatic represenation of the recursion. For particles transforming in non-trivial representations of the little group, the BCFW shift (3.114) has to be extended to the polarization vectors as well [114], and the boundary terms vanish if the shift vector l_μ is chosen to align with the polarization vector of one of the shifted particles, $l_\mu = e_{1\,\mu}$. In this case the sum over partitions in the BCFW recursion relation (3.116) also includes a sum over a complete set of propagating states, labeled for example by their polarization data for gluons or gravitons.

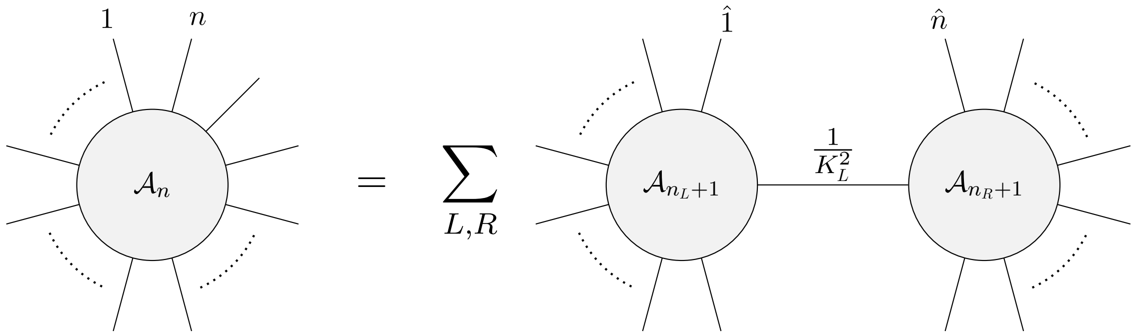

Figure 3.2: A diagrammatic representation of the BCFW relation (3.116).

The recursion (3.116) has been a useful tool to prove novel amplitude representations. In particular, it guarantees that any expression satisfying factorization[6] and the boundary condition (3.115) is a representation of the amplitude. In §3.4.1 we adapt the shift to our formulae, in

[6] including the correct 3-particle amplitudes

§3.4.2, we show that our amplitudes factorize correctly. The proof of the vanishing of boundary terms for gauge theory and gravity follows a similar strategy to the one employed to show factorization and we refer the interested reader to the original paper [2].

3.4.1 Shift

Fundamental spinors. In the 6d spinor-helicity formalism, we introduce a BCFW shift dependent on the chiral polarization data of the shifted particles by

$$\hat{\kappa}^a_{1A} = \kappa^a_{1A} + z\,\epsilon^a_1\,\epsilon_{nA}\,, \qquad \hat{\kappa}^a_{nA} = \kappa^a_{nA} + z\,\epsilon^a_n\,\epsilon_{1A}\,. \tag{3.117}$$

This shift evidently leaves the polarization spinors ϵ_A invariant, but shifts the spinors $\langle v_1 \kappa_{1\,A}\rangle$ and $\langle v_n \kappa_{n\,A}\rangle$ featuring in the polarised scattering equations by a term proportional to the polarization spinor of the other particle. The invariance of the polarization spinors $\epsilon_{1,n}$ ensures that the shift is well-defined, in the sense that the 'shift-spinors' $\delta\kappa_{1,n}$ are themselves unaffected. In terms of momenta, the spinorial deformation (3.117) corresponds to a standard BCFW shift,

$$\hat{k}_{1AB} = k_{1AB} + z\,l_{AB}\,, \qquad \hat{k}_{nAB} = k_{nAB} - z\,l_{AB}\,. \tag{3.118}$$

In contrast to the usual construction however, the shift vector l_{AB} is composed of the polarization spinors of both particles 1 and n,

$$l_{AB} = \epsilon_{n\,[A}\epsilon_{1\,B]}\,. \tag{3.119}$$

In addition to preserving momentum conservation and being orthogonal to the momenta of the shifted particles, this choice of shift vector is also orthogonal to the polarization of the shifted particles, which guarantees the vanishing of boundary terms for Yang Mills and gravity.

For generic polarization data of the particles 1 and n, the BCFW shift (3.119) differs from the standard BCFW shift for Yang-Mills theory and gravity [114], as well as the 6d spinorial shift of [98]. There the shift vector for gluons and gravitons is chosen to align with the polarization of one of the shifted particles, $l_\mu = e_{1\,\mu}$, to ensure that the boundary terms vanish. This setup can be recovered from the shift (3.117) only if the polarization spinors for particles 1 and n are related in a specific way, so the two shifts are in general inequivalent.

Anti-fundamental spinors. The anti-fundamental shift

$$\hat{k}_1^{AB} = k_1^{AB} + z\, l^{AB}\,, \qquad \hat{k}_n^{AB} = k_n^{AB} - z\, l^{AB}\,, \tag{3.120}$$

is of course related to the chiral one via $l^{AB} = \varepsilon^{ABCD} q_{CD}$, but this does not fully determine the shift of the anti-chiral spinors $\hat{\kappa}^A_{\dot{a}}$. We can use this freedom to choose a BCFW shift where both deformations $\delta\kappa^A_{1\dot{a}}$ and $\delta\kappa^A_{n\dot{a}}$ are proportional to the same spinor $\tilde{\epsilon}^A$:

$$\hat{\kappa}_1{}^A_{\dot{a}} = \kappa_1{}^A_{\dot{a}} - z\,\tilde{\epsilon}^A \left(\epsilon_{n\,B}\kappa_1{}^B_{\dot{a}}\right)\,, \tag{3.121a}$$

$$\hat{\kappa}_n{}^A_{\dot{a}} = \kappa_n{}^A_{\dot{a}} - z\,\tilde{\epsilon}^A \left(\epsilon_{1\,B}\kappa_n{}^B_{\dot{a}}\right)\,. \tag{3.121b}$$

The spinor $\tilde{\epsilon}^A$ is constructed such that it is a valid choice for $\tilde{\epsilon}_1^A = \tilde{\epsilon}^A$ and $\tilde{\epsilon}_n^A = \tilde{\epsilon}^A$,

$$\tilde{\epsilon}^A = \epsilon_{1\,a}\kappa_n{}^A_{\dot{a}} \left(\kappa_n{}^B_{\dot{a}}\,\kappa_1{}^a_B\right)^{-1} + \epsilon_{n\,a}\kappa_1{}^A_{\dot{a}} \left(\kappa_1{}^B_{\dot{a}}\,\kappa_n{}^a_B\right)^{-1}\,. \tag{3.122}$$

The first term is just the canonical choice for $\tilde{\epsilon}_1^A$ and pure gauge for particle n, and vice versa for the second term so that:

$$\tilde{\epsilon}^A\,\kappa^a_{1A} = \epsilon^a_1\,, \qquad \tilde{\epsilon}^A\,\kappa^a_{nA} = \epsilon^a_n\,. \tag{3.123}$$

The anti-fundamental BCFW deformation then leads to the shift (3.120) for the momenta, where the shift vector q is again determined by the chiral polarization spinors of both shifted particles. The polarization spinor $\tilde{\epsilon}^A$ as well as the shift-spinors $\delta\kappa_1{}^A_{\dot{a}}$ and $\delta\kappa_n{}^A_{\dot{a}}$ are invariant under the BCFW deformation, and the shift (3.121) is well-defined.

Shifting the supermomenta. In the R-symmetry preserving supersymmetry representation, the supershift is not implemented via a linear shift in the fermionic variables, but rather by a multiplicative exponential factor

$$\mathcal{I}_n \to \hat{\mathcal{I}}_n = \mathcal{I}_n \exp\left(-z\, q_{1I} q_{nJ} \Omega^{IJ}\right) . \tag{3.124}$$

This is the fermionic Fourier transform of the standard linear super-BCFW shift in the little-group preserving representation, see e.g. [115]. As expected, the Fourier transform interchanges linear shifts of the variables in z with a multiplication by an exponential factor.

3.4.2 Factorisation

In this section we want to show how the singularities of the amplitude (2.62) appear in the limit $k_I^2 \to 0$, where I is a subset of $\{1, ...n\}$, and that in this limit the amplitude factorises as:

$$\mathcal{A}_n = \sum{}' \mathcal{A}_{n_L+1} \frac{1}{k_L^2} \mathcal{A}_{n_R+1} \tag{3.125}$$

with $n_L + n_R = n$ and $\sum'$ indicates a sum over polarization states or a 'supersum' over states in the supermultiplet when considering superamplitudes.

The integrands in (2.62) are polynomials in the kinematic variables. The singularities can then only arise from the boundary of the moduli space $\partial\mathfrak{M}_{0,n}^{\text{pol}}$. This is the moduli space encoding the locations of the punctures σ_i as well as the values for u_i, v_i, modulo the symmetry group $\mathrm{SL}(2,\mathbb{C})_\sigma \times \mathrm{SL}(2,\mathbb{C})_u$. Moreover, the permutation invariance of the reduced determinant $\det' H$ guarantee that for Yang Mills and supergravity the singularities actually come from the boundary of the moduli space of the Riemann sphere, $\partial\widehat{\mathfrak{M}}_{0,n} \subset \partial\mathfrak{M}_{0,n}^{\text{pol}}$. Here $\partial\widehat{\mathfrak{M}}_{0,n}$ denotes the Deligne-Mumford compactification of the moduli space of marked Riemann surfaces, obtained by adding nodal surfaces to ensure compactness [116], [93].

This boundary of the moduli space corresponds to separating degenerations that split the sphere Σ into two components, Σ_L and Σ_R, and that partition the punctures as $n = n_L + n_R$,

$$\partial\widehat{\mathfrak{M}}_{0,n} \simeq \widehat{\mathfrak{M}}_{0,n_L+1} \times \widehat{\mathfrak{M}}_{0,n_R+1} . \tag{3.126}$$

The boundary $\partial\widehat{\mathfrak{M}}_{0,n}$ can be parametrised by gluing two Riemann spheres Σ_L and Σ_R as follows. Choose a marked point on each sphere, $\sigma_R \in \Sigma_R$ and $x_L \in \Sigma_L$, and remove the disks $|\sigma - \sigma_R| < \varepsilon^{1/2}$ and $|x - x_L| < \varepsilon^{1/2}$, where ε is the parameter governing the degeneration. Then we can form a single Riemann surface by identifying,

$$(x - x_L)\,(\sigma - \sigma_R) = \varepsilon\,. \tag{3.127}$$

The boundary of the moduli space $\partial\widehat{\mathfrak{M}}_{0,n}$ corresponds to the limiting case $\varepsilon \to 0$.
First of all we can establish the correspondence between degenerations of the punctured Riemann sphere and the factorisation channels of the amplitude.
Using the parametrization of the boundary (3.127), one can see that the spinor $\lambda(\sigma)$ induces spinors $\lambda^{(L)}(\sigma)$ and $\lambda^{(R)}(\sigma)$ on Σ_L and Σ_R respectively and that these stay of order one throughout the degeneration and so do the polarised scattering equations. In particular we have:

$$\prod_{i=1}^{n} \delta^4\left(\mathcal{E}_i\right) = \prod_{i\in L} \delta^4\left(\mathcal{E}_i^{(L)}\right) \prod_{p\in R} \delta^4\left(\mathcal{E}_p^{(R)}\right)\,. \tag{3.128}$$

One then finds that in the degeneration limit the propagator

$$k_L := \sum_{i\in L} k_i \tag{3.129}$$

goes on shell, i.e.

$$k_L^2 = O(\varepsilon) \tag{3.130}$$

as $\varepsilon \to 0$, which corresponds to a factorisation channel.
It is then possible to track the behaviour of the measure of integration in the degeneration and to verify that it mirrors the behaviour of the boundary of the moduli space [3] [117]. This results in:

$$d\mu_n^{\text{pol}} = \frac{\varepsilon^{2(n_L-1)}}{\prod_{i\in L} x_{iL}^4}\frac{d\varepsilon}{\varepsilon}\int_{(\kappa)} \frac{d^8\kappa_A^a}{\text{vol}\,\text{SL}(2,\mathbb{C})}\, d\mu_{n_L+1}^{\text{pol}}\, d\mu_{n_R+1}^{\text{pol}}\,. \tag{3.131}$$

The delta-functions $\delta\left(k_L^2 - \varepsilon\mathcal{F}\right)$ enforcing that $\varepsilon \sim k_L^2 \sim k_R^2$ are part of the momentum conservation contained in the polarised measure.

It remains to spell out the behaviour of the integrands. We find for the Parke-Taylor factor:

$$\mathrm{PT}(\alpha) = \varepsilon^{-(n_L-1)} \prod_{i\in L} x_{iR}^2 \;\, \mathrm{PT}(\alpha_L)\mathrm{PT}(\alpha_R)\,, \tag{3.132}$$

where the particles in the subset L are all consecutive in α, the ordering of the external particles (for a given color-ordered amplitude). If the particles in L are not consecutive in α, the Parke-Taylor contributes with more powers of ε, thus cancelling the pole. It is indeed this integrand that selects the singularities corresponding to planar diagrams of a given color ordering.
As for the reduced determinant, we find

$$\det{}'H = \varepsilon^{-(n_L-1)} \langle v_L v_R\rangle [\tilde{v}_L\tilde{v}_R] \prod_{i\in L} x_{iR}^2 \; \det{}'H_L \det{}'H_R\,. \tag{3.133}$$

For amplitudes involving only gluons in the top state of the super Yang Mills superfield (2.81), we expect the amplitude to factorise as[7]:

$$\begin{aligned}\sum_{\text{states}} \mathcal{A}_{n_L+1}\mathcal{A}_{n_R+1} &= \epsilon_{ab}\,\mathcal{A}^a_{n_L+1}\mathcal{A}^b_{n_R+1} = v_L^{[a}\epsilon_L^{b]}\,\mathcal{A}^a_{n_L+1}\mathcal{A}^b_{n_R+1} \\ &= \frac{1}{\langle \epsilon_L\epsilon_R\rangle}\mathcal{A}_{n_L+1}(\epsilon_L)\mathcal{A}_{n_R+1}(\epsilon_R) + \underbrace{\langle \epsilon_L\epsilon_R\rangle \mathcal{A}_{n_L+1}(v_L)\mathcal{A}_{n_R+1}(v_R)}_{=0}\,,\end{aligned} \tag{3.134}$$

where the second factor vanishes because the subamplitudes have a single particle in the bottom state. The polarised scattering equations impose $\langle v_L v_R\rangle = \langle \epsilon_L\epsilon_R\rangle^{-1}$, so that (3.131),(3.132) and (3.133) together determine the correct factorisation behaviour.

Sum over states. We discuss here what behaviour we expect under factorization from the susy factors. In general, supersymmetric invariance determines the 'gluing factor' $G(q_L, q_R)$ that is responsible for the sum over states in a factorization channel,

$$\mathcal{A}_n = \frac{1}{k_L^2}\int d^{2N}q_L d^{2N}q_R \;\mathcal{A}_{n_L+1}\mathcal{A}_{n_R+1}\; G(q_L,q_R)\,. \tag{3.135}$$

[7]We look here at $\mathcal{N}=(1,0)$, the result for the other chirality is analogous.

This can be seen as follows: acting on the LHS with the full susy generator Q_{AI} we need to have

$$Q_{AI}\mathcal{A}_n = 0 \qquad Q^{(L)}_{AI}\mathcal{A}_{n_L+1} = Q^{(R)}_{AI}\mathcal{A}_{n_R+1} = 0\,. \tag{3.136}$$

Using that

$$Q_{AI}\mathcal{A}_{n_L+1} = -Q_{LAI}\mathcal{A}_{n_L+1}\,, \qquad Q_{AI}\mathcal{A}_{n_R+1} = -Q_{RAI}\mathcal{A}_{n_R+1}\,, \qquad Q_{AI}G(q_L,q_R) = 0\,, \tag{3.137}$$

we find that

$$G(q_L,q_R) = \langle \epsilon_L\epsilon_R\rangle^N \exp\left(i\langle v_L v_R\rangle\, q_{LI}q_{RJ}\Omega^{IJ}\right) \tag{3.138}$$

solves this, where the normalization is fixed by comparison with the purely bosonic case. To conclude the proof of factorization, it can be shown that the supersymmetry representation factorizes as:

$$e^{F_N} = \langle \epsilon_L\epsilon_R\rangle^{2N}\int d^{2N}q_L d^{2N}q_R\; e^{F_L+F_R}\; e^{i\langle v_L v_R\rangle\, q_{LI}q_{RJ}\Omega^{IJ}}\,. \tag{3.139}$$

3.5 Discussion

In this chapter we have argued that the polarized scattering equations provide a natural generalization of the twistor and ambitwistor supersymmetric formulae from four dimensions. They lead to formulae for a full spectrum of supersymmetric gauge, gravity and brane theories in six-dimensions. These formulae are furthermore shown to factorize properly as a consequence of properties of the polarized scattering equations themselves, as described in §3.4. This led to a proof of the main formulae by BCFW recursion.

There remain issues that are not optimally resolved in our framework. Because the solutions to the polarized scattering equations themselves depend on the polarization data, it is no longer obvious that the formulae we obtain are linear in each polarization vector as they need to be, although the proof is relatively straightforward. As shown in §3.1, there is an $n+2$ dimensional vector space of potential solutions to the polarized scattering equations whose dimensionality is then reduced by choice of polarization spinors. It should be possible to develop this further to produce formulae that are manifestly linear in the polarization data, or alternatively with free little-group indices as is more usually in higher-dimensional spinor-helicity frameworks. Further avenues are as follows.

Grassmannians, polyhedra, and equivalence with other formulations. In four dimensions, twistor-string formulae for amplitudes, and indeed general BCFW terms, can be embedded as $2n-4$-dimensional cycles in the Grassmannian $G(k,n)$ for amplitudes with k negative helicity particles, [118, 119].

In [82] it was similarly shown that their 6d formulae could be embedded into a Lagrangian Grassmannian, i.e., the Grassmannian $LG(n,2n)$ of Lagrangian n-spaces in a symplectic $2n$-dimensional vector space. Ref. [111] further discussed how the polarized scattering equation formulation of [1, 2] can also be embedded in the same Grassmannian, allowing one to see that the two formulations are essentially gauge equivalent representations. In the formulation in this chapter, an element of the Grassmannian can be represented as an $n \times 2n$ matrix C_l^{ia} with a being the little group index for k_i and l being also a particle index.[8] The symplectic form is given by $\Omega_{iajb} = \varepsilon_{ab}\delta_{ij}$ and the condition that C_l^{ia} defines an element of the Lagrangian Grassmannian is that

$$C_l^{ia} C_m^{jb} \Omega_{iajb} = 0 \,. \tag{3.140}$$

This skew form is natural in the sense that it arises from momentum conservation in the form

$$\kappa_{iA}^{a} \kappa_{jB}^{b} \Omega_{iajb} = 0 \,. \tag{3.141}$$

The Grassmannian integral formula then takes the form

$$\int_\Gamma d\mu \, \mathcal{I} \int \prod_j \delta^4(C_j^{ia} \kappa_{iaA}) \,. \tag{3.142}$$

Here $\mathcal{I}$ is a theory dependent integrand, Γ a cycle in the Grassmannian of dimension $4n-6$,

[8] For [82, 111] this l-index is replaced by ak where a is the global little group index, and $k = 0, \ldots, (n-2)/2$ indexes a basis in the space of polynomials on $\mathbb{C}$ of degree $(n-2)/2$.

and $d\mu$ a measure on Γ. Our data embeds into the Grassmannian by

$$C_j^{ai} = \frac{\langle u_i u_j \rangle}{\sigma_{ij}} \epsilon_i^a - \delta_j^i v_i^a \,, \tag{3.143}$$

with Γ parametrized by (σ_i, u_i, v_i) subject to the constraints $\langle v_i \epsilon_i \rangle = 1$ and modulo the Möbius transformations on the σ_i, and SL(2) on the u_i. A different parametrization[9] for Γ is given in [82], and in [111] it was argued that the two representations are gauge equivalent in $LG(n, 2n)$.

In this chapter in §3.2.4, the argument for linearity of the reduced determinants in the polarization data relies on a map between solutions to the polarized scattering equations that have different polarization data. This map should therefore similarly arise from an analogous gauge transformation in the Grassmannian $LG(n, 2n)$.

Polyhedra such as the amplituhedron [120] emerge when BCFW cycles in a Grassmannian are united into one geometric object whose combinatorics are determined by a certain positive geometry. The original amplituhedron was adapted to momentum twistor or Wilson-loop descriptions of $N = 4$ super Yang-Mills amplitudes [121–123], but there is, at least as yet, no analogue of this in six dimensions. The version of the 4d amplituhedron ideas that are most natural in the context of the Grassmannian descriptions here is that described in [112], a $2n-4$-dimensional space. It follows from the above that the analogue in 6d should therefore be a $4n - 6$ dimensional space. In our context this space will then be naturally embedded in $\mathbb{R}^{4n}$ (perhaps projected onto some quotient) as the image of the positive Lagrangian Grassmannian $LG_+(n, 2n)$ under the map

$$Y_{lA} = C_l^{ia} \kappa_{iaA} \,. \tag{3.144}$$

There is of course an anti-chiral version also. It remains to explore these frameworks.

Worldsheet models in 6d. Another gap in our description is to identify ambitwistor string models that underly the formulae. Ambitwistor-string models that admit vertex operators that yield the polarized scattering equations and supersymmetry factors were introduced in [1], together with worldsheet matter that provides the reduced determinants. However, these were

[9] In the notation of those references, the $4n-6$-cycles are parametrized by (σ_i, w_{ia}^b) subject to a normalization of the determinants of the W_{ia}^b in terms of the σ_i.

chiral, and combining both chiralities to produce the gauge and gravity formulae has so far proved problematic: there are constraints needed to identify the two otherwise independent chiral halves. However, as seen here such constraints don't seem to matter too much at the level of the formulae. The chiral models would seem to be a better bet for the various $(N, 0)$ theories, but for these the worldsheet matter required to provide the integrands has yet to be identified. The issues facing the 6d worldsheet models are resolved on reduction as we will describe in later chapters.

Higher dimensions. Representations of ambitwistor space, in terms of twistor coordinates with little-group indices exist in higher dimensions also. Furthermore, naive ambitwistor models in those coordinates lead to higher-dimensional analogues of the polarized scattering equations. A discussion of such models was given in [124]. Again one can obtain supersymmetric ampltude formulae without worrying too much about the detailed implementation of the models. In particular, there are many more constraints required to restrict the representation to ambitwistor space as in the space of null geodesics, and again these were not implemented in any systematic way. Indeed closely related models were proposed over the years by Bandos and coworkers [109, 125–129]. Bandos takes the attitude that the additional constraints should not be imposed, and instead that it should be possible to find genuine M-theory physics in these extra degrees of freedom [125, 130, 131].

Symmetry reduction Although the six dimensional models that inspired the formulae of the present chapter still present issues, they were successfully reduced to five dimensions where matter systems for gauge theory and gravity were found in [85]. Both the six dimensional formulae presented here and the ones of [82] have been dimensionnally reduced to five and four dimensions, including to derive expressions for amplitudes on the Coulomb branch of $\mathcal{N} = 4$ SYM. Because the reduction was only performed at the level of the formula, one still needs to input by hand kinematic data that solves momentum conservation. We will show in the next two chapters that it is possible to write ambitwistor string models of massive particles that automatically assign values to the masses. These will be first formulated as intrinsically four dimensional theories in chapter 4, by implementing a twistorial description of massive particles in a worldsheet model. In chapter 5 we will present them as symmetry reductions of the five dimensional models of [85], adding RNS-type models we obtained in a similar fashion.

Chapter 4

Massive models in four dimensions

In their original form, all ambitwistor-string theories and associated worldsheet formulae appear to be tightly restricted to theories and amplitudes involving only massless particles. The underlying approach suggests that, if one wishes to compute amplitudes for theories with massive particles, one should consider quantum field theories of holomorphic strings whose target is the complexified phase space of massive particles. The massless case has also shown that, in order to incorporate fermions simply, we should use a twistor representations of the phase space.

Nearly 50 years ago, Roger Penrose, followed by Zoltan Perjes, gave a twistor description of massive particles in terms of a set of two or more twistors up to an internal symmetry group [132, 133]. He proposed the *twistor-particle programme* based on the twistor quantization of this description. In particular, it was hoped that the representation theory of the internal symmetry group should classify elementary particles; see for example [134–137] and references therein. Although this programme has not been pursued further by the twistor community, the framework was taken up by other authors in the particle physics community. For these authors, quantization via a worldline Lagrangian approach was used leading to studies of the spectrum of such twistor particle models, often incorporating supersymmetry. Two-twistor particle models include [138–140], see also [141, 142] for more recent studies that have a good number of references to the evolution of the subject, including [143, 144].[1] Such worldline actions are a stepping stone to ambitwistor-string formulations. These are holomorphic strings whose target

[1] Note also a two-twistor model along the lines of an ambitwistor string [145, 146], but focussed on massless particles.

is a complexified phase space; the action being built directly from the holomorphic symplectic potential on such a complexified phase space [43].

A particular advantage of the two-twistor representation of the phase space of massive particles is that it reduces to the nonlinear massive phase space via a symplectic quotient from the vector space of a pair of twistors. Such a symplectic quotient can be done via BRST in the quantum field theory, and all computations can be performed in a linear free-field quantum field theory on the Riemann surface. However, a key lesson from the massless cases is that, even if one is only interested in bosonic Yang-Mills or gravity, fermionic symmetries are needed on the worldsheet and supersymmetries on space-time to obtain simple uniform formulae incorporating all relevant helicities. We will see that these supersymmetries can also be introduced in the massive case, leading to simple compact formulae for amplitudes in otherwise complicated, non-linear gauge and gravity theories.

In its simplest approach, massive particles were understood in terms of a pair of 4d twistors $(Z_a, \bar{Z}_a) \in \mathbb{T} \times \mathbb{T}$, $a = 1, 2$. Each twistor $Z \in \mathbb{T}$ has four complex components, that according to more recent (and less Penrosian) conventions are written as $Z = (\lambda_\alpha, \mu^{\dot\alpha})$ i.e., as a pair of 2-component spinors; $\bar{Z}$ is the $\mathrm{SU}(2,2)$ complex conjugate of Z, defining a dual twistor by $\bar{Z} = (\bar\lambda_{\dot\alpha}, \bar\mu^{\alpha})$. This description of massive particles was defined up to an internal symmetry group $\mathrm{SU}(2) \times \mathbb{C}$, where the $\mathrm{SU}(2)$ acts conformally invariantly on the a index. It can be understood in more conventional terms as the stabilizer of the massive momentum in the Lorentz group, the little group, see for example the massive spinor-helicity framework of [12].[2] The factor of $\mathbb{C}$ in the symmetry group breaks conformal invariance and determines the particle masses.

Here we complexify twistors so that the complex conjugate twistor $\bar{Z}$ becomes a dual twistor $\tilde{Z}$ independent of $\bar{Z}$ giving the pair $Y = (Z, \tilde{Z}) \in \mathbb{T}_{\mathbb{C}} := \mathbb{T} \times \mathbb{T}^*$. We can also think of such a complexified twistor as a Dirac twistor $Y = (\lambda_A, \mu^A)$, given as a pair of 4-component Dirac spinors $\lambda_A = (\lambda_\alpha, \tilde\lambda_{\dot\alpha})$ and $\mu^A = (\tilde\mu^\alpha, \mu^{\dot\alpha})$. We will also incorporate supersymmetry extending $Y \to \mathcal{Y} = (\lambda_A, \mu^A, \eta^I)$ with $\mathcal{N}$ additional fermionic components η^I. This description gives a natural inner product $\mathcal{Y} \cdot \mathcal{Y} := Z \cdot \tilde{Z} + \eta_I \eta^I$ from the duality between $\mathbb{T}$ and $\mathbb{T}^*$ and skew

[2]More generally, n-twistor descriptions were considered with symmetry groups containing $\mathrm{SU}(n)$; in the twistor particle programme, particle multiplets were to be understood via the representation theory of such internal symmetry groups. The quantization of massive worldline models based on these descriptions has been studied by a number of authors, see [141, 147] and references therein.

form Ω_{IJ}.

Our 4d massive twistorial models are given by holomorphic maps from the Riemann surface Σ to the complexified two-twistor description of massive particles. They consist of a pair of complexified twistor fields $\mathcal{Y}_a(\sigma) = (\mathcal{Z}_a(\sigma), \widetilde{\mathcal{Z}}_a(\sigma))$, $a = 1,2$ taking values in worldsheet spinors $K_\Sigma^{1/2}$. To reduce to the twistor representation of the massive particle phase space, we also gauge the currents $(\mathcal{Y}_a \cdot \mathcal{Y}_b,\, \lambda^2,\, \tilde{\lambda}^2)$ that generate the (complexified) internal symmetry group $\mathrm{SL}(2) \times \mathbb{C} \times \tilde{\mathbb{C}}$. Here $\lambda^2 := \det(\lambda)$ and its conjugate $\tilde{\lambda}^2$ determine the squared mass of the massive momentum $P_{\alpha\dot{\alpha}} = \lambda_\alpha^a \tilde{\lambda}_{\dot{\alpha}}^b \epsilon_{ab}$. Thus we arrive at the model

$$S_{4d} = \int_\Sigma \mathcal{Y}^a \cdot \bar{\partial} \mathcal{Y}_a + A_{ab} \mathcal{Y}^a \cdot \mathcal{Y}^b + A(\lambda^2 - j^H) + \tilde{A}(\tilde{\lambda}^2 - j^H) + S_m \,. \tag{4.1}$$

Here S_m is some theory dependent additional worldsheet matter that in particular can give rise to a current j^H associated to some symmetry generator H. The $A_{ab} = A_{(ab)}$ are gauge fields for the $\mathrm{SL}(2)$ little group, and $(A, \tilde{A})$ gauge the $\mathbb{C} \times \tilde{\mathbb{C}}$ part of the internal symmetry group; they are also Lagrange multipliers relating the values of the particle masses to their charges under H.

Although this two-twistor massive model is a string whose target is the complexified two-twistor description of massive particles of [133], it can be identified with the dimensional reduction of the 6d and 5d ambitwistor strings in [85]. The contractions of the massive spinor helicity variables correspond to two components of the internal momentum when embedding the massive variables in a six dimensional massless momentum. The two-twistor string produces correlators that localize on delta functions that fix the values of internal momenta in terms of charges under H and $\tilde{H}$ for all particles involved. In addition to this, the correlators are further localized by delta functions imposing a polarised version of the scattering equations as in [1,2].

The models above not only allow us to derive formulae involving any number of massive particles, but also give an alternative formulation of the massless models in [83]. This is of particular importance as it presents a framework in which a massless field can be *deformed* to go off-shell, which is a necessary prerequisite for defining a gluing operator in the four dimensional twistorial model and producing loop amplitudes.

In the next section we introduce the two-twistor geometry of the massive particle phase space. We then briefly present the Penrose transform and its complexification. This can be used in a two-twistor string that computes amplitudes for theories with massive particles in four dimensions. Such formulae incorporate fermions and supersymmetry, generalizing the massless case of [83]. They are based on the polarised scattering equations that have already been introduced and studied in six and five dimensions [2, 85, 124]; these can also be related [111] to the formulae of [82]. We focus on a model adapted to the Coulomb branch of $\mathcal{N} = 4$ SYM; this contains a gauge field, fermions and scalars that have a vacuum expectation value to give masses to some of the particles, analogous to the standard model. Nevertheless, as in the models of [85, 117], we can write down a full range of models for particles of spin 0, spin 1 and spin 2 following the double copy, although the scope for introducing masses into the gravity models are limited. We will not provide a discussion of amplitude formulae in this chapter as they have considerable overlap with the ones of §5.4.

4.1 Massive particles

We first review the twistor description of massive particles in terms of a pair of twistors with redundancy described by the two-twistor internal symmetry group $\mathrm{SU}(2)\times\mathbb{C}$. This framework ties in directly with the (more recent) spinor-helicity formalism for expressing polarization data for massive particles. Anticipating the string model we then complexify the two-twistor description, and introduce the Penrose transform for massive momentum eigenstates.

4.1.1 Review of twistor internal symmetry groups for massive particles

Massless particles: As described in [103, 132–134], a general twistor $Z^{\mathcal{A}} = (\lambda_\alpha, \mu^{\dot\alpha}) \in \mathbb{T}$ determines a massless particle whose momentum $P_{\alpha\dot\alpha}$ and angular momentum $M^{\mu\nu} = M^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}+$ c.c. about the origin, can be assembled into the *angular momentum twistor* given by

$$L^{\mathcal{AB}} := \begin{pmatrix} 0 & P_\alpha^{\dot\beta} \\ P_\beta^{\dot\alpha} & M^{\dot\alpha\dot\beta} \end{pmatrix} = \begin{pmatrix} 0 & \lambda_\alpha\bar\lambda^{\dot\beta} \\ \lambda_\beta\bar\lambda^{\dot\alpha} & \bar\lambda^{(\dot\alpha}\mu^{\dot\beta)} \end{pmatrix} = Z^{(\mathcal{A}}I^{\mathcal{B})\mathcal{C}}\bar Z_{\mathcal{C}}\,. \tag{4.2}$$

In this formula, the *infinity twistor* breaks conformal invariance and is defined by

$$I^{\mathcal{A}\mathcal{B}}\bar{Z}_{\mathcal{B}} = (0, \bar{\lambda}^{\dot{\alpha}})\,, \qquad I_{\mathcal{A}\mathcal{B}}Z^{\mathcal{B}} = (0, \lambda^{\alpha})\,,$$

extending the spinor contractions to degenerate inner products $\langle Z_1 Z_2\rangle := I_{\mathcal{A}\mathcal{B}}Z_1^{\mathcal{A}}Z_2^{\mathcal{B}} = \langle\lambda_1\lambda_2\rangle$ on twistor space. The angular momentum twistor is invariant under the internal symmetry transformation $Z \to \mathrm{e}^{i\theta}Z$, which we can identify as the little group rotating the phase of the constituent spinors of the massless momentum $P_{\alpha\dot{\alpha}} = \lambda_\alpha\bar{\lambda}_{\dot{\alpha}}$.

