



General relativity and topological string duality through Penrose–Ward transform

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Abstract This paper discusses the relation between topological M-theory, self-dual Yang–Mills and general relativity. We construct a topological membrane field action from Witten’s cubic string field theory, which reduces to topological Yang–Mills on a one-parameter family of conifolds. It turns out that this can be interpreted as the twistor space of the four-dimensional Lagrangian submanifold M for large momenta. From the viewpoint of the target, we find that A-model and B-model on M unify in the topological membrane theory through the Penrose–Ward transform. The partition function is constructed and it is shown that, in the weak-coupling regime, it is equal to the partition function of Donaldson–Witten theory. Additionally, homological mirror symmetry, background independence as well as role of knot cobordisms as topological two-branes is discussed. It is outlined that all types of Floer homology are part of the topological membrane theory. Additionally, we find evidence that in the non-perturbative regime, the partition function of the membrane field action and that of the partially twisted (2,0) $SU(N)$ superconformal field theory on the worldvolume of N topological fivebranes must coincide.

1 Introduction

The field theoretic approach to Donaldson theory on four-manifolds, which is based on topological K-theory, using the moduli space of instantons, was introduced by Witten [1] following a series of papers, where he described Morse theory in the framework of perturbative supersymmetric quantum field theory [2]. Later Floer constructed invariants on three-manifolds Y utilizing tools of gauge theory to construct an infinite-dimensional Morse homology out of the Chern–Simons functional, which led to the realization [3] that the extension of the Donaldson polynomials to manifolds with boundary was possible under the condition that

they take values in the dual of the Floer groups. In [4] the author showed that the action of Donaldson–Witten theory could be derived by BRST quantization of a single topological term. We utilize the identification of three-dimensional Einstein gravity with the Chern–Simons functional [5,6] and show that at least in the maximal symmetric case the dynamics of the scale factor is described by the topological term with the usual (anti)-self dual gauge by identifying the Einstein–Hilbert action on $Y \times \mathbb{R}$ with self-dual Yang–Mills. This agrees with the findings of several older papers about self-dual gravity and the connection to loop quantum gravity and the Palatini action [7–9]. However, in the maximal symmetric case we can take a different route to string theory by the interpretation of the theta-vacuum structure as the gravitational background of a stack of N D-branes in congruence with the characteristic of other string/gauge dualities, where branes disappear to deform the background metric of the dual description. One version of these dualities of the topological string inspires another main emphasis of this paper. In [10] Witten showed that the topological A-model string with Calabi–Yau target T^*Y is a $U(N)$ Chern–Simons theory. We use the close relation between topological Yang–Mills and Chern–Simons to construct a membrane field theory, where the membrane fields describe maps from the worldvolume of a 2-Brane into an eight-dimensional target $X = T^*Y \times T^*\mathbb{R}_t$ subject to a Courant sigma-model. Its boundary theories are the A- and B-model topological string. Moreover, the topological M-model is a unification of both topological strings and possess a U-duality reflected in a simultaneous S-duality and T-duality in form of the weak/strong duality and homological mirror symmetry, respectively. The Lagrangian submanifolds are Lagrangian cobordisms between those of the topological string embedded inside X . It is shown that this membrane field theory reduces to Donaldson–Witten theory on a one-parameter family of Calabi–Yau three-varieties in the self-dual gauge and that complex and symplectic structures

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unify in form of the Penrose–Ward transform. This seems also evident by the fact that its partition function counts stable holomorphic structures over the twistor space of the Lagrangian cobordism of the form $M = Y \times \mathbb{R}$, which are in bijection with self-dual connections on M . Another interesting property is that it relates different CFTs with each other. Moreover, the cobordisms described by instantons relating two critical points of the Chern–Simons functional are the result of the deformation of one conformal background into another. One can find all different types of Floer homology, – instanton-, symplectic and instanton knot homology – inside this membrane field theory, which is an encouraging aspect. Additionally, one should be able to view homological mirror symmetry from the setting of this theory in an interesting new light, in that it emerges from the parent topological membrane model.

The paper focuses on the physical aspects of the theories discussed and is structured as follows: In the first two section we identify pure general relativity of maximal symmetric spacetimes with topological Yang–Mills theory by using tools of foliation and Floer homology. This is possible when we view the scale factor as a parametrization of the time coordinate of the gauge potential and follow that the vacuum structure can be interpreted as a stack of 3-branes. The main part of the paper is concerned with the topological membrane theory. In particular, its construction with ∞ -Chern–Weil theory in Sect. 4 and unification of the A- and B-model from the viewpoint of the target in form of the Penrose–Ward transform. Then we focus on the worldvolume/worldsheet in that we outline the underlying sigma-model and its connection to the AKSZ Poisson structures that describe the worldsheet theories of the topological strings. I give some more evidence of the similarity with Donaldson–Witten theory by calculating the partition function in analogy to Donaldson–Thomas theory on Calabi–Yau three-folds and analyse the connection between the different couplings and the corresponding behaviour of symplectic and complex structures in the target. There is also a categorical interpretation in form of a stable $(1, \infty)$ -category, which is shortly discussed. The two following sections are dedicated to homology mirror symmetry and the consequences for background independence, where we find further evidence that the membrane fields are instantons/coherent sheaves and paths in the stratified space of superconformal field theories with $N = 2$. Additionally, we analyse how general relativity, as described in the first section, arises from the membrane field theory, while incorporating closed/open string duality into the framework. Finally, we discuss how the AGT correspondence occurs, when we vary from the non-perturbative regime into the perturbative. This seems to imply that topological M-theory must also include the topological 6d SCFT, which we find is true.

2 General relativity as (A)SD Yang–Mills

Our goal in this section is to summarize and clarify what was already shown in [11], which is to connect $SO(4)$ Yang–Mills gauge potentials $A = A_\mu dx^\mu$ to homogeneous and isotropic solutions $g_{\mu\nu}$ of the four-dimensional pure Einstein–Hilbert action by reduction to a system with one degree of freedom. We start with the Yang–Mills action functional on a smooth manifold M as the cylinder $M \simeq Y \times \mathbb{R}$ in temporal gauge $A_0 = 0$.

$$S_{YM}[A] = \frac{1}{2g^2} \int d^4x \operatorname{tr} F_{\mu\nu}^2 = \frac{1}{g^2} \int d^4x \operatorname{tr} (\vec{E}_i^2 + \vec{B}_i^2), \quad (1)$$

where $F_{\mu\nu}$ is the usual $SO(4)$ -Lie-algebra-valued two-form defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig [A_\mu, A_\nu]. \quad (2)$$

This theory can be split into self-dual and anti-self-dual parts, which can be understood topologically via the clutching construction. The principal $SO(4)$ -bundle defined over $M \simeq Y \times \mathbb{R} \simeq S^4 \simeq \mathbb{R}^4 \cup \{\infty\}$ is decomposed by the definition of good open covers \mathfrak{U}_i , which overlap, where one defines the clutching map $g_{12} : \mathfrak{U}_1 \cap \mathfrak{U}_2 \rightarrow SU(2)$. It is convenient to split the bundle into two $SU(2)$ sub-bundles. We define two pairs of two open covers

$$\mathfrak{U}_1 := S^4 - \{+\infty\}$$

$$\mathfrak{U}_2 := B_{+\infty}$$

and

$$\mathfrak{U}_3 := S^4 - \{-\infty\}$$

$$\mathfrak{U}_4 := B_{-\infty}.$$

$B_{\pm\infty}$ are small neighbourhoods of the north and south poles. The odd-labelled cover the whole sphere except for the north and south pole, which are points at infinity. From homology and clutching construction now follows that $SO(4)$ instantons are classified by maps

$$\mathfrak{U}_1 \cap \mathfrak{U}_2 \times \mathfrak{U}_3 \cap \mathfrak{U}_4 \rightarrow SO(4)$$

The intersection is isomorphic to

$$\mathfrak{U}_1 \cap \mathfrak{U}_2 \simeq S^3,$$

which means that

$$h = g_{12} \cdot g_{34} : S^3 \times S^3 \rightarrow SO(4),$$

where $g_{12} : \mathfrak{U}_1 \cap \mathfrak{U}_2 \rightarrow SU(2)$ and since $S^3 \simeq SU(2)$

$$h : SU(2) \times SU(2) \rightarrow SO(4).$$

It is well known that the isomorphism classes of principal $SO(4)$ -bundles on S^4 are in bijection with homotopy classes

of continuous functions S^4 into the classifying space for $SO(4)$ -bundles and so

$$\begin{aligned} & \{\mathfrak{U}_1 \cap \mathfrak{U}_2 \times \mathfrak{U}_3 \cap \mathfrak{U}_4 \rightarrow SO(4)\}_{\text{homotopy}} \\ & \simeq \pi_3(S^3 \times S^3) = \mathbb{Z} \oplus \mathbb{Z}. \end{aligned}$$

The next step is to include principal connections and the corresponding curvature forms. The first intuition is, that one has four $\mathfrak{su}(2)$ -curvature forms that are locally defined on the four charts \mathfrak{U}_i . But since on the quadruple overlap, we have that $h = g_{12} \cdot g_{34} = g_{14}$, the $\mathfrak{so}(4)$ -valued one-form A decomposes into

$$\begin{aligned} A^+ & \in \Omega^1(\mathfrak{U}_1, \mathfrak{su}_L(2)) \\ A^- & \in \Omega^1(\mathfrak{U}_4, \mathfrak{su}_R(2)). \end{aligned}$$

We see that we can find the reason why the gravitational instanton corresponds to two Yang–Mills instantons in the bundle topology. Hence, the $\mathfrak{so}(4)$ -valued curvature form F_A is not defined globally on all S^4 , but is rather a gluing of the $F_{A^\pm} = F^\pm$, which are locally defined on the charts \mathfrak{U}_1 and \mathfrak{U}_4 , which we just call \mathfrak{U}_\pm in the following. The six-dimensional space of two forms $\Omega^2(M, \mathfrak{so}(4))$ decomposes into three-dimensional vector spaces

$$\Omega^2(M, \mathfrak{so}(4)) = \Omega^{2+}(\mathfrak{U}_+, \mathfrak{su}_L(2)) \oplus \Omega^{2-}(\mathfrak{U}_-, \mathfrak{su}_R(2)).$$

Via the principle of extremal action, the equations of motion unfold to be

$$\mathcal{D}_\mu F^{\mu\nu} = 0. \quad (3)$$

Since

$$\mathcal{D}_\mu * F^{\mu\nu} = 0$$

where $\frac{1}{2}\epsilon_{\alpha\beta\mu\nu}F_{\alpha\beta} = *F^{\mu\nu}$ is the dual field strength, equation (3) can be expressed as a first order system

$$F^{\mu\nu} = \pm *F^{\mu\nu}. \quad (4)$$

(Anti)-self-dual BPS states solving (4) are called instantons and they saturate the bound

$$S \geq \frac{8\pi^2}{g^2} |k| \quad (5)$$

with the winding number k of the configuration and action S . These equations are automatically satisfied if the electric and magnetic fields associated to the gauge connection are parallel, which means that in Euclidean space they satisfy

$$E_i = i B_i. \quad (6)$$

The important point is that a Yang–Mills instanton $A \in \Omega^1(Y \times \mathbb{R}, \mathfrak{su}(2))$ in temporal gauge can be equally defined as a one-parameter family of gauge connections on Y

$$t \rightarrow A_t \in \Omega^1(S^3, \mathfrak{su}(2))$$

$$\vec{A} = \sum_{i=1}^3 A_i dx^i. \quad (7)$$

It is easy to show that the (anti)-self-dual Yang–Mills equation can be cast into the form

$$\frac{d}{dt} A_t = \partial_t CS = \pm *_3 F_{A_t} \in \Omega^1(Y, \mathfrak{su}(2)). \quad (8)$$

with $*_3\alpha = *(dt \wedge \alpha)$, where α is a one-form on Y . This expression tells us that Yang–Mills instantons are the gradient flow lines of the Chern–Simons functional

$$CS[A] = \frac{1}{8\pi^2} \int_Y \text{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \quad (9)$$

on the space of principal $\mathfrak{su}(2)$ -connections. One can define a chain complex and the corresponding homology is known as Floer homology [12, 13], which is essentially Morse theory of the functional above. Let us denote the space of connections of $P \rightarrow Y$ by \mathbb{A} , and the set of the sections of the adjoint bundle $ad(P)$ as \mathcal{G} . The elements are gauge transformations g , which are bundle automorphisms that cover the identity. The goal is to construct a Morse chain complex out of \mathbb{A}/\mathcal{G} . Following Floer and Donaldson, we consider the tangent space of the subspace of irreducible connections \mathbb{A}^* at a connection B given by

$$T_B \mathbb{A}^* \cong \Omega^1(Y, ad(P)). \quad (10)$$

We can do the same with an equivalence class $[B] \in \mathbb{A}/\mathcal{G}$. The metric on Y induces a Hodge star operator $*_3$, with which we can define the adjoint covariant derivative $d_B^* = -*_3 d_B *_3$ with respect to B . It turns out that the tangent space is the kernel of d_B^* .

$$T_{[B]}(\mathbb{A}^*/\mathcal{G}) \cong \ker(d_B^*). \quad (11)$$

The critical points of the Chern–Simons action functional on \mathbb{A}^*/\mathcal{G} are the gauge equivalence classes of flat connections. In this infinite-dimensional setting the difference of the index of critical points of the Morse function is replaced by the spectral flow of operators of a path between two flat connections. The difference of the set of eigenvalues that cross zero along the two paths between both connections in opposite directions is denoted by $sf(B_0 B_1)$. Consider now the space of (anti)-self dual connections A^\pm on $P \times \mathbb{R}$ over $Y \times \mathbb{R}$ moduli automorphisms $\text{Aut}(P \times \mathbb{R})$ and the \mathbb{R} -action by shifting the t variable. By the perturbation $\varepsilon > 0$ of the CS-action, we create a smooth oriented manifold \mathcal{M}_ε with

$$\dim \mathcal{M}_\varepsilon = sf(A^+, A^-) - 1. \quad (12)$$

If its dimension is zero, \mathcal{M}_ε is compact and we can count its points by signs. We can define the boundary operator

$$\begin{aligned} \partial A_\pm &= \sum_{\substack{A_\pm \in \mathbb{A}_f, \\ sf(A^+, A^-) = 1}} \# \mathcal{M}_\varepsilon(A^+, A^-) \cdot A_\pm \\ & \quad \end{aligned} \quad (13)$$

and the Floer complex $CF_\bullet(Y)$, which is independent of the perturbation as well as the metric. Additionally, the boundary operator is nilpotent, and the corresponding homology is defined as

$$HF_\bullet(N) = \frac{\ker \partial}{\dim \partial}. \quad (14)$$

We will return to the legacy of Andreas Floer when we discuss its possible role in the stable $(\infty, 1)$ -category of Lagrangian submanifolds with Lagrangian branes as morphisms in the context of homological mirror symmetry.

