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An Investigation into Supersymmetric Flux Backgrounds and their Moduli via Generalised Geometry

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November 26, 2020

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Submitted in part fulfilment of the requirements for the degree of Doctor of Philosophy in Physics of Imperial College London

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Declaration

I hereby certify that, to the best of my knowledge, all of the material in this dissertation which is not my own work has been properly acknowledged. This research described has been done in collaboration with Daniel Waldram, Anthony Ashmore and Charles Strickland-Constable. The presentation closely follows our papers [1, 2].

Abstract

We provide a detailed analysis of flux backgrounds of string and M-theory that preserve minimal supersymmetry in terms of (exceptional) generalised geometry. The geometry in each case is conveniently described in terms of generalised G-structures, where the integrability conditions are equivalent to the Killing spinor equations. Interestingly, there seems to be a common structure among the G-structures, in that they are described by an involutive complex subbundle of the generalised tangent bundle, and a vanishing moment map. We call these structures '*Exceptional Complex Structures*' (ECS) because of their similarity to (generalised) complex structures. In analysing the integrability conditions we find interesting links to 'Geometric Invariant Theory' (GIT) which may have important consequences for unsolved problems in conventional geometry. The moment map picture also provides a systematic way of studying the moduli. We use the relation between symplectic quotients and complexified quotients to analyse the moduli, giving exact results in a broad range of cases.

We start with backgrounds of heterotic string theory with a 4-dimension external Minkowski space. We show how the Hull-Strominger system can be reinterpreted as an integrable $\mathrm{SU}(3) \times \mathrm{Spin}(6+n) \subset \mathrm{O}(6, 6+n)$ structure. We provide expressions for the superpotential and the Kähler potential in this new language and analyse the moment map involved in the integrability conditions. This moment map interpretation of the Hull-Strominger system is an important step in applying GIT to prove the existence of solutions, given certain constraints. This extension of Yau's theorem to particular non-Kähler manifolds has been of interest to mathematicians for some time and our work may indicate possible new approaches to solving it. We also analyse the moduli of the Hull-Strominger system and recover the results of others.

The next chapter focuses on M-theory backgrounds with a 5-dimensional external space. While it does not describe the full geometry, we focus on the $\mathrm{SU}^*(6) \subset \mathrm{E}_{6(6)} \times \mathbb{R}^+$ structure present in the supergravity solution. We find the most generic local form for exceptional complex structures in this case, classifying them as either 'type 0' or 'type 3'. This classification is only pointwise, as there can be type-changing solutions. Using the general form, we are able to find the moduli of all constant-type exceptional complex structures, as well as all those that satisfy a 'generalised $\partial\bar{\partial}$-lemma'. Interestingly, these results hold for AdS solutions. We analyse these and show that they are always of constant type 3. Hence, we are able to reinterpret the spectrum of a given CFT_4 that is dual to some $\mathrm{AdS}_5 \times M_6$ in terms of cohomology groups related to some integrable distribution $\Delta \subset T_{\mathbb{C}}$.

We then look at backgrounds of M-theory and type IIB with a 4-dimensional Minkowski external space. We are able to reinterpret both G_2 backgrounds and GMPT backgrounds in terms of integrable $\mathrm{SU}(7) \subset \mathrm{E}_{7(7)} \times \mathbb{R}^+$ structures. We are also able to give an expression for

the superpotential and the Kähler potential for generic backgrounds using this new language. Once again, we study the implications of the moment map picture and find interesting links with GIT. We highlight how this may be used to find a form of stability for G_2 structures. Again, we provide a method of systematically finding the moduli of these flux backgrounds and apply it to the G_2 and the GMPT cases. For G_2 we recover the known results, while for GMPT we are able to find the exact moduli, extending work that has been done in the past.

Finally, we analyse the exceptional complex structures via Hitchin functionals. The Kähler potentials in each case provide a natural candidate for the extension of Hitchin functionals to exceptional geometry. Following the work of Pestun and Witten [3], we find the second variation of the Kähler potentials under complexified generalised diffeomorphisms and quantise that quadratic action for $SU^*(6)$ and $SU(7)$ structures. We suggest possible applications as 1-loop corrections to certain terms in the effective M-theory action in 5 and 4 dimensions respectively.