Massive particles: In order to describe massive particles, we introduce a sum over two twistors $Z_a^{\mathcal{A}}$, $a = 1, 2$ with complex conjugates $\bar{Z}^a_{\mathcal{A}}$. These yield the angular momentum twistor

$$L^{\mathcal{A}\mathcal{B}} = Z_a^{(\mathcal{A}}I^{\mathcal{B})\mathcal{C}}\bar{Z}^a_{\mathcal{C}}\,. \tag{4.3}$$

In particular, the momentum is given by

$$P_{\alpha\dot{\alpha}} = \lambda_{a\alpha}\bar{\lambda}^a_{\dot{\alpha}}\,, \tag{4.4}$$

and so we can identify the indices a, b as the $\mathrm{SU}(2)$ little-group indices that stabilizes the massive momentum $P_{\alpha\dot{\alpha}}$ inside the Lorentz group.

Penrose and Perjes [132, 133] define the two-twistor internal symmetry group to be the Poincaré invariant transformations that preserve the angular momentum twistor. This group is $\mathrm{SU}(2)\times\mathbb{C}$ where the $\mathrm{SU}(2)$ acts as the massive little group, and the factor of $\mathbb{C}$ is given by the complex transformations

$$\delta Z_a^{\mathcal{A}} \propto I^{\mathcal{A}\mathcal{B}}\bar{Z}^b_{\mathcal{B}}\epsilon_{ba}\,.$$

These symmetries all preserve the symplectic form and potential [148]

$$\Omega_m := d\Theta_m, \qquad 2\Theta_m := iZ_a^{\mathcal{A}}d\bar{Z}^a_{\mathcal{A}} - i\bar{Z}^a_{\mathcal{A}}dZ_a^{\mathcal{A}}\,.$$

The internal symmetry group action with respect to the potential Θ_m is generated by the Hamiltonians

$$Z^{\mathcal{A}}_{(a}\cdot\bar{Z}_{b)\mathcal{A}}\,,\quad \lambda^2:=\frac{1}{2}\lambda^a_\alpha\lambda^\alpha_a=\frac{1}{2}\langle Z_a,Z^a\rangle=\frac{1}{2}I_{\mathcal{AB}}Z^{\mathcal{A}}_aZ^{\mathcal{B}a}\,,$$

for the factors of SU(2) and $\mathbb{C}$ respectively. We can therefore define the phase space $\mathcal{P}_m$ for particles of mass m as the symplectic quotient

$$\mathcal{P}_m=\{Z_a\in\mathbb{T}\times\mathbb{T}|Z_{(a}\cdot\bar{Z}_{b)}=0,\langle Z_aZ^a\rangle=m\}/\{\mathrm{SU}(2)\times\mathbb{C}\}\,. \tag{4.5}$$

It is easy to see that this is a 6 real-dimensional symplectic manifold with symplectic potential Θ_m.

4.1.2 Dirac spinors and spinor-helicity for massive particles

As remarked above, the SU(2) of the internal symmetry group is the massive particle 'little group', the subgroup of the spin double cover of the Lorentz group that preserves a time-like momentum $P_{\alpha\dot\alpha}$; the representations of this little group are naturally identified with the polarization states of massive particles as follows. For a massive particle of momentum $k_{\alpha\dot\alpha}$ we write as above

$$k_{\alpha\dot\alpha}=\kappa_{a\alpha}\tilde\kappa^a_{\dot\alpha}\,,$$

where $a=1,2$ is an SU(2) little group index raised and lowered by $\varepsilon_{ab}=\varepsilon_{[ab]},\varepsilon_{12}=1$. In the real case $\tilde\kappa^a_{\dot\alpha}$ can be taken to be the complex conjugate of $\kappa_{\alpha a}$ reducing the little group to SU(2). We denote little group contractions by:

$$(v_1v_2):=v_{1a}v_{2b}\varepsilon^{ab}\,.$$

The mass m is given by $k^2=m^2=\det(k_{\alpha\dot\alpha})=\det\kappa\det\tilde\kappa$; so defining

$$\det\kappa=M\,,\qquad\det\tilde\kappa=\tilde M\,, \tag{4.6}$$

we have $M\tilde M=m^2$ and although we can fix the phases of κ and $\tilde\kappa$ so that $M=\tilde M=\pm m$, later we will want to keep them independent before they are fixed by the model.

Massive particles are not chiral, and two-component spinors necessarily double up with their conjugates. For a more compact notation, we introduce Dirac 4-component spinors with indices denoted by capital Roman letters from the beginning of the alphabet as

$$\psi_A = (\psi_\alpha, \tilde{\psi}^{\dot\alpha}), \qquad \psi^A = \varepsilon^{AB}\psi_A := (\psi^\alpha, \tilde{\psi}_{\dot\alpha}), \qquad \psi_{1A}\psi_2^A = \psi_{1\alpha}\psi_2^\alpha + \tilde{\psi}_1^{\dot\alpha}\tilde{\psi}_{2\dot\alpha},$$

and we will raise and lower indices with ε^{AB}, ε_{AB}, $\varepsilon^{AB}\varepsilon_{AC} = \delta_C^B$. Also note the γ_5 matrix defined by

$$\gamma_{5A}^{\ B}\psi_B = i(\psi_\alpha, -\tilde{\psi}^{\dot\alpha}) .$$

The mass-m Dirac operator $D^m_{AB} = D^m_{[AB]}$ in this notation is

$$D^m_{AB} := -i\nabla_{AB} + m\varepsilon_{AB} = -i\begin{pmatrix} 0 & \nabla_\alpha^{\ \dot\beta} \\ -\nabla^{\dot\alpha}_{\ \beta} & 0 \end{pmatrix} + m\varepsilon_{AB} .$$

The spin s massive field equations for $\Psi^{A_1\dots A_{2s}} = \Psi^{(A_1\dots A_{2s})}$ becomes

$$D^m_{BA_1}\Psi^{A_1\dots A_{2s}} = 0 . \tag{4.7}$$

At spin $s = 1$, we obtain $F_{AB} = F_{(AB)}$ whose 2×2 block-decomposition contains the 2-form curvature spinors along the diagonal and $mA_{\alpha\dot\beta}$ on the off-diagonal, where $A_{\alpha\dot\alpha}$ is the one-form potential.

Introducing a little group spinor ϵ_a, the general plane wave on Minkowski space of spin-s can be decomposed into Dirac spinor wave functions as

$$\Psi_{A_1\dots A_{2s}} = \epsilon_{A_1}\dots\epsilon_{A_{2s}}\mathrm{e}^{ik\cdot x}, \qquad \epsilon_A = \epsilon_a\kappa_A^a =: (\epsilon\kappa_A), \qquad \kappa_{Aa} = (\kappa_{\alpha a}, \tilde{\kappa}_a^{\dot\alpha}) . \tag{4.8}$$

For spin $1/2$, this is an ordinary massive Dirac field momentum eigenstate with polarization ϵ_a; for spin $s = 1$ this describes a massive field with potential $A_{\alpha\dot\alpha} = \frac{1}{m}\epsilon_{ab}\,\kappa^a_\alpha\tilde{\kappa}^b_{\dot\alpha}\,\mathrm{e}^{ik\cdot x}$, with polarization $\epsilon_{(ab)} = \epsilon_a\epsilon_b$. In general, spin-$s$ massive particles transform as the symmetric part of rank $2s$ tensors of the massive little group SU(2), with polarization data $\epsilon_{a_1\dots a_{2s}} = \epsilon_{(a_1\dots a_{2s})}$. Note that the polarization in (4.8) is taken to be simple to tie in with later supersymmetric expressions, corresponding to a null polarization vector. We refer to [12, 52, 149] for more extended recent discussions of spinor-helicity for massive particles.

To reduce to the massless case, we can take half the spinor components to vanish $\kappa_{1\alpha} = 0 = \tilde{\kappa}_0^{\dot{\alpha}}$, whereupon the little group spinor components ϵ_0 and ϵ_1 parametrize the positive and negative helicity states respectively.

4.1.3 The complexified particle phase space and Penrose transform

In all (massless) ambitwistor strings [43], the target space is the complexification of the massless particle phase space, often referred to as ambitwistor space and denoted by $\mathbb{A}$. To define this we first introduce complexified twistor space $\mathbb{T}_{\mathbb{C}}$ by

$$Y = (Z, \tilde{Z}) \in \mathbb{T}_{\mathbb{C}} := \mathbb{T} \times \mathbb{T}^* .$$

Then the complexified phase space of massless particles in four dimensions has become known as ambitwistor space $\mathbb{A}$, defined non-projectively as the holomorphic symplectic quotient

$$\mathbb{A} = \{Y \in \mathbb{T}_{\mathbb{C}} | Y \cdot Y := Z \cdot \tilde{Z} = 0\} / \{Z \cdot \partial_Z - \tilde{Z} \cdot \partial_{\tilde{Z}}\} , \tag{4.9}$$

with respect to the symplectic structure

$$\Omega_{\mathbb{A}} = d\Theta_{\mathbb{A}} , \qquad \Theta_{\mathbb{A}} := iZ \cdot d\tilde{Z} - i\tilde{Z} \cdot dZ .$$

This is the target of the original twistor strings [22, 23, 31] and the closely related ambitwistor strings [83].

In analogy with the massless case, here we take the target space to be $\mathcal{P}_m^{\mathbb{C}}$, the complexification of the massive particle phase space $\mathcal{P}_m$. We represent $\mathcal{P}_m^{\mathbb{C}}$ as the holomorphic symplectic quotient analogue of (4.5) as

$$\mathcal{P}_m^{\mathbb{C}} := \{Y_a \in \mathbb{T}^{\mathbb{C}} \times \mathbb{T}^{\mathbb{C}} | Z_{(a} \cdot \tilde{Z}_{b)} = 0 \, , \, \langle Z_a Z^a \rangle = [\tilde{Z}_a \tilde{Z}^a] = m\} / \mathrm{SL}(2, \mathbb{C}) \times \mathbb{C} \times \tilde{\mathbb{C}} . \tag{4.10}$$

One of the oldest applications of twistor theory has been to provide solutions to the free field equations. In the massless case this is achieved via the Penrose transform, which represents zero-rest mass helicity-h fields as twistor cohomology classes $H^1(\mathbb{PT}, \mathcal{O}(2h-2))$. Using the identifications $\mathbb{A} = T^*\mathrm{P}\mathbb{T} = T^*\mathrm{P}\mathbb{T}^*$, representatives of these cohomology classes can also be pulled back to ambitwistor space. While two-twistor descriptions in the literature [137] lead to H^2 representatives by building on the real massive particle phase space, we use the complexification $\mathcal{P}_m^{\mathbb{C}}$ to obtain representatives in (Dolbeault) cohomology classes $H^1(\mathcal{P}_m^{\mathbb{C}}, \mathcal{O}(2s-2))$ that couple naturally to the worldsheet. Here we will focus on the scalar case $s = 0$, the extension to spinning particles can be achieved most straightforwardly via supersymmetry and is discussed in §4.1.4.

To represent the plane wave (4.8) with momentum $k_{\alpha\dot\alpha} = \kappa_{\alpha a}\tilde\kappa^a_{\dot\alpha}$ on $\mathbb{T}^{\mathbb{C}} \times \mathbb{T}^{\mathbb{C}}$ it will be convenient to reorganise the spinor constituents of Y_a as a 'Dirac twistor'

$$Y_a = (\lambda_{aA}, \mu_a^A)\,, \qquad \lambda_{aA} := (\lambda_{a\alpha}, \tilde\lambda_a^{\dot\alpha}), \qquad \mu_a^A := (\mu_a^{\dot\alpha}, \tilde\mu_{\alpha a})\,.$$

Writing $\kappa_{Aa} = (\kappa_{\alpha a}, \tilde\kappa_a^{\dot\alpha})$, we define the corresponding cohomology representative in $H^1(\mathcal{P}_m^{\mathbb{C}}, \mathcal{O}(-2))$ by introducing four auxiliary complex variables u_a, v_a;

$$\Phi_\kappa(Y_a) = \int d^2u\, d^2v\, \bar\delta^4((u\lambda_A) - (v\,\kappa_A))\bar\delta((v,\epsilon) - 1)\, \exp\left((u\mu^A)\epsilon_A\right)\,. \tag{4.11}$$

Here the line bundle $\mathcal{O}(n)$ is the bundle of homogeneity degree n in the Y_a, and for a complex variable z we define $\bar\delta(z)$ to be the distributional $(0,1)$-form

$$\bar\delta(z) = \bar\partial \frac{1}{2\pi i z} = \delta(\Re z)\delta(\Im z) d\bar z\,.$$

After the u_a, v_a integrals have been performed, $\Phi_\kappa(Y_a) \in H^1(\mathcal{P}_m^{\mathbb{C}}, \mathcal{O}(-2))$ is indeed a $(0,1)$-form as desired, and the integration over u ensures invariance under the SL$(2,\mathbb{C})$ little group. For the Penrose transform, we take $\det(\kappa)$ and $\det(\tilde\kappa)$ to be unconstrained; in the two-twistor string these quantities will be constrained to agree with the particle masses determined by the underlying theory.

To see that this indeed corresponds to a plane-wave on space-time, we impose the incidence relations $\mu_a^{\dot\alpha} = ix^{\alpha\dot\alpha}\lambda_{a\alpha}$ and $\tilde\mu_a^{\alpha} = -ix^{\alpha\dot\alpha}\tilde\lambda_{a\dot\alpha}$. Then on the support of the delta functions we have $(u\lambda_A) = (v\kappa_A)$ and $(v\epsilon) = 1$, giving [3]

$$(u\mu^A)\epsilon_A = ix^{\alpha\dot\alpha}(v\kappa_\alpha)(\epsilon\tilde\kappa_{\dot\alpha}) - ix^{\alpha\dot\alpha}(v\tilde\kappa_{\dot\alpha})(\epsilon\kappa_\alpha) = ix\cdot k\,. \tag{4.12}$$

The parameters (u_a, v_a) can be integrated against the delta functions to yield the single delta function $\bar\delta(k\cdot P - m\lambda^2 - m\tilde\lambda^2)$ where $P_{\alpha\dot\alpha} = \lambda_{a\alpha}\tilde\lambda^a_{\dot\alpha}$; this gives the Dolbeault representation for a simple pole. This delta function imposes a massive analogue of the *scattering equations* that play a key role in the CHY formulae for massless amplitudes [29,30,33,83]. A version of these *massive scattering equations* has been studied for massive amplitudes in the CHY representation [75], see the discussion in §4.3 for more details. The delta-functions in Φ_κ have become known as the *polarized scattering equations* due to their dependence on a choice of polarization spinor.[4] These serve to define additional and unique parameters u_a, v_a on the support of the massive scattering equations that will play a key role later.

4.1.4 Supersymmetric extension

We will aim to generate supersymmetric formulae for two reasons. Amplitudes for theories with a variety of spins become drastically simpler in the supersymmetric case because many or all the particles can be expressed as one multiplet. This leads to uniform formulae from which different sectors with particles of different spins can be read off. Moreover, amplitudes for non-supersymmetric theories can be extracted from these superamplitudes at tree-level and at one-loop [117, 153]. A more structural reason is that all (ambi-) twistor string models that describe gauge theory and gravity require space-time supersymmetry to be anomaly free—the supersymmetric extension of twistor space includes additional fermionic variables that cancel

[3]This Penrose transform is closely related to the (indirect) 6d Penrose transform [104,150]. The twistor space for 6d is $\mathbb{T}_{\mathbb{C}}|_{Y\cdot Y=0}$ and the plane wave (4.8) is represented by

$$\Psi_\kappa(Y) = \int (\epsilon v)^n \frac{ds}{s^{n-1}}\,(vdv)\,\bar\delta^4(s\lambda_A - (v\kappa_A))\exp\left(s\mu^A\epsilon_A/(\epsilon v)\right) \;\in\; H^2(\mathbb{PA}, \mathcal{O}(n-2))\,.$$

Following [85,106,151,152], the massive $\Phi_\kappa(Y_a)$ in (4.11) can then be constructed via $\Phi_\kappa(Y_a) = \int \Psi_\kappa((u,Y))(udu)\,.$

[4]The polarization spinor will play a more prominent role in amplitude formulae based on the polarized scattering equations, because the path integrals introduce ϵ-dependence in $\lambda_A(\sigma)$.

anomalies from the bosonic variables. Thus we introduce a supersymmetric extension of $\mathcal{P}_m^{\mathbb{C}}$, as well as the plane wave Φ_κ.

On the Coulomb branch of $\mathcal{N}=4$ super Yang-Mills, some scalars acquire a vacuum expectation value, effectively breaking the gauge group from SU$(N+M)$ down to SU$(N)\times$ SU(M). The states can then be organised into two types of multiplets; a massless vector multiplet transforming in the adjoint of the residual gauge group, and a massive vector multiplet in the bifundamental of SU$(N)\times$ SU(M). Massive multiplets are in the so-called $1/2$-BPS, ultrashort massive representations of $\mathcal{N}=4$ with central extension $Z_{IJ}=2M\Omega_{IJ}$, with Sp$(\mathcal{N}/2)$ R-symmetry, with skew form Ω_{IJ} and indices $I,J=1,\ldots,\mathcal{N}=4$. For massless multiplets on the other hand, R-symmetry is enhanced to a full SU(4) by the vanishing of the central extension as $m\to 0$. Employing the notation of the previous section, we can combine the supercharges into a Dirac spinor $Q_I^A=(Q_{\alpha I},\,Q_I^{\dagger\dot\alpha})$, such that the supersymmetry algebra takes the compact form

$$\{Q_{AI},Q_{BJ}\}=2\Omega_{IJ}D^m_{AB}\,.$$

In the massless case, the structure of the supersymmetry algebra greatly simplifies as the only non vanishing component of the Dirac operator is $D^0_{\alpha\dot\alpha}=\nabla_{\alpha\dot\alpha}$.

The action of the supercharges arranges the states in multiplets as follows. The massive multiplet is composed of a massive spin one field F_{AB}, five massive scalars ϕ_{IJ} and four massive Weyl-Majorana spinors Ψ_A^I:

$$\mathscr{F}^m=(\phi_{IJ}=\phi_{[IJ]},\Psi_I^A\,,F^{AB}=F^{(AB)})\,,\qquad \phi_{IJ}\Omega^{IJ}=0\,. \tag{4.13}$$

The massless multiplet is:

$$\mathscr{F}^0=(\phi_{IJ}=\phi_{[IJ]}\,,\Psi^I_\alpha,\tilde\Psi_{I\dot\alpha}\,,F_{\alpha\beta},F_{\dot\alpha\dot\beta})\,, \tag{4.14}$$

where the R-symmetry indices now label the fundamental of SU(4) and can therefore no longer be raised and lowered. It contains the two familiar ±1 helicity states of the massless spin-1, six real massless scalars ϕ_{IJ} and eight massless gluino states via the chiral parts of $\Psi^I_\alpha,\tilde\Psi_{I\dot\alpha}$. We note that the massless scalars ϕ_{IJ} are no longer trace-free; the extra 6th component arises from the loss of one of the polarization degrees of freedom going from the massive spin-1 field F_{AB} to the massless case.

For momentum eigenstates with space-time dependence $\phi = \exp(ik{\cdot}x)$, the supersymmetry generators reduce to the massive little group as

$$Q_{AI} = \kappa_A^a Q_{aI}\,, \qquad \{Q_{aI}, Q_{bJ}\} = 2\Omega_{IJ}\varepsilon_{ab}$$

where κ_a^A is defined by (4.8), because the Dirac operator reduces as $D^m_{AB}\phi = (\kappa_A\kappa_B)\phi$. In the massless limit we have the natural embedding of the little group via $\kappa_{1\alpha} = 0 = \tilde{\kappa}_{0\dot{\alpha}}$, $\kappa_{0\alpha} = \kappa_\alpha, \quad \tilde{\kappa}_{1\dot{\alpha}} = \tilde{\kappa}_{\dot{\alpha}}$.

Both the massive and massless multiplets are annihilated by half of the supercharges so that their 8 bosonic and 8 fermionic states can all be encoded into the exterior powers of $\mathcal{N} = 4$ fermionic supermomenta q_I, $I = 1, \dots 4$. These are defined to be the eigenvalues of an anticommuting subset of the Q_{Ia}. To define this subset, we introduce a basis (ϵ_a, ξ_a) of the fundamental representation of $SL(2)$ so that the supermomenta are defined by the action of the supercharges on functions on on-shell superspace via:

$$Q_{aI}\tilde{\mathscr{F}}(\kappa, q) = \left(\xi_a q_I + \epsilon_a \Omega_{IJ}\frac{\partial}{\partial q_J}\right)\tilde{\mathscr{F}}(\kappa, q)\,. \tag{4.15}$$

The massive and massless multiplets are expanded on on-shell superspace as follows:

$$\begin{aligned}\tilde{\mathscr{F}}^{(m)}_{(\kappa,q)} &= F^{\epsilon\epsilon}(\kappa) + q_I\Psi^{\epsilon I}(\kappa) + q^2F^{\epsilon\xi}(\kappa) + \frac{1}{2}q_Iq_J\Phi^{IJ}(\kappa) + q^2q_I\Psi^{\xi I}(\kappa) + q^4F^{\xi\xi}(\kappa)\\ \tilde{\mathscr{F}}^{(0)}_{(\kappa,q)} &= g^h(\kappa) + q_I\Psi^{\epsilon I}(\kappa) + \frac{1}{2}q_Iq_J\varphi^{IJ}(\kappa) + q^2q_I\Psi^{\xi I}(\kappa) + q^4g^{-h}(\kappa)\,,\end{aligned} \tag{4.16}$$

with $q^4 = (q_Iq_J)(q^Iq^J)$ and $(q^3)^I_a = \partial q^4/\partial q^a_I$.

It is then standard procedure to encode such multiplets in superfields on a supersymmetric extension of Minkowski space satisfying (4.15) and to derive a supersymmetric Penrose transform by establishing a supergeometric correspondence with super-twistor space. We can bypass some of this by studying the action of supersymmetry on super-twistors.

Supertwistors and the Penrose transform. We extend the bosonic complexified twistor $Y \in \mathbb{T}^{\mathbb{C}}$ with $\mathcal{N}$ fermionic coordinates η^I, $I = 1, \ldots, \mathcal{N}$ to give $\mathcal{Y} = (\lambda_A, \mu^A, \eta^I) \in \mathcal{T}^{\mathbb{C}}$, using Dirac-spinor notation. These Fermionic coordinates allow the supersymmetry to act geometrically as

$$Q_{AI} = \lambda_A \frac{\partial}{\partial \eta^I} + \eta^J \Omega_{JI} \frac{\partial}{\partial \mu^A}\,, \qquad \{Q_{AI}, Q_{BJ}\} = 2\Omega_{IJ} \lambda_{[A} \frac{\partial}{\partial \mu^{B]}}\,,$$

where the anticommutator now generates the action of translations on $\mathcal{T}^{\mathbb{C}}$. This extends in the obvious way to the two-twistor description of supersymmetric massive particles in terms of $\mathcal{Y}_a$ with sums over the a-index in each term. Again, the supersymmetric extension for the $\mathcal{Y}_a$ becomes:

$$\mathcal{Y}_a = (\lambda_{aA},\ \mu_a^A,\ \eta_a^I)\,,$$

with again $I = 1, \ldots, \mathcal{N}$. The plane wave representative for particles with spinor helicity data $\kappa_{Aa} = (\kappa_{\alpha a}, \tilde{\kappa}_a^{\dot\alpha})$, supermomentum q and polarization data ϵ_a will take the form:

$$\Phi_{(\kappa,q)}(\mu, \lambda) = \int d^2u\, d^2v\, w\, \bar\delta^4((u\lambda_A) - (v\,\kappa_A))\, \bar\delta((\epsilon v) - 1)\, e^{iu_a\left(\mu^{Aa}\epsilon_A + q_I \eta^{Ia}\right) - \frac{1}{2}(\xi v)\, q^2} \tag{4.17}$$

Here w is a function of weight 2 in $\mathcal{Y}_a$(or -2 in u); as far as the Penrose transform is concerned, this can be taken to be $w = (\lambda_\alpha \tilde\lambda_{\dot\alpha}) e^{\alpha\dot\alpha}$ where $e^{\alpha\dot\alpha}$ is a polarization vector for the spin-1 1-form. In the string model however, w plays an important role in the vertex operators, and will be constructed differently. This representative indeed satisfies

$$Q_{AI} \Phi_{\kappa,q}(\mathcal{Y}_a) = ((\kappa_A \xi) q_I + \epsilon_A \Omega_{IJ} \frac{\partial}{\partial q_I}) \Phi_{\kappa,q}(\mathcal{Y}_a)\,,$$

and we can then read off the Penrose transform for the component fields from the action of the supersymmetry generators.

4.2 Massive two-twistor string

The significance of the twistor representations of spaces of massless and massive particles is that they are represented as symplectic quotients of vector spaces. This means that in order to construct a theory of maps from a Riemann surface $\Sigma \to \mathcal{P}_m^{\mathbb{C}}$, we can start with a quantum field theory of maps $\Sigma \to \mathbb{T}^{\mathbb{C}}$ in the massless case and $\Sigma \to \mathbb{T}^{\mathbb{C}} \times \mathbb{T}^{\mathbb{C}}$ in the massive case; in both cases, by virtue of the twistor representations, these are free field theories on the worldsheet Σ. We then realize the symplectic quotient in the Lagrangian framework by gauging the Hamiltonian symmetries as we shall describe below. These gauge symmetries are then dealt with via BRST in the quantum field theory. In both cases, the free field theory action is based on the restriction of the symplectic potential Θ to $T_\Sigma^{0,1}$. This has the consequence that the worldsheet commutators and OPEs encode the symplectic structure Ω_m on $\mathcal{P}_m^{\mathbb{C}}$.

We first briefly review the massless case; although the construction for the massive two-twistor string will be analogous, but with target $\mathcal{P}_m^{\mathbb{C}}$ and the different massive supersymmetry representation. In the next section we explain how the models allow us to construct amplitudes as correlation functions of vertex operators in these models.

The massless case. The twistor strings of Witten & Berkovits [22, 23] and the 4d ambitwistor string of [83] are theories of holomorphic maps $\mathcal{Y} = (\mathcal{Z}, \tilde{\mathcal{Z}}) : \Sigma \to \mathbb{T}^{\mathbb{C}}$ gauged by $\mathbb{C}^*$ where we now use supertwistors $\mathcal{Z} = (\lambda_\alpha, \mu^{\dot\alpha}, \eta^I) \in \mathbb{T} = \mathbb{C}^{4|\mathcal{N}}$, $I = 1, \ldots, \mathcal{N}$ and their complexification $\mathbb{T}^{\mathbb{C}} = \mathbb{T} \times \mathbb{T}^*$; this is the complexification of the four-dimensional massless Ferber superparticle [154].

The four-dimensional ambitwistor string of [83] is closest to the massive case, being with worldsheet fields twisted to take values in $\Omega_\Sigma^{1/2,0}$ and so we briefly review it here. It is a theory of holomorphic maps from a Riemann surface $\mathcal{Y} : \Sigma \to \mathbb{T}_{\mathbb{C}} \otimes \Omega_\Sigma^{1/2,0}$, so that the coordinates $\mathcal{Y}$ are worldsheet spinors. The reduction to ambitwistor space is enforced by gauging the little-group Hamiltonian $\mathcal{Y} \cdot \mathcal{Y} := \mathcal{Z} \cdot \tilde{\mathcal{Z}}$ with the worldsheet gauge field $A \in \Omega_\Sigma^{0,1}$. The basic bosonic 4d ambitwistor action in conformal gauge[5] is based on the symplectic potential

$$S_{4d}^0 = \int_\Sigma \widetilde{\mathcal{Z}} \cdot \bar\partial \mathcal{Z} - \mathcal{Z} \cdot \bar\partial \widetilde{\mathcal{Z}} + A\, \mathcal{Y} \cdot \mathcal{Y}\,. \tag{4.18}$$

[5]The full action would start with a term eT where $e \in T_\Sigma^{1,0} \otimes \Omega_\Sigma^{0,1}$ is a Beltrami differential thought of as a gauge field parametrizing complex structures on Σ up to coordinate transformations and $T \in (\Omega_\Sigma^{1,0})^2$ is the holomorphic stress energy tensor; this is then gauge fixed, giving rise to ghosts $(b, c) \in ((\Omega_\Sigma^{1,0})^2, T_\Sigma^{1,0})$ and BRST operator $Q = \oint cT + bc\partial c/2$.

Classically, A is a Lagrange multiplier that enforces the constraint $\mathcal{Y}\cdot\mathcal{Y}=0$ and the quotient by its Hamiltonian vector field arises because $(Z,\tilde{Z})\to(\alpha Z,\alpha^{-1}\tilde{Z})$ are gauge symmetries of the action when accompanied by the gauge transformations $A\to A+\bar{\partial}\log\alpha$. Thus the holomorphic symplectic quotient to $\mathbb{A}$ in (4.9) is realized in this Lagrangian framework by the gauge field A. In the QFT this is implemented via BRST quantization. The models of [83] also include additional worldsheet matter fields but these are much as described for the massive case below.

Massive models. In order to have target space $\mathcal{P}_m^{\mathbb{C}}$, we start with maps $\mathcal{Y}_a:\Sigma\to\mathbb{T}_{\mathbb{C}}\times\mathbb{T}_{\mathbb{C}}$, with the reduction to $\mathcal{P}_m^{\mathbb{C}}$ obtained by gauging the complexified two-twistor massive internal symmetry group. Thus, our theory is one of maps $\mathcal{Y}_a:\Sigma\to\mathbb{T}_{\mathbb{C}}\otimes\mathbb{C}^2\otimes\Omega_\Sigma^{1/2,0}$ with action (again in conformal gauge)

$$S_{4d}=\int_\Sigma \mathcal{Y}^a\bar{\partial}\mathcal{Y}_a+A_{ab}\mathcal{Y}^a\cdot\mathcal{Y}^b+A(\lambda^2-j^H)+\tilde{A}(\tilde{\lambda}^2-j^H)+S_m\,. \tag{4.19}$$

Here $a=1,2$ is the little group index, and $(A_{ab}=A_{(ab)},A,\tilde{A})$ are worldsheet $(0,1)$-forms that act as Lagrange multipliers for the constraints, and as gauge fields for the internal two-twistor symmetry group. With this symmetry, we no longer have the freedom to allow worldsheet fields of different degrees as we did for the twistor-string. In order to describe specific space-time theories, the basic action must be supplemented by further worldsheet fields such as a current algebra for gauge theory and some analogue of worldsheet supergravity for gravity with details given below. Here we assume that it contains a current-algebra that gives rise to a $(1,0)$-form j_H on the worldsheet that generates some symmetry.

To be more explicit, in quantizing the fields $\mathcal{Y}_a(\sigma)=(\mathcal{Z}_a(\sigma),\tilde{\mathcal{Z}}_a(\sigma))$, for σ a coordinate on Σ, the only non-trivial OPEs are

$$\mathcal{Z}_a^{\mathcal{A}}(\sigma)\tilde{\mathcal{Z}}_{b\mathcal{B}}(0)=\frac{\delta^{\mathcal{A}}_{\mathcal{B}}}{\sigma}\varepsilon_{ab}+\dots\,.$$

reflecting the Poisson brackets. These OPEs can lead to anomalies for the little group SL$(2,\mathbb{C})$ generated by $J^{ab}=\mathcal{Y}^a\cdot\mathcal{Y}^b$. For a consistent model these anomalies have to vanish, which requires judicious choices for the worldsheet matter S_m.

The fields $a, \tilde{a}$ gauge the constraints $\lambda^2 - j^H = 0 = \tilde{\lambda}^2 - j^H$. These equations constrain the mass operators

$$\lambda^2 := \frac{1}{2}\lambda_\alpha^a \lambda_a^\alpha = \det(\lambda_\alpha^a)\,, \qquad\qquad \tilde{\lambda}^2 := \frac{1}{2}\tilde{\lambda}_{\dot\alpha}^a \tilde{\lambda}_a^{\dot\alpha} = \det(\tilde{\lambda}_a^{\dot\alpha})\,,$$

to be given by a $(1,0)$-form j^H on the worldsheet Σ. We write j^H to indicate that this will be taken to be the current associated to the element $h \in \mathfrak{g}$, living in the Cartan subalgebra of some symmetry of the system. This j^H will be constructed from the matter fields and, through the constraints above, will determine the masses of the particles. For a given matter content, different choices of j^H correspond to different distributions of masses within the models. The massless models (4.18) are recovered from these massive ones when $j^H = 0$ by reducing the path integral.

Worldsheet matter. A variety of physically interesting models can be constructed from different choices of S_m. These will be made up of current algebras, whose action will be denoted by S_C, and worldsheet fermions providing a supersymmetric extension of the worldsheet gauge algebra, denoted by S_ρ. The latter will play a similar role to worldsheet supergravity in the superstring, and is requried for models describing gauge theory and supergravity.

A worldsheet current algebra is a theory on the worldsheet from which one can construct worldsheet currents $j^a \in \Omega^{1,0}_\Sigma \otimes \mathfrak{g}$ for some Lie algebra $\mathfrak{g}$, satisfying the OPE

$$j^{\mathfrak{a}}(\sigma) j^{\mathfrak{b}}(0) \sim \frac{l\,\delta^{\mathfrak{ab}}}{\sigma^2} + \frac{f^{\mathfrak{ab}}_{\mathfrak{c}}\, j^{\mathfrak{c}}}{\sigma}$$

where $\mathfrak{a}, \mathfrak{b}$ are Lie-algebra indices, $l \in \mathbb{Z}$ is the level and $f^{\mathfrak{ab}}_{\mathfrak{c}}$ the structure constants of $\mathfrak{g}$. Such current algebras can be constructed in a number of ways, most easily for SO(n) and SU(n) by 'real' or 'complex' free fermions on the worldsheet. See also [117] for a construction referred to as a comb-system, with level zero and novel properties that allow the construction of Einstein-Yang-Mills amplitudes. We will not specify the action S_C explicitly, but merely assume that we have the currents $j^{\mathfrak{a}}$ in the theory.