For the moment this is enough information to associate the Yang–Mills action functional with the Einstein–Hilbert action. We start with the Lorentzian Yang–Mills action

$$\begin{aligned} S_{YM} &= - \int_{Y \times \mathbb{R}} |F|^2 d\mu = - \frac{1}{2} \int d^4x \operatorname{Tr} F_{\mu\nu}^2 \\ &= - \int d^4x \operatorname{tr} (\vec{E}_i^2 + \vec{B}_i^2) \end{aligned} \quad (15)$$

where, thanks to the temporal gauge, the electric and magnetic fields have a simple interpretation:

$$E_i = F_{0i} = - \frac{\partial A_i(\vec{x}, t)}{\partial t} \quad (16)$$

can be interpreted as the velocity vector on the path A_t , whereas the magnetic field

$$B_i = \frac{1}{2} \epsilon_{ijk} F_{jk} \quad (17)$$

is identified with the curvature of A_t on S^3 . Now we will parametrize t by the collective coordinate $\phi(t)$. Notice that the family $A_i^{(\phi)}$ is invariant under the choice of $\phi(t)$. We will see that this function is the scale factor of an FLRW-metric. Let us continue by calculating

$$-E_i = \partial_0 A_i^{(\phi)} = \frac{\partial A_i(\vec{x}, \phi(t))}{\partial \phi} \dot{\phi}(t) \quad (18)$$

Hence, we write (15) as

$$\begin{aligned} S_{YM} &= - \int d^4x \operatorname{tr} (\vec{E}_i^2 + \vec{B}_i^2) \\ &= - \int dt \left(\int \operatorname{tr} \left(\frac{\partial \vec{A}}{\partial \phi} \right)^2 d^3x \dot{\phi}^2(t) + \int \operatorname{tr} \vec{B}^2 d^3x \right). \end{aligned} \quad (19)$$

We require that

$$m(\phi) = -2 \int \operatorname{tr} \left(\frac{\partial \vec{A}}{\partial \phi} \right)^2 d^3x = 1 \quad (20)$$

and identify the second term, which is the curvature density integrated over S^3 , with the potential

$$2V(\phi) = - \int \operatorname{tr} \vec{B}^2 d^3x. \quad (21)$$

The expression (20) is a path-dependent mass, which is in general a positive number. Since the gauge potential is independent of the choice of parametrization, the tunnelling rate R does not know about this quantity as well. Additionally, at the minima of the potential $V(\phi)$, we have also $m(\phi) = 0$. We assume that the gauge potential is independent of the space coordinate. With the Pauli vector $\vec{\sigma}$ we can write it as

$$\vec{A} = f(\phi(t)) \vec{\sigma}.$$

Equation (20) then gives us the relation between the volume V_Y and the function $f(\phi(t))$.

$$c_1 \left(\frac{\partial f(\phi(t))}{\partial \phi} \right)^2 V_Y = 1,$$

where c_1 is a positive and real constant from taking the trace. If the gauge potential is linear in the function ϕ , then we have that

$$c_1 V_Y = 1.$$

Following from the above, the Yang–Mills action on $Y \times \mathbb{R}$ becomes the action of a one-dimensional particle in configuration space of principal connections modulo gauge transformations inside $V(\phi)$

$$S_{YM} = \int dt \left(\frac{1}{2} \dot{\phi}^2(t) - V(\phi) \right). \quad (22)$$

Since instantons are paths for which the tunnelling rate R becomes maximal they satisfy

$$\vec{E}(\vec{x}, \phi(t)) = \pm \vec{B}(\vec{x}, \phi(t)) \quad (23)$$

because then we have equality in the gauge independent quantity

$$e^{-R} \leq e^{-2 \int_{t_1}^{t_2} |\vec{E} \cdot \vec{B}| d^4x} = e^{-8\pi^2 |k|}. \quad (24)$$

How is this related to the Einstein–Hilbert action of a FLRW-metric? First, we want to consider the Riemannian four-manifold as one-parameter families of metrics on Y , similar to how we considered instantons as families of gauge fields on Y . We follow the description in [14] and we will write bold variables to distinguish tensors and vectors from scalars when written in non-coordinate form and to avoid confusion when these variables are used later in a different context. It is convenient to specify how exactly we seek to foliate (M, g) . We can embed hypersurfaces into M by a map

$$\mathfrak{E} : \tilde{\Sigma} \rightarrow M \quad (25)$$

such that the image $Y = \mathfrak{E}(\tilde{\Sigma})$ and the preimage $\tilde{\Sigma}$ are homeomorphic. It also defines a push forward map \mathfrak{E}^* between the respective tangent spaces $T(M)$ and $T(Y)$. The metric induced on the hypersurface is then $\mathbf{h} = \mathfrak{E}^*g$. Specifying $x^i = (x, y, z)$, the components of \mathbf{h} are $h_{ij} = g_{ij}$. Furthermore, there is the Levi-Civita connection \mathfrak{D} on Y satisfying

$\mathcal{D}\mathbf{h} = 0$. The deduced Riemann curvature tensors on M and Y are

$${}^{(4)}R^\gamma_{\mu\alpha\beta}w^\gamma = [\nabla_\alpha, \nabla_\beta]w^\gamma \quad (26)$$

and

$${}^{(3)}R^k_{lij}v^l = [\mathfrak{D}_i, \mathfrak{D}_j]v^k \quad (27)$$

respectively, where $w \in T(M)$ and $v \in T(Y)$ and ∇ is the spacetime connection. We define the Ricci tensor by ${}^{(4)}R_{\alpha\beta} = {}^{(4)}R^\mu_{\mu\alpha\beta}$ and the Ricci scalar ${}^{(4)}R = g^{\mu\nu}R_{\mu\nu}$ as its contraction. Additionally, to the intrinsic curvature there is also the notion of extrinsic curvature, which depends on the embedding. We will only consider spacelike hypersurfaces. We can define the variation of the normal \mathbf{n} along a vector \mathbf{v} tangent to Y via ∇ by $\mathcal{S} : \mathbf{v} \rightarrow \nabla_\mathbf{v}\mathbf{n}$. Spacelike hypersurfaces are defined by $\mathbf{n} \cdot \mathbf{n} = -1$. The extrinsic curvature $\mathbf{K} : (\mathbf{u}, \mathbf{v}) \rightarrow -\mathbf{u} \cdot \mathcal{S}(\mathbf{v})$ is a bilinear form defined on the tangent spaces of Y . The trace $K = h^{ij}K_{ij}$. In the following assume that M has no closed timelike curves. As for the instantons we want to consider continuous sets of hypersurfaces $(Y_t)_{t \in \mathbb{R}}$ and a foliation of the globally hyperbolic spacetime M such that its topology is $Y \times \mathbb{R}$. That is, Y is a Cauchy surface. We define the lapse function N by $\mathbf{n} = N\vec{\nabla}t$ and the normal evolution vector $\mathbf{m} = N\mathbf{n}$. Notice that we have $\mathbf{m} \cdot \mathbf{m} = -N^2$. The evolution of the three-metric is given by its Lie derivative along \mathbf{m} .

$$\mathcal{L}_m h_{ij} = -2NK_{ij} \quad (28)$$

It is possible to express the Riemann tensor of M in terms of quantities associated to the hypersurfaces Y_t . To do this we need to write down the Gauss relation first, where one more quantity needs to be defined. We write the orthogonal projector onto the hypersurfaces $\vec{\gamma}$, which satisfies $\vec{\gamma}(\mathbf{n}) = 0$ and $\vec{\gamma}(\mathbf{v}) = \mathbf{v}$. Furthermore, $\vec{\gamma}\nabla = \mathfrak{D}$ and $\mathcal{L}_m\vec{\gamma} = 0$. It is also important to note that

$$\nabla_\beta n_\alpha = -K_{\alpha\beta} - \mathfrak{D}_\alpha \ln N n_\beta. \quad (29)$$

Then the contracted Gauss relation is

$$\begin{aligned} \gamma^\mu_{\alpha} \gamma^\nu_{\beta} {}^{(4)}R_{\mu\nu} + \gamma_{\alpha\mu} n^\rho \gamma^\nu_{\beta} n^\sigma {}^{(4)}R^\mu_{\rho\nu\sigma} \\ = R_{\alpha\beta} + KK_{\alpha\beta} - K_{\alpha\mu} K^\mu_{\beta}. \end{aligned} \quad (30)$$

Writing the projector in components, one can show that

$$\begin{aligned} \gamma_{\alpha\beta} n^\rho \gamma^\nu_{\beta} n^\sigma {}^{(4)}R^\mu_{\rho\nu\sigma} \\ = -K_{\alpha\sigma} K^\sigma_{\beta} + \frac{1}{N} \mathfrak{D}_\alpha \mathfrak{D}_\beta \gamma^\mu_{\alpha} \gamma^\nu_{\beta} n^\sigma \nabla_\sigma K_{\mu\nu} \\ = \mathcal{L}_m K_{\alpha\beta} + \frac{1}{N} \mathfrak{D}_\alpha \mathfrak{D}_\beta N + K_{\alpha\mu} K^\mu_{\beta}. \end{aligned} \quad (31)$$

Combining both relations, we get

$$\begin{aligned} \gamma^\mu_{\alpha} \gamma^\nu_{\beta} {}^{(4)}R_{\mu\nu} = -\frac{1}{N} \mathcal{L}_m K_{\alpha\beta} - \frac{1}{N} \mathfrak{D}_\alpha \mathfrak{D}_\beta N \\ + R_{\alpha\beta} + KK_{\alpha\beta} - 2K_{\alpha\mu} K^\mu_{\beta}. \end{aligned} \quad (32)$$

Now contracting with $\gamma^{\alpha\beta}$ we get

$${}^{(4)}R + {}^{(4)}R_{\mu\nu} n^\mu n^\nu = {}^{(3)}R + K^2 - \frac{1}{N} \mathcal{L}_m K - \frac{1}{N} \mathfrak{D}_i \mathfrak{D}^i N, \quad (33)$$

which can be further simplified by the Gauss relation to

$${}^{(4)}R = {}^{(3)}R + K^2 + K_{ij} K^{ij} - \frac{2}{N} \mathcal{L}_m K - \frac{2}{N} \mathfrak{D}_i \mathfrak{D}^i N. \quad (34)$$

This separation serves the identification of constraints and true dynamical variables and is the right starting point for the Hamiltonian formulation of general relativity. For a patch V of M delimited by two spacelike hypersurfaces at constant t_1 and t_2 , we can write the Einstein Hilbert action as

$$S_{EH} [g_{\mu\nu}] = \int_V \sqrt{g} {}^{(4)}R d^4x \quad (35)$$

with $\sqrt{g} = N\sqrt{h}$

$$\begin{aligned} S_{EH} [g_{\mu\nu}] = \int_V N\sqrt{h} \left({}^{(3)}R + K^2 + K_{ij} K^{ij} \right. \\ \left. - \frac{2}{N} \mathcal{L}_m K - \frac{2}{N} \mathfrak{D}_i \mathfrak{D}^i N \right) d^4x. \end{aligned} \quad (36)$$

The last two terms are total derivatives and so they won't contribute to the equations of motions but ensure that we have a well-defined variation principle when dealing with Dirichlet boundary conditions. One can show that they are equal to the Gibbons–York–Hawking [15] boundary term $2 \int_Y \varepsilon \sqrt{h} K d^3x$. Hence, we can write the above action as an integral over the hypersurface.

$$S_{EH} [g_{\mu\nu}] = \int_{t_1}^{t_2} \int_{Y_t} N\sqrt{h} \left({}^{(3)}R + K_{ij} K^{ij} - K^2 \right) d^3x. \quad (37)$$

For hypersurfaces with constant curvature, we have the FLRW-metric

$$g_{\mu\nu} = -N^2(t)dt^2 + \phi^2(t)\gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \quad (38)$$

where ϕ is the scale factor and β is the shift defined by $\partial_t = \mathbf{m} + \beta$. The Riemann curvature tensor for the synchronous slicing with $\beta = 0$, $N = 1$ is

$${}^{(3)}R_{ijkl} = \frac{R}{6} (\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}), \quad (39)$$

and the Ricci-scalar

$${}^{(3)}R_{ij} = 2k\gamma_{ij}. \quad (40)$$

We can easily calculate that

$$K_{ij} K^{ij} - K^2 = -6 \left(\frac{\dot{\phi}}{\phi} \right)^2 \quad (41)$$

$${}^{(3)}R = 6 \frac{k}{\phi^2} \quad (42)$$

$$\sqrt{h} = \phi^3. \quad (43)$$

Henceforth, the Einstein–Hilbert action becomes a one-dimensional problem, where the dynamical variable is the scale factor.

$$S_{EH} = \frac{1}{2} \int dt 6\phi \left(-\dot{\phi}^2 + k \right). \quad (44)$$

The crucial step is now to view the curvature parameter k of $Y \times \mathbb{R}$ as a one-parameter family of curvature parameters of Y_t , so that it also depends on the scale factor $\phi(t)$. We see that if

$$\dot{\phi}^2(t) = k(\phi(t)), \quad (45)$$

then $(Y \times \mathbb{R}, g_{\mu\nu})$ is a gravitational instanton because the energy is minimized, and it satisfies

$${}^{(4)}R_{\mu\nu} = 0. \quad (46)$$

This becomes clearer when we use the Hamiltonian approach.

$$L(q, \dot{q}) = N\sqrt{h} \left({}^{(3)}R + K_{ij}K^{ij} - K^2 \right). \quad (47)$$

The canonical conjugate momentum is

$$\pi^{ij} := \frac{\partial L}{\partial \dot{h}_{ij}} = \sqrt{h} \left(K h^{ij} - K^{ij} \right). \quad (48)$$

Its Legendre transformation gives the Hamiltonian density

$$\mathcal{H} = -N\sqrt{h} \left({}^{(3)}R + K_{ij}K^{ij} - K^2 \right) - 2\beta^i \left(\mathcal{D}_i K - \mathcal{D}_j K_i^j \right). \quad (49)$$

The Hamiltonian

$$H = - \int_{\Sigma_t} N\sqrt{h} \left({}^{(3)}R + K_{ij}K^{ij} - K^2 \right) - 2\beta^i \left(\mathcal{D}_i K - \mathcal{D}_j K_i^j \right) d^3x \quad (50)$$

simplifies in synchronous gauge to

$$H = - \int_{\Sigma_t} N\sqrt{h} \left({}^{(3)}R + K_{ij}K^{ij} - K^2 \right) d^3x. \quad (51)$$

If Eq. (45) holds, the Hamiltonian constraint $H = 0$ is satisfied, while the momentum constraint is automatically satisfied in case of a FLRW-metric. Thus, we can consider the action of which (45) are the equations of motion.

$$S[\phi] = \frac{1}{2} \int dt \left(\dot{\phi}^2(t) - k(\phi) \right). \quad (52)$$

The minima associated to the potential are Ricci-flat spaces and for multiple vacua there are interpolating solutions as flow lines of the corresponding gradient flow. For more complicated spacetimes, we cannot write the Einstein–Hilbert action as a Yang–Mills action. The connection formulation needed is called Palatini action.

