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A Study of Noncommutative Instantons

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A Thesis presented for the degree of

Doctor of Philosophy

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Dedicated to

Mum and dad

A Study of Noncommutative Instantons

Andrew Iskauskas

Submitted for the degree of Doctor of Philosophy

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Abstract

We consider the properties and behaviour of 2 U(2) noncommutative instantons: solutions of the NC-deformed ADHM equations which arise from U(2) 5d Yang-Mills theory, where the underlying space is $\mathbb{R}^2_{\text{NC}} \times \mathbb{R}^2_{\text{NC}}$. The ADHM construction allows us to find all such solutions, which form a moduli space of allowed configurations. We derive the metric for such a space, and consider the dynamics of the instantons on this space using the Manton approximation. We examine the reduction of this system to lower-dimensional soliton theories, and finally consider the effect of adding a Higgs field to the SYM theory, resulting in a potential on the instanton moduli space.

Declaration

The work in this thesis is based on research carried out at the Centre for Particle Theory, Department of Mathematical Sciences, Durham, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

The material in Chapters 1 and 2 is a review of existing literature, with the exception of Section 2.2.1, which is taken from my own research [1]. The material in the remaining chapters is based in original research in collaboration with my supervisor, Dr. Douglas J. Smith [2].

Acknowledgements

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To those people that were brave enough to proofread the thesis, I extend my thanks. A hearty cheers goes to my supervisor Dr. Douglas Smith, Caroline Walters, Olivia Kent, and my father for looking over the "finished" product and offering sage advice and suggestions for improvement.

Finally, I'd like to extend the greatest thanks to my parents, for encouraging and supporting me and believing that I could make it this far in the world of research. For that, I owe you more than I shall ever be able to repay.

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Introduction

"Lasciate ogni speranza, voi ch'entrate!"

– Dante, *The Inferno*

Instantons have long provided a fertile testing ground for exploring aspects of Yang-Mills theory [3], and can play an important role in determining the behaviour of non-perturbative effects in supersymmetric Yang-Mills theory (SYM) [4, 5, 6]. In comparison with other solitons, however, little is known about their dynamics. In particular, when compared to monopoles, this paucity of information is most apparent. We aim to elucidate some of the aspects of charge 2 instantons in a U(2) gauge theory, which allows an insight into the underlying Yang-Mills and string theoretical pictures.

The motivation for studying instantons begins with a consideration of superstring theory. These theories admit dynamical extended objects known as D-branes, arising from applying Dirichlet conditions on open string endpoints [7]. The D-branes can be shown to source the magnetic and electric Ramond-Ramond charges, which are necessary in order to guarantee superstring invariance under T-duality [8]. The

coupling of a Dp-brane (that is, a D-brane extended in p spatial dimensions) is of Dirac-Born-Infeld (DBI) form [9]:

$$S_{Dp} = -\mu_p \int \mathrm{d}^{p+1}x \mathrm{Tr}\Big(e^{-i\Phi}(-\det\sqrt{G_{ab} + B_{ab} + 2\pi\alpha' F_{ab}})\Big),$$

where G_{ab} and B_{ab} are the components of the NS-NS fields parallel to the brane, F_{ab} is the gauge field on the brane, Φ is the dilaton, and μ_p is some p-specific coupling constant. In the limit of small separation between a stack of N Dp-branes, such an action can be seen to reduce to the theory of maximally supersymmetric SU(N) super-Yang-Mills, a point that we shall revisit shortly.

The presence of D-branes has a profound effect on the superstring theory, forcing spontaneous symmetry breaking of some of the supersymmetries. In fact, D-branes leave exactly half of the supersymmetries unbroken; they are BPS states of the theory [10]. This is true for Dp-branes of any dimensionality, p. The study of BPS states of a theory is common in a number of fields because of their relative simplicity compared to a generic state of the theory: due to their fecundity in string theories and general gauge theories, BPS states can be studied to calculate black hole entropy [11], determine stability properties of Calabi-Yau manifolds [12], and more abstractly may be used in the consideration of Teichmuller spaces [13]. Finding a class of BPS solutions is a useful step in gaining understanding about the properties of any theory that admits them.

To demonstrate the connection between BPS states in superstring theory and instantons, we turn to the action of N coincident D4-branes. In the low-energy limit (that is, when the strings stretched between such D-branes have low mass and hence their contribution to the DBI action are subleading), the coupling takes the form of

a Chern-Simons term in the action:

$$S_{CS} = \frac{1}{2}(2\pi\alpha')^2\mu_4 \int C_1 \wedge \mathrm{tr}(F_2 \wedge F_2),$$

where F_2 is the 2-form field strength on the D4 world-volume. The integrated trace term is simply $8\pi^2 c_2$, where $c_2 \in \mathbb{Z}$ is the second Chern number [14]. Hence, for a given integer c_2, one may rewrite this action as

$$S_{CS} = c_2(4\pi\alpha')^2\mu_4 \int C_1,$$

which, with the identification $\mu_0 = (4\pi\alpha')^2\mu_4$ is the low-energy action of c_2 D0-branes. We also note that the integral of the Chern term appears in another context: namely 5d super-Yang-Mills $U(N)$ theory, where F_2 is the field strength of the gauge field A [15]. Hence the low-energy dynamics of D-branes is described by a super-Yang-Mills theory, and any BPS states present in the string theory also arise in SYM, representing D0-branes charged under a $U(N)$ gauge field [16].

Nevertheless, the study of Yang-Mills theory, while simpler than string theory, is still non-trivial. One may use arguments first presented in [10] to demonstrate that BPS states in SYM arise as self-dual solutions to the static equations of motion: that is, solutions for which $\star F = \pm F$, where $\star$ is the Hodge dual, $\star F_{\mu\nu} = 1/2\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$. These solutions are called instantons and are $1/2$ BPS, cementing the correspondence between BPS states from D-branes and those in Yang-Mills theory. Moreover, an algebraic formulation of such solutions exists via the ADHM construction [17], generating the class of all instantons with a given "instanton charge" c_2 and for a given gauge group $U(N)$. These are solutions in the 4-dimensional Euclidean Yang-Mills theory and may be identified with the static solutions of the full $4 + 1$-dimensional

theory.

Away from the coincident limit in the D-brane theory, one may still obtain BPS solutions. By introducing a Higgs field (and breaking yet more of the supersymmetries), one may consider 1/4-BPS states. From the point of view of the D4-D0 theory, Higgsing the branes introduces bound states between D0s and fundamental strings; from the perspective of the Yang-Mills BPS states, this is equivalent to introducing a non-zero scalar field in the action [18]. The instantons arising from such a theory behave as if under the influence of an external electric charge and are referred to as dyonic instantons [5].

A further connection facilitated by the study of instantons has been seen in M-theory [19]. Consider the $(2,0)$ superalgebra generated by the M5-brane. Under dimensional reduction of M-theory along, say, the x^5 direction, one obtains a charge

$$Z_5 \equiv -\frac{1}{8g_{YM}^2} \int \mathrm{d}^4x \, \mathrm{tr}(F_{ij}F_{kl}\epsilon_{ijkl}),$$

which for consistency of supersymmetry must be identified with the Kaluza-Klein (KK) momentum, P_5, corresponding to the compactified direction [9]. Again, we observe that the instanton charge appears in this context, and the tower of KK states is classified by the instanton charge, $P_5 = c_2/R_5$, where $R_5 \equiv g_{YM}^2/4\pi^2$ is the radius of compactification in the x^5 direction. Crucially, in order for index calculations of the number of degenerate BPS states to agree in both the M-theory and superstring theory cases, the instanton contribution must be taken into account [20]. In fact, the duality between the D4-brane theory and the compactified M-theory is only UV-complete with the addition of the instanton contributions [21, 22]. Localisation techniques have been employed to examine such index calculations from the point of view of dyonic instantons [23], where the contributions to the index are

centred around the zeroes of the instanton potential. For the case of the single U(N) instanton, the result was found to agree with the explicit calculation in maximally supersymmetric $\mathcal{N} = 8$ SQM [24]. Thus, a consideration of instantons may allow us to scry into the behaviour of the M5-brane.

There are also a large number of identifications that can be made between instantons and other solitonic solutions in reduced dimensions. It is known that noncommutative instantons in gauge group SU($2N$) displaying SO(3) invariance can produce a class of non-Abelian vortices [25, 26] and it is believed that a more general class of vortices coupled non-trivially to a gauge field can be obtained by considering a dimensional reduction of noncommutative dyonic instantons [27]. As an extension, a large body of material is dedicated to the study of vortex systems with impurities, thus providing an entry point into problems considered in condensed matter physics: see, for example, [28]. By considering instantons whose ADHM data has circle invariance, one can obtain monopoles in hyperbolic space with platonic symmetries [29, 30] and in a similar vein, Skyrmions may be constructed (in Euclidean [31] or hyperbolic [32] space) by computing the holonomy of SU(2) instantons.

It is not straightforward to gain a deep understanding of the dynamics of instantons on the full field theory. Instead, it proves fruitful to employ an observation due to Manton [33] and study the motion of instantons as geodesics on the moduli space of solutions. The moduli space is a $4kN$-dimensional space made up of all instanton solutions for a given gauge field U(N) and topological charge k. Configurations within this moduli space can be seen as minimum energy solutions of the field theory and, should we perturb such a solution by a small velocity, we expect that it will remain in (or energetically close to) the moduli space. It transpires that one may view the dynamics of slow-moving instantons as geodesic motion on this moduli space, endowed with a suitable hyperKähler metric, and it then becomes

feasible to consider low-energy scattering and evolution of the field theory.

The moduli space of instantons constructed contains singularities arising from instantons of zero size. Such "small" instantons have a dual picture in the string theory of a transition between the Higgs and Coulomb branches of the D0 theory [6]. The Coulomb branch of the theory corresponds to D0 branes separated from the D4s: the moduli describe the positions of the D0s transverse to the D4s. The Higgs branch corresponds to the D0s 'dissolved' in the D4s, and their moduli are precisely the moduli of instantons in the Yang-Mills theory. The singularity in the metric of this moduli space, attained when the instantons hit zero-size, then corresponds to the transition point between the two branches. To circumvent this problem, it is possible to use a noncommutative framework in which a minimum bound is placed on the instantons' size via the introduction of a Fayet-Iliopoulos term [34]. This modification to the theory smooths out the moduli space singularities, and it has been seen explicitly that the metric takes Eguchi-Hanson form in the case of a single U(1) instanton [35]. The ADHM procedure applied to a noncommutative system returns the expected results: namely, solutions are self-dual and maintain integer charge [36].

The dynamics of commutative dyonic instantons with gauge group SU(2) have been studied for a single instanton and two well-separated instantons [37], and more recently an extensive analysis of the dynamics has been studied for two instantons with arbitrary separation [38]. A free single instanton may evolve into a configuration where its size ρ can vanish, resulting in the small instanton. The introduction of a potential term guarantees that this singular point can not be reached for an instanton that starts with a non-zero angular momentum and a bounded, non-zero, size. Specifically, it will remain in a stable orbit with conserved angular momentum. In the case of multiple instantons, however, this may not hold: the instantons may

trade angular momentum with each other, allowing one instanton to grow in size at the expense of its counterpart, approaching the zero-size singularity in finite time. This was shown in [38]. The zero-size singularity still exists, therefore, for more than a single dyonic instanton; we must consider a noncommutative deformation to the space in order to conclusively remove the singularity.

The outline of this thesis is as follows. In Chapter 2, we will review the construction of instanton solutions as solutions to the self-dual Yang-Mills field equations. A consideration of solitonic solutions, via the Bogomolny argument, leads one naturally to an algebraic formulation of instantons for a given topological charge, k. The results extend to noncommutative spaces; we summarise the connection between noncommutative function space and the quantum mechanical analogue. Having constructed solutions, we consider the parameter space of the charge k instantons as furnishing a moduli space of allowed configurations, and may derive an algebraic formalism for determining the metric on this moduli space. This allows us to analyse the dynamics of two instantons via the Manton approximation [33]. Finally, we consider the effect of introducing a non-zero electric charge, or potential, on the moduli space in a similar manner.

In Chapter 3, we proceed to explicitly derive the solutions for 2 U(2) instantons in both the commutative and noncommutative frameworks. The presence of noncommutativity perturbs the known solutions in a non-trivial manner, and by finding an expedient parametrisation for this perturbation we may calculate the metric of the noncommutative 2-instanton system. Due to the induced complexity of solutions, it is not an easy task to find a description of the full, 16-dimensional moduli space. However, we may make use of some global symmetries of the system to consider a geodesic submanifold of this space. With this reduction, explicit results may be obtained. We consider the results and, as expected, we find that the

manifold generated is in fact smooth and singularity-free, unlike in the commutative case. This is indicative of the results gained in [35] where the single instanton moduli space was seen to correspond to the Eguchi-Hanson metric, which contains no orbifold singularities.

In Chapter 4, we use the results gained to consider dynamics, and in particular scattering, of the two noncommutative instantons. The presence of a non-zero Fayet-Iliopoulos term in the overarching field theory has profound consequences for the results gained: most strikingly, right-angled scattering (a distinguishing feature of most soliton dynamics) is no longer the natural behaviour, even for a vanishing Higgs field. In fact, a wide range of behaviours are present, of which scattering at $\pi/2$ is only one possible outcome. We use a variety of consistency checks to ensure the validity of these results, and via identification with other soliton solutions find agreement with the expected behaviours. Finally, we consider the association between Yang-Mills instantons and the lower-dimensional non-Abelian vortices, where the presence of noncommutativity represents a non-trivial gauge coupling to the U(1) gauge field in a U(N) vortex theory.

In Chapter 5, we extend the analysis of the previous chapter to dyonic instantons. The results obtained herein suggest that one may consider the noncommutativity to function as an ersatz effective potential on the moduli space of commutative instantons. The dynamics of two commutative instantons admits orbiting solutions, where the attractive force of the potential is balanced by the natural repulsive force of the instantons. In the noncommutative picture, we find an analogous result, with some interesting modifications: previously stable orbiting configurations can become unstable in finite time, demonstrating scattering not seen in the commutative case, with varying noncommutative strength. We then briefly consider the possible vortex behaviour that would arise from a dimensional reduction of a noncommutative

dyonic instanton configuration.

In Chapter 6, we present some partial results encompassing many aspects of the instanton theory. We suggest a means by which the maximally non-Abelian vortex behaviour could be reproduced via a dimensional reduction of dyonic non-commutative instantons. We then examine some adjustments that will occur in the index calculation of BPS states in the corresponding $\mathcal{N} = 4$ SQM. Using an index scaling argument, one can see that the zeroes of the potential correspond to the bound states of the D0 quantum mechanical system and include a class of zeroes not present in the single instanton cases previously studied [24]. Finally, we suggest a feasible method for generating solutions to the U(3) instanton ADHM data via recourse to the well-established U(2) results. Finally, in Chapter 7, we summarise our results and outline future directions.

2

The construction of instantons

In this section, we review instantons in $(4+1)$-dimensional Yang-Mills theory. This will encompass both 'free', 1/2-BPS, instantons in pure Yang-Mills theory and their dyonic 1/4-BPS counterparts. We outline the ADHM construction for such a field theory, which reduces the problem of finding self-dual solutions to the Yang-Mills field equations to a set of algebraic equations on allowed configurations. The free parameters in the solved ADHM data can be seen to correspond to a set of collective coordinates on a moduli space of instanton solutions. We proceed to consider this moduli space and observe that the geodesic approximation provides a means of analysis of slow-moving instantons. We finally consider the key differences between the commutative and noncommutative formulations.

2.1   Instantons in 5d Yang-Mills

We first review the underlying string theoretical interpretation of Yang-Mills theory. The low-energy dynamics of a stack of N coincident D4-branes may be identified with an $\mathrm{SU}(N)$ super-Yang-Mills field theory [7]. Such a system preserves one

half of the supercharges, and is thus described by an $\mathcal{N} = 2$ SUSY theory in five dimensions. Open strings stretched between the D4-branes give rise to a U(N) world-volume gauge symmetry, with associated gauge field A_μ, $\mu = 0, 1, \ldots, 4$. The theory also contains five adjoint scalars X^I, $I = 5, 6, \ldots, 9$, describing the branes' relative positions in the transverse directions. By factoring out the centre of mass from the theory we obtain 5-dimensional super-Yang-Mills.

For the purposes of considering instantons, we henceforth consider only the bosonic sector of the theory, with (for convenience) gauge coupling set to one. This analysis works, however, for the more general fermionic set-up. The associated action is

$$S = -\int \mathrm{d}^5 x \, \mathrm{Tr} \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu X^I D^\mu X^I + \frac{1}{4} [X^I, X^J]^2 \right), \qquad (2.1)$$

where the covariant derivative is given in standard form

$$D_\mu X^I = \partial_\mu X^I - i[A_\mu, X^I]$$

and the field strength is

$$F_{\mu\nu} = \partial_{[\mu} A_{\nu]} - i[A_\mu, A_\nu].$$

While the construction of instantons is valid for all choices of gauge group U(N), the calculational complexity vastly increases with larger gauge groups. We consider only a stack of two D4-branes, so that the gauge group is U(2). As well as the world-volume and transverse indices outlined above, we will also use the indices i, j to denote the purely spatial directions of the 5d theory.

We may choose to set just one of the transverse scalar fields, $X^5 \equiv \phi$, to be non-zero. The induced Higgs VEV, $\langle \phi \rangle$, will then correspond to the separation of

the branes in the X^5 direction. This is equivalent to any other choice of transverse brane separation up to some SO(5) rotation of the X^I, and in choosing a particular direction we break the R-symmetry of the full Yang-Mills theory. However, this does not affect the validity of the analysis (and, in fact, will be crucial in certain identifications with lower-dimensional solitonic theories). The energy of the system is

$$E = \int \mathrm{d}^4 x \mathrm{Tr}\left(\frac{1}{2} F_{i0} F_{i0} + \frac{1}{4} F_{ij} F_{ij} + \frac{1}{2} D_0 \phi D_0 \phi + \frac{1}{2} D_i \phi D_i \phi \right). \tag{2.2}$$

In order to obtain solitonic solutions, we seek to find minimum energy solutions to the bosonic Yang-Mills theory. The requirement for finite energy solutions is tantamount to requiring that the gauge field strength F_{ij} vanishes at spatial infinity. This can be guaranteed by demanding that the gauge field becomes pure gauge: that is

$$A_i = -\partial_i g^\infty (g^\infty)^{-1}$$

as $|x| \to \infty$. The map $g^\infty : S^3_\infty \to \mathrm{SU}(2)$ defines a winding number from the sphere at infinity to the gauge group, the degree of which is given by the second Chern number $c_2 \in \mathbb{Z}$. We define, for identification, the following quantities:

$$\begin{aligned} k &\equiv -\frac{1}{8\pi^2} \int \mathrm{d}^4 x \epsilon_{ijkl} \mathrm{Tr}(F_{ij} F_{kl}), \\ \mathcal{Q}_E &\equiv \int \mathrm{d}^4 x \mathrm{Tr}(D_i \phi F_{i0}). \end{aligned} \tag{2.3}$$

These are to be interpreted as the topological charge and electric charge, respectively, of the theory. The topological charge is equivalent to the winding number, and so for a given $k \in \mathbb{Z}$ we may consider the family of all instantons with winding k. The electric charge arises as the Noether charge associated to the maximally unbroken U(1) of the SU(2) global gauge symmetry of the Yang-Mills Lagrangian. Such

solutions may smoothly deform into one another due to the presence of the map g, but must remain in this k-sector of the theory. Hence, in the instanton picture, each successive value of k decouples from all others and this will allow us to consider evolution and scattering of k-instanton solutions for a particular k. We also note that a consideration of negative k is equivalent to that of positive k from the point of view of the field theory energy: the instanton solutions for a given $k > 0$ also provide the $k < 0$, which we denote as anti-instantons. The correspondence between the two types will be solidified shortly.

Employing the standard Bogomolny argument [10] to bound the energy, we find

$$
\begin{aligned}
E = \int \mathrm{d}^4 x \mathrm{Tr} \Big(&\frac{1}{4}(F_{ij} \pm \star F_{ij})^2 \mp \frac{1}{2} F_{ij} \star F_{ij} \\
&+ \frac{1}{2}(F_{i0} \pm D_i \phi)^2 \mp F_{i0} D_i \phi + \frac{1}{2}(D_0 \phi)^2 \Big),
\end{aligned}
\tag{2.4}
$$

where $\star F_{ij} \equiv \frac{1}{2} \epsilon_{ijkl} F_{ijkl}$ is the Hodge dual of the field strength. The choices of sign in this expression are correlated within each line, but independent between the two lines. Then the energy is bounded by

$$
E \geq 2\pi^2 |k| + |\mathcal{Q}_E|,
$$

and this Bogomolny bound is saturated when

$$
\begin{aligned}
F_{ij} &= \pm \star F_{ij}, \\
F_{i0} &= \pm D_i \phi, \\
D_0 \phi &= 0,
\end{aligned}
\tag{2.5}
$$

where, again, the choices of sign are independent. These are the BPS equations

for dyonic U(2) instantons. The first equation requires that the field strength be (anti-)self-dual, and the second and third are satisfied when the fields are static and $A_0 = \pm\phi$. Since each k-sector decouples from all others, and the anti-self-dual solution (corresponding to anti-instanton) data is in some sense contained within the self-dual data, we need only consider the self-dual case, solutions of which we henceforth refer to as instantons. If we consider a non-zero scalar field ϕ, it will still be necessary to satisfy the background field equations for the Yang-Mills theory; namely

$$D^2\phi = 0, \tag{2.6}$$

and this requirement will be important in the consideration of dyonic instantons.

The Bogomolny equations, while simpler than those of the full Yang-Mills theory, do not trivially admit analytic solutions. Fortunately, the ADHM construction [17] relates these differential constraints on the gauge field into purely algebraic ones. This will allow us to explicitly construct classes of self-dual instantons whose induced gauge field automatically satisfies the Bogomolny equations (2.5). Before we can apply the ADHM construction, however, it is necessary to consider the noncommutative analogue.

2.2   Noncommutative $\mathbb{R}^4$

As described above, the study of instantons allows us to find non-trivial solutions to the Yang-Mills field equations in (static) Euclidean $\mathbb{R}^4$ that would otherwise be occluded. In the previous section, the spatial $\mathbb{R}^4$ (consisting of x_i, $i = 1, \ldots, 4$) admits trivial commutation relations between each direction. For reasons that shall become apparent, we may introduce an underlying noncommutative geometry to

the theory by making some, or all, of these commutation relations non-zero. This is equivalent to choosing a preferred complex structure on the space. We stipulate the following generic commutation relations:

$$[x_i,\, x_j] = i\theta_{ij} \tag{2.7}$$

where θ_{ij} is a real, antisymmetric matrix. Without loss of generality, we may break the underlying SO(4) symmetry of the space of express θ in a simpler form [35]

$$(\theta_{ij}) = \begin{pmatrix} 0 & -\theta_1 & 0 & 0 \\ \theta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\theta_2 \\ 0 & 0 & \theta_2 & 0 \end{pmatrix} \tag{2.8}$$

for θ_1 and θ_2 real. Classically, if both of the θ_i are non-zero then we may scale the two coordinate directions corresponding to, say, θ_1 such that the noncommutativity parameters have equal magnitude. The condition that θ_{ij} is self- or anti-self-dual is equivalent to requiring that $\theta_1 - \theta_2 = 0$ or $\theta_1 + \theta_2 = 0$, respectively. It transpires that the choice of duality in the background $\mathbb{R}^4$ will affect our ability to obtain consistent self- or anti-self-dual solutions in the Yang-Mills theory; in particular, to obtain self-dual YM solutions one must consider an anti-self-dual background, and vice versa [36]. We will return to this point in due course. As befits our earlier choice, we consider only the anti-self-dual background case, and so define

$$\theta_1 = \zeta = -\theta_2$$

for $\zeta > 0$.

From the perspective of the Yang-Mills field theory, the introduction of a non-commutative background induces a deformation in the notion of multiplication: one now must consider functions multiplied using the Moyal-$\star$ product. For functions $f(x)$ and $g(x)$ valued in $\mathbb{R}^4_{NC}$, we have [39]

$$f \star g(x) = \exp\left(\frac{i}{2}\theta_{ij}\partial_i\partial_j'\right)f(x)g(x')\bigg|_{x'=x}. \tag{2.9}$$

This gives an expansion in powers of θ:

$$f \star g(x) = f(x)g(x) + \frac{i}{2}\theta^{ij}\partial_i f(x)\partial_j g(x) + \mathcal{O}(\theta^2).$$

In this noncommutative framework, the gauge field A_i transforms as

$$A_i \mapsto g^{-1} \star A_i \star g + g^{-1} \star \partial_i g,$$

where g takes values in U(N). The field strength is correspondingly adjusted as

$$F_{ij} = \partial_{[i}A_{j]} - i[A_i,\, A_j]_\star,$$

where we denote the commutator with $\star$ to emphasise the non-standard multiplication therein.

From the point of view of finding solutions to the Bogomolny equations (2.5), working in the noncommutative framework allows for a greater range of instanton configurations, circumventing Derrick's theorem [40] due to the additional length scale $[\zeta] = \text{length}^2$. However, with the above formalism, one would have to calculate such solutions to all orders in ζ. With the exception of in the simplest cases, such an approach is severely non-trivial and prevents any meaningful analysis. We may

proceed due to an isomorphism between the algebra of functions with the $\star$-product and the algebra of operators on some Hilbert space, as demonstrated in [41]. This identification will allow us to utilise the ADHM procedure in the noncommutative framework.

2.2.1 Noncommutativity in General Spacetimes

We briefly set aside discussion of the ADHM construction for instantons to describe the correspondence between function space under the Moyal $\star$-product and operators on a Hilbert space. To illustrate the procedure we consider, for simplicity, a noncommutative theory in $\mathbb{R}^2$, giving a single non-trivial spatial commutation relation $[x_1, x_2] = i\theta_{12}$. Then for a generic function on this space, we have the associated Fourier transform

$$\tilde{f}(\alpha_1, \alpha_2) = \int \mathrm{d}^2x \, e^{i(\alpha_1 x_1 + \alpha_2 x_2)} f(x_1, x_2),$$

where α_i are the conjugate momenta to x_i. We may then define an operator $\hat{\mathcal{O}}_f(\hat{x}_1, \hat{x}_2)$ on the Hilbert space of functions of the analogous quantum system:

$$\hat{\mathcal{O}}_f(\hat{x}_1, \hat{x}_2) = \frac{1}{(2\pi)^2} \int \mathrm{d}^2\alpha \, U(\alpha_1, \alpha_2) \tilde{f}(\alpha_1, \alpha_2) \tag{2.10}$$

where $U(\alpha_1, \alpha_2) = \exp(-i(\alpha_1 \hat{x}_1 + \alpha_2 \hat{x}_2))$. Using Baker-Campbell-Hausdorff, we see that

$$U(\alpha_1, \alpha_2) U(\beta_1, \beta_2) = U(\alpha_1 \beta_1 + \alpha_2 \beta_2) e^{-\frac{i}{2}\theta_{12}(\alpha_1 \beta_2 - \alpha_2 \beta_1)}.$$

This map defines the correspondence between function multiplication on a noncommutative space and quantum mechanical commutation relations. We now seek an

expression for $\hat{\mathcal{O}}_f \hat{\mathcal{O}}_g$:

$$
\begin{aligned}
\hat{\mathcal{O}}_f \hat{\mathcal{O}}_g =& \frac{1}{(2\pi)^4} \int \mathrm{d}^2\alpha \, \mathrm{d}^2\beta \, U(\alpha_1, \, \alpha_2) U(\beta_1, \, \beta_2) \tilde{f}(\alpha_1, \, \alpha_2) \tilde{g}(\beta_1, \, \beta_2) \\
=& \frac{1}{(2\pi)^4} \int \mathrm{d}^2\alpha \, \mathrm{d}^2\beta \, U(\alpha_1 + \beta_1, \, \alpha_2 + \beta_2) e^{-\frac{i}{2}\theta_{12}(\alpha_1\beta_2 - \alpha_2\beta_1)} \tilde{f}(\alpha_1, \, \alpha_2) \tilde{g}(\beta_1, \, \beta_2).
\end{aligned}
$$

Under a suitable change of variables

$$
\alpha_i \to \frac{1}{2}\gamma_i + \delta_i, \quad \beta_i \to \frac{1}{2}\gamma_i - \delta_i,
$$

the integration measure is unchanged and the integral becomes

$$
\begin{aligned}
\hat{\mathcal{O}}_f \hat{\mathcal{O}}_g =& \frac{1}{(2\pi)^4} \int \mathrm{d}^2\gamma \, \mathrm{d}^2\delta \, U(\gamma_1, \, \gamma_2) e^{\frac{i}{2}\theta_{12}(\gamma_1\delta_2 - \gamma_2\delta_1)} \\
& \cdot \tilde{f}\left(\frac{\gamma_1}{2} + \delta_1, \, \frac{\gamma_2}{2} + \delta_2\right) \tilde{g}\left(\frac{\gamma_1}{2} - \delta_1, \, \frac{\gamma_2}{2} - \delta_2\right).
\end{aligned}
$$

At this point, we note that the Fourier transform of the Moyal $\star$-product (2.9) in two spatial dimensions can be put into the form

$$
\widetilde{f \star g}(\gamma_1, \, \gamma_2) = \int \mathrm{d}^2\delta \, e^{\frac{i}{2}\theta_{12}(\gamma_1\delta_2 - \gamma_2\delta_1)} \tilde{f}\left(\frac{\gamma_1}{2} + \delta_1, \, \frac{\gamma_2}{2} + \delta_2\right) \tilde{g}\left(\frac{\gamma_1}{2} - \delta_1, \, \frac{\gamma_2}{2} - \delta_2\right),
$$

and so

$$
\begin{aligned}
\hat{\mathcal{O}}_f \hat{\mathcal{O}}_g =& \frac{1}{(2\pi)^2} \int \mathrm{d}^2\gamma \, U(\gamma_1, \, \gamma_2) \widetilde{f \star g}(\gamma_1, \, \gamma_2) \\
=& \hat{\mathcal{O}}_{f \star g}.
\end{aligned}
$$

This identification will hold provided the original $\star$-product, $f \star g$, has a well-defined series expansion. If the expansion of the exponent in (2.9) is not convergent, then

the above considerations break down and the associated quantum operators on the Hilbert space will not necessarily be valid.

