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Particles with Negative Energies in Nonrelativistic and Relativistic Cases

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Abstract: States of particles with negative energies are considered for the nonrelativistic and relativistic cases. In the nonrelativistic case it is shown that the decay close to the attracting center can lead to the situation similar to the Penrose effect for a rotating black hole when the energy of one of the fragments is larger than the energy of the initial body. This is known as the Oberth effect in the theory of the rocket movement. The realizations of the Penrose effect in the non-relativistic case in collisions near the attracting body and in the evaporation of stars from star clusters are indicated. In the relativistic case similar to the well known Penrose process in the ergosphere of the rotating black hole it is shown that the same situation as in ergosphere of the black hole occurs in rotating coordinate system in Minkowski space-time out of the static limit due to existence of negative energies. In relativistic cases differently from the nonrelativistic ones, the mass of the fragment can be larger than the mass of the decaying body. Negative energies for particles are possible in the relativistic case in cosmology of the expanding space when the coordinate system is used with a nondiagonal term in metrical tensor of the space-time. Friedmann metrics for three cases: open, close and quasieuclidian, are analyzed. The De Sitter space-time is shortly discussed.

Keywords: Penrose effect; black hole; metric; expanding Universe

1. Negative Energies and the Penrose Effect in Nonrelativistic Case

It is well known that in the non-relativistic case, energy is determined with accuracy to the additive constant. If the energy of a particle resting on infinity is put equal to zero then the sign of the energy defines movement of the particle in Kepler problem being either limited (in the case of negative sign) or nonlimited, when the energy is positive or zero [1]. If the body arriving in the region with nonzero gravitational field of some other object decays there in two parts, so that the velocity of one part is smaller than the second cosmic velocity in the point of decay, then its energy will be negative. So the energy of the second part becomes larger than the energy of the initial body. This is some realization of the Penrose effect [2,3] on getting the energy from the rotating black hole due to the decay of some body in the ergosphere.

Let us give some evaluations for the case when the initial body with mass m and the velocity v_0 on infinity decays on the distance r from the attracting center of mass M on two fragments so that the fragment with mass m_1 is flying relative to the first part of mass $m - m_1$ in the opposite direction to

the initial one with the velocity u . The velocity of the initial body at the distance r from the attracting mass M is

$$v_r = \sqrt{v_0^2 + \frac{2GM}{r}}, \quad (1)$$

where G is the gravitational constant. Let us find the velocities of the fragments after the decay using the conservation of the momentum. So one gets for the velocity of fragment with mass $m - m_1$

$$v_2 = v_r + \frac{m_1}{m}u. \quad (2)$$

The projection of the velocity of the fragment with mass m_1 on the direction of the initial movement in the point of decay is

$$v_1 = v_r - \left(1 - \frac{m_1}{m}\right)u. \quad (3)$$

It is easy to prove the identity

$$E_2 + E_1 = E_0 + E_f, \quad (4)$$

where

$$E_2 = (m - m_1) \left(\frac{v_2^2}{2} - \frac{GM}{r} \right) \quad (5)$$

is the energy of the fragment with mass $m - m_1$ in the gravitational field of the attracting body.

$$E_1 = m_1 \left(\frac{v_1^2}{2} - \frac{GM}{r} \right) \quad (6)$$

is the energy of the fragment with mass m_1 ,

$$E_0 = \frac{mv_0^2}{2} \quad (7)$$

is the kinetic energy of the initial body far from the attracting body. Note that in the nonrelativistic case one has in the equation of energy conservation for the considered process (4) the term

$$E_f = \frac{m_1 u^2}{2}. \quad (8)$$

This is the energy necessary for the flight of the fragment with the relative velocity u . It is evident interpretation that m_1 is the mass of the fuel flying from the nozzle of the rocket and E_f is the energy of the fuel.