Nevertheless, in the prescribed case general relativity is reduced to the scale factors particle like motion through configuration space feeling the potential $k(\phi)$. Upon identifying $k(\phi(t)) = 2V(\phi(t))$,

$$k(\phi(t)) = 2V(\phi(t)), \quad (53)$$

we see that the scale factor is essentially a parametrization of the path described by the instantons in field configuration space of equivalence classes of $SU(2)$ -Lie algebra valued principal connections modulo gauge transformations. It follows immediately, that

$$\dot{\phi} = \pm \sqrt{k(\phi)} \iff \vec{E}(\vec{x}, \phi(t)) = \pm \vec{B}(\vec{x}, \phi(t)) \quad (54)$$

because with $k(\phi) = \left(\frac{dW}{d\phi} \right)^2$ and $V(\phi) = \frac{1}{2} \left(\frac{dW}{d\phi} \right)^2$ the Hamiltonians can be written as

$$\begin{aligned} H_g &= \frac{1}{2} \int dt \left(\dot{\phi}^2 \mp \left(\frac{dW}{d\phi} \right)^2 \right) + [W(\phi^+) - W(\phi^-)] \\ H_{YM} &= \frac{1}{2} \int dt \left(\dot{\phi}^2 \mp \left(\frac{dW}{d\phi} \right)^2 \right) + [W(\phi^+) - W(\phi^-)] \end{aligned} \quad (55)$$

The last term is topological and has an integral representation

$$\int_{\phi^-}^{\phi^+} \sqrt{2V(\phi)} d\phi = \int_{-\infty}^{\infty} \sqrt{2V(\phi)} \dot{\phi} dt, \quad (56)$$

which in the Yang–Mills case we recognize as proportional to the second Chern-class

$$Q = \int_{-\infty}^{\infty} \sqrt{2V(\phi)} \dot{\phi} dt = \int \text{tr} \vec{E} \vec{B} d^4x = \int F_{\mu\nu} * F^{\mu\nu} d^4x. \quad (57)$$

The Ricci tensor associated to the path is

$$\begin{aligned} {}^{(3)}R_{ij}(t) &= 2k(\phi) \gamma_{ij} \\ &= 2\dot{\phi}^2(t) \gamma_{ij} \end{aligned} \quad (58)$$

where γ_{ij} is the maximally symmetric metric on Y . We can show not only that the above expression is the Einstein equations with a positive cosmological constant on Y , but that it hides self-dual and anti-self-dual equations of $SU(2)$ Yang–Mills, which follows directly from (45) (Fig. 1). We can write

$$\dot{\phi}^2(t) = k(\phi) = \int_{S^3} |B|^2 = 2V(\phi(t)), \quad (59)$$

where we scaled away the unimportant constant factor. Splitting this into kink and anti-kink denoted by the superscript + and – respectively through taking the square root gives

$$\begin{aligned} \dot{\phi}^+(t) &= \sqrt{2V(\phi(t))} \\ \dot{\phi}^-(t) &= -\sqrt{2V(\phi(t))} \end{aligned} \quad (60)$$

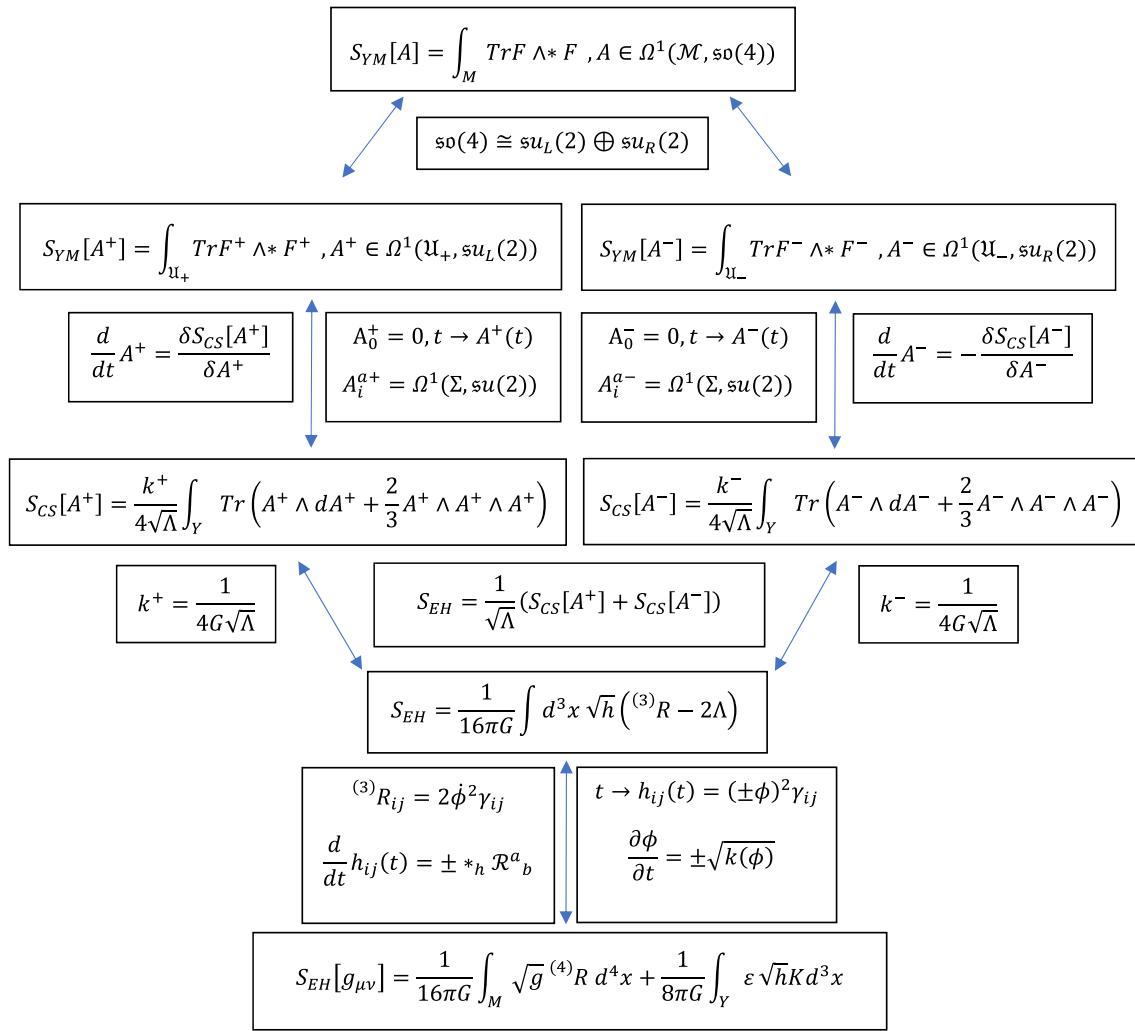


Fig. 1 Connection between gravitational instantons in the FLRW framework and (anti)-self-dual Yang–Mills

Because of (16) and (17) this is equal to writing

$$-E_i = \frac{A_i^\pm(\vec{x}, \phi(t))}{\partial \phi} \dot{\phi}^\pm(t) = \pm \frac{1}{2} \epsilon_{ijk} F_{jk} \quad (61)$$

$$\iff \partial_0 A_i^{\pm(\phi)} = \pm *_3 F_{A_i^\phi} \in \Omega^1(S^3, su(2)) \quad (62)$$

$$\iff F_{\mu\nu}^\pm = \pm * F_{\mu\nu}^\pm, \quad (63)$$

where the last equation is defined on $Y \times \mathbb{R}$. This concludes that the pure Einstein equations for an isotropic and homogeneous space $Y \times \mathbb{R}$ with $k(\phi) \geq 0$ have an interpretation as the combination of anti-self and self-dual $SU(2)$ -instanton equations on $Y \times \mathbb{R}$. This subset of solutions to the pure Einstein equations exists in Yang–Mills. The non-trivial solutions (with non-zero winding) are characterized by vanishing four-dimensional curvature and never vanishing three-curvature. We can express the gradient flow equation (62) in terms of vielbeins ε_i^α as

$$h_{ij}(t) = \varepsilon_i^\alpha(t) \varepsilon_j^\beta(t) \delta_{\alpha\beta} \quad (64)$$

$$\varepsilon_i^\alpha(t) = \phi(t) \gamma_i^\alpha$$

$$\partial_0 \varepsilon_i^\alpha(t) \varepsilon_j^\beta(t) \delta_{\alpha\beta} = \dot{\phi}^2(t) \gamma_{ij} = *_3 \mathcal{R} = (^3)R_{ij}, \quad (65)$$

where $\mathcal{R} = d\omega + \omega \wedge \omega$ is the curvature-form of an $\mathfrak{so}(4)$ -connection ω and $\alpha, \beta \in 1, 2, 3$. In a similar manner as the Yang–Mills instantons are the gradient flow lines of the Chern–Simons action functional, the gravitational instantons are essentially the gradient flow lines of three-dimensional Einstein–Hilbert action with a positive cosmological constant, yet they are neither self nor anti-self-dual. The gravitational action for an FLRW-space with $1 \geq k(\phi) \geq 0$

$$S_g[g_{\mu\nu}] = \int_{t_1}^{t_2} \int_{\Sigma_t} N \sqrt{h} ((^3)R + K_{ij} K^{ij} - K^2) d^3x. \quad (66)$$

written down in the language of connection one-forms is

$$S_{YM}[A] = -\frac{1}{2} \int_M tr F \wedge *F + \frac{\theta}{16\pi^2} \int_M tr F \wedge F, \quad (67)$$

where $A \in \mathcal{Q}^1(M, \mathfrak{so}(4))$. The choice of the gauge group is interesting in so far that there is a FLRW-metric associated to every gauge field in the theory, whose isometry group is $SO(4)$ for $k(\phi) > 0$. Classically, this theory has no propagating degrees of freedom. Instead, there are only non-trivial tunnelling processes. The topological Yang–Mills action enjoys a much larger gauge group than the ordinary action. Under the action of this enlarged gauge group, the curvature two-form takes the role of a gauge field. So, we need to impose a gauge fixing condition of the field strength as well in such a theory, but that is exactly what we did to be able to understand general relativity as a Yang–Mills theory. The requirement was that the field strength of every connection is (anti)-self dual, expressed by the scale factor saturating the Bogomolny-bound and non-abelian magnetic and electric field being parallel. As a matter of this fact the theory we described is actually defined by

$$S_T[A] = \int_M \text{tr} F \wedge F$$

$$F = \pm *F. \quad (68)$$

We can summarize this section by the statement that the Einstein–Hilbert action on the subspace of superspace of FLRW-metrics has the structure of (A)SD Yang–Mills theory. BRST-quantization of (68) involves two additional gauge fixings. In Witten 1988, a topological QFT was analysed, which in fact can be reproduced by BRST-quantization of the topological Yang–Mills action. It was also shown that the BRST-quantized action is deeply related to the computation of Donaldson polynomials and Floer groups. Moreover, it is basically the quantum field theoretic approach to the computation of Donaldson polynomials, taking their values in the dual of the Floer homology at the boundary. The fully gauge fixed action (68) has the form

$$S_{DW} = \int_M d^4x \sqrt{g} \text{tr} \left(\frac{3}{8} F_{\mu\nu} F^{\mu\nu} + \frac{3}{8} F_{\mu\nu} *F^{\mu\nu} \right. \\ \left. + \frac{1}{2} \varrho D_\mu D^\mu \lambda - i\eta D_\mu \psi^\mu + i D_\mu \chi^{\mu\nu} - \frac{i}{8} \varrho [\chi_{\mu\nu}, \chi^{\mu\nu}] \right. \\ \left. - \frac{i}{2} \lambda [\psi_\mu, \psi^\mu] - \frac{i}{2} \varrho [\eta, \eta] - \frac{1}{8} [\varrho, \lambda]^2 \right), \quad (69)$$

where D_μ is the covariant derivative, ϱ, λ and ψ, η, χ are bosonic fields and fermionic fields, respectively, and χ is self-dual. The correlation function with respect to this action computes the Donaldson polynomials.

$$\langle \mathcal{O}_{a_1} \cdots \mathcal{O}_{a_d} \rangle = \int e^{-\frac{S_{DW}(\Phi)}{\hbar}} \prod_{i=1}^d \mathcal{O}_{a_i}(\Phi) D[\Phi], \quad (70)$$

with nice functionals \mathcal{O} in the fields, which are collectively denoted by Φ . The partition function is invariant under the change of the metric and the strength of the coupling as long as it is non-zero. Following Witten, for cycles

$\gamma \in H^{k_\gamma}(M, \mathbb{Z})$, where $k_\gamma = 0, \dots, 4$, one defines differential forms W of degree k_γ as

$$W_0 = \frac{1}{2} \text{tr}(\varrho \wedge \varrho)$$

$$W_1 = \text{tr}(\varrho \wedge \psi)$$

$$W_2 = \text{tr} \left(\frac{1}{2} \psi \wedge \psi + i \varrho \wedge F \right)$$

$$W_3 = i \text{tr}(\psi \wedge F)$$

$$W_4 = -\frac{1}{2} \text{tr}(F \wedge F). \quad (71)$$

The observables of interest are

$$\mathcal{O}^{(\gamma)} = \int_\gamma W_{k_\gamma}, \quad (72)$$

to which one associates $4-k_\gamma$ forms Θ^γ on the Uhlenbeck-compactified moduli space of instantons $\overline{\mathcal{M}}_{Inst}$. The Donaldson polynomials are

$$\langle \mathcal{O}^{r_1} \cdots \mathcal{O}^{r_d} \rangle = \int e^{-S_{DW}(\Phi)/\hbar} \prod_{i=1}^d \mathcal{O}^{(r_i)}(\Phi) D[\Phi]$$

$$= \int_{\overline{\mathcal{M}}_{Inst}} \prod_{i=1}^d \Theta^{\gamma_i}. \quad (73)$$

The findings in this section can be easily generalized to three-manifolds, which are simply connected, compact and boundaryless, since they are homeomorphic to the three-sphere. Then the potential would still control the constant curvature and its minima would still be corresponding to Ricci-flat spaces. Further generalisation is also possible as long as the resulting four-dimensional gradient flow lines are orientable and admit a Riemannian metric but then the construction in terms of the scale factor fails and it is unclear if there is a four-dimensional action from which the instanton equations can be derived by variational principles. It would be more likely that we have to impose the (anti)-self-duality condition by hand as for example in type IIB string theory or that we have to go over to singular instantons, where the singularities are knot cobordisms (Sect. 7). However, these are rampant assumptions.