To demonstrate the subtlety of the procedure, we turn to the simplest 'alternate' parametrisation of $\mathbb{R}^2$; namely, polar $\mathbb{R}^2 = \text{span}\{r, \phi\}$. If we begin with the functions $f = r$, $g = \phi$ and attempt to promote them to quantum operators using the method above, then by series expansion in ζ we see that

$$[r, \phi]_\star \equiv r \star \phi - \phi \star r = -\frac{i\zeta}{r} \sum_{n=0}^{\infty} \frac{(4n)!}{(2n+1)} \left(\frac{\zeta}{4r^2}\right)^{2n}, \tag{2.11}$$

which is clearly not convergent. This starting point, however, was used in [42] in order to examine the BTZ black hole in polar coordinates, and the relation obtained was

$$[\hat{r}, \hat{\phi}] = -i\zeta \hat{r}^{-1},$$

which has a worrisome singularity at the origin. In order to have a consistent description of noncommutative polar $\mathbb{R}^2$, one can instead promote the function space (r^2, ϕ) to operators. This gives a terminating series expansion of $r \star \phi$, yielding the well-defined commutation relation $[\hat{r}^2, \hat{\phi}] = 2i\zeta$. Hence any attempt to use polar noncommutativity has no valid quantum operator $\hat{r}$. This simple example outlines the importance of rigorously verifying the validity of the starting conditions, and could have utility in deriving other, more complicated, noncommutative spaces. Given the interest in using noncommutative space to generate 'quantum' gravity theories via the Seiberg-Witten map [43], it would be a key consideration were one to take, say, Schwarzchild or AdS space as the starting point for a noncommutative theory.

These caveats aside, we may derive spatial commutation relations of functions

on $\mathbb{R}^2_{\text{NC}}$ as operator relations on a Hilbert space of operators, and vice versa. In the following work, we simply use the standard 'rectangular' $\mathbb{R}^2$ in each factor of $\mathbb{R}^4 = \mathbb{R}^2_{NC} \times \mathbb{R}^2_{NC}$ and therefore regularity is guaranteed. Hence, this correspondence provides a consistent definition of the ADHM operators in noncommutative scenarios.

Interpreting the spatial dimensions as in $\mathbb{R}^4$ or $\mathbb{C}^2$, then, the ADHM procedure can be seen to follow in precisely the same manner as the commutative analogue but for the fact that the underlying gauge group of the gauge field is U(2), rather than SU(2) (in fact, the term $A_i^4 1_2$ can be considered in the commutative case, but decouples from the theory and therefore has no impact on the analysis). This stems from the fact that the 'simpler' gauge group SU(2) is not closed under the Moyal $\star$-product multiplication [44]. Explicitly, we have

$$A_i = A_i^a \frac{\sigma^a}{2} + A_i^4 \frac{1_2}{2},$$

where σ^a provide the normal Pauli matrix representation of SU(2). This isomorphism validates the use of the ADHM toolbox, to which we now turn, in a noncommutative framework.

As a final point, note that one need not necessarily consider a 'full' noncommutative $\mathbb{R}^4$. The canonical form (2.8) can effectively be seen to split into two non-interacting $\mathbb{R}^2_{NC}$ pieces. In considerations of the reduction from instantons to monopoles or vortices, one must consider a compactification of one or two (respectively) of the spatial directions, usually along with a requirement of circle invariance. It is, therefore, possible to consider a semi-noncommutative space whose commutation relations comprise only $[x_1, x_2] = i\theta_1$ and compactify in the $x_3 - x_4$ directions. This is anticipated to yield other, lower dimensional, noncommutative

solitons. Conversely, one may compactify in the noncommutative directions. When considering the monopole reduction, it appears that the solitonic solutions obtained become extended objects in the resultant D-brane geometry [24]. One finds that under a suitable compactification, a form of S-duality exists in 5d which allows for an identification between the (instanton) Kaluza-Klein tower and a tower of monopoles. This was demonstrated in [45] via identification with the standard BPS SU(2) monopoles. However, it is not clear how to make an appropriate identification in a general instantonic configuration, and indeed what the corresponding duality is. Such considerations would be of great interest in generating consistent monopole configurations from noncommutative instanton solutions.

2.3 The ADHM construction

The ADHM construction allows for a class of algebraic constraints to be explicitly formulated for a given instanton number k and gauge group U(N) [17]. Formally, the ADHM data Δ is a $(2k + N) \times 2k$ complex-valued matrix restricted by unitarity and symmetry considerations, whose free parameters form a moduli space of allowed U(N) k-instanton configurations. In practice, and given our restriction to U(2), we may use the fact that in even gauge groups the ADHM data may be represented in block quaternion form [14]. We may use a basis of quaternions $q = q_i e_i$ where

$$e_a = i\sigma_a, \ a = 1, 2, 3,$$

$$e_4 = 1_2,$$

and σ_a are the generators of SU(2). Of course, the resulting ADHM constraints are equivalent were we to parametrise Δ with entries in $\mathbb{C}$.

To construct a charge k U(2) instanton solution, then, we consider a $(k+1) \times k$ quaternion-valued matrix Δ which must satisfy

$$\Delta^\dagger \Delta = 1_2 \otimes f^{-1}(x), \tag{2.12}$$

where $f^{-1}(x)$ is a Hermitian, invertible $k \times k$ matrix. Δ may be put into so-called 'canonical form' [46]:

$$\Delta(x) = a - bx$$

for $x \equiv x_i e_i$ being the spatial directions of the underlying $\mathbb{R}^4$ and a, b being x-independent quaternion-valued block matrices

$$a = \begin{pmatrix} L \\ M \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1_{2k} \end{pmatrix}. \tag{2.13}$$

In these conventions, L is a quaternionic row vector and M a $k \times k$ quaternion-valued matrix. The ADHM constraints (2.12) then become

$$L^\dagger L + M^\dagger M + |x|^2 1_{2k} - (M^\dagger bx + \bar{x} b^\dagger M) = 1_2 \otimes f^{-1}(x).$$

The ADHM constraint first requires us to demand the reality of the terms linear in x. Using the quaternion block form of M, this implies that

$$-\big((M_i)^T \bar{e}_i x + M_i \bar{x} e_i\big),$$

is real, and hence $M = M^T$; that is, M is symmetric. This, in turn, forces the reality of $f^{-1}(x)$. Then the constraints reduce, in the commutative case, to a simple

x-independent constraint

$$a^\dagger a = 1_2 \otimes \mu \tag{2.14}$$

for μ real, x-independent and invertible. Note, for future reference, that this result only holds in the case where $|x|^2 = \bar{x}x \propto 1_2$. In the noncommutative case this will be non-trivially modified.

Having found solutions to (2.14), we then seek to find a set of normalised quaternionic vectors in the null space of Δ [38],

$$\Delta^\dagger U = 0, \; U^\dagger U = 1_2. \tag{2.15}$$

The gauge field may then be constructed from U via

$$A_i = iU^\dagger \partial_i U. \tag{2.16}$$

With this definition of A_i, the associated field strength will naturally satisfy the self-duality condition in (2.5). It is instructive to demonstrate this fact explicitly, as the calculation bears many of the hallmarks of any calculation involving Δ and U. We make use of the following identities:

$$\begin{aligned}
U^\dagger \partial_i U = -(\partial_i U^\dagger)U, \\
(\partial_i \Delta^\dagger)U = -\Delta^\dagger \partial_i U,
\end{aligned} \tag{2.17}$$

which may be derived from (2.15) straightforwardly, as well as the completeness relation

$$UU^\dagger + \Delta f \Delta^\dagger = 1_4 \tag{2.18}$$

and the derivative of Δ

$$\partial_i \Delta = -b\partial_i x = -b\sigma_i. \tag{2.19}$$

Then, expanding F:

$$F_{ij} = \partial_{[i} A_{j]} - i[A_i, A_j]$$
$$= i(\partial_{[i} U^\dagger \partial_{j]} U) + i(U^\dagger \partial_{[i} U)(U^\dagger \partial_{j]} U).$$

Application of (2.17) to the second term yields

$$F_{ij} = i(\partial_{[i} U^\dagger \partial_{j]} - \partial_{[i} U^\dagger U U^\dagger \partial_{j]} U),$$

and replacing the $UU^\dagger$ expressions using the completeness relation (2.18) and moving the derivatives onto Δ using (2.17) gives

$$F_{ij} = i(\partial_{[i} U^\dagger \partial_{j]} U - \partial_{[i} U^\dagger (1_4 - \Delta f \Delta^\dagger) \partial_{j]} U)$$
$$= iU^\dagger \partial_{[i} \Delta f \partial_{j]} \Delta^\dagger U.$$

All that remains is to differentiate Δ using (2.19) and note that $f \propto 1_2$ commutes with the σ_i to obtain

$$F_{ij} = iU^\dagger b\sigma_{[i} f \bar\sigma_{j]} b^\dagger U$$
$$= iU^\dagger b f (\sigma_i \bar\sigma_j - \sigma_j \bar\sigma_i) b^\dagger U.$$

Then self-duality follows from the self-duality of $\sigma_i \bar\sigma_j - \sigma_j \bar\sigma_i$, as can be readily established using the properties of the Pauli matrices.

It will be crucial to note that the ADHM data Δ is defined only up to some

gauge transformation of the form [6]

$$U \to QU,$$

for Q suitably defined. There exist, therefore, equivalent instanton solutions for any given explicit solution of the ADHM data, and such redundancies must be removed before considering the moduli space. Explicitly, we may seek transformations of the form

$$\Delta \to Q\Delta R^{-1} = \begin{pmatrix} q & 0 \\ 0 & R \end{pmatrix} \Delta R^{-1}, \tag{2.20}$$

where R mixes the quaternion entries in M and $q \in \mathrm{U}(2)$. Due to this redundancy, while initially the ADHM data for k $\mathrm{U}(2)$ instantons contains $2k(k+1)$ free quaternionic parameters, the combination of the ADHM constraints and the redundancies remove $8k^2$ parameters. This leaves $8k$ free parameters, in agreement with the general $4kN$ result for N charge k instantons.

The extension to noncommutativity is relatively straightforward. The key change is that the expression $|x|^2$ in the ADHM constraints is no longer proportional to 1_2. If we write the quaternion representation of the spatial components in complex matrix form:

$$x = x_i q_i = \begin{pmatrix} x_4 + ix_3 & x_2 + ix_1 \\ -x_2 + ix_1 & x_4 - ix_3 \end{pmatrix} = \begin{pmatrix} z_2 & z_1 \\ -\bar{z}_1 & \bar{z}_2 \end{pmatrix}$$

then by expanding and using the commutation relations (2.7), it can be seen that

$$
\begin{aligned}
|x|^2 = \bar{x}x &= \begin{pmatrix} \bar{z}_2 & -z_1 \\ \bar{z}_1 & z_2 \end{pmatrix} \begin{pmatrix} z_2 & z_1 \\ -\bar{z}_1 & \bar{z}_2 \end{pmatrix} \\
&= \begin{pmatrix} \bar{z}_2 z_2 + z_1 \bar{z}_1 & 0 \\ 0 & \bar{z}_1 z_1 + z_2 \bar{z}_2 \end{pmatrix} \\
&= (\bar{z}_1 z_1 + \bar{z}_2 z_2)1_2 + \begin{pmatrix} -2\zeta & 0 \\ 0 & 2\zeta \end{pmatrix}.
\end{aligned}
\tag{2.21}
$$

This additional piece not proportional to 1_2 must be absorbed into the solution of the ADHM data. This complication notwithstanding, it is still possible to solve the relevant constraints. Another subtlety of the noncommutative instanton construction is that for given choices of duality, the completeness relation (2.18) may fail to hold [36]. Viewing the complex coordinates z_i and $\bar{z}_i$ as creation and annihilation operators, we may generate a Fock space, $\mathcal{H}$, of a pair of simple harmonic oscillators. A generic state in $\mathcal{H}$ may be written as $|n_1, n_2\rangle$, such that $z_i|n_1, n_2\rangle = \sqrt{\zeta}\sqrt{\delta_{ij}n_j + 1}|n_1 + \delta_{i1}, n_2 + \delta_{i2}\rangle$. We write $P \equiv \Delta f \Delta^\dagger$ as the Hermitian projector on $\mathcal{H}$, which is a map from the $(N + 2k)$ dimensional ADHM data to a subset thereof via $P : \mathcal{H}^{N+2k} \to P\mathcal{H}^{N+2k} \subset \mathcal{H}^{N+2k}$. Since P is clearly Hermitian, all eigenvalues of P are either zero or one. Denoting the corresponding zero and one eigenstates by $|\Psi^p\rangle$ and $|\Phi^r\rangle$, we have

$$
P|\Psi^p\rangle = 0, \ |\Psi^p\rangle \in \mathcal{H}^{N+2k}, \ \langle\Psi^p|\Psi^q\rangle = \delta^{pq},
$$

$$
P|\Phi^r\rangle = |\Phi^r\rangle, \ |\Phi^r\rangle \in \mathcal{H}^{N+2k}, \ \langle\Phi^r|\Phi^s\rangle = \delta^{rs}.
$$

Due to the completeness of eigenstates, we may write a generic state in the Hilbert

space as

$$1 = \sum_p |\Psi^p\rangle\langle\Psi^p| + \sum_r |\Phi^r\rangle\langle\Phi^r|$$
$$= \sum_p |\Psi^p\rangle\langle\Psi^p| + \Delta f \Delta^\dagger,$$

where we have used the fact that all non-zero eigenvalues of P are equal to one. The completeness relation will only hold if we may write

$$\{|\Psi^p\rangle\} = \{U^{pu}|s_u\rangle\} \tag{2.22}$$

for $|s_u\rangle$ arbitrary states in $\mathcal{H}$. If this is true, then it follows naturally that $\sum |\Psi^p\rangle\langle\Psi^p| = UU^\dagger$, and so the completeness relation is guaranteed.

The above considerations follow automatically in the commutative case, as may be readily verified. However, in the noncommutative case, while a non-zero state of the form of (2.22) will be a normalised zero-mode of P, it is no longer true that all zero-modes may be written in this form. If there are zero-modes that cannot be expressed in this way, then the completeness relation is not satisfied and the (A)SD construction of instantons via the ADHM data breaks down. This subtlety means that if we wish to obtain self-dual instantons, we must consider an anti-self-dual background, and vice versa. If one starts with a self-dual background then, while solutions to the self-dual ADHM constraints may be found, one cannot obtain a normalisable U that satisfies the completeness relation. Conversely, if one constructs a normalisable U, the self-dual ADHM constraints will not necessarily admit non-trivial solutions.

2.4 The pure instanton moduli space

We now turn our attention to the moduli space acquired from a solution of the ADHM data. Initially, we focus our attention on instanton solutions in the absence of a potential term (that is, with $\langle \phi \rangle = 0$); we shall visit the construction of dyonic instantons shortly. The remainder of this section holds for both commutative and noncommutative instantons, up to suitable notational definitions (the discussion of which we postpone until Chapter 3).

The moduli space, $\mathcal{M}_k$, of instantons describes the space of all distinct instantons solutions of instanton number k. Due to topological considerations, the full moduli space $\mathcal{M}$ is expected to comprise disconnected moduli spaces for every possible instanton number, i.e., $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \ldots$, so that it is sensible to consider only one particular instanton charge and be confident in the validity of results thus gained. By 'distinct', we emphasise that the gauge transformation in (2.20) does not necessarily generate new solutions to any given set of ADHM data, and it will be necessary to quotient out such redundant transformations before considering geodesics on the moduli space. We shall denote the quotiented moduli space of k-instantons by $\widehat{\mathcal{M}}_k$; for reasons of convention, we shall maintain the global gauge transformations corresponding to the overall U(2) gauge symmetry. The coordinates in $\widehat{\mathcal{M}}_k$ are furnished by the free variables in the ADHM data and provide an $8k$-dimensional parameter space of all possible unique instanton solutions of charge k. Varying the parameters of Δ is equivalent to smoothly varying the position in $\widehat{\mathcal{M}}_k$, and under certain conditions one can regard this as representing geodesic motion on the moduli space. We may obtain an induced metric for the moduli space using the ADHM data as follows. The topic (for the commutative case) is addressed in more detail in [38], with only the most salient points addressed here.

Prior to quotienting, the moduli space admits a natural inner product:

$$g(\delta_r A_i \delta'_r A_i) = \int d^4 x \text{Tr}(\delta_r A_i \delta'_r A_i). \tag{2.23}$$

Here the index r enumerates the $8k$ parameters in Δ or, equivalently, the coordinate directions on $\widehat{\mathcal{M}}_k$. The δ_r, then, represent tangent vectors along coordinate directions in the moduli space. In order for the δ_r to truly correspond to coordinate motion on the moduli space, it will be necessary to remove any tangent vectors in a direction orthogonal to the moduli space. This can be achieved by demanding that all tangent vectors to the space are orthogonal to local (redundant) gauge transformations:

$$D_i \delta_r A_i = 0.$$

Variations satisfying the Bogomolny equations (2.5) (which, in absence of a Higgs field ϕ simply reduce to requiring self-duality) and this gauge-fixing condition will be referred to as zero modes. Labelling the free ADHM parameters (moduli space coordinates) as z^r for $r = 1, 2, \ldots, 8k$, then the zero modes may be written as

$$\delta_r A_i = \partial_r A_i - D_i \epsilon_r \tag{2.24}$$

and ϵ_r must be chosen such that

$$D_i(\delta_r A_i) = 0. \tag{2.25}$$

Once we have a class of ϵ_r for every tangent vector $\delta_r A_i$ of the moduli space, we may then consider the g arising from (2.23) as the metric on the moduli space.

In order to consider scattering, it is necessary to reintroduce some time depen-

dence into the system. These 'dynamic' solutions will not, generically, lie in $\widehat{\mathcal{M}}_k$ and hence will not describe instanton configurations, but for sufficiently small velocities will lie close to the instantonic minimum energy solutions and so represent a good approximation to the movement and interaction of instantons [33]. The consideration of small velocities also makes the calculation tractable, as it allows us to consider only the linearised version of the resulting field equations rather than the full dynamic theory. The self-dual field equations become

$$D_i(\delta_r A_j) - D_j(\delta_r A_i) = \epsilon_{ijkl} D_k(\delta_r A_l). \tag{2.26}$$

When introducing time-dependence, we only consider geodesic motion on the moduli space of ADHM data and hence allow the instantons to vary in time only in the moduli space parameters $z^r \to z^r(t)$, so that for a solution to the Yang-Mills equations we have $A_i(\mathbf{z}(t); \mathbf{x})$. Then we require

$$D_i F_{i0} = D_i(D_i A_0 - \dot{z}^r \partial_r A_i) = 0.$$

This may be solved by choosing $A_0 = \dot{z}^r \epsilon_r$ for ϵ_r restricted as in (2.25). Then

$$F_{i0} = -\dot{z}^r \delta_r A_i,$$

where the zero modes are defined as in (2.24). We obtain an effective action for motion on the moduli space

$$S = \frac{1}{2} \int \mathrm{d}^5 x \mathrm{Tr}(F_{i0} F_{i0}) = \frac{1}{2} \int \mathrm{d}t g_{rs} \dot{z}^r \dot{z}^s, \tag{2.27}$$

with associated metric

$$g_{rs} = \int \mathrm{d}^4 x \, \mathrm{Tr}(\delta_r A_i \delta_s A_i). \tag{2.28}$$

This linearised approximation will only hold when the magnitude of $\dot{z}^r$ is small; that is, where the velocity of the instantons is small. The linearisation of the field theory, under the small velocity stipulation, allows us to generate the metric determining instanton dynamics which, at every point on a given geodesic, approximate a 'snapshot' of a truly dynamical Yang-Mills solution.

2.5 Dyonic instantons

We now consider the effect of introducing a non-zero electric charge, $\mathcal{Q}_E$, on the metric of $\widehat{\mathcal{M}}_k$. While turning on the scalar ϕ will not change the structure of the metric, the electric charge will vary across the space. This is equivalent to considering an induced potential on the space.

As in Section 2.4, we introduce time-dependence into the collective coordinates z^r of $\widehat{\mathcal{M}}_k$. Denoting the scalar field as ϕ, this perturbs the Gauss' law constraint as

$$D_i F_{i0} + [D_0 \phi, \, \phi] = 0. \tag{2.29}$$

Note that this perturbed constraint is no longer satisfied by the ansatz $A_0 = \dot{z}^r \epsilon_r$. Therefore we perturb A_0 away from the free case via

$$A_0 = \phi + \dot{z}^r \epsilon_r$$

and obtain

$$F_{i0} = -(\dot{z}^r \delta_r A_i - D_i \phi). \tag{2.30}$$

Since ϕ must satisfy the background field equation (2.6), it is straightforward to see that the perturbation term vanishes in $D_i F_{i0}$ and we satisfy the vanishing of this quantity as in the free case. The commutator term in (2.29), however, is non-zero. To surmount this problem, we may first note that $D_i \phi$ ipso facto satisfies the conditions for a zero mode: it solves the linearised self-dual equations (2.26) and the background field equation (2.6) by construction. This allows us to express it in the basis of zero modes of $\widehat{\mathcal{M}}_k$,

$$D_i\phi = |q|K^r\delta_r A_i, \tag{2.31}$$

for some vector K^r. This additional term may be absorbed into F_{i0} via an appropriate coordinate transformation, so that

$$F_{i0} = -(\dot{z}^r - |q|K^r)\delta_r A_i \equiv -\dot{y}^r \delta_r A_i,$$
$$y^r = z^r - |q|K^r t.$$

With this transformation, the effective action is simply

$$S = \frac{1}{2}\int \mathrm{d}t\, g_{rs}\dot{y}^r\dot{y}^s - |q|^2 g_{rs}K^r K^s, \tag{2.32}$$

where we have neglected terms $\mathcal{O}(\dot{z}^2|q|^2)$. This action, then, forms a good approximation to the low-energy dynamics of dyonic instantons when both the velocity, $\dot{z}$, and the strength of the potential, $|q|$, are much less than unity. This second requirement has a reasonable interpretation: just as in the considerations of low velocity we expect the results to be good approximations of motion on $\widehat{\mathcal{M}}_k$ provided we deviate only slightly from the moduli space, so too must we ensure that our

potential is sufficiently 'shallow' compared to the energy gradient away from the moduli space that the resulting field configurations are only slightly higher energy than the minimum energy, instanton, configurations.

We now have all the information required to calculate the metric of dyonic commutative and noncommutative U(2) instantons for any given topological charge k. In principle, we may find and classify Killing vectors of the derived metric, identify conserved quantities and ascertain geodesic motion. In practice, however, the complexity and scale of the data makes it hard to proceed with such analysis without resorting to explicit parametrisations of the zero modes. Fortunately, we are able to calculate the metric without explicit recourse to the somewhat abstract zero mode procedure described above, via the ADHM construction.

2.6 A derivation of the moduli space

We may now consider generating the moduli space metric for instantons with a given gauge group U(N) and topological charge k. The general method described above is, unfortunately, intractable in all but the simplest cases. However, given a parametrisation of the ADHM data for an instanton configuration, we may algebraically derive the metric on the moduli space via a method developed by Osborn [46].

We first recall that the gauge field, A_i, is constructed from the ADHM data as

$$A_i = iU^\dagger \partial_i U, \quad \text{where} \quad \Delta^\dagger U = 0, \ U^\dagger U = 1.$$

It is possible to calculate the derivative of U with respect to a coordinate of the

moduli space, z^r, using the ADHM relations in Section 2.3:

$$\partial_r A_i = -iU^\dagger \partial_r \Delta f \bar{e}_i b^\dagger U + iU^\dagger b e_i f \partial_r \Delta^\dagger U + D_i(iU^\dagger \partial_r U). \tag{2.33}$$

It is clear from this expression that the final term is some gauge transformation of the data. However, it is not guaranteed at this stage that the remaining terms satisfy the conditions for a zero-mode. We thus seek to perform appropriate transformations on Δ so that all the gauge transformation components are collected into one term, and then the remainder can be viewed as a bona fide zero-mode on the moduli space. This is equivalent to projecting out the gauge data onto some gauge-fixed subspace of the moduli space. Recall that the ADHM data is invariant under a transformation

$$\Delta \mapsto Q\Delta R \quad U \mapsto QU.$$

We have a different Δ at each point in the moduli space, and may then consider the gauge transformation Q as being dependent on the collective coordinate position in $\widehat{\mathcal{M}}_k$; explicitly

$$\Delta(z^r(t)) \mapsto \exp(t\delta_r Q)\Delta \exp(t\delta_r R).$$

By construction, such a parametrisation leaves A_i invariant but we may now write its z^r derivative (2.33) as

$$\partial_r A_i = iU^\dagger C_r f \bar{e}_i b^\dagger U + iU^\dagger b e_i f C_r^\dagger U + D_i(iU^\dagger \partial_r U + iU^\dagger \delta_r QU) \tag{2.34}$$

where, for economy of notation, we have defined

$$C_r \equiv \partial_r \Delta + \delta_r Q\Delta + \Delta\delta_r R. \tag{2.35}$$

This freedom inherited from the redundancy of the ADHM data allows us to 'tune' $\delta_r Q$ and $\delta_r R$ until the only components of $\partial_r A_i$ that correspond to gauge transformations reside in the final term. To achieve this, we must apply the zero-mode conditions on the tangent vectors of first two terms of A_i:

$$\delta_r A_i = -iU^\dagger C_r f \bar{e}_i b^\dagger U + iU^\dagger b e_i f C_r^\dagger U. \tag{2.36}$$

We may solve (2.36) in terms of a constraint on C_r: the requirement for a zero-mode is that (see [46] or [38] for a proof) C_r must be independent of x and the combination $\Delta^\dagger C_r$ be symmetric. Expressing this in terms of the ADHM data, we require

$$a^\dagger C_r = (a^\dagger C_r)^\mathrm{T},$$
$$b^\dagger C_r = (b^\dagger C_r)^\mathrm{T}. \tag{2.37}$$

All that remains is to fix the forms of $\delta_r Q$ and $\delta_r R$ such that this condition is satisfied. Henceforth, we restrict ourselves to the case of $N = 2$ and $k = 2$, but this procedure applies to any choice of gauge group and topological charge.

Note that in canonical form, b has very simple structure:

$$b = \begin{pmatrix} 0 \\ 1_{2k} \end{pmatrix}.$$

Then the generic form of the transformations is restricted by the requirement that $QbR = b$ and $\delta b = 0$, allowing us to determine $\delta_r Q$ entirely in terms of $\delta_r R$. The resulting constraint on $\delta_r R$ is as follows:

$$a^\dagger \delta a - (a^\dagger \delta a)^\mathrm{T} = a^\dagger b \delta R b^\dagger a - b^\dagger a \delta R a^\dagger b + \mu^{-1} \delta R + \delta R \mu^{-1}, \tag{2.38}$$

where $\mu^{-1} \equiv a^{\dagger}a$. This is now a purely algebraic constraint on allowed zero-modes.