For gauge and gravity theories, we need a supersymmetric extension of the worldsheet gauge algebra. This plays a similar role to the worldsheet supergravity of the conventional RNS models, see also [43] for the ambitwistor-string version. The supersymmetric extension of the bosonic gauge algebra $\mathfrak{sl}_2 \times \mathbb{C}^2$ is constructed by introducing the worldsheet fermions $(\rho_A, \tilde{\rho}^A) \in \Omega^0(\Sigma, K_\Sigma^{1/2})$ with action

$$S_\rho = \int_\Sigma \tilde{\rho}^A \bar{\partial}\rho_A + b_a \left(\gamma_5^{AB}\lambda_A^a \rho_B\right) + \tilde{b}_a\, \lambda_A^a \tilde{\rho}^A \,. \tag{4.20}$$

Here the $(b^a, \tilde{b}^a)$ are fermionic gauge fields and so are $(0,1)$-forms on the worldsheet. They are Lagrange multipliers that impose the constraints $\gamma_5^{AB}\lambda_A^a\rho_B = \lambda_A^a\tilde{\rho}^A = 0$ and their gauge transformations translate μ_a^A in the direction of $(\rho^A, \tilde{\rho}^A)$. The only non-trivial OPE's of the constraints are given by

$$\left(\gamma_5^{AB}\lambda_A^a\rho_B\right)(z)\, \left(\lambda_B^b\tilde{\rho}^B\right)(w) \sim \frac{\varepsilon_{ab}}{z-w}\left(\lambda^2 - \tilde{\lambda}^2\right). \tag{4.21}$$

These symmetries thus give a supersymmetric extension of the two-twistor internal symmetry group $\mathbb{C} \times \mathfrak{sl}_2 \ltimes H(0,4)$, where H denotes the Heisenberg Lie superalgebra[6].

Models. With these ingredients, models without SL$(2,\mathbb{C})$-anomalies can be constructed by combining a pair of worldsheet matter systems, much along the lines of the double copy for the RNS ambitwistor strings as in [117] as follows:

massive bi-adjoint scalar	$S^{\text{BAS}} = S_{4d} + S_C + S_{\tilde{C}}\,,$
super Yang-Mills on the Coulomb branch	$S^{\text{CB}} = S_{4d} + S_\rho + S_C\,,$
super-gravity	$S^{\text{sugra}} = S_{4d} + S_{\rho_1} + S_{\rho_2}\,.$

In this construction two points are worth highlighting:

(i) The closure of the constraint algebra requires that both constraints $\lambda^2 - j^H = 0 = \tilde{\lambda}^2 - j^H$

[6] The Heisenberg superalgebra $H(m_b, m_f)$ has a central element z, as well as $2m_b$ even and m_f odd generators, $H = \langle x_1, \ldots, x_{2m_b}, z\rangle \oplus \langle \psi_1, \ldots, \psi_{m_f}\rangle$. The generators satisfy the 'usual' commutation relations

$$[x_i, x_{2i}] = z\,, \qquad \{\psi_r, \psi_s\} = 2\delta_{rs}\, z\,.$$

involve the *same* current j^H for super Yang-Mills, whereas a more general construction is possible for the bi-adjoint scalar.

(ii) Unlike the twistor- and ambitwistor models for 4d massless theories, these models fit neatly into the double copy format [37] expressed directly in the CHY formulae [33] and in the corresponding RNS ambitwistor strings [43]. However, it is harder to find a j^H to endow our particles with mass in the gravitational case because there is no additional current algebra, and with $j^H = 0$ our models are massless. We also note that as in [43, 85, 155], both S^{CB} and S^{BAS} also contain a gravity sector, but it is of higher order and remains massless.

BRST and anomalies: Gauge fixing the action via BRST generates ghost systems, the well-known $(b, c) \in (\Omega^1_\Sigma)^2 \times T_\Sigma$ for worldsheet diffeomorphisms, as well as additional fermionic ghosts associated to internal two-twistor symmetry group, and bosonic ghosts for the fermionic currents in S_ρ. The BRST operator takes the usual form:

$$Q = \oint c^i (T^m_i + \frac{1}{2} T^g_i)\,, \tag{4.22}$$

where the sum runs over all sets of ghosts, and T^m and T^g are the matter and ghost parts of the currents respectively. By construction $Q^2 = 0$ classically, but in the QFT double contractions (or worldsheet bubble diagrams with two external gauge fields) can lead to anomalies so that $Q^2 \neq 0$ with a potential obstruction arising from any of the gauged symmetries. Here we briefly summarize the results of such calculations.

The models above only have a vanishing $\text{SL}(2,\mathbb{C})$ anomaly (corresponding to the two-twistor internal symmetry group) for maximal space-time supersymmetry, as evident from the anomaly coefficient

$$\mathfrak{a}_{\text{SL}(2)} = \sum_i (-1)^{F_i} \text{tr}_{R_i}(t^k t^k) = \begin{cases} 4\,\text{tr}_{\text{F}}(t^k t^k) - \text{tr}_{\text{adj}}(t^k t^k) = 0 & \text{bi-adjoint scalar} \\ \frac{3}{4}\,(4 - \mathcal{N}) & \text{Coulomb branch} \\ \frac{3}{4}\,(8 - \mathcal{N}) & \text{supergravity}\,. \end{cases}$$

The anomaly coefficient vanishes trivially for the bi-adjoint scalar, and for maximal supersymmetry in the case of gauge theory and gravity. The sum here runs over all fields that transform non-trivially under the internal two-twistor symmetry group $\mathrm{SL}(2,\mathbb{C})$. Similarly, the Virasoro central charge can be calculated for all models, giving

$$\mathfrak{c}^{\mathrm{BAS}} = -40 + \mathfrak{c}_j\,, \qquad \mathfrak{c}^{\mathrm{CB}} = -32 + \mathcal{N} + \mathfrak{c}_j\,, \qquad \mathfrak{c}^{\mathrm{sugra}} = -20 + \mathcal{N}\,,$$

where $\mathfrak{c}_j$ denotes the central charge of the internal current algebra. The conformal anomaly thus vanishes for suitable choice of S_j, with $\mathcal{N} = 4$ and $\mathfrak{c}_j = 28$ for Yang-Mills theory on the Coulomb branch, and $\mathfrak{c}_j = 40$ for the bi-adjoint scalar. The Virsaoro anomaly for the supergravity model also vanishes if we include a central charge term $\mathfrak{c}_{6d} = 12$ arising from six compactified dimensions. After BRST gauge-fixing, all such models are free worldsheet theories with vanishing anomalies.[7] In the next chapter we explain how to obtain n-point amplitudes from these models.

4.3 Summary and discussion

We have seen that a chiral string whose target is the complexification of Penrose's two-twistor representation of the massive particle phase space yields theories of massive particles in four dimensions. The spectrum of these models includes massive particles, and correlators give amplitude formulae for super Yang-Mills on the Coulomb branch among other theories. These string models represent the confluence of two separate developments: the twistor-particle program of the 70's describing massive particles, and the more recent ambitwistor string models describing scattering amplitudes for massless particles. In the latter approach a chiral or holomorphic string whose target is the complexification of the space of massless particles yields amplitudes for theories of massless fields. Here we have seen that the logic extends naturally to massive particles. The significance of Penrose's twistor description is that it provides a canonical representation of the space as the symplectic quotient of a vector space modulo

[7] Strictly speaking, the Lie algebra element H should also be null or the current algebra should have level $l = 0$ as in the comb systems of [42].

a Hamiltonian group action allowing the BRST quantization of a free quantum string. The twistorial description furthermore facilitates the incorporation of fermions and supersymmetry.

The models described in this chapter, and related ones, can also be derived via a symmetry reduction of the higher-dimensional ambitwistor string models, as we will show in the next chapter. This alternative interpretation of the two-twistor string highlights that these worldsheet models describe a subset of massive models, where the particle masses are related to their (higher-dimensional) charges under a symmetry. While this may appear restrictive, it includes many theories of immediate interest; in particular, all massive particles that we encounter in the standard model arise from the Higgs mechanism that can be obtained by symmetry reduction.

Chapter 5 will contain full details of the fixed vertex operators and picture changing operators that we omitted for brevity. Mirroring the close relation of the models, the resulting amplitude formulae for massive particles are also closely related to those obtained previously by dimensional reduction from six and five dimensions [1, 2, 85]. These can further be related to the formulae of [82] by a change of gauge choice of an embedding inside a Lagrangian grassmannian [111]. At low point orders, these expressions match the results obtained in [12, 50, 71, 72, 156, 157] by BCFW recursion.

CHAPTER 5

Massive models from symmetry reduction

Symmetry reduction has long been used in the study of differential equations, with numerous applications across mathematical physics, including the twistorial take on integrability [158]. In string theory, Kaluza-Klein (KK) reductions are the result of compactification followed by a truncation of massive modes and they constitute a simple example of symmetry reduction where the theory is taken to be independent of one or several spacetime dimensions:

$$\Phi(x^\mu, z) = \Phi(x^\mu)\,. \tag{5.1}$$

The work of Cremmer, Scherk and Schwarz [159, 160] instructs us on how to generalise the procedure in order to introduce masses into the system. The idea is to 'twist' the KK cylinder condition by the action of some symmetry of the original theory:

$$\Phi(x^\mu, z) = g_z(\Phi(x^\mu))\,, \tag{5.2}$$

giving rise to a mass matrix via the group element g_z. The aim of this chapter is to show how this kind of reduction can be implemented at the level of the ambitwistor string.

In the previous chapter we have introduced massive models in four dimensions as theories of maps into the phase space of the complexified massive particles in its twistorial description. We take here a different approach and present them as symmetry reductions of models in five dimensions [85]. The formalism we employ is along the lines of the dimensional reduction to five dimension, with the difference that the extra components of momentum along which we are reducing are combined with currents of some symmetry of the system.

As discussed in the introduction, Dolan & Goddard and Naculich [3,75] had conjectured a massive form of the scattering equations (1.18). Their work was based solely on the original CHY formulae for massless scattering, so that it wasn't clear at that point whether the full CHY formulae described consistent amplitudes and what theories these would correspond to. More importantly, the formulae didn't give any indication of what values the masses should take, other than that internal momentum should be conserved. This raises the question of whether the full CHY formulae would admit a consistent description of amplitudes for massive particles, such that masses of both external and internal propagating particles would correspond to masses in some field theory.

In this chapter we will take a 'top-down' approach. We define models of massive particles in the ambitwistor string as symmetry reductions of higher dimensional massless theories. Because the models and amplitudes that we employ in higher dimensions have been proven to correspond to the known massless theories, the massive amplitudes are also verified, provided that we have an understanding of the corresponding theory as a symmetry reduction. It is then natural to expect that the amplitude formulae factorise on poles corresponding to masses in the spectrum of the theory we are representing. The first few sections will be dedicated to showing the consistency of factorisation in the amplitudes formulae we derive. The massive polarised scattering equations on which correlators in the twistorial models localise are closely related to (1.18). After the path-integral we will find that λ_{Aa} is given in (5.81) and we can compute [1,2]:

$$P_{AB}(\sigma) := (\lambda_A \lambda_B) = \sum_i \frac{K_{iAB}}{\sigma - \sigma_i}\, d\sigma\,, \tag{5.3}$$

where K_i is defined as in (1.17). Because we noted below (4.12) that the polarise scattering equations imply $K_{AB} \cdot P^{AB} = 0$, on their support we have:

$$0 = K_{iAB}\, P^{AB}(\sigma_i) = \sum_{j \neq i} \frac{k_i \cdot k_j - \mathsf{k}_i \cdot \mathsf{k}_j}{\sigma_i - \sigma_j}\,, \tag{5.4}$$

which are precisely the massive scattering equations predicted by Naculich. As it was the case for massless scattering, singularities both in the RNS and twistorial ambitwistor string stem from degenerations of the system of equations (5.4). Therefore we introduce an RNS-type model for biadjoint scalars in section 5.2 in order to carefully study the mass assignment of internal particles. We show how to introduce masses by gauging currents more generally in the original ambitwistor string [43] and we give an argument for the consistency of the mass assignment for internal propagating particles using factorisation and symmetry invariance in §5.3.

To describe more interesting supersymmetric models, the second part of the chapter (see §5.4) focuses on the four-dimensional models such as the ones presented in chapter 4, where special attention is given to a model for the Coulomb branch of $\mathcal{N} = 4$ SYM in §5.5. For this model we establish several explicit examples and interesting properties of the amplitudes.

In §5.6 we present symmetry reductions that exploit the R-symmetry of maximally supersymmetric theories in five dimensions. These are CSS reductions of the type described in [159, 160]. Theey produce a range of massive theories with various degrees of residual supersymmetry and massless gluons and gravitons in sYM and gravity respectively. We study the spectrum of the reduced theory and present instances of double copy that produce gauged supergravities in four dimensions, such as the ones that were studied in [161].

5.1 Symmetry reduction

The aim of this chapter is to show how the procedure of symmetry reduction can be integrated in the ambitwistor strings of [43, 83] to produce models of massive particles and derive amplitude formulae that naturally rely on a massive version of the scattering equations. In this section we review the basics of symmetry reduction and introduce the toy model that we will use to implement it in the ambitwistor string.

The simplest example of a symmetry reduction that we will consider is a trivial dimensional reduction

$$\partial_d \Phi = 0, \tag{5.5}$$

where Φ stands for any field in the theory. This arises as the limit of a compactified theory as the size of the compact dimension goes to zero. More generally, translations in the extra dimension can be combined with the action of some symmetry G of the theory to impose, schematically:

$$(\partial_d + H) \cdot \Phi = 0 \, . \tag{5.6}$$

Here H is an element in a choice of Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

In the supergravity literature, reductions of this type arise in the limit of vanishing compact dimensions for what are known as (Cremmer-)Scherk-Schwarz reductions. These were first studied in [159, 160] as a way to derive effective supergravities in four dimensions with spontaneous breaking of supersymmetry at various scales. They generalize dimensional reduction, here along the direction z, by twisting the periodicity condition under the action of some symmetry of the theory:

$$\Phi(x^\mu, z + 2\pi R) = g_z(\Phi(x^\mu, z)) \qquad g_z = g(z) = e^{iHz}, \qquad H \in \mathfrak{h} \, , \tag{5.7}$$

The field can then be expanded in a basis of eigenfunctions for the compact dimensions, modulo the action of g_z:

$$\Phi(x^\mu, z) = e^{iHz} \sum_n \Phi_n(x^\mu) e^{in\frac{z}{R}} \, . \tag{5.8}$$

In the limit as the compactification radius goes to zero, the tower of massive modes decouples and we can write:

$$\Phi(x^\mu, z) = g_z(\Phi(x^\mu)) \, . \tag{5.9}$$

Several remarks are due. The periodicity condition is equivalent in the $R \to 0$ limit to:

$$\partial_z \Phi(x^\mu, z) = iH\Phi(x^\mu, z) \, . \tag{5.10}$$

This is what we schematically wrote as (5.6) and we can see that it produces mass terms such as:

$$(\partial_z \Phi(x^\mu, z))^2 = H^2 \Phi(x^\mu)^2 \, , \tag{5.11}$$

for all fields that are charged under H. The most important consequence of the decomposition (5.9) is that the whole dependence on the compactified dimension is carried by g_z and hence drops out on account of the symmetry of the theory.

Bi-adjoint scalar The simplest model we will construct describes bi-adjoint scalars, a theory of scalar fields $\Phi = \phi^{a\dot{a}}T^a\tilde{T}^{\dot{a}}$, in the adjoint of two Lie algebrae $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ with structure constants $f, \tilde{f}$. We recall the lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi^{a\dot{a}}\partial_\mu\phi_{a\dot{a}} + \frac{\lambda}{3}f^{abc}\tilde{f}^{\dot{a}\dot{b}\dot{c}}\phi^{a\dot{a}}\phi^{b\dot{b}}\phi^{c\dot{c}} = \mathrm{Tr}\left(\frac{1}{2}\partial^\mu\Phi\partial_\mu\Phi + \frac{\lambda}{3}[\Phi,\Phi]\Phi\right), \tag{5.12}$$

One can reduce the theory above from $d+1$ to d dimensions via a symmetry reduction:

$$\frac{\partial}{\partial x^d}\phi^{a\dot{a}} = H^a_b\phi^{b\dot{a}} + \tilde{H}^{\dot{a}}_{\dot{b}}\phi^{a\dot{b}}, \tag{5.13}$$

where $H, \tilde{H}$ are in the Cartan subalgebra of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ respectively. Upon this reduction, the kinetic term $(\partial_d\phi^{a\dot{a}})^2$ of the bi-adjoint scalar action gives rise to a mass-matrix for the theory. The reduction (5.13) fixes the dependency of the fields on the x^d coordinate. For the field $\Phi = \phi^{a\dot{a}}T^aT^{\dot{a}}$, such a reduction (say $\tilde{H} = 0$) dictates that $\Phi(x^\mu, x^d) = e^{Hx^d}\Phi(x^\mu)e^{-Hx^d}$ so that the x^d dependency drops out of the acti.

In the case of a generic flavour group, the element H defining the reduction can be any linear combination of elements of the Cartan subalgebra, where the mass parameters are the $\mathrm{rank}(\mathfrak{g})$ coefficients of the linear combination. For instance, if we take $G\times\tilde{G} = SU(N)\times SU(\tilde{N})$ with $N = \tilde{N} = 2$, there is only one element in the Cartan subalgebra, so we can write:

$$H = m\,\mathrm{diag}(1,0,-1) \qquad \tilde{H} = \tilde{m}\,\mathrm{diag}(1,0,-1), \tag{5.14}$$

Expanding the mass matrix, one can read off the mass of the various states in the theory.

5.2 Massive model for bi-adjoint scalar

The theory we introduced above will be the playground for us to establish the mechanism of symmetry reduction in the ambitwistor string: introducing masses in the simple context of the biadjoint scalar theory, we will derive the massive scattering equations and study the factorization properties of the massive correlators. These are universal to all formulae derived from the ambitwistor string and we will then be able to extend our description to more interesting theories in the following sections.

5.2.1 Worldsheet model for the massive bi-adjoint scalar

Our aim is to construct a massive model by performing a symmetry reduction of the massless bi-adjoint scalar worldsheet action described in §2.2. To ease into the implementation of (5.9), let us first take a look at how a trivial dimensional reduction can be realised by gauging translations in the $d-1$ direction:

$$S = S^{m=0} + \int_\Sigma a P_{d-1}\,. \tag{5.15}$$

This action is invariant under the gauge transformation:

$$\delta a = \bar\partial \alpha, \quad \delta X^{d-1} = \alpha, \quad \delta P = 0\,, \tag{5.16}$$

which indeed generates translations along the $d-1$ axis. Because of the additional term in the action, BRST gauge fixing imposes the constraint $K_{d-1} = 0$ on the kinematics of the vertex operators.

When performing a symmetry reduction, we can then gauge a current that is the sum of one component of momentum with an element j^H that acts as a symmetry on the original higher dimensional theory. The natural candidate in the case of the bi-adjoint scalar is the symmetry associated to the current algebra[1]:

$$S = S^{m=0} + \int_\Sigma a(P_{d-1} - j^H) \tag{5.18}$$

Here j^H is the current corresponding to the element H in the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Notice

[1] We can combine this with the action of both current algebrae:

$$S = S^{m=0} + \int_\Sigma a(P_{d-1} - j^H - j^{\tilde H})\,, \tag{5.17}$$

but we will use the action above for the sake of readability.

that it would be impossible to set P_{d-1} equal to a constant mass M through this construction, as the gauged current needs to be a one-form.

The action above is invariant under the gauge transformation:

$$\delta a = \bar{\partial}\alpha, \quad \delta X^{d-1} = \alpha, \quad \delta P = 0, \quad \delta S_j = \int_\Sigma j^H(\sigma)\bar{\partial}\alpha\,, \tag{5.19}$$

corresponding to a translation in the $d-1$ direction composed with the action of the symmetry element H. We can recognise the variation of S_j as the definition of the current j^H. In addition to the usual (b, c) and $(\tilde{b}, \tilde{c})$ ghosts, we have to introduce fermionic $\pi \in \Pi\Omega^0(\Sigma, K_\Sigma)$ and $\xi \in \Pi\Omega^0(\Sigma)$ ghosts. The BRST operator contains an additional term:

$$\oint \frac{1}{2}cT_a^{\text{gh}} + \xi(P_{d-1} - j^H + \frac{1}{2}T_a^{gh})\,, \tag{5.20}$$

with $T_a^{\text{gh}} = \partial\xi\pi$. For the massive bi-adjoint scalar we take vertex operators that as in (2.13), with kinematical data $K_i = (k_i, \kappa_i)$ as in (1.17):

$$c_i\tilde{c}_i(T \cdot j)(\tilde{T} \cdot \tilde{j})e^{iK_i \cdot X(\sigma_i)}\,. \tag{5.21}$$

Here k_i are the physical components of momentum while κ_i are an artifact of the construction via symmetry reduction: BRST invariance of the model constrains their value, thus assigning masses to the external particles. Indeed BRST gauge fixing in the presence of vertex operators produces $n-1$ so-called *picture changing operators*, one for each modulus of the gauge field a. This procedure was outlined in the review §2.2 and it is analogous to the twistorial case that we will treat in detail in §5.4.3. The result of gauge fixing are $n-1$ delta functions enforcing:

$$\delta(\text{Res}_{\sigma_i}(P_{d-1} - j^H))\,. \tag{5.22}$$

On color eigenstates,

$$j^H(\sigma) \cdot V_i(\sigma_i) \sim \frac{h_i}{\sigma - \sigma_i}V_i(\sigma_i)\,, \tag{5.23}$$

where h_i is the charge of the external state under H, $h_i : [H, T^{a_i}] = h_iT^{a_i}$. Anticipating the localisation of the path integral in (5.29), we can evaluate the residues to find: Then the delta functions constrain internal momentum as:

$$\delta(\kappa_i - h_i)\,, \tag{5.24}$$

BRST invariance still requires the states to be on-shell $K_i^2 = 0$. Then, although we will keep writing the full momentum K, the conditions above tell us that the vertex operators describe massive states in $d-1$ dimensions:

$$K_i = (k_i|\kappa_i) \qquad k_i^2 = \kappa_i^2 = h_i^2 =: m_i^2\,. \tag{5.25}$$

Massive amplitudes as correlators We derive amplitude formulae as n-point correlators in this model:

$$\mathcal{A}_n = \left\langle \prod_{k=2}^{n} \delta(\mathrm{Res}_{\sigma_k}(P_{d-1} - j^H)) \prod_{l=1}^{3} c_l\tilde{c}_l(T_l \cdot j(\sigma_l))(\tilde{T}_l \cdot \tilde{j}(\sigma_l))e^{iK_l\cdot X(\sigma_l)} \prod_{i=4}^{n} \int_\Sigma \mathcal{V}_i \right\rangle \tag{5.26}$$

where $\mathcal{V}_i$ are the integrated vertex operators in (2.15) with the replacement $k_i \to K_i$. These contain the $n-3$ delta functions $\delta(K_i \cdot P(\sigma_i))$ that give rise to the scattering equations.

We can integrate out the X field in the path integral by incorporating the plane waves factors into an effective action, which then contains all the dependence on X in the form:

$$\frac{1}{2\pi}\int_\Sigma (P\cdot\bar{\partial}X)(\sigma) + 2\pi i\sum_{i=1}^{n}\bar{\delta}(\sigma-\sigma_i)K_i\cdot X(\sigma)\,, \tag{5.27}$$

When performing the path integral for the X field, the zero modes contribute delta functions $\delta^{d-1}(\sum k_i)\delta(\sum \kappa_i)$, conserving both momentum and internal momentum. The rest of the integration generates delta functions enforcing the equations:

$$\bar{\partial}P_\mu = 2\pi i\sum_i K_{i\mu}\bar{\delta}\left(\sigma-\sigma_i\right)\,. \tag{5.28}$$

As in the massless case, at genus zero these are solved by:

$$P_\mu(\sigma) = \sum_{i=1}^{n} \frac{K_{i\mu}}{\sigma - \sigma_i}\,. \tag{5.29}$$

so that P is completely localised on this solution.

What we obtain is a worldsheet formula for the scattering amplitude of n states in a theory of massive and massless scalar fields. The amplitude is decomposed into a sum of color ordered amplitudes over all color structures:

$$\mathcal{A}_n = \sum_{\alpha\in S_n/Z_n}\sum_{\beta\in S_n/Z_n} \mathrm{tr}\left(T^{\alpha(1)}\dots T^{\alpha(n)}\right)\mathrm{tr}\left(\tilde{T}^{\beta(1)}\dots \tilde{T}^{\beta(n)}\right) m_n(\alpha|\beta)\,, \tag{5.30}$$

The double partial amplitudes is given by the correlator as:

$$m_n(\alpha|\beta) = \delta^{d-1}\left(\sum_i k_i\right)\delta\left(\sum_i \kappa_i\right)\prod_{k=2}^{n}\delta(\mathsf{k}_k - h_k)\int\frac{\prod_i' \mathrm{d}\sigma_i\bar{\delta}(E^i)}{\mathrm{Vol}(SL(2,\mathbb{C}))}\,\mathrm{PT}(\alpha)\,\mathrm{PT}(\beta)\,, \tag{5.31}$$

localised on the solutions to the massive scattering equations:

$$E^i = \sum_{j\neq i}\frac{k_i\cdot k_j - \mathsf{k}_i\cdot\mathsf{k}_j}{\sigma_{ij}}\,, \tag{5.32}$$

with masses assigned via:

$$\mathsf{k}_i = h_i \qquad [H, T^{a_i}] = h_i T^{a_i} \qquad |\mathsf{k}_i| = m_i\,. \tag{5.33}$$

These equations here simply arise by inserting the solution for $P_\mu(\sigma)$ in $K_i\cdot P(\sigma_i) = 0$. Recall that, as explained in the massless case, the delta functions imposing these equations come from fixing the moduli of the gauge field $\tilde{e}$ in the presence of n vertex operators, that is they enforce the constraint $P^2 = 0$ in the original d dimensional space.

The massive scattering equations E^i are in the form of the equations proposed by Naculich (1.15), with $\Delta_{ij} = -\mathsf{k}_i\cdot\mathsf{k}_j$ These equations are invariant under simultaneous $SL(2,\mathbb{C})$ transformations of the punctures $\{\sigma_i\}$ by conservation of the internal momentum,

$$\sum_i \kappa_i = 0\,, \tag{5.34}$$

which guarantees that the constraints (1.16) are satisfied:

$$-\sum_{i\neq j} \kappa_i \cdot \kappa_j = m_i^2.$$

In this derivation we provided a worldsheet description of massive theories, together with amplitude formulae and mass assignments that follow automatically from the model. We will show in the next section that the invariance of the scattering amplitude under the original global symmetry guarantees the consistency of mass assignments also for particles propagating in the internal channels.

5.2.2 Consistency of the bi-adjoint scalar massive model

While Naculich's work had shown that the scattering equations could be modified to include massive particles, it wasn't obvious that the full CHY formulae would describe amplitudes fot an underlying massive field theory. The model that we have described above for biadjoint scalars and that we generalise in the next section provides us with a well established worldsheet description of known massive theories. To make the matter more concrete, in this section we will show that (5.31) gives a consistent representation of scattering amplitudes in the massive theory described in section 5.1. In order to do so we want to show that also the particles propagating in the factorisation channels have masses corresponding to their charges under the flavor group. For instance, in the case of an $SU(2)$ flavor group we expect the masses of internal particles to take value in the spectrum of the corresponding lagrangian theory, namely 0 and m, and the interactions are consistent with that lagrangian. We make this manifest by showing that also for the particles propagating in the factorization channels, internal momentum κ is assigned as their flavour charge. The mass assignment then follows straightforwardly. Before we go through the argument in more detail, we briefly recall the factorisation properties of

CHY-type amplitude formulae, as outlined in the introduction.

Factorization in the CHY formalism The tree level scattering equations relate the boundaries of the moduli space of the n-punctured Riemann sphere to factorization channels of the amplitude. The boundaries correspond to configurations where the Riemann sphere degenerates into two subspheres joint at a node and can be parametrized as:

$$\sigma_i = \sigma_I + \varepsilon x_i + \mathcal{O}(\varepsilon^2) \qquad \text{for } i \in I\,, \tag{5.35}$$

where I is the subset of $\{1, \dots n\}$ labelling the punctures on one of the two subspheres. The degeneration corresponds to the limit $\varepsilon \to 0$. In this limit the scattering equations tell us that $K_i \cdot P_I(x_i) = \mathcal{O}(\varepsilon)$, where P_I is the original one form restricted to the I component of the degenerate Riemann sphere. This entails:

$$K_I^2 := (\sum_{i\in I} K_i)^2 = \frac{1}{2}\sum_{i,j\in I} K_i \cdot K_j = \sum_{i,j\in I, i\neq j} \frac{x_i K_i \cdot K_j}{x_i - x_j} = \sum_{i\in I} x_i K_i \cdot P_I(x_i) = \mathcal{O}(\varepsilon)\,, \tag{5.36}$$

i.e. in the limit $\varepsilon \to 0$, the particle 'I' goes on-shell. The boundary described above indeed corresponds to a factorization channel on the support of the scattering equations.

Furthermore the scattering equations factorise on each of these channels: as $\varepsilon \to 0$ they reduce to two sets of constraints on the two subspheres with an additional puncture I on each, corresponding to the node by which they are joint and such that the residue of the one form on that puncture is $\pm K_I$. For theories of interest, this factorization property is carried over to the whole amplitude, allowing to write [3]:

$$K_I^2\,\mathcal{A}_n^{\text{CHY}} \to \mathcal{A}_{|I|}^{\text{CHY}}\mathcal{A}_{|\bar{I}|}^{\text{CHY}} \qquad \text{as } K_I^2 \to 0\,, \tag{5.37}$$

for all factorization channels I.

Masses of the internal particles When considering a symmetry reduction, the factorization channels correspond to:

$$0 = \big(\sum_{i \in I} k_i\big)^2 - \big(\sum_{i \in I} \mathsf{\kappa}_i\big)^2 = k_I^2 - \mathsf{\kappa}_I^2 \,. \tag{5.38}$$

If the values of $\mathsf{\kappa}$ are assigned consistently, we must have $\mathsf{\kappa}_I^2 = m_I^2$, where m_I is in accordance with the spectrum of the theory, for any factorization channel that can appear in the amplitude. In other words, we want the mass of the particle propagating in the internal channel as defined via conservation of internal momentum on the subspheres to match with the mass we expect for the propagator of that particle from the lagrangian description of the given theory. We will see that this is automatic for internal momenta assigned via symmetry reduction and we find $\mathsf{\kappa}_I^2 = h_I^2 = m_I^2$. We want to show that all internal particles in (5.31) are consistent with the lagrangian theory (5.12) reduced via (5.13). The masses in the amplitude formula are assigned via:

$$\mathsf{\kappa}_i = h_i \qquad h_i : [H, T^{a_i}] = h_i T^{a_i} \,. \tag{5.39}$$

For any collection of generators $\{T^{a_1}, \dots T^{a_n}\}$ we can write:

$$0 = \mathrm{Tr}\Big[[H, T^{a_1} \cdots T^{a_n}] \Big] = \Big(\sum_i h_i \Big) \, \mathrm{Tr}\Big[T^{a_1} \cdots T^{a_n} \Big] \,. \tag{5.40}$$

This means that either the sum of all the internal momenta vanishes or the corresponding trace structure does, i.e. all the partial amplitudes that give a non vanishing contribution obey overall conservation of charge under H.

Now we can exploit what we know about the factorization properties of (5.31). Singularities arise as a collection of punctures $\{\sigma_i\}_{i \in I}$ become degenerate. The leading pole in the degeneration parameter ε corresponds to subsets I of the external labels that are adjacent in both orderings α and β. By the considerations that lead to (5.36), we have that these singularities correspond to momentum $K_I = \sum_{i \in I} K_i$, going on shell. The particle propagating in the internal channel has internal momentum $\sum_{i \in I} \mathsf{\kappa}_i$. In order to show that $\sum_{i \in I} \mathsf{\kappa}_i$ is the H eigenvalue for the state propagating in the internal line, we manifest the factorisation of the amplitude formula. In the limit $K_I^2 \to 0$ we have:

$$(k_I^2 - (\sum_{i\in I} \mathsf{K}_i)^2)\,\mathcal{A}_n$$
$$\to \sum_{\alpha_I,\alpha_{n-I}} \sum_{\beta_I,\beta_{n-I}} \mathrm{Tr}[T^{a_{\alpha(1)}} \cdots T^{a_{\alpha(n)}}]\mathrm{Tr}[\tilde{T}^{b_{\beta(1)}} \cdots \tilde{T}^{b_{\beta(n)}}] m_{|I|+1}(\alpha_I|\beta_I) m_{|\bar{I}|+1}(\alpha_{\bar{I}}|\beta_{\bar{I}})\,,$$

where the orderings α_I, β_I, count $|I| + 1$ indices each, including the one for the extra puncture inserted at the node. We refer to appendix B.1 for details of the factorization of the trace structure, which follows from the completeness relation for $SU(N)$. Then the amplitude (5.30) factorizes as:

$$(k_I^2 - (\sum_{i\in I} \mathsf{K}_i)^2)\,\mathcal{A}_n$$
$$\to \sum_{a_I,b_I} \sum_{\alpha_I,\beta_I} \mathrm{Tr}[a_{\alpha_I} a_I]\tilde{\mathrm{Tr}}[b_{\beta_I} b_I] m_{|I|+1}(\alpha_I|\beta_I) \sum_{\alpha_{\bar{I}},\beta_{\bar{I}}} \mathrm{Tr}[a_I a_{\alpha_{\bar{I}}}]\tilde{\mathrm{Tr}}[b_I b_{\beta_{\bar{I}}}] m_{|\bar{I}|+1}(\alpha_{\bar{I}}|\beta_{\bar{I}})\,,$$

where the trace factors are $\mathrm{Tr}[a_\alpha; a_I] = \mathrm{Tr}[T^{a_{\alpha(i_1)}} \cdots T^{a_{\alpha(i_{|I|})}} T^{a_I}]$. The argument (5.40) for this trace implies $h_I = -\sum_{i\in I} h_i$. From the form of the amplitude (5.31), we can see that the delta functions enforce conservation of internal momentum and $\kappa_i = h_i$ for all but one particles in the subamplitude, which we take to be particle I. Then putting all these conditions together, we can write:

$$\mathsf{K}_I = -\sum_{i\in I} \mathsf{K}_i = -\sum_{i\in I} h_i = h_I\,, \tag{5.41}$$

where in the first equality we have used internal momentum conservation in the subamplitude, in the second one we have used the $|I|$ 'mass assigning' delta functions in $m_{|I|+1}(\alpha_I|\beta_I)$ and the last equality follows from charge conservation in the subamplitude. This proves that the particles propagating in the internal channels all have masses corresponding to their charges under the $SU(N)$ flavor group. By the same argument, we can take $|I| = n-1$ to show that the constraint $\mathsf{K}_i - h_i = 0$ holds for all n external particles despite the fact that the delta functions only enforce it on $n-1$ of them.