For more complicated spacetimes not only depending on an overall volume deformation we cannot write the Einstein–Hilbert action as a Yang–Mills action. The corresponding connection formulation is the Palatini action and from its self-dual version the constraint equations of general relativity can be expressed in terms of Ashtekar variables, which are the starting point of loop quantum gravity. However, three-dimensional general relativity with cosmological constant can be expressed as a $SO(4)$ -Chern–Simons theory, which might be as satisfying considering the fact we can represent spacetimes as families of three-dimensional leaves. As described, the flat connections are the critical points, which from the point of view of general relativity

are Ricci-flat spaces. Additionally, we have instantons representing cylinders $Y \times \mathbb{R}$. So, a general four-dimensional theory might not be needed when we restrict to (anti)-self-dual one-parameter families. Concerning the generalisation to the gauge group $SO(3, 1)$, Witten argued that $2 + 1$ -dimensional gravity with positive cosmological constant equals $SO(3, 1)$ Chern–Simons theory. Accordingly, gradient flow lines of the functional would be $SO(3, 1)$ Lie algebra-valued instantons but then we had four-dimensional manifolds with two-time directions or the instantons as a parameter family of a spatial coordinate. The $SO(4)$ stems from the fact that we consider 3-dimensional gravity instead of $2 + 1$ -dimensional, such that the associated instantons are $SO(4)$ Lie algebra valued. This might be interpreted by the fact that every gauge field represents a FLRW-metric with semi-definite curvature parameter, which isometry group is $SO(4)$.

3 General relativity as a stack of N D3-branes

As an aside on instantons, in the $SO(4)$ situation we have

$$A^{(k=k_L-k_R)} = \begin{pmatrix} A^{SU_L(2)} & 0 \\ 0 & A^{SU_R(2)} \end{pmatrix} \\ = \frac{if(x)}{2} \begin{pmatrix} \eta_{\mu\nu}^a \sigma^a x^\nu & 0 \\ 0 & \bar{\eta}_{\mu\nu}^a \sigma^a x^\nu \end{pmatrix}, \quad (74)$$

where $f(x)$ solves the cubic wave equation $(\Delta - \partial_t^2)f - f^3 = 0$, $\eta_{\mu\nu}^a$, σ^a are the t’Hooft and Pauli matrices respectively and $k_L > 0$, $k_R < 0$ are the windings of the respective sector. The action is

$$S = \frac{8\pi^2}{g^2} (k_L - k_R) \quad (75)$$

In the semiclassical approximation the Yang–Mills vacuum decomposes into enumerable topological sectors connected by tunnelling in configuration space. So, in principle there should be a map from the moduli space of the theory we just established into the space of Riemann four-surfaces of genus and in a naive path integral approach by summing over the field configuration space we would also sum over a subspace of Riemann surfaces. This looks like four-dimensional quantum gravity at least for a subset of maximally symmetric, positively curved surfaces. It is also worth pointing out that the trivial vacua of the YM action are gradient flow lines of the Chern–Simons action functional as well because the instanton equation becomes trivial. One might say that the Minkowski metric is a trivial gravitational instanton not leaving pure gauge. It is therefore an identity component gauge transformation under which the Chern–Simons functional is invariant. More generally we take a $SU(2)$ -path $A(t)$ on $S^3 \times [t_i, t_f]$. Since A_{t_i} and A_{t_f} are related by a gauge transformation, we can define a connection \hat{A} on a bundle over

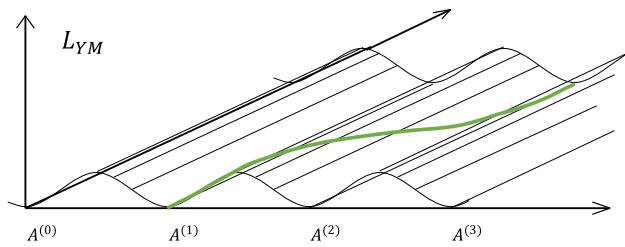


Fig. 2 The semiclassical vacuum structure of Yang–Mills theory

the torus $S^3 \times S^1$. In the case of $SO(4)$ it follows that

$$(CS(A_{t_f}) - CS(A_{t_i})) \times (CS(A_{t_i}) - CS(A_{t_f})) \\ = -\frac{1}{2} \int_{S^3 \times S^1} \langle F_{\hat{A}} \wedge F_{\hat{A}} \rangle \times \frac{1}{2} \int_{S^3 \times S^1} \langle F_{\hat{A}} \wedge F_{\hat{A}} \rangle dt \quad (76)$$

Expression (76) is an element of $\mathbb{Z} \oplus \mathbb{Z}$ and zero for identity component gauge transformation because start and endpoint of the paths are identical (Fig. 2).

The theta-vacuum picture is of course far from the true ground state of quantum Yang–Mills, which is a strongly coupled theory. However, it tells us that in the weak coupling regime the fluctuations of the vacuum sectors tunnel through the wells and all sectors should be described as a single self-interacting system. This vacuum structure looks like—and can be interpreted as—the gravitational background formed by N D3-branes because the theta-vacuum is essentially their gravitational imprint in the sense that the three-curvature forms a periodic pattern. What we have established is thus a field theory in the background of N D3 branes. Astonishingly, we came from general relativity of maximally symmetric positively curved spaces to a landscape of branes. However, it fits very nicely with the duality between open bosonic string theory that requires the presence of D-branes and closed string theory, where the branes act as sources of Kähler fluxes. Additionally, the quantized topological Yang–Mills theory can be regarded as a twisted $N = 2$ SUSY Yang–Mills theory in four dimensions, which is identical to two out of three possibilities of twisting its $N = 4$ cousin, which lives on a stack of D3 branes. We propose therefore that general relativity for the subset of solutions discussed above is dual to a large N topological open bosonic string theory. The large N duality between open and closed topological string theory is due to Vafa and Gopakumar. To make sense of the limit from the gauge theory perspective [16], one keeps the t’Hooft coupling $\lambda = g_{YM}^2 N$ fixed while taking $N \rightarrow \infty$. Obviously the t’Hooft limit is classical. The propagator of the gauge particles is

$$\langle A_{\mu j}^i(x) A_{\nu l}^k(y) \rangle = \Delta_{\mu\nu} (x - y) \delta_l^i \delta_j^k \quad (77)$$

giving rise to the double line notation. In the so-called planar limit, only the graphs are dominant, which wrap around the sphere. In 1997 it was conjectured by Maldacena [17] and

others that $\mathcal{N} = 4$ $SU(N)$ SYM is dual to Type IIB supergravity on $AdS_5 \times S^5$. All these arguments fit quite well with the fact that string theory at long distances gives rise to a theory of quantum gravity. However, for our theory here there is already a correspondence between the gauge fields and four-dimensional manifolds as cobordisms.

4 Self-dual general relativity on $Y \times \mathbb{R}$ as a membrane field theory

It is a nice matter of fact that the sphere on which our theory lives has a special relationship with open and closed string field theory. The starting point is the topological non-linear sigma model from a twist of a $N = 2$ superconformal sigma model introduced by Witten living on a two-dimensional Riemann surface Σ_g with genus g . The two possibilities of twisting are known as the A-model and B-model. In this paper we will be particularly concerned with the former. The target space in our specific case is a Calabi–Yau manifold $X = T^*S^3$, which is Kähler. This so-called deformed conifold admits a Ricci-flat Kähler metric G_{ij} . We follow closely [10, 18, 19].

One version of the twisted σ -model, the A-model, which we define in more detail for convenience, can be compactly written as

$$\mathcal{S} = -i \{ \mathcal{Q}, \mathcal{V} \} + t \int_{\Sigma_g} \varphi^* (J), \quad (78)$$

where the last term is topological, depending on the homotopy of the map $\varphi : \Sigma_g \rightarrow X$. It vanishes for the deformed conifold according to the vanishing theorem. \mathcal{V} is defined as

$$\mathcal{V} = t \int_{\Sigma} d^2 z G_{I\bar{J}} \left(\psi_{\bar{z}}^I \partial_z \varphi^{\bar{J}} + \partial_{\bar{z}} \varphi^I \psi_z^{\bar{J}} \right). \quad (79)$$

It is important to note that there is a one-to-one correspondence of the \mathcal{Q} -cohomology with the deRham cohomology of the Kähler target X and that with the restriction to (1,1)-forms the A-model computes deformations of the Kähler moduli. Meaning, that \mathcal{Q} has an interpretation of an exterior derivative taking p -forms to $(p+1)$ -forms. Picking local coordinates Φ^i on X , we have

$$\begin{aligned} \mathcal{S} = 2t \int_{\Sigma_g} dz d\bar{z} & \left[\frac{1}{2} G_{ij} \partial_z \Phi^i \partial_{\bar{z}} \Phi^j + i G_{I\bar{J}} \psi_{\bar{z}}^I \mathcal{D}_z \chi^{\bar{J}} \right. \\ & \left. + G_{\bar{I}J} \psi_z^{\bar{I}} \mathcal{D}_{\bar{z}} \chi^J - R_{I\bar{I}J\bar{J}} \psi_{\bar{z}}^I \psi_z^{\bar{J}} \chi^J \chi^{\bar{J}} \right], \end{aligned} \quad (80)$$

where \mathcal{D}_a is the covariant derivative, $\chi \in \varphi^*(TX)$ are Grassmann fields and $\psi_{\bar{z}}^I \in \varphi^*(T^{(1,0)}X)$ and $\psi_z^{\bar{I}} \in \varphi^*(T^{(0,1)}X)$ are the only non-zero components of a Grassmannian one-form. Most importantly

$$T_{\alpha\beta} = \left\{ \mathcal{Q}, \frac{\delta V}{\delta g^{\alpha\beta}} \right\} = \{ \mathcal{Q}, b_{\alpha\beta} \}, \quad (81)$$

for the theory is formally topological. So that after parametrizing the worldsheet by σ, τ with $0 \leq \sigma \leq \pi, -\infty < \tau < \infty$ and additionally choosing the metric $ds^2 = d\sigma^2 + d\tau^2$, we get the Hamiltonian

$$\begin{aligned} \{ \mathcal{Q}, b_0 \} = L_0 = & \int_0^\pi d\sigma T_{00} \\ = & \int_0^\pi d\sigma \left(-\frac{1}{t} G^{ij} \frac{\delta^2}{\delta \varphi^i(\sigma) \delta \varphi^j(\sigma)} \right. \\ & \left. + t G^{ij} \frac{d\varphi^i d\varphi^j}{d^2 \sigma} \right) + \text{fermions}. \end{aligned} \quad (82)$$

This follows immediately from the self-duality of $\psi = \psi_\sigma d\sigma + \psi_\tau d\tau$, because it enables us to express ψ_σ in terms of ψ_τ . Reading of the canonical commutation relation from the action

$$\begin{aligned} \left[\frac{d\varphi^i}{d\tau}(\sigma), \varphi^j(\sigma') \right] &= \frac{1}{it} G^{ij} \delta(\sigma - \sigma') \\ \{ \psi_\tau(\sigma), \chi(\sigma') \} &= \frac{1}{t} \delta(\sigma - \sigma') \end{aligned} \quad (83)$$

we can write $d\varphi/d\tau$ through $\delta/\delta\varphi$. We should also mention that the large t behaviour is classical and exact. It is now required that the boundary condition preserves the BRST symmetry, so that it should be mapped into a Lagrangian submanifold L_i of the target Calabi–Yau, which in our case is the sphere. In this way we can wrap N D3-branes around S^3 . The string functional Ψ of bosonic open string theory therefore contains constant maps from the boundary of the worldsheet into the targets submanifold, as well as commuting and anticommuting zero modes. It takes values in a \mathbb{Z} -graded algebra \mathfrak{A} such that the degree $N_g = 1$ and that for every gauge parameter Λ with degree 0, Ψ is invariant under the very large set of gauge transformations

$$\Psi \rightarrow \delta \Psi = \mathcal{Q} \Lambda + g_s (\Psi \star \Lambda - \Lambda \star \Psi). \quad (84)$$

The star product is a map $\star : \mathfrak{A} \otimes \mathfrak{A} \rightarrow \mathfrak{A}$ such that N_g is additive, which is the ghost number. The action functional of the string field is

$$S = \frac{1}{g_s} \int \Psi \star \mathcal{Q} \Psi + \frac{2}{3} \Psi \star \Psi \star \Psi, \quad (85)$$

where the integral has ghost number $N_g = -3$. From the Hamiltonian follows in particular that

$$\Psi = c(q) + \chi^a A_a(q) + \sum_{p=1}^3 \chi^{a_1} \dots \chi^{a_p} A_{a_1 \dots a_p}^{(p)}, \quad (86)$$

where thanks to the D-branes the functional is $U(N)$ Lie-algebra valued. In contrast to the physical open bosonic string, in the large t limit the energy eigenmodes for the

topological string decouple and the field is cut down to a single excitation

$$\Psi = \chi^a A_a(q). \quad (87)$$

This lowest lying mode is nothing else than an $u(n)$ -valued one form, on which the BRST charge acts as an exterior derivative. Hence the string field reduces to a point. This induces the dictionary

$$\star \longrightarrow \wedge$$

$$Q \longrightarrow d$$

$$\int \longrightarrow \int_Y$$

for the A-model reduces to ordinary Chern–Simons theory with

$$g_s = \frac{2\pi}{k+N}, \quad (88)$$

where k is the level of the CS-theory. It was now first postulated by Gopakumar and Vafa, what is known as GV-duality, that the large N open topological string on the deformed conifold is dual to the closed topological string on the resolved conifold. The effect of the D-branes wrapping the 3-cycle is the creation of a Kähler flux $N g_s$ on the S^2 linking the 3-cycle. To explain this, we take the Lagrangian submanifold L to be intersected by the 3-cycle. Then the 2-cycle, which is the boundary of S^3 links the submanifold. Wrapping N branes creates a Kähler flux through the otherwise homologically trivial boundary

$$t = \int_{S^2} \omega = N g_s. \quad (89)$$

It follows that the D-branes act as a delta source for the flux. They can be imagined as sitting on the tip of the deformed conifold. Making the geometric transition to the resolved conifold, the nontrivial S^2 on the tip has volume $t = N g_s$ and due to the lack of nontrivial cycles the branes have disappeared. The flux of the D-branes created a Kähler deformation of the metric leaving the complex structure untouched. Hence, closed and open theories are dual and connected by a geometric transition of the conifold. Their partition functions are identical.