To proceed, we now need an explicit form for the inner product of two such zero modes. This problem is overcome by exploiting an identity due to Corrigan et. al. [47]: if we have zero modes in the form (2.36), then

$$\mathrm{Tr}(\delta_r A_i \delta_s A_i) = -\frac{1}{2}\partial^2 \mathrm{Tr}\big(C_r^{\dagger}(1+P)C_s f\big), \tag{2.39}$$

where P is the projection operator on the space of ADHM data, $P = UU^{\dagger} = 1_4 - \Delta f \Delta^{\dagger}$. A proof of this identity may be found in [48]. This identity removes the difficulty of integrating the zero-modes to find the inner product, turning the integral into a trivial evaluation of the fields at the boundary. Due to the normalisation of the ADHM nullspace vector U, this is straightforward to evaluate (even in the noncommutative case), and we obtain

$$g_{rs} = \int \mathrm{d}^4 x \, \mathrm{Tr}(\delta_r A_i \delta_s A_i)$$
$$= 2\pi^2 \mathrm{Tr}\big(C_r^{\dagger} P_{\infty} C_s + C_r^{\dagger} C_s\big)$$

where P_{∞} is the projector at spatial infinity. Expanding in terms of $\partial_r a$ and $\delta_r R$ provides the final result for the metric:

$$g_{rs} = 2\pi^2 \mathrm{Tr}\big(\partial_r a^{\dagger}(1+P_{\infty})\partial_s a - \big(a^{\dagger}\partial_r a - (a^{\dagger}\partial_r a)^{\mathrm{T}}\big)\delta_s R\big). \tag{2.40}$$

With this expression, one need only solve the ADHM data in order to obtain the metric of the moduli space.

2.7 The ADHM construction for dyonic instantons

In Section 2.5 we saw that it is possible to introduce a potential into the effective action for instantons via a consideration of zero-modes. As above, we may turn the zero-mode considerations into something entirely dependent on the ADHM data.

Recall that the effect of turning on a scalar, Φ, corresponding to some Higgs VEV gave consistent zero-modes on the moduli space of instantons provided the background field equation of Yang-Mills, $D^2\Phi = 0$, is satisfied. The method by which we may achieve this from the ADHM perspective is to introduce an ansatz for Φ [48]; namely,

$$\Phi = iU^\dagger \mathcal{A} U, \quad \mathcal{A} = \begin{pmatrix} q & 0 \\ 0 & P \end{pmatrix}, \tag{2.41}$$

where U is the null vector obtained in the ADHM construction, q is a pure quaternion and P is a $k \times k$ real antisymmetric matrix. The VEV of the field Φ is determined by iq.

Now we must consider the constraint that arises from the background field equation. Using the identities (2.17)-(2.19), and after a rafale of lengthy but uncomplicated manipulation (see the Appendix for details), we obtain

$$D^2\Phi = -4iU^\dagger \{bfb^\dagger, \mathcal{A}\}U + 4iU^\dagger bf \mathrm{Tr}_2(\Delta^\dagger \mathcal{A}\Delta) fb^\dagger U. \tag{2.42}$$

The trace in the second term is a quaternion trace: for a generic quaternion $q = q_i e_i$, we have $\mathrm{Tr}_2 q = 2\Re(q)$. Since $\mathcal{A}$ is in block diagonal form, the first term reduces to $-4iU^\dagger \{f, P\}U$. For the second term, we may use the form of Δ in (2.13), defining

$M' \equiv M - 1_k x$, to rewrite the trace as

$$\mathrm{Tr}_2(L^\dagger q L) + \tfrac{1}{2}\mathrm{Tr}_2\big([M'^\dagger,\, P]M' - M'^\dagger[M',\, P] + \{P,\, f^{-1}\} - \{P,\, L^\dagger L\}\big).$$

In the commutators, since x appears only linearly and multiplied by 1_k, the x-dependence vanishes. Collecting all of this together, (2.42) becomes

$$\begin{aligned}
D^2 \Phi = -4i\big(U^\dagger\{f,\, P - \tfrac{1}{2}\mathrm{Tr}_2(P)\}U \\
+ U^\dagger b f\big(\mathrm{Tr}_2(L^\dagger q L) - [M_i,\, [M_i,\, P]] - \{P,\, L_i^{\mathrm{T}} L_i\}\big)f b^\dagger U\big),
\end{aligned}$$

where the $i,\, j$ indices denote the quaternion indices, and not the matrix components. Finally, since P is real, $\mathrm{Tr}_2(P) = 2P$ so the first term vanishes. All that remains is the required constraint on P:

$$\mathrm{Tr}_2(L^\dagger q L) - [M_i,\, [M_i,\, P]] - \{P,\, L_i^{\mathrm{T}} L_i\} = 0. \tag{2.43}$$

In this manner, one may find a scalar field Φ by solving the constraint on P in terms of the ADHM data for a given instanton configuration.

There is, however, a different way to ascertain the potential on the moduli space, detailed in [24] for the single instanton case (we will study this in somewhat more detail in Chapter 5). Consider the VEV as given in the ansatz (2.41):

$$q = \mathrm{diag}(v_1,\, v_2,\, \ldots,\, v_N)$$

for a U(N) theory. Assuming that the ansatz is consistent, the potential can be

written in an alternative form to that above:

$$V \sim \mathrm{Tr}(p^\dagger |q|^2 p - p^\dagger q^\dagger p P),$$

where for convenience we have denoted by p the collection of 'quaternion' entries in Δ. Transforming to Calabi form, with $L \equiv \sum x_i$ where the x_i are the relative positions of the instantons, we find that

$$
\begin{aligned}
V &\sim \sum x_i v_i^2 - \frac{1}{L}\left(\sum x_i v_i\right)^2 \\
&= \frac{1}{L}\sum_{i<j} x_i x_j (v_i - v_j)^2 \\
&= C^{AB} 2(v_A - v_N) 2(v_B - v_N),
\end{aligned}
$$

where C^{AB} is the Calabi metric on the space. Here, we see that every Killing direction, φ_A, is identified with a factor $2(v_A - v_N)$. This is equivalent to the term gained by an F-string connecting two D4-branes. This allows us to express each Killing vector G as

$$G = \sum_A^{N-1} 2(v_A - v_N)\frac{\partial}{\partial \varphi}.$$

In our case, where we have only two D-branes, if we have a non-zero VEV (equivalently separated branes), the potential receives non-trivial contributions from the Killing vectors corresponding to global rotations of the data and, around the zeros of the potential, these are the maximal contributions. In practice, this allows us to consider the potential as arising from the overall gauge rotation of the data via

$$V = \frac{1}{2}g_{rs}G^r G^s = \frac{1}{2}g_{\Theta\Theta}v^2, \tag{2.44}$$

where g_{rs} is the instanton moduli space metric and Θ in our case will represent the sum of the gauge angles of the instantons. It can be seen that, using this formalism in the commutative SU(2) case, one recovers the same result as the above considerations. The two formulations each possess their own merits: in the calculation of the noncommutative instanton, it proves fruitful to use the Killing vector identification. In classifying the zeroes of the potential (a calculation necessary to the determination of the index of BPS spectra in the quantum theory), it is instead more useful to consider the general form provided by the constraint (2.43).

This ends the required preliminaries; we now turn our attention to finding an explicit solution for the ADHM data and metric of two noncommutative U(2) instantons. Such a solution can be obtained, and we shall be able to consider scattering of instantons in this space.

3

Noncommutative U(2) instantons

In this section, we turn our attention to finding explicit solutions to the noncommutative ADHM constraints for two U(2) instantons. This will allow us to generate the moduli space metric, consider scattering, and analyse the symmetries of the data. While to consider geodesics on the moduli space of the full data (comprising 16 free parameters, 4 of which are trivial centre of mass coordinates and will not be considered) is computationally expensive, we may use the symmetries inherent in the metric to consider geodesic submanifolds of the space $\widehat{\mathcal{M}}_2$. Finally, we may solve the constraints on allowed scalar fields Φ to find the analogous dyonic result.

3.1 The commutative $k = 2$ data

We first record, for comparison, the commutative $k = 2$ data presented in [38]. We consider centred instantons so that the 4 geometrically trivial centre of mass coordinates need not be considered. The blocks of Δ are written explicitly in terms

of quaternions as

$$L = \begin{pmatrix} v_1 & v_2 \end{pmatrix},$$

$$M = \begin{pmatrix} \tau & \sigma \\ \sigma & -\tau \end{pmatrix},$$

which satisfy the symmetry requirements of the ADHM constraints

$$\Delta^\dagger \Delta = f^{-1}(x) \otimes 1_2. \tag{3.1}$$

The remainder of the ADHM constraints, namely $a^\dagger a = \mu^{-1} 1_2$, split into two parts. The diagonal elements yield $|v_a|^2 + |\tau|^2 + |\sigma|^2 + |x|^2$, where we define $|q|^2 \equiv \bar{q}q = q_i^2 1_2$ and the index a ranges over 1, 2. These are, therefore, trivially satisfied in the commutative case. The off-diagonal constraints in (3.1) give us

$$\bar{v}_1 v_2 + \bar{\tau}\sigma - \bar{\sigma}\tau = 0$$

and its conjugate. These constraints may be combined as $\bar{v}_1 v_2 - \bar{v}_2 v_1 = 2(\bar{\sigma}\tau - \bar{\tau}\sigma)$ and solved, in general, by [46]

$$\sigma = \frac{\tau}{4|\tau|^2}(\bar{v}_2 v_1 - \bar{v}_1 v_2) + \lambda \tau \tag{3.2}$$

for $\lambda \in \mathbb{R}$ arbitrary. The parameter λ encodes the O(2) symmetry arising from the matrix R in the ADHM redundancies (2.20), corresponding to local gauge transformations of the instantons: it will play an important role in determining the symmetries of the moduli space metric. For calculational ease, we may choose to break this symmetry to a discrete subgroup thereof by setting $\lambda = 0$. Heuristically, then,

our ADHM data contains only three independent quaternion terms and some centre of mass coordinates (suppressed inside τ), which furnishes the full 16-dimensional space, up to some concrete identification of gauge parameters which we consider shortly.

The metric for such data has already been calculated in [38] and we will not revisit it in detail here. The salient points of the analysis are that the metric splits into two parts: a 'flat' and an 'interacting' part:

$$
\begin{aligned}
\frac{\mathrm{d}s^2}{8\pi^2} &= \mathrm{Tr}\big(\mathrm{d}s_{\text{flat}}^2 + \mathrm{d}s_{\text{int}}^2\big) \\
&= \mathrm{Tr}\bigg(\big(\mathrm{d}v_1^2 + \mathrm{d}v_2^2 + \mathrm{d}\tau^2 + \mathrm{d}\sigma^2\big) - \frac{\mathrm{d}k^2}{N_A} \bigg),
\end{aligned}
\tag{3.3}
$$

where

$$
dk = \bar{v}_1 \mathrm{d}v_2 - \bar{v}_2 \mathrm{d}v_1 + 2(\bar{\tau}\mathrm{d}\sigma - \bar{\sigma}\mathrm{d}\tau),
$$

$$
N_A = |v_1|^2 + |v_2|^2 + 4\big(|\tau|^2 + |\sigma|^2\big)
$$

and $\mathrm{d}q^2 \equiv \mathrm{d}q \cdot \mathrm{d}q = \frac{1}{2}\mathrm{Tr}(\mathrm{d}\bar{q}\mathrm{d}q)$. This may be further simplified by explicitly writing $\sigma = \sigma(v_1, v_2, \tau)$ and application of a series of quaternion trace identities. Unfortunately, the noncommutative case is not so clear.

With this commutative data, one may consider scattering. Unlike in the single instanton case (where geodesic motion can avoid the singularity at zero-size by starting with non-zero angular momentum) the interactions between the two instantons can, and do, allow one instanton to shrink to zero-size in finite time [37]. In the single instanton case, the singularity can be smoothed out by considering noncommutativity on the space, and the resulting moduli space is Eguchi-Hanson [35]. We wish to achieve the same smoothing in the two instanton case.

3.2 The noncommutative deformation

Given the above, it is natural to wonder if one could deform the commutative data to encompass the effect of the noncommutativity. This is reinforced by various expected limits of the noncommutative metric: the singularity at v_1, v_2, $\tau \to 0$ should be resolved; it should reduce smoothly to the singular, commutative, metric as we reduce the noncommutativity parameter to zero; and in the limit of large separation (that is, in the zero interaction limit) the metric should reduce to the direct product of two distinct single noncommutative instanton Eguchi-Hanson metrics. Given these considerations, we deform the commutative data as follows.

To temporarily avoid confusion with the 'vanilla' data above, we begin by writing the unconstrained ADHM data in the form

$$L = \begin{pmatrix} w_1 & w_2 \end{pmatrix}$$

$$M = \begin{pmatrix} t & s \\ s & -t \end{pmatrix}.$$

Recall that with a noncommutative $\mathbb{R}^4$, the diagonal terms will no longer be automatically proportional to the identity, but instead receive a term from $\bar{x}x$:

$$\bar{x}x = |x|^2 1_2 + \begin{pmatrix} -2\zeta & 0 \\ 0 & 2\zeta \end{pmatrix}.$$

The off-diagonal terms in the ADHM constraint, having no x-dependence, remain the same. Hence we may still express s in a similar form to the expression for σ in (3.2):

$$s = \frac{t}{4t^\dagger t}\left(w_2^\dagger w_1 - w_1^\dagger w_2\right). \tag{3.4}$$

We use the $\dagger$ notation to reinforce that the entries in Δ need no longer be quaternionic.

The solution of these new constraints now results in a choice of how to perturb the quaternion parts of Δ_{comm}. The most expedient choice is to retain the quaternionic nature of t, which we will return to labelling as τ, and absorb the noncommutativity into the w_a as follows:

$$w_a = v_a M_a,$$

$$M_a = \frac{1}{\sqrt{|v_a|^2}} \begin{pmatrix} \sqrt{|v_a|^2 + \alpha\zeta} & 0 \\ 0 & \sqrt{|v_a|^2 - \alpha\zeta} \end{pmatrix}, \tag{3.5}$$

where $\alpha = \alpha(\tau, v_1, v_2)$ is some function of the commutative parameters to be determined. One notes that the expression for s is also no longer quaternionic, due to its form in (3.4). The constraint on α is given by requiring that the non-identity proportional parts of the nonquaternionic data,

$$\left(w_a^\dagger w_a + s^\dagger s\right) - \begin{pmatrix} 2\zeta & 0 \\ 0 & -2\zeta \end{pmatrix} \propto 1_2 \tag{3.6}$$

for $a = 1, 2$ and the solution, while non-trivial (a derivation is given in the Appendix), is given by

$$\alpha = \frac{32|\tau|^2 |v_1|^2 |v_2|^2}{16|\tau|^2 |v_1|^2 |v_2|^2 + |\bar{v}_2 v_1 - \bar{v}_1 v_2|^2 (|v_1|^2 + |v_2|^2)}. \tag{3.7}$$

It is perhaps clear at this stage why the calculation of the noncommutative metric is so much more computationally expensive than that of the commutative case. Even something as simple as the 'flat' $\mathrm{d}w_a^2$ is a non-trivial multi-term expansion of all

of the moduli space parameters due to the renegade additional factors arising from x. In practice, however, we can avoid some of the complications inherent in the noncommutative metric by treating α as a parameter in its own right, and deriving a geodesic equation for α. The α geodesic equation contains no actual dynamical content, but represents a partial reparametrisation of the moduli space parameters that we remove after the metric calculation by demanding satisfaction of the $\mathrm{d}\alpha$ equation at all times. From a simulational perspective, we will allow the instanton solutions to undergo "geodesic motion" and only consider the results gained to be valid if the constraint on α is maintained under this geodesic flow.

Even with this simplification, calculating the metric for noncommutative instantons is not easy. Consider first the 'flat' term $\mathrm{d}w_1^2$. The derivative is given by

$$\mathrm{d}w_1 = \mathrm{d}v_1 M_1 + v_1 \frac{\bar{v}_1 \cdot \mathrm{d}v_1}{|v_1|^2}(M_1^{-1} - M_1).$$

Even in the free sector of the metric, we obtain additional terms proportional to M_a. These will have minimal impact for small noncommutativity or large instantons, but in the regime where $\zeta \sim |v_a|$ the additional noncommutative effects will be dominant. This complexity of the deformation, even for the 'free' metric terms, prevents us in all but the most simple cases from using the properties of quaternion products

$$\bar{e}_i e_j = -\bar{e}_j e_i + 2\delta_{ij},$$
$$e_i q \bar{e}_i = 2\mathrm{Tr}(q)1_2,$$
$$e_i q e_i = \bar{e}_i q \bar{e}_i = -2\bar{q}$$

and quaternion trace identities. In particular, a key result during the calculation of the commutative metric relied on the tracelessness of pure quaternions and the

identity

$$\mathrm{Tr}(p\bar{q}\bar{r}s) = \mathrm{Tr}(\Re(p\bar{q})\Re(\bar{r}s) - \Im(q\bar{p})\Im(\bar{r}s))$$

$$= 2p_i q_i r_j s_j - 2(\epsilon_{ijkl} - \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})p_i q_j r_k s_l.$$

A possible avenue of exploration in order to utilise such identities may be to consider the commutation relations between the quaternions and the noncommutative deformations M_a. Note that

$$[M_a, \, e_\beta] = iP_a\epsilon_{\beta\gamma}e_\gamma \quad \text{for } \beta, \, \gamma = 1, 2,$$

$$[M_a, \, e_i] = 0 \qquad\qquad\qquad \text{else,}$$

where

$$P_a = \sqrt{|v_a|^2 + \alpha\zeta} - \sqrt{|v_a|^2 - \alpha\zeta}.$$

We may write these commutation relations schematically as

$$[M_a, \, e_i] = iP_a\epsilon_{ij}e_j, \tag{3.8}$$

where it is understood that, in this notation, $\epsilon_{3i} = \epsilon_{4i} = \delta_{3i} = \delta_{4i} = 0$. Then we may use (3.8) to collect together the factors of M_a in the derived s. The result is

$$s = \sigma M_1 M_2 + \frac{\tau}{4|\tau|^2}\frac{(\bar{v}_2 v_1)^i}{|v_1||v_2|}(i(|v_2|P_2 M_1 + |v_1|P_1 M_2)\epsilon_{ij} - 2P_1 P_2\delta_{ij})e_j.$$

While this does make clearer the additional factors introduced into s as a result of the noncommutativity, the sheer level of maladdress inherent in this expression prevents it from providing a clear path to an explicit form for the metric without

choosing a definite parametrisation.

Before we select a relevant parametrisation for the metric, we first consider the metric derivation presented in [38]. We may write the metric in terms of the ADHM data as

$$g_{rs} = 2\pi^2 \mathrm{Tr}\big(\partial_r a^\dagger (1 + P_\infty)\partial_s a - \big(a^\dagger \partial_r a - (a^\dagger \partial_r a)^{\mathrm{T}}\big)\delta_s R\big), \tag{3.9}$$

where P_∞ is the projector at infinity, given in our case by $\mathrm{diag}(1, 0, 0)$, and the variation δR, where R is the gauge transformation in (2.20), is determined by the symmetry of the theory and the constraint

$$a^\dagger \delta a - (a^\dagger \delta a)^{\mathrm{T}} = a^\dagger b \delta R b^\dagger a - b^\dagger a \delta R a^\dagger b + \mu^{-1}\delta R + \delta R \mu^{-1}. \tag{3.10}$$

We now consider the deformation of each term under the introduction of noncommutativity. The redundancy (2.20) now requires $q \in \mathrm{U}(2)$, rather than $\mathrm{SU}(2)$. The 'flat' terms possess no redundancy under the $\mathrm{SU}(2)$ piece, as in the commutative case, but there is an isometry corresponding to the additional $\mathrm{U}(1)$ factor that needs to be gauged away. Generically, we have a transformation

$$w_a \rightarrow w_a e^{i\xi} \,, \ \mathrm{d}w_a \rightarrow (\mathrm{d}w_a + i\mathrm{d}\xi w_a)e^{i\xi},$$

for $\xi \in \mathbb{R}$. In computing $\mathrm{d}w_a^\dagger \mathrm{d}w_a$, we must identify the conjugate momentum, p_ξ, associated to this isometry and set it to zero. For arbitrary data w_a, after completing the square we obtain

$$\mathrm{d}w_a^2 = \mathrm{d}\bar{w}_a \mathrm{d}w_a + |w_a|^2 \bigg(\mathrm{d}\xi + \frac{\kappa}{2|w_a|^2}\bigg)^2 - \frac{\kappa^2}{4|w_a|^2}, \tag{3.11}$$

where $\kappa = \mathrm{d}\bar{w}_a w_a - \bar{w}_a \mathrm{d}w_a$. The second term is equivalent to $|w_a|^2 p_\xi^2$, and so must

vanish. The additional U(1) factor has nevertheless induced an additional factor in the flat instanton pieces. We note, at this stage, that in the limit of large separation, only the flat part of the metric contributes and s vanishes. We then find that an explicit parameterisation of the w_a,

$$
w_a = \begin{pmatrix} \sqrt{\rho_i^2 + \alpha\zeta}\, u_{a1} & -\sqrt{\rho_i^2 - \alpha\zeta}\, \bar{u}_{a2} \\ \sqrt{\rho_i^2 + \alpha\zeta}\, u_{a2} & \sqrt{\rho_i^2 - \alpha\zeta}\, \bar{u}_{a1} \end{pmatrix},
$$

for $u_{a1} = \cos\theta_a e^{i(\psi_a + \phi_a)}$ and $u_{2a} = \sin\theta_a e^{i(\psi_a - \phi_a)}$, results in two copies of the Eguchi-Hanson metric using the result in (3.11), as expected. In the commutative case, the expression κ vanishes in the final metric due to the vanishing of the deformation and the presence of the trace in the metric calculation.

In the 'interacting' part, the redundancy symmetries in (2.20) to be parametrised by δR now lie in U(2) rather than O(2), as explained in [46]. The constraint (3.10) is then modified accordingly. In the commutative case, the multiplicative factors around δR were proportional to the identity, and therefore $\delta R \propto a^\dagger \delta a - (a^\dagger \delta a)^{\mathrm{T}}$ naturally followed. In the noncommutative case, this no longer occurs. A solution is still obtainable, however: one may use the explicit w_a and τ dependence of the data a and b to (anti-)commute them through δR and explicitly multiply by the inverse of the matrix multiplicative factor. For the sake of completeness, symbolically we have

$$
\delta R = \left(w_1^\dagger \mathrm{d}w_2 - w_2^\dagger \mathrm{d}w_1 + 2(\bar{\tau}\mathrm{d}s - s^\dagger \mathrm{d}\tau) \right) \left(w_1^\dagger w_1 + w_2^\dagger w_2 + 4(\bar{\tau}\tau + s^\dagger s) \right)^{-1},
$$

and so the interacting part of the metric follows trivially, and is given by:

$$-\mathrm{Tr}\left(\left(w_1^\dagger \mathrm{d}w_2 - w_2^\dagger \mathrm{d}w_1 + 2(\bar{\tau}\mathrm{d}s - s^\dagger \mathrm{d}\tau)\right)^2 \left(w_1^\dagger w_1 + w_2^\dagger w_2 + 4(\bar{\tau}\tau + s^\dagger s)\right)^{-1}\right).$$

It is possible, at this point, to expand s in terms of w_a and τ and calculate the inverse but the resulting expression is not illuminating. Instead, we now exploit the symmetries of the metric to obtain tractable results.

3.3   Complex restriction of the moduli space

As we have seen, the noncommutative framework causes a number of complications in determining a useful form of the metric. Taking a generic parametrisation of w_1, w_2 and τ via, for example, Euler angles or complex matrices would be the easiest way to generate a full metric for the instantons, but this has proven to be computationally expensive. We may, instead, consider whether any valid geodesic submanifolds of the data exist that admit a sensible parametrisation and tractable metric calculation. Such a submanifold can be generated by certain fixed points of a symmetry of the metric. Consider the unexpanded form of $\mathrm{d}s^2$:

$$\mathrm{d}s^2 = \mathrm{Tr}\left(\mathrm{d}w_1^\dagger \mathrm{d}w_1 + \mathrm{d}w_2^\dagger \mathrm{d}w_2 + \mathrm{d}\bar{\tau}\mathrm{d}\tau + \mathrm{d}s^\dagger \mathrm{d}s - N_A^{-1}\mathrm{d}k^2\right), \tag{3.12}$$

where N_A is the multiplicative factor defined in [38]. The key symmetry that we wish to consider is conjugation of the data by a unit quaternion, p:

$$w_1 \to pw_1\bar{p}, \quad w_2 \to pw_2\bar{p}, \quad \tau \to p\tau\bar{p}.$$

The imposition of invariance under such an SO(2) symmetry has the effect of fixing a plane in the $\mathbb{R}^4$ instanton world-volume, and the resulting data can be viewed as living on the transverse directions to this plane. In the commutative picture, the invariance of the metric under such a transformation was guaranteed as the corresponding transformation rule for σ, that is $\sigma \to p\sigma\bar{p}$, is naturally respected. It is not as simple in the noncommutative case, due to the commutation relations (3.8). In order to apply the same analysis, we may only consider conjugation symmetries whose direction commutes with the direction of the noncommutativity. Clearly, then, this symmetry is valid only for $p = e_3$ in the noncommutative picture; the choice of noncommutativity has removed some of the underlying symmetries of the space, as would be anticipated. Our valid geodesic submanifold, then, is composed of τ, v_1, $v_2 \in \text{Span}\{e_3, 1_2\}$. Note that this complexification is in agreement with the arguments put forward in [26], where the e_3-e_4 plane is chosen in order to break the correct subgroup of the ADHM symmetries (we will examine this in more detail in Section 4.4).

We thus consider an explicit complex parametrisation of the form

$$v_a = \rho_a(\cos\theta_a 1_2 + \sin\theta_a e_3),$$

$$\tau = \omega(\cos\chi 1_2 + \sin\chi e_3).$$

Here, as in [38], we will interpret ρ_a as the size of the a-th instanton and θ_a as its internal gauge orientation in the U(1) subgroup of the overall SU(2). The parameters in τ will play a part in determining the separation and angle of incidence of the instantons from the origin (the parameters arising from τ have a less straightforward interpretation, related to the instanton separation, which we discuss shortly). Due

to the commuting nature of the deformation in this submanifold, then, we obtain

$$s = sM_1M_2 = M_1M_2s,$$

and in this parametrisation s and the noncommutative deformation function α take on a simpler form:

$$
\begin{aligned}
s &= \frac{\sin \phi}{2\omega}\sqrt{\rho_1^2\rho_2^2 + \alpha^2\zeta^2}\, e_3 + \frac{\alpha\zeta(\rho_1^2 + \rho_2^2)\sin \phi}{2\omega\sqrt{\rho_1^2\rho_2^2 + \alpha^2\zeta^2}}1_2, \\
\alpha &= \frac{8\omega^2}{4\omega^2 + \sin \phi(\rho_1^2 + \rho_2^2)},
\end{aligned}
\tag{3.13}
$$

where we now define $\phi \equiv \theta_1 - \theta_2$ to be the relative gauge angle on the moduli space. We also define $\Theta \equiv \theta_1 + \theta_2$, corresponding to the total gauge angle.