As it is seen from (4) in case $E_1 - E_f < 0$ one has the analogy with the Penrose process. The full mechanical energy of the fragment with mass $m - m_1$ (the rocket without fuel) is larger than the initial mechanical energy of the initial object E_0 . From (2)–(8) one can find the conditions of the realization of such process

$$E_2 - E_0 = E_f - E_1 = m_1 \left[v_2 u \left(1 - \frac{m_1}{m} \right) - \frac{v_0^2}{2} \right]. \quad (9)$$

This expression is positive for small r . Note that formally the profit in energy is unlimited, however it is evident that r must be larger than the Schwarzschild radius $r_g = 2GM/c^2$ where c is the light velocity. As one can see from (9) to get the energy profit comparable with the relativistic rest mass of the fuel $m_1 c^2$ one needs relativistic values of the fuel expiration velocity and use of gravitational field close to the gravitational radius. Surely in this case one must do calculations using relativistic theory (changing the Newton potential on the Schwarzschild metric and taking into account that relative velocities are close to the velocity of light).

Note that for $m_1 \ll m$ and $v_0^2 \ll v_2 u$ formula (9) shows that the profit in the energy of the rocket in the engine start with the big velocity in the region of movement is explained by the fact that the engine traction with the fuel expiration relative velocity u is constant and the work A done in this case is proportional to the velocity of the rocket movement:

$$A \approx \frac{um_1}{\Delta t} v_2 \Delta t = v_2 u m_1. \quad (10)$$

It was H. Oberth [4] in 1929 who was the first to propose this way to increase the efficiency of the rocket engine by starting the engine in the periaster of the trajectory. It is used in astronautics in gravitational maneuvers using the Moon and inter planet flights.

Another example of realization of the Penrose effect in nonrelativistic case is the following. Let on two meeting circular orbits around the body with attracting mass M two particles 1, 2 with masses $m_1 \gg m_2$ are rotating on the distance r . Then in case of absolute elastic collision the particle with mass m_2 will move from the first particle with relative velocity $2\sqrt{GM/r}$. Its velocity in the system of rest will be $3\sqrt{GM/r}$ and so the energy will be $7GMm_2/(2r)$ increasing in $4GMm_2/(r)$. It is evident that the energy of the first particle decreases in the same value. Note that the profit in energy of the order close to $m_2 c^2$ is possible only near the gravitational radius of the attracting body.

If three or more particles are interacting one also can have process similar to the Penrose process. For example such decays occur for three interacting stars or evaporation of stars clusters [5].

2. Negative Energies in Rotating Coordinates

In relativity theory negative energies are usually absent.

In the nonrelativistic limit in the Kepler problem of the massive body with mass m moving around gravitating mass M on the distance r the full energy of this body taking into account the rest mass energy can be less than zero if this distance is very small

$$E = mc^2 + \frac{mv^2}{2} - G\frac{mM}{r} < 0 \Rightarrow r < \frac{GM}{c^2} = \frac{r_g}{2}. \quad (11)$$

Here r_g is the gravitational radius.

However physics in the ergosphere of the rotating black hole shows that, as it was discovered by R. Penrose [2], relativistic particles can have negative energies due to dependence of the energy on the angular velocity of the body rotating around the black hole. In our papers [6,7], it was shown that similar effect occurs in rotating coordinates in Minkowski space out of the static limit defined by us in analogy with the ergosphere.

The interval in Minkowski space in rotating cylindrical coordinates is

$$ds^2 = (c^2 - \Omega^2 r^2) dt^2 + 2\Omega r^2 d\varphi dt - dr^2 - r^2 d\varphi^2 - dz^2, \quad (12)$$

where Ω is the angular rotation velocity.

Let us consider this formula for the case of our solar system when the Earth is at rest (nonrotating around its axis) as ancient Greeks thought. The use of such rotating coordinate system is important because after all our observatories and observers are at rest in this case. If $\Omega_{\oplus} \approx 7.29 \cdot 10^{-5} \text{ s}^{-1}$ is the angular rotational velocity then from (12) one has $g_{00} = 0$ for $r_s = c/\Omega_{\oplus} = 4.11 \cdot 10^9 \text{ km}$. It is some distance between the orbits of Uranus and Neptune. For distance $r > r_s$ the coefficient of metric $g_{00} < 0$ but this does not mean that $ds^2 < 0$ because of the presence of nondiagonal term. For $ds^2 > 0$ physical motion with the velocity $v < c$ is still possible and no breaking of causality occurs. We call the distance r_s the static limit in analogy of the corresponding surface in the Boyer-Lindquist coordinates of rotating black hole [8]. For $r > r_s$ no body can remain at rest in the rotating coordinate system.