The goal of this section is now to formulate a string field theory that reduces to the topological Yang–Mills theory (68). We start by pointing the reader’s attention towards the fact that the classical configurations of the above topological σ -model are pseudo-holomorphic maps. From the Hamiltonian (82) followed that string fields were functionals of

$$\varphi : I \longrightarrow T^* S^3 \quad (90)$$

such that ∂I is mapped to S^3 . Instead, we want to consider a one-parameter family of such maps

$$t \longrightarrow \varphi(t),$$

$t \in \mathbb{R}$. This is a path in the infinite-dimensional loop space LS^3 and associated to it is the one-parameter family of string functionals $\Psi_t(\varphi(t), \dots)$, which is a path in the infinity dimensional space of string functionals. These families should be subject to some membrane σ -sigma model. Hence, it is a point in the path space of string functionals of loops PS^3 . Therefore, we have the worldsheet metric

$$ds^2 = \delta_{ij} d\sigma^i d\sigma^j \quad (91)$$

with $(\sigma^0, \sigma^1) = (\sigma, \tau)$ and the Euclidean metric tensor δ_{ij} as a foliation of the worldvolume metric

$$ds^2 = dt^2 + \delta_{ij} d\sigma^i d\sigma^j. \quad (92)$$

The claim is now that we can write $\varphi(t)$ as a map from the one-parameter family of worldsheets into a one-parameter path of deformed conifolds inside a manifold X complex dimension four.

$$\varphi_t : I \times \mathbb{R} \longrightarrow T^* (S^3 \times \mathbb{R}). \quad (93)$$

This means we consider the worldsheet coordinates to be depending on the parameter t . We can imagine this as lifting the dimension from a string to a membrane. The construction of the “membrane field action” is in our case closely connected to the background independence of the topological string. To see this, [20,21] we have to observe that the open string field theory has a cyclic L_∞ -structure when a fixed representation in terms of $N \times N$ matrices is chosen. Moreover, the string field action (85) can be viewed as the action functional of a ∞ -Chern–Simons theory. It can be defined by the invariant polynomial on the ∞ -Lie algebroid inducing a ∞ -Chern–Weil homomorphism. This means the string field is a connection on a ∞ -bundle and the string field Lagrangian sends it to the n -circle bundle. Let \mathfrak{l} denote the L_∞ -algebra and $\mathfrak{W}(\mathfrak{l})$ its Weil algebra. We can write the transgression between $(-) \in \mathfrak{W}(\mathfrak{l})$ and the cocycle as

$$d_{\mathfrak{W}(\mathfrak{l})} \text{CS} = \mathcal{Q} \left(\Psi \star \mathcal{Q} \Psi + \frac{2}{3} \Psi \star \Psi \star \Psi \right) = (-, -). \quad (94)$$

Since $\mathfrak{W}(\mathfrak{l})$ has trivial cocycle cohomology there is a $\text{CS} \in \mathfrak{W}(\mathfrak{l})$ for every $(-)$. More generally, we may write the Chern–Simons form with a n -ary bracket $[-, \dots, -]_n : \mathfrak{l}^{\otimes n} \longrightarrow \mathfrak{l}$ as

$$\text{CS}(A) = \frac{1}{2} (\Psi \star \mathcal{Q} \Psi) + \sum_{n=3}^{\infty} \frac{1}{n!} (\Psi [\Psi, \dots, \Psi]_n, \Psi). \quad (95)$$

In analogy with ordinary Chern–Weil theory the Lagrangian of Witten’s open bosonic string field theory is generated by

$$d_{\mathfrak{W}(\mathfrak{l})} \text{CS} = \mathfrak{F} \star \mathfrak{F} \quad (96)$$

Due to the fact that this is part of the Weil-algebra, it is also BRST-closed. This motivates us to define

$$S = \int \mathfrak{F} \star \mathfrak{F}, \quad (97)$$

where $\mathfrak{F} = \mathcal{Q}\Psi + g_s \Psi \star \Psi$. This new string field, however, is now a functional of pseudoholomorphic maps from the worldvolume of a 2-Brane Σ_3 into an eight-dimensional target $T^*(S^3 \times \mathbb{R})$ such that we have maps $\varphi_t : \partial \Sigma_3 \rightarrow T^*(S^3 \times \mathbb{R})$ and $\partial I \times \mathbb{R}$ is mapped to $S^3 \times \mathbb{R}$. Thus, we have also one-parameter families of three-dimensional Lagrangian submanifolds foliating four-dimensional submanifolds $M = S^3 \times \mathbb{R}$. We define the integral to have the ghost number of the vacuum, which is $-4\chi(\Sigma)$, where $\dim_{\mathbb{C}}(X) = 4$ is the complex dimension of the target. The string theory is not anymore defined on a disc but on a cylinder of the disk crossed with the real line. Therefore, we have to soak up an additional zero mode in that direction giving a total of four zeromodes. As mentioned earlier, the target is a one-parameter family of deformed conifolds. We define two string field theories $S_0[\Psi^{(0)}]$ and $S_1[\Psi^{(1)}]$ on each of the boundary components of this cylinder, respectively.

$$\partial T^*(M) = \overline{T^*S^3} \cup T^*S^3. \quad (98)$$

The two theories are equivalent up to homotopy

$$S_0[\Psi^{(0)}] = S_1[\Psi^{(1)}] + \int_{=const} \mathfrak{F} \star \mathfrak{F}. \quad (99)$$

In particular, there should be an intertwining solution relating both theories for which (97) is a positive constant. With the same arguments that reduced the A-model to Chern–Simons theory on the deformed conifold, (97) reduces to the action of topological Yang–Mills. We fix \mathfrak{F} through the introduction of an operator σ , which maps elements of the graded A_∞ -algebra with ghost number N_g to elements with ghost number $d - N_g$, where d is the dimension of the submanifold. We employ that

$$\mathfrak{F} \pm \sigma \mathfrak{F} = 0. \quad (100)$$

When the ghost number reduces to the rank of forms, the operator σ reduces to the Hodge star and (100) to the (anti-)self-dual Yang–Mills equations in the case when $d = 4$. We have also that

$$\mathcal{Q}^* = -(-1)^{d(N_g-1)} \sigma \mathcal{Q} \sigma \quad (101)$$

$$\langle \Psi, \mathcal{Q} \star \Xi \rangle = \langle \mathcal{Q}\Psi, \Xi \rangle \quad (102)$$

A dynamical string field satisfying (100) is also a solution of

$$\mathcal{Q}\mathfrak{F} + g_m \mathfrak{F} \star \Psi = 0. \quad (103)$$

In the limit where the membrane coupling vanishes, the equation of motion becomes

$$\mathcal{Q}^2 \Psi = 0, \quad (104)$$

which is the nilpotency condition of the BRST charge. The question is now how we get back to (anti)-self dual $SO(4)$ Yang–Mills and general relativity. Obviously, the first thing is that the branes have to disappear. Henceforth, in analogy with the GV-duality, we make the geometric transition and arrive at a one-parameter family of resolved conifolds. The family of S^2 on the tip of the resolved conifolds over $M = S^3 \times \mathbb{R}$ parametrizes the constant complex structure J on the tangent bundle TM . Moreover, we view the resolved conifold as a spherical fibration over S^3

$$o : S^2 \times Y \rightarrow S^3. \quad (105)$$

Such that the family of conifolds is the bundle

$$E(M) : Y \times \mathbb{R} \times S^2 \rightarrow Y \times \mathbb{R}. \quad (106)$$

Each fibre of $E(M)$ is a sphere S^2 . We can view them as fibres of a $SO(4)/U(2)$ bundle, which is the associated bundle to the principal bundle P of orthonormal frames over M . The bundle P is the $SO(4)$ bundle of orthonormal frames on M . The associated bundle E

$$E(M) = P \times_{SO(4)} SO(4)/U(2) \quad (107)$$

is actually the twistor space \mathcal{P} of M . It is an interesting fact that the Penrose–Ward transformation [22–24] relates solution to the (anti)-self-dual Yang–Mills equations on M and solutions to the holomorphic Chern–Simons equations on the complex twistor space over M . We only considered the A-model in this work. However, in this context it should be noted that under similar circumstances previously described, the B-model reduces to holomorphic Chern–Simons theory and the membrane field action could be equally derived by considering coherent sheaves instead of instantons. Therefore, one might expect that the relation of the A- and B-model unfolds on the target through a lift of the $SO(4)$ bundle to a holomorphic bundle over the complex space $\mathcal{P} = (E(M), \mathfrak{J})$, where \mathfrak{J} is an integrable almost complex structure on the twistor space. Hence, the Penrose–Ward transform should relate A- and B-model topological strings and branes. In Sect. 6 we delve deeper into the relation from the viewpoint of the worldvolume. It implicates a deep relation between Donaldson–Witten theory and the topological string and membrane models.

5 Relating A- and B-model topological strings through Penrose–Ward transform

I want to get into more detail about the relation of the two different topological string theories mentioned in the last sec-

tion. For a detailed discussion of string theories and twistor spaces see [25] and the references therein. We saw that the family of resolved conifolds are the twistor space over M , which is the associated bundle to an $SO(4)$ bundle. This space has an almost complex structure \mathfrak{J} as the tensor sum of vertical and horizontal subspaces of the tangent space TE . We define the holomorphically trivial bundle \mathfrak{e} over the complex twistor space \mathcal{P} . There is a projection

$$\pi : \mathcal{P} \longrightarrow M \quad \forall x \in M. \quad (108)$$

We have two commuting, two-dimensional spinor representations usually denoted by $\mu = \mu_\alpha$ and $\lambda = \lambda_\alpha$. We will now briefly describe the ambitwistor space as a third order thickening of the Klein-quadric. There are two ways to parametrize the twistor space \mathcal{P} either we demand that $\lambda_{\dot{\alpha}} \neq (0, 0)^T$ or $\mu_\alpha \neq (0, 0)^T$. It follows that we have the coordinates $(\omega^\alpha, \lambda_{\dot{\alpha}}, \eta^{\dot{\alpha}}, \mu_\alpha)$ on $\mathcal{P} \times \mathcal{P}_*$. By covering the sphere with two patches, we have

$$\begin{aligned} \mathcal{P} \times \mathcal{P}_* &= \mathcal{O}(1) \oplus \mathcal{O}(1) \times \mathcal{O}(1) \oplus \mathcal{O}(1) \\ &\longrightarrow \mathbb{C}P^1 \times \mathbb{C}P^1. \end{aligned} \quad (109)$$

The ambitwistor space is just the gluing of the twistor space and its dual. Next, we split the principal $SO(4)$ bundle into self-dual and anti-self-dual sub-bundles W^\pm with their respective (anti)-self-dual connections A^\pm . We have the identification $\mathcal{P} = \mathcal{O}(1) \oplus \mathcal{O}(1) \longrightarrow \mathbb{C}P^1$. With the help of (108) we define $E^- = \pi^* W^-$ through $\mathcal{A}^{(0,1)} = \pi^* A^-$ and $\mathcal{F}^{(0,2)} = \pi^* F^-$, such that

$$\mathcal{F}^{(0,2)} = d\mathcal{A}^{(0,1)} + \mathcal{A}^{(0,1)} \wedge \mathcal{A}^{(0,1)} = 0. \quad (110a)$$

Likewise, for the ASD bundle

$$\mathcal{F}^{(2,0)} = d\mathcal{A}^{(1,0)} + \mathcal{A}^{(1,0)} \wedge \mathcal{A}^{(1,0)} = 0 \quad (110b)$$

on the dual twistor space \mathcal{P}_* . This can be constructed by projectors onto the respective holomorphic sectors. Equations (110) are obtained by variation of the holomorphic Chern–Simons functional

$$\int_P \Omega \wedge CS(\mathcal{A})^{(0,3)}, \quad (111)$$

where Ω is a $(3,0)$ form. Hence, we have a correspondence between solutions of the (anti)-holomorphic CS action on the twistor space \mathcal{P} and its dual \mathcal{P}_* and the (A)SD $SU(2)$ Yang–Mills equations on M . If the B-model-part inside the topological M-theory reduces to (110) on $M = Y \times \mathbb{R}$ and the A-model-part to (A)SD Yang–Mills, then they are related by the Penrose–Ward transform, which involves a holomorphic lift of the associated bundle, and dual from the viewpoint of the target. In the bulk the distinction between the A-model and the B-model vanishes, which is reflected by the fact that every (anti)-self-dual connection on the target of the M-model corresponds to a pseudo-holomorphic structure on the complex bundle over the twistor space. The topological membrane

theory splits at the boundary Calabi–Yau threefold because of the possibility to blow up the singularities in two different ways, and with it to wrap the respective topological branes around the cycles. One possibility to do this is controlled by a complex parameter, while the other is controlled by a Kähler parameter. We may describe this as the collapse of the respective cycles of a toric fibration [26,27] that describes certain algebraic manifolds including the conifold. The M-model is sensitive to which cycle collapses reflecting the sensitivity of the B-model regarding the complex structure and on the other side the A-model’s dependence on the Kähler moduli of the target. Another way to understand this, is to look at the behaviour of the couplings, which we do in the next section. Although the discussion focused on the different resolutions of the conifold, the statements made in this section hold for a general four-manifold M as long as it is orientable, and it admits a Riemannian metric. Furthermore, one follows that the discussion in the last section can be generalized to cotangent spaces of such M . This points also to the existence of holomorphic general relativity.

6 The membrane sigma-model

We consider again the membrane field theory with action

$$S = \int \mathfrak{F} \star \mathfrak{F}, \quad (112)$$

where \mathfrak{F} is a vector in a homotopy Lie algebra L_∞ . In general, it is defined by

$$\mathfrak{F} = \sum_{n=0}^{\infty} \frac{1}{n!} [\Psi^n], \quad (113)$$

with the membrane family of string fields. Integral and composition operations are given by

$$\begin{aligned} \int \Psi &= \int \mathcal{D}\varphi(\sigma(t)) \prod_{0 \leq \sigma(t) \leq \frac{\pi}{2}} \delta[\varphi(\sigma(t))] \\ &\quad - \varphi(\pi - \sigma(t)) \Psi[\varphi(\sigma(t))], \end{aligned} \quad (114)$$

which is just an extension of the usual string field operation over the real line. Similarly, we can define a star product that glues different membranes.

$$\begin{aligned} &\int \Psi_1 \star \dots \star \Psi_N \\ &= \int \prod_{i=1}^N \mathcal{D}\varphi_i(\sigma(t)) \prod_{i=1}^N \prod_{0 \leq \sigma(t) \leq \frac{\pi}{2}} \delta[\varphi_i(\sigma(t))] \\ &\quad \delta[\varphi_i(\sigma(t)) - \varphi_{i+1}(\pi - \sigma(t))] \Psi_i[\varphi_i(\sigma(t))]. \end{aligned} \quad (115)$$

We set all higher products except for the two lowest equal to zero and define the operator $\mathcal{Q}' = \mathcal{Q} + \Psi$. The immense

gauge invariance of (112) is expressed as

$$\delta\Psi = \mathcal{Q}'\Lambda = \mathcal{Q}\Lambda + \Psi \star \Lambda \quad (116)$$

$$\delta\mathfrak{F}(\Psi) = \mathcal{Q}'(\delta\Psi) = \Lambda \star \mathfrak{F}. \quad (117)$$