It is now possible to calculate the metric on this 6-dimensional submanifold. Defining, for convenience, the following quantities:

$$
\begin{aligned}
\rho_{i\pm}^2 &\equiv \rho_i^2 \pm \alpha\zeta, \\
P_i &\equiv \rho_i^4 - \alpha^2\zeta^2, \\
\Omega_\pm &\equiv \rho_1^2\rho_2^2 \pm \alpha^2\zeta^2, \\
N_\pm &\equiv 4\omega^2 + \rho_1^2 + \rho_2^2 \pm 2\alpha\zeta + \frac{1}{\omega^2}\rho_{1\pm}^2\rho_{2\pm}^2\sin^2\phi,
\end{aligned}
$$

we find the flat part to be

$$
\begin{aligned}
\mathrm{d}s_{\text{flat}}^2 =& \frac{1}{P_1}\left(\rho_1^4 + \frac{\rho_1^2\Omega_-\sin^2\phi}{4\omega^2}\right)\mathrm{d}\rho_1^2 + \frac{1}{P_2}\left(\rho_2^4 + \frac{\rho_2^2\Omega_-\sin^2\phi}{4\omega^2}\right)\mathrm{d}\rho_2^2 \\
&+ (\mathrm{d}\omega^2 + \omega^2\mathrm{d}\chi^2)\left(1 + \frac{\Omega_+\sin^2\phi}{4\omega^4}\right) + \frac{1}{4}(\rho_1^2 + \rho_2^2 - \frac{1}{2}\alpha^2\zeta^2)(\mathrm{d}\Theta^2 + \mathrm{d}\phi^2) \\
&+ \frac{1}{2}(\rho_1^2 - \rho_2^2)\mathrm{d}\Theta\mathrm{d}\phi + \frac{\Omega_+\cos^2\phi}{4\omega^2}\mathrm{d}\phi^2 - \frac{\Omega_+\sin 2\phi}{4\omega^4}\omega\mathrm{d}\omega\mathrm{d}\phi \\
&+ \frac{\rho_1\rho_2\sin^2\phi}{2\omega^4}\left(\omega^2\mathrm{d}\rho_1\mathrm{d}\rho_2 - \omega\mathrm{d}\omega(\rho_1\mathrm{d}\rho_2 + \rho_2\mathrm{d}\rho_1)\right) \\
&+ \frac{\rho_1\rho_2\sin 2\phi}{4\omega^2}(\rho_2\mathrm{d}\rho_1 - \rho_1\mathrm{d}\rho_2)\mathrm{d}\phi \\
&+ \alpha\mathrm{d}\alpha\zeta^2\left(\frac{\rho_1\mathrm{d}\rho_1}{P_1}\left(\frac{(\rho_1^2 - \rho_2^2)\sin^2\phi}{4\omega^2} - 1\right) + \frac{\rho_2\mathrm{d}\rho_2}{P_2}\left(\frac{(\rho_2^2 - \rho_1^2)\sin^2\phi}{4\omega^2} - 1\right)\right. \\
&\left. - \frac{1}{4\omega^2}(2\omega\mathrm{d}\omega\sin^2\phi - \omega^2\sin 2\phi\mathrm{d}\phi)\right) \\
&+ \frac{\mathrm{d}\alpha^2\zeta^2\sin^2\phi}{16\omega^2 P_1 P_2}\left(\Omega_-(\rho_1^2 + \rho_2^2) - 2\alpha^2\zeta^2(\rho_1^4 - 2\alpha^2\zeta^2 + \rho_2^4) + 4\zeta^2\omega^2\Omega_-(\rho_1^2 + \rho_2^2)\right)
\end{aligned}
$$

and the interacting part, similarly, is

$$
\begin{aligned}
\mathrm{d}s_{\text{int}}^2 =& \frac{(\cos\phi(\rho_{1-}\mathrm{d}\rho_{2-} - \rho_{2-}\mathrm{d}\rho_{1-}) - 2\rho_{1-}\rho_{2-}\sin\phi(\mathrm{d}\Theta - 2\mathrm{d}\chi))^2}{8\rho_{1-}\rho_{2-}N_-} \\
&+ \frac{(\cos\phi(\rho_{1+}\mathrm{d}\rho_{2+} - \rho_{2+}\mathrm{d}\rho_{1+}) - 2\rho_{1+}\rho_{2+}\sin\phi(\mathrm{d}\Theta - 2\mathrm{d}\chi))^2}{8\rho_{1+}\rho_{2+}N_+}.
\end{aligned}
$$

The form of the metric is perhaps not particularly simple, but one can verify the anticipated properties. In the limit of $\zeta \to 0$, we see that $\Omega_\pm \to \rho_1^2\rho_2^2$, $P_i \to \rho_i^4$, $\rho_{i\pm}^2 \to \rho_i^2$ and so $N_\pm \to N_A$, where N_A is the multiplicative factor defined in [38]. With the vanishing of the final three lines in $\mathrm{d}s_{\text{flat}}$, it is then easy to see that one recovers the commutative metric of two instantons in this limit.

We may also verify the expected result at the large separation limit: as ω becomes large, the interacting term is subleading and $\alpha \to 2 \Rightarrow \mathrm{d}\alpha \to 0$. Ignoring the flat

space $\mathrm{d}\omega^2 + \omega^2 \mathrm{d}\chi^2$ term, we obtain

$$\mathrm{d}s_{sep}^2 = \frac{\mathrm{d}\rho_1^2}{1 - 4\zeta^2/\rho_1^4} + \left(1 - \frac{4\zeta^2}{\rho_1^4}\right)\rho_1^2 \mathrm{d}\theta_1^2 + \frac{\mathrm{d}\rho_2^2}{1 - 4\zeta^2/\rho_2^4} + \left(1 - \frac{4\zeta^2}{\rho_2^4}\right)\rho_2^2 \mathrm{d}\theta_2^2. \quad (3.14)$$

This is two copies of the Eguchi-Hanson metric restricted to the complex subspace, which was demonstrated to be the metric of a single instanton in $\mathrm{U}(N)$ gauge groups [24, 26].

Finally, before examining the symmetries of the metric in more detail, we note that the noncommutative metric still permits the Killing vectors ∂_Θ and ∂_χ. The second vector corresponds to the overall SO(2) symmetry of the flat (ω, χ) space geometry which, under the addition of a VEV, will remain unbroken. The vector ∂_Θ, as justified in [24], will contribute to the potential as

$$V = \frac{1}{2}g_{rs}G^r G^s = \frac{v^2}{2}g_{\Theta\Theta}, \quad (3.15)$$

where v is the strength of the potential. Hence, for later reference, we may read off the potential term for the complexified noncommutative metric:

$$
\begin{aligned}
V ={}& \frac{1}{2}v^2 g_{\Theta\Theta} \\
={}& \frac{1}{4}v^2 \left(\rho_1^2 + \rho_2^2 - \frac{1}{2}\alpha^2\zeta^2 - 4\omega^2 \right. \\
& \left. + \frac{2\omega^2(\rho_1^2 + \rho_2^2 + 4\omega^2 - 2\alpha\zeta)}{N_-} + \frac{2\omega^2(\rho_1^2 + \rho_2^2 + 4\omega^2 + 2\alpha\zeta)}{N_+}\right).
\end{aligned}
\quad (3.16)
$$

In the limit as $\zeta \to 0$, this agrees with the 2-instanton commutative complexified potential given in [38], and in the single instanton limit we obtain agreement with the complexified version of the U(1) potential obtained in [24]. We may similarly derive the angular momentum, L, of the instantons, given by $g_{\Theta i}\dot{z}^i$, which we expect

to be conserved in any subsequent geodesic motion:

$$
\begin{aligned}
L = {}& \frac{2}{v^2} V \dot{\Theta} + \frac{1}{4}(\rho_1^2 - \rho_2^2)\dot{\phi} + \sin^2 \phi \left(\frac{\rho_{1+}^2 \rho_{2+}^2}{N_+} + \frac{\rho_{1-}^2 \rho_{2-}^2}{N_-} \right)\dot{\chi} \\
& + \frac{1}{4}\sin(2\phi)\left(\left(\frac{\rho_{2+}^2}{N_+} + \frac{\rho_{2-}^2}{N_-} \right)\rho_1\dot{\rho}_1 + \left(\frac{\rho_{1+}^2}{N_+} + \frac{\rho_{1-}^2}{N_-} \right)\rho_2\dot{\rho}_2 \right).
\end{aligned}
\tag{3.17}
$$

We will explicitly verify in the following chapter that this is a conserved quantity.

3.4 Symmetries of the noncommutative metric

Before taking refuge in the cote of numerical simulation, we conclude this chapter with a brief analysis of the symmetries of the noncommutative moduli space. The solution for s in the ADHM constraints allowed some freedom over a choice of constant τ term; explicitly we found

$$
s = \frac{\tau}{4|\tau|^2}(w_2^\dagger w_1 - w_1^\dagger w_2) + \lambda\tau,
\tag{3.18}
$$

for $\lambda \in \mathbb{C}$. A particular choice of λ breaks the O(2) gauge symmetry, represented by the ADHM transformation $\Delta \to Q\Delta R^{-1}$, down to a discrete subgroup. These discrete symmetries are quotiented when considering the moduli space metric: the fixed points of these symmetries will, upon quotienting, give rise to orbifold singularities in the moduli space. Indeed, in the commutative case, it can be seen that the zero-size singularity corresponds to such fixed points. We must consider the nature of such symmetries to ensure that the noncommutative moduli space is singularity-free, and the resulting manifold smooth.

Having absorbed the U(1) factor into the derivation of the noncommutative moduli space metric derivation, the residual symmetries generated by R may be

considered as reflections or rotations of the ADHM data. We therefore have the following ADHM-invariant transformations of the data:

$$\tilde{w}_1 = w_1 \cos\theta \mp w_2 \sin\theta,$$

$$\tilde{w}_2 = w_1 \sin\theta \pm w_2 \cos\theta,$$

$$\tilde{\tau} = (\cos^2\theta - \sin^2\theta)\tau \mp 2\cos\theta\sin\theta s, \qquad (3.19)$$

$$\tilde{s} = \pm(\cos^2\theta - \sin^2\theta)s + 2\cos\theta\sin\theta\tau.$$

Such transformations clearly leave the expression $\tilde{w}_2^\dagger \tilde{w}_1 - \tilde{w}_1^\dagger \tilde{w}_2$ invariant. However, to leave $\lambda = 0$ invariant we must have either $\cos^2\theta - \sin^2\theta = 0$ or $\cos\theta\sin\theta = 0$. Hence, the remaining discrete symmetries of R are described as rotations or reflections of Δ with $\theta = n\pi/4$, $n = 0, 1, \ldots, 7$, namely the elements of the dihedral group D_4.

Now we consider each group of transformations in turn, and its action on the ADHM data.

- $\cos\theta = \pm 1$, $\sin\theta = 0$. These transformations preserve τ and s, and preserve or negate the signs of w_1 and w_2. The fixed point of the non-trivial symmetry occurs when $w_i = -w_i$, that is when $w_i = 0$. This is the conical singularity encountered in the commutative case. Note that in the noncommutative picture, for generic $\zeta \neq 0$ this fixed point no longer lies on the moduli space of instantons, as the noncommutative parameter bounds the instanton size from below as $|w_i| \geq \sqrt{\alpha\zeta}$[1]. The action of this symmetry, therefore, does not give rise to a singularity under quotienting in the noncommutative picture, as anticipated.

---

[1]Note that as $\omega \to 0$, $\alpha \to 0$ and it would appear that the instantons may attain zero-size. In this limit, however, s is the dominant term describing in the metric and the instanton sizes are more correctly described by $|w_1 \pm w_2|^2/2$, which remain bounded.

- $\cos\theta = 0$, $\sin\theta = \pm 1$. Such transformations again preserve τ, swap the roles of w_1 and w_2 (potentially with a sign change), and may negate s. This corresponds to the indistinguishability of the two instantons on the moduli space. We may reinterpret this as a simple invariance under the relabelling of instantons $1 \leftrightarrow 2$, and obtain the previous case. The fixed points of these symmetries are, for this reason, the same zero-size instanton points as above, and may be safely discounted for the same reasons.

- $\cos\theta = \pm\frac{1}{\sqrt{2}}$, $\sin\theta = \pm\frac{1}{\sqrt{2}}$. This is equivalent to swapping τ and s, and redefining the w_i as some linear combination of each other. The only fixed point of this symmetry is the 'trivial' fixed point, $w_1 = w_2 = \tau = s = 0$. As we will see in Section 4.1, this fixed point has a geometric interpretation on the moduli space, and the 'singularity' obtained has no effect on the smoothness of the underlying metric.

We may now justify the claim that noncommutativity 'smooths out' the moduli space: the orbifold singularities present as a result of quotienting global gauge transformations of the ADHM data no longer appear in the noncommutative moduli space due to the new, ζ-dependent, form of the w_i. This is exactly what one expects [49]. From the D4-D0 perspective, a commutative solution describes D0s dissolved in D4s; the "small instanton" singularities arise from the transition between the (dissolved) Higgs branch, describing Yang-Mills theory, and (separated) Coulomb branches of the D-brane theory. In the noncommutative framework, the Coulomb branch is frozen out of the world-volume field theory, and the $\zeta \neq 0$ theory allows one to describe both dissolved and separated D0 branes without passing through the small instanton singularity.

Finally, we note that the form of the metric gained above appears to have a sin-

gularity as $\omega \to 0$. This is a particular concern, as we expected that the introduction of noncommutativity would remove all metric singularities. However, we now need to consider the interpretation of the parameter ω. We may note that s is inversely proportional to ω: thus the consideration of $\omega \to 0$ is equivalent to the consideration of $|s| \to \infty$. This, coupled with the symmetries presented above, show that the parameters of τ and s in conjunction describe the separation and orientation of the instantons, and so the points $\omega \to 0$ and $\omega \to \infty$ are equivalent: both are attained when the instantons are located at spatial infinity. The potential singularity of the metric at this point, therefore, is not a feature of the moduli space. This could also be demonstrated via a reparametrisation

$$\omega \to \frac{1}{\sqrt{2}} \left(\omega + \frac{\sin \phi}{2\omega} \sqrt{\rho_1^2 \rho_2^2 + \alpha^2 \zeta^2 + \frac{\alpha^2 \zeta^2 (\rho_1 + \rho_2)^2}{\rho_1 \rho_2 + \alpha^2 \zeta^2}} \right), \tag{3.20}$$

that is, $|\tau| \to (|\tau| + |s|)/\sqrt{2}$, which would resolve the worrisome $1/\omega$ dependence of the metric.

This concludes the derivation and analysis of the noncommutative instanton moduli space. Via a deformation of the ADHM data, solutions to the noncommutative instanton field theory can be generated, and shown to behave as expected. While it has not been possible to find a concise, explicit form for the full 16-dimensional metric for 2 instantons, nevertheless a geodesic submanifold of the metric still exists and one may reliably consider the evolution and behaviour of instantons on this reduced, 6-dimensional, space. The zero-size singularity is no longer a feature of this moduli space, achieving correspondence with the overarching D-brane picture. We may now turn to more interesting aspects of this instanton solution: evolution and scattering.

4

Noncommutative instanton dynamics

In this chapter, we examine the geodesic motion of two noncommutative instantons on the moduli space $\widehat{\mathcal{M}}_2$. While the metric, and induced geodesic equations, on the complexified moduli space obtained in the previous chapter do not admit analytic solutions in all but the simplest considerations, a numerical approach may be taken to simulate scattering, orbiting and general behaviour of the two instantons. We first consider the case where $\langle \phi \rangle = 0$ before looking at the dyonic extension to the moduli space in Chapter 5. The noncommutative framework admits some surprising results, particularly with regard to stable configurations of the instantons. Finally, we briefly discuss our results in the context of the non-Abelian vortex picture and find agreement with the results described in [50].

4.1 Instanton scattering

In order to consider the effect of scattering, we first turn to the common observation of soliton dynamics [38, 14]: two solitons colliding head-on at small velocities often results in right-angled scattering. We first note that in the metric presented in the previous section, the magnitude of τ, ω, admits a natural interpretation as the instanton separation. However, it is not unique in this respect. In particular, the gauge transformations that leave ADHM data Δ invariant admit an equivalent ADHM solution of the form

$$\Delta' = \begin{pmatrix} \frac{1}{\sqrt{2}}(w_1 + w_2) & \frac{1}{\sqrt{2}}(w_1 - w_2) \\ s & \tau \\ \tau & -s \end{pmatrix}. \tag{4.1}$$

Hence, we may state that s has equal claim to describing the separation of the instantons. This statement is further supported by the structure of s. For large τ, the magnitude of s is small and so in this regime the separation is adequately described by the parameter ω. Conversely, for small τ it is the s term that will dominate. In the case where the two parameters are of similar size, neither interpretation truly holds. More formally, the separation of the instantons is given by the eigenvalues of the lower block, M, of Δ:

$$\lambda_\pm = \pm\sqrt{\tau^2 + s^2}. \tag{4.2}$$

Note that the terms in the square root are not equivalent to $q^\dagger q$. We interpret these eigenvalues as follows. For τ large, the eigenvalues are approximately $\pm\tau$ and so we identify the configuration as that of two instantons whose centres are at $\pm|\tau|$. As τ reduces, the size of s is less suppressed, until we approach the point where τ and s are of equal magnitude. At this point, we note that the separation (4.2) vanishes since

τ and s are related by an imaginary phase in the commutative case. Passing beyond this point, as we reduce τ further then s becomes the dominant parameter controlling separation. Right-angled scattering arises due to this interchange between τ and s, coupled with the imaginary multiplicative factor which causes a phase difference of $\pi/2$ between the τ-dominated and s-dominated regimes of parameter space. In the noncommutative picture, this is not as clear. The presence of the parameter α in the expression for s makes the zero-eigenvalue requirement more complicated, and the results are dependent on the magnitude of ζ. In the complexified moduli space, the vanishing of the eigenvalues (4.2) for non-zero ζ now requires the satisfaction of

$$4 \sin^2 \phi (\rho_1^2 \rho_2^2 - \alpha^2 \zeta^2) = \omega^4,$$

which has a larger class of solutions than its commutative counterpart, and hence the relationship between s and τ which causes zero separation need not result in $\pi/2$ scattering. Additionally, we see that a zero-separation solution does not require vanishing instanton sizes, which justifies once more the removal of the zero-size singularity.

The scattering scenario is shown in Figure 4.1. Using the complexified metric derived previously, we identify ρ_i with the size of the i-th instanton. The angle χ defines the angle of incidence of scattering relative to the axis (so that an angle of $\chi = 0$ represents head-on scattering) and ω the initial separation[1]. Due to the discontinuous jump that occurs at zero separation (representing the symmetry between w_1 and w_2), a naïve numerical simulation breaks down at the point of

---

[1]This interpretation of the parameters as describing two separate particles holds when the instantons are not overlapping: that is, when their separation ω is greater than their sizes ρ_a. When this requirement does not hold, the instantons are better interpreted as a single charge 2 object around the origin, and in the limit of the zero separation the configuration is axially symmetric and annular. This is a justifiable interpretation, due to the additional symmetries between τ and s possessed in the coincident limit.

collision. We thus follow [38] and reparametrise the variables in τ as $\omega = \sqrt{x^2 + b^2}$ and $\chi = \arctan(b/x)$. Then we may interpret x as the initial separation along the axis and b as an impact parameter. A head-on collision will occur when the impact parameter goes to 0 but can be approximately observed for sufficiently small, non-zero, b. All simulations were run using Mathematica 9.0 [51]; a representative sample of the code is presented in Appendix B.
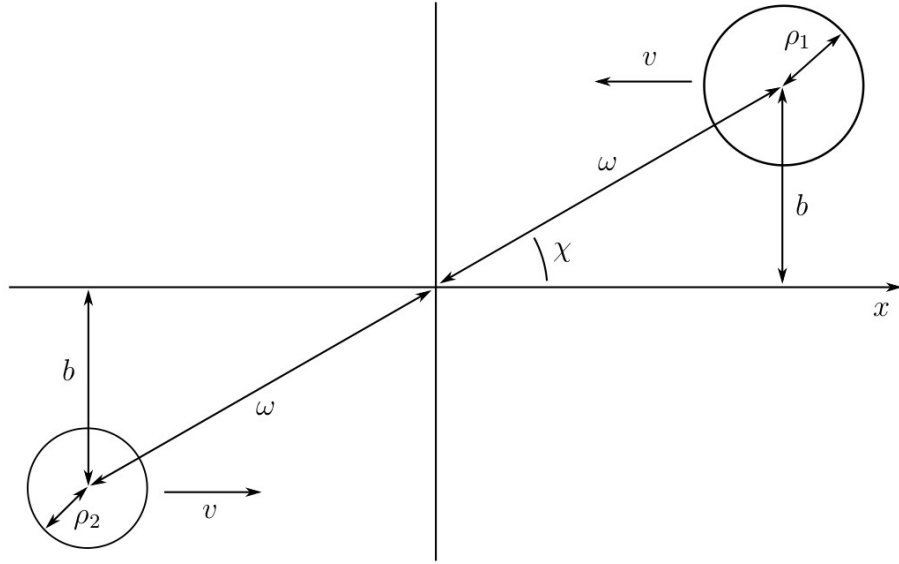

Figure 4.1: The relevant parameter set-up for scattering simulations. The general separation of the instantons is described by $1/\sqrt{2}(|\tau| + |s|)$, and this is what the "x" and "y" axes describe. In subsequent plots, where it is helpful, we plot the sizes of the instantons at regular t-intervals to demonstrate size evolution and instanton speed.

4.2 Head-on Collisions

We first consider the results of such a 'head-on' collision in both the commutative ($\zeta = 0$) and noncommutative ($\zeta = 0.1$) systems, as shown in Figure 4.2. The presence of right-angled scattering is perhaps heartening, as this agrees with the expected soliton behaviours. The key point, however, lies in the size plots. While

Head-on scattering for $\zeta = 0$.

Head-on scattering for $\zeta = 0.1$.

Size evolution of the commutative scattering setup.

Size evolution of the noncommutative scattering setup.
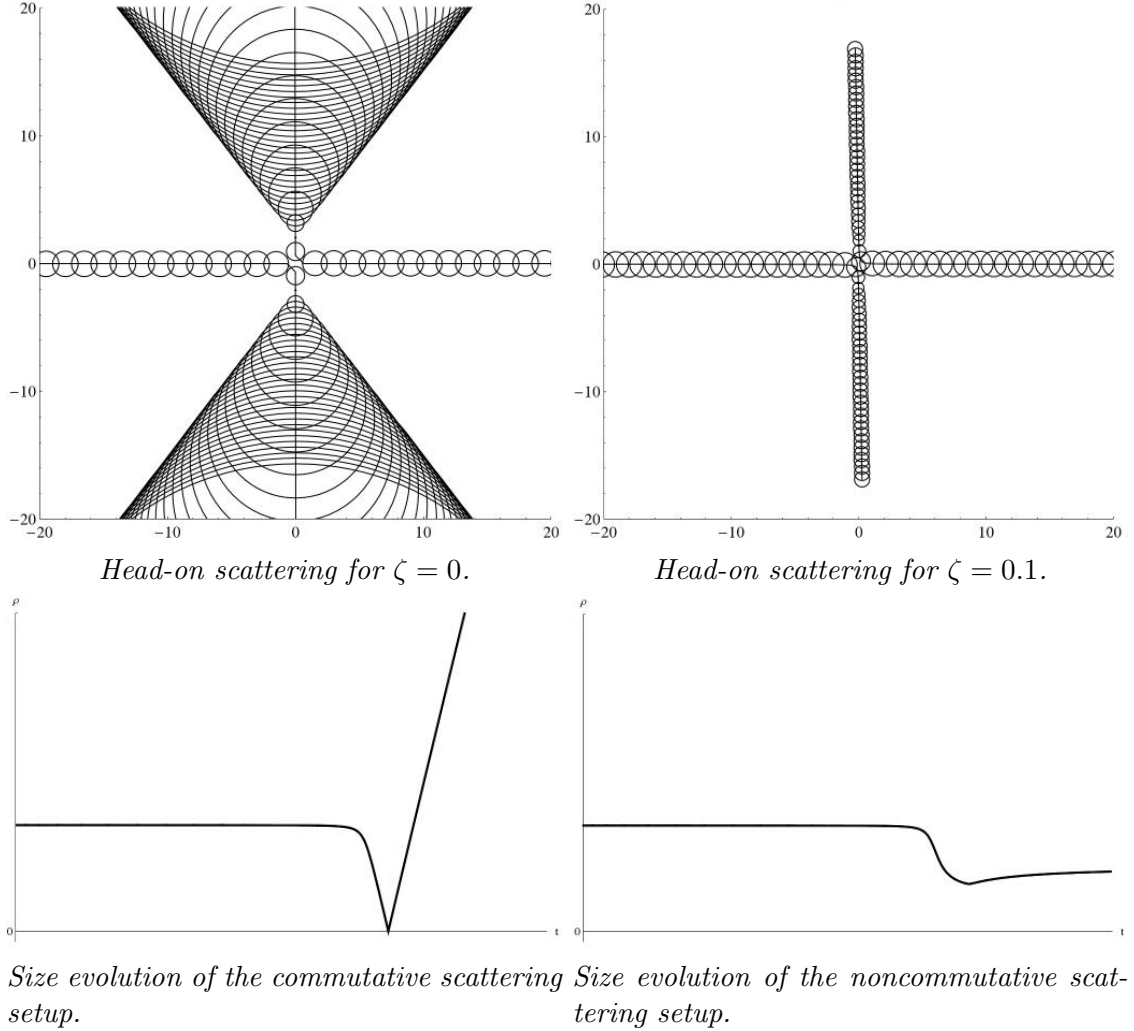
Figure 4.2: A comparison of commutative (left) $\zeta = 0$ and noncommutative (right) $\zeta = 0.1$ instanton scattering for given initial conditions $\phi = \pi/2$ (corresponding to the instantons having opposite orientations), $b = 0.001$, $x = 30$ and $\rho_1 = \rho_2 = 1$. The second row of plots demonstrates the evolution of the instanton sizes. Right-angled scattering is still a valid behaviour in the noncommutative case for small impact parameter. We note that, as anticipated, the instanton sizes do not vanish at the point of collision, thus avoiding the moduli space singularity attained in the commutative case.

in the commutative framework the instanton sizes reach the zero-size singularity, no such problem exists in the noncommutative analogue. This is as expected, since one anticipated that the noncommutativity would smooth out the singular point encountered in the commutative picture. It can be verified that the minimum of

the size is attained just after collision, and with the parameters evaluated, this minimum is precisely $\sqrt{\alpha\zeta}$. This agrees with our expectations: the noncommutative deformation to the metric took the form $\rho_i^2\rho_j^2 - \alpha^2\zeta^2$ for $i = 1, 2$ and $j = 1, 2$ so the singularities at $\rho_i = 0$ are replaced by a circle around the ρ_i parameter spaces of size $\sqrt{\alpha\zeta}$. This trend is shown in Figure 4.3.
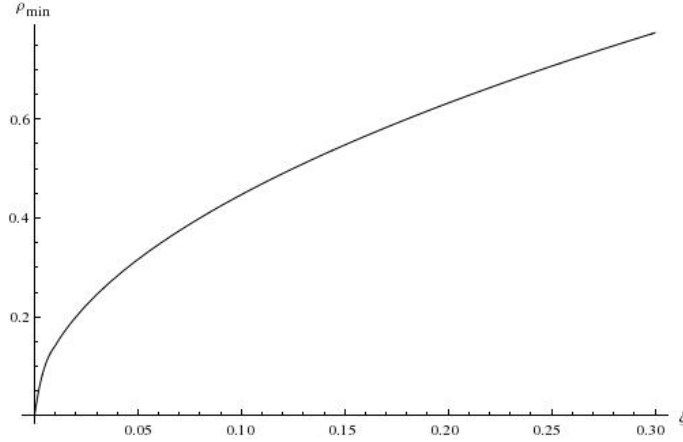
Figure 4.3: Minimum instanton size achieved via head-on scattering with varying ζ. The dependence of $\rho_{\min}$ on ζ is given by $\sqrt{\alpha\zeta}$, where α is evaluated at the point of collision.
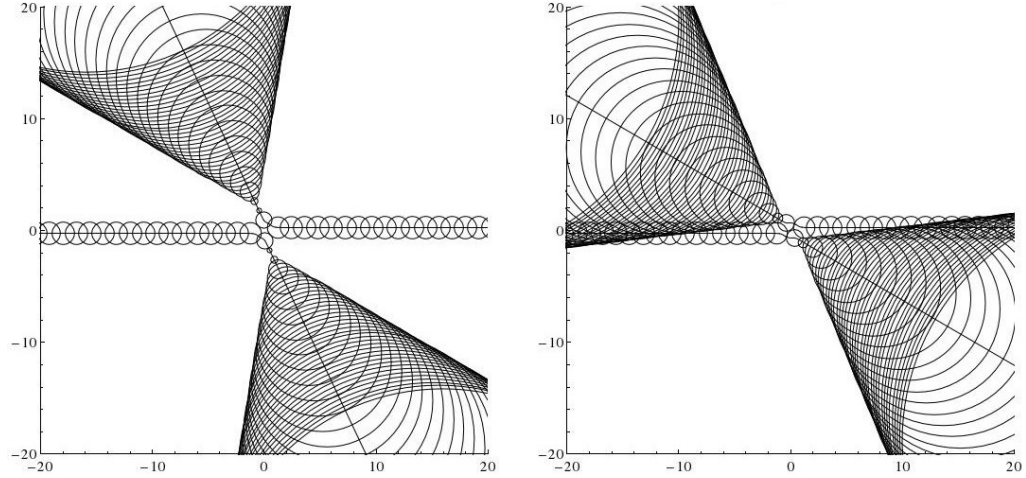
Figure 4.4: Commutative and noncommutative scattering for $b = 0.25$. Scattering occurs in the noncommutative case, but at a modified angle to that of the commutative instantons.

The above demonstrates that the 'attractiveness' of the noncommutative bound

state displays a large sensitivity to the value of the impact parameter, b. As one varies the impact away from head-on, we obtain scattering, although the presence of $\zeta \neq 0$ deforms the scattering solutions away from the commutative scattering angle. This behaviour under introduction of noncommutativity appears to be a generic feature of all soliton systems which arise from reductions of noncommutative instantons: in considerations of non-Abelian vortices (where a Fayet-Iliopoulos parameter serves to couple the Abelian U(1) non-trivially to the rest of the gauge group), this attractive behaviour is also manifest [50]. It is natural to ask whether such an attractive force on the moduli space could be interpreted as an induced potential on the space, even for the free instanton moduli space. This is a question that we will revisit in due course.