In cylindrical coordinates of Minkowski space 4-vector of energy-momentum

$$p'_i = \left(\frac{E'}{c}, -p'^r, -L'_z, -p'^z \right), \quad (13)$$

where

$$L'_z = r^2 p'^\varphi = mr^2 \frac{d\varphi'}{d\tau} = \frac{E'}{c^2} r^2 \frac{d\varphi'}{dt} \quad (14)$$

is the projection of the angular momentum on the z axis. Transforming (13) to the rotating coordinates one obtains

$$p_i = \left(\frac{E' + \Omega L'_z}{c}, -p'^r, -L'_z, -p'^z \right). \quad (15)$$

Therefore, the energy in these coordinates is

$$E = E' + \Omega L'_z. \quad (16)$$

So the energy E can be negative depending on the sign of $L'_z = L_z$.

As it is known the energy can be written with the help of the Killing vector ζ^i as

$$E^{(\zeta)} = \int_{\Sigma} T_{ik} \zeta^i d\sigma^k, \quad (17)$$

where $\{\Sigma\}$ is some set of spacelike hypersurfaces orthogonal to ζ^i and T_{ik} is the energy-momentum tensor of some matter. Note that negative values of the energy $E^{(\zeta)}$ can be obtained for the positive energy density T_{ik} if the Killing vector becomes spacelike as it is the case for the ergosphere of the rotating black hole and in our case in rotating coordinates in region out of the static limit. This means that the conditions of the Penrose-Hawking theorem on singularities in cosmology are not broken in spite of existence of particles with negative energies considered in this paper. Another remark concerns negative energy of the galaxy on the surface of the expanding sphere with homogeneous density of matter inside it resembling the closed Universe in Newton's approximation. In the exact relativistic case it is not the energy but the energy density of matter is present in Einstein's equations. Energy is not conserved in expanding Universe. However as one can see in next part of our paper this non conserved energy of the particle can have in some situations negative sign.

For a pointlike particle with the mass m located at point x_p one obtains (see [6])

$$T^{ik}(x) = \frac{mc^2}{\sqrt{|g|}} \int ds \frac{dx^i}{ds} \frac{dx^k}{ds} \delta^4(x - x_p) \quad (18)$$

and

$$E^{(\zeta)} = mc^2 \frac{dx^i}{ds} g_{ik} \zeta^k = c(p, \zeta). \quad (19)$$

For the rotating coordinates (12) and $\zeta = (1, 0, 0, 0)$ we obtain the energy $E^{(\zeta)}$ is equal to (16).

One can easily obtain the condition for a negative energy beyond the static limit [7]

$$L_z < -\sqrt{\frac{p_z^2 c^2 + m^2 c^4}{\Omega^2 - (c/r)^2}}, \quad \frac{v}{c} > \frac{c}{r\Omega}, \quad r > c/\Omega. \quad (20)$$

Some experiment to observe the consequences of the Penrose process for our solar system was proposed in [7]. The idea of the experiment is to show the possibility of such processes which can be interpreted differently in two different coordinate systems: the inertial one and the noninertial rotating coordinates. The particle beyond the static limit decays on two fragments one with the positive energy, the other with the negative one. The observers on the Earth can not see the particle with negative energy because it exits only out of the static limit. However, they can see the fragment with positive

energy which in rotating coordinates is larger than the energy of the initial decaying particle. From the inertial point of view the energies of all particles are positive but the energy of particle falling on the Earth is larger than the energy of the initial particle if one takes into account the rotation of the Earth around its axis. All quantitative estimates one can find in [7].

Further we shall discuss the third possibility of existence of particle states with negative energy, that of the expanding Universe.

3. Negative Energies and Static Limit in Expanding Universe

Expanding Universe in the standard model is described in the Friedmann-Robertson-Walker form in synchronous frame [9] as

$$ds^2 = c^2 dt^2 - a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right), \quad (21)$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$, $k = 1$ for closed cosmological model, $k = -1$ for open model and $k = 0$ for quasi-Euclidean flat model.

The metric (21) can be written also in other coordinates

$$ds^2 = c^2 dt^2 - a^2(t) \left(d\chi^2 + f^2(\chi) d\Omega^2 \right), \quad (22)$$

where

$$f(\chi) = \begin{cases} \sin \chi, & k = 1, \\ \chi, & k = 0, \\ \sinh \chi, & k = -1 \end{cases}$$

under replacing $r = f(\chi)$. In closed model χ is changing from 0 to π , in cases $k = 0, -1$ one has $\chi \in [0, +\infty)$. The radial distance between points $\chi = 0$ and χ in metric (22) is $D = a(t)\chi$ and it's the same in the metric (21). If t is fixed then the maximal value of D is $D_{\max} = \pi a(t)$. In open and flat models D is non limited.