The membrane field Ψ is a functional of maps of the underlying membrane σ -model from the worldvolume of the 2-brane Σ_3 into the target space X . By the standard AKSZ construction [28] of a higher Poisson structure, the σ -model is constituted of Courant algebroid-valued differential forms $\xi : \Sigma_3 \rightarrow X$, a one-form $\alpha \in \Omega^1(\Sigma_3, \xi^* \mathfrak{E})$ and a two-form $\mathcal{F} \in \Omega^2(\Sigma_3, \xi^* X)$, where $\mathfrak{E} = TX \oplus T^*X = \mathcal{L}_+ \oplus \mathcal{L}_-$ is the second order bundle with coordinates (I, \tilde{J}) , where $I = 1, \dots, 2d$ and $\tilde{J} = 1, \dots, 4d$. We are also choosing the generalized metric $\eta_{\tilde{I}\tilde{J}}$ and define the projection of the anchor $\rho_J^I = (\rho_J^I, \tilde{\rho}^{J\tilde{I}})$ of \mathfrak{E} by a map $\rho : \mathcal{L}_+ \rightarrow TX$ with $\rho_J^I = \rho_J^I + \eta_{JK}\tilde{\rho}^{IK}$. There is also the three-form $T_{\tilde{I}\tilde{J}\tilde{K}}(e_{\tilde{I}}e_{\tilde{J}}e_{\tilde{K}})$. After projection onto the DFT vectors, the $O(d, d, \mathbb{Z})$ -invariant action reads

$$\begin{aligned} S_{\text{Courant}} = & \int_{\Sigma_3} \mathcal{F}_I \wedge d\xi^J + \eta_{IJ}\alpha^I \wedge d\alpha^J - \rho_J^I(\xi)\alpha^J \wedge \mathcal{F}_I \\ & + \frac{1}{6}T_{IJK}\wedge\alpha^I\wedge\alpha^J\wedge\alpha^K \\ & + \int_{\partial\Sigma_3} \frac{1}{2}\mathfrak{g}_{IJ}(\xi)\alpha^I \wedge *\alpha^J, \end{aligned} \quad (118)$$

where \mathfrak{g} is the DFT-projection of the metric on \mathfrak{E} and $*$ is the Hodge-operator on the worldsheet. Since Courant algebroids are in bijection with the AKSZ sigma-model, the above expression is an enlargement of the AKSZ construction. We do not want to go into more detail about the technicalities, which can be found in [29, 30]. Furthermore, we define again $\Sigma_3 = (I \times \mathbb{R}) \times \mathbb{R}_t$ and $\partial\Sigma_3 := \Sigma_2 = I \times \mathbb{R}_t$ and $X = C_3 \times T^*\mathbb{R}_t$. The complex dimension of the target $\dim_{\mathbb{C}}(X) = 4$, because we choose $C_3 = T^*Y$ to be a Calabi–Yau three-fold (C_3, ω) and $T^*\mathbb{R}_t \simeq \mathbb{C}$. X then has the symplectic structure $\omega \oplus \omega_{\mathbb{R}^2}$. We demand that $\partial\Sigma_3$ is mapped onto Lagrangian submanifold $M \subset X$ with $\dim_{\mathbb{C}}(M) = 2$ and $M = Y \times \mathbb{R}_t$. One can understand M as Lagrangian cobordisms [31, 32] between lower Lagrangian submanifolds $L_i = Y_i$. With some important additional requirements such as monotony and support, they form a stable $(\infty, 1)$ -category, which was conjectured to be identical to the partially wrapped Fukaya category of C_3 [33]. In this way the instanton Floer homology, Lagrangian Floer homology and instanton knot homology are part of (112). It is comfortable to introduce some additional structure on Y that allows for composition, a \mathbb{Z} -graded Floer homology and the possibility to orient moduli spaces of polygons in Floer theory. That is, we regard the Y and M as Lagrangian branes. Returning to the maps ξ of the three-dimensional σ -model, as prescribed, the t-boundary $\Sigma_2 = I \times \mathbb{R}_t$ is mapped to the Lagrangian cobordisms, but there are also the maps $\varphi : \Sigma_2 \rightarrow C_3$ with $\Sigma_2 = I \times \mathbb{R}$,

such that ∂I is mapped to Y , where $I := \partial\Sigma_2 = [0, \pi]$. In general, the holomorphic maps $\varphi : \partial\Sigma_3 \rightarrow C_3$ are classified by the homology class $H_2(C_3, \mathbb{Z})$ and are subject to the A-twisted nonlinear sigma-model, whose path integral computes Gromov–Witten invariants. They count the virtual number of curves inside the moduli space of φ . On the other side we have the B-twisted model on $\partial\Sigma_3$ that does not count stable maps but objects in the derived category of coherent sheaves $D^b\text{Coh}(C_3)$ on C_3 . In particular $D^b\text{Coh}(C_3)$ can be thought of the category of D-branes of the B-model, while the Fukaya category $\text{Fuk}(C_3)$, a derived version of an A_∞ -category of certain Lagrangian submanifolds of C_3 , is the category of the A-branes. Kontsevich’s homological mirror symmetry [34] states that for two mirror Calabi–Yau varieties A and B it holds that $\text{Fuk}(A) \cong D^b\text{Coh}(B)$. The deformation of objects in the derived category of coherent sheaves is a L_∞ -algebra. Moreover, the B-model computes the virtual number of points inside the moduli stack of semi-stable objects in $D^b\text{Coh}(B)$, which are ideal sheaves. This number is actually an invariant of the underlying symplectic variety called Donaldson–Thomas invariant. The correspondence of Gromov–Witten and Donaldson–Thomas invariants on C_3 is a different manifestation of the duality between the two topological string theories. In particular it is the manifestation of an S-duality since the GW partition function is evaluated when the string coupling $g_m|_{\partial\Sigma_3} = g_s$ is small, while the partition function for the DT invariants is evaluated in $q = e^{-g_s}$ and valid when $1/g_s$ is small [35]. Obviously both models should arise as boundary theories of the membrane sigma-model, and this is indeed the case. After gauge fixing in the bulk described in [36], A- and B-model arise in the framework of generalized complex geometry. Given a generalized complex structure

$$\mathbb{J}^{IJ} = \begin{pmatrix} J_j^i & \pi^{ij} \\ 0 & -J_j^i \end{pmatrix}, \quad (119)$$

where π^{ij} is the Poisson bivector, the doubled Courant sigma-model with the DFT projection $\alpha^I = \left(\frac{1}{g_m}q^i, g_m p_i\right)$

$$\begin{aligned} S_{\text{DFT}} = & \int_{\Sigma_3} g_m \left(\mathcal{F}_i d\xi^i - p_i dq^i + J_j^i \mathcal{F}_i q^j + \partial_i J_i^k q^i q^j p_k \right) \\ & + \frac{1}{g_m} \left(\pi^{ij} \mathcal{F}_i p_j - \frac{1}{2} \partial_i \pi^{jk} q^i p_j p_k \right) \end{aligned} \quad (120)$$

reproduces the AKSZ description of the A-model and the B-model on the boundary after introducing the membrane coupling $g_m = 1$ and setting $J = 0$ and $\pi = 0$, respectively. The Hamiltonian is given by

$$\begin{aligned} \mathcal{H}_{J, \pi} = & \frac{1}{g_m} \left(\pi^{ij} \mathcal{F}_i p_j - \frac{1}{2} \partial_i \pi^{jk} q^i p_j p_k \right) \\ & + g_m \left(J_j^i \mathcal{F}_i q^j + \partial_i J_i^k q^i q^j p_k \right) \end{aligned} \quad (121)$$

It holds that

$$g_m = g_A = \frac{1}{g_B} \quad (122)$$

We see that for $g_m \ll 1$ and $g_m \gg 1$ the boundary theories are also reducing to the AKSZ forms of the A- and B-model respectively. Namely the Poisson and the complex Courant sigma model. This means that modulating the coupling has the same effect as continuously exchanging complex and Poisson structure, which is a manifestation of a topological U-duality. The two Kähler forms of the boundary models are related through the membrane coupling

$$g_m k_B = k_A. \quad (123)$$

While the last section drew the unification of both models from the viewpoint of the target in form of the Penrose–Ward transform, the membrane sigma-model shows this unification on the side of the worldsheet/worldvolume. The partition functions of the boundary theories are just the generating functions for the GW and DT-invariants, and we know at least what the membrane field theory computes on a Calabi–Yau fourfold of the form $X = T^*Y \times T^*\mathbb{R}_t$: The Donaldson–Polynomials. Moreover, the partition function of the membrane sigma-model and the Donaldson–Witten model should coincide. In the last section we will see that this is only true in the perturbative regime.

7 Homological mirror symmetry

It should be interesting to what extend we can gain new insights into the mirror duality of the boundary models. Mirror symmetry is a kind of T-duality first observed by physicists in the context of topological string theory, which is characterized by exchanging the symplectic structure of the target with a complex structure. At infinity resolved and deformed geometries as well as the singular are topological equivalent. As described previously, they look like a sphere bundle over the base manifold that we can associate with its twistor space. In general, the B-branes are represented by holomorphic vector bundles. The previous sections suggest that the topological membrane model links certain Calabi–Yau three-folds with their mirror. That is, while the objects of the respective derived categories are linked by the mirror transform of the underlying target space, the morphisms between them, which are subject to the parent membrane theory, are in bijection through the Penrose–Ward transform. Moreover, in this sense topological M-theory does not only relate different Fukaya-categories to each other in terms of Lagrangian correspondences, it also relates the corresponding categories of derived sheaves. Therefore, it is helpful to analyse the stable $(\infty, 1)$ -category with the Lagrangian cobordism as morphisms through the instanton Floer homol-

ogy similar to how the symplectic Floer homology is implemented into the Fukaya category. The new important ingredient in our reasoning is the Atiyah–Floer conjecture that states

$$H^*(Y) \cong HF\left(R\left(H_g^1\right), R\left(H_g^2\right)\right). \quad (124)$$

Here Y is a homology 3-sphere, H_g^i are the handle bodies satisfying $Y = H_g^1 \cup_{\Sigma_g} H_g^2$ produced by Heegaard splitting along the surface Σ_g and $R\left(H_g^i\right)$ is the space of flat connections on Σ_g that extend into the H_g^i . The invariant $H^*(Y)$ is determined by the Floer homology that is the homology of the chain complex defined in Sect. 1. Roughly speaking, the conjecture tells us that symplectic Floer homology and instanton Floer homology are in congruence. A proof is still missing due to some obstructions related to the singularity of the spaces of flat connections and the resulting problems in defining the right side of (124) in a rigorous way [37, 38]. On the level of our membrane field theory this conjecture is restated by saying that the stable $(\infty, 1)$ -category of L with Lagrangian cobordisms as morphisms M is identified with the partially wrapped Fukaya category. In particular the additional structure includes a vector bundle E , and to the flat connections A we associate the Lagrangian branes, and the Lagrangian cobordisms M correspond to (anti)-self-dual instantons that are the flow lines of the Chern–Simons action functional. This association became evident in the first section. On the other hand, we have the Fukaya category utilizing symplectic Floer homology and Gromov–Witten theory, where the flow lines are holomorphic strips, and the critical points are the intersection points of certain Lagrangian submanifolds. The equivalence of both categories includes the statement of the Atiyah–Floer conjecture.

The next step in our reasoning is the fact that the gradient flow equations on the four-manifold M , that are the (anti)-self-dual Yang Mills equations, are equivalent to the Hermitian–Yang Mills equations.

$$F = \pm * F \iff \mathcal{F}^{(2,0)} = \mathcal{F}^{(0,2)} = 0, \quad k \wedge F = 0. \quad (125)$$

This was basically the statement of the last section because there is no difference to holomorphic Chern–Simons in the six-dimensional case. A solution \mathcal{A} to these equations is a Hermitian–Einstein connection and corresponds to a pseudo-holomorphic structure over the twistor space. In this sense an (anti)-self-dual connection A is also a Hermitian–Einstein connection \mathcal{A} . The Kobayashi–Hitchin correspondence [39–41] ensures that every connection \mathcal{A} corresponds to a semi-stable holomorphic bundle ε . The sections of such a slope-semi-stable holomorphic vector bundle form the bounded derived category of coherent sheaves $D^b\text{Coh}(C_3)$, where the states satisfying the slope stability condition, which are better known among physicists as BPS-states. This establishes

the equivalence of A-branes and B-branes through the category involving the Lagrangian cobordism in the sense that the morphisms of both categories are related by the Kobayashi–Hitchin correspondence and the Penrose–Ward transform. The equivalence is also embodied in the generalized twistor correspondence, which transfers data on the manifold M ((anti)-self-dual bundles) to holomorphic data on the twistor space over M (holomorphic bundles over the twistor space). The T-duality together with the S-duality discussed in the last section point towards a topological U-duality of the topological M-model. This means that while changing the coupling of the 2-branes continuously we describe a path in the compactified space of $N = 2$ superconformal field theories $\bar{\mathcal{M}}_{N=2}$ in the two limits the sigma-model degenerates into the two boundary topological strings. Obviously, the topological membrane model relates different topological string theories on different backgrounds with each other. This can either be two A-models, two B-models or A-model and B-model. It turns out that the above description of the topological M-model enables a very clear geometric picture of mirror symmetry, which reproduces the approach in [42]. The boundary of $\bar{\mathcal{M}}_{N=2}$ is the stratified space of CFTs with Calabi–Yau target. The coupling g_m divides it into three-sectors: The orbit of constant $g_m = 0$ describes the A-stratum, while for $g_m = \infty$ we are in the B-stratum. Additionally, at $0 < g_m < \infty$ we have the “M-stratum” with a special coincidence orbit at $g_m = 1$. The A-stratum is parametrized by equivalence classes $(J_W \mathbb{R}^+ \cdot g_W, B_W)$, where g is a Calabi–Yau metric and B is a B-field. Likewise, $(J_{W^\vee} \mathbb{R}^+ \cdot g_{W^\vee}, B_{W^\vee})$ are classes parametrizing the B-stratum. Last but not least in the M-stratum we can pick $([g_W], B_W, [g_{W^\vee}], B_{W^\vee})$ as local coordinates, where $[g]$ is a class of metrics. A mirror transformation is nothing else than the continuous path $\gamma(g_m)$ in $\partial\bar{\mathcal{M}}_{N=2}$ with $g_m \in [0, \infty]$ connecting the A-stratum and the B-stratum. The variation of g_m is actually equal to the action of an additive semigroup because with

$$g_m = g_A = \frac{1}{g_B}$$

and so, with $(g_A, g_B) \in \mathbb{R}^+ \times \mathbb{R}^+$, the action

$$\begin{aligned} & ([G_X], B_X, [G_{X^\vee}], B_{X^\vee}) \\ & \mapsto (e^{g_A} [G_X], B_X, e^{g_B} [G_{X^\vee}], B_{X^\vee}). \end{aligned} \quad (126)$$

varies the size of two tori of a Narain lattice $\Gamma^{d,d}$ defined by \mathbb{Z}^{2d} with the quadratic form Ω

$$\Omega(x_i, y_i) = \sum_{i=1}^n x_i y_i. \quad (127)$$

We can write it as the sum of two lattices of rank d , $\Gamma^{d,d} = L \oplus L^\vee$. The symmetry operations of $\Gamma^{d,d}$ are T-dualities and

are identical to the symmetries of the covariant DFT theory

$$Aut\left(\Gamma^{d,d}, \Omega\right) = O(d, d, \mathbb{Z}). \quad (128)$$

Thus, membrane fields describe geometric transitions in terms of paths $\gamma(g_m(t))$. The geometric picture so far did not include branes, but it turns out that they transform as half-spinor representation under $O(4, 4, \mathbb{Z})$. Consider the brane charge μ as the generalized Chern character of the bundle E associated to a brane. We can describe the lattice of branes by a group K^i in K-theory because the map μ describes a correspondence between branes and classes in K-theory. Hence the full lattice is $\Gamma^{d,d} \oplus K^i$. In particular even and odd-dimensional branes transform as conjugate spinor representations of the U-duality group $G = O(5, 5, \mathbb{Z})$

$$\begin{aligned} K^0 &\cong \bigwedge^{even} L^\vee \\ K^1 &\cong \bigwedge^{odd} L^\vee \end{aligned} \quad (129)$$