5.3 Massive amplitudes from gauged currents

In this section we summarise the results we obtained for the bi-adjoint scalar theory in the previous section in a language that extends naturally to all theories possessing an ambitwistor string representation that factorises on singularities. We consider symmetry reductions of the type we described in section 5.1. They are obtained in the ambitwistor string by gauging currents that generate a combination of translations along the $d-1$ dimension and transformations under an internal symmetry group G of the theory:

$$S^{\mathrm{red}} = \int_\Sigma a(P_{d-1} - j^{\mathcal{H}}) \tag{5.42}$$

where P_{d-1} is the d-th component of the spacetime vector P_μ and it is a $(1,0)$-form on the worldsheet. The current $j^{\mathcal{H}}$ is a $(1,0)$-form on the worldsheet valued in the Cartan subalgebra of some internal symmetry algebra $\mathfrak{g}$. Under the transformation:

$$\delta a = \bar\partial\alpha, \quad \delta X^{d-1} = \alpha, \quad \delta P = 0\,, \tag{5.43}$$

the total action $S = S^{\mathrm{bos}} + S^{\mathrm{red}} + S^{\mathrm{m}}$ has an overall variation:

$$\delta S = \int_\Sigma j^{\mathcal{H}}(\sigma)\bar\partial\alpha\,, \tag{5.44}$$

where $\alpha \in \Omega^0(\Sigma)$ is a bosonic gauge parameter. This can be compensated by a symmetry transformation generated by $j^{\mathcal{H}}$. Then $a \in \Omega^{0,1}(\Sigma)$ gauges translations in the $d-1$ direction up to a symmetry transformation H as in (5.9).

The quantisation of the action in the presence of the term S^{red} involves introducing fermionic ghosts $\pi \in \Pi\Omega^0(\Sigma, K_\Sigma)$ and $\xi \in \Pi\Omega^0(\Sigma)$ associated to the gauge field a and additional terms in the BRST operator:

$$\oint \xi(P_{d-1} - j^{\mathcal{H}}) \subset Q \tag{5.45}$$

The vertex operators are of the form (2.13):

$$c_i\tilde{c}_i\, w_i\, e^{iK_i\cdot X}\,. \tag{5.46}$$

Here w_i carries a representation of the algebra $\mathfrak{g}$, so that :

$$j^{\mathcal{H}}(\sigma)\cdot w_i(\sigma_i) \sim \frac{h_i}{\sigma-\sigma_i} w_i(\sigma_i)\,. \tag{5.47}$$

BRST invariance puts the states on-shell and gauge fixing in the presence of n vertex operators, via the additional term (5.45) in the BRST operator, generates delta functions giving an assignment of $\kappa = K^{d-1}$ to $n-1$ of the K^μ-momentum eigenstates:

$$\delta(\mathrm{Res}_{\sigma_i}(P_{d-1} - j^{\mathcal{H}})) = \delta(\kappa_i - h_i)\,. \tag{5.48}$$

The element $\mathcal{H}$ lives in the Cartan subalgebra of $\mathfrak{g}$:

$$\mathcal{H} = \sum_{k=1}^{\mathrm{rank}(\mathfrak{g})} m_k H_k\,, \tag{5.49}$$

where H_i are the Cartan generators and l_i, the coefficients of the linear combination, correspond to mass parameters.

The computation of correlators is analogous to the case of the bi-adjoint scalar, in particular the X path integral can be performed explicitly and the tree level amplitudes are found to be completely localised on the solutions to the massive scattering equations:

$$E^i = \sum_{j\neq i} \frac{k_i\cdot k_j - \kappa_i\cdot\kappa_j}{\sigma_{ij}}\,, \tag{5.50}$$

with masses assigned as $m_i = |\kappa_i| = |h_i|$. The contributions from the matter systems are computed exactly as in the massless case, then evaluated on the reduced kinematics.

Additionally, we find that the amplitudes are only non vanishing on the support of a $\kappa-$conserving delta function. This has two important consequences: on the one hand internal momentum conservation guarantees $SL(2,\mathbb{C})$ invariance of the massive scattering equations. Moreover it plays an essential role in showing the consistency of the masses of particles propagating in the internal channels, generalising the argument in the previous section. In section 5.2.2 we recalled the factorisation properties of amplitudes obtained as correlators in the massless ambitwistor string. For the massive models obtained by symmetry reduction it follows that:

$$(k_I^2 - \big(\sum_{i\in I} \kappa_i\big)^2\big)\, \mathcal{A}_n^{C_{i_1}\dots C_{i_n}} \to \sum_{C_I,\dots} \mathcal{A}_{|I|+1}^{C_{i_1}\dots C_I} \mathcal{A}_{|\overline{I}|+1}^{C_I\dots C_{i_n}} \qquad \text{as} \qquad (k_I^2 - \big(\sum_{i\in I} \kappa_i\big)^2\big) \to 0\,, \tag{5.51}$$

for all factorization channels I. Here C_i label the quantum number associated to the symmetry group G. The sum above runs over all possible values of quantum numbers of the internal particle, including C_I. In appendix B.2 we review how the symmetry of the original system implies charge conservation $\sum_j h_j = 0$ on the scattering amplitudes. This argument applied to sub-amplitudes tells us that the only non vanishing contributions to the channel come from intermediate states I such that $-\sum_{i\neq I} h_i = h_I$ where h_I is the $\mathcal{H}$ eigenvalue of the propagating state. One can then use κ-conservation in the subamplitude and the $|I|$ delta functions fixing $\kappa_i = h_i$ to find:

$$\kappa_I = -\sum_{i\neq I} \kappa_i = -\sum_{i\neq I} h_i = h_I\,, \tag{5.52}$$

thus showing that the assignment of masses given by the symmetry reduction is consistent: also for the states propagating in the internal channels the masses are assigned consistently as charges under the internal symmetry group.

One should notice that charge conservation $\sum_i h_i = 0$ does not come automatically out of the computation of the correlator. Indeed we can run the argument above on the full amplitude: the delta functions in the amplitude fix $\kappa_i = h_i$ for $n-1$ particles. Then we have $\kappa_n = -\sum_{i\neq n} \kappa_i = -\sum_{i\neq n} h_i$ and in order to show that κ_n is indeed an eigenvalue under the original symmetry group, one needs to derive the equation $\sum_i h_i = 0$ from an argument like the one in appendix B.2.

In principle we could apply this type of reduction to the heterotic ambitwistor string. However, as we discussed in the introduction the twistorial realisations of the ambitwistor string offer a better representation of supersymmetry. In the following sections we will then consider massive theories obtained by symmetry reduction from the five dimensional models of [85]. As we argued earlier, although additional moduli are introduced, the amplitudes are supported on the scattering equations obtained in the dimension-agnostic description and the singularity structure is still determined by the boundaries of $\mathfrak{M}_{0,n}$ so that this argument remains valid.

5.4 Two-twistor string as a symmetry reduction

In the first part of this chapter we have described symmetry reductions and their implementation in the context of the RNS ambitwistor string. We have used the vectorial models to study the singularity structure of the amplitude formulae and the mass assignment for internal propagating particles. In this section we implement the symmetry reduction in the twistorial models of [85] to obtain more interesting massive models in four dimensions with various degrees of supersymmetry. We begin by studying how the symmetry reduction acts on five dimensional momentum space, which will lead us to the embedding of massive and massless spinor helicity variables and ambitwistors in four dimensions.

5.4.1 From five to four dimensions

Our starting point are models such as the ones outlined in §2.4.5. We pick another fixed non-null vector Ω_2 ($\Omega_1 \cdot \Omega_2 = 0$). In a non-dynamical setting, the reduction we seek constrains five dimensional momenta to obey:

$$\Omega_2 \cdot K = M \,, \tag{5.53}$$

where M is fixed by the worldsheet model in the way we will detail in the next section. The choice of Ω_2 breaks the spin group $\mathrm{Sp}(4,\mathbb{C}) \to \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \simeq \mathrm{Spin}(4,\mathbb{C})$, each factor acting separately on positive and negative chirality spinors. The index A of $\mathrm{Sp}(4,\mathbb{C})$ decomposes accordingly into $(\alpha, \dot{\alpha})$, one for each factor of the spin group. In the language of symmetry reduction we used in part one, the component $\Omega_2 \cdot K$ is the *internal* momentum κ. By the discussion above, five dimensional momentum also satisfies $\Omega_1 \cdot K = 0$ so that we can pick a frame such that:

$$K_{AB} = \begin{pmatrix} K \cdot \Omega_2 \epsilon_{\alpha\beta} & k_\alpha^{\dot\beta} \\ k_\beta^{\dot\alpha} & K \cdot \Omega_2 \epsilon^{\dot\alpha\dot\beta} \end{pmatrix}, \tag{5.54}$$

where $k_{\alpha\dot\beta} = k_\alpha^{\dot\gamma}\varepsilon_{\dot\gamma\dot\beta}$ is the spinorial form of the four dimensional massive momentum with mass $m = |\Omega_2 \cdot K| = |M|$, as expected in the $(2,2)$ representation of the spin group. The spinor helicity decomposition of the massive momentum, as in [12], follows from the decomposition of K_{AB}:

$$k_{\alpha\dot\alpha} = \kappa_\alpha^a \tilde\kappa_{\dot\alpha a} \quad \det\kappa = \frac{1}{2}(\kappa_\alpha, \kappa^\alpha) = M = K \cdot \Omega_2 = \det\tilde\kappa\,, \tag{5.55}$$

where $a = 1, 2$ is an $SL(2,\mathbb{C})$ massive little group index raised and lowered by $\epsilon_{ab} = \epsilon_{[ab]}, \epsilon_{12} = 1$. These spinor helicity variables consistently require $k^2 = m^2 = \det(k_{\alpha\dot\alpha}) = \det\kappa\det\tilde\kappa = M^2$. As before we denote little group contractions by $(\cdot\,,\,\cdot)$ and contractions of undotted and dotted indices as $\langle\cdot\,,\,\cdot\rangle$ and $[\cdot\,,\,\cdot]$ as is standard for four dimensional spinor-helicity.

Coming from higher dimension, it is natural to build representations out of the four dimensional Dirac spinor representation labelled by A. We understand this as a reflection of the fact that massive particles are not chiral, so that for physical states Weyl spinors double up with their conjugates. The polarization states of massive particles are understood as representations of the massive little group. Spin-s massive particles transform as the symmetric part of rank $2s$ tensors of the massive little group, with polarization data $\epsilon_{a_1\ldots a_{2s}} = \epsilon_{(a_1\ldots a_{2s})}$. In particular we can write

$$\psi_A = (\epsilon\kappa_A)\mathrm{e}^{ik\cdot x} = \epsilon_a\kappa_A^a\,\mathrm{e}^{ik\cdot x}\,, \qquad \kappa_{Aa} = (\kappa_{\alpha a}, \tilde\kappa_{\dot\alpha a}) \tag{5.56}$$

for a massive Dirac field momentum eigenstate with polarization ϵ_a. Similarly, a massive spin-1 field has polarization given by a symmetric $\epsilon_{(ab)}$:

$$F_{AB} = \epsilon_{(a}\tilde\epsilon_{b)}\kappa_A^a\kappa_B^b\,\mathrm{e}^{ik\cdot x}\,. \tag{5.57}$$

For generic spin-s massive fields we will take the decomposition in Dirac indices:

$$\Psi_{A_1\ldots A_{2s}} = \epsilon_{(a_1\ldots a_{2s})}\kappa_{A_1}^{a_1}\cdots\kappa_{A_{2s}}^{a_{2s}}\,\mathrm{e}^{ik\cdot x}\,. \tag{5.58}$$

Massless particles are naturally embedded in this description by taking $M = m = 0$, which can be achieved with the above spinor helicity variables by restricting to $\kappa_{2\alpha} = 0 = \kappa_1^{\dot\alpha}$, so that the little group indices $a = 1, 2$ correspond to self-dual and anti-self-dual polarizations.

5.4.2 Massive models from symmetry reduction

In section 5.3 we have seen that we can implement a symmetry reduction in a worldsheet model by gauging constraints that relate the extra components of momentum to currents for an internal symmetry of the higher-dimensional theory. Here we apply the same procedure to the five dimensional models (2.59) and show that we obtain massive models in four dimensions such as the ones described in chapter 4.

In order to perform the symmetry reduction, we want to gauge a constraint that will implement (5.53) at the level of the worldsheet. Because P has weight $(1,0)$, the mass 'M' needs to be implemented by a $(1,0)-$form on the worldsheet that we will take to be a current j^H for some symmetry of the five-dimensional model we are reducing. In twistorial variables, the constraints to be implemented for the reduction[2] are:

$$J_{\Omega_1} = \frac{1}{2}(\lambda^2 - \tilde\lambda^2) \qquad J_{\Omega_2} = \frac{1}{2}(\lambda^2 + \tilde\lambda^2) - j^H\,, \tag{5.59}$$

where $\lambda^2 = \frac{1}{2}(\lambda_\alpha \lambda^\alpha) = \det(\lambda_\alpha^a)$ and similarly for $\tilde\lambda$. Here j^H denotes a current generator associated to the element $H \in \mathfrak{g}$ in the Cartan subalgebra of some symmetry of the theory in five dimensions. It is generally constructed out of fields appearing in the model. Here we will first discuss the case where H is in the Cartan of the color group of a (S)YM theory so that it is an element of the current algebra in the matter model, as described below.

We can rearrange the currents in a more symmetric form by taking combinations $J_{\Omega_\pm} = J_{\Omega_2} \pm J_{\Omega_1}$, leading to:

$$S = \int_\Sigma \mathcal{Z}^a \cdot \bar\partial_e \mathcal{Z}_a + A_{ab}\mathcal{Z}^a \cdot \mathcal{Z}^b + A(\lambda^2 - j^H) + \tilde A(\tilde\lambda^2 - j^H) + S_m\,. \tag{5.60}$$

The supertwistor fields $\mathcal{Z}^a$ are worldsheet spinors as in the five-dimensional models. Little

[2]including the trivial reduction from six dimensions that was included in (2.59)

group transformations are gauged by the fields $A_{ab} = A_{(ab)} \in \Omega^{0,1}(\Sigma, \mathfrak{sl}_2)$ and deformations of the worldsheet complex structure $\bar{\partial}_e = \bar{\partial} + e\partial$ by the Beltrami differential $e \in \Omega^{0,1}(\Sigma, T_\Sigma)$. The fields $A, \tilde{A} \in \Omega^{0,1}(\Sigma)$ are Lagrange multipliers imposing the constraints $J_{\Omega_\pm} = 0$ and effectively enforcing the symplectic reduction to the target space of the model. We recognize this as the phase space of the complexified massive particle by comparison with the construction we presented in chapter 4. The action S_m is composed of matter systems such as the ones described in [85], which we briefly review below.

Symmetry reduction The fields $a, \tilde{a}$ gauge transformations that combine translations along the Ω_2 and Ω_1 directions with the action of the symmetry group G via H. These transformations are:

$$\delta A = \bar{\partial}\alpha \qquad \delta\mu^{Aa} = \alpha(\Omega_2^{AB} + \Omega_1^{AB})\lambda_B^a \qquad \delta\lambda_A^a = 0 \qquad \delta S_m = \int_\Sigma \bar{\partial}\alpha\, j^H\,, \tag{5.61}$$

and

$$\delta\tilde{A} = \bar{\partial}\tilde{\alpha} \qquad \delta\mu^{Aa} = \tilde{\alpha}(\Omega_2^{AB} - \Omega_1^{AB})\lambda_B^a \qquad \delta\lambda_A^a = 0 \qquad \delta S_m = \int_\Sigma \bar{\partial}\tilde{\alpha}\, j^H\,. \tag{5.62}$$

Without the need to specify the action of the symmetry on the matter fields in S_m, we can specify the variation of the action by the definition of the Noether current j^H. It is then the symmetry of the theory specified by S_m that guarantees the consistency of the reduction. In terms of the original constraints (5.59), these disentangle into translations in the Ω_1 direction and combinations of translations in the Ω_2 direction and transformations under H, as we expect in the optic of the reduction $6 \to 5 \to 4$d.

Here the phase space of complexified massive particles arises as a symplectic quotient of the five dimensional ambitwistor space parametrized as in (2.61) with the additional constraints $J_{\Omega_\pm} = 0$ and the gauging of transformations that combine translations along the $\Omega_\pm$ direction and the action of the symmetry element H. The five dimensional little group combines with the gauging along the two reduced dimension to form the internal symmetry group of the complexified massive particle. We can then identify the six dimensional supertwistor $\mathcal{Z}$

with the Dirac supertwistor $\mathcal{Y}$ we used to construct the four dimensional models in (4.19). As expected, these have $\mathcal{N} = \mathcal{N}^{(4d)} = 2N$ fermionic components η^I. Through the reduction, we obtain the correspondence with four dimensional Minkowski superspace $\mathcal{M}_{4|4\mathcal{N}}$ parametrized by $(x_{\alpha\dot\alpha}, \theta^{AI})$, with $\theta^{AI} = (\theta^I_\alpha, \tilde\theta^{\dot\alpha I})$ in the usual notation and $I = 1, \dots \mathcal{N}$ via:

$$\mu^A = \begin{pmatrix} 0 & x^{\alpha\dot\beta} \\ -x^{\dot\alpha\beta} & 0 \end{pmatrix} \lambda_B + \omega_{IJ}\theta^{AI}\eta^J, \quad \eta^I = \theta^{AI}\lambda_A\,, \tag{5.63}$$

Worldsheet matter For the models we will consider, the relevant matter systems were presented in §4.2. One is a current algebra, whose action we denote S_C and recall the defining relations for the currents $j^{\mathfrak{a}} \in \Omega^0(\Sigma, K_\Sigma \otimes \mathfrak{g})$:

$$j^{\mathfrak{a}}(\sigma) j^{\mathfrak{b}}(0) \sim \frac{l\delta^{\mathfrak{ab}}}{\sigma^2} + \frac{f^{\mathfrak{ab}}_{\mathfrak{c}} j^{\mathfrak{c}}}{\sigma}\,. \tag{5.64}$$

The second type of matter we need to consider in order to build gauge and gravity models is the system of worldsheet fermions $(\rho_A, \tilde\rho^A) \in \Pi\Omega^0(\Sigma, K_\Sigma^{1/2})$ with action given by (4.20).

Out of these two types of systems, we can construct models without $SL(2,\mathbb{C})$ anomalies for bi-adjoint scalars, gauge theory and gravity as follows:

massive bi-adjoint scalar	$S^{\text{BAS}}_m = S_C + S_{\tilde C}\,,$
super Yang-Mills on the Coulomb branch	$S^{\text{CB}}_m = S_\rho + S_C\,,$
super-gravity	$S^{\text{sugra}}_m = S_{\rho_1} + S_{\rho_2}\,.$

Algebra of constraints One might be tempted to generalize the models in (5.60) by taking different currents j^H and $j^{\tilde H}$ in the constraints. This would correspond to performing a less trivial reduction directly from the six dimensional model, where translations are combined with the action of a symmetry in both extra dimensions. However, as we showed in (4.21), in order to guarantee the closure of the algebra of constraints in the presence of worldsheet supersymmetry S_ρ we need to take $j^H = j^{\tilde H}$. This fact is consistent with the lack of well functioning models for gauge and gravity theories in six dimensions.

Gauge fixing The BRST gauge fixing of the model (5.60) has been discussed in (4.2), where we introduced fermionic ghosts associated to the gauge fixing of worldsheet diffeomorphisms, the little group and the internal symmetry group $SL(2,\mathbb{C}) \times \mathbb{C} \times \tilde{\mathbb{C}}$, as well as bosonic ghosts associated to the fermionic constraints in S_ρ. The gauge field e can be fixed to zero and, in the absence of operator insertions, so can the fields A_{ab}, A, $\tilde{A}$, b^a, $\tilde{b}^a$.

We defined the BRST charge and verified the possible obstructions to the vanishing of Q^2 at the level of the QFT. These can in principle arise from an $\mathfrak{sl}_{2,\mathbb{C}}$ anomaly or a conformal anomaly. The $\mathfrak{sl}_{2,\mathbb{C}}$ anomaly coefficient was shown to vanish for maximally supersymmetric gauge theory and gravity. The central charge vanishes for a suitable choice of S_j such that $c_j = 28$ for Super Yang-Mills, $c_j = 40$ for biadjoint scalars and the residual central charge in the gravitational theory can be understood as coming from six compactified dimensions.

Plane wave representatives We will consider scattering of plane wave representatives on ambitwistor space as in [85]. For scalar states, these are obtained from the indirect Penrose transform in six dimensions as elements of $H^1(\mathbb{PA}, \mathcal{O}(-2))$ that naturally give the $H^1(\mathcal{P}_m^{\mathbb{C}}, \mathcal{O}(-2))$ representatives described in chapter 4 upon dimensional reduction. For scalar external states of kinematics κ_A^a we have:

$$\Phi_\kappa\left(Y_a\right) = \int d^2u d^2v \bar{\delta}^4\left((u\lambda_A) - (v\kappa_A)\right) \bar{\delta}((v\,\epsilon) - 1) \exp\left(\left(u\mu^A\right)\epsilon_A\right) \,, \tag{5.65}$$

where u^a, v^a are four auxiliary complex variables. As noted in [5], the components $\det(\kappa)$ and $\det(\tilde{\kappa})$ of the external kinematics are unconstrained in the eyes of the Penrose transform, and are only fixed by the model via BRST to give the signed mass of the state.

Supersymmetric extension In order to describe supersymmetric theories, we take the extension of (5.65) as in §4.2 following [5, 85]. When considering momentum eigenstates of momentum $K_{AB} = (\kappa_A \kappa_B)$, the supersymmetry algebra $\{Q_{AI}, Q_{BJ}\} = 2\Omega_{IJ} P_{AB}$ of six and five dimensions reduces to the little group as:

$$Q_{AI} = \kappa_A^a Q_{aI}, \quad \{Q_{aI}, Q_{bJ}\} = 2\Omega_{IJ}\varepsilon_{ab} \tag{5.66}$$

This leads to the four dimensional supersymmetry algebra with a central extension $Z_{IJ} = 2\det(\kappa)\Omega_{IJ}$. We are primarily interested in describing the Coulomb branch as a symmetry reduction, in which case the symplectic form Ω_{IJ} is preserved and the action of the supercharges organizes the states in massive and massless supermultiplets as we will detail in §5.5. For now, let it suffice to say that both kinds of multiplets are annihilated by half of the supercharges so that on shell superspace can be parametrized by $\mathcal{N} = 4$ fermionic supermomenta q_I, $I = 1, \cdots 4$. These are taken to be eigenvalues of an anticommuting subset of the supercharges Q_{aI}, thus necessarily breaking either the action of either the little group or the R-symmetry. In line with previous work, we employ the R-symmetry preserving representation and define supermomenta q_I as:

$$Q_{aI}\tilde{\mathscr{F}}(\kappa, q) = \left(\xi_a q_I + \epsilon_a \Omega_{IJ}\frac{\partial}{\partial q_J}\right)\tilde{\mathscr{F}}(\kappa, q)\,, \tag{5.67}$$

where (ϵ_a, ξ_a) define a basis of the little group fundamental representation and $\tilde{\mathscr{F}}(\kappa, q)$ is a function on on-shell superspace.

We will detail later how the states of multiplets on the Coulomb branch of $\mathcal{N} = 4$ SYM are encoded in the exterior powers of the supermomenta. The standard procedure would then imply expanding the multiplets in superfields on superspacetime and establish a super Penrose transform to ambitwistor space. Here we simply present the supersymmetric extension of the plane wave representative (5.65) on superambitwistor space parametrized by $\mathcal{Y}_a = (\lambda_{Aa}, \mu_a^A, \eta_a^I)$. On this space, the supercharges act geometrically as $Q_{AI} = \lambda_A \frac{\partial}{\partial \eta^I} + \eta^J \Omega_{JI} \frac{\partial}{\partial \mu^A}$, so that the function:

$$\Phi_{(\kappa,q)}(\mathcal{Y}_a) = \int d^2u d^2v \bar{\delta}^4\left((u\lambda_A) - (v\kappa_A)\right)\bar{\delta}((\epsilon v) - 1)e^{iu_a\left(\mu^{Aa}\epsilon_A + q_I\eta^{Ia}\right) - \frac{1}{2}(\xi v)q^2} \tag{5.68}$$

obeys the correct intertwining relations between superambitwistor space and on-shell momentum superspace.

5.4.3 Vertex operators and pictures

In building correlators we will need to define various types of vertex operators, whose form is determined by how much residual gauge freedom they have left. Most well known are the effects of gauge fixing diffeomorphisms on the vertex operators, namely the distinction between *fixed* and *integrated* vertex operators. We will not detail how this distinction arises as it is well known in the literature and refer the reader to [93], where integrated vertex operators are derived from the treatment of the moduli space of metrics. Here we will content ourselves in saying that at tree level the c-ghosts associated to the gauge field e have three zero modes that need to be saturated in the correlator, calling for three fixed and $n-3$ integrated vertex operators as described below. We will structure the discussion around vertex operators for symmetry reductions of maximal Super Yang-Mills in five dimensions. Because these involve both a current algebra and worldsheet fermions it is easy to carry the discussion over to the biadjoint scalar and supergravity case via the double copy.

On the worldsheet, we build $(1,1)$-form vertex operators by combining the plane wave representative (5.68) with a theory-specific factor $w \in \Omega^0(\Sigma, K_\Sigma^2)$ via a product $\circ$, which should be understood as a convolution so that w may depend on u. Gauge fixing worldsheet diffeomorphisms distinguishes fixed and integrated vertex operators as:

$$V = c\, w \circ \Phi_{(\kappa,q)}(\sigma) \qquad \mathcal{V} = \int d\sigma\, w \circ \Phi_{(\kappa,q)}(\sigma)\,, \tag{5.69}$$

where $\Phi_{(\kappa,q)}(\sigma) \in H^1(\Sigma, T_\Sigma)$ is the pullback to the worldsheet of the plane wave representative (5.68).

Fixed vertex operators A similar distinction between fixed and integrated vertex operators appears when gauge fixing the other gauge fields present in the models. We take here the perspective whereby fixed vertex operators are fundamental objects and the integrated ones are derived from the integration of moduli associated to the gauge field e. We will say more generally that the fixed vertex operators are in picture -1 and the integrated ones in picture 0, in analogy with the fermionic symmetries in ordinary string theory. There is one distinct picture number for each of the gauge fields we are fixing: $w^{(p_b,p_{\tilde{b}})}_{(p_a,p_{\tilde{a}})}$. For these symmetries, we will show in more detail how to gauge fix them in the presence of punctures and how integrated vertex operators arise. We begin by writing fully fixed vertex operators:

$$w^{(\text{-}1,\text{-}1)}_{(\text{-}1,\text{-}1)} = t\,\tilde{t}\,\delta\left((u\gamma)\right)\delta\left((u\tilde{\gamma})\right)\,\mathfrak{t}_{\mathfrak{a}}j^{\mathfrak{a}}\,. \tag{5.70}$$

Here $t\,\tilde{t}$ are the c-ghosts associated with the gauging of the A and $\tilde{A}$ fields, while the γ ghosts are associated with the fermionic gauge fields $b^a, \tilde{b}^a$[3]. Because the vertex operator is automatically invariant under the u-projected fermionic currents $(u\lambda_A)\rho^A$ and $(u\lambda_A)\tilde{\rho}^A$, the vertex operator above is BRST invariant, without the need to force the corresponding components of γ to vanish.

Picture changing operators In this section we discuss in more detail the BRST gauge fixing of the fields $A, \tilde{A}$ as well as the fermionic $b^a, \tilde{b}^a$ in the presence of vertex operators. The approach is analogous to the one described in [85, 99] and it produces so called *picture changing operators*. For each of the gauge fields we introduce a gauge fixing term in the action of the form:

$$\{Q_B, b\,F(\phi)\}\,, \tag{5.71}$$

where $F(\phi) = \phi - \phi^{GF}$ is the gauge fixing condition and b is the associated antighost (here not referring to diffeomorphisms).[4]

Having already fixed worldsheet diffeomorphisms, the gauge transformations associated to $a, \tilde{a}$ are as in (5.61) and (5.62), where the variations are required to vanish at the vertex operators insertion points. This means that we are not able to gauge fix the fields to zero and these are only allowed to vary within a cohomology class of $H^{0,1}\left(\Sigma, \mathcal{O}\left(-\sigma_1 - \ldots - \sigma_n\right)\right)$. The gauge fixed fields can then be expanded in a basis h_i of $(0,1)$-forms on the worldsheet that span this

[3] These shouldn't be confused with the antighost b for the gauging of worldsheet diffeomorphisms.

[4] The reader might notice that here we give a general prescription to gauge fix all residual gauge transformations after having gauge fixed worldsheet diffeomorphisms, but we do not discuss the gauge fixing of the little group via the fields A_{ab} in the same manner. While the vertex operators (5.68) are well understood from the Penrose transform, we expect that there should also be a way to make sense of the polarised scattering equations and additional moduli integrations via a description such as the one outlined above for the other gauge fields.

$n-1$ dimensional cohomology group:

$$A^{\mathrm{GF}} = \sum_{i=1}^{n-1} h_i A_i \qquad \tilde{A}^{\mathrm{GF}} = \sum_{i=1}^{n-1} h_i \tilde{A}_i \,. \tag{5.72}$$

The off-shell BRST transformations of the fields are:

$$\begin{gathered} \delta_B\, A = \bar{\partial} t \qquad \delta_B\, A_i = \alpha_j \qquad \delta_B\, s = N \\ \delta_B\, \alpha_j = 0 \qquad \delta_B\, N = 0 \,, \end{gathered} \tag{5.73}$$

where N is the Nakanishi-Lautrup field that acts as a Lagrange multiplier for the gauge fixing condition. Then the gauge fixing term for A (and similarly for $\tilde{A}$) can be expanded as follows:

$$\int_\Sigma \{Q_B, s\,(A - A^{\mathrm{GF}})\} = \int_\Sigma N(A - A^{\mathrm{GF}}) + s\bar{\partial} t + \sum_{i=1}^{n-1} \alpha_i \int_\Sigma s\, h_i \tag{5.74}$$

Integrating out the auxiliary field N enforces the gauge fixing condition and produces a term of the form

$$\sum_{i=1}^{n-1} A_i \int_\Sigma h_i\, J_{\Omega_+} \tag{5.75}$$

Then integrating out the fermionic and bosonic moduli α_i, A_i we obtain $n-1$ insertions of:

$$\Xi_i = \delta\left(\int_\Sigma h_i J_{\Omega_+}\right)\left(\int_\Sigma h_i\, s\right) \,, \tag{5.76}$$

as well as analogous contributions $\tilde{\Xi}_i$ from $\tilde{A}$. The basis elements h_i can be chosen to extract residues at given points.

The treatment of the fermionic gauge fields is analogous and was presented in [85]. By the invariance of the vertex operators under half of the Heisenberg superalgebra, the components of $b^a, \tilde{b}^a$ that are parallel to u can be gauge fixed to zero, while the orthogonal ones develop moduli and produce $n-2$ picture changing operators [5]:

$$\Upsilon\,(z_l) = \delta(\langle \hat{u}\beta\rangle)\,\langle \hat{u}\lambda_A\rangle\,\rho^A, \quad \tilde{\Upsilon}\,(z_l) = \delta(\langle \hat{u}\tilde{\beta}\rangle)\,\langle \hat{u}\lambda_B\rangle\,\tilde{\rho}^B\,, \tag{5.77}$$

where $\hat{u}$, u form a local basis for the little group.

As both the Ξ_i, $\tilde{\Xi}_i$ and Υ_j, $\tilde{\Upsilon}_j$ come in pairs, in addition to (5.70) we will only need vertex operators in pictures $(0,0;-1,-1)$ and $(0,0;0,0)$ in order to compute correlators. These are obtained as the limit as $\sigma \to \sigma_i$ of the OPE $\quad \text{PCO}(\sigma)\cdot w(\sigma_i), \quad$ and we obtain:

$$w^{\text{(-1,-1)}}_{\text{(0,0)}} = \delta\big(\text{Res}_{\sigma_i}(\lambda^2 - j^H)\big)\;\delta\big(\text{Res}_{\sigma_i}(\tilde{\lambda}^2 - j^H)\big)\delta\left((u\gamma)\right)\delta\left((u\tilde{\gamma})\right)\;\mathsf{t}_{\mathfrak{a}} j^{\mathfrak{a}}\,, \tag{5.78}$$

$$w^{\text{(0,0)}}_{\text{(0,0)}} = \delta\big(\text{Res}_{\sigma_i}(\lambda^2 - j^H)\big)\;\delta\big(\text{Res}_{\sigma_i}(\tilde{\lambda}^2 - j^H)\big)\left(\frac{\langle \hat{u}\lambda_A\rangle\,\epsilon^A}{\langle u\hat{u}\rangle} + \epsilon^A\epsilon_B\rho_A\tilde{\rho}^B\right)\;\mathsf{t}_{\mathfrak{a}} j^{\mathfrak{a}}\,. \tag{5.79}$$

From this derivation we observe that the term

$$Q_m := \oint t(\lambda^2 - j^H) + \tilde{t}(\tilde{\lambda}^2 - j^H)\,.$$

in the BRST operator is responsible for fixing the masses of the external particles via the delta functions in (5.78). The mass is assigned as the residue of the current j^H acting on the external state as an OPE. This action depends on the choice of current j^H and we will detail it later on as we consider specific theories.

5.4.4 Massive amplitudes as correlators

We compute scattering amplitudes as correlators in the models described above. Because the ghost zero modes need to be saturated[6] and the residual gauge symmetry fixed, the only non trivial correlators with n insertions must contain vertex operators in the various pictures as:

$$\mathcal{A}_n = \left\langle V_{1\,\text{(-1,-1)}}^{\;\text{(-1,-1)}} V_{2\,\text{(0,0)}}^{\;\text{(-1,-1)}} V_{3\,\text{(0,0)}}^{\;\text{(0,0)}} \prod_{i=4}^{n} \mathcal{V}_{i\,\text{(0,0)}}^{\;\text{(0,0)}} \right\rangle\,.$$

[5]The picture changing operators actually contain other terms involving mixed ghost products that are generated by the term $\{Q,\beta^a\}F_a(b^a)$, where (β^a,γ^a) is the ghost system for the gauging of b^a. These however don't contribute to the scattering amplitude as they either vanish on the support of the delta functions or they have the wrong ghost number.