Given the modifications to the scattering behaviour under the introduction of a non-zero ζ, it is instructive to examine the scattering angles obtained. The results are shown in Figure 4.5 for equal size instantons (since this provides right-angled scattering in the commutative case) and $\zeta = 0.1$. We note that as we vary the impact parameter, the scattering angle varies accordingly from standard scattering to a scattering angle greater than π. This demonstrates that, far from being the standard result, right-angled scattering is one possible outcome from the collisions of noncommutative instantons.

The presence of a 'maximal' scattering angle raises more questions about the behaviour of the instantons. From Figure 4.5 one can see that there appears to be a "critical" tuning between b and ζ which maximises the final scattering angle. Such a tuning exists for all possible values of b within the interaction region (or equivalently, ζ within the Manton approximation), as can be seen in Figure 4.6. These configurations correspond to an intermediate state between scattering regimes, where the instantons retain enough repulsive force to temporarily overcome
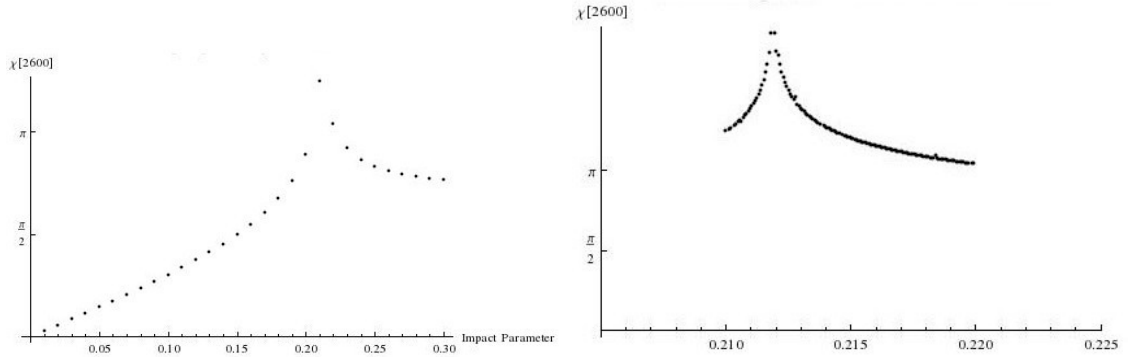
Figure 4.5: Scattering with varying b and $\zeta = 0.1$. A 'critical' point in configuration space exists at $b \sim 0.21$, where the instantons temporarily orbit before scattering. This is shown more clearly in the second plot, where we have zoomed in around the critical point.

the attractive ζ force, generating an unstable, short-lived, orbit around the centre. An example of such behaviour is shown in Figure 4.7, where the transition between the two scattering regimes is clear.
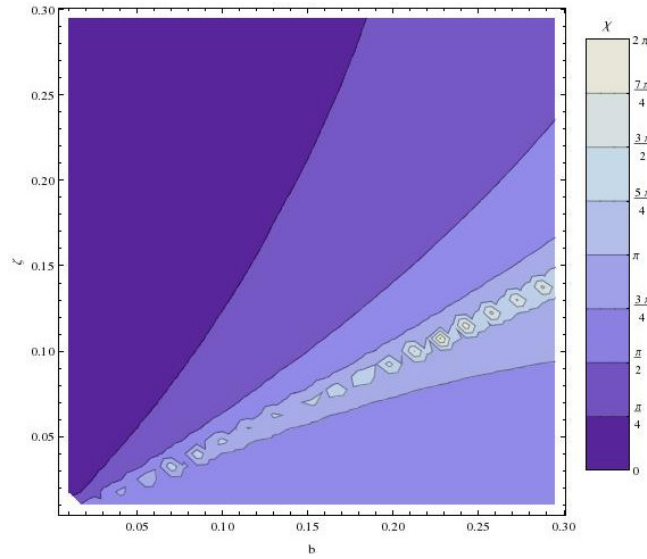

Figure 4.6: Contour plot of final scattering angle with varying ζ and b. The region bounded by the contour $\chi_{\mathrm{scat}} = \pi$ contains configurations with the unstable orbit characteristics, demonstrated in Figure 4.7.

These results are perhaps surprising: right-angled scattering does appear, but is not the most general result for close to head-on collisions between two noncommu-
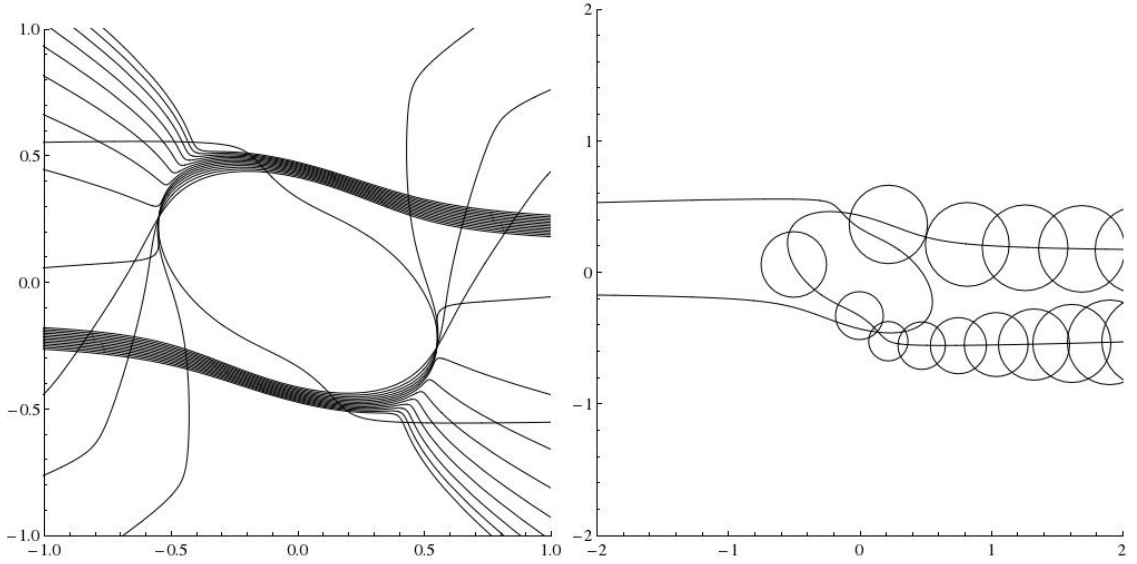
Figure 4.7: Collisions for a range of b and $\zeta = 0.1$. The configuration that maximises the scattering angle (the right figure) corresponds to a "slingshot", where the instantons orbit each other before returning whence they came. In the right-hand plot (where the size of one instanton has been suppressed for clarity), the right instanton approaches from above the x-axis with stable size and speed and leaves more slowly, but with an increasing size.

tative instantons. In fact, it naturally arises from a consideration of the symmetries of the ADHM data and the expression for the separation (4.2). The more involved form of s, coupled with the presence of the parameter α in the data, allows for a greater range of initial data causing the τ-s identification change. As a result, one can obtain scattering in a range of scenarios and scattering angles, of which right-angled is but one aspect.

4.3   Validity of Simulations

Before proceeding further, it is natural to wonder whether the above results represent a sensible conclusion. The behaviour of the noncommutative instantons is markedly different from that of their commutative counterparts, and the interpretation of such

results is not clear. It may be helpful, therefore, to examine expected features of the numerical results and ensure that the obtained behaviour is valid.

In the 'free' instanton picture, one does not anticipate the velocity of motion on the moduli space to affect the scattering behaviour. This is due to the fact that the evolution of our instantons is not true dynamical motion, per se, but geodesic motion through allowed instanton configurations on $\widehat{\mathcal{M}}_2$. Increasing the velocity on the moduli space, from this point of view, is equivalent to increasing the time-step in the numerical simulations. Hence, provided the velocity is within the ranges allowed by the Manton approximation, one expects that the geodesic motion obtained will be unchanged if one varies v. As Figure 4.8 demonstrates, the atypical noncommutative scattering behaviour is independent of the chosen velocity, up to slight fluctuations as one moves away from the valid range of the Manton approximation.
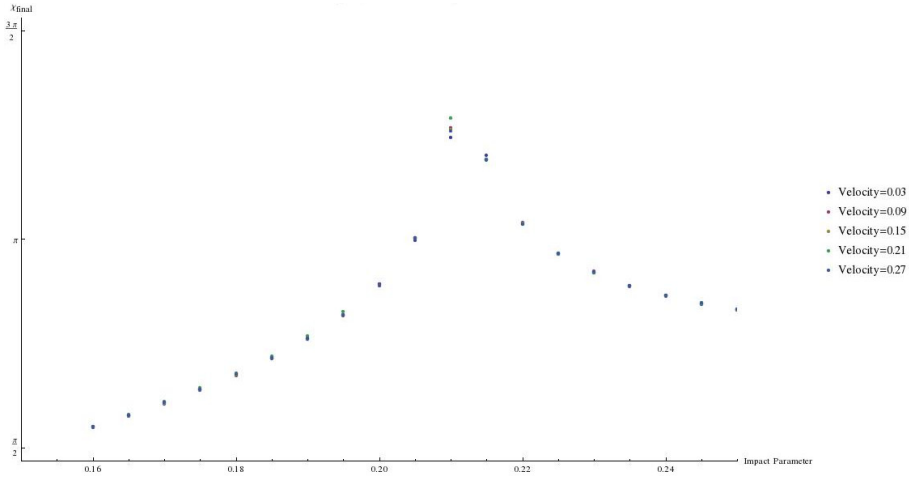
Figure 4.8: Scattering behaviour with varying 'velocity' ($v = 0.03$-0.27). The results gained are identical over the range of allowed velocities, up to small numerical differences, resulting from numerical error in the geodesic approximation.

We may also verify that the angular momentum (3.17) is conserved: the period of greatest volatility is around the point of collision. In this regime, the difference between initial angular momentum and that of the scattering configuration varies

only slightly, and well within expected numerical error. The "change" in angular momentum is shown in Figure 4.9.
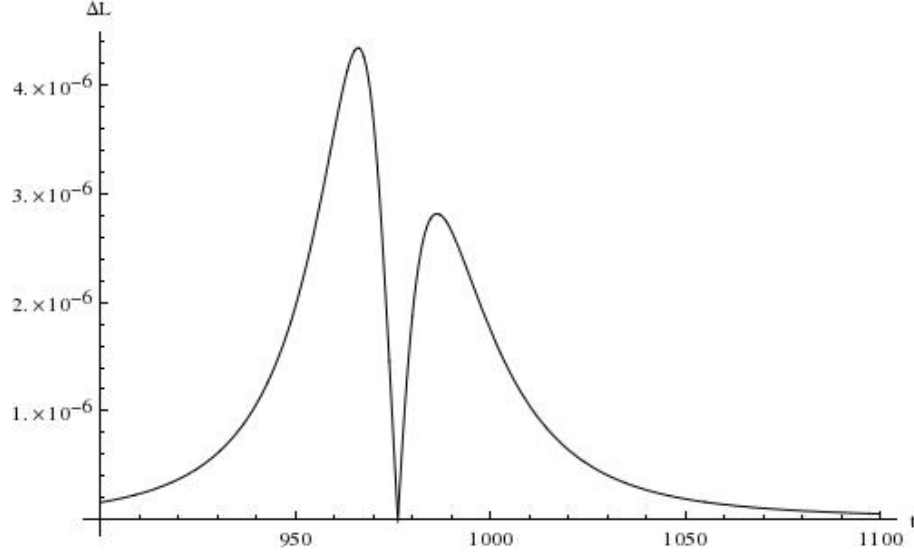

Figure 4.9: The variation in angular momentum of the system around the point of collision for a configuration of instantons with total starting angular momentum 1. The difference between the initial angular momentum and that of the evolved configuration never exceeds $\mathcal{O}(10^{-5})$, well within numerical error. Outside of the scattering region, the difference drops to $\mathcal{O}(10^{-8})$.

Finally, a diagnostic test of the solution method for the geodesic equations is in order. To this end, we may calculate the residuals of each geodesic equation at every timestep in the process of numerically solving the equations, and compare the results with those obtained in the commutative set-up (and indeed, for later comparison, with those obtained in the vortex picture). The numerics in the commutative and vortex simulations are valid up to errors $\sim \mathcal{O}(10^{-4})$ as checked in [38] and [50] respectively: as Figure 4.10 indicates, the errors in the numerical analysis of the noncommutative instantons is comparable in order to the previous results. It transpires that the critical scattering case is the most susceptible to numerical errors of the three possible scattering outcomes: the scattering configurations exhibit numerical error $\mathcal{O}(10^{-5})$ or smaller in all cases.
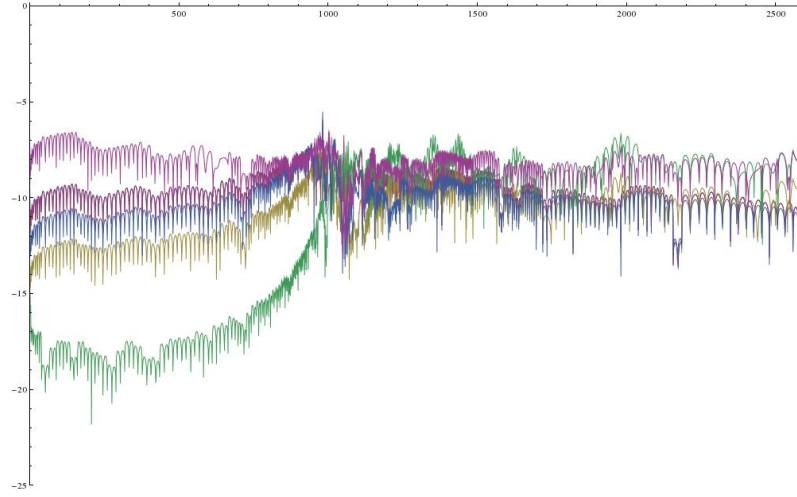
(a) Residuals for commutative head-on scattering.
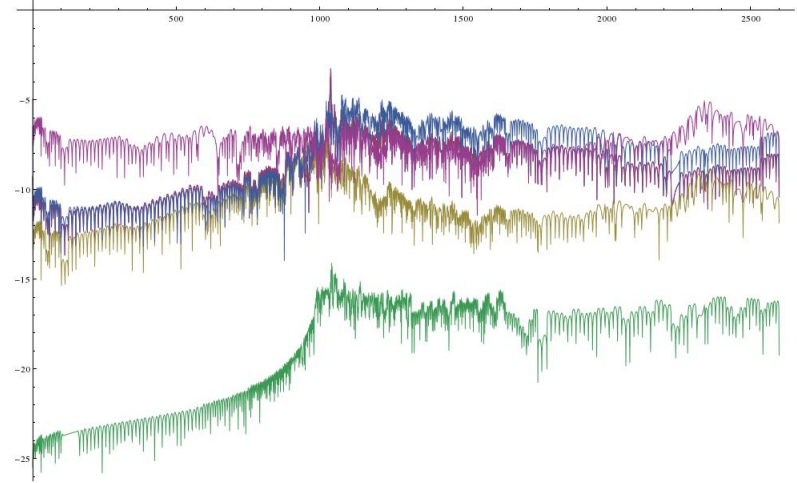
(b) Residuals for noncommutative critical scattering.

Figure 4.10: Log-plotted residuals of the commutative head-on and critical scattering situations. The maximum numerical error (attained shortly before collision) is $\mathcal{O}(10^{-4})$.

4.4 The connection to vortices

The results gained for instantons have wider reach to other solitonic systems. The $(4+1)$-dimensional Yang-Mills theory can be dimensionally reduced in a number of ways to obtain other lower-dimensional theories. Accordingly, instantons (as solutions to bosonic Yang-Mills theory) can be dimensionally reduced to produce

monopole and vortex solutions.

The vortex picture is an interesting one: the vortices are static solutions to a (2+1)-dimensional maximally supersymmetric $\mathcal{N} = 4$ field theory. To guarantee the existence of vortex solutions, the bosonic Lagrangian of such a theory can be adapted to contain a Fayet-Iliopoulos parameter, which modifies the D-term constraints and ensures symmetry breaking of the vacuum. The introduction of such a term in a U(N) vortex theory mediates the coupling between the SU(N) gauge symmetry and the remnant U(1) symmetry, in a similar vein to the instanton picture (see [26] for a fuller description) and hence the vortex solutions obtained can be considered to be non-Abelian [52]. The equivalence between the instanton and vortex deformations is not quite straightforward, however.

To make clear the connection, we must consider the symmetries of the instanton data [26]. The full symmetry group of the ADHM data for U(N) instantons is

$$G_{\text{inst}} = \text{SO}(5) \times \text{U}(N) \times \text{SU}(2) \times \text{U}(1),$$

where the SO(5) rotates the transverse scalars X^I, the U(N) is the overall flavour symmetry (corresponding to the ADHM redundancies) and the SU(2)×U(1) symmetry is the unbroken parts of the world-volume SO(4) symmetry after the introduction of noncommutativity. The vortex theory arises via a symmetry breaking of a subgroup of G_{inst} that leaves the matter content and SUSY structure equivalent to that of the vortices. To achieve this, we weakly gauge a U(1) factor inside SO(5). We can interpret this in a more concrete sense via the ADHM data and corresponding moduli space. The U(1) gauge field is tantamount to a circle action on the moduli space, which will have a corresponding triholomorphic Killing vector $\hat{k}$. Gauging by this S^1 action leads to a potential term in the instanton Lagrangian, with mass term

proportional to $\hat{k}^2$. Now, considering the fixed points of the circle action (equivalently, all points in the moduli space for which $\hat{k} = 0$) gives us exactly the vortex moduli space. To ensure isometry between the two sets of theories, one must relate the instanton noncommutative parameter, ζ, to the gauge coupling of the vortex theory; namely,

$$\zeta = \frac{\pi}{2e^2}. \tag{4.3}$$

There are a number of open questions in this analysis, most of which are unfortunately beyond the scope of this work. The FI parameter in the vortex theory already guarantees the existence and smoothness of vortex solutions, unlike in the overarching instanton theory. Due to the identification between ζ and the gauge coupling of the vortex U(1), descending to a theory of vortices from noncommutative instantons may, rather than resolving the moduli space, lead to singularities not present in the original theory [26]. More work on this aspect of the analysis, including classifying such potential singularities, would be helpful.

The scope of vortex solutions, a priori, appears to be larger than those configurations that would arise from the instanton reduction. The instantons, when dimensionally reduced, provide a 'critically coupled' non-abelian vortex theory and in fact, one can see from (4.3) that in the commutative limit the U(1) part of U(N) is frozen out of the theory. However, the theory of vortices may also admit its own FI parameter as well as the U(1) gauge coupling. It would appear that, as ζ is in some sense determined by the coupling e, that the instanton theory says nothing about the noncommutative structure of the vortex theory. It seems incongruous to assert that the vortices have additional freedom not possessed by the instantons, but a clear justification of the converse would be preferred. As it stands, we may only consider equivalence of our solutions to this critically coupled theory.

A point that naturally stems from the above discussion is related to dyonic instantons. If we choose a potential for the dyonic instantons in a direction orthogonal to the $U(1) \subset SO(5)$, then we should anticipate some form of 'dyonic' vortices to appear. The nature of such a theory is not clear, but work is being done to include a Higgs field to the vortex picture (e.g. [53]). The key stumbling block of such an identification stems from the nature of the potential term in the instanton theory. Symmetry requirements, and the satisfaction of the BPS equations and corresponding ADHM constraints, forced us to select a particular form for the potential term. Such restrictions do not exist in the vortex theory, and so there is no guarantee that a dyonic vortex theory would become an allowable dyonic instanton theory under a dimensional 'lift'. The converse, however, should be true: a dyonic instanton should find a natural, lower-dimensional analogue in the vortex picture. The behaviour and properties of such solutions would be vastly illuminating.

We postpone discussion of such issues except for the dyonic question, which we revisit briefly in Chapter 6. The aim here is to ensure that our solutions to the non-commutative ADHM equations agree, upon reduction, with the known behaviours of non-abelian vortices [54]. We note that, in the 'free' instanton case, we have no real restriction on the choice of $U(1) \subset SO(5)$ to gauge. We may also note that when complexifying the moduli space of instantons, we required the data to be unchanged under conjugation by the unit quaternion e_3. The data we have obtained, then, is fixed under the circle action generated by e_3 and hence viable as a starting point for the comparison with vortices.

Figure 4.11 shows the results of this vortex limit. We reproduce the results gained in [54] from the instanton data and observe the expected behaviour: the non-abelianisation is shown in the different behaviours with varying gauge orientation $\phi \equiv \theta_1 - \theta_2$. Of course, this is just one aspect of non-Abelian vortex scattering, but
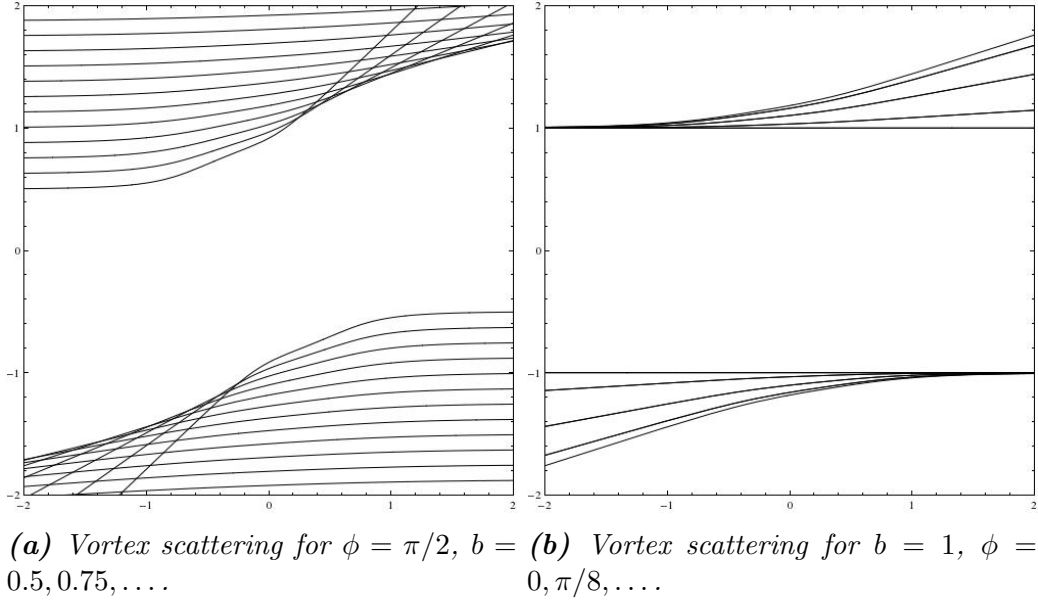
(a) Vortex scattering for $\phi = \pi/2$, $b = 0.5, 0.75, \ldots$ *(b) Vortex scattering for $b = 1$, $\phi = 0, \pi/8, \ldots$*

Figure 4.11: Vortices from a reduction of the instanton moduli space.

nonetheless it is encouraging to see the scattering behaviour exactly reproduced in the context of instantons.

We have now seen the qualitative differences between commutative and noncommutative scattering of 2 instantons in U(2). While, in the absence of a Higgs field, the commutative head-on configuration follows the expected soliton result (right-angled scattering), this is not the standard behaviour for $\zeta \neq 0$. Depending on the strength of the noncommutativity, one may obtain a large variety of scattering angles not present in the commutative case. The instanton minimum size is bounded below by the magnitude of the noncommutativity parameter, as one would have anticipated. One may obtain other, more interesting, scattering results not present in the commutative case. For a given noncommutative strength, the instantons may display unstable orbiting behaviour, where their latent repulsive force is temporarily balanced by the attractive ζ effect. Via a parameter search, such configurations may be seen to exist for any value of ζ within the Manton approximation.

Such surprising results should be (and were) treated with caution. We analysed the validity of the results, via recourse to known geodesic symmetries and conserved quantities, and found agreement in all cases. The numerics were subjected to similar analysis, and found to be valid to a level of numerical error comparable to that of the commutative results. Finally, we utilised the correspondence between dimensionally reduced instantons and vortices to verify that the expected behaviour was reproduced.

Having seen the effect that a non-zero ζ has on 'free' instanton scattering, it is now time to turn on a potential strength and discuss the nature of the scattering behaviour therein. We shall see that the noncommutativity has a pronounced and non-trivial impact on the stability and general properties of dyonic instanton scattering behaviour.

5

Dyonic Noncommutative

Instantons

In this chapter, we consider the effect on the dynamics of noncommutative instantons under the addition of a potential term. The ADHM construction in Chapter 3 demonstrated that the potential term does not remain unchanged after we consider a noncommutative space. We would expect, then, that the dynamics of such instanton solutions should change accordingly.

We first recall the form of the potential term (3.16) for two noncommutative U(2) instantons:

$$
\begin{aligned}
V = &\frac{1}{4}v^2\bigg(\rho_1^2 + \rho_2^2 - \frac{1}{2}\alpha^2\zeta^2 - 4\omega^2 \\
&+ \frac{2\omega^2(\rho_1^2 + \rho_2^2 + 4\omega^2 - 2\alpha\zeta)}{N_-} + \frac{2\omega^2(\rho_1^2 + \rho_2^2 + 4\omega^2 + 2\alpha\zeta)}{N_+}\bigg),
\end{aligned}
\tag{5.1}
$$

where $N_\pm \equiv 4\omega^2 + \rho_1^2 + \rho_2^2 \pm 2\alpha\zeta + \frac{1}{\omega^2}\rho_{1\pm}^2\rho_{2\pm}^2\sin^2\phi$ and $\rho_{i\pm} \equiv \sqrt{\rho_i^2 \pm \alpha\zeta}$. Consider first the extremal limit as noncommutativity becomes comparable to instanton size, that is $\alpha\zeta \sim \rho^2$, $N_+ \to 4\omega^2 + 4\rho^2 + 2\rho^2/\omega^2$ and $N_- \to 4\omega^2$. Then as $\omega \to 0$, we

see that the terms containing $N_\pm$ become negligible and the effect of the potential term on the full dynamics is dominated by the ρ_a^2 and $\alpha^2\zeta^2$ terms. Since the instanton sizes are bounded below by the noncommutativity parameter in any geodesic motion (as we demonstrated in Figure 4.3), the contribution from the ζ^2 term will also be subdominant. Conversely, as previously mentioned (and studied in [38]), the commutative limit gives a similar picture: both $N_\pm$ terms are subleading in the scattering limit. Hence any substantive effects of the introduction of noncommutativity are not to be found in straightforward scattering. Nevertheless, the difference between dyonic instantons and their regular counterparts may be seen in some aspects of scattering in a neighbourhood around $\omega = 0$, and we may consider those. Moreover, dyonic instantons may exhibit a feature not present in the free case: it is possible to 'tune' the latent repulsive force of the instantons and the attractive potential force to obtain stable orbiting solutions. We shall examine whether such solutions are an option in the noncommutative framework.

5.1 The attractiveness of noncommutativity

Before moving on to consider dyonic noncommutative instantons, we may analyse the effect of noncommutativity on the free instanton picture. We noted that the presence of non-zero ζ seems to introduce an attractive effect to the normal instanton scattering that, for sufficiently high ζ, overrides the normal repulsion of the two instantons. Then, before we concern ourselves with an additional potential force, we should investigate whether we may view the noncommutative effect as a genuine attractive effect. If so, then we would expect the transition to dyonic instantons to be unremarkable: the same scattering solutions will exist, but each solution will correspond to a two-parameter space spanned by the strength of the potential, v,

and the noncommutativity.

The clearest possible test of this is the following. We set our instantons at a finite distance apart such that in the commutative $v = 0$ case the repulsive behaviour is manifest. Sending the two initially at right angles to the line of separation, we would expect a deviation away from $\pi/2$ for a small time, until the instantons are suitably far away that interaction effects cease to dominate. We may then repeat this for some appreciably large value of ζ. The results are shown in Figure 5.1 for unit-size instantons and initial separation 0.9. Crucially, the separation is chosen such that the extent of the instantons initially overlap, and so interaction effects are the dominant initial contribution to the instanton dynamics.
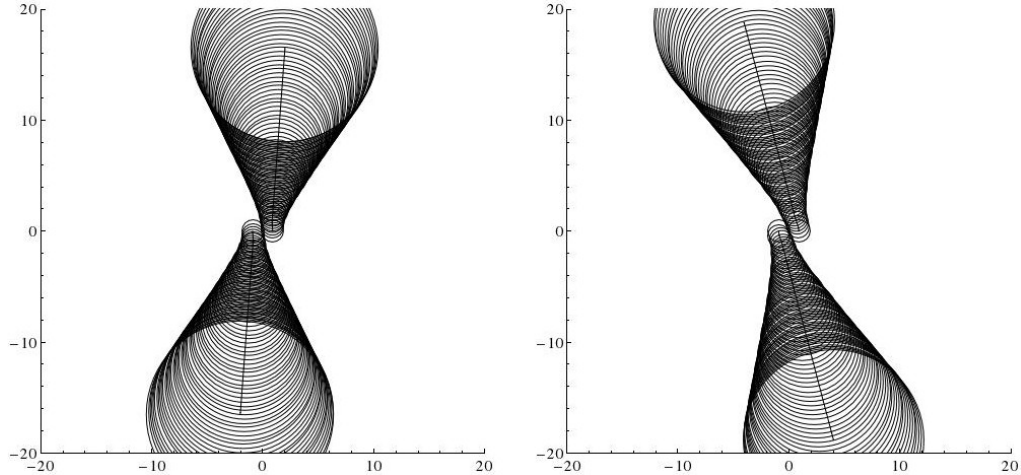
Figure 5.1: A demonstration of the attractive effect of noncommutativity: overlapping instantons with initial motion at an angle $\pi/2$ to the x-axis. On the left, for $\zeta = 0.05$. repulsive behaviour dominates short-scale interactions; on the right, for $\zeta = 0.3$, attractive behaviour is the key feature.