To map the Ext -groups of $D^b Coh(C_3)$ to the Hom -spaces of $Fuk(C_3)$ we observe bundles of the three and five-branes wrapping cycles in the dual geometries, which are extended over the real line such that they describe wrapped four and six-dimensional worldvolumes inside $X = C_3 \times T^* \mathbb{R}_t$. Hence, we have a four-cycle $M = Y \times \mathbb{R}_t$ with self-dual bundle E of rank N and a six-dimensional $\mathcal{P} = Y \times \mathbb{R}_t \times S^2$ with holomorphically trivial bundles E , which are stable coherent sheaves in the given topological class. We can associate the respective moduli spaces $\mathcal{M}_{Inst}(M)$ and $\mathcal{M}_{Coh}(\mathcal{P})$, which are Hilbert schemes. The Penrose–Ward transformation is a bijective map $f : \mathcal{M}_{Inst}(M) \rightarrow \mathcal{M}_{Coh}(\mathcal{P})$. Accordingly, for the Hilbert spaces of BPS states it holds that

$$\mathcal{H}_{M-brane}(\mu) = H^*(\mathcal{M}_{Inst}(M)) = H^*(\mathcal{M}_{Coh}(\mathcal{P})). \quad (130)$$

Modifying the brane coupling the topological M-brane at $g_m = 1$ degenerates into topological A-branes and B-branes. They are its weak and strong coupling limits. Furthermore, we can expect that the topological branes and topological strings are related by topological U-duality. Moreover, we will expect that μ and p are related by U-duality transformations

$$\mathcal{H}_{t-brane}(\mu) = \mathcal{H}_{t-string}(p), \quad (131)$$

where $p \in \Gamma^{4,4}$ is the momentum the unimodular lattice $\Gamma^{d,d}$. Therefore, we have a relation of A-model topological and B-model topological string in a geometric picture.

8 Background independence

There are two ways of describing a SFT on different backgrounds: Either we deform the conformal field theory on the worldsheet, or we expand the string field action around an infinitesimal solution of the shifted equations of motion. For example, we can have two CFTs given by $S_{CFT,1}$ and $S_{CFT,2}$ on the worldsheets $\Sigma_{2,1}$ and $\Sigma_{2,2}$, which are related by a marginal deformation

$$\delta S_{CFT} = \frac{1}{2\pi} \int d^2z o(z, \bar{z}), \quad (132)$$

where o is a primary (1,1) operator, such that

$$S_{CFT,1} = S_{CFT,2} + \delta S_{CFT}. \quad (133)$$

In correspondence we can formulate the two SFT of dynamical string fields Ψ_1, Ψ_2 associated to the CFTs. They are related by the membrane field Ψ

$$S[\Psi_1] = S[\Psi_2] + \int F_\Psi \star F_\Psi, \quad (134)$$

where the worldvolume Σ_3 is bounded by $\Sigma_{2,1} \cup \Sigma_{2,2}$. Alternatively, one can show that the membrane field action, which is defined on the space of CFTs, is reparameterization invariant. Similar to Chern–Simons theory we have large gauge transformations relating different vacua, which, however, in this case are conformal field theories. In the target this amounts to relating two vacua of Chern–Simons theory with each other. As the worldsheets Σ_2 bound the brane worldvolume the associated flat spaces bound the gravitational instantons and so it is not hard to guess that the field theories on the intertwining worldvolume Σ_3 correspond to semi-classical solutions of the gradient flow equations of the Einstein–Hilbert action or equivalently the Chern–Simons functional. As the metric on the brane is a one-parameter family of surface metrics, so is the target space metric. The deformation between two different conformal backgrounds results in a deformation of the target space metric. This might be just an expansion or contraction in the linear case but can also be a non-linear deformation depending on which direction we are moving the space of metrics. In fact, this is similar to the statement that the interpolating metric can be arbitrary as long as it asymptotically approaches two Ricci-flat spaces at infinity. Let's take as simple and concrete example the A-model on the deformed conifold T^*S^3 . We have N A-branes wrapped around the Lagrangian submanifold S^3 and open strings stretched between them. These are the Hom -spaces of the Fukaya category. Assuming the correctness of the Atiyah–Floer conjecture we can instead consider instantons on $S^3 \times \mathbb{R}_t$. To connect two branes, we deform the CFT by a path $\gamma(g_m(t))$ in $\partial\mathcal{M}_{N=2}$. In this geometry the membrane coupling controls the radius of the zero locus. Starting at $g_m(t = -\infty) = 0$, the radius R of S^3 is infinite

and so are the branes. Now we move towards the coincidence orbit at $g_m = 1$, where the S^3 a non-zero minimal radius. If we would increase g_m further, there would be the S^2 growing and we had the resolved geometry. Instead, we decrease g_m again such that $g_m(-\infty) = 0$ and the sphere grows until it is infinite and completely flat. This is what we described in the first section but with the N wrapped branes forming a theta vacuum structure. Similarly, we could have started at the opposite stratum connecting different 5-dimensional B-branes through stable coherent sheaves constituting the Ext -groups in the resolved geometry. This is very interesting in my opinion. A deformation of the CFT induces a gradient flow of the string field action functional, that shapes the geometry of the target space. The flow lines of the gradient flow equation of cubic string field theory are the membrane fields. With this knowledge it is rather obvious that, while the A-model reduces to Chern–Simons theory, the membrane fields reduce to instantons. Now we should also be able to explain the duality between the D-branes and the theta-vacuum structure. The dual A-model closed string geometry is the $U(2)$ -bundle E over S^3 . The closed strings wrap the spheres and are one-dimensional subspace $S^1 \subset S^2$ of them. We define now a standard one-form connection A as a projection from the tangent bundle $T_{S^1}E$ onto vertical subspaces V_{S^1} of points $p \in S^1$ at the fibre F_x over a basepoint $x \in S^3$. Hence, the string induces the principal $U(2)$ -valued principal connection A , which in turn induces a connection on the $SO(3)$ -bundle of trace-free, skew-adjoint automorphisms. We conclude that $U(N)$ -Chern–Simons theory in the large N limit arising in the open string geometry is dual to a A-model closed string geometry that is a $SO(3)$ -principal bundle. However, this is only one half of the story. It seems that we also have to assume a stack of N anti-D-branes inside of the A-model open string geometry such that we have $U(N) \times U(N)$ -Lie algebra-valued connections. The topological T-duals then are $SU(2) \times SU(2)$ connections. Since the base manifold Y is three-dimensional, the $SU(2) \times SU(2)$ -Chern–Simons theory is the Einstein–Hilbert action on Y with cosmological constant. As known from section one, the gradient flow lines with the form of an FLRW-metric are also solutions to four-dimensional general relativity. All this arises and is understandable in the framework of topological M-theory as established within this work. Additionally, we have seen in this section how the membrane field theory produces its own background.

9 Knot cobordisms as topological 2-branes in membrane field theory

Wilson loop operators along knots are a crucial ingredient in Chern–Simons theory and they play the role of open strings in the closed string background. Obviously, we need

to construct Lagrangian submanifolds, specifying the boundary conditions to incorporate them in our description. The construction of the submanifolds is due to Ooguri and Vafa [43] and we follow it closely. For every knot \mathcal{K} in Y there is a Lagrangian submanifold $\mathcal{C}_{\mathcal{K}}$ in T^*Y . We parametrize the knot by $q(s)$. We define

$$\mathcal{C}_{\mathcal{K}} = \left\{ (q(s), p) \in T^*Y \mid \sum_a \frac{\partial q_a}{ds} p_a = 0, 0 \leq s \leq 2\pi \right\} \quad (135)$$

as the conormal bundle, where p_a are the coordinates on the cotangent bundle. This can be understood as the twistor space over the knot \mathcal{K} . Hence, it has the topology $S^1 \times S^2$ and intersects the zero locus at \mathcal{K} . Wrapping M branes on this space creates an $U(M)$ Chern–Simons theory on $\mathcal{C}_{\mathcal{K}}$. Now we have the maps $\phi \in \mathcal{K}$. Of course, this opens a new sector of strings, which can be shown to be described by a complex scalar, which ends are charged under the $U(N)$ and $U(M)$ gauge fields on the branes. In the resolved geometry these branes survive the geometric transition and describe the open string sector in the large N duality. One can show that there is a map between the \mathcal{K} and Lagrangian submanifolds $\mathcal{C}_{\mathcal{K}}$ in $\mathcal{O}(1) \oplus \mathcal{O}(1) \rightarrow \mathbb{C}P^1$ sending $\mathcal{K} \rightarrow \mathcal{C}_{\mathcal{K}}$ and $b_1(\mathcal{C}_{\mathcal{K}}) = 1$. The open string amplitudes are the same as the knot invariants computed in CS theory.

Again, we can derive what role they play in the topological membrane theory. Let us consider two knots $\mathcal{K}_1, \mathcal{K}_2$ in Y_1 and Y_2 of two different leaves T^*Y_1 and T^*Y_2 inside $X = T^*Y \times T^*\mathbb{R}_t$ of constant t at t_1 and t_2 . There is a knot cobordism $S \subset M$, where M is a Lagrangian cobordism between Y_1 and Y_2 with $\partial S = \mathcal{K}_1 \cup \mathcal{K}_2$. We construct the conormal bundle over S simply as

$$\mathcal{C}_S = \left\{ (q(s(t)), p(t)) \in \mathcal{X} \mid \sum_a \frac{\partial q_a}{ds} p_a = 0, 0 \leq s(t) \leq 2\pi \right\}, \quad (136)$$

which has the topology of $U(2)$ -bundle over S , where $t \in [t_1, t_2]$. Furthermore, the knot cobordism related the Khovanov homology Kh of \mathcal{K}_1 to that of \mathcal{K}_2 . In [44] the author describes how one can calculate Kh from instantons. This comes naturally within the membrane field theory. Let's consider as before the Lagrangian submanifolds Y_i as critical points of the Chern–Simons functional with singularities of codimension 1, where the locus is a knot \mathcal{K} and the cobordism M as an instanton. This instanton has singularities of codimension 2, where the locus is the embedded surface defined by the knot cobordism S . These are the conormal bundles over the knots and knot cobordisms. The key point is now to consider the Chern–Simons action functional over the space of singular instantons modulo gauge transformation. Let P

be a sphere-bundle over Y . Let \mathfrak{B} be the space of smooth connection of $ad(P)$ and $\mathfrak{B} \setminus \mathcal{C}_{\mathcal{K}}$ the subspace of smooth connections in the restriction of $ad(P)$ to $Y \setminus \mathcal{K}$ with a singularity along \mathcal{K} . The group of determinant-one gauge transformation $\mathcal{G}(\mathcal{K})$ acts on this space freely and one defines

$$\mathfrak{B}_{\mathcal{K}} = \mathfrak{B} / \mathcal{G}(\mathcal{K}). \quad (137)$$

Define further the set $\mathfrak{R}(Y \setminus \mathcal{K})$ of homomorphisms $\varsigma : \pi_1(Y \setminus \mathcal{K}) \rightarrow SU(2)$. Then the critical points C of the CS functional are

$$C = \mathfrak{R}(Y \setminus \mathcal{K}) / SU(2). \quad (138)$$

So, when we consider the membrane field theory with knots, we have to consider the twistor space with codimension-2 singularities as sphere bundles over knot cobordisms in the Lagrangian cobordisms. This will modify the membrane field action on the family of conifolds. With knots, the classical topological membrane theory (112) descends to the Chern–Weil formula

$$S_T[A] = \int_{M \setminus S} \text{tr} F \wedge F = -\frac{1}{4} p_1(P_{\Delta}) [M_{\Delta}] + \frac{1}{16} I_s \quad (139)$$

where we have singular bundle data $P_{\Delta} \rightarrow M_{\Delta}$ as in [45] and $I_s = S \cdot S$ is the self-intersection number, while the first term is the sum of instanton and monopole numbers. Thus, the partition function will be modified as well and it is to be expected that the expectation values along the knot cobordisms will coincide with the expectation values of the topological 2-branes in the membrane model, similar to how Wilson loop operators along knots in CS-theory are identical to open string amplitudes.

Here lies an unforeseen connection with an interesting and somewhat mysterious aspect of M-theory. Even though we did not discuss it explicitly, it was an implicit part of the topological membrane model when we discussed N fivebranes wrapped around the twistor space $\mathcal{P} = M \times S^2$.