[6]The c ghosts have 3 zero modes, the $t,\tilde{t}$ have one each, and the $\gamma^a,\tilde{\gamma}^a$ have two each.

After gauge fixing, all the fields are free. The evaluation of the scattering amplitude is analogous to the one in [5, 85], so we only discuss here the main features of the formulae we obtain:

$$\mathcal{A}_n = \prod_{i=2}^{n} \delta(\kappa_i^2 - h_i)\, \delta(\tilde{\kappa}_i^2 - h_i) \int d\mu_n^{\text{pol}}\, \mathcal{I}_n\, e^{F_{\mathcal{N}}}\,, \tag{5.80}$$

The $2 \times (n-1)$ delta functions sitting in front of the formula fix the mass parameters κ_i^2 and $\tilde{\kappa}_i^2$ of the external particles to be equal to the eigenvalue h_i under the group by which we are reducing. These come from the picture changing operators $\Xi, \tilde{\Xi}$, where the basis elements h_i are chosen to extract residues at given points. As in [85], when computing the path integral one finds that it localizes on the solution

$$\lambda_A^a(\sigma) = \sum_{i=1}^{n} \frac{u_i^a \epsilon_{iA}}{\sigma - \sigma_i}\,. \tag{5.81}$$

This way we can extract the residue in the delta functions in (5.78) on the support of the polarised measure that we describe below:

$$\text{Res}_{\sigma_i}(\lambda^2 - j^H) = \kappa_i^2 - h_i\,. \tag{5.82}$$

Here we write h_i to indicate the eigenvalue of the external state under the action of the element j^H. It is important to note that h_i is not the mass but rather a 'signed mass': we will refer to the mass as $m_i = |h_i|$. In general, the vertex operator will carry some representation of the symmetry group so that:

$$j^H(\sigma) \cdot V_i(\sigma_i) \sim \frac{h_i}{\sigma - \sigma_i} V_i(\sigma_i)\,. \tag{5.83}$$

More specifically, in the case of the Coulomb branch massive external states carry a factor $\mathfrak{m}_a j^a$ and j^H is an element of the current algebra. We will discuss this in more detail around (5.96). This way the delta function enforce the mass-shell condition for $n-1$ particles.

The integration measure is strongly reminiscent of the six and five dimensional one:

$$d\mu_n^{\text{pol}} := \frac{\prod_j d\sigma_j\, d^2 u_j\, d^2 v_j}{\text{vol}\,\text{SL}(2,\mathbb{C})_\sigma \times \text{SL}(2,\mathbb{C})_u} \prod_{i=1}^{n} \bar{\delta}^4\Big((u_i \lambda_A(\sigma_i)) - (v_i \kappa_{iA})\Big)\bar{\delta}\big((v_i \epsilon_i) - 1\big)\,, \tag{5.84}$$

localised on the solutions to the polarized scattering equations:

$$\mathcal{E}_{iA} = (u_i\lambda_A(\sigma_i)) - (v_i\kappa_{iA}) = \sum_{j\neq i} \frac{(u_iu_j)\epsilon_{jA}}{\sigma_i - \sigma_j} - (v_i\kappa_{iA}) = 0\,, \tag{5.85}$$

as described in detail in [1,2]. As shown in [5], these equations imply the massive scattering equations for the σ_i that we have seen arise in the symmetry reduced RNS models of section 5.3 and that were originally conjectured by Naculich [4] and Dolan & Goddard [3]:

$$\sum_{j\neq i} \frac{k_i\cdot k_j - h_ih_j}{\sigma_i - \sigma_j} = 0\,. \tag{5.86}$$

The $5n$ delta functions are used to localise the u_i, v_i, σ_i integrations but, because three of the us and three of the σs are already fixed by the gauge, overall six delta functions remain after integration. These impose conservation of the six dimensional momentum (5.54) and ultimately lead to the consistency of the mass assignments. Indeed they give $\kappa_1^2 = -\sum_{i=2}^n \kappa_i^2 = -\sum_{i=2}^n h_i = h_1$, where the last equality is given by charge conservation. As discussed in §5.3 this guarantees that the amplitudes vanish unless $\sum h_i = 0$.

As described in [5], we obtain the following integrands:

$$\mathcal{I}_n^{\text{BAS}} = \text{PT}(\alpha)\,\text{PT}(\beta)\,, \qquad \mathcal{I}_n^{\text{CB}} = \text{PT}(\alpha)\,\text{det}'H\,, \qquad \mathcal{I}_n^{Grav} = \text{det}'H\,\text{det}'\tilde{H}\,, \tag{5.87}$$

where $\text{PT}(\alpha)$ denotes the Parke-Taylor factor and the reduced determinant is obtained from the evaluation of the $\rho\tilde{\rho}$ system (c.f. [85]):

$$\text{det}'H := \frac{1}{(u_1u_2)}\,\text{det}\,H^{[12]}_{[12]}\,,$$

where, the $n\times n$ matrix H is defined by

$$H_{ij} = \frac{\epsilon_{iA}\epsilon_j^A}{\sigma_{ij}}, \qquad H_{ii} = -e_i^{AB}(\lambda_A\lambda_B)\,(\sigma_i)\,,$$

and the sub- and superscripts indicate that both the rows and the columns 1 and 2 have been removed.

The exponential factors in the supersymmetric plane wave give rise to the term

$$e^{F_{\mathcal{N}}} := \exp\left(\sum_{j<k} \frac{(u_j u_k) q_j \cdot q_k}{\sigma_j - \sigma_k} - \frac{1}{2}\sum_{j=1}^{n} (\xi_j v_j) q_j^2\right) . \tag{5.88}$$

All the dependence on the supermomenta is contained in this factor and when expanding it in different powers of q one can read off it the various component amplitudes as we will detail below. We note that, while in the reduction that leads to the Coulomb branch supersymmetry is preserved and all states in the original multiplets have the same mass, there are other ways of performing a reduction, such as the R-symmetry reduction described below in §5.6, that break the original supersymmetry and give rise to smaller supermultiplets. In these cases we can still read component amplitudes off this formula, but we should keep in mind that states in the same *higher dimensional* multiplet can have different masses.

5.5 Coulomb branch

We have claimed that the model we focused on in the previous sections describes the Coulomb branch of $\mathcal{N} = 4$ SYM. In this section we will justify our claim, showing that the Lagrangian theory can be defined via a symmetry reduction of 5d $N = 2$ SYM. This procedure imposes a specific dependence of fields on the extra dimension, which can be eliminated by a gauge transformation at the price of giving a vacuum expectation value to a scalar field, thus producing the more familiar formulation of this theory. We will begin by a review of the usual description of the Coulomb branch, show the equivalence to the model derived via symmetry reduction and identify the spectrum of this theory of massive particles. We will then implement the symmetry reduction in the worldsheet model for maximal super Yang-Mills in five dimensions and derive amplitude formulae.

Coulomb branch via VEV'd scalars The Coulomb branch of $\mathcal{N} = 4$ SYM with Lie algebra $\mathfrak{g}$ is usually described by assigning a vacuum expectation value to some of the scalar fields. The theory at the origin of the moduli space is a theory of massless particles describing a vector potential field A_μ, six real scalars Φ^a transforming in the **6** of the $SO(6)$ R-symmetry group and four Majorana spinors Ψ^I_A in the fundamental of $SU(4) \simeq SO(6)$. All the fields transform in the adjoint representation of the gauge group.

In the simplest case, a gauge group $U(N+M)$ is spontaneously broken to $U(N) \times U(M)$ by the vacuum configurations of some of the scalars, e.g.:

$$\langle \Phi^1 \rangle = iH \sim iv \cdot \mathrm{diag}(\mathbb{1}_N, 0_M) \qquad \langle \Phi^a \rangle = 0 \quad a \neq 6\,. \tag{5.89}$$

Writing the scalars as the antisymmetric product of two fundamentals of $SU(4)$, this is equivalent to:

$$\langle \Phi_{IJ} \rangle = \Omega_{IJ} H\,, \tag{5.90}$$

where Ω has two dimensional Levi-Civitas on the off diagonal blocks and zeros in the diagonal blocks. Then the theory on the Coulomb branch has a residual $\mathrm{Sp}(4) \subset SU(4)$ R-symmetry, preserving the bilinear form Ω.

Because the fields transform in the adjoint representation, they are represented by $(N+M) \times (N+M)$ matrices. Under this symmetry breaking, the $(N+M)^2$ generators of the original gauge group reduce to the generators for the residual $U(N)$ and $U(M)$, together with $2NM$ broken generators:

$$\mathrm{ad}_{N+M} \to (\mathrm{ad}_N, 1) \oplus (1, \mathrm{ad}_M) \oplus (\mathbf{N}, \bar{\mathbf{M}}) \oplus (\bar{\mathbf{N}}, \mathbf{M}) = \begin{pmatrix} A^{ab}_\mu & W^{a\dot{b}} \\ \bar{W}^{\dot{a}b} & A^{\dot{a}\dot{b}}_\mu \end{pmatrix}. \tag{5.91}$$

One can see the mass terms arise upon replacing $\Phi^I \to H + \phi^I$ in the Lagrangian of $\mathcal{N}=4$ SYM. The decomposition (5.91) of the adjoint under symmetry breaking and the form of H (5.89) tell us that the generators of the residual $U(N) \times U(M)$ symmetry remain massless, whereas the broken generators W and $\bar{W}$ aquire a mass proportional to v. The fermions and the scalars also live in the adjoint representation of the gauge group, and decompose similarly to the vector. In addition to this, however, they also transform non-trivially under the R-symmetry group, which is broken to $\mathrm{Sp}(4)$. Under this residual symmetry, the six scalars w transform in a $\mathbf{5}$ plus a singlet, consisting of the trace $\Omega_{IJ} w^{IJ}$. This component is absorbed by the gluons and becomes a polarization state of the massive spin-one field via the Higgs mechanism. Then the massive scalars are $w_{12}, w_{13}, w_{34}, w_{24}$ and the combination of w_{14} and w_{23} that is orthogonal to the longitudinal boson, i.e. $(w_{14} + w_{23})/\sqrt{2}$.

Spectrum The procedure outlined above leaves two types of states with respect to the color group. The first corresponds to elements $\mathfrak{t} \in \mathfrak{u}_N \times \mathfrak{u}_M$, that commute with H and therefore correspond to massless states. The second are the off-diagonal blocks consisting of elements $\mathfrak{m} \in \mathbb{C}^N \otimes (\mathbb{C}^M)^* \oplus \mathbb{C}^M \otimes (\mathbb{C}^N)^*$ for which $[H, \mathfrak{m}] = M^H_{\mathfrak{m}}\, \mathfrak{m}$ so that they define massive states with mass $|M^H_{\mathfrak{m}}|$.

Supersymmetry The action of the supercharges casts the massless states in a vector multiplet transforming in the adjoint of the residual gauge group:

$$\mathscr{F}^0 = (\phi_{IJ} = \phi_{[IJ]}\, , \Psi^I_\alpha, \tilde{\Psi}_{I\dot{\alpha}}\, , F_{\alpha\beta}, F_{\dot{\alpha}\dot{\beta}})\, . \tag{5.92}$$

For these multiplets the R-symmetry is enhanced to a full SU(4), so that the fundamental indices can no longer be raised and lowered. The multiplet contains the two familiar ± 1 helicity states of the massless spin-1, six real massless scalars ϕ_{IJ} and eight massless gluino states via the chiral parts of $\Psi^I_\alpha, \tilde{\Psi}_{I\dot{\alpha}}$.

The massive supermultiplets are the so-called $1/2$-BPS, ultrashort massive representations of $\mathcal{N} = 4$ with central extension $Z_{IJ} = 2M\Omega_{IJ}$, with $\mathrm{Sp}(\mathcal{N})$ R-symmetry, with skew form Ω_{IJ} and indices $I, J = 1, \ldots, \mathcal{N} = 4$. They are bifundamentals of $U(N) \times U(M)$, composed of a massive W-boson (3 bosonic d.o.f.) F_{AB}, five massive scalars ϕ_{IJ} and the fermionic partners, four massive Weyl-Majorana spinors Ψ^I_A (8 fermionic d.o.f.):

$$\mathscr{F}^m = (\phi_{IJ} = \phi_{[IJ]}, \Psi^A_I\, , F^{AB} = F^{(AB)})\, , \qquad \phi_{IJ}\Omega^{IJ} = 0\, . \tag{5.93}$$

Coulomb branch as a symmetry reduction We show now that there is an alternative derivation of this theory as a symmetry reduction of five dimensional maximally supersymmetric Yang-Mills. The gauge field $\mathcal{A}^{(4)}$ on the Coulomb branch of $\mathcal{N} = 4$ SYM can be embedded in a five dimensional gauge field, where the extra component is the vev'd scalar Φ^1, so that $\mathcal{A}^{(5)} = \mathcal{A}^{(4)} + (iH + \phi)\mathrm{d}x^4$. The field ϕ has vanishing vev and neither ϕ nor $\mathcal{A}^{(4)}$ have any dependence on the coordinate x^4, i.e. in this gauge $\partial_4\mathcal{A}^{(5)} = 0$.

Under a gauge transformation $\mathcal{U} = \exp(iHx^4)$, the connection transforms as

$$\mathcal{A}^{(5)'} = \mathcal{U}\mathcal{A}^{(5)}\mathcal{U}^\dagger + \mathcal{U}\partial_4\mathcal{U}^\dagger dx^4 = \mathcal{U}\mathcal{A}^{(5)}\mathcal{U}^\dagger - iH\mathrm{d}x^4 = \mathcal{U}\mathcal{A}^{(4)}\mathcal{U}^\dagger + \mathcal{U}\phi\mathcal{U}^\dagger\mathrm{d}x^4. \tag{5.94}$$

The gauge transformation has eliminated the non zero vev of the scalar. The price is the introduction of an explicit dependence on the x_4 coordinate:

$$\partial_4\mathcal{A}^{(5)'} = \partial_4\mathcal{U}\mathcal{A}^{(5)}\mathcal{U}^\dagger + \mathcal{U}\mathcal{A}^{(5)}\partial_4\mathcal{U}^\dagger = i[H,\mathcal{U}\mathcal{A}^{(5)}\mathcal{U}^\dagger] = i[H,\mathcal{A}^{(5)'}]\,. \tag{5.95}$$

This equation defines a symmetry reduction from $N = 2$ SYM in five dimensions to the Coulomb branch. In this description the mass terms are derived from the kinetic terms of the five-dimensional theory via (5.95) and the dependency of the fields on x_4 is fixed in such a way that this drops out of the action. This is sketched in appendix B.4.

From Lagrangian theory to ambitwistor model The model presented in [85] has been shown to reproduce tree level scattering amplitudes for 5d $N = 2$ SYM. The model we built in the previous section (5.60) is built as a symmetry reduction in the ambitwistor string. The elements $\mathfrak{t} \in \mathfrak{u}_N \times \mathfrak{u}_M$ and $\mathfrak{m} \in \mathbb{C}^N \otimes (\mathbb{C}^M)^* \oplus \mathbb{C}^M \otimes (\mathbb{C}^N)^*$ of the broken algebra described above define states in the worldsheet theory via the current algebra generators: massless $\mathfrak{t}\cdot j(\sigma)$ and massive $\mathfrak{m}\cdot j(\sigma)$. This is consistent with the mass assignments for these states as the OPEs (5.83) of the respective currents take the following form:

$$j^H(\sigma)\;\mathfrak{t}\cdot j(\tilde\sigma) \sim 0, \qquad\qquad j^H(\sigma)\;\mathfrak{m}\cdot j(\tilde\sigma) \sim \frac{M^H_{\mathfrak{m}}}{\sigma - \tilde\sigma}\mathfrak{m}\cdot j\,, \tag{5.96}$$

as by (5.64). We can thus identify vertex operators built from currents $\mathfrak{t}\cdot j$ with the massless vector multiplet transforming in the adjoint of the residual $U(N)\times U(M)$ gauge group, whereas vertex operators built from $\mathfrak{m}\cdot j$ describe the massive vector multiplet.

Expanded on the on-shell superspace we described in (5.67), the supermultiplets are organized as follows:

$$\begin{aligned}\tilde{\mathscr{F}}^{(m)}_{(\kappa,q)} &= F^{\epsilon\epsilon}(\kappa) + q_I \Psi^{\epsilon I}(\kappa) + q^2 F^{\epsilon\xi}(\kappa) + \frac{1}{2} q_I q_J \Phi^{IJ}(\kappa) + q^2 q_I \Psi^{\xi I}(\kappa) + q^4 F^{\xi\xi}(\kappa) \\ \tilde{\mathscr{F}}^{(0)}_{(\kappa,q)} &= g^h(\kappa) + q_I \Psi^{\epsilon I}(\kappa) + \frac{1}{2} q_I q_J \varphi^{IJ}(\kappa) + q^2 q_I \Psi^{\xi I}(\kappa) + q^4 g^{-h}(\kappa)\,,\end{aligned} \tag{5.97}$$

with $q^4 = (q_I q_J)(q^I q^J)$ and $(q^3)^I_a = \partial q^4/\partial q^a_I$. This tells us that the leading term in the amplitude involves n (massive or massless) bosons, which is what we expect from the way we constructed vertex operators. Component amplitudes can be read off the exponential factor in the superamplitude by matching the corresponding powers in the multiplets.

5.5.1 Amplitudes

The formula for the superamplitude of $\mathcal{N}=4$ SYM on the Coulomb branch is given by:

$$\mathcal{A}^{\text{CB}}_n(\alpha, \{k_i\}, \{M_i\}, \{q_i\}) = \int \mathrm{d}\mu_n \,\text{PT}(\alpha) \det{}' H^{\text{CB}}\, \mathrm{e}^{F_{\mathcal{N}}}\,. \tag{5.98}$$

We begin by considering amplitude involving only the leading vector component of either the massive or massless supermultiplet (5.97), i.e. $\mathcal{A}^{\text{CB}}_n(\alpha, \{k_i\}, \{M_i\}, \{q_i = 0\})$. This is either a massive W boson ($W^{\epsilon\epsilon} = \epsilon_a \epsilon_b W^{(ab)}$) or a gluon of helicity h dictated by the polarization ϵ_i. In this case the amplitude is simply:

$$\int \mathrm{d}\mu_n \,\text{PT}(\alpha) \det{}' H^{\text{CB}} = \delta\left(\sum_j k_j\right) \delta\left(\sum_j M_j\right) \sum_{i=1}^{(n-3)!} \text{PT}(\alpha) \det{}' H^{\text{CB}} \frac{(\sigma_{ij}\sigma_{jk}\sigma_{ki})^2}{\det \Phi^{ijk}_{ijk}}\,, \tag{5.99}$$

with:

$$|\Phi_{ij}| := |\partial E_i/\partial \sigma_j| = \begin{cases} \frac{(k_i+k_j)^2 - (M_i+M_j)^2}{\sigma_{ij}^2} & i \neq j \\ \sum_{k\neq i} \frac{(k_i+k_k)^2 - (M_i+M_k)^2}{\sigma_{ik}^2}, & i = j \end{cases} \tag{5.100}$$

Four vector bosons At four points, taking the ordering $\alpha = (1234)$, the expression above reads:

$$A_4 = \frac{1}{(u_1 u_2)(u_3 u_4)} \frac{\sigma_{12}\sigma_{34}}{((k_1+k_2)^2 - (M_1+M_2)^2)} \left(\epsilon_{1A} \epsilon_3^A \epsilon_{2B} \epsilon_4^B - \frac{\sigma_{31}\sigma_{42}}{\sigma_{41}\sigma_{32}} \epsilon_{1A} \epsilon_4^A \epsilon_{2B} \epsilon_3^B \right)\Bigg|_* \tag{5.101}$$

where $*$ indicates that we are evaluating the expression on the unique solution:

$$\sigma_1 = [(1,0)] \quad \sigma_2 = [(1,1)] \quad \sigma_3 = \left[\left(1, -\frac{((k_1+k_3)^2 - (M_1+M_3)^2)}{((k_1+k_2)^2 - (M_1+M_2)^2)}\right)\right] \quad \sigma_4 = [(0,1)] \tag{5.102}$$

$$(u_1u_2) = -\frac{\varepsilon^{ABCD} k_{1AB}\epsilon_{3C}\epsilon_{4D}}{\varepsilon^{ABCD}\epsilon_{1A}\epsilon_{2B}\epsilon_{3C}\epsilon_{4D}} \qquad (u_3u_4) = -\frac{\varepsilon^{ABCD} k_{3AB}\epsilon_{1C}\epsilon_{2D}}{\varepsilon^{ABCD}\epsilon_{1A}\epsilon_{2B}\epsilon_{3C}\epsilon_{4D}} \tag{5.103}$$

We obtain the generic formula for amplitudes involving gluons and W bosons[7]:

$$\mathcal{A}_4 = \frac{(\varepsilon^{ABCD}\epsilon_{1A}\epsilon_{2B}\epsilon_{3C}\epsilon_{4D})^2}{((k_1+k_2)^2 - (M_1+M_2)^2)((k_1+k_4)^2 - (M_1+M_4)^2)}\,. \tag{5.104}$$

From this expression one can extract four point amplitudes for specific states by assigning the correct kinematics, polarization and mass to the external particles. W-bosons have massive momenta, decomposed into massive spinor-heliciy variables, and generic polarization, together with the following assignment of mass parameters[8]:

$$M^W = m \qquad M^{\bar{W}} = -m \tag{5.105}$$

Gluons, on the other hand, have massless momenta, whose spinor helicity variables are embedded in the massive ones as explained in section (5.4.1). They have $M^g = 0$ polarization vectors are:

$$\epsilon_a^{+1} = (1,0) \qquad \epsilon_a^{-1} = (0,1)\,. \tag{5.106}$$

From the discussion in §5.3, it is clear that the amplitude vanishes unless $\sum_i M_i = 0$, i.e. unless W and $\bar{W}$ come in pairs, with any number of gluons. Let us consider for example the amplitude for a $W\bar{W}$ pair and two negative helicity gluons. From (5.104) we get:

$$A_4(W,\bar{W},g^-,g^-) = \epsilon_{1a}\epsilon_{1b}\epsilon_{2c}\epsilon_{2d}\frac{[1^a2^c][1^b2^d]\langle 34\rangle^2}{s_{12}(s_{14}-m^2)}\,, \tag{5.107}$$

[7]The details of this evaluation can be found in the derivation of the four point amplitude in $6d$ SYM, in section 5.2 of [2].

[8]One should keep in mind that M^H denotes the eigenvalue under the symmetry by which we are reducing and it corresponds to a *signed* mass parameter. Here m is the (positive) mass of the W-bosons.

where we have used $s_{ij} = (k_i + k_j)^2$. In line with previous work by the authors, the amplitudes obtained are contracted into arbitrary polarisation data and one can deduce the amplitude in the standard form with free little group indices by stripping it of the polarization 2-vectors ϵ_i:

$$A_4(W^{ab}, \bar{W}^{cd}, g^-, g^-) = \frac{[1^a 2^c][1^b 2^d]\langle 34\rangle^2}{s_{12}(s_{14} - m^2)} + \text{symmetrize (a,b);(c,d)}\,. \tag{5.108}$$

A particularly compact notation was introduced in [12], which we will employ from here on. Massive spinor-helicity variables are written in **bold**, to indicate that they carry completely symmetrized little group indices. We can then rewrite (5.107) one more time:

$$A_4(W, \bar{W}, g^-, g^-) = \frac{[\mathbf{12}]^2\langle 34\rangle^2}{s_{12}(s_{14} - m^2)}\,. \tag{5.109}$$

Similarly one can obtain expressions for different orderings and helicity assignments:

$$A_4(W, \bar{W}, g^-, g^+) = \frac{(\langle \mathbf{1}3\rangle[\mathbf{2}4] - \langle \mathbf{2}3\rangle[\mathbf{1}4])^2}{s_{12}(s_{14} - m^2)} \qquad A_4(W, g^-, \bar{W}, g^-) = \frac{[\mathbf{13}]^2\langle 24\rangle^2}{(s_{12} - m^2)(s_{14} - m^2)}\,.$$

Finally, for four W bosons:

$$A_4(W, \bar{W}, W, \bar{W}) = \frac{1}{s_{12}s_{14}} \cdot \Big(\langle \mathbf{12}\rangle[\mathbf{34}] + [\mathbf{12}]\langle \mathbf{34}\rangle - \langle \mathbf{13}\rangle[\mathbf{24}] - [\mathbf{13}]\langle \mathbf{24}\rangle + \langle \mathbf{14}\rangle[\mathbf{23}] + [\mathbf{14}]\langle \mathbf{23}\rangle\Big)^2$$

Similar expressions were obtained in [82], by dimensional reduction, and in [50, 71] by BCFW recursion.

We verified numerically that the amplitude formula reproduces at five point the n-point formula obtained by recursion in [50, 71, 72] for a pair of $W\bar{W}$ bosons and $n-2$ same-helicity gluons:

$$A_5(W, \bar{W}, g^+, g^+, g^+) = \frac{\langle \mathbf{12}\rangle^2[3|(m^2 + (k_4 + k_5 + k_1)(k_2 + k_3 + k_4))|5]}{\langle 34\rangle\langle 45\rangle((k_2 + k_3)^2 - m^2)((k_2 + k_3 + k_4)^2 - m^2)}\,. \tag{5.110}$$

The existence of compact expressions at n point give hopes that a simplification might occur in the scattering equations when this particular set of polarisation data is chosen, reminiscent of MHV in four dimensions. Despite our efforts we were unable to reproduce the n-point result as the formula appears to have support on all $(n-3)!$ solutions of the scattering equations. We will return to this point briefly in §5.7.

Massive scalars and gluons In order to obtain amplitudes for states further down the multiplet, one needs to consider the expansion in supermomenta (5.97) and take the corresponding derivatives of the exponential factor. The massive and massless supermultiplet have a similar structure, with the main distinction that in the massless multiplet the component $\sim q^2\Omega_I J$ is a scalar state. Let us consider for instance the amplitude for two massive scalars and two gluons. This component amplitude is extracted from the superamplitude as follows:

$$A_4\left(w_{IJ}, g, g, \bar{w}_{KL}\right) = \mathcal{A}_4 \frac{\partial}{\partial q_1^I}\frac{\partial}{\partial q_1^J}\frac{\partial}{\partial q_4^K}\frac{\partial}{\partial q_4^L} e^{F_{\mathcal{N}}}\bigg|_{q_i=0}\,, \tag{5.111}$$

where $\mathcal{A}_4$ is the leading amplitude (5.104) and the exponential is given by (5.88). The only term contributing is the quadratic one in the expansion of the exponential. Since for massive scalar states $\Omega_{IJ}w^{IJ}=0$, the terms $\sim q_j^2$ do not contribute and the derivatives bring down a factor of $U_{14}^2(\Omega_{IK}\Omega_{JL}+\Omega_{IL}\Omega_{JK})$ in front of the amplitude, with $U_{ij}=\frac{u_{ij}}{\sigma_{ij}}$. Evaluated on the solution to the scattering equations, this gives:

$$A_4\left(w_{IJ}, g, g, \bar{w}_{KL}\right) = (\Omega_{IK}\Omega_{JL}+\Omega_{IL}\Omega_{JK})\frac{(\varepsilon^{ABCD}\epsilon_{ab}\kappa^a_{1A}\kappa^b_{1B}\epsilon_{2C}\epsilon_{3D})^2}{(s_{12}-m^2)s_{14}}\,. \tag{5.112}$$

We can now evaluate the amplitude for different helicity assignments:

$$\begin{aligned} A_4\left(w_{IJ}, g^+, g^-, \bar{w}_{KL}\right) &= (\Omega_{IK}\Omega_{JL}+\Omega_{IL}\Omega_{JK})\frac{\langle 3|k_1|2]^2}{(s_{12}-m^2)s_{14}}\,,\\ A_4\left(w_{IJ}, g^+, g^+, \bar{w}_{KL}\right) &= (\Omega_{IK}\Omega_{JL}+\Omega_{IL}\Omega_{JK})\frac{m^2[23]^2}{(s_{12}-m^2)s_{14}}\,. \end{aligned} \tag{5.113}$$

These expressions match the results obtained in [12, 162].

Massive quarks We can perform the symmetry reduction to generate the symmetry breaking $SU(N+1)\to SU(N)\times U(1)$. From the discussion in the previous section, we know that the massive states are in the fundamental of $SU(N)$. At tree level, when looking at amplitudes involving only gluons and massive fermions, the truncation of the theory is consistent with the standard model description of massive quarks in QCD. We can generate amplitudes that match the results in the literature [72]:

$$A_4(\psi,\bar{\psi},g^+,g^+) = m\frac{\langle \mathbf{12}\rangle[34]^2}{s_{12}(s_{14}-m^2)} \qquad A_4(\psi,\bar{\psi},g^-,g^+) = \frac{\langle 3|k_1|4]([\mathbf{1}4]\langle \mathbf{2}3\rangle - [\mathbf{2}4]\langle \mathbf{13}\rangle)}{s_{12}(s_{14}-m^2)}\,,$$

5.5.2 Supersymmetry "Ward identities"

At n-points, while we cannot simplify the formula further, we can establish relations between the different component amplitudes for one pair of massive particles and all other particles gluons of positive (negative) helicity. Indeed, for this specific configuration the α ($\dot{\alpha}$) component of the polarized scattering equations is particularly simple because only the massive particles contribute to it, so it decouples from the rest of the system:

$$U_{12}\epsilon_{2\alpha} = (v_1\kappa_{1\alpha}) \qquad U_{12} = \frac{m}{\langle 12\rangle} \tag{5.114}$$

We know that the component amplitudes are related for example by:

$$A_n(\psi,\bar{\psi},g^+,\dots g^+) = A_n(W,\bar{W},g^+,\dots g^+)\partial_{q_1}\partial_{q_2}\exp(\mathcal{F}_{\mathcal{N}}) = U_{12}A_n(W,\bar{W},g^+,\dots g^+)$$

This identification follows from the expansion of the supermultiplets on on-shell superspace (5.97). This establishes a supersymmetry Ward identity:

$$A_n(\psi,\bar{\psi},g^+,\dots g^+) = \frac{m}{\langle 12\rangle}A_n(W,\bar{W},g^+,\dots g^+)\,, \tag{5.115}$$

which is consistent with the n-point formulae obtained in [162] and [72]. Similar relations hold for scalars:

$$A_n(w,\bar{w},g^+,\dots g^+) = \Big(\frac{m}{\langle 12\rangle}\Big)^2 A_n(W,\bar{W},g^+,\dots g^+)\,, \tag{5.116}$$

which can be compared to [157]. We can see this coming from the requirement of supersymmetry by writing:

$$\mathcal{A}_n = A_n(W, \bar{W}, g^+, \ldots g^+)\big(1 + \sum_{j<k} U_{jk} q_j \cdot q_k - \frac{1}{2}\sum_j (\xi_j v_j) q_j^2 + \mathcal{O}(q^4)\big)\,, \tag{5.117}$$

for a supersymmetric amplitude involving two massive and $n-2$ massless states. Then by the definition of on-shell superspace (5.67):

$$\begin{aligned}[Q_{AI}, \mathcal{A}_n] = \sum_i [Q_{iAI}, \mathcal{A}_n] = A_n(W, \bar{W}, g^+, \ldots g^+) \sum_i \big((\kappa_{iA}\xi_i) q_{iI} - \epsilon_{iA}(\xi_i v_i) q_{iI} \\ + \sum_{j\neq i} \epsilon_{jA} U_{ji} q_{jI} + \mathcal{O}(q^3)\big)\,.\end{aligned} \tag{5.118}$$

Then at order $\mathcal{O}(q_1)$, supersymmetry imposes:

$$(\kappa_{1A}\xi_1) + \sum_{k\neq 1} U_{k1}\epsilon_{kA} - (\xi_1 v_1)\epsilon_{1A} = 0\,, \tag{5.119}$$

Taking the α components and contracting into ϵ_1^α, we see that this is equivalent to the polarised scattering equation (5.114), using that ξ is normalised against ϵ as $(\epsilon_i \xi_i) = 1$.

5.6 R-symmetry reduction

The machinery of symmetry reduction only requires the choice of a symmetry group. It can thus be applied to obtain less familiar theories of massive particles. We illustrate this here by performing a reduction of maximally supersymmetric gauge and gravity theories via a generator of the R-symmetry. This principally affects the scalars and spinors that are in nontrivial R-symmetry representations, thus producing a theory with less than maximal supersymmetry. This type of reductions are known in the supergravity literature as (Cramer)-Scherk-Schwarz reductions and have been shown to generate gauged supergravities in four dimensions. We will present a few examples of these theories in our formalism and point out a peculiar instance of double copy at the level of the worldsheet.

5.6.1 Reducing SYM

We want to study the reduction of 5d $N = 2$ SYM to four dimensions, obtained by associating one component of momentum to charges under the $R-$symmetry group:

$$\frac{\partial}{\partial x^4}(A_m, \Phi_{IJ}, \Psi^I_A) = (0, H^K_{[I}\Phi_{J]K}, H^K_I\Psi_{AK})\,, \tag{5.120}$$

where H^K_I is in the fundamental representation of $\mathrm{Sp}(4)_R$. Expanding the kinetic terms under (5.120), one obtains mass terms for the fermions and the scalars as well as some interaction terms. The reduced theory contains one massless vector A_μ, two massless scalars, four massive scalars and four massive Majorana fermions. Details of the reduction are given in appendix B.5.