On the left hand side, with $\zeta = 0.05$, we observe the expected behaviour from the commutative case. The instantons temporarily repel, before maintaining a steady course. On the right hand side, for $\zeta = 0.3$, a very different picture emerges. Far from repelling, the short-distance behaviour is attractive, before the instantons separate too far for interaction effects to dominate. If the noncommutativity is

strong enough, then the tendency is for the instantons to come together rather than pull apart. Figure 5.2 shows the changing angle of exit with varying ζ for some values of the separation, where the crossover point between repulsion and interaction becomes clear.
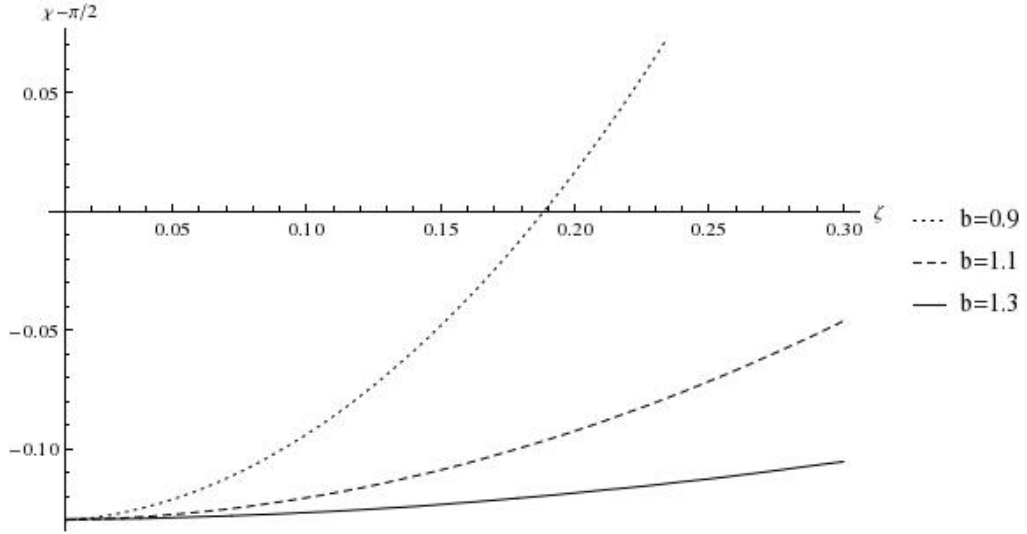

Figure 5.2: The attractive/repulsive interface for noncommutative instantons. For $\chi - \pi/2 < 0$, repulsion occurs. For suitably small initial separation, one can instead obtain $\chi - \pi/2 > 0$ (attraction).

Figure 5.2 also shows the sensitivity of such behaviour to the initial separation. We plot the value of the final scattering angle $\chi - \pi/2$ against ζ for a variety of impact parameters. When the plots remain below the x-axis, the instantons are scattering repulsively; when they cross the axis, this demonstrates the transition to attractive scattering. For large initial separation, the generic instanton repulsion is the only notable effect on the dynamics due to the subleading nature of the ζ modifications to the metric, and the crossover between repulsion and attraction is not evinced. Note that the trajectories of the plots suggests that the case $b = 1.1$ will eventually cross the transition point. However, the value of ζ at which it does so is outside the valid parameter regime for the geodesic approximation and therefore

cannot be considered to be a feature of the system. Nevertheless, it can be seen that the introduction of a noncommutative parameter to the moduli space can cause an attractive, rather than repulsive, effect.

The above strongly suggests that, for instantons initially positioned close together, the noncommutative effect is dominant and attractive. We now see the important effects of a noncommutative parameter on instanton scattering. Far from a simple modification to scattering angle, we may observe very different behaviours. For an initially small impact parameter, we may recover the standard results of soliton scattering. However, for off-centre scattering configurations, the presence of ζ can effect an attractive force between the two instantons, greatly modifying their scattering behaviour. We now turn on an actual potential force in the metric, and consider the twin effects of the two attractions.

5.2   The dyonic picture

Now we introduce a non-zero potential strength, v. The results of Section 5.1 were suggestive of a potential-like force on the moduli space arising from the noncommutativity. A potential of the form [37, 38] is also useful, however, as it may allow us to examine whether the slow-roll instability as $\rho \to \infty$ exists in the noncommutative case. It may also shed some light on the BPS spectra, via an analysis of the zeroes of the potential [24].

The results are shown in Figure 5.3. This demonstrates quite different characteristics: for relatively low potential strength, the instantons can attract and form a stable orbit (of which we will see more shortly), with the potential force and repulsive force balanced. Even if one breaks the Manton approximation by allowing $|v|^2 > 1$, the 'instanton' solutions attract so strongly that the configuration resembles that of

a head-on collision. There is no configuration that envinces attractive behaviour of the form seen in the pure (that is, static with $\mathcal{Q}_E = 0$) noncommutative instanton case.
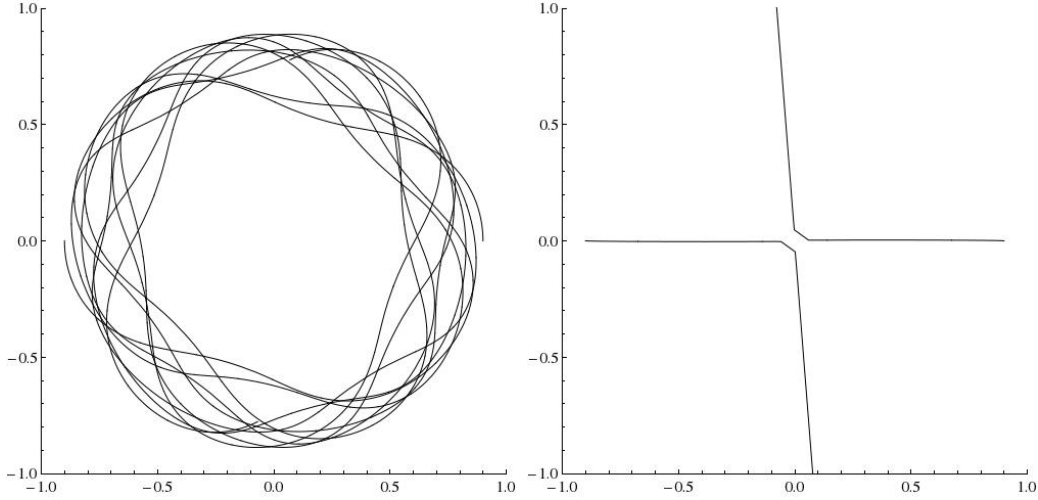
(a) *Orbiting from perpendicular instantons, for $v = 0.07$.*

(b) *Beyond Manton scattering: potential strength $v = 10$. Objects scatter as if propelled inwards to begin with.*

Figure 5.3: The attraction options for commutative dyonic 'instantons'. The instantons either attract and reside in a fixed orbit, or attract with such force that scattering occurs. No intermediate behaviour is demonstrated.

This is interesting, but perhaps not surprising. The dynamics of noncommutative instantons are resulting from purely geodesic motion: that is, any scattering effect arises due to the geometry of the moduli space. Since the key feature of the noncommutative moduli space is that the singularity at zero is smoothed out, stable valid geodesics exist that may pass arbitrarily near the origin, and so the instantons are more likely to stably orbit at small separation. This justifies the appearance of the 'slingshot' dynamics. In the dyonic commutative picture, the singularity at zero-separation remains, and the instantons are unable to replicate this behaviour. Any deviation from the geodesic motion effected by the potential or a velocity will not overcome the singularity at the origin.

This aside, we consider the available solutions under the influence of both ζ and v. In the search for interesting results, we ignore some regions of the parameter space: the addition of an attractive potential term is not going to change the scattering behaviour for small impact parameter. Rather, we will focus on the regions of parameter space where scattering did occur in the free noncommutative picture and analyse any changes that arise in those situations. In the following, we consider a range of initial impact parameters, $0.32 \leq b \leq 0.52$, and stipulate that the combined 'strength' of the noncommutativity and potential are fixed. Figure 5.4 shows the results for different partitions of $\zeta + v = 0.15$, where this partition and strength are chosen in order to demonstrate the salient qualitative behaviours.

In the first case, we consider pure noncommutativity. This is a familiar result: we have a modified scattering picture. As we dial down ζ and dial up v, we see very different behaviours. While ζ dominates, the pure noncommutative picture is still approximately valid; as the effect of the potential dominates, then scattering is guaranteed, albeit with the expected changes to the final scattering angle. Somewhere around the midpoint of this transition (demonstrated in Figure 5.4 for $\zeta = 0.05$ and $v = 0.1$), the behaviour becomes more interesting. A zoomed out version of this plot is shown in Figure 5.5, and shows the presence of unstable orbits even without the initial conditions chosen by [38].

One final point to make with regards to these results is that the qualitative difference between configurations with similar initial conditions can be considerable. The moduli space is incredibly sensitive to any adjustments to impact parameter and potential strength, in particular. This is not surprising: given the respective instabilities inherent in both the dyonic commutative and free noncommutative instanton configurations, a combination thereof allows for a greater range of unstable dynamical systems in the ζ-v parameter space. It is possible that even more exotic
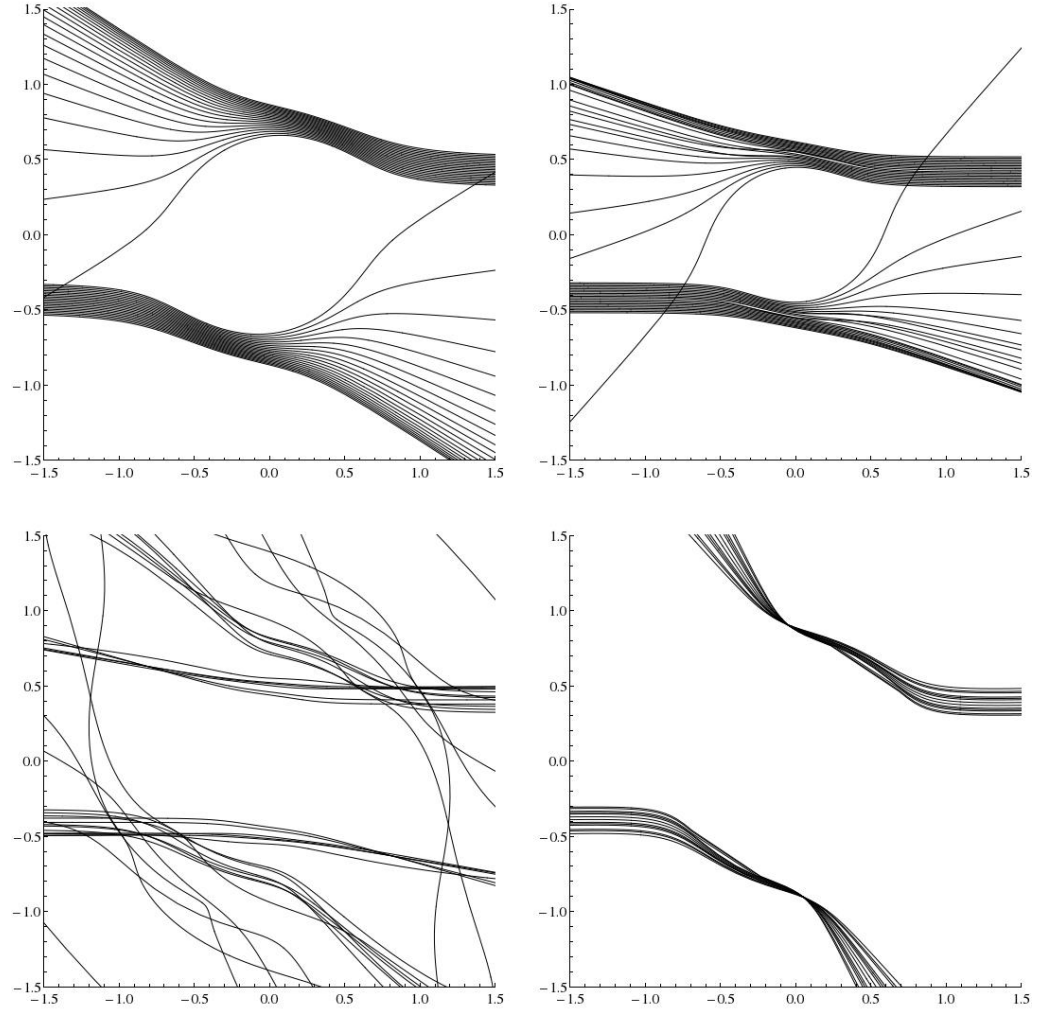
Figure 5.4: Dyonic noncommutative scattering for $\zeta + v = 0.15$. The free noncommutative result $\zeta = 0.15$ is shown in the top left, followed by $\zeta = 0.1$, $v = 0.05$; $\zeta = 0.05$, $v = 0.1$ and $v = 0.15$ (commutative dyonic) respectively.

behaviour can be demonstrated for careful tunings of the initial instanton configurations, but given the computational expense in undertaking a complete parameter search for the dyonic noncommutative instantons, we leave this consideration for further work.
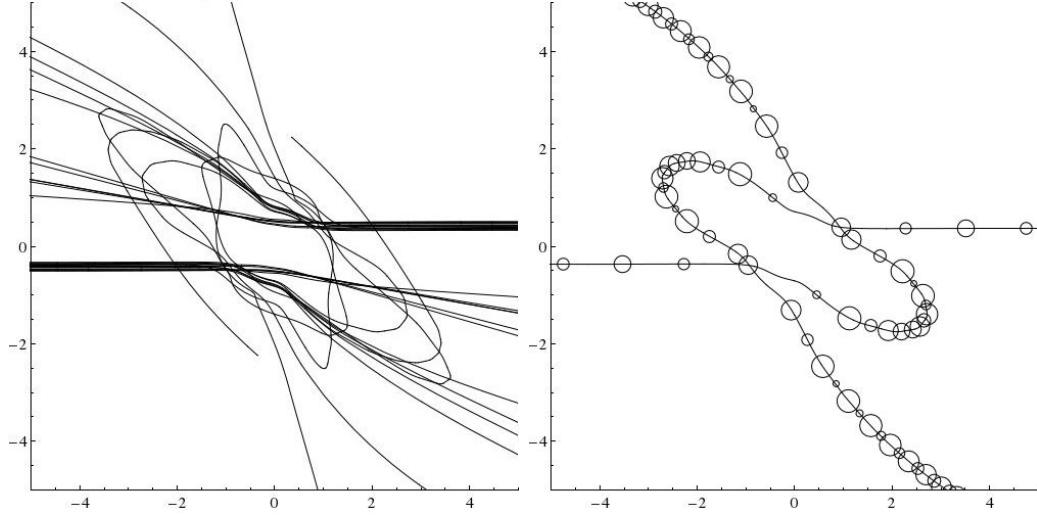
Figure 5.5: A zoomed out plot of the $\zeta = 0.05$, $v = 0.1$ configuration in Figure 5.4, and one particular unstable orbit from the initial plot with size oscillation shown.

5.3 Dyonic Orbits

Despite the observed instability of certain scattering scenarios as in Section 5.2, and the atypical behaviour of some 'non-scattering' situations as in Section 5.1, we may examine whether the stable orbits known to exist in the commutative picture remain in the noncommutative analogue. Such orbits existed at a point of equilibrium between the attractive and repulsive forces of the potential and the instanton effect, respectively. Given our previous considerations, it is not clear whether such a situation may be replicated for noncommutative instantons.

One key point in the search for such systems is that of longevity: the presence of a non-zero ζ has introduced the possibility of attraction and scattering for previously normal scattering scenarios, if ζ is large enough. This option is still possible if we start with a stable orbit and turn on noncommutativity, but the time taken to demonstrate the behaviour may be much longer. With this in mind, all numerical simulations run to investigate the possibility of orbits have been run for around 5 times longer than those in [38] to rule out eventual scattering. We again consider

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the interplay between the noncommutative parameter ζ and the strength of the potential v.

The first question is whether naïvely adding a non-zero ζ to previously known stable orbits affects the qualitative results. We take the stable orbit previously determined and turn on some amount of noncommutativity. The differences are shown in Figure 5.6 and Figure 5.7, where we record the evolution of the trajectories of the instantons and their sizes. In the commutative case, the instantons oscillate in a regular fashion, trading size as they sweep out an annulus in the moduli space. The maximum (minimum) combined size $\rho_1 + \rho_2$ is reached on the outer (inner) edge of the annulus, as one would expect from the 'free' scattering data. This orbit is stable, and exhibits no interesting features beyond those shown in Figure 5.6.
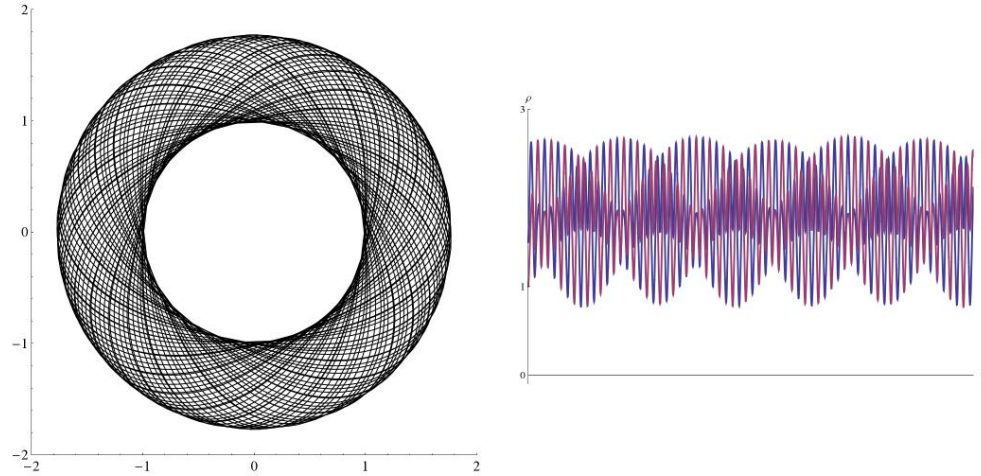
Figure 5.6: A commutative dyonic orbit. The sizes and separation are bounded above and below, due to the annulus swept out in the τ plane. The right-hand plot charts the sizes ρ_1 (blue) and ρ_2 (red) of each instanton as they orbit.

The noncommutative equivalent is less aesthetically pleasing, though it still exhibits a stable configuration. The instantons begin as in the commutative case (as can be seen most clearly in the size plots) before starting to trade sizes in an irregular fashion. This results in a more irregular orbit, but it remains stable for an
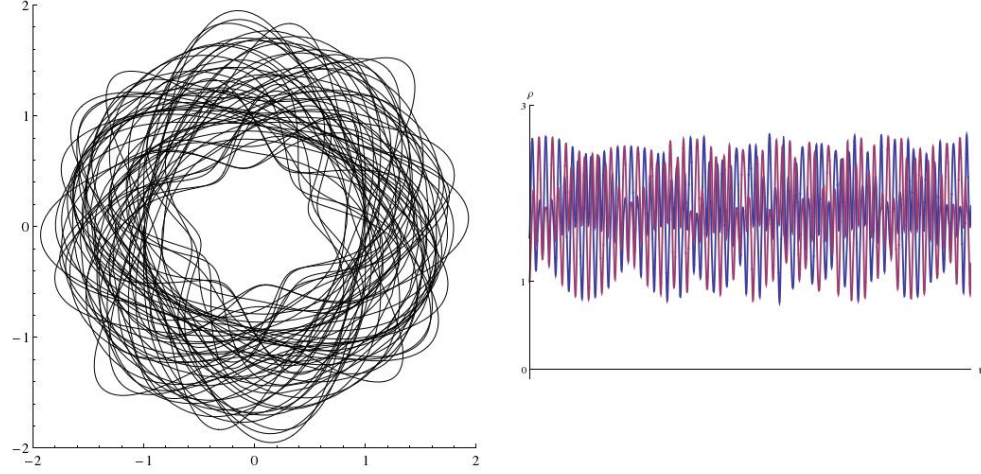
Figure 5.7: The same initial conditions as in Figure 5.6 with $\zeta = 0.1$. Again we also monitor the size evolution of the instantons. The separation and size behaviour is more chaotic.

indefinite period of time. The minimum distance between the two instantons is also reduced: this agrees with the results gained from the free case, where the removal of the singularity in the moduli space allows for the instantons to comfortably reside in more tightly bound configurations.

The difference in allowed sizes is worthy of discussion. Whereas in the free instanton case, the introduction of noncommutativity placed a lower bound on the instanton size (as anticipated in the avoidance of the zero-size singularity), we see that in the dyonic case the noncommutativity reduces the lower bound of the instanton size in stable orbits. This is demonstrated by Figure 5.8, where we see that the sizes of the noncommutative instantons diverges from that of the commutative case as they orbit. This, too, is the expected result: in the commutative theory the potential is introduced 'by hand' to prevent the zero-size singularity, and so ζ need not play the role of stabiliser. Instead, for stable orbits where no scattering, and hence zero separation, occurs, the presence of the attractive ζ-force allows tighter orbits, which cause the bound on minimum size to be reduced. One may consider

taking this to its logical conclusion: for sufficiently strong noncommutativity (potentially outside the Manton limit), the bound on instanton size could hit zero, but this would occur in configurations where the attractive effect causes scattering of the instantons. Hence the already established 'free' size constraint would take effect in this limit, and the zero-size singularity will not play a part.
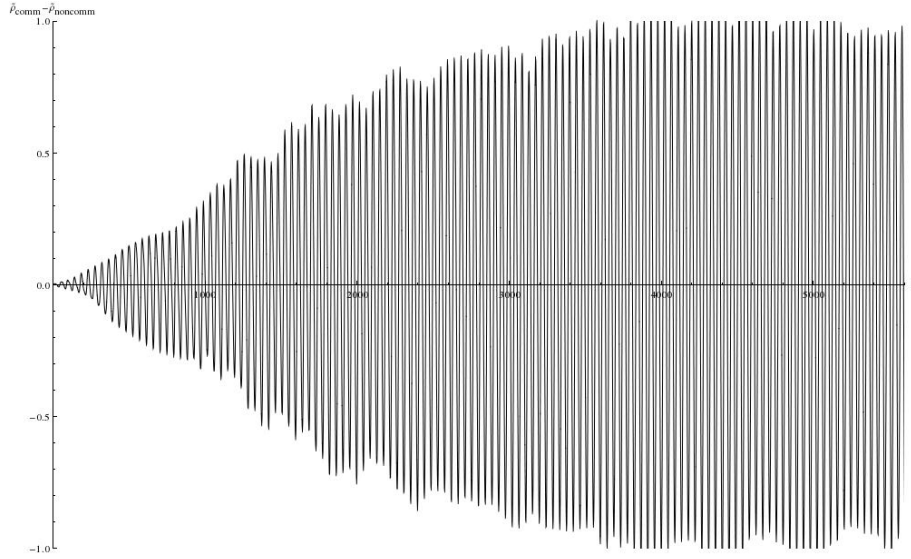

Figure 5.8: The difference in average size, $\tilde{\rho} = (\rho_1 + \rho_2)/2$, in the commutative and noncommutative orbiting configurations. Due to the greater range of allowed separation in the noncommutative picture, the sizes correspondingly oscillate over a greater range of sizes until stabilising.

Of course, this behaviour should not be assumed to be a generic feature of noncommutatively deformed orbits. As in Section 5.2, we may choose to maintain the combined effect of noncommutativity and potential, and consider the interplay between the two parameters. Figure 5.9 demonstrates the two possible options for instability.

These results underline the variety of dynamical outcomes that may occur due to the presence of the additional parameter ζ. It would be overweening to suggest that the set of results above is exhaustive: it is feasible to imagine some carefully

(a) *Long-lived unstable orbit for $\zeta = 0.1$, $v = 0.2$.*

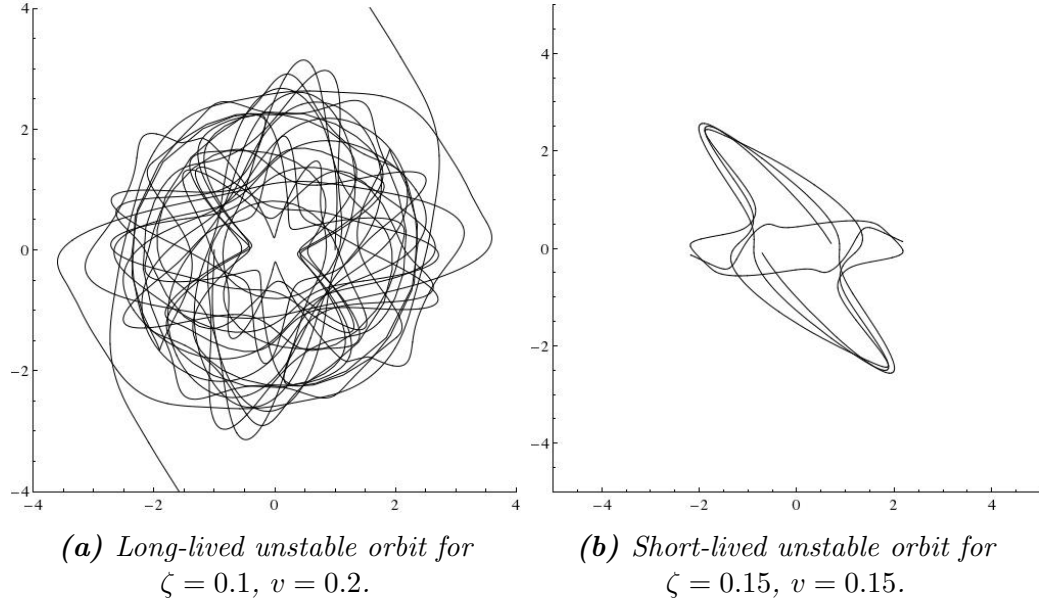**(b)** *Short-lived unstable orbit for $\zeta = 0.15$, $v = 0.15$.*

Figure 5.9: Unstable orbit evolution. Depending on the initial conditions, the configuration can degenerate from a 'stable' orbit to a scattering scenario in vastly different timescales.

tuned system that undergoes orbit, scattering and reorbit. However, the vastness of the parameter space, coupled with the computational intensity of the numerical simulations, makes a full characterisation of the space unwieldy.

This concludes our discussion of dyonic noncommutative instantons. We have seen that we may view the noncommutative effect as an attractive force on the moduli space, and the addition of a potential creates a number of interesting possible scenarios for interaction. In the commutative case, the scattering behaviour was modified only slightly in the final scattering angle χ_{final} by the addition of a potential. With the inclusion of noncommutativity, as in the 'free' noncommutative instanton case we may have a modifed scattering picture, which persists regardless of the strength of the potential. This may also occur in the previously stable orbiting configurations, where the effect of non-zero ζ can cause instability in the orbiting behaviour.

6

Partial Results and Future Directions

In this chapter, we present a set of partial results to open questions on the study of instantons. There are many such interesting directions that could be taken in extending the work done here: we highlight those that should be tractable while possessing merit in terms of the present research in the field.

6.1 Non-Abelian vortices from dyonic instantons

We previously examined the reduction of the instanton moduli space to the $(2+1)$-dimensional non-Abelian vortex theory in Section 4.4, where the noncommutative parameter in the instanton picture translates into a gauge coupling between the $U(1)$ and $SU(2)$ gauge groups. We observed that there appeared to be dynamical vortex configurations that could not be reproduced in the instanton model: the solution of the vortex equations allows an additional Fayet-Iliopoulos parameter that finds no apparent analogue in the higher dimensional theory. Moreover, there

is no straightforward correspondence between a potential in the vortex theory (where there are no symmetry constraints or ansatz of the form (2.41)) and the 'canonical' potential term. However, one would anticipate that a Higgs field can be chosen in the vortex theory that would correspond to the dimensional reduction of our potential term V in the instanton theory. We may then consider the behaviours we would expect in the vortex theory from such a reduction.