10 6d $N = (2, 0)$ superconformal field theory

The worldvolume theory of such a stack of branes in M-theory is believed to be a six-dimensional $(2, 0)$ superconformal field theory, which is a quantum theory of non-abelian gerbes. Hence, there is good reason that its topologically twisted version should be part of our considerations of topological M-theory and that it somehow is described by the membrane field action (112). Indeed, there is a beautiful connection, which involves the previously discussed knot cobordisms in the four-manifold M and the AGT correspondence [46, 47]. A review can be found in [48]. The AGT correspondence relates a conformal field theory on a Riemann surface of genus with n punctures $C_{g,n}$ to a $N = 2$ superconformal field theory on a complex surface $M_{\mathbb{C}}$. Both theories

are reductions of the twisted $(2, 0)$ theory on $M_{\mathbb{C}} \times C_{g,n}$ and the precise content of either is determined by the parameters of the other. For example, the four-dimensional theory depends on the punctures and complex structure on C . We saw before from the membrane field action (112) that topological M-theory on T^*M reduces to Donaldson-Witten theory on M , which is a topological twisted $N = 2$ quantum field theory. From section 4 we can extract that there is most likely an equivalence between the Nekrasov partition function and the partition function of topological M-theory. Without digressing further, we note that the interesting point here is that, when we discussed the geometric picture of mirror symmetry, the twistor space \mathcal{P} was wrapped by a stack of N B-branes inside the M-stratum for $1 \leq g_m < \infty$. Their worldvolume theory can be described by the topological twisted six-dimensional $(2, 0)$ superconformal field theory on $\mathcal{P} = M \times \mathbb{CP}^1$, because the topological M5-branes only degenerate into D5-branes at $g_m = \infty$. The knot cobordisms inside M can be interpreted as codimension 2 defects from the 4d perspective and as such they are equivalent to a degenerate vertex operator of the CFT. Since we identified them with topological M2-branes it is clear that they are half-BPS states and preserve half of the original supersymmetry of the 6d theory. This is exactly the requirement that they also preserve the supersymmetry in the 4d theory. Moreover, it follows that they are inserted at the singularities of a one-form $a_z dz$ on a bundle over C modulo gauge transformation. We pick local coordinates $z \in \mathbb{CP}^1$ and x to parametrize the fibres of $T^*\mathbb{CP}^1$. Furthermore, we define a N -fold cover Σ of \mathbb{CP}^1 with a one-form $\lambda = x dz$ through

$$\Sigma = \left\{ (z, x) \mid \langle \det(\lambda - a_z) \rangle \right. \\ \left. = \lambda^N + \sum_{k=2}^N \alpha_k(z) \lambda^{N-k} = 0 \right\} \subset T^*\mathbb{CP}^1. \quad (140)$$

It is determined by the vacuum configuration. Specifically, by the VEVs $v_k(z) = \langle P_k(a_z) \rangle dz^k = \alpha_k(z) dz^k$ of the six-dimensional SCFT, in the case $g = \text{su}(N)$. Apart from that, $a_z = a_6 + i a_7$ is the only non-zero component of the adjoint-valued scalars a_i of the rotation group of the transverse coordinates to the branes in the ambient spacetime, which parametrize the vacua of the desired $(2, 0)$ superconformal field theory. Moreover, when analysing the embedding of the anti-chiral supercharges of the four-dimensional theory into the six-dimensional, one recognizes the restriction the invariant polynomials constructible from a_i to depend only on the coordinate z . Thus, Σ is interpreted as the Seiberg-Witten curve of a $N = 2$ 4d theory specified by a Lie algebra, a surface and puncture data $(g, C, (\rho_i, \mu_i))$, where ρ_i are Lie algebra homomorphisms and $\mu_i \in \mathfrak{g}_C$ are mass parameters. With its help one can derive the prepotential of the Coulomb branch as well as the masses of the BPS states. Let's assume

that \mathbb{CP}^1 has full tame punctures at z_i . It follows that a_z has first order poles $a_z \sim \mu_i (z - z_i)^{-1} dz + O(1)$ defined up to conjugation. Such punctures correspond to a flavour symmetry, which is coupled to the constant value of the background vector multiplet scalar, specified through the mass parameters. However, they can collide to form higher order poles, which results in a non-conformal behaviour of the conformal blocks. Assuming that we have two wild punctures $z_i = 0, \infty$ and $M = \mathbb{C}^2$. In the limit $z \rightarrow \infty$ the puncture at infinity translates into poles $v_2 \sim z^{-4} dz^2$ and hence into wild singularities of the SW curve. These correspond to the residue of the Hitchin field on C up to conjugation, or equivalently, to the quadratic differential $v_2 = \alpha_2(z) dz^2$, which has no other poles than the z_i . Hence, the SW curve Σ is defined by $x^2 = \alpha_2$, and corresponds to the spectral curve of the Hitchin system on C , translating into periods of the SW prepotential $F_M(\vec{a}; q)$ that describes the IR effective action of the $N = 2$ theory. In our case of the two punctured \mathbb{CP}^1 , the Hitchin system is the Toda integrable system with chiral algebra WA_{N-1} . The instanton contribution to the prepotential is given by the Nekrasov partition function

$$F_{\mathbb{C}^2}^{inst}(\vec{a}; q) = \lim_{\epsilon_1, \epsilon_2 \rightarrow 0} Z_{\mathbb{C}^2}^{inst}(\epsilon_1, \epsilon_2, \vec{a}; q), \quad (141)$$

taking the form of a conformal block

$$Z_{\mathbb{C}^2}^{inst}(\epsilon_1, \epsilon_2, \vec{a}; q) = \langle \psi_{\mathbb{C}^2}, q^{L_0} \psi_{\mathbb{C}^2} \rangle. \quad (142)$$

Here $\psi_{\mathbb{C}^2}$ is a Whittaker vector in the completion of graded components of an infinite-dimensional $\mathbb{Z}_{\geq 0}$ -graded Hilbert space $H_N(\mathbb{C}^2)$ and q^{L_0} acts as multiplication by q^n on each of the components. $\psi_{\mathbb{C}^2}$ is a state in the representation corresponding to a topological two-brane. It was shown that any $N = 2$ gauge theory on M corresponds to such a Donaldson-Witten integrable system [49], which we just prescribed. In this way, we find the AGT correspondence within topological M-theory. If we analyse the full geometric set-up, we observe that topologically twisted M-theory on a compact eight-manifold X with $Spin(7)$ holonomy is equivalent to M-theory on $X \times S^3$, which is isomorphic to a $SU(2)$ bundle over X . The $Spin(7)$ -instantons on X take values in the $SU(2)$ Lie algebra. We can decompose X as $M \times Q \times \mathbb{R}^3$ with $Q = T^*C$, a four-dimensional hyperkähler manifold and place coincident M5-branes on $M \times C$ inside it. This setup is equivalent to 6d $(2, 0)$ SCFT with gauge group $SU(N)$ on the worldvolume of the M5-branes on $M \times C$. In this way, topological M-theory on X should be equal to the prescribed partially twisted SCFT on $M \times C$. When collecting all the findings of this paper, we have that in the weak-coupling regime

$$\mathcal{Z}_{T^*M}(\Psi(\xi), q) \\ = \int \mathcal{D}\Psi e^{-\int \mathfrak{F}^* \mathfrak{F}} = \sum_{k=0}^{\infty} q^k \int_{\mathcal{M}_{G,k}(M)} \mathfrak{C}(P \rightarrow M), \quad (143)$$

where Ψ is a functional of maps $\xi: \Sigma_3 \rightarrow X$, $M_{G,k}$ is the moduli space of G -valued, anti-self-dual connections with topological charge k . The characteristic class \mathcal{C} of the bundle P is a topological invariant of a Riemannian manifold $M \subset X$ and $q = e^{-2\pi g_m}$. \mathcal{Z}_{T^*M} is the partition functions of twisted $N=2$ gauge theory, which is the cohomological Yang-Mills theory that computes the Donaldson-invariants. In the first term (143) is written as the path integral of the membrane field action that must be gauge fixed appropriately. As prescribed, for $X = T^*M$ it is equal to the Nekrasov partition function of $N=2$ $SU(N)$ gauge theory because $\mathfrak{F} \rightarrow F = dA + A \wedge A$.

$$\begin{aligned} \mathcal{Z}_M^{inst}(\Psi(\xi), q) &= \int \mathcal{D}\Psi e^{-\int \mathfrak{F}^* \mathfrak{F}} = \int \mathcal{D}A e^{-\frac{1}{2} \int F \wedge F} \\ &= \sum_{k=0}^{\infty} q^k \int_{\mathcal{M}_{N,k}(M)} 1. \end{aligned} \quad (144)$$

Yet, the partition function for (112) is much more general considering the possible content of the membrane functional, determined by the values it takes at the boundary. In the case we discussed explicitly, when all higher modes become massive and decouple and when in the weak coupling regime, the intertwining membrane field is just an $U(N)$ instanton. Assuming $C = \mathbb{CP}^1$ without punctures, the Penrose-Ward transform relates the instantons in the infrared regime to holomorphic vector bundles over \mathcal{P} in a bijective manner. The transition from objects in the perturbative regime to objects in the non-perturbative regime is geometric. Applying this to the partition function, we have to replace instantons by stable coherent sheaves \mathcal{E} of rank r and the instanton moduli space $\mathcal{M}_{N,k}$ by the moduli space of torsion-free sheaves $\mathcal{M}_{r,k}$ over \mathcal{P} with $ch_3 = -k$.

$$\begin{aligned} \mathcal{Z}_{\mathcal{P}}^{coh}(\Psi(\xi), q) &= \int \mathcal{D}\Psi e^{-\int \mathfrak{F}^* \mathfrak{F}} \\ &= \sum_{k=0}^{\infty} q^k \int_{[\mathcal{M}_{r,k}(\mathcal{P})]^{virt}} 1. \end{aligned} \quad (145)$$

In this way we get

$$\mathcal{Z}_M^{inst}(\Psi(\xi), q) = \mathcal{Z}_{\mathcal{P}}^{coh}(\Psi(\xi), q). \quad (146)$$

remembering that this is an IR-UV duality. Hence,

$$\mathcal{Z}_{\mathcal{P}}^{coh}(\Psi(\xi), q) = \mathcal{Z}_{top-(2,0) SCFT}. \quad (147)$$

This is an explicit example of the general statement concerning the limits of the partition function \mathcal{Z} of (112).

$$\begin{aligned} \mathcal{Z}_{g_m \rightarrow 0} &= \mathcal{Z}_{top-A} \\ \mathcal{Z}_{0 < g_m < 1} &= \mathcal{Z}_{top-N=2} \\ \mathcal{Z}_{g_m \rightarrow \infty} &= \mathcal{Z}_{top-B} \end{aligned} \quad (148)$$

In the finite non-perturbative regime, topological M-theory on $M \times T^*C$ is partially twisted 6d (2,0) superconformal

field theory on $M \times C$.

$$\mathcal{Z}_{1 < g_m < \infty} = \mathcal{Z}_{top-(2,0) SCFT} \quad (149)$$

because g_m controls the size of the Riemann surface C , which shrinks to a point in the limit $g_m \searrow 1$, where the SCFT becomes $N=2$ gauge theory but which has finite size in the regime $1 < g_m < \infty$. However, the string field is in general a linear combinations of gauge fields of a higher principal bundle. They decouple because of the topological nature of the sigma-models discussed, making them fairly easier to handle. Such principal n -bundles can be expressed as Lie crossed modules of n -tuples of Lie groups G_n with group homomorphisms, usually denoted as $t: G_{n+1} \rightarrow G_n$ satisfying the Pfeiffer identity and another identity, which renders them as G-homomorphisms. They are also understood as categorification of Lie-groups, constructable with higher Čech-cohomology. Throughout the paper we considered topological strings, where only zero modes contributed, and the string field had ghost number one. This made it possible to connect the four-dimensional twisted infrared theory to its topological six-dimensional superconformal UV-completion via the Penrose-Ward transform, because we considered only self-dual connections on a 1-principal bundle. Now we also consider the two-form B and the three-form C as higher gauge connections and excitations, together with their respective curvature forms H and G . Henceforth, the string field

$$\Psi = (A, B, C) \quad (150)$$

can be interpreted as a section of a principal 3-bundle with structure 3-group $(G_3 \rightarrow G_2 \rightarrow G_1, \succ, \{\bullet, \bullet\})$. Here we should note that the principal bundle is expressed on a Lie 2-crossed module, where G_1 acts smooth \succ on G_2 and G_3 via automorphisms and the smooth map $\{\bullet, \bullet\}: G_2 \times G_2 \rightarrow G_1$ is the so-called Pfeiffer lifting. Furthermore, as one can show, this is a non-abelian generalisation of the principal 2-bundle with abelian 2-gerbes, e. g. when G_3 is trivial. This trivialisation can be seen as the mathematical process for the decoupling of the higher excitations, where only the lowest lying mode on the principal 1-bundle survives. It follows, that the membrane field will now contain higher self-dual connections in the infrared regime, where we extend the principal 3-bundle over Y onto $Y \times \mathbb{R}$. Then the vector \mathfrak{F} is the triple (F, H, G) with

$$\begin{aligned} F &= dA + A \wedge A - \underline{B} \\ H &= dB + \underline{A \wedge B} \\ G &= dC + \underline{A \wedge C} + \{B, B\} \end{aligned} \quad (151)$$

where the line indicates that the quantity was acted on by t . Note that one needs to specify the wedge product between forms valued in the different groups. There will be a higher Penrose-Ward transform relating non-abelian self-dual gerbes to holomorphic vector and principal 3-bundles,

for which there are interconnecting self-duality relations between the curvatures in (151). Higher Penrose–Ward transforms were constructed in [50]. Throughout the paper we just considered the case where the associated groups of these higher structures are trivial. However, in general, increasing of the coupling into the regime of strong coupling will associate these higher structures to each other through a 3-Penrose–Ward transform. Hence, the action (112) is a unification of interesting theories, depending on the content of the membrane field functional, which is determined by a sigma-model, and the coupling strength. The interconnections between the theories discussed are for a great part of conjectural nature. Although strong evidence exists, proving them in general is often very hard. As we saw, they are incarnation of the same topological theory.

11 Conclusion

We have shown that the Einstein–Hilbert action over the space of FLRW-metrics takes the form of (A)SD $SO(4)$ Yang–Mills. The underlying theory is the Donaldson–Witten theory. A supersymmetric topological theory computing the Donaldson polynomials of the underlying manifold. The construction of the membrane field action incorporated the ∞ -extension of Chern–Weil theory and it reduced to a topological Yang–Mills theory on a four-dimensional manifold, which upon (A)SD gauge fixing becomes a (A)SD Yang–Mills theory. It was shown how the mirror duality of complex and symplectic structures is reflected in the membrane model by the Penrose–Ward transform as well as how the membrane coupling is related to the string couplings. From this we were able to derive the partition function for the model without Wilson loops in the boundary and found that it counts slope-stable bundles over the twistor space, which are related by the Karenbeck–Donaldson–Yau theorem and the Penrose–Ward transform to the counting of instantons over Lagrangian cobordisms, which is closely related to Donaldson theory. Furthermore, we discussed homological mirror symmetry from the viewpoint of the membrane field theory, where we could utilize again the connection between self-dual Yang Mills and slope-stable complex bundles to relate the morphisms of either categorical description of the respective topological string models. At the end it was shown how the inclusion of knots into the boundary theory transfers into the membrane model by drawing the connection to instanton knot homology and Khovanov homology. We find that the knots are the boundaries of 2-Branes embedded into the Lagrangian cobordisms described by knot cobordisms and that the expectation values along the cobordisms correspond to the expectation values of 2-branes in the topological M-model. All this is related to general relativity by the fact alone that the Chern–Simons functional is

three dimensional Einstein–gravity with cosmological constant. For me personally the most important question remaining is to what extend the membrane field action captures the non-perturbative framework of physical M-theory in a similar way as string field theory captures the dynamics of both the physical and topological string. The underlying DFT is a promising hint because it is known that supergravity theories can be formulated this way and they are the low energy limit of M-theory. Additionally, from the topological viewpoint we note that we were able to formulate string phenomenology in physically relevant dimensions without the need of compactified dimensions. In the end, the AGT correspondence seems to be a central object inside topological M-theory, relating the superconformal theory in six-dimensions to $N = 2$ gauge theories. The latter is topological M-theory on X in the weak coupling regime, where the target looks like $X = T^*M$, degenerating into the A-model on $T^*\partial M = T^*Y$ for $g_m \rightarrow 0$. The former is topological M-theory in the non-perturbative regime, where $X = M \times T^*C$.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: There will be no additional data deposited due to the theoretical nature of the findings presented in this work.]

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