Spectrum For $\mathrm{Sp}(4)$, in the Cartan-Weyl basis we can write a linear combination of the two Cartan elements in the fundamental as:

$$H = \mathrm{diag}(m_1, m_2, -m_1, -m_2)\,, \tag{5.121}$$

where m_i are the parameters of the linear combination. In this basis the symplectic matrix Ω is

$$\Omega = \begin{pmatrix} 0 & \mathbb{1}_2 \\ -\mathbb{1}_2 & 0 \end{pmatrix} \tag{5.122}$$

Then we can take as independent scalars $\Phi_{12}, \Phi_{13}, \Phi_{23}, \Phi_{24}, \Phi_{34}$, with $\Phi_{13} = -\Phi_{24}$. Expanding the mass term for the scalars we find that Φ_{12} and Φ_{34} have mass $|m_1 + m_2|$ while Φ_{14} and Φ_{23} have mass $|m_1 - m_2|$. Similarly, we find that $\Psi_1 + i\Psi_3$ is a massive Dirac spinor of mass $|m_1|$, while $\Psi_2 + i\Psi_4$ has mass $|m_2|$. The scalars Φ_{13} and ϕ, coming from the extra component of the five dimensional vector, remain massless.

We note that for the choice $m_1 = m = m_2$ one obtains a theory of one massless vector, four massless scalars, four massive fermions of mass m and two massive scalars of mass $2m$. We will refer to this theory as $\mathcal{N} = 0^*$ SYM.

Supersymmetry It is clear from these mass assignments that the original $SO(5)$ R-symmetry is broken. Counting on shell degrees of freedom one sees that there can be no residual supersymmetry in this theory for both $m_{1,2}$ non vanishing. However it is possible to preserve $\mathcal{N}=2$ supersymmetry by performing the reduction only along one of the two directions in the Cartan, i.e. taking for example $m_2=0$, $m_1=m$. In this case half of the fermions are massless and we can group the on shell degrees of freedom into one massive ultrashort matter multiplet Ψ^m and one massless vector multiplet $\mathcal{V}^0$, with:

$$\begin{aligned}\Psi^m \quad &= \quad \text{1 massive Dirac fermion } \Psi_1^B + i\Psi_3^B,\\ &\quad\quad \text{2 massive complex scalars } \Phi_{12}+i\Phi_{23} \text{ and } \Phi_{14}+i\Phi_{34}\end{aligned}$$

$$\begin{aligned}\mathcal{V}^0 \quad &= \quad \text{1 vector } (F_{\alpha\beta}, F_{\dot\alpha\dot\beta})\\ &\quad\quad \text{2 Weyl fermions } (\Psi_2,\Psi_4)\\ &\quad\quad \text{1 complex scalar } \phi + i\Phi_{24}\,,\end{aligned}$$

where we have named $F_{\alpha\beta}, F_{\dot\alpha\dot\beta}$ the vector components of the five-dimensional vector in 4d and ϕ the scalar component. One can then verify that:

$$\begin{aligned}&\left[Q_{AJ},P^5\right](\Psi_1^B+i\Psi_3^B,\Phi_{12}+i\Phi_{23},\Phi_{14}+i\Phi_{34})=0\\ &\left[Q_{AJ},P^5\right](F_{\alpha\beta},F_{\dot\alpha\dot\beta},\Psi_2,\Psi_4,\phi+i\Phi_{24})=0\,,\end{aligned} \tag{5.123}$$

for $J=2,3$ with the choice of basis in (5.121) and $m_2=0$. Then the states of the reduced theory sit in $\mathcal{N}=2$ supermultiplets generated in this basis by the action of $Q_{2,3}$, whereas the two remaining supercharges of the higher dimensional theory bring us from massless to massive states and viceversa.

We recognize the reduced theory as the $\mathcal{N}=2^*$ theory that one obtains by giving a mass to the adjoint $\mathcal{N}=2$ hypermultiplet sitting inside the $\mathcal{N}=4$ massless vector. More generally, for a reduction with p non-vanishing mass parameters we obtain a theory with $\mathcal{N}=4-2p$ residual supersymmetries.

Worldsheet model The operators:

$$J_{IJ} = \eta_{(I} \cdot \eta_{J)} \tag{5.124}$$

are generators for the $\mathrm{Sp}(4)$ R-symmetry acting on worldsheet operators. We can then take the current:

$$j_H = \eta_{(I} \cdot \eta_{J)} H^I_K \Omega^{JK} = m_1 \eta_{(1} \cdot \eta_{3)} + m_2 \eta_{(2} \cdot \eta_{4)} \,, \tag{5.125}$$

to construct an ambitwistor string model such as the ones described in section 5.4 for the class of theories above. This element spans the Cartan subalgebra for different values of $m_{1,2}$ and does not spoil the closure of the algebra of constraints [9].

One can then read the mass assignment for various states in the multiplet by considering the OPE:

$$j_H(\sigma) \cdot \Phi_{\kappa,q}(\sigma_i) \sim \frac{1}{\sigma - \sigma_i} \sum_{s=1}^{2} m_s \big(q_{i;s} \frac{\partial}{\partial q_{i;s}} - q_{i;s+2} \frac{\partial}{\partial q_{i;s+2}} \big) \Phi_{\kappa,q} \tag{5.127}$$

where $\Phi_{\kappa,q}$ is the supersymmetric plane wave representative as in (5.68). Following the discussion around (5.83), this gauging assigns masses:

$$m_1 \big(q_1 \frac{\partial}{\partial q_1} - q_3 \frac{\partial}{\partial q_3} \big) + m_2 \big(q_2 \frac{\partial}{\partial q_2} - q_4 \frac{\partial}{\partial q_4} \big) \,, \tag{5.128}$$

which corresponds to the spectrum we derived above from the lagrangian mass terms.

Formulae From these models we obtain amplitude formulae for theories with various amounts of residual supersymmetry, such as the $\mathcal{N} = 2^*$ theory discussed above. The peculiarity in these expressions is that the full superamplitude remains expressed as an expansion in the original (now broken) $\mathcal{N} = 4$ superspace. One can then extract amplitudes involving the desired massive or massless particles by reading them off the appropriate coefficients. As we have seen above, the mass operator contains derivatives with respect to specific components of supermomenta, so massive and massless multiplets of the reduced theory can be embedded together in

[9] The only non-trivial OPE is:

$$(\lambda^2 - j_H) \circ (\mathcal{Z}^{(a} \cdot \mathcal{W}^{b)}) \sim 0 \tag{5.126}$$

the larger (broken) superspace, at the price of introducing derivatives in the scattering equations.

5.6.2 CSS gauged supergravities

The same kind of symmetry reduction can be carried out on the gravitational model, exploiting the $\mathrm{Sp}(8)$ R-symmetry of five dimensional maximal supergravity. In the supergravity literature this procedure goes by the name of CSS reduction, after Cremmer, Scherk and Schwarz [159, 160], as outlined in §5.1. It has been shown [163] that the result of a CSS reduction of five dimensional supergravity by an element H of the $E_{6(6)}$ is a gauged supergravity in four dimensions. We will then equivalently refer to such models as CSS reductions or CSS gaugings. When taking the element H in the maximal compact subgroup $\mathrm{USp}(8)$ of $E_{6(6)}$, the gauge group is called 'flat' and the theory has Minkowski vacua. It depends on four independent mass parameters, corresponding to the four elements of the Cartan subgroup of $\mathrm{Sp}(8)$. These fix the scale of the spontaneous supersymmetry breaking, which produces a theory with residual $\mathcal{N} = 8 - 2p$ supersymmetry, with p the number of non-vanishing mass parameters.

Maximal supergravity in five dimensions has a gravity multiplet whose content is summarized in table 5.1. All particles but the graviton transform non trivially as $\mathrm{Sp}(8)$ antisymmetric

Spin	dof	$\mathrm{Sp}(4)_R$
2	5_B	**1**
$\frac{3}{2}$	$8 \times 4_F$	**8**
1	$27 \times 3_B$	**27**
$\frac{1}{2}$	$48 \times 2_F$	**48**
0	$42 \times 1_B$	**42**

Table 5.1: Degrees of freedom and R-symmetry representation for the states in the gravity multiplet in 5d $N = 4$ supergravity.

traceless[10] tensors. As a consequence, under an R-symmetry reduction

$$\partial_4 \Phi_{I_1 I_2 \ldots I_k} = H^J_{[I_1} \Phi_{I_2 \ldots I_k] J} \tag{5.129}$$

[10]This is intended as $\Phi_{I_1 I_2 \ldots I_k} \Omega^{I_i I_j} = 0$.

some of the gravitinos, graviphotons, gravi-photinos and scalars aquire a mass. For $\mathrm{Sp}(8)$, choosing the same basis as above for $\mathrm{Sp}(4)$, we can write a generic element of the four dimensional Cartan subalgebra as:

$$H = \mathrm{diag}(m_1, m_2, m_3, m_4, -m_1, -m_2, -m_3, -m_4)\,. \tag{5.130}$$

Then, for a $k-$index tensor, the mass assignment is:

$$|\sum_{i=1}^{k} h_{I_i}|\,. \tag{5.131}$$

These theories are described by worldsheet models of the form:

$$S = \int_\Sigma \mathcal{Z}^a \cdot \bar{\partial}_e \mathcal{Z}_a + A_{ab}\mathcal{Z}^a \cdot \mathcal{Z}^b + a\left(\lambda^2 - j^{H_R}\right) + \tilde{a}\left(\tilde{\lambda}^2 - j^{H_R}\right) + S_{\rho_1} + S_{\rho_2}\,, \tag{5.132}$$

where the $\mathrm{Sp}(8)$ current is given by:

$$j^{H_R} = \eta_I \cdot \eta_J H^I_K \Omega^{JK}\,. \tag{5.133}$$

Similarly to the massive gauge models of section 5.6.1, this current generates masses only for part of the original supermultiplet by acting as a derivative in the supermomenta:

$$j^{H_R}(\sigma)\cdot\Phi_{\kappa,q}(\sigma_i) \sim \frac{1}{\sigma-\sigma_i}\sum_{s=1}^{4} m_s\big(q_{i;s}\frac{\partial}{\partial q_{i;s}} - q_{i;s+4}\frac{\partial}{\partial q_{i;s+4}}\big)\Phi_{\kappa,q} =: \frac{\mathcal{D}q_i}{\sigma-\sigma_i}\Phi_{\kappa,q}\,, \tag{5.134}$$

As a consequence, the formulae we obtain are superamplitudes containing both massive and massless component amplitudes. It is only once we specify the external states that we can talk about the scattering equations that localise the correlator being massive or massless:

$$\delta\Big(\sum_{j=1}^{n}\frac{k_i\cdot k_j - \mathcal{D}q_i\mathcal{D}q_j}{\sigma_{ij}}\Big)\mathrm{e}^{F_\mathcal{N}}\,. \tag{5.135}$$

CSS gauging with $\mathcal{N}=6$. We begin by considering the case $m_1 = m$, $m_{2,3,4} = 0$. Taking into account the tracelessness conditions, we summarize below the massive and massless spectrum of the reduced theory. We find that the reduction preserves $\mathcal{N} = 6$ supersymmetry and the states make up one massless gravity multiplet and one massive ultrashort gravitino multiplet. This corresponds to a model with $j^H = m\,\eta_{(1} \cdot \eta_{5)}$.

Spin	**Mass**	
	0	m
2	2_B	
$\frac{3}{2}$	$6 \times 2_F$	$2 \times 4_F$
1	$16 \times 2_B$	$12 \times 3_B$
$\frac{1}{2}$	$26 \times 2_F$	$28 \times 2_F$
0	$30 \times 1_B$	$28 \times 1_B$
	$\mathcal{H}^0_{\mathcal{N}=6}$	$2\mathcal{X}^m_{\mathcal{N}=6}$

Table 5.2: Spectrum of the reduced theory under the choice $H = \mathrm{diag}(m, 0, 0, 0, -m, 0, 0, 0)$.

CSS gauging with $\mathcal{N} = 4$. By the same procedure we can obtain theories with residual $\mathcal{N} = 4$ supersymmetry. Taking $m_1 = m = -m_2$, $m_{3,4} = 0$, we obtain one massless graviton multiplet, four massless vector multiplets, four massive gravitino multiplets with mass m and two massive vector multiplets of mass $2m$. All massive multiplets are ultrashort representations. The corresponding model has $j^H = m\,(\eta_{(1} \cdot \eta_{5)} - \eta_{(2} \cdot \eta_{6)})$.

Spin	**Mass**		
	0	m	$2m$
2	2_B		
$\frac{3}{2}$	$4 \times 2_F$	$4 \times 4_F$	
1	$10 \times 2_B$	$16 \times 3_B$	$2 \times 3_B$
$\frac{1}{2}$	$20 \times 2_F$	$24 \times 2_F$	$8 \times 2_F$
0	$26 \times 1_B$	$16 \times 1_B$	$10 \times 1_B$
	$\mathcal{H}^0_{\mathcal{N}=4} \oplus 4\mathcal{V}^0$	$4\mathcal{X}^m_{\mathcal{N}=4}$	$2\mathcal{V}^{2m}$

Table 5.3: Spectrum of the reduced theory under the choice $H = \mathrm{diag}(m, -m, 0, 0, -m, m, 0, 0)$.

5.6.3 Double copy

The gauged supergravities described above have been the object of recent work by Chiodaroli, Günaydin, Johansson and Roiban [161,164,165], who have studied how they can be obtained as double copies of massive gauge theories. Worldsheet models in the ambitwistor string and the formulae they produce have an explicit double copy structure, whereby one chooses a left and a right systems, which can be combined in any pairing. Having constructed models for gauged supergravities, we observe an instance of double copy where one supergravity theory can be arise from several different left/right pairs. Here below we describe this novel 'worldsheet' double copy, we illustrate it with examples and we relate it to the spacetime double copy of [161].

On the worldsheet, we establish a prescription for double copying gauge theory models. We start with two models that are composed of one set of worldsheet fermions S_ρ and one current algebra S_j. We also consider the η system to come as part of the matter action[11] and to incorporate supersymmetry breaking terms such as the ones in the R-symmetry reductions discussed above. Altogether the models take the form:[12]

$$S^{\eta}_{\mathcal{N}}(H^R) + S_\rho + S_j + \int_\Sigma a(\lambda^2 - j^H_{CB}) + \tilde{a}(\tilde{\lambda}^2 - j^H_{CB})\,, \tag{5.136}$$

where j^H_{CB} is associated to an element of the color group as before for the Coulomb Branch. Here we take:

$$S^{\eta}_{\mathcal{N}}(H^R) = \int_\Sigma \eta_i \cdot \bar{\partial}\eta^i + A_{ab}\eta^a_i\eta^{bi} + a(\lambda^2 - \eta_i \cdot \eta_j (H^R)^i_k \Omega^{jk}) + \tilde{a}(\tilde{\lambda}^2 - \eta_i \cdot \eta_j (H^R)^i_k \Omega^{jk})\,,$$

so that $\mathcal{N} = 4-2p$, with p the number of non-vanishing mass parameters, indicates the amount of residual supersymmetry after SSB and the R-symmetry indices run up to 4.

From two models of this type we can form a gravitational model by discarding the current algebrae (and associated Coulomb-Branch-like gaugings) and combining the worldsheet

[11] This is justified by the criticality of the models.

[12] When double copying two models such as this, we discard the greyed out part of the action and combine the two fermionic systems as described below.

fermion systems to write:

$$S^{\eta}_{\mathcal{N}_1+\mathcal{N}_2}\Big(\begin{pmatrix} H_1^R & 0 \\ 0 & H_2^R \end{pmatrix}\Big) + S_{\rho_1} + S_{\rho_2}\,. \tag{5.137}$$

We identify $\eta^I = (\eta^i_1, \eta^i_2)$ as well as $\Omega_{IJ} = \begin{pmatrix} \Omega_1^R & 0 \\ 0 & \Omega_2^R \end{pmatrix}$, so that schematically we can write:

$$\mathrm{SYM}(H_1) \otimes \mathrm{SYM}(H_2) \sim \mathrm{sugra}(H_1 \oplus H_2) \tag{5.138}$$

The charge associated to the symmetry reduction is the sum of the charges of the two gauge theories, through $j^H_{\mathrm{sugra}} = j^{H_1}_{SYM} + j^{H_2}_{SYM}$. This indicates that on the worldsheet the double copy as prescribed here doesn't need the mass spectra of the *left* and *right* theories to match. There is in fact a lot of different pairings that should in principle produce the same double copy. Not only can we have different values of $\mathcal{N}_1$ and $\mathcal{N}_2$ summing to $\mathcal{N}$, but we are also free to add any Coulomb-branch-like reductions to both gauge models, see Table 5.4.

As an example, let us consider the CSS gauged supergravity with residual $\mathcal{N} = 6$ supersymmetry. Here we have no choice but to take one of the two models to be $\mathcal{N} = 4$ SYM and the other the $\mathcal{N} = 2^*$ massive theory of section 5.6.1. Both of these models are in principle free to be on the Coulomb branch. The multiplets and respective R-symmetry charges of the left and right theories combine as follows:

$$\mathcal{V}^0 \otimes \mathcal{V}^0 \to \mathcal{H}^0 \qquad \mathcal{V}^0 \otimes \Psi^{\pm m} \to \mathcal{X}^{\pm m}\,, \tag{5.139}$$

so that overall they double copy to $\mathcal{N} = 6^*$ supergravity with one gravity multiplet and two massive gravitino multiplets:

$$\underbrace{\mathcal{V}^0}_{\mathcal{N}=4} \otimes \underbrace{(\mathcal{V}^0 \oplus \Psi^{+m} \oplus \Psi^{-m})}_{\mathcal{N}=2^*} \to \underbrace{(\mathcal{H}^0 \oplus \mathcal{X}^{+m} \oplus \mathcal{X}^{-m})}_{\mathcal{N}=6^*}\,. \tag{5.140}$$

We insist on the fact that this is a double copy prescription *on the worldsheet*. In order to understand what this implies for spacetime amplitudes, it is easier to consider a component amplitude instead of the full superamplitude, so that the equations (5.135) become proper massive or massless scattering equations. We pick a component amplitude involving two massive gravitinos and all massless top state gravitons, so that the kinematic variables involved in the scattering equations are massive for $i = 1, 2$ and massless for the rest. We obtain the desired amplitude on spacetime by evaluating the correlator on the solutions to the scattering equations:

$$\sum_{\{\sigma_i,u_i,v_i\}} U_{12}\frac{(\sigma_{ij}\sigma_{jk}\sigma_{ki})^2}{\det\Phi^{ijk}_{ijk}}\det{}'H\det{}'H\,, \tag{5.141}$$

where the factor U_{12} comes from the supersymmetry exponential factor. This expression still presents a double copy structure, where each of the two reduced determinants can be taken as a contribution from a gauge theory model. However, both sub-integrands are evaluated on the solutions to scattering equations that are massive in particles 1 and 2, so that the mass spectrum has to match between the two gauge theories, contrary to the worldsheet double copy. In particular, here a spin $3/2$ massive state has to come as a double copy of a massive spin $1/2$ with a *massive* spin 1. Repeating the same reasoning with the rest of the states, we find that *on spacetime* the amplitude comes from double copying:

$$\underbrace{(\mathcal{V}^0\oplus\mathcal{W}^m)}_{\text{CB }\mathcal{N}=4}\otimes\underbrace{(\mathcal{V}^0\oplus\Psi^{+m}\oplus\Psi^{-m})}_{\mathcal{N}=2^*}\to\underbrace{(\mathcal{H}^0\oplus\mathcal{X}^{+m}\oplus\mathcal{X}^{-m})}_{\mathcal{N}=6^*}\,. \tag{5.142}$$

That is, the massive scattering equations require the $\mathcal{N} = 4$ theory to be on the Coulomb branch. Then evaluating the correlator on spacetime selects one pair of theories out of all the candidates for the worldsheet double copy. This result corresponds to what was observed in [161] using BCJ numerators.

This phenomenon is even more explicit in the case of the $\mathcal{N} = 4^*$ CSS supergravity. Here the worldsheet double copy allows both:

$$\underbrace{(\mathcal{V}^0\oplus\Psi^{+m}\oplus\Psi^{-m})}_{\mathcal{N}=2^*}\otimes\underbrace{(\mathcal{V}^0\oplus\Psi^{+m}\oplus\Psi^{-m})}_{\mathcal{N}=2^*}\to\underbrace{(\mathcal{H}^0\oplus4\mathcal{V}^0\oplus4\mathcal{X}^m\oplus2\mathcal{V}^{2m})}_{\mathcal{N}=4^*}\,. \tag{5.143}$$

and

$$\underbrace{(\mathcal{V}^0 \oplus \mathcal{W}^m \oplus \mathcal{W}^{2m})}_{\text{CB } \mathcal{N}=4} \otimes \underbrace{(4\psi^m \oplus A^0 \oplus 2\phi^{2m} \oplus 4\phi^0)}_{\mathcal{N}=0^*} \to \underbrace{(\mathcal{H}^0 \oplus 4\mathcal{V}^0 \oplus 4\mathcal{X}^m \oplus 2\mathcal{V}^{2m})}_{\mathcal{N}=4^*} . \tag{5.144}$$

where $\mathcal{N} = 0^*$ is an R-symmetry reduction of Super-Yang Mills with no residual supersymmetry as described in section 5.6.1. On spacetime, on the other hand, double copying two $\mathcal{N} = 2^*$ we couldn't possibly produce states of mass $2m$ because of the requirement of mass matching. The only way we can obtain the desired spectrum is by double copying the $\mathcal{N} = 0^*$ with $\mathcal{N} = 4$ on the Coulomb branch with color symmetry breaking pattern $SU(3N) \to SU(N) \times SU(N) \times SU(N)$. We expect this to hold in the formalism of [161].

Left	**Right**			
	$\mathcal{N} = 2^*$	$\mathcal{N} = 2^*$ on CB	$\mathcal{N} = 4$	$\mathcal{N} = 4$ on CB
$\mathcal{N} = 0^*$	$\mathcal{N} = 2^*$	$\mathcal{N} = 2^*$	$\mathcal{N} = 4^*$	$\boldsymbol{\mathcal{N} = 4^*}$
$\mathcal{N} = 0^*$ on CB	$\mathcal{N} = 2^*$	$\mathcal{N} = 2^*$	$\mathcal{N} = 4^*$	$\mathcal{N} = 4^*$
$\mathcal{N} = 2^*$	$\mathcal{N} = 4^*$	$\mathcal{N} = 4^*$	$\mathcal{N} = 6$	$\mathcal{N} = 6$
$\mathcal{N} = 2^*$ on CB	$\mathcal{N} = 4^*$	$\mathcal{N} = 4^*$	$\mathcal{N} = 6$	$\boldsymbol{\mathcal{N} = 6}$

Table 5.4: Double copy on the worldsheet. Within each coloured block, all resulting CSS supergravities are the same, so multiple left/right gauge theories double copy to the same supergravity on the world-sheet. The (unique) space-time double copy is highlighted in bold-face.

5.7 Summary and discussion

In this chapter we have shown how symmetry reduction can be implemented in the ambitwistor string to obtain models and amplitude formulae involving massive particles. This opens many possibilities in a framework that seemed until recently to be intrinsically massless. The RNS ambitwistor string of §5.3 is a solid foundation for the factorisation properties of the amplitudes formulae in all models, both vectorial or twistorial, that one can obtain as symmetry reductions. We have shown how the two-twistor string of chapter 4 gives a rich ground to develop new models from maximally supersymmetric theories in five dimensions. These include both Coulomb-branch-type theories and more unusual CSS reductions.

From the path integral of these models, we have arrived at the compact formulae (5.80), supplemented by (5.84) and (5.87), supported on a massive version (5.85) of the polarised scattering equations and with manifest supersymmetry for appropriate gauge and gravity theories including massive particles. Like all twistor-string, CHY and ambitwistor-string amplitude formulae, all the integrations are saturated against delta functions so that these are really residue formulae summing contributions from the $(n-3)!$ solutions to a massive extension of the scattering equations discussed further below. As shown in chapter 3, the extra data in the polarised extension is uniquely obtained by linear equations on the support of these scattering equations and the amplitude formulae are linear in the polarization data.

Contrary to the massless four-dimensional formulae of [83], in which the double copy properties are hidden in the measure, the expressions derived here present the standard structure with two half integrands that can be combined to form amplitudes for scalars, spin-one and spin-two particles as in the CHY formulae and corresponding RNS models of [42, 43]. In the context of R-symmetry reductions we presented a novel instance of worldsheet double copy between gauge theories with massive matter and various degrees of supersymmetry and gauged supergravities.

To conclude this chapter, we will give an overview of open research directions.

Reductions along several dimensions and 'spectrum' of accessible theories While here we have only considered reductions from $(d+1)$ to d dimensions, the formalism is expected to extend to more complicated reductions from $(d+M)$ dimensions. In the twistorial models of chapters 4 and 5 we are limited in this regard. While in principle we could perform reductions along two extra dimensions (i.e. coming down from six), we have mentioned that the six dimensional models of [85] are not consistent in their present state. One way of circumventing this issue would be to start with the ten dimensional pure spinor model of [166,167]. Nevertheless, in the RNS models it is clear how one should proceed — and in fact we have written the discussion of §5.3 in a notation that automatically extends to the more general case where κ_i is an $M-$dimensional vector. Reductions from higher dimensions could be either successive circle reductions or even more complicated non-abelian patterns dictated by a choice of compact manifold.

At the level of the formulae one might hope to obtain from new models of this type, the most appealing feature would be the additional freedom in the assignment of masses. Because we have seen that symmetry reduction relates masses to charges under a symmetry group of the original theory, all the formulae we obtained had to satisfy 'signed mass conservation'. What we really were saying was that symmetry reduction relates *internal momentum* to charges, so that when this is an M dimensional vector its norm is equal to the mass. Charge conservation constrains each of its components but the norm is overall less constrained. For instance we could have a massless momentum k in d dimensions with an internal momentum that has several non-vanishing components. This allows kinematic configurations with odd numbers of external massive particles (such as the four-point amplitudes considered in [12]) or even all massless external kinematics with propagating massive states such as the massive graviton exchange of [168]. We outline an example in appendix B.6.

As the models stand, we have only performed symmetry reductions along the color and R symmetry generators. While coming down from more than one dimension we would have the choice to gauge larger subgroups, we could also explore whether other types of symmetries could be exploited, for instance to generate massive spin-2 particles. Following this thought further, we would like to know what is the 'spectrum' of theories that we are able to describe by symmetry reduction and which remain unattainable — together with this should also come a better understanding of the physical relevance of the theory we can represent.

R-symmetry and double copy In section 5.6 we have presented symmetry reductions performed along the $R-$symmetry generators in maximally supersymmetric Yang-Mills and gravity. For gauged supergravities obtained as (CSS) symmetry reductions, we have given a novel form of massive double copy on the worldsheet. Going back to the table 5.4, we have seen that there is a whole family of left and right pairs for one given supergravity theory and that the amplitude formula automatically selects one of these pairs on spacetime via the scattering equations. Looking at this from a different perspective we see that for all the other possible pairs we don't have a way of deriving double copy constructions *on spacetime*. It seems obvious that for the pairs that don't have matching mass spectra there shouldn't be a valid double copy but it would be interesting to see if this extends to all discarded pairs.

There is a well established formalism for the construction of gauged supergravities (see [169] for a review). For further investigations of these theories in the context of the ambitwistor string, it would be good to make contact with the relevant objects in that description such as the embedding tensor and the symplectic frame.

Solutions to the polarised scattering equations and Ward identities The amplitude formulae we have derived rely on the polarised scattering equations (5.85). Following our remarks in chapter 3, we evaluate the integrands on the solutions $\{\sigma_i, u_i, v_i\}$ by first solving the massive scattering equations (5.50) and then solving the polarised ones on top of those. Finding analytic solutions becomes a challenging task already at five point. An interesting approach to the evaluation of formulae that rely on the scattering equations has been developed in [170–172]. It relies on the (computationally costly) construction of a Gröbner basis of the ideal generated by the scattering equations and allows one to evaluate the amplitude formula without the need to know the individual solutions to the scattering equations. It would be interesting to gain a better understanding of how this is reflected at the level of the path integral.

The massless polarised scattering equation in four dimensions exhibit a special structure (see section 2.4.2) whereby solutions are split into MHV sectors, so that for instance at $n-$point and MHV degree only one solution for $\{\sigma_i\}$ contributes to the MHV amplitude and this can be evaluated explicitly [173]. For massive particles, even in the case of two massive particles and $n-2$ massless gluons of the same helicity, we can checked numerically that the amplitude has support on all the $(n-3)!$ solutions for $\{\sigma_i\}$.

In recent years several $n-$point formulae have been derived by BCFW recursion [72, 73]. We have seen in 5.5.2 that the supersymmetry of the full superamplitude can be exploited to obtain Ward identities relating the different component amplitudes. This can be done for any superamplitude formula and one obtains relations where the coefficients are the polynomials in the moduli U_{ij}. What is interesting is that we have found a case where we could easily solve for one particular coefficient U_{ij} without the need to solve the full system, thus relating different $n-$point amplitudes that had previously appeared in the literature [72, 162]. We have hope that further investigation of special configurations could lead to more cases where we can solve for a subset of coefficients U_{ij} to find more Ward identities.

Loops The two-twistor string also provides an alternative formulation of the massless ambitwistor string [83], but in a framework in which a massless field can be *deformed* to go off-shell. This allows us to adapt the elegant method of deriving loop amplitudes in [84] via a *gluing operator* but now to theories with fermions and supersymmetry such as super Yang-Mills theory. This construction arises from the vector model of the ambitwistor string, where it has been shown that loop integrands can be localized on a *nodal sphere* rather than the torus that more usually arises in string theory [117, 174]. At the level of the worldsheet model, this nodal structure of loop correlators is realized in the worldsheet CFT as the *gluing operator* Δ, which encodes the propagator of the target-space field theory. One-loop amplitudes then have two equivalent descriptions; a string-inspired one as correlators on a torus, and an alternative representation as $g = 0$ correlators in the presence of a gluing operator. We will extend this construction to the twistorial models in chapter 6.

CHAPTER 6

Loops from the gluing operator in four dimensions

Wen and Zhang [175] have recently presented D3 brane loop amplitude formulae derived as forward limits of M5 brane tree level amplitudes in six dimensions. Starting from the twistorial formulae of [108] and [2], $n-$point loop amplitudes in four dimensions were derived from an $(n+2)-$point tree level amplitude in six. This procedure was understood in the CHY formalism [176] as coming from the nodal Riemann sphere description of loop integrands in the RNS ambitwistor string [117, 174]. Whereas loop-level correlators have long been an active field of research in the RNS ambitwistor string [99,117,174,176–198], progress for the twistorial models has been rather limited [175, 199, 200]. The cause for this discrepancy seems to be two-fold; in the $\mathcal{N} = 4$ (ambi-)twistor string, superconformal gravity states propagating in the loop make it difficult to extract the $\mathcal{N} = 4$ super Yang-Mills integrand, whereas for $\mathcal{N} = 8$ supergravity the absence of a bc-system complicates the calculation of torus correlators [31]. However, the progress in the RNS ambitwistor string model suggests that both of these difficulties can be resolved by adopting a different approach, where loop integrands arise from a *nodal sphere* rather than a torus in the worldsheet model.

This relies on a property specific to the ambitwistor string models: the one-loop integrands, in addition to being modular invariant, are fully localized on a loop-level extension of the scattering equations [99]. They can therefore be simplified by a residue theorem on the moduli space [174], effectively trading one of the scattering equations for a localization on the non-separating boundary divisor, where the torus degenerates to a nodal sphere. The resulting amplitude formulæ over the nodal sphere are compact, manifestly rational, and can be extended from 10d supergravity to a variety of other theories and dimensions [117, 176, 178]. Moreover, extensions of this argument remain valid at two loops, recasting the two-loop integrand as a moduli integral over the two-nodal sphere [185, 186].

At the level of the CFT, the simple structure of the loop correlators originates from the presence of a so-called *gluing operator* Δ, which encodes the propagator of the target-space field theory [84]. As an off-shell object, Δ cannot be a local operator in the CFT, [1] and it contains (in addition to a genuinely non-local factor) a pair of local operators — corresponding to the off-shell states of the propagator — inserted at two special points σ_+ and σ_-. If the two marked points lie on different worldsheets, Δ functions as a standard tree-level propagator, and can be used to formulate the BCFW recursion at the level of the underlying CFT [84]. However, if both $\sigma_\pm$ lie on the same sphere[2] the correlators reproduce precisely the one-loop integrand formulæ localized on the nodal sphere. In the ambitwistor string, one-loop integrands can thus be recovered from $g = 0$ correlators in the presence of a gluing operator.

Here we propose that many of the issues plaguing the twistorial models at loop level can be resolved by following this latter strategy of defining a gluing operator and working directly on the nodal sphere. In an important distinction from the RNS model however, it turns out that the inherently on-shell nature of the 4d twistorial ambitwistor strings hinders our ability to define a gluing operator [22, 23, 31, 83].[3] This can be understood intuitively from the degeneration from genus one to the nodal sphere: as discussed above, the residue theorem trades one of the scattering equations for the localization on the nodal sphere, and therefore $P^2 \neq 0$ on the nodal sphere. At the level of the CFT, this arises from the non-local component of Δ, which modifies the effective gauged current from P^2 to $P^2 - \ell^2\omega_{+-}^2$. The twistorial models, on the other hand, *solve* the constraint $P^2 = 0$, and thus cannot account for the deformations away from $P^2 = 0$ necessary for the definition of the gluing operator.

The models of chapters 4 and 5 give an alternative massless model in four dimension. Because it is embedded in higher dimensions, it allows for more degrees of freedom. While all

[1] to be precise, it cannot be local and BRST-invariant, but we'd like to retain the latter

[2] corresponding to a non-separating boundary divisor of the genus-one moduli space

[3] Here, we mean by 'on-shell' that the constraint $P^2 = 0$, which is gauged in the RNS ambitwistor string, is explicitly solved in the twistor models, with $P_{\alpha\dot\alpha} = \lambda_\alpha\tilde\lambda_{\dot\alpha}$.

external particles remain on-shell in 4d, we will see that this introduces enough 'off-shell' aspects into the model to allow for a gluing operator to be defined in close analogy with [84]. We begin this chapter by discussing the action for the massless worldsheet model, its origin from the massive model in §5.4.2, and its relation to the familiar 4d ambitwistor string. The extra currents in this model play an important role in the construction of the gluing operator, and we highlight the differences and similarities to Δ_{RNS} of [84]. We conclude by calculating n-point correlators involving Δ, which give rise to the twistorial one-loop integrand formulæ of [175].