Work on dyonic non-Abelian vortices [53] demonstrated that a vortex theory with some non-zero Higgs field $\langle \phi \rangle = \mathrm{diag}(m_1, \ldots, m_N)$ modified the Bogomolny equations and gave a stricter Bogomolny bound on minimum energy states:

$$E_{\text{vortex}} \geq |2\pi^2 k v^2| + \left| \sum Q_i m_i \right|.$$

Contrast this with the result for dyonic instantons:

$$E_{\text{instanton}} \geq |2\pi^2 k| + |\mathcal{Q}_E|.$$

It seems feasible that the two theories may be equivalent under dimensional reduction, with some identification of the electric charge, $\mathcal{Q}_E$, in the instanton theory with the additional mass term, $\sum m_i Q_i$, in the vortex theory. We leave this identification for future work.

In [53, 54], a number of dynamical vortex considerations were considered explicitly: while we lack the tools to do the same here, we nevertheless consider the naïve reduction of the dyonic instanton to the 2+1-dimensional theory. The results gained thus far are interesting and hint at a deeper relationship between the dyonic instantons and vortices. Figure 6.1 shows the typical scattering characteristics, plotted in the same manner as Figure 4.11. We note that there is a general trend towards

attraction: the behaviour is suggestive of (though not as strong as) the results in [54] when the vortices are subject to a maximised non-Abelian effect. This is a curious result since, as already mentioned in Section 4.4, the vortices possess additional degrees of freedom in the parameter space that the instanton picture seems to neglect.
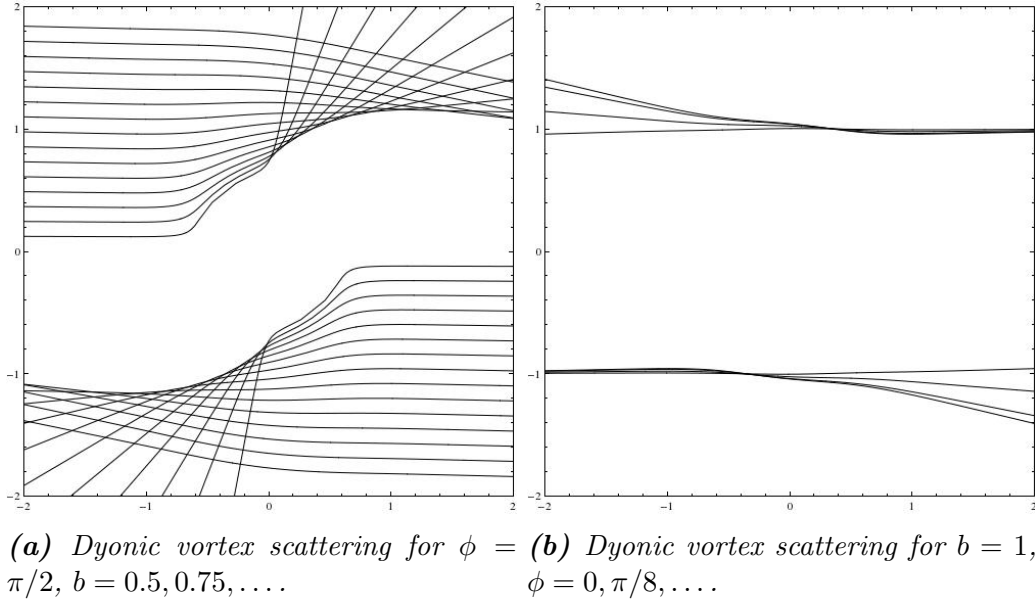

(a) *Dyonic vortex scattering for $\phi = \pi/2$, $b = 0.5, 0.75, \ldots$.*

(b) *Dyonic vortex scattering for $b = 1$, $\phi = 0, \pi/8, \ldots$.*

Figure 6.1: 'Dyonic' vortices from a reduction of the instanton moduli space. We note the attractive behaviour, reminiscent of the behaviour of maximally non-Abelian dyonic vortices.

It would be interesting to examine the parameter space of the reduced instanton moduli space more closely, particularly with regards to the induced charge arising from the potential term. It is possible that there exists, within the bounds of the Manton approximation, a configuration that produces the maximally non-Abelian behaviour known to occur in [54]. It is especially interesting that the dyonic potential in the instanton theory seems to result in a modification to the non-Abelian effect, rather than a potential in the vortex theory. This strange result merits further study.

6.2 Index Counting From Dyonic Instantons

In this section, we discuss the index counting mechanism that connects the instanton picture to that of string theory and, eventually, M-theory. It is very much unclear what the full picture is in this case, but we present the results gained so far. The index counting mechanism, as we shall see, allows one to count the difference of fermionic and bosonic BPS states in the SYM quantum mechanics via the zeroes of the dyonic instanton potential. Due to the difficulty of producing a full parametrisation of the instanton moduli space, we may not systematically describe and categorise the space of zeroes possessed by the U(2) noncommutative 2-instanton potential. We may, however, make some general comments as to the form of some of the zeroes of the potential. In particular, we shall see that, unlike in the commutative case, a class of such zeroes follow a broadly similar form to that of [24]. We first describe the SUSY QM on which the index is defined, and henceforth follow the conventions of [20].

The Lagrangian of a SUSY QM system with 8 supercharges is given as follows:

$$
\mathcal{L} = \frac{1}{2}\Big(g_{mn}\dot{z}^m\dot{z}^n + ig_{mn}\bar{\psi}^m\gamma^0 D_t\psi^n + \frac{1}{6}R_{mnpq}\bar{\psi}^m\psi^n\bar{\psi}^p\psi^q \\
- g_{mn}G_I^m G_I^n - i\nabla_m G_{in}\bar{\psi}^m(\Omega^I\psi)^n \Big),
$$

(6.1)

where ψ^m is a two-component Majorana spinor, $\gamma^0 = \sigma_2$, $\gamma^1 = i\sigma_1$, $\gamma^2 = -i\sigma_3$, and $\bar{\psi} = \psi^T\gamma^0$. The Ω are defined as $\Omega_4 = \delta_n^m\gamma_{\alpha\beta}^1$, $\Omega_5 = \delta_n^m\gamma_{\alpha\beta}^2$ and $\Omega_s = iJ_n^{(s)m}\delta_{\alpha\beta}$ for $s = 1, 2, 3$. The manifold on which the theory is defined must be hyperKähler, so that there exist three covariantly constant complex structures $J^{(s)}$ satisfying

$$
J^{(s)}J^{(t)} = -\delta^{st} + \epsilon^{stu}J^{(u)},
$$

and the Killing vector field G should be triholomorphic:

$$\mathcal{L}_G g = 0, \quad \mathcal{L}_g J^{(s)} = 0.$$

Defining $\varphi \equiv (\psi_1^m - i\psi_2^m)/\sqrt{2}$ and $Q \equiv (Q_1 - iQ_2)/\sqrt{2}$, where Q_α are the associated supercharges, then we may define Z_2 gradings on this theory:

$$\tau_\pm \equiv \prod(\sqrt{i} e_m^E \varphi^m \pm \sqrt{-i} e_m^E \varphi^{*m}), \tag{6.2}$$

where e_m^E is the vielbein of the manifold. These operators are only non-zero in the event that a scalar field is turned on in the theory (which without loss of generality we may choose to be G^5), and they, along with the corresponding Dirac operators $D_\pm \equiv iQ \pm Q^\dagger$, define Witten indices $\mathcal{I}_\pm$ counting the number of BPS states for a given central charge. This index is, a priori, non-trivial to calculate from the field theory, but given the non-zero scalar field we may use properties of the index to relate this calculation to a tractable one.

Provided care is taken when choosing the definition of the Hilbert space, Dirac operators $D_\pm$ and involutions $\tau_\pm$, we may utilise a scaling argument to restrict the points of contribution to the index. Consider a space $L_\pm$ as the restriction of the space to eigenvalues ± 1 of D, and Hilbert spaces E_1 and E_2 as the spaces of sections of line bundles with finite graph and L_2 norms, respectively. Then it can be shown that, including a deformation of the metric, that the Dirac operators $D_\pm$ are Fredholm on the appropriate spaces and there exist a continuous family of such operators related by quasiisometries acting on $D_\pm$ [20]. Then the index calculation is unchanged by such deformations. In practial terms, such quasiisometries and metric scalings allow us to scale the potential term, arising from $G^5 \neq 0$, such that

the contributions to the index are localised around the zeroes of V (for details, and a proof of this scaling argument, one may consult [23, 20, 24]).

This result allows an entry point into the index calculation of the QM theory via the theory of instantons. We note that, by construction, the instanton moduli space manifold is automatically hyperKähler, and the potential naturally triholomorphic on the space. Then we may scale the potential (arising from non-zero $\langle \phi \rangle = X^5$) such that the index calculation is localised around its zeroes. This has utility in computing the index of complicated SUSY quantum mechanical systems, where the index calculation from the field theory side is non-trivial. This method was used in [24] to calculate the index of the single noncommutative U(N) instanton, and agreement was found between that calculation and the corresponding field theory index.

In practical terms, it is not straightforward to extend this to the 2-instanton case. The problems discovered in Section 3.2 mean that it is not easy to examine a fully-parametrised potential term in the 16-dimensional moduli space. However, it is straightforward to see that the method of calculating the potential in [38] remains valid in the noncommutative case, albeit with the caveats previously discussed when deriving the noncommutative metric. Explicitly, the potential term takes the form

$$V \sim v^2 \left(w_1^\dagger w_1 + w_2^\dagger w_2 - (w_1^\dagger w_1 + w_2^\dagger w_2 + 4(\bar{\tau}\tau + s^\dagger s))^{-1} |w_2^\dagger \hat{q} w_1 - w_1^\dagger \hat{q} w_2|^2 \right), \quad (6.3)$$

where the quaternion q has magnitude $|q| \equiv v$. We now wish to analyse any possible zeroes in both the commutative and noncommutative cases.

The first, simplest, option is when $w_1^\dagger w_1 = w_2^\dagger w_2 = 0$. This yields a set of zeroes over the 4-dimensional τ space. While, initially, it may appear that the index calculation would be ill-defined, we note that the quasiisometries allow us

to reduce the τ plane to a point, and therefore the calculation is regular. In the commutative case, this corresponds to $|v_1|^2 = |v_2|^2 = 0$ which has a single zero, namely the zero-size singularity, which is known in the field theory to correspond to a non-renormalisable field theory. In the noncommutative case, we recall that the additional U(1) symmetry means that the expression $w_i^\dagger w_i$ produces terms of the form

$$\rho_i^2 - \frac{\alpha^2 \zeta^2}{\rho_i^2} \cos^2 \theta_i,$$

where θ_i is the parameter corresponding to the e_3 direction. For generic non-zero ζ, and up to scalings of τ, we have two zeroes for each instanton v_i: the north and south poles of the circle $\rho_i^2 = \alpha \zeta$ [1]. Hence, in the noncommutative case, we obtain 4 zeroes in this sector. This is not surprising: the single instanton calculation in [24] found 2 zeroes in a similar vein. The system under consideration here is one in which both instantons vanish independently, and so one would expect to obtain the equivalent solution for two well-separated instantons.

The second option is one in which the two terms are of equal magnitude: explicitly,

$$w_1^\dagger w_1 + w_2^\dagger w_2 = \left(w_1^\dagger w_1 + w_2^\dagger w_2 + 4(\bar{\tau}\tau + s^\dagger s))^{-1} |w_2^\dagger \hat{q} w_1 - w_1^\dagger \hat{q} w_2|^2 \right). \tag{6.4}$$

This option is not present in the single instanton case. The general solution to this requirement is not clear, but we may see that one class of zeroes occur due to the possibility of bound 2-instanton states in a small-separation regime. In this regime, we see that the second term in the potential simplifies considerably. It can be seen

---

[1]We note, as in Section 3.4, that naïvely it may be possible for $\alpha \to 0$ when $\tau \to 0$, giving the zero corresponding to the zero-size singularity. However, we note as before that in this limit, the system is more adequately parametrised by $\tau \to \frac{1}{\sqrt{2}}(\tau + \sigma)$, and α remains non-vanishing

quickly that zeroes of the potential in this regime are given by

$$w_2 = \hat{q}w_1.$$

This gives a group of zeroes parametrised by the free variables in w_1 and the direction quaternion, $\hat{q}$. In any such theory under consideration, the quaternion $\hat{q}$ is fixed by the symmetry requirements of the dyonic potential term and so does not correspond to an additional class of zeroes. However, again we need consider the symmetries of the instanton metric: this set of zeroes is equivalent to $w_1 = w_2$ up to a global U(2) rotation of the data, and then from the field theory view this corresponds to a quasiisometrical deformation of the index. There is then, in both the commutative and noncommutative cases, only one interesting possibility for zeroes in the small-separation limit in this sector, which is represented in the instanton system by an axially symmetric 2-instanton bound state.

This is by no means an exhaustive analysis of the possible zeroes of the potential in the 2-instanton case: it may be possible to satisfy the vanishing of V for other, more general, configurations where neither the well-separated or small-separation limits are attained. However, we simply wish to demonstrate that the well-known results from the single instanton analysis are recovered in the noncommutative 2-instanton picture, along with additional interesting contributions to the index that occur as a result of 'interacting' instantons. It would be interesting to extend this analysis, and complete the index calculation in the manner of [23, 24] as a means of understanding more about the underlying field theory. We note that, given the imposition of a U(2) gauge group in our work, all contributions to the index are weighted by the same factor $v \equiv |v_2 - v_1|$, so after a complete characterisation of the class of zeroes of the potential, the index calculation should follow straightforwardly.

This would not be the case were one to extend the analysis to, say, noncommutative SU(3) instantons. Finally, it was mentioned in [24] that the single U(N) instanton index calculation possessed N^3 scaling degrees of freedom: this is indicative of the connection to M-theory, where the same scaling appears to be observed for a stack of N M5-branes. More work on this calculation would be useful.

6.3 SU(3) Instantons

Finally, we approach the question of a larger gauge group, U(N). The cases where N is even have long been known to be tractable [55], as one may maintain the quaternionic nature of the entries in the ADHM matrix Δ. It is less clear, however, on how to proceed in the case of odd gauge group. A set of solutions for SU(3) instantons possessing cylindrical symmetry is well-established [56], but a general solution has been elusive. Such a general solution for gauge group SU(3) would be illuminating, as it may give an indication on the most general method for generating such instanton solutions.

Concretely, the SU(3) ADHM data, Δ, is a 7×4 matrix of complex entries satisfying the usual ADHM constraints. We note that the only adjustment to the dimension of the induced moduli space comes from the upper block, L, and so it is still possible to view the lower block of Δ as in quaternionic form. In the spirit of the U(2) noncommutative deformation we have presented, we seek to introduce a deformation, P, to the v_i such that $v_i^\dagger v_j$ is unchanged for $i, j = 1, 2$. This would guarantee satisfaction of the ADHM constraints in the same way as in the SU(2) case. In particular, this means that

$$P^\dagger P = 1_2.$$

For an indication of how the solution may proceed, we write P as $P = (z_1, z_2)$ for $z_i \in \mathbb{C}^3$. Then we require z_1 and z_2 be orthogonal unit vectors. One may verify that the parameter space is equivalent to an S^3 fibration over S^5, and comprises 8 free variables. Given that the moduli space of 2 SU(3) instantons should contain 24 independent variables, it is not unreasonable to aver that such a deformation could encompass all the additional ADHM parameters arising from a consideration of gauge group SU(3) rather than SU(2). The calculation of the metric on the moduli space would follow in a broadly similar manner to that presented here and in [38], with the notable caveat that the matrix R in the gauge redundancies is now O(3), rather than O(2) (in the commutative case) or U(2) (in the noncommutative case) [46].

The key to achieving anything with such an approach would be to find a useful parametrisation of this data. We may note that there are a number of straightforward examples of such matrices, corresponding to the three canonical independent subgroups of SU(2) in SU(3):

$$P_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad P_\beta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Choosing any of these P, along with a general SU(2) gauge transformation, would give all possible embeddings of the SU(2) instanton data into SU(3). it is unlikely, however, that this would give the whole story. A general parametrisation, for use in the ADHM data, is not easily obtainable. If such a parametrisation could be found that allowed for tractable computations of $\mathrm{d}w_i$, the metric and dynamics would be relatively simple to calculate. It is interesting that the above considerations implic-

itly transform each 'instanton' datum v_i by the same matrix. The fact that there are 8 free parameters, exactly in line with the expectations of an SU(3) solution, suggests that the relative gauge orientations of the two SU(3) instantons should be adequately demonstrated in a good parametrisation of the matrix P. Alternatively, it is possible that, as in the SU(2) case, we are looking only at some relative gauge angle sums and differences, and that a reparametrisation of the matrix P would shed some light on this result.

Such a consideration would be valuable in the context of the connection to D-branes. In the U(2) sector, we are implicitly considering just 2 D-branes and so any string configurations must stretch between the two, or be localised on one. The lift to SU(3) would allow for a richer class of possible string configurations, including those where a bound state passes through a D-brane. In particular, an index calculation for such a theory could have profound effects on the understanding of the M5-brane.

As a final point, the noncommutative extension of the SU(3) instantons may be straightforward, for suitable choices of the deformation matrix P. Metrical complexities aside, with an understanding of the noncommutative deformation and the even to odd gauge group deformation, it may be possible to generalise to arbitrary SU(N) and U(N) for any gauge group.

In this chapter, we have outlined a number of reasonable extensions to the present work, for which we may make some meaningful comments. The qualitative difference under the addition of a potential in the vortex picture appears more pronounced than that of the pure instanton case, where the changed behaviour is suggestive of the maximally non-Abelian vortex picture shown in [54]. This is broadly independent of the strength of noncommutativity (at least, when considering parameters within the Manton range), and would benefit from further study. We have outlined some of the key differences between the index calculations for single and multiple

instantons, where new zeroes of the Higgs field appear as a result of multi-instanton bound states. A result of this kidney was expected, but there are still a number of unanswered questions about the full set of zeroes for 2-instanton systems, and this remains a potentially fertile area of examination. Finally, we have given some indication of an avenue of exploration with regards to SU(3) commutative instantons, whence interesting string theoretical behaviours could spring. There are always more aspects of the instanton theory that can be considered. The relevance of instantons to a large number of currently relevant topics in mathematics and physics will always mean that certain aspects of the theory have been neglected. However, the above considerations suggest that some interesting things can be said about a number of different aspects of string theory and solitons given the work done here, and that the research is feasible.

7

Conclusions and Outlook

In this work, we have examined the construction, properties and behaviour of 2 noncommutative dyonic instantons arising as minimum energy, static solutions to a U(2) Yang-Mills theory. The ADHM procedure allows us to translate the BPS equations for such a theory, comprising the requirement of self-duality of the fields, into a purely algebraic set of constraints on allowed instanton data. The remaining distinct instanton solutions, taking into account any gauge redundancies, furnish a 16-dimensional 'moduli space', upon which we may embed a metric. By definition, this metric is hyperKähler and admits a triholomorphic Killing vector, which we identify with a potential term on the moduli space.

The added complexity stemming from considering an underlying noncommutative $\mathbb{R}^4$ means that an illuminating, explicit form for the metric is presently beyond our reach. However, in the noncommutative picture there still remain valid geodesic symmetries of the moduli space metric corresponding to non-singular fixed points of the full metric. This allows (after discounting the 4 centre of mass coordinates in the 16d space) a consideration of a 6-dimensional submanifold, upon which scattering and dynamics can be analysed via the Manton approximation of slow-moving

instantons. The dynamics of noncommutative instantons can behave in a very different way to their commutative counterparts: stable bound states may exist due to the presence of the non-zero noncommutative parameter ζ as a stabiliser for the minimum size of the instantons. Consequently, the moduli space is free of the so-called 'small instanton' singularity and in the limit of large separation, one obtains two copies of the Eguchi-Hanson metric. This is the only available 4-dimensional hyperKähler manifold admitting a triholomorphic Killing vector, and so the result here agrees with heuristic arguments about the nature of the 2-instanton noncommutative moduli space as well as the results demonstrated in [35, 24].

The scattering, in general, of noncommutative instantons displays some intriguing traits beyond that of the stable bound states. Unlike in the commutative case, where right-angled scattering is the outcome of a head-on collision, this is far from the only possibility in the presence of noncommutative space. One may, for some value of ζ, obtain a variety of scattering outcomes, and the angle of scattering for instantons is no longer fixed by the choice of impact parameter. Indeed, one may find that for interacting instanton configurations, a transition between two different scattering regimes persists as one increases the scattering offset between the instantons. Of course, in the limit as $\zeta \to 0$, the standard commutative scattering behaviour is reproduced in all cases.

The geodesic submanifold allowed by the introduction of noncommutativity readily lends itself to an identification with vortices: 2+1-dimensional solitons of a U(N) theory related to instantons via a dimensional reduction in two of the spatial directions. A large body of work has been dedicated to the study of non-Abelian vortices, and the noncommutativity parameter in the instanton theory descends to determine the coupling strength between the SU(N) and U(1) factors of the vortex [27, 28, 26, 50, 52, 25]. Aspects of the analysis of vortex dynamics in [54] can be

reproduced in the instanton model, as we have shown. However, the full picture of vortex scattering is unlikely to stem purely from the 'free' noncommutative instanton: in particular, the maximally non-Abelian case is beyond the reach of tuning of ζ in the instanton theory. However, we have also seen that the dimensional reduction of dyonic noncommutative instantons displays traits indicative of the maximally non-Abelian results, and more study on this, in conjunction with the works of, e.g., [53] on dyonic non-Abelian vortices may allow more concrete statements to be made about the correspondence.

Links to vortices notwithstanding, the transition to dyonic noncommutative instantons evinces further interesting deviations from the commutative picture. Configurations that resulted in scattering in the commutative case need not remain for $\zeta \neq 0$; orbiting configurations (where the attractive potential strength balances the repulsive force of the instantons) need not be stable. Conversely, one can find a new class of stable orbits for the noncommutative case, where the (admittedly more chaotic) orbits can reside in a smaller stable orbit. This is not general behaviour: one may have quasi-stable orbits where the instantons scatter after a finite orbiting time. This option was not as common in the commutative picture.

Finally, one may use the instanton potential to make some comments about the BPS index calculation of the underlying $\mathcal{N} = 4$ quantum mechanical theory. One may use a well-understood scaling argument [20] to deduce that any non-trivial contributions to the index are localised around the zeroes of the dyonic potential. While the complexity of the 16-dimensional moduli space (and hence potential) calculation mean that we have not been able to fully categorise all possible zeroes of the $k = 2$ U(2) dyonic potential, it can be seen that alongside zeroes of the form given in [24] for each individual instanton, there exist zeroes of the potential corresponding to a charge 2 single instanton bound state which will non-trivially

affect any index calculation.

There are, of course, a number of open questions that would benefit from greater analysis. The first, and most important, is the need to find a definite parametrisation of the full, 16-dimensional, moduli space for the noncommutative instantons. This would allow a full analysis of the symmetries of the moduli space, and would allow us to make a concrete identification between the space and some 16-dimensional hyperKähler metric. We have demonstrated a few key features that this full moduli space should possess: it remains to find and verify such properties once a parametrisation can be determined. Another question that the full moduli space would be able to shed light on is the subject of geodesic completeness. In the commutative case, it is clear that the moduli space is not geodesically complete, as there are instanton configurations that can reach the singular point at the origin. In the noncommutative framework, it has been demonstrated [6, 34] that the singularities present for $\zeta = 0$ are resolved, which is suggestive (but not proof) that the moduli space of noncommutative instantons are geodesically complete. An explicit, 16-dimensional, description of the $k = 2$ U(2) noncommutative instanton moduli space should lend more evidence to the assertions about completeness. The complexification of the moduli space also forced us to consider the two instantons where their internal gauge orientations differed by some U(1) factor in the full U(2). It would be interesting to observe the allowed dynamics when the instantons possess relative angles valued in the full U(2) theory. Similarly, a full understanding of the metric on this space would yield a parametrised form of the dyonic potential for two noncommutative instantons: a classification of all possible zeroes of the potential would be crucial to an understanding of the index calculation, and associated scaling degrees of freedom, of the SQM theories. Given the relationship between such theories and M-theory, this would have far-reaching utility in understanding the M5-brane. As

a final point on the moduli space approximation, it would be useful to compare the results of the Manton-approximated system against those of the full Yang-Mills field theory. This is a very computationally intensive aim, due to the complexity of the noncommutative U(2) SYM theory, and remains out of reach at present. It would be interesting as to whether designing a module for computing the full metric in C or Python to deal with quaternions explicitly would be more efficient from a calculational standpoint: work on this is in progress.

Given the known connections between instantons and other, lower-dimensional solitonic theories, there are a number of different reductions one should be able to make given the results presented here aside from the previously considered vortex theory. A similar story to the vortices emerges in the case of monopoles. In fact, monopoles also appear on the string theoretical side as bound states between fundamental strings and D0-branes [57], so an identification between instantons and monopoles may be more readily verified on both the soliton and field theory sides. A configuration of commutative circle-invariant instantons can be seen, under dimensional reduction, to correspond to hyperbolic monopoles with certain properties [30]: whether this connection can be realised in a noncommutative context remains to be seen. Similarly, it has been seen that well-separated monopoles obey Kepler-like laws, corresponding to some central force governing their behaviour [58]. It would not be surprising if this were also true for instantons, and should be a relatively straightforward way to verify whether the instanton-monopole identification persists with $\zeta \neq 0$.

As a final point, the clearest extension of this work would be to increase the size of either the gauge group of the number of instantons in the theory. While there has been a large amount of work done on the properties of commutative SU($2N$) instantons (where the ADHM data naturally admits a parametrisation in terms of

quaternions), the noncommutative analogue has not been studied for $k > 1$. In these theories, it would not be surprising to see a similar noncommutative deformation to the one presented in this work, and the only barrier to progress would be that of computational complexity. Instanton theories for odd gauge group are even less well-understood, due to the awkwardness of parametrising the ADHM data. In particular, the commutative and noncommutative SU(3) 2-instanton cases would be interesting, as from the D-brane picture, after Higgsing the D4-branes, these could represent configurations where the fundamental strings do not begin and end on the same brane. This could give an eventual insight, via index calculations, into the strange scaling behaviour of M-theory, where the scaling of allowed states of M2-branes ending on M5-branes is N^3 rather than the expected N^2. The ADHM data, however, is much more complicated and a illustrative parametrisation has thus far been elusive. A possible avenue of exploration is, in the spirit of the noncommutative deformation, to find a matrix, containing the additional moduli space parameters, that deforms the v_1 and v_2 of the SU(2) instanton configurations without changing the satisfaction of the ADHM constraints. Such a deformation exists: the next stage will be to find an expedient parametrisation of the deformation to make clear the effect of the SU(3) gauge group. Finally, an extension to greater instanton charge k may allow scattering of, say, two 2-instanton bound states. At first glance, this would appear to be a reasonable extension, as the data readily admits a quaternionic parametrisation and the ADHM constraints may still take similar form, albeit with more quaternion entries. The case of $k = 3$ would be the obvious starting point in any such endeavour.