Take the new coordinates t, D, θ, φ (see also [10,11]). Then

$$dD = \frac{\dot{a}}{a} D dt + a d\chi, \quad d\chi = \frac{1}{a} \left(dD - \frac{\dot{a}}{a} D dt \right) \quad (23)$$

and the interval (22) becomes

$$ds^2 = \left(1 - \left(\frac{\dot{a}}{a} \frac{D}{c} \right)^2 \right) c^2 dt^2 + 2 \frac{\dot{a}}{a} D dD dt - dD^2 - a^2 f^2(D/a) d\Omega^2. \quad (24)$$

The interval (24) is the special case of the more general interval

$$ds^2 = g_{00}(dx^0)^2 + 2g_{01}dx^0 dx^1 + g_{11}(dx^1)^2 + g_{\Omega\Omega}d\Omega^2, \quad (25)$$

where $g_{11} < 0$, $g_{\Omega\Omega} < 0$ and $g_{00}g_{11} - g_{01}^2 < 0$.

As it was shown in our paper [12] for the energy of the particle one has

$$E = p_0 c = mc^2 g_{0k} \frac{dx^k}{ds} = mc^2 \frac{dx^0}{ds} \left(g_{00} + g_{01} \frac{dx^1}{dx^0} \right). \quad (26)$$

This energy due to limitation from causality $ds^2 > 0$

$$\frac{g_{01} - \sqrt{g_{01}^2 - g_{11}g_{00} - g_{11}g_{\Omega\Omega} \left(\frac{d\Omega}{dx^0} \right)^2}}{-g_{11}} \leq \frac{dx^1}{dx^0} \leq \frac{g_{01} + \sqrt{g_{01}^2 - g_{11}g_{00} - g_{11}g_{\Omega\Omega} \left(\frac{d\Omega}{dx^0} \right)^2}}{-g_{11}} \quad (27)$$

is limited by inequality

$$\begin{aligned} \frac{mc}{-g_{11}} \frac{dx^0}{d\tau} \left(g_{01}^2 - g_{11}g_{00} - |g_{01}| \sqrt{g_{01}^2 - g_{11}g_{00} - g_{11}g_{\Omega\Omega} \left(\frac{d\Omega}{dx^0} \right)^2} \right) \leq E \leq \\ \frac{mc}{-g_{11}} \frac{dx^0}{d\tau} \left(g_{01}^2 - g_{11}g_{00} + |g_{01}| \sqrt{g_{01}^2 - g_{11}g_{00} - g_{11}g_{\Omega\Omega} \left(\frac{d\Omega}{dx^0} \right)^2} \right). \end{aligned} \quad (28)$$

As one can see the states with zero and negative energy are possible in the region where $g_{00} < 0$ (out of the static limit). In the chosen coordinate system in this region no physical body can be at rest. In this region movement will be observed either from the observer if $g_{01} > 0$ corresponding to the expanding Universe (24) with $\dot{a} > 0$ or to the observer for $g_{01} < 0$ in case of contracting Universe (24) with $\dot{a} < 0$.

The “static” limit for the metric (24) is

$$D_s = \frac{c}{|h(t)|}, \quad (29)$$

where $h(t) = \dot{a}/a$ is the Hubble parameter. This corresponds to the radius of the so-called light sphere [10]. In open and quasi-Euclidean models due to non limited D the region always exists with $D > D_s$, where particles can't be at rest in used coordinates and states with negative energies are possible. In closed model $D \leq \pi a(t)$ and the condition for the existence of such region is

$$|\dot{a}(t)| > c/\pi. \quad (30)$$

To understand the meaning of the limit (30) consider closed Universe with dust matter. Then (see [9])

$$a = a_0(1 - \cos \eta), \quad t = \frac{a_0}{c}(\eta - \sin \eta), \quad \eta \in (0, 2\pi), \quad a \in (0, 2a_0), \quad (31)$$