6.1 The model

Let us first introduce an ambitwistor worldsheet model that (i) agrees with the familar 4d ambitwistor string models for tree-level correlators involving only vertex operators and (ii) allows for the definition of a gluing operator, i.e. contains gauged currents that allow for deformations with $P_{4\text{d}}^2 \neq 0$. One way of achieving this in the twistorial models is to reduce a higher-dimensional model (e.g. the 5d worldsheet model of [85]) and dimensionally reducing it to 4d. While in these models $P^2 = 0$ is still solved exactly, the 4d part $P_{4\text{d}}$ can now satisfy $P_{4\text{d}}^2 \neq 0$. Equivalently, we may start with the massive model of section 5.4.2, but restrict to the massless case where $j^H = 0$. This gives the action:

$$S_{4\text{d}}^0 = \int_\Sigma \mathcal{Z}^a \cdot \bar{\partial}_e \mathcal{Z}_a + A_{ab} \mathcal{Z}^a \cdot \mathcal{Z}^b + A\lambda^2 + \tilde{A}\tilde{\lambda}^2 + S_m \,, \tag{6.1}$$

where $\mathcal{Z}$ are as before the equivalent of the Dirac supertwistors $\mathcal{Y} = (\lambda_A, \mu^A, \eta^I)$ we considered in the models of [5]. That this reduces the target space to $\mathbb{A}_4$ is intuitively clear from the mass relations (4.6). In appendix C we show this by explicit integration of the additional degrees of freedom in the path integral.

We also include the same worldsheet matter S_m as in the massive case, with

$$S_m^{\text{sYM}} = S_\rho + S_j \,, \qquad\qquad S_m^{\text{sugra}} = S_\rho + S_{\tilde{\rho}} \,, \tag{6.2}$$

where S_j is a current algebra, and S_ρ lifts the worldsheet gauge algebra to a super gauge algebra, which can be identified as $\mathfrak{sl}_2 \ltimes H(0,2p)$, where H is the Heisenberg Lie superalgebra, and $p=1$ for super Yang-Mills and $p=2$ for supergravity respectively.

Since the model (6.1) is a special case of the massive model discussed in the previous section, it is clear that the BRST gauge-fixing is only modified trivially so that:

$$Q = Q_{\rm m}\Big|_{j^H=\tilde{j}^{\tilde{H}}=0} \tag{6.3}$$

The anomaly counting is also unaffected by taking $j^H = \tilde{j}^{\tilde{H}} = 0$, so we still have

$$\mathfrak{a}^{\rm sYM}_{\rm SL(2)} = \frac{3}{4}\left(4-\mathcal{N}\right)\,, \qquad\qquad \mathfrak{a}^{\rm sugra}_{\rm SL(2)} = \frac{3}{4}\left(8-\mathcal{N}\right)\,, \tag{6.4}$$

for the gauge-theory and gravity models respectively, and the SL(2) gauge anomaly vanishes for maximal supersymmetry. The models also maintain the same central charges,

$$\mathfrak{c}^{\rm sYM} = -32+\mathcal{N}+\mathfrak{c}_j\,, \qquad\qquad \mathfrak{c}^{\rm sugra} = -20+\mathcal{N}\,. \tag{6.5}$$

As in the massive case, the models are therefore critical if we include the central charge from six compactified dimensions, as well as a current algebra of central charge $\mathfrak{c}_j = 16$ for super Yang-Mills.

Vertex operators and tree-level correlators. As before, the vertex operators take the form (5.68). Because we take $j^H=0$, the BRST cohomology only contains massless states

$$Q\circ V(\sigma_i) \supset \left(A\kappa^2 + \tilde{A}\tilde{\kappa}^2\right) V(\sigma_i) = 0\,, \tag{6.6}$$

which is reflected in the delta functions produced by the picture changing operators as in (5.78). Tree-level correlators take the form:

$$\mathcal{A}_n = \prod_{i=2}^{n} \delta(\kappa_i^2)\,\delta(\tilde{\kappa}_i^2) \int d\mu_n^{\rm pol}\,\mathcal{I}_n\, e^{F_{\mathcal{N}}}\,, \tag{6.7}$$

These are trivial dimensional reductions of the six and five dimensional formulae to four dimensional massless kinematics: in chapter 2 they have been shown to agree with the familiar amplitudes in maximal super Yang-Mills and maximal supergravity, as obtained from the twistor or ambitwistor string.

6.2 Gluing operator

While the 4d ambitwistor string and the worldsheet model (6.1) are equivalent at tree-level, a gluing operator may readily be defined in the latter, but not the former. The reason for this is that the gluing operator plays the role of a target-space propagator, and is therefore an inherently off-shell object. The constraint $P^2 = 0$ is solved rather than gauged in the original twistorial models, thus preventing any deformation that leads to $P^2 \neq 0$. For S^0_{4d} on the other hand, P_{AB} is reduced from six to four dimensions via the gauged currents λ^2 and $\tilde{\lambda}^2$, so a non-local operator can lead to $P^2_{4d} \neq 0$ by deforming them.

We may also see the need for a non-local operator from a different perspective, as stressed in [84]: since the propagator is off-shell, the gluing operator cannot both be local and BRST invariant. We will follow the approach of ref. [84], and construct the gluing operator as a non-local, but BRST invariant object. We thus require that the gluing operator Δ:

(i) encodes a target-space propagator, i.e. includes two local operators $\mathcal{O}_\pm$, which are extensions of the vertex operators V to off-shell momentum $\pm\ell$, as well as the appropriate sum over states

(ii) is BRST invariant.

From these requirements, we write the following general form of the gluing operator (c.f. [84]),

$$\Delta(\sigma_+, \sigma_-) = \int \frac{d^D\ell}{\ell^2}\, W(\sigma_+, \sigma_-) \sum_{\text{states}} \mathcal{O}_+(\sigma_+)\mathcal{O}_-(\sigma_-)\,. \tag{6.8}$$

For the operators $\mathcal{O}_\pm$, we will need two off-shell, back-to-back momenta $\pm\ell$, which we parametrize in the massive spinor-helicity formalism as[4]

$$\ell_{\alpha\dot\alpha} = (\kappa_{+\alpha}\tilde\kappa_{+\dot\alpha})\,, \qquad -\ell_{\alpha\dot\alpha} = (\kappa_{-\alpha}\tilde\kappa_{-\dot\alpha})\,, \qquad \kappa^a_{-A} = (-1)^a \kappa^a_{+A}\,, \tag{6.9}$$

where the mass parameter $M_\ell =: L$ is defined as usual via

$$\ell^2 = L^2\,, \qquad \det(\kappa_\pm) = \pm L\,. \tag{6.10}$$

We may then define the operators $\mathcal{O}_\pm$ as the (trivial) extension of the vertex operators V to an off-shell momentum,

$$\mathcal{O}_\pm = V\Big|_{k\to\pm\ell}\,. \tag{6.11}$$

While this is enough to satisfy condition (i) above, other choices of $\mathcal{O}_\pm$ may in principle be possible that also satisfy $\mathcal{O}_\pm\big|_{\pm\ell\to k} = V$. We will verify below that for the choice (6.11), there exists a W such that the gluing operator Δ is BRST-invariant. To be explicit, this gives

$$\mathcal{O}_\pm(\sigma_\pm) = \int d^2u\, d^2v\; \bar\delta^4\big((u\lambda_A) - (v\,\kappa_{\pm A})\big)\,\bar\delta\big((\epsilon_\pm v) - 1\big)\, w\, e^{iu_a\left(\mu^{Aa}\epsilon_{\pm A} + q_{\pm\, I}\eta^{Ia}\right) - \frac{1}{2}(\xi v) q_\pm^2}\,. \tag{6.12}$$

The sum over states depends on the model in question. For both super Yang-Mills and supergravity, it will be convenient to take $(\epsilon_+\epsilon_-) = 1$. For the S_ρ matter system, the sum over states can then conveniently be performed by a fermionic integral over the supermomenta $q_\pm$ of the propagating particle, whereas for the current algebra S_j the colour-flow through the propagator takes the form $\delta_{\mathfrak{a}\mathfrak{b}}$. For super Yang-Mills, we thus have

$$\sum_{\text{states}} \mathcal{O}_+(\sigma_+)\mathcal{O}_-(\sigma_-) = \int d^{\mathcal{N}}q_+\, d^{\mathcal{N}}q_-\; \delta_{\mathfrak{a}\mathfrak{b}}\; \mathcal{O}_+^{\mathfrak{a}}(\sigma_+)\mathcal{O}_-^{\mathfrak{b}}(\sigma_-)\, e^{iq_+\cdot q_-}\,, \tag{6.13}$$

and similarly for supergravity.

At this stage, it is easy to verify explicitly that the operator (6.13) is not BRST-closed due to the off-shell momentum ℓ. The failure to be BRST-closed must be compensated by the operator $W(\sigma_+,\sigma_-)$ in (6.8), which is therefore genuinely non-local. Using the BRST-closure to define

[4] To be explicit, the last equation implies that we use the following convenient choice for the relation between the spinors of the back-to-back momenta: $\kappa^0_{-A} = \kappa^0_{+A}$ and $\kappa^1_{-A} = -\kappa^1_{+A}$.

W, we find

$$W(\sigma_+, \sigma_-) = \exp\left(\pm \int_\Sigma (L\,A + L\,\tilde{A})\,\omega_{+-} \right) , \tag{6.14}$$

where ω_{ij} is the differential with simple poles at the marked points, $\omega_{ij} = \frac{\sigma_{ij}\, d\sigma}{(\sigma-\sigma_i)(\sigma-\sigma_j)}$. Let us see explicitly that this achieves the objective, and that Δ is now in the BRST cohomology. Since W depends on the gauge fields A and $\tilde{A}$, it modifies the BRST operator to an *effective BRST operator* Q_{eff}. After BRST quantization and integrating out the gauge fields in the presence of the gluing operator, this effective BRST operator takes the form[5]

$$Q_{\text{eff}} \supset \oint cT + \nu \left(\lambda^2 - L\,\omega_{+-}\right) + \tilde{\nu} \left(\tilde{\lambda}^2 - L\,\omega_{+-}\right) , \tag{6.15}$$

where we have only given the currents affected by the presence of W. We see that the effective BRST operator contains precisely the correct terms to render the gluing operator BRST-closed,

$$Q_{\text{eff}} \circ \Delta = 0 \,. \tag{6.16}$$

6.3 One-loop amplitudes

Having constructed the gluing operator Δ as the BRST-closed operator encoding the propagator, we can now calculate loop amplitudes (here for super Yang-Mills) as correlators including Δ on a single Riemann sphere:

$$\int_{\mathfrak{M}_{1,n}} \langle \mathcal{V}_1(\sigma_1) \cdots \mathcal{V}_n(\sigma_n) \rangle_\Sigma = \int_{\mathfrak{M}_{0,n+2}} \langle \Delta(\sigma_+, \sigma_-)\, \mathcal{V}_1(\sigma_1) \cdots \mathcal{V}_n(\sigma_n) \rangle_\Sigma \,. \tag{6.17}$$

As proposed in [84] for the RNS ambitwistor string, this will calculate one-loop amplitudes. We will see that the expressions precisely match the one-loop amplitudes obtained in [175] from a back-to-back forward limit of the 6d spinorial amplitude formulæ.

From the form of the gluing operator in the previous section we can write the amplitude

[5] The calculation here mirrors [84] closely, and many additional details can be found there.

(6.17) as:

$$\int \frac{\mathrm{d}^4\ell}{\ell^2}\,\mathrm{d}^{\mathcal{N}}q_+\,\mathrm{d}^{\mathcal{N}}q_-\,\mathrm{e}^{iq_+\cdot q_-}\int_{\mathfrak{M}_{0,n+2}} W(\sigma_+,\sigma_-)\delta_{\mathfrak{a}\mathfrak{b}}\left\langle \mathcal{V}_1\left(\sigma_1\right)\cdots\mathcal{V}_n\left(\sigma_n\right)\mathcal{O}_+^{\mathfrak{a}}\left(\sigma_+\right)\mathcal{O}_-^{\mathfrak{b}}\left(\sigma_-\right)\right\rangle_\Sigma . \tag{6.18}$$

Here the factor $W(\sigma_+,\sigma_-)$ acts as we described in the previous section to provide an effective 'mass term' for punctures $\sigma_\pm$ associated with the on-shell momentum. Then the correlator is computed as an $(n+2)-$point correlator with two off-shell particles with back-to-back momenta. This formula is an analogue of the ones we derived for tree level scattering in the previous chapter, with adjacent particles $\pm$ in the color-ordering because of the sum over states dictated by the gluing operator. Because of the special kinematic configuration involved, the scattering equation and the spin 1 integrand can be simplified as follows.

Polarized scattering equations. We embed the spinors $\kappa_\pm$ and $\tilde{\kappa}_\pm$ as usual into 6d kinematics $\kappa^a_{\pm A}$, so that the $4D$ part of the loop momentum ℓ is now off-shell, c.f. (6.10). Similarly, we embed the external momenta (massless in $4D$) via

$$\kappa^a_\alpha = (0,-\kappa_\alpha)\ , \qquad \tilde{\kappa}^a_{\dot\alpha} = (\tilde{\kappa}_{\dot\alpha},0)\ , \tag{6.19}$$

and $4D$ polarization data can be incorporated naturally via

$$\epsilon_{ia} = (0,-1) \quad i\in -\,, \qquad\qquad \epsilon_{pa} = (1,0) \quad p\in +\,, \tag{6.20}$$

corresponding to the usual $\pm$ helicity eigenstates. In particular, this implies that

$$\epsilon_i^\alpha := (\epsilon_i\,\kappa_i^\alpha) = \kappa_i^\alpha \qquad \tilde{\epsilon}_i^{\dot\alpha} := (\epsilon_i\,\tilde{\kappa}_i^{\dot\alpha}) = 0 \qquad i\in -\,, \tag{6.21a}$$

$$\epsilon_p^\alpha := (\epsilon_p\kappa_p^\alpha) = 0 \qquad \tilde{\epsilon}_p^{\dot\alpha} := (\epsilon_p\tilde{\kappa}_p^{\dot\alpha}) = \tilde{\kappa}_p^{\dot\alpha} \qquad p\in +\,. \tag{6.21b}$$

As in $6D$, the polarized scattering equations for any particle $i\in\{1,2,\ldots,n,+,-\}$ are then given by

$$\mathcal{E}_{i\alpha} := (u_i\lambda_\alpha(\sigma_i)) - (v_i\kappa_{i\alpha}) \qquad \tilde{\mathcal{E}}_{i\dot\alpha} := (u_i\tilde{\lambda}_{\dot\alpha}(\sigma_i)) - (v_i\tilde{\kappa}_{i\dot\alpha})\ . \tag{6.22}$$

where, as before, λ and $\tilde{\lambda}$ are defined by

$$\lambda^a_\alpha(\sigma) = \sum_{i\in h_-} \frac{u^a_i \epsilon_{i\alpha}}{\sigma - \sigma_i} + \frac{u^a_+ \epsilon_{+\alpha}}{\sigma - \sigma_+} + \frac{u^a_- \epsilon_{-\alpha}}{\sigma - \sigma_-}\,, \tag{6.23a}$$

$$\tilde{\lambda}^a_{\dot\alpha}(\sigma) = \sum_{p\in h_+}^{n} \frac{u^a_p \tilde{\epsilon}_{p\dot\alpha}}{\sigma - \sigma_p} + \frac{u^a_+ \tilde{\epsilon}_{+\dot\alpha}}{\sigma - \sigma_+} + \frac{u^a_- \tilde{\epsilon}_{-\dot\alpha}}{\sigma - \sigma_-}\,. \tag{6.23b}$$

Here, we have used that half the ϵ's vanish, (6.21), and wlog, we can choose the polarization of the loop momentum to be

$$\epsilon_{+a} = (1,0)\,, \qquad \epsilon_{-a} = (0,1)\,, \tag{6.24}$$

i.e. in the conventions of [175]:

$$\epsilon^\alpha_+ := (\epsilon_+\,\kappa^\alpha_+) = \kappa^{\alpha 0}_+ \qquad \tilde{\epsilon}^{\dot\alpha}_+ := (\epsilon_+\,\tilde{\kappa}^{\dot\alpha}_+) = \tilde{\kappa}^{\dot\alpha 0}_+ \qquad \text{for} \quad +\ell\,, \tag{6.25a}$$

$$\epsilon^\alpha_- := (\epsilon_-\,\kappa^\alpha_-) = \kappa^{\alpha 1}_- = -\kappa^{\alpha 1}_+ \qquad \tilde{\epsilon}^{\dot\alpha}_- := (\epsilon_-\,\tilde{\kappa}^{\dot\alpha}_-) = \tilde{\kappa}^{\dot\alpha 1}_- = -\tilde{\kappa}^{\dot\alpha 1}_+ \qquad \text{for} \quad -\ell\,. \tag{6.25b}$$

This implies in particular that we can express the loop momentum ℓ as

$$\ell^{\alpha\dot\alpha} = (\kappa^\alpha_+ \tilde{\kappa}^{\dot\alpha}_+) = \epsilon^\alpha_- \tilde{\epsilon}^{\dot\alpha}_+ - \epsilon^\alpha_+ \tilde{\epsilon}^{\dot\alpha}_-\,. \tag{6.26}$$

Integrands. The only non-trivial part comes from $\det{}'H$, which we can simplify in this one-loop set-up. With data as above, H is given by

$$H_{ij} = H^-_{ij} = \frac{\langle \epsilon_i \epsilon_j\rangle}{\sigma_{ij}} \qquad H_{pq} = H^+_{pq} = \frac{[\tilde{\epsilon}_p \tilde{\epsilon}_q]}{\sigma_{pq}} \qquad H_{ip} = 0 \tag{6.27a}$$

$$H_{i\pm} = \frac{\langle \epsilon_i \epsilon_\pm\rangle}{\sigma_{i\pm}} \qquad H_{p\pm} = \frac{[\tilde{\epsilon}_p \tilde{\epsilon}_\pm]}{\sigma_{p\pm}}\,, \tag{6.27b}$$

Here, we take $i,j \in h_-$ and $p,q \in h_+$. We have the freedom to define the reduced determinant by removing the $\pm$ rows and columns from H. With this choice, the resulting determinant is block-diagonal, and the result has the appealing form

$$\det{}'H = \frac{1}{(u_+u_-)^2} \det H^{[+-]}_{[+-]} = \frac{1}{(u_+u_-)^2} \det H^+ \,\det H^-\,, \tag{6.28}$$

reminiscent of tree-level. Indeed, this form makes it obvious that the integrand behaves correctly on a single cut.

Putting everything together, we can write the loop amplitude (6.18) as:

$$\mathcal{A}_n^{1-\text{loop}} = \int \frac{\mathrm{d}^4\ell}{\ell^2}\, \mathrm{d}^{\mathcal{N}} q_+\, \mathrm{d}^{\mathcal{N}} q_- \mathrm{e}^{iq_+\cdot q_-} \int \mathrm{d}\mu_{n+2}^{\text{pol}} \frac{1}{(u_+ u_-)^2} \det H^+ \det H^- \mathrm{PT}(\alpha, \sigma_+, \sigma_-)\, \mathrm{e}^{F_{\mathcal{N}}}\,,$$

where the polarisation measure is familiar from the six-dimensional tree level formulae, with the polarised scattering equations as described above. A similar expression can be found for supergravity and both agree with the formulae presented in [175].

6.4 Comparison to the gluing operator in the RNS ambitwistor string

In section 6.2, we constructed the gluing operator Δ_{4d} following the same guiding principles used for Δ_{RNS} in ref. [84]; both are built from two local operators that trivially extend the vertex operators off-shell, and are BRST-closed. In this section, we compare the two gluing operators, and discuss similarities and differences. As we will see below in more detail, Δ_{RNS} can be constructed directly in the 10d model, but requires the constraint P^2 to be gauged rather than solved, and thus does not exist in spinorial models. For clarity, we will compare the two gluing operators in the RNS model reduced to $d < 10$, where both constructions are well-defined and lead to equivalent gluing operators.

Δ_{RNS} : Let us start by reviewing briefly the gluing operator Δ_{RNS} as constructed in [84]. Following the same motivation as given above, the gluing operator takes the form

$$\Delta_{\text{RNS}}(\sigma_+, \sigma_-) = \int \frac{d^D\ell}{\ell^2}\, W(\sigma_+, \sigma_-) \sum_{\text{states}} \mathcal{O}_+(\sigma_+)\mathcal{O}_-(\sigma_-)\,, \tag{6.29}$$

where $\mathcal{O}_\pm$ are again off-shell extensions of the vertex operator, obtained by replacing the on-shell momentum k by the off-shell $\pm\ell$ respectively as in (6.11). [6] In the RNS ambitwistor string,

[6]For the bi-adjoint scalar $\mathcal{O}_+^{a\dot{a}} = c\tilde{c}\, j^a \tilde{j}^{\dot{a}}\, e^{i\ell\cdot X}$, with the sum over states implemented via $\Delta_{ab\dot{a}\dot{b}} = \delta_{ab}\delta_{\dot{a}\dot{b}}$.

BRST invariance requires W to be the following Wilson-line-like operator,

$$W_{\text{RNS}}(\sigma_+,\sigma_-)=\exp\left(\frac{\ell^2}{2}\int_\Sigma \tilde{e}\,\omega_{+-}^2\right)\,. \tag{6.30}$$

After BRST quantization, this leads to an effective BRST operator of the form

$$Q_{\text{eff}}=\oint cT+\frac{\tilde{c}}{2}\left(P^2-\ell^2\omega_{+-}^2\right)\,. \tag{6.31}$$

Note that this operator is well-defined in $D=10$ dimensions, and no dimensional reduction has been necessary in its derivation. As discussed in the beginning of the section, this reflects that in the RNS ambitwistor string, $P^2=0$ is a *gauged* constraint, which can be deformed by the Wilson-line-like operator W.

In order to compare Δ_{RNS} to the gluing operator in the twistorial model, we reduce it to 4d. Due to the absence of non-trivial winding modes, the toroidal compactification is trivial in the RNS ambitwistor string [117], and the formula (6.29) remains valid, but with the loop momentum $\ell_{(4d)}$ reduced to 4d. This extends straightforwardly to the BRST operator:

$$Q_{\text{eff}}^{(4d)}=\oint cT+\frac{\tilde{c}}{2}\left(P_{(4d)}^2-\ell_{(4d)}^2\omega_{+-}^2\right)\,. \tag{6.32}$$

$\boldsymbol{\Delta_{4d}}$: It is helpful to transpose the construction of the last section from the twistorial to the RNS model. In analogy with (6.1), we toroidally compactify five dimensions, and *gauge* the reduction from 5d to 4d by including the following term in the action,

$$S\supset\int_\Sigma a\,P\cdot\Omega\,. \tag{6.33}$$

Here Ω_1 is the vector pointing in the 'fifth' dimension, and the constraint both restricts tangent vectors to 4d and identifies different parallel 4-planes as explained in §2.4.5. While we may still define W as in (6.30), we can now alternatively achieve BRST invariance of the gluing operator by taking

$$W_{\text{RNS}}^{\text{4d}}(\sigma_+,\sigma_-)=\exp\left(|\ell|\int_\Sigma a\,\omega_{+-}\right)\,. \tag{6.34}$$

This is the RNS equivalent of W_{4d} in the twistorial model. Note that in contrast to W_{RNS}, this construction is only possible when dimensionally reducing to $D < 10$. On the other hand, it has the advantage of being applicable in models where the $P^2 = 0$ constraint is solved rather than gauged, as we have seen explicitly in the preceding section.

Despite the slightly differing constructions, both gluing operators give the same effective BRST operator after quantization;

$$Q_{\text{eff}}^{(4d)} = \oint cT + \frac{\tilde{c}}{2} P_{(5d)}^2 = \oint cT + \frac{\tilde{c}}{2} \left(P_{(4d)}^2 - \ell_{(4d)}^2 \omega_{+-}^2 \right) . \tag{6.35}$$

In the second equality, we have integrated out the gauge field a to find $P \cdot \Omega = |\ell_{(4d)}| \, \omega_{+-}$, as dictated by the inclusion of the effective term in (6.34).

6.5 Discussion

In this chapter we have shown that the formalism that we developed in chapter 5 gives an alternative massless model for the ambitwistor string in four dimensions, which can be used to derive loop amplitudes via a gluing operator. The construction makes clear that the model (6.1) can only encode nodal operators at one and two loops. This can be seen from comparison to the Q-cut formalism [180], where D-dimensional linearized loop propagators arise from massless propagators in $(D + g)$ dimensions, where g is the loop level. To be able to replicate the linear propagators via a gluing operator, the model similarly needs to start in $D + g$ dimensions. Because the models (6.1) should be considered to come down from five dimensions rather than six, as discussed in the previous chapters, we don't expect this description to hold at two loops. Even in the RNS ambitwistor string, the nodal operator has currently only been formulated at one loop, with important new features appearing at two loops [185, 186]. As such, any discussion of loops beyond $g = 1$ is beyond the scope of this thesis.

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APPENDIX A

Mathematical toolbox

This appendix is meant to clarify some mathematical language that is employed throughout the thesis. The aim is not to give rigorous definitions and it is not intended for an audience of mathematicians but it might result useful to the reader who is unfamiliar with the subject in order to understand various notations. Excellent discussions of the material of the first two paragraphs can be found in [201].

Complex projective spaces Complex projective space $\mathbb{CP}^n$ is the space of complex lines through the origin in $\mathbb{C}^{n+1}$:

$$\mathbb{CP}^n = \left(\mathbb{C}^{n+1} \setminus \{0\}\right) / \, \mathbb{C}^* \, , \tag{A.1}$$

with the action of $\mathbb{C}^*$ defined as:

$$[X_0, X_1 \ldots, X_n] \sim [\lambda X_0, \lambda X_1 \ldots, \lambda X_n] \qquad \lambda \in \mathbb{C}^* \, , \tag{A.2}$$

thus identifying non-zero multiples in $\mathbb{C}^{n+1}$. The coordinates $\{X_i\}$ are called *homogeneous coordinates*.

Line bundles on complex projective spaces Throughout the thesis we will be interested in many examples of complex projective spaces. Both twistor and ambitwistor space are projective spaces and so is the worldsheet, the compact Riemann surface that describes the embedding of the string in its target space. We want to describe objects that live on these spaces, namely 'functions' of their coordinates that are projectively well defined. This requires us to consider, rather than maps from a projective space X to $\mathbb{C}$, maps into *line bundles* over X. These are complex vector bundles with one-dimensional fibers and holomorphic transition functions. Over $\mathbb{CP}^n$, the most natural line bundle to construct is the tautological line bundle J, whose fiber over a point is the line it represents in $\mathbb{C}^{n+1}$ as in (A.1). More generally, line bundles over $\mathbb{CP}^n$ are classified according to their transition functions as powers of the tautological line bundle J and its dual, the hyperplane line bundle H. We get lots of other bundles by taking powers of J and H, with J^d and H^d having transitions functions that are dth powers of the transition functions for J and H. The tautological line bundle has inverse transition functions with respect to H so $J = H^{-1}$ and we can label line bundles by positive or negative dth powers of H, often written $\mathcal{O}_{\mathbb{CP}^n}(d)$. Their global sections are homogeneous polynomials of degree d.

An important line bundle we will often consider is the canonical line bundle $K_{\mathbb{CP}^n}$. This is the (holomorphic) line bundle of holomorphic $(n, 0)-$forms on $\mathbb{CP}^n$. On Σ, these are holomorphic $(1, 0)-$forms and we identify K_Σ with the (one-dimensional) cotangent bundle T^*_Σ, with negative powers of the canonical bundle indicating powers of the tangent bundle T_Σ. The canonical bundle K_Σ is the line bundle $\mathcal{O}_\Sigma(-2)$.

Conformal Field Theory Ambitwistor strings are two-dimensional CFTs. Their actions are expressed in terms of fields on a closed Riemann surface that we refer to as the worldsheet. The (primary) fields are characterised by their statistics and conformal weight.[1] This is a pair of half-integer indices $(h, \bar{h})$ that label the field's behaviour under two-dimensional conformal transformations. Because in two dimensions conformal transformations are equivalent to holomorphic coordinate transformations, we can identify the conformal weight with the form degree. This implies that we take fields $\Phi(\sigma)$ to be valued in powers of the holomorphic and antiholomorphic canonical line bundles:

$$\Phi \in \Omega^0(\Sigma, K_\Sigma^h \otimes \bar{K}_\Sigma^{\bar{h}}) \,. \tag{A.3}$$

[1]We also often refer to the sections of line bundles $\mathcal{O}_X(d)$ as having 'weight' d in the projective scale of X. It should always be clear from the context what weight we are referring to but if not specified it is usually the conformal weight.

The CFTs that we encounter in the ambitwistor string are chiral CFTs: these are free theories described by the $bc-$system for anticommuting fields and by the $\beta\gamma-$system for commuting fields. We describe the two systems jointly as a $BC-$system, using $\epsilon = \pm 1$ to keep track of bosonic/fermionic statistics. This discussion is standard in string theory and CFT textbooks such as [202], here we only list the main properties of these systems that we use throughout the thesis. The action of the chiral CFT in conformal gauge is:

$$S = \frac{1}{2\pi} \int \mathrm{d}^2 z B \bar{\partial} C \,, \tag{A.4}$$

The fields have weights:

$$B \sim (h, 0) \qquad\qquad C \sim (1-h, 0) \,, \tag{A.5}$$

and OPEs:

$$C(z)B(w) \sim \frac{1}{z-w} \qquad\qquad B(z)C(w) \sim -\frac{\epsilon}{z-w} \,, \tag{A.6}$$

Noether's theorem gives the holomorphic stress energy tensor:

$$T = -hB\partial C + (1-h)(\partial B)C \,, \tag{A.7}$$

The central charge is twice the coefficient of the fourth order pole in the OPE $T(z)T(w)$:

$$\mathfrak{c} = 2\epsilon(6h^2 - 6h + 1) \tag{A.8}$$

The Riemann-Roch theorem implies that the number of zero modes at genus g for the two fields is given by:

$$n_C - n_B = (2h-1)(1-g) \,, \tag{A.9}$$

for the minimal total number of zero modes.

APPENDIX B

Symmetry reductions

This appendix contains a collection of diverse material in support of chapter 5.

B.1 Factorization of the trace structure

Consider only one $SU(N)$ ordering:

$$\mathcal{A}_n = \sum_{\alpha} \mathrm{Tr}[T^{a_{\alpha(1)}} \cdots T^{a_{\alpha(n)}}] A_n(\alpha(1) \cdots \alpha(n))\,. \tag{B.1}$$

In the limit $K_I^2 \to 0$ the color ordered amplitude $A_n(\alpha(1) \cdots \alpha(n))$ has a non vanishing residue on the pole K_I^{-2} only if the labels in the subset $I \subset \{i\}_1^n$ are consecutive in the ordering α. If we split the indices as $\{1, ...n\} = I \cup \overline{I} = \{k\}_{k=1}^{|I|} \cup \{l\}_{l=1}^{|\overline{I}|}$, we can write the total amplitude $\mathcal{A}_n$ in the limit $K_I^2 \to 0$ as:

$$K_I^2 \cdot \mathcal{A}_n \to \sum_{\beta,\gamma} \mathrm{Tr}[T^{a_{\beta(k_1)}} \cdots T^{a_{\beta(k_{|I|})}} T^{a_{\gamma(l_1)}} \cdots T^{a_{\gamma(l_{|\overline{I}|})}}] A_{|I|+1}(\beta(k_1) \cdots \beta(k_{|I|} I)) A_{|\overline{I}+1|}(I \gamma(l_1) \cdots \gamma(l_{|\overline{I}|}))\,, \tag{B.2}$$

where I is the particle propagating on the internal leg going on shell. We can use the completeness relation of $SU(N)$ to factorize the trace structure:

$$\mathrm{Tr}[T^{a_\beta} T^{a_\gamma}] = \sum_{a_I} \mathrm{Tr}[T^{a_\beta} T^{a_I}] \mathrm{Tr}[T^{a_I} T^{a_\gamma}] + \frac{1}{N} \mathrm{Tr}[T^{a_\beta}] \mathrm{Tr}[T^{a_\gamma}]\,, \tag{B.3}$$

where we have used the shorthand notation $\mathrm{Tr}[T^{a_\beta}] = \mathrm{Tr}[T^{a_{\beta(k_1)}} \cdots T^{a_{\beta(k_{|I|})}}]$. The first term gives the correct term for the factorization of the amplitude:

$$K_I^2 \cdot \mathcal{A}_n \to \mathcal{A}_{|I|}\mathcal{A}_{|\bar{I}|}\,, \tag{B.4}$$

so it remains to show that the second term vanishes when summed over all permutations β, γ, i.e.:

$$\sum_{\beta,\gamma} \mathrm{Tr}[T^{a_\beta}]\mathrm{Tr}[T^{a_\gamma}]A_{|I|+1}(\beta(k_1)\cdots\beta(k_{|I|})I)A_{|\bar{I}+1|}(I\gamma(l_1)\cdots\gamma(l_{|\bar{I}|})) = 0\,. \tag{B.5}$$

In order to show this, let us recall that the color-ordered amplitudes are cyclic and they obey the $U(1)$ decoupling identity:

$$A_n[1,2,\dots n] + A_n[2,1,\dots n] + \cdots + A_n[2,3,\dots n-1,1,n] = 0 = A_n[\mathrm{Cycles}\{2,\dots \mathrm{n}\},1]\,. \tag{B.6}$$

We can split the sum over permutations β of the labels in I into permutations $\tilde{\beta}$ of all labels in I but k_1 which we choose to fix combined with a sum over cycles of $\{k_1, \tilde{\beta}\}$ for a given ordering $\tilde{\beta}$:

$$\sum_{\beta} = \sum_{\tilde{\beta}} \sum_{\mathrm{Cycles}\{k_1,\tilde{\beta}\}}\,. \tag{B.7}$$

Because of the ciclicity of the trace we have:

$$\sum_{\beta} \mathrm{Tr}[T^{a_\beta}]A_{|I|+1}(\beta(k_1)\cdots\beta(k_{|I|}I)) = \sum_{\tilde{\beta}} \mathrm{Tr}[T^{a_{k_1}}T^{a_{\tilde{\beta}}}] \sum_{\mathrm{Cycles}\{k_1,\tilde{\beta}\}} A_{|I|+1}(k_1\tilde{\beta}I) = 0\,. \tag{B.8}$$

B.2 Invariance argument

In showing the consistency of the factorization channels in the reduced theory, we have invoked the invariance of the scattering amplitudes under the action of the group G to claim that the sum of the charges of all the particles involved in the process should vanish. Here we review how the invariance of the theory provides the argument generalising (5.40).