Appendix A

Supplementary Calculations

A.1 The Dyonic Potential Constraint

We seek to turn the background field equation of Yang-Mills for a scalar field Φ, $D^2\Phi = 0$, into an algebraic constraint in terms of the ADHM data. We use the ansatz

$$\Phi = iU^\dagger \mathcal{A}U, \quad \mathcal{A} = \begin{pmatrix} q & 0 \\ 0 & P \end{pmatrix},$$

where q is a generic quaternion and $P \in \mathrm{U}(2)$. Crucially for what follows, $\mathcal{A}$ is x-independent. Using the form of the gauge covariant derivative and the expression for A_i in terms of the ADHM data, $A_i = iU^\dagger \partial_i U$, along with the following identities:

$$\partial_i U^\dagger U = -U^\dagger \partial_i U$$

$$\partial_i \Delta^\dagger U = \Delta^\dagger \partial_i U$$

$$UU^\dagger = 1 - \Delta f \Delta^\dagger \qquad\qquad \text{(A.1.1)}$$

$$\partial_i \Delta = -be_i$$

and their Hermitian conjugates, one obtains

$$D_i\Phi = \partial_i(iU^\dagger\mathcal{A}U) - i[iU^\dagger\partial_iU, \, iU^\dagger\mathcal{A}U]$$

$$= i\partial_iU^\dagger\mathcal{A}U + iU^\dagger\mathcal{A}\partial_iU - i\partial_iU^\dagger UU^\dagger\mathcal{A}U - iU^\dagger\mathcal{A}UU^\dagger\partial_iU$$

$$= i\partial_iU^\dagger\mathcal{A}U + iU^\dagger\mathcal{A}\partial_iU - i\partial_iU^\dagger(1 - \Delta f\Delta^\dagger)\mathcal{A}U - iU^\dagger\mathcal{A}(1 - \Delta f\Delta^\dagger)\partial_iU$$

$$= -iU^\dagger\partial_i\Delta f\Delta^\dagger\mathcal{A}U - iU^\dagger\mathcal{A}\Delta f\partial_i\Delta U$$

$$= iU^\dagger\big(be_if\Delta^\dagger\mathcal{A} + \mathcal{A}\Delta f\bar{e}_ib^\dagger\big)U.$$

In calculating $D^2\Phi = D_iD_i\Phi$, we expand each term resulting from the second covariant derivative separately for ease of reading. The first term gives

$$\partial_i(D_i\Phi) = i\partial_iU^\dagger\big(be_if\Delta^\dagger\mathcal{A} + \mathcal{A}\Delta f\bar{e}_ib^\dagger\big)U + U^\dagger\big(be_if\Delta^\dagger\mathcal{A} + \mathcal{A}\Delta f\bar{e}_ib^\dagger\big)\partial_iU$$

$$- iU^\dagger\big(be_if\bar{e}_ib^\dagger\mathcal{A} + \mathcal{A}be_if\bar{e}_ib^\dagger\big)U.$$

The second term, since f commutes with the quaternions, becomes $-4iU^\dagger\{bfb^\dagger, \, \mathcal{A}\}$. The remaining terms require further expansion:

$$iU^\dagger be_if\Delta^\dagger\mathcal{A}\partial_iU + i\partial_iU^\dagger\mathcal{A}\Delta f\bar{e}_ib^\dagger U + i\partial_iU^\dagger be_if\Delta^\dagger\mathcal{A}U + iU^\dagger\mathcal{A}\Delta f\bar{e}_ib^\dagger\partial_iU$$

$$= iU^\dagger be_if\bar{e}_ib^\dagger\mathcal{A} + iU^\dagger\mathcal{A}be_if\bar{e}_ib^\dagger U + i\partial_iU^\dagger be_if\Delta^\dagger\mathcal{A}U + iU^\dagger\mathcal{A}\Delta f\bar{e}_ib^\dagger\partial_iU$$

$$= 4iU^\dagger\{bfb^\dagger, \, \mathcal{A}\}U + i\partial_iU^\dagger be_if\Delta^\dagger\mathcal{A}U + iU^\dagger\mathcal{A}\Delta f\bar{e}_ib^\dagger\partial_iU,$$

and so the first term of this expression and the previously calculated part cancel. All that remains, then is

$$\partial_i(D_i\Phi) = i\partial_iU^\dagger be_if\Delta^\dagger\mathcal{A}U + iU^\dagger\mathcal{A}\Delta f\bar{e}_ib^\dagger\partial_iU. \tag{A.1.2}$$

The commutator term of $D^2\Phi$ is more involved, and we make liberal use of the identities (A.1.1) throughout.

$$-i[A_i,\, D_i\Phi] = -i[iU^\dagger\partial_i U, iU^\dagger(be_i f\Delta^\dagger\mathcal{A} + \mathcal{A}\Delta f\bar{e}_i b^\dagger)U]$$
$$= -i\Big(\partial_i U^\dagger(be_i f\Delta^\dagger\mathcal{A} + \mathcal{A}\Delta f\bar{e}_i b^\dagger)U$$
$$+ U^\dagger(be_i f\Delta^\dagger\mathcal{A} + \mathcal{A}\Delta f\bar{e}_i b^\dagger)\partial_i U$$
$$- \partial_i U^\dagger\Delta f\Delta^\dagger(be_i f\Delta^\dagger\mathcal{A} + \mathcal{A}\Delta f\bar{e}_i b^\dagger)U$$
$$- U^\dagger(be_i f\Delta^\dagger\mathcal{A} + \mathcal{A}\Delta f\bar{e}_i b^\dagger)\Delta f\Delta^\dagger\partial_i U\Big).$$

We now evaluate the first two lines of this expression:

$$-i\Big(\partial_i U^\dagger be_i f\Delta^\dagger\mathcal{A}U + \partial_i U^\dagger\mathcal{A}\Delta f\bar{e}_i b^\dagger U + U^\dagger be_i f\Delta^\dagger\mathcal{A}\partial_i U + U^\dagger\mathcal{A}\Delta f\bar{e}_i b^\dagger\partial_i U\Big)$$
$$= -i\partial_i U^\dagger be_i f\Delta^\dagger\mathcal{A}U - iU^\dagger\mathcal{A}\Delta f\bar{e}_i b^\dagger\partial_i U - 4iU^\dagger\{bfb^\dagger,\mathcal{A}\}.$$

The first two terms cancel with (A.1.2). All that remains to calculate is the final two lines of the commutator term:

$$i\Big(U^\dagger be_i f\Delta^\dagger be_i f\Delta^\dagger\mathcal{A}U + U^\dagger\mathcal{A}\Delta f\bar{e}_i b^\dagger\Delta f\bar{e}_i b^\dagger U$$
$$+ U^\dagger be_i\Delta^\dagger\mathcal{A}\Delta\bar{e}_i fb^\dagger U + U^\dagger bfe_i\Delta^\dagger\mathcal{A}\Delta\bar{e}_i fb^\dagger U\Big).$$

We note that the terms in the first line are Hermitian conjugates of each other. The expression $be_i f\Delta^\dagger be_i f\Delta^\dagger$ is anti-Hermitian, so the first line vanishes. Noting that $e_i M\bar{e}_i = 2\mathrm{Tr}_2(M)$, where Tr_2 is a quaternion trace, we arrive at the final result:

$$D^2\Phi = -4iU^\dagger\{bfb^\dagger,\, \mathcal{A}\} + 4iU^\dagger bf\,\mathrm{Tr}_2(\Delta^\dagger\mathcal{A}\Delta)fb^\dagger U.$$

A.2 The Algebraic Constraint on Zero-modes

We state that the expression

$$\delta_r A_i = -U^\dagger C_r f \bar{e}_i b^\dagger U + i U^\dagger b e_i f C_r^\dagger U,$$

for $C_r \equiv \partial_r \Delta + \delta_r Q \Delta + \Delta \delta_r R$, is a zero-mode if C_r is x-independent and $\Delta^\dagger C_r$ is Hermitian. We now demonstrate this. Consider the term, comprising part of $\delta_r A_i$, given by $\psi_i \equiv U^\dagger b f e_i$. Treating this as a vector in the fundamental representation, the covariant derivative is

$$D_i \psi_j = \partial_i \psi_j - i A_i \psi_j$$
$$= U^\dagger e_i b f \Delta^\dagger b f e_j + U^\dagger b f (\bar{e}_i b^\dagger \Delta + \Delta^\dagger b e_i) f e_j,$$

where the identities (A.1.1) have been employed. Then in the expression $\Delta^\dagger b$, we may write explicitly $\Delta^\dagger b \equiv l_i e_i$ where the l_i are generically complex-valued (in the commutative case, these would simply be real-valued matrices). Then we have

$$D_i \psi_j = U^\dagger b f l_k f (e_i \bar{e}_k e_j + \bar{e}_i e_k e_j + \bar{e}_k e_i e_j)$$
$$= -U^\dagger b f l_k f (e_i \bar{e}_j e_k - 2\delta_{jk} e_i - 2\delta_{ik} e_j).$$

With $D_i \psi_j$ in this form, it is clear that the expression satisfies the zero-mode conditions $D_{[i} \psi_{j]} = \frac{1}{2} \epsilon_{ijkl} D_k \psi_l$ and $D_i \psi_i = 0$. Now we consider the full expression for the putative zero-mode:

$$D_i(\delta_r A_j) = -i(D_i U^\dagger) C_r \psi_j + i \psi_j C_r^\dagger (D_i U) - i U^\dagger C_r (D_i \psi_j) + i(D_i \psi_j) C_r^\dagger U$$
$$= -i U^\dagger b f (e_i \Delta^\dagger C_r \bar{e}_j - e_j C_r^\dagger \Delta \bar{e}_i) f b^\dagger U - i U^\dagger C_r (D_i \psi_j^\dagger) + i(D_i \psi_j) C_r^\dagger U,$$

where, amongst other things, we treat U as a vector in the fundamental representation, so that $D_i U = U^\dagger e_i b f \Delta^\dagger$. By the previous consideration, we need not worry about the final two terms as they already satisfy the zero-mode constraint. The first term merits more consideration: for it to satisfy the zero-mode conditions we must have

$$e_{[i}\Delta^\dagger C_r \bar{e}_{j]} - e_{[j}C_r^\dagger \Delta \bar{e}_{i]} = \frac{1}{2}\epsilon_{ijkl}(e_k \Delta^\dagger C_r \bar{e}_l - e_l C_r^\dagger \Delta \bar{e}_k)$$

$$e_i \Delta^\dagger C_r \bar{e}_i - e_i C_r^\dagger \Delta \bar{e}_i = 0.$$

The second requirement automatically holds. The first requirement holds iff the expression inside the quaternions is Hermitian: explicitly,

$$(\Delta^\dagger C_r)^\dagger = \Delta^\dagger C_r.$$

This requirement may, due to the form of Δ, be broken up into two separate constraints:

$$a^\dagger C_r = (a^\dagger C_r)^\dagger,$$

$$b^\dagger C_r = (b^\dagger C_r)^\dagger.$$

This is the form used in our calculations.

A.3 The Calculation of α

The noncommutative deformation of the SU(2) ADHM data has the form

$$w_i = M_i v_i,$$

where

$$M_i = \frac{1}{\sqrt{|v_i|^2}} \begin{pmatrix} \sqrt{|v_i|^2 + \alpha\zeta} & 0 \\ 0 & \sqrt{|v_i|^2 - \alpha\zeta} \end{pmatrix}.$$

α is, as yet, unconstrained. However, the diagonal ADHM constraints force the e_3 part of $|w_i|^2 + |\tau|^2 + |s|^2$ to be equal to 2ζ. Hence we may use this constraint to obtain α in terms of the quaternionic data.

The key calculation is that of $|s|^2$. We have

$$|s|^2 = \frac{1}{16|\tau|^2} |\bar{v}_2 M_2 M_1 v_1 - \bar{v}_1 M_1 M_2 \bar{v}_2|^2.$$

Because of the quaternionic nature of the v_i, and the squaring of s, we may pull through the M_i deformations through the v_i, and any additional terms arising from the commutator of s with the e_i will not contribute. Then we have

$$|s|^2 = \frac{M_1^2 M_2^2}{16|\tau|^2} |\bar{v}_2 v_1 - \bar{v}_1 v_2|^2.$$

Calculating the deformation term:

$$M_1 M_2 = \frac{1}{|v_1^2||v_2|^2} \left(\left(|v_1|^2|v_2|^2 + \alpha^2\zeta^2\right) 1_2 + \alpha\zeta(|v_1|^2| + v_2|^2) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right).$$

Since the ADHM constraints are defined up to some overall scalar factor in f^{-1},

the additional terms proportional to the identity can be absorbed via a redefinition, and thus do not merit consideration. The final term is the important one, as this is constrained by the total noncommutative deformation in x. The s parameter, then, contributes to the noncommutative part of the ADHM constraints with a factor

$$\frac{|\bar{v}_2 v_1 - \bar{v}_1 v_2|^2}{16|\tau|^2} \frac{\alpha\zeta}{|v_1|^2 |v_2|^2} (|v_1|^2 + |v_2|^2).$$

We may insert this into either of the diagonal ADHM constraints, knowing that the e_3 part of $|w_i|^2$ is simply $\alpha\zeta$ for $i = 1$, 2. The result is

$$\alpha\left(1 + \frac{(|v_1|^2 + |v_2|^2)|\bar{v}_2 v_1 - \bar{v}_1 v_2|^2}{16|\tau|^2 |v_1|^2 |v_2|^2}\right) = 2.$$

This equation can be trivially rearranged to give the required expression for α, namely

$$\alpha = \frac{32|\tau|^2 |v_1|^2 |v_2|^2}{16|\tau|^2 |v_1|^2 |v_2|^2 + (|v_1|^2 + |v_2|^2)|\bar{v}_2 v_1 - \bar{v}_1 v_2|^2}.$$

This form, while complicated in terms of calculating expressions such as $\mathrm{d}w_i$, has the advantage of being completely general in its derivation. Such a procedure should apply in the consideration of noncommutative instantons for any gauge group or instanton charge.

Appendix B

Mathematica Code

We present, for completeness, the Mathematica [51] code produced to calculate the metric of two noncommutative instantons, and a minimal working example of the simulation code.

B.1   Calculation of the metric

The calculation code takes the data for w_1, w_2 and τ and calculates the corresponding σ. From this, we derive the flat and interacting parts of the metric, along with the induced potential term. The geodesic equations are calculated using the normal geodesic equation, and the result is exported to an external file to avoid problems with kernel quitting.

```
(* Assumptions... *)
$Assumptions = Element[{α[t], ρ1[t], ρ2[t], θ1[t], θ2[t],
    θ[t], φ[t], ρ1₊[t], ρ2₊[t], ρ1₋[t], ρ2₋[t], ω[t], χ[t], ξ}, Reals];

(* Declare independent elements of Δ *)
v1 = {{ρ1₊[t] eⁱ θ1[t], 0}, {0, ρ1₋[t] e⁻ⁱ θ1[t]}};
v2 = {{ρ2₊[t] eⁱ θ2[t], 0}, {0, ρ2₋[t] e⁻ⁱ θ2[t]}};
τ = ω[t] {{eⁱ χ[t], 0}, {0, e⁻ⁱ χ[t]}};

(* Induced σ *)
σ =
    1
  ──────── τ.(ConjugateTranspose[v2].v1 - ConjugateTranspose[v1].v2) // FullSimplify;
  4 ω[t]²

(* Substitutions for parameters ρ₊, ρ₋, and angle sums *)
ρsubs = {ρ1₊'[t] → ρ1[t] ──────────────────── , ρ1₋'[t] → ρ1[t] ──────────────────── ,
                              ρ1'[t] + ξ α'[t] / 2                  ρ1'[t] - ξ α'[t] / 2
                                    ρ1₊[t]                                ρ1₋[t]

    ρ2₊'[t] → ρ2[t] ──────────────────── , ρ2₋'[t] → ρ2[t] ──────────────────── };
                       ρ2'[t] + ξ α'[t] / 2                  ρ2'[t] - ξ α'[t] / 2
                             ρ2₊[t]                                ρ2₋[t]

anglesubs = {θ1[t] → ── (θ[t] + φ[t]), θ2[t] → ── (θ[t] - φ[t]),
                      1                          1
                      2                          2

    θ1'[t] → ── (θ'[t] + φ'[t]), θ2'[t] → ── (θ'[t] - φ'[t])};
             1                            1
             2                            2

(* Differentials *)
dv1 = D[v1, t] //. Join[ρsubs, anglesubs] // FullSimplify;
dv1bar = D[ConjugateTranspose[v1] // Simplify, t] //.
    Join[ρsubs, anglesubs] // FullSimplify;
dv2 = D[v2, t] //. Join[ρsubs, anglesubs] // FullSimplify;
dv2bar = D[ConjugateTranspose[v2] // Simplify, t] //.
    Join[ρsubs, anglesubs] // FullSimplify;
dτ = D[τ, t] // FullSimplify;
dτbar = D[ConjugateTranspose[τ] // FullSimplify, t] // FullSimplify;
dσ = D[σ, t] //. Join[ρsubs, anglesubs] // FullSimplify;
dσbar =
  D[ConjugateTranspose[σ] // Simplify, t] //. Join[ρsubs, anglesubs] // FullSimplify;
Λ = ConjugateTranspose[v2].v1 - ConjugateTranspose[v1].v2 // FullSimplify;

(* Parts of the interacting metric expression *)
dk = Simplify[ConjugateTranspose[v1]].dv2 - Simplify[ConjugateTranspose[v2]].dv1 +
    2 (Simplify[ConjugateTranspose[τ]].dσ - Simplify[ConjugateTranspose[σ]].dτ) //.
    anglesubs // FullSimplify;
N_A = Inverse[ConjugateTranspose[v1].v1 + ConjugateTranspose[v2].v2 +
      4 (ConjugateTranspose[τ].τ + ConjugateTranspose[σ].σ)] //.
    anglesubs // FullSimplify;

(* Explicit substitutions for ρ₊ and ρ₋ *)
ξsubs = {ρ1₊[t] → Sqrt[ρ1[t]^2 + α[t] ξ], ρ1₋[t] → Sqrt[ρ1[t]^2 - α[t] ξ],
    ρ2₊[t] → Sqrt[ρ2[t]^2 + α[t] ξ], ρ2₋[t] → Sqrt[ρ2[t]^2 - α[t] ξ]};
```

```
(* Final form for interacting part *)
dk = dk //. ζsubs;
dθ = -dk.N_A //. ζsubs // FullSimplify;
dint = dk.dθ // FullSimplify;

(* Final form for flat part *)
flatpart = Simplify[dv1bar.dv1] +
    Simplify[dv2bar.dv2] + Simplify[dτbar.dτ] + Simplify[dσbar.dσ];
dflat = ComplexExpand[flatpart] //. ζsubs;

(* Full metric *)
metric = 1 / 2 Tr[dflat] + 1 / 2 Tr[dint];

(* Conversion from the metric form to a matrix *)
metricInMatrixForm[metric_, coords_] := ParallelTable[If[i == j, 1, 1 / 2] * Simplify[
      Coefficient[metric, coords[[i]] '[t] * coords[[j]] '[t]]], {i, 1, 6}, {j, 1, 6}];

coords = {ρ1, ρ2, θ, ϕ, ω, χ};

met = metricInMatrixForm[metric, coords];

(* The potential term *)
V = 1 / 2 v^2 met[[3, 3]] // FullSimplify;
```

```
(* Calculating the geodesic equations *)
Options[CalculateEoMs] = {
   ShowTime → False
   };
CalculateEoMs[g_, V_, coords_, OptionsPattern[]] := Module[{
     dim = Length[coords],
     EoMs, startTime = AbsoluteTime[], timeRemaining,
     Vnot = V /. {a_[t] → a}
   }, (
    EoMs = Table[0, {i, 1, dim}];
    If[OptionValue[ShowTime],
     Print["Time remaining: ", Dynamic[timeRemaining], " seconds"]];
    Do[(
      Do[
       EoMs[[i]] += -2 D[g[[i, j]] * coords[[j]]'[t], t],
       {j, 1, dim}
       ];

      Do[
       Do[
        EoMs[[i]] +=
         D[g[[j, k]], coords[[i]][t]] * coords[[j]]'[t] * coords[[k]]'[t],
        {k, 1, dim}
        ],
        {j, 1, dim}
       ];

      EoMs[[i]] += - D[Vnot, coords[[i]]] /. Map[# → #[t] &, coords];

      timeRemaining = (AbsoluteTime[] - startTime) / i * (dim - i);
      ), {i, 1, dim}];
     EoMs
    )];

EoMs = CalculateEoMs[met, V, coords, ShowTime → False];

(* Export to file (for stability) *)
EoMs >> ~/Documents/EoMs.txt;
```

B.2 Simulations

The simulation code first imports the previously calculated geodesic equations of motion (twice, once with explicit t dependence in order to plot residuals). We then define an evolution function which takes the equations of motion and incorporates initial conditions on the parameters, along with the coordinate transformation from (ω, χ) to (x, b), before passing to Mathematica's NDSolve function. This allows us to plot trajectories and size evolution of scattering scenarios for any combination of the $(\rho_1, \rho_2, \theta, \phi, \omega, \chi, \zeta, v)$ parameters. We present two examples: "head-on" collisions for inwards-travelling instantons and small impact parameter, and "orbiting" configurations, along with their residuals. Of course, a much greater range of simulations, tests and parameter searches were performed using the simulational building blocks here. A full version of this notebook may be found at `maths.dur.ac.uk/~kzcg21/documents/MathematicaCode.tar.gz`, along with the text file containing the geodesic equations. In order to optimise the speed of commutative simulations, we calculated two sets of geodesic equations: one for the commutative ($\zeta = 0$) case, and one for the noncommutative case. The commutative geodesic equations are, of course, equivalent to the noncommutative counterparts under the substitution $\zeta \rightarrow 0$.

```
(* Loads required packages for NDSolve *)
Needs["DifferentialEquations`InterpolatingFunctionAnatomy`"];

(* Import EoMs *)
EoMs = << ~/Documents/EoMs.txt;
(* Re-import with t dependence for residuals *)
residuals[t_] = << ~/Documents/EoMs.txt;

(* Conditions for NDSolve *)
initialrange = 2600;
EvolveSingleSystem[EoMs_, coords_,
    initialConditions_, {start_, end_}] := Module[
    {
     system, solutions
    },
    system = Join[
      Map[EoMs[[#]] == 0 &, Range[1, Length[EoMs]]],
      Map[#[[1]][start] == #[[2]] &, initialConditions]
     ];
    solutions = NDSolve[system, coords, {t, start, end},
      StepMonitor :> (Set[k, t]; steps++), MaxSteps → 50 000];
    If[Head[solutions] === NDSolve,
     $Failed,
     solutions
    ]
   ];
WithImpactParameter[conditions_, impactParameter_] := (
  Join[conditions, {
     ω → Sqrt[b^2 + x^2] ,
     χ → ArcTan[b / x],
     ω' → (x * x' + b * b') / Sqrt[b^2 + x^2],
     χ' → (x * b' - b * x') / (b^2 + x^2)
    } /. impactParameter]
 )

(* Solving for given values of impact parameter,
noncommutativity and potential strength *)
Sols[ξ0_, impact_, pot_, range_] := EvolveSingleSystem[
    EoMs /. {ξ → ξ0, v → pot},
    {ρ1, ρ2, θ, ϕ, ω, χ},
    WithImpactParameter[
     {ρ1 → 1, ρ1' → 0, ρ2 → 1, ρ2' → 0, θ → 0, θ' → 0, ϕ → π / 2, ϕ' → 0},
     {x → 30, x' → -3/100, b → impact, b' → 0}], {0, range}][[1]];
```

```
(* Commutative test: ζ=0 *)
steps = 0;
k = 0;
Print["Steps: ", Dynamic[steps]];
Print["t=", Dynamic[k]];
sols = Quiet[Sols[0, 0.01, 0, 2600], NDSolve::ndsdtc];
(* Parametric plot of NDSolve results *)
(* If system encounters singularity, guarantee that plotting doesn't crash *)
solrange = InterpolatingFunctionDomain[sols[[1]]][[1, 2]][[1, 1]][[2]];

dynamics = ParametricPlot[{
     {
      -ω[t] Cos[χ[t]],
      -ω[t] Sin[χ[t]]
     },
     {
      ω[t] Cos[χ[t]],
      ω[t] Sin[χ[t]]
     }
    } /. sols, {t, 0, solrange},
   PlotRange → 20, PlotStyle → {Black}, AxesOrigin → {-20, -20},
   Epilog → Table[{
      {Black, Circle[{ω[t] Cos[χ[t]], ω[t] Sin[χ[t]]} /. sols, ρ1[t] /. sols]},
      {Black, Circle[{-ω[t] Cos[χ[t]], -ω[t] Sin[χ[t]]} /. sols, ρ2[t] /. sols]}
     }, {t, 0, solrange, 40}],
   ImageSize → 200,
   TicksStyle → 14
  ];

sizes = Plot[{ρ1[t], ρ2[t]} /. sols, {t, 0, solrange},
   PlotStyle → {Thickness[0.004], Black}, ImageSize → 200,
   Axes → {True, True}, PlotRange → {{0, 1500}, {3, -0.1}},
   Ticks → {None, {0, 1, 3}}, AxesLabel → {"t", "ρ"}];
Rasterize[Show[GraphicsGrid[{{dynamics, sizes}}]], ImageResolution → 144]
```
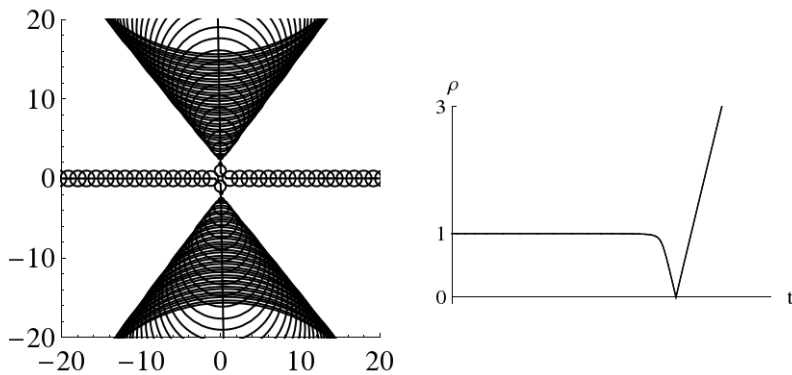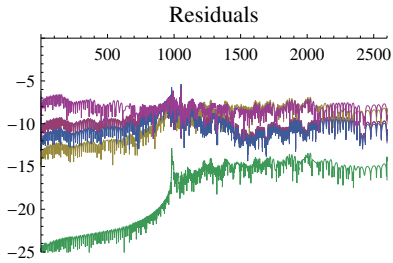
```
resbase = Join[sols, {ξ → 0, b → 0.01, v → 0}];
Plot[Evaluate[RealExponent[residuals[t] /. resbase]],
 {t, 0, 2600}, AxesOrigin → {0, 0}, ImageSize → 200,
 PlotRange → {{0, 2600}, {0, -25}}, PlotLabel → "Residuals"]
```



```
(* Orbiting solutions for given ξ and v *)
SolsOrbit[ξ0_, pot_, range_] := EvolveSingleSystem[
    EoMs /. {ξ → ξ0, v → pot},
    {ρ1, ρ2, θ, ϕ, ω, χ},
    WithImpactParameter[
     {ρ1 → 1.5, ρ1' → 0, ρ2 → 1, ρ2' → 0, θ → 0, θ' → 0.3, ϕ → π / 3, ϕ' → 0},
     {x → 1, x' → 0, b → 0, b' → 0.098}], {0, range}][[1]];

(* Commutative orbit test *)
steps = 0;
k = 0;
Print["Steps: ", Dynamic[steps]];
Print["t=", Dynamic[k]];
orbitsols = Quiet[SolsOrbit[0, 0.3, 5600], NDSolve::ndsdtc];
solrange = InterpolatingFunctionDomain[orbitsols[[1]]][[1, 2]][[1, 1]][[2]];

dynamics = ParametricPlot[{
     {
      -ω[t] Cos[χ[t]],
      -ω[t] Sin[χ[t]]
     },
     {
      ω[t] Cos[χ[t]],
      ω[t] Sin[χ[t]]
     }
    } /. orbitsols, {t, 0, solrange},
   PlotRange → 2, PlotStyle → {Black, Thick}, AxesOrigin → {-2, -2},
   Epilog → Table[{
      (*{Black,Circle[{ω[t]Cos[χ[t]],ω[t]Sin[χ[t]]}/.sols,ρ1[t]/.sols]},
      {Black,Circle[{-ω[t]Cos[χ[t]],-ω[t]Sin[χ[t]]}/.sols,ρ2[t]/.sols]}*)
     }, {t, 0, solrange, 40}],
   ImageSize → 200,
   TicksStyle → 14
  ];
```
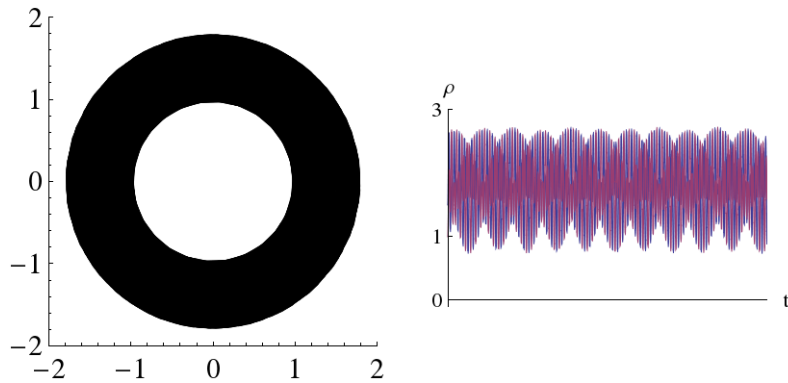
```
sizes = Plot[{ρ1[t] /. orbitsols, ρ2[t] /. orbitsols},
    {t, 0, solrange}, PlotStyle → {Thickness[0.004]}, ImageSize → 200,
    Axes → {True, True}, PlotRange → {{0, solrange}, {3, -0.1}},
    Ticks → {None, {0, 1, 3}}, AxesLabel → {"t", "ρ"}];
Rasterize[Show[GraphicsGrid[{{dynamics, sizes}}]], ImageResolution → 144]
```
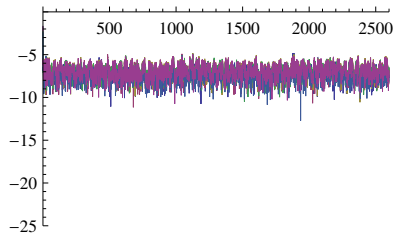


```
resbaseorbit = Join[orbitsols, {ξ → 0, v → 0.3}];
Plot[Evaluate[RealExponent[residuals[t] /. resbaseorbit]], {t, 0, 2600},
 AxesOrigin → {0, 0}, ImageSize → 200, PlotRange → {{0, 2600}, {0, -25}}]
```

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