$D_s = a|\tan(\eta/2)|$ and (30) becomes

$$\left| \tan \frac{\eta}{2} \right| < \pi. \quad (32)$$

So that in region $\eta \in (0, 2\pi)$ one has

$$0 < \eta < 2 \tan^{-1} \pi, \quad 2(\pi - \tan^{-1} \pi) < \eta < 2\pi. \quad (33)$$

The scale factor in region (33) is in the limits

$$0 < a < \frac{2\pi^2 a_0}{1 + \pi^2} \quad (34)$$

and

$$D_s = \frac{D_{\max}}{\pi} \left| \tan \frac{\eta}{2} \right| \leq D_{\max}, \quad (35)$$

i.e., the region of existence of negative energies in closed dust Universe is changing from the almost entire Friedman universe except of the observer vicinity at $\eta \rightarrow 0, 2\pi$ to empty set at η outside the intervals (33).

Note that for closed dust Universe for $\eta \ll 1$ one has

$$a \approx \left(\frac{9c^2 a_0}{2} \right)^{1/3} t^{2/3}. \quad (36)$$

As it seen from (33) for this case one has the region out of the static limit with states with negative energies.

Static limit in the considered model at $\eta \lesssim 0.74\pi$ lies in the domain of the particle's horizon

$$l_p = a(t) \int_0^t \frac{dt'}{a(t')} = a\eta. \quad (37)$$

This means that the region with negative particle energies intersects with the region of causally connected phenomena for the observer at the origin of coordinate system.

The energy of the freely moving particle is

$$E = E' \left(1 - \frac{\dot{a}^2 D^2}{a^2 c^2} + \frac{\dot{a}}{a} \frac{D}{c^2} \frac{dD}{dt} \right), \quad (38)$$

where $E' = mc^2 dt/d\tau$. So in these coordinates particles with negative energies are moving in coordinate $D > D_s$ so that the velocity is slower than some definite value

$$\frac{dD}{dt} < \frac{c^2 a}{D \dot{a}} \left(\frac{D^2}{D_s^2} - 1 \right). \quad (39)$$

Let us rewrite the inequality (39) in terms of coordinates t, χ, θ, φ . Then we obtain in the case $\dot{a} > 0$

$$v = a \frac{d\chi}{dt} < -c \frac{D_s}{D}, \quad D > D_s. \quad (40)$$

This has a meaning similar to that obtained by us for the case of rotating coordinate frame t, χ, θ, φ : particles with negative energies close to the static limit ($D \rightarrow D_s$) must move with velocities close to the light velocity in direction of the observer in expanding Universe.

The necessary condition for the possibility of observing processes involving particles with negative energy is $D_s < L_H$, where

$$L_H = a(t)c \int_t^{t_{\max}} \frac{dt'}{a(t')} \quad (41)$$

is cosmological event horizon, t_{\max} is the life time of the universe. In the case of the de Sitter Universe with the scale factor $a = a_0 \exp Ht$, where a_0 and H are constants, $t_{\max} = \infty$, the Hubble constant is constant and the static limit $D_s = c/H$ is constant and it is equal to the cosmological event horizon. So processes (Penrose processes) with particles with negative energy are not seen by the observer in this case. This is analogous to situation of nonrotating black holes where particles with negative energy exist only inside the event horizon [13]. Note that in the limit $t \rightarrow \infty$, the standard cosmological Λ CDM (cold dark matter) model tends to the de Sitter stage.

However for models with $k = 0$ and scale factors $a = a_0 t^\alpha$ ($0 < \alpha < 1$) the cosmological horizon $L_H = \infty$ and some visible consequences of existence of particles with negative energies can be observed. In the closed Universe with dust matter (31) $t_{\max} = 2\pi a_0/c$ and we have

$$L_H = a(\eta)(2\pi - \eta). \quad (42)$$

The condition $D_s < L_H$ is reduced to

$$\left| \tan \frac{\eta}{2} \right| < 2\pi - \eta. \quad (43)$$

This inequality in particular is true over the entire interval of the existence of the static limit in the era of expansion of the closed universe $0 < \eta < 2 \tan^{-1} \pi$ and also at the end of the compression era. Thus some processes with the particles with negative energies can be observed in the models of closed universe also. However, unlike the de Sitter universe, the energy of the particle is not conserved in these cases. This makes the manifestation of the Penrose effect less obvious.

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