Let us write a generic $n-$point amplitude as:

$$\mathcal{A}_n = \langle \Phi_1^{R_1} \cdots \Phi_n^{R_n} | S | 0 \rangle \,. \tag{B.9}$$

The transformed amplitude under the action of an element of the Cartan subgroup:

$$\begin{aligned} \mathcal{A}_n^{'} &= \langle \Phi_1^{'R_1} \cdots \Phi_n^{'R_n} | S^{'} | 0 \rangle \\ &= \langle (\mathbb{1} + (\sum_{i=1}^{\mathrm{rank}(\mathfrak{g})} l_i H_i) + \ldots) \Phi_1^{R_1} \cdots (\mathbb{1} + (\sum_{i=1}^{\mathrm{rank}(\mathfrak{g})} l_i H_i) + \ldots) \Phi_n^{R_n} | S | 0 \rangle \,. \end{aligned} \tag{B.10}$$

To first order in l:

$$\mathcal{A}^{'} = \mathcal{A} + \mathcal{H}^{R_1} \langle \Phi_1^{R_1} \cdots \Phi_n^{R_n} | S | 0 \rangle + \cdots + \langle \Phi_1^{R_1} \cdots \mathcal{H}^{R_n} \Phi_n^{R_n} | S | 0 \rangle \,. \tag{B.11}$$

By the definition taken in (5.48):

$$\mathcal{A}^{'} = \mathcal{A} + (h_1 + \cdots + h_n) \langle \Phi_1^{R_1} \cdots \Phi_n^{R_n} | S | 0 \rangle \,. \tag{B.12}$$

Then invariance of the amplitude $\mathcal{A} = \mathcal{A}^{'}$ implies

$$0 = (h_1 + \cdots + h_n) \langle \Phi_1^{R_1} \cdots \Phi_n^{R_n} | S | 0 \rangle \,. \tag{B.13}$$

So that for all non vanishing amplitudes the internal momenta of the particles defined as (5.48) involved sum up to zero.

B.3 Scalars in maximal SYM in five and four dimensions

In this appendix we describe the representations of scalars in maximal super Yang Mills upon reduction from 10 dimensions. When reducing this theory from 10 to d dimensions, the $l = 10 - d$ extra components of the connection are reduced to l real scalars fields $\{\Phi_a\}_{a=1}^{l}$. The theory is invariant under rotations of the scalars, i.e. these transform in the fundamental representation of $SO(l)$ R-symmetry transformations. We can alternatively write these scalars in terms of representations of the spin covering group $\mathrm{Spin}(l)$.

In four dimensions we have $\mathrm{Spin}(6) \simeq SU(4)$ and the 6 scalars transform in the antisymmetric tensor $\Phi_{IJ} = -\Phi_{JI}$ ($I, J = 1, \dots 4$), satisfying the self-duality condition $\star\Phi = \Phi^\dagger$, with:

$$(\star\Phi)^{IJ} = \frac{1}{2}\epsilon^{IJKL}\Phi_{KL}\,. \tag{B.14}$$

We can construct such a representation from $\{\Phi^a\}_{a=1}^6$ as follows. We label the six components $\Phi^a = (\phi^1, \phi^2, \phi^3, \tilde{\phi}^1, \tilde{\phi}^2, \tilde{\phi}^3)$ and construct Φ_{IJ} via:

$$\Phi_{mn} = \epsilon_{mnp}(\tilde{\phi}^p - i\phi^p) \qquad \phi_{m4} = \tilde{\phi}_m + i\phi_m \qquad m, n = 1, 2, 3\,. \tag{B.15}$$

In five dimensions $\mathrm{Spin}(5) \simeq Sp(4)$ and we can use Euclidean gamma matrices to relate the fundamental of $SO(5)$ to the antisymmetric tensor of $Sp(4)$:

$$\Phi_I^J = (\Gamma^a)_I^J\Phi_a\,. \tag{B.16}$$

$Sp(4)$ indices are raised and lowered via the matrix Ω_{IJ}.

B.4 The Coulomb Branch as a symmetry reduction from the Lagrangian

In this picture, mass terms for the Coulomb branch arise in the symmetry reduction from the kinetic terms in the action of $N = 2$ SYM:

$$\begin{aligned} S = \int d^5x \mathrm{Tr}\Big[&-\frac{1}{4}F_{mn}F^{mn} - \frac{1}{2}D_m\Phi^j D^m\Phi^j \\ &+\frac{1}{4}[\Phi^j, \Phi^k]^2 + \frac{i}{2}\bar{\Psi}^{AI}(\gamma^m)_A^B D_m\Psi_{BI} - \frac{1}{2}\bar{\Psi}^{AI}(\Gamma^j)_I^K[\Phi_j, \Psi_{AK}]\Big]\,, \end{aligned} \tag{B.17}$$

where m runs from 0 to 4 and i runs from 1 to 5. γ^m and Γ^i are respectively Lorentzian and Euclidean gamma matrices in five dimensions. The spinors result from the reduction of the 16_+

representation of $SO(1,9)$ which decomposes under the subalgebrae $SO(1,4)\times SO(5)$ as

$$16_+ \rightarrow (4,4) \quad : \quad \psi^I_A\,, \tag{B.18}$$

where the 4 is the fundamental of $Sp(4)\simeq SO(5)$. They are subject to a Majorana condition. One can easily check that the condition:

$$\partial_4(A_\mu,\Phi^{IJ},\Psi^I_A) = [H,(A_\mu,\Phi^{IJ},\Psi^I_A)]\,, \tag{B.19}$$

produces equivalent masses to the ones generated by vev'd scalars in $N=4$ SYM.

B.5 R-symmetry reduction

In this appendix we describe the R-symmetry reduction of maximal super Yang-Mills from five to four dimensions at the level of the lagrangian. We begin expanding the kinetic terms under (5.120). Starting with the field strength:

$$F_{mn}F^{mn} = F_{\mu\nu}F^{\mu\nu} + 2D_\mu\phi D^\mu\phi\,, \tag{B.20}$$

with $\phi = A_4$. This is the kinetic term for a four dimensional vector plus the kinetic term for an extra scalar. The kinetic term for the scalars Φ_{IJ} is:

$$\begin{aligned}\frac{1}{\alpha}D_m\Phi^i D^m\Phi^i = D_m\Phi_{IJ}D^m\Phi^{IJ} =& D_\mu\Phi_{IJ}D^\mu\Phi^{IJ} + H^M_{[I}\Phi_{J]M}H^{[I}_P\Phi^{J]P} - iH^M_{[I}\Phi_{J]M}[\phi,\Phi^{IJ}]\\ &- i[\phi,\Phi_{IJ}]H^{[I}_M\Phi^{J]M} - [\phi,\Phi_{IJ}][\phi,\Phi^{IJ}]\,.\end{aligned} \tag{B.21}$$

Here we have the kinetic term for the 5 scalars in four dimensions, a mass term for the 5d scalars, a cubic and a quartic interaction term between the 5d scalars and the extra scalar ϕ.

Now the kinetic term for the fermions:

$$\bar\Psi^{AI}(\gamma^m)^B_A D_m\Psi_{BI} = \bar\Psi^{AI}(\gamma^\mu)^B_A D_\mu\Psi_{BI} + H^J_I\bar\Psi^{AI}(\gamma^4)^B_A\Psi_{BJ} - i\bar\Psi^{AI}(\gamma^4)^B_A[\phi,\Psi_{BI}]\,, \tag{B.22}$$

corresponding to the four dimensional kinetic term, mass terms for the spinors and a Yukawa interaction term between the spinors and the extra scalar coming out of the vector.

However, the fermions here are still in the spinor representation of $SO(1,4)$, whereas we'd like to write these as spinors in four dimensions. First, let's remind ourselves that the original $10-$dimensional Weyl spinors obey a Majorana condition:

$$\Psi_+^{\mathrm{T}} C = \bar{\Psi}_+ \, . \tag{B.23}$$

This condition reads

$$\Psi_{AI} C^{AB} \Omega^{IJ} = \Psi^{AI} (\gamma_0)_A^B \delta_I^J = \bar{\Psi}^{BJ} \, , \tag{B.24}$$

for spinors in 5 spacetime dimensions. Now we want to further bring this down to four dimensions. We need make a choice for the 4d gamma matrices in terms of the five dimensional ones, and in particular we will do that so that the charge conjugation matrix is the same as the five dimensional one. This is possible because both in 4 and 5 dimensions the C-matrix is antisymmetric. However in five dimensions the C-matrix is C_-, i.e. the five dimensional gamma matrices have the following symmetry:

$$C \gamma_\alpha C^{-1} = \gamma_\alpha^{\mathrm{T}} \qquad \alpha = 0, \ldots 4 \, . \tag{B.25}$$

In four dimensions there are in principle two choices for the C-matrix but only C_+ is compatible with the Majorana condition, so that the symmetry property of gamma matrices in this basis differs in four and five dimensions:

$$C G_\mu C^{-1} = -G_\mu^{\mathrm{T}} \qquad \mu = 0, \ldots 3 \, . \tag{B.26}$$

On the other hand the chiral matrix $G_5 = iG_0 \cdots G_3$ in four dimensions has the same symmetry properties as the five dimensional gamma matrices so we can take $G_5 = \gamma_4$. Then for the other four dimensional gamma matrices it's easy to verify that $G_\mu = -i\gamma_\mu\gamma_4$ satisfy (B.26).

We can further choose to write the four dimensional gamma matrices in the Weyl basis:

$$G_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix} \qquad G_5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} . \tag{B.27}$$

In this basis the charge conjugation matrix can be written

$$C = iG_0G_2 = -i \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix} . \tag{B.28}$$

The Majorana condition (B.24) can then be written:

$$\Psi_{AI} C^{AB} \Omega^{IJ} = i \bar{\Psi}^{CJ} (G_5)_C^{\ B} , \tag{B.29}$$

or equivalently

$$\Psi_{AI} = iC^{-\mathrm{T}} \cdot G_5^{-\mathrm{T}} \cdot G_0^{-\mathrm{T}} \Psi^* = -(G_0 \cdot G_2 \cdot G_5 \cdot G_0)_{AC} \cdot \Psi^*_{CJ} \Omega_{JI} = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}_{AC} \Psi^*_{CJ} \Omega_{JI} , \tag{B.30}$$

where we kept the notation $A, ...$ for the four dimensional Dirac spinor indices. We can further look at the condition on the projected left and right components of the fermion:

$$\begin{aligned} \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}_{AC} ((\frac{\mathbb{1} \pm G_5}{2})_C^{\ D} \Psi_{DJ})^* \Omega_{JI} &= (\frac{\mathbb{1} \mp G_5}{2})_A^{\ B} \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}_{BD} (\Psi_{DJ})^* \Omega_{JI} \\ &= (\frac{\mathbb{1} \mp G_5}{2})_A^{\ D} \Psi_{DI} , \end{aligned} \tag{B.31}$$

So we have:

$$\Psi_{R/L;I} = \sigma_2 \Psi^*_{L/R;J} \Omega_{JI} . \tag{B.32}$$

Now the mass term for the fermions reads:

$$\begin{aligned} H_I^J \bar{\Psi}^{AI} (G_5)_A^{\ B} \Psi_{BJ} &= -iH_I^J \Omega^{KI} \Psi_{LK} \cdot \sigma_2 \cdot \Psi_{LJ} - iH_I^J \Omega^{KI} \Psi_{RK} \cdot \sigma_2 \cdot \Psi_{RJ} \\ &= -iH_I^J \Omega^{KI} (\Psi_{LK} \cdot \sigma_2 \cdot \Psi_{LJ} - \Psi^*_{LM} \cdot \sigma_2 \cdot \Psi^*_{LN} \Omega_{MK} \Omega_{NJ}) , \end{aligned} \tag{B.33}$$

in terms of Majorana spinors in four dimensions $\Psi_I = (\chi_I \;\; \sigma_2\chi^*_J\Omega_{JI})^T$.

If we write out the scalar mass terms explicitly we find:

$$\frac{1}{2}H^M_{[I}\Phi_{J]M}H^{[I}_P\Phi^{J]P} = \Phi_{12}\Phi^{12}(m_1{+}m_2)^2{+}\Phi_{13}\Phi^{13}(m_1{-}m_2)^2{+}\Phi_{24}\Phi^{24}(m_1{-}m_2)^2{+}\Phi_{34}\Phi^{34}(m_1{+}m_2)^2\,, \tag{B.34}$$

whereas for the spinors we simply have:

$$\begin{aligned} H^J_I\Omega^{KI}\Psi_{LK}\cdot\sigma_2\cdot\Psi_{LJ} =& - m_1(\Psi_{L1}\cdot\sigma_2\cdot\Psi_{L4}+\Psi_{L4}\cdot\sigma_2\cdot\Psi_{L1}) \\ &+ m_2(\Psi_{L2}\cdot\sigma_2\cdot\Psi_{L3}+\Psi_{L3}\cdot\sigma_2\cdot\Psi_{L2})\,, \end{aligned} \tag{B.35}$$

and similarly for the conjugate term.

Overall the lagrangian describes one massless vector A_μ, two massless scalars ϕ, Φ_{14}, four massive scalars and four massive Majorana fermions:

$$\begin{aligned} S = \int d^4x\, \mathrm{Tr}[&- \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D_\mu\phi D^\mu\phi - \frac{\alpha}{2}D_\mu\Phi_{IJ}D^\mu\Phi^{IJ} + \frac{i}{2}\bar\Psi^{AI}(\gamma^m)^B_A D_m\Psi_{BI} \\ &+ \frac{1}{2}H^J_I\Omega^{KI}(\Psi_{LK}\cdot\sigma_2\cdot\Psi_{LJ} - \Psi^*_{LM}\cdot\sigma_2\cdot\Psi^*_{LN}\Omega_{MK}\Omega_{NJ}) \\ &- \frac{\alpha}{2}H^M_{[I}\Phi_{J]M}H^{[I}_P\Phi^{J]P} \\ &- \frac{1}{2}\bar\Psi^{AI}(\Gamma^j)^K_I[\Phi_j,\Psi_{AK}] - \frac{1}{2}\bar\Psi^{AI}(G_5)^B_A[\phi,\Psi_{BI}] \\ &+ \frac{\alpha}{2}iH^M_{[I}\Phi_{J]M}[\phi,\Phi^{IJ}] + \frac{\alpha}{2}i[\phi,\Phi_{IJ}]H^{[I}_M\Phi^{J]M} \\ &+ \frac{1}{4}[\Phi^j,\Phi^k]^2 + \frac{\alpha}{2}[\phi,\Phi_{IJ}][\phi,\Phi^{IJ}])]\,. \end{aligned} \tag{B.36}$$

B.6 Massive amplitudes with a single massive particle

Performing a symmetry reduction along several higher dimensions allows for the presence of massless particles with non vanishing internal momenum. In this section we will illustrate how, starting from a theory of massless particles in six dimensions with signature $(2,4)$, we can perform a symmetry reduction along two null directions. We will detail the procedure in the simple case of the biadjoint scalar theory in six dimensions.

It is instructive to start with an example of an amplitude we would wish to construct: a four point amplitude with one massive particle and three massless paricles. This case is simple enough that we can assign by hand values of internal momenta to each of the particles in such a way that internal momenum is conserved and each particle has the desired mass:

$$K_1 = (k_1, \kappa, i\kappa) \qquad K_2 = (k_2, \kappa, -i\kappa) \qquad K_3 = (k_3, 0, 0) \qquad K_4 = (k_4, 2\kappa, 0)\,. \tag{B.37}$$

It is easy to see that for this assignment of kinematic data, with k_i four dimensional momenta summing to zero, the full six dimensional momentum is conserved and particles $1, 2$ and 3 are massless whereas particle 4 is massive.

We can accomplish this configuration of masses by picking two elements in a Cartan of the group G, under which the original theory is invariant, and assign masses as:

$$\partial_+ \Phi = m\, H_1 \cdot \Phi \qquad \partial_- \Phi = m\, H_2 \cdot \Phi\,, \tag{B.38}$$

where:

$$\partial_+ = (0,0,0,0,1,1) \cdot \partial \qquad \partial_- = (0,0,0,0,1,-1) \cdot \partial\,. \tag{B.39}$$

In the case of the biadjoint scalar theory, we write:

$$\partial_+ \phi^{a\dot{a}} = m H_1^{ab} \phi^{b\dot{a}} + \tilde{m} \tilde{H}_1^{\dot{a}\dot{b}} \phi^{a\dot{b}} \qquad \partial_- \phi^{a\dot{a}} = m H_2^{ab} \phi^{b\dot{a}} + \tilde{m} \tilde{H}_2^{\dot{a}\dot{b}} \phi^{a\dot{b}}\,, \tag{B.40}$$

where $h_i, \tilde{h}_i$ are elements of the Cartan of the groups $SU(N), SU(\tilde{N})$. This assigns values to the internal momentum (here taking $\tilde{m} = 0$):

$$\begin{pmatrix} \kappa_1^s \\ \kappa_2^s \end{pmatrix} = \begin{pmatrix} m(h_1^s + h_2^s) \\ m(h_1^s - h_2^s) \end{pmatrix}\,, \tag{B.41}$$

where the notation is in line with sections 5.3-5.2.2.

For instance, in the case $N = \tilde{N} = 3$, we can take:

$$H_1 = \,\mathrm{diag}(1, -1, 0) \qquad H_2 = (0, 1, -1)\,, \tag{B.42}$$

in the fundamental representation, and $H_1^{\mathrm{Ad}} \cdot \Phi = [H_1, \Phi]$. It's easy to see we can give rise to the internal momenta given above for the four point amplitudes with these assignment of charges.

It is clear that in order to perform this construction we need a Lie group whose Cartan subalgebra has at least dimension 2. When performing this type of reduction on $\mathcal{N} = (1,1)$ SYM in six dimensions, we can obtain more general patterns of symmetry breaking $U(N) \to U(N_1) \times U(N_2) \times U(N - N_1 - N_2)$ on the Coulomb branch of $\mathcal{N} = 4$ SYM.

APPENDIX C

Embedding of the massless models

We want to show here that the massless model (6.1) that we introduced to describe loops is equivalent to the massless models presented in [83].

We start with the action:

$$S = \int_{\Sigma} (\lambda^a \cdot \bar{\partial}\mu^b + \mu^a \cdot \bar{\partial}\lambda^b)\epsilon_{ab} + \frac{a}{2}\lambda^2 + \frac{\tilde{a}}{2}\tilde{\lambda}^2 + A_{ab}\lambda^a \cdot \mu^b + S_m \,. \tag{C.1}$$

The kinetic term can be expanded as:

$$\langle\lambda^1\bar{\partial}\tilde{\mu}^2\rangle - \langle\lambda^2\bar{\partial}\tilde{\mu}^1\rangle + \langle\tilde{\mu}^1\bar{\partial}\lambda^2\rangle - \langle\tilde{\mu}^2\bar{\partial}\lambda^1\rangle + [\lambda^1\bar{\partial}\tilde{\mu}^2] - [\lambda^2\bar{\partial}\tilde{\mu}^1] + [\tilde{\mu}^1\bar{\partial}\lambda^2] - [\tilde{\mu}^2\bar{\partial}\lambda^1] \,. \tag{C.2}$$

And the gauged currents:

$$\frac{\lambda^2}{2} = a\langle\lambda^1\lambda^2\rangle \tag{C.3}$$

$$\frac{\tilde{\lambda}^2}{2} = \tilde{a}[\tilde{\lambda}^1\tilde{\lambda}^2] \tag{C.4}$$

$$\lambda^1 \cdot \mu^1 = \langle\lambda^1\tilde{\mu}^1\rangle + [\tilde{\lambda}^1\mu^1] \tag{C.5}$$

$$\lambda^2 \cdot \mu^2 = \langle\lambda^2\tilde{\mu}^2\rangle + [\tilde{\lambda}^2\mu^2] \tag{C.6}$$

$$\lambda^1 \cdot \mu^2 + \lambda^2 \cdot \mu^1 = \langle\lambda^1\tilde{\mu}^2\rangle + [\tilde{\lambda}^1\mu^2] + \langle\lambda^2\tilde{\mu}^1\rangle + [\tilde{\lambda}^2\mu^1] \tag{C.7}$$

Integrating out the fields $a, \tilde{a}$ fixes:

$$\lambda^1 = t\lambda^2 \qquad \tilde{\lambda}^1 = \tilde{t}\tilde{\lambda}^2 \,, \tag{C.8}$$

More precisely, this is how we integrate out the constraints. We can expand both λ^1_α and $\tilde{\lambda}^{1\dot{\alpha}}$ as:

$$\lambda^1_\alpha = \frac{a}{\langle\lambda^2\xi\rangle}\lambda^2_\alpha - \frac{b}{\langle\lambda^2\xi\rangle}\xi\,, \tag{C.9}$$

and similarly for the tilded version. Here ξ is some auxiliary spinor orthogonal to λ^2. Then we make a change of variables from λ^1_α to a, b. The jacobian is -1. Then the path integral is:

$$\int D\lambda^1_\alpha \delta(\langle\lambda^1\lambda^2\rangle)(\cdots) \to \int DaDb\delta(b)(\cdots) \to \int \langle\lambda^2\xi\rangle DtDb\delta(b)(\cdots)\,. \tag{C.10}$$

We can further perform the change $a \to t = \frac{a}{\langle\lambda^2\xi\rangle}$, with jacobian $\langle\lambda^2\xi\rangle$. We can replace everywhere $\lambda^1 = t\lambda^2 - \frac{b}{\langle\lambda^2\xi\rangle}\xi$ and enforce the delta function $b = 0$. The kinetic term then reads:

$$t\langle\lambda^2\bar{\partial}\tilde{\mu}^2\rangle - \langle\lambda^2\bar{\partial}\tilde{\mu}^1\rangle + \langle\tilde{\mu}^1\bar{\partial}\lambda^2\rangle - t\langle\tilde{\mu}^2\bar{\partial}\lambda^2\rangle - \bar{\partial}t\langle\tilde{\mu}^2\lambda^2\rangle + \tilde{t}[\tilde{\lambda}^2\bar{\partial}\mu^2] - [\tilde{\lambda}^2\bar{\partial}\mu^1] + [\mu^1\bar{\partial}\tilde{\lambda}^2] - \tilde{t}[\mu^2\bar{\partial}\tilde{\lambda}^2] - \bar{\partial}\tilde{t}[\mu^2\tilde{\lambda}^2] -$$

This suggests the field redefinition:

$$\tilde{\mu}^2 = t^{-\frac{1}{2}}\tilde{\mu}'^2\,, \tag{C.11}$$

$$\tilde{\mu}^1 = t^{\frac{1}{2}}\tilde{\mu}'^1\,, \tag{C.12}$$

$$\lambda^2 = t^{-\frac{1}{2}}\lambda'^2\,, \tag{C.13}$$

Which modifies the kinetic term to:

$$\begin{aligned}
&\langle\lambda^2\bar{\partial}\tilde{\mu}^2\rangle + t^{\frac{1}{2}}\bar{\partial}(t^{-\frac{1}{2}})\langle\lambda^2\tilde{\mu}^2\rangle - \langle\lambda^2\bar{\partial}\tilde{\mu}^1\rangle - t^{-\frac{1}{2}}\bar{\partial}(t^{+\frac{1}{2}})\langle\lambda^2\tilde{\mu}^1\rangle + \langle\tilde{\mu}^1\bar{\partial}\lambda^2\rangle + t^{\frac{1}{2}}\bar{\partial}(t^{-\frac{1}{2}})\langle\tilde{\mu}^1\lambda^2\rangle \\
&\quad -\langle\tilde{\mu}^2\bar{\partial}\lambda^2\rangle - t^{\frac{1}{2}}\bar{\partial}(t^{-\frac{1}{2}})\langle\tilde{\mu}^2\lambda^2\rangle - t^{-1}\bar{\partial}t\langle\tilde{\mu}^2\lambda^2\rangle \\
&\qquad = \langle\lambda^2\bar{\partial}\tilde{\mu}^2\rangle - \langle\lambda^2\bar{\partial}\tilde{\mu}^1\rangle + \langle\tilde{\mu}^1\bar{\partial}\lambda^2\rangle - \langle\tilde{\mu}^2\bar{\partial}\lambda^2\rangle - \frac{1}{2}t^{-1}\partial t(\langle\lambda^2\tilde{\mu}^2\rangle - \langle\tilde{\mu}^2\lambda^2\rangle - 2\langle\tilde{\mu}^2\lambda^2\rangle + \langle\lambda^2\tilde{\mu}^1\rangle + \langle\tilde{\mu}^1\lambda^2\rangle) \\
&\qquad = \langle\lambda^2\bar{\partial}\tilde{\mu}^2\rangle - \langle\lambda^2\bar{\partial}\tilde{\mu}^1\rangle + \langle\tilde{\mu}^1\bar{\partial}\lambda^2\rangle - \langle\tilde{\mu}^2\bar{\partial}\lambda^2\rangle
\end{aligned}$$

The transformation has determinant 1.

In order to check if the t and $\tilde{t}$ path integrals decouple, one also needs to see how the remaining gauge currents behave under the field redefinition:

$$J^{11} = \langle\lambda^1\tilde{\mu}^1\rangle + [\tilde{\lambda}^1\mu^1] \to t\langle\lambda^2\tilde{\mu}^1\rangle + \tilde{t}[\tilde{\lambda}^2\mu^1] \tag{C.14}$$

$$J^{22} = \langle\lambda^2\tilde{\mu}^2\rangle + [\tilde{\lambda}^2\mu^2] \to t^{-1}\langle\lambda^2\tilde{\mu}^2\rangle + \tilde{t}^{-1}[\tilde{\lambda}^2\mu^2] \tag{C.15}$$

$$J^{12} = \langle\lambda^1\tilde{\mu}^2\rangle + [\tilde{\lambda}^1\mu^2] + \langle\lambda^2\tilde{\mu}^1\rangle + [\tilde{\lambda}^2\mu^1] \to \langle\lambda^2\tilde{\mu}^2\rangle + [\tilde{\lambda}^2\mu^2] + \langle\lambda^2\tilde{\mu}^1\rangle + [\tilde{\lambda}^2\mu^1] \tag{C.16}$$

One can then rewrite the gauge fields:

$$\alpha + \beta + \gamma = A_{12} \qquad \frac{\alpha}{\tilde{t}} + \frac{\beta}{t} = -A_{11} \qquad \alpha\tilde{t} + \beta t = -A_{22}\,, \tag{C.17}$$

so that the corresponding contribution to the action is:

$$\alpha(J^{12} - \frac{1}{\tilde{t}}J^{11} - \tilde{t}J^{22}) + \beta(J^{12} - \frac{1}{t}J^{11} - tJ^{22}) + \gamma J^{12}\,, \tag{C.18}$$

with

$$J^{\alpha} = J^{12} - \frac{1}{\tilde{t}}J^{11} - \tilde{t}J^{22} = (\frac{t-\tilde{t}}{t\tilde{t}})\langle\lambda^2(\tilde{t}\tilde{\mu}^2 - t\tilde{\mu}^1)\rangle\,, \tag{C.19}$$

$$J^{\beta} = J^{12} - \frac{1}{t}J^{11} - tJ^{22} = (\frac{\tilde{t}-t}{t\tilde{t}})[\tilde{\lambda}^2(t\mu^2 - \tilde{t}\mu^1)]\,, \tag{C.20}$$

$$J^{\gamma} = J^{12} = \langle\lambda^2\tilde{\mu}^2\rangle + [\tilde{\lambda}^2\mu^2] + \langle\lambda^2\tilde{\mu}^1\rangle + [\tilde{\lambda}^2\mu^1]\,, \tag{C.21}$$

The determinant of this transformation is $|\frac{\partial A}{\partial(\alpha\beta\gamma)}| = \frac{(t-\tilde{t})(t+\tilde{t})}{t\tilde{t}}$. We can also rescale $\alpha' = \frac{t\tilde{t}}{t-\tilde{t}}\alpha$ and $\beta' = \frac{t\tilde{t}}{t-\tilde{t}}\beta$, with determinant $-(\frac{t-\tilde{t}}{t\tilde{t}})^2$.

One can then integrate out the fields α' and β', enforcing:

$$\delta(\langle\lambda^2(\tilde{t}\tilde{\mu}^2 - t\tilde{\mu}^1)\rangle)\,, \tag{C.22}$$

$$\delta([\tilde{\lambda}^2(t\mu^2 - \tilde{t}\mu^1)])\,, \tag{C.23}$$

We can first define $(\tilde{\mu}^2)' = \tilde{t}\tilde{\mu}^2 - t\tilde{\mu}^1$, $(\mu^2)' = t\mu^2 - \tilde{t}\mu^1$ with determinant $\frac{1}{t^2\tilde{t}^2}$ (notice that this change should be carried out in the rest of the action too but we'll do that below). Then decompose $\mu^2, \tilde{\mu}^2$ on a basis (λ^2, ξ) and tilded as we did for λ^1.

$$\tilde{\mu}^1_\alpha = \frac{a}{\langle\lambda^2\xi\rangle}\lambda^2_\alpha - \frac{b}{\langle\lambda^2\xi\rangle}\xi\,. \tag{C.24}$$

Then the path integral is:

$$\int D\tilde{\mu}^2 D\mu^2 \delta(\langle\lambda^2\tilde{\mu}^2\rangle)\delta([\tilde{\lambda}^2\mu^2])(\cdots) \to \int DaDb\delta(b)DcDf\delta f(\cdots) \to \int \langle\lambda^2\xi\rangle[\tilde{\lambda}^2\tilde{\xi}]DmDnDb\delta(b)Df\delta(f)(\cdots)\,.$$

We can then replace everywhere:

$$\tilde{\mu}^2 = \frac{t}{\tilde{t}}\tilde{\mu}^1 + m'\lambda^2\,, \tag{C.25}$$

$$\mu^2 = \frac{\tilde{t}}{t}\mu^1 + n'\tilde{\lambda}^2\,, \tag{C.26}$$

and further multiply by $t\tilde{t}$ for $m, n \to m'n'$.

The kinetic terms become:

$$\begin{aligned}&\langle\lambda^2\bar{\partial}\tilde{\mu}^2\rangle - \langle\lambda^2\bar{\partial}\tilde{\mu}^1\rangle + \langle\tilde{\mu}^1\bar{\partial}\lambda^2\rangle - \langle\tilde{\mu}^2\bar{\partial}\lambda^2\rangle + [\tilde{\lambda}^2\bar{\partial}\mu^2] - [\tilde{\lambda}^2\bar{\partial}\mu^1] + [\mu^1\bar{\partial}\tilde{\lambda}^2] - [\mu^2\bar{\partial}\tilde{\lambda}^2] = \\ &= (\frac{t-\tilde{t}}{\tilde{t}})(\langle\lambda^2\bar{\partial}\tilde{\mu}^1\rangle - \langle\tilde{\mu}^1\bar{\partial}\lambda^2\rangle) + \bar{\partial}(\frac{t}{\tilde{t}})\langle\lambda^2\tilde{\mu}^1\rangle + (\frac{\tilde{t}-t}{t})([\tilde{\lambda}^2\bar{\partial}\mu^1] - [\mu^1\bar{\partial}\tilde{\lambda}^2]) + \bar{\partial}(\frac{\tilde{t}}{t})[\tilde{\lambda}^2\mu^1]\,,\end{aligned}$$

And the remaining current:

$$J^{12} = (\frac{t+\tilde{t}}{\tilde{t}})\langle\lambda^2\tilde{\mu}^1\rangle + (\frac{t+\tilde{t}}{t})[\tilde{\lambda}^2\mu^1]\,. \tag{C.27}$$

Let us write (note that this is just notation, not a change of variables):

$$y = \frac{t-\tilde{t}}{\tilde{t}} \qquad \bar{y} = \frac{t+\tilde{t}}{\tilde{t}} \qquad z = \frac{\tilde{t}-t}{t} \qquad \bar{z} = \frac{t+\tilde{t}}{t}\,. \tag{C.28}$$

So that:

$$\bar{\partial}\frac{t}{\tilde{t}} = \bar{\partial}y \qquad \bar{\partial}\frac{\tilde{t}}{t} = \bar{\partial}z \qquad \frac{t+\tilde{t}}{t-\tilde{t}} = \frac{\bar{y}}{y} = -\frac{\bar{z}}{z}\,. \tag{C.29}$$

Then we can write the kinetic terms:

$$(\langle\lambda^2\bar{\partial}(y\tilde{\mu}^1)\rangle - \langle(y\tilde{\mu}^1)\bar{\partial}\lambda^2\rangle + [\tilde{\lambda}^2\bar{\partial}(z\mu^1)] - [(z\mu^1)\bar{\partial}\tilde{\lambda}^2]\,. \tag{C.30}$$

And the current:

$$\bar{y}\langle\lambda^2\tilde{\mu}^1\rangle + \bar{z}[\tilde{\lambda}^2\mu^1]\,. \tag{C.31}$$

This suggest the field redefinition:

$$\mu^1 \to z^{-1}\mu^1 \qquad \tilde{\mu}^1 \to y^{-1}\tilde{\mu}^1 \qquad \gamma \to \frac{y}{\bar{y}}\gamma = -\frac{z}{\bar{z}}\gamma\,. \tag{C.32}$$

Overall we obtain the effective action:

$$S^{eff} = \int_\Sigma \langle\lambda^2\bar{\partial}\tilde{\mu}^1\rangle - \langle\tilde{\mu}^1\bar{\partial}\lambda^2\rangle + [\tilde{\lambda}^2\bar{\partial}\mu^1] - [\mu^1\bar{\partial}\tilde{\lambda}^2] + \gamma(\langle\lambda^2\tilde{\mu}^1\rangle + [\mu^1\tilde{\lambda}^2])\,, \tag{C.33}$$

which is the original action for massive models in four dimensions, with $Z = (\lambda^2, \mu^1)$ and $W = (\tilde{\mu}^1, \tilde{\lambda}^2)$. The determinant is:

$$\langle\lambda^2\xi\rangle^2[\tilde{\lambda}^2\tilde{\xi}]^2\frac{1}{t\tilde{t}}\,. \tag{C.34}$$

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