Acknowledgements

This thesis is the culmination of years of work and there have been many people who I would like to thank for their support along the way.

Most importantly, I would like to thank my supervisor, Daniel Waldram, whose help was invaluable in this work. His patience with my endless questions, and his insight into physics made this thesis possible and I am extremely grateful. I would also like to thank my collaborators Anthony Ashmore and Charles Strickland-Constable for many important discussions and their help through much of my PhD.

I want to extend my appreciation to all of my friends and colleagues at Imperial who have helped me with my physics problems, as well as providing fun and laughter throughout the 4 years. I would particularly like to thank Edward Tasker. He was an incredibly bright physicist, a wonderful person, and a great friend. We had many discussions about our work that helped me immensely through my research. His puns also brought so much laughter to group and he will be sorely missed. Special thanks also goes to Julius, Arianna, Alex, and Matthew for their support.

Finally, I would like to thank the Shlads and Dogs, my family, and my partner El for their love and support. They have always provided welcome distraction from the world of physics and have kept me sane through the insane world of academia.

This work was supported by a PhD studentship from the STFC Doctoral Training Programme ST/N504336/1.

Contents

Chapter 1

Introduction

In this introductory chapter, we will give a brief historical survey of theoretical physics through the 20$^{\text{th}}$ century to provide motivation for the rest of the thesis. We will introduce string theory and see why it has been such an active area of research for over 50 years. We will describe some of the key properties of string theory, including the troublesome requirement of 10 dimensions. Removing these unobserved dimensions to retrieve something more physical is the main motivation behind studying Minkowski backgrounds of string theory - the main focus of this thesis.

1.1 String Theory and Supergravity

Physics in the 20$^{\text{th}}$ saw a rapid progression in our understanding of the fundamental nature of the universe. This progress was advanced along two crucial, yet so far irreconcilable branches. On the one hand, Einstein's theory of general relativity revolutionised the way we view gravity - replacing a static universe with a dynamic spacetime whose curvature creates geodesics that resemble gravitational attraction. On the other hand, quantum mechanics and quantum field theory were being developed to explain the curious behaviour of subatomic interactions. These eventually led to the standard model, a unified description of all forces of nature except gravity. Both branches have been tested and, time and again, predictions prove to be correct. However, all attempts to provide a quantum theory of gravity that also produces the gauge groups and representations of the standard model have, so far, proven fruitless. It is this gap in understanding that string theory tries to bridge.

Quantum field theory was born early in the development of quantum theory when people tried to apply the same techniques to electromagnetic radiation that they had used to describe the atom [4]. The photon is massless and therefore cannot be described by a non-relativistic approximation like quantum mechanics. QFT was therefore born out of a requirement to merge quantum mechanics and special relativity. Unfortunately, while these methods seem to work well for free theories, once they are applied to higher order interactions, most calculations return an infinite answer. This can be understood as coming from loops arising in the expansion in the coupling constant. One has to integrate over all momenta running in loops, and these can often diverge.

While this initially caused much confusion, today we understand this as reflecting the fact that we should think of all theories as effective theories. This means that the theory is only a valid description of the physics at scales $|p| < \Lambda$, above which something new occurs. In this Wilsonian picture [5–8], Λ is called the cut-off and the coupling constants change as we lower Λ. In this process, previously vanishing couplings may turn on, so we should include all terms consistent with the symmetry of the theory in any effective description. Once we have done this the Lagrangian contains an infinite number of interaction terms and coupling constants. However, to any particular order in the expansion there are only a finite number of terms that need to be fixed through experiment. Hence, effective theories are still predictive for any theory to a given accuracy, at a given energy scale.

Soon, people tried to apply quantum techniques to Einstein's general relativity [9–14]. In a quantum description, we restrict ourselves to describing perturbations around some fixed background. That is, we write

$$g_{\mu\nu} = \hat{g}_{\mu\nu} + \kappa h_{\mu\nu} \tag{1.1}$$

where $\hat{g}$ is fixed and h is the perturbation, or graviton. The coupling constant goes like $\kappa^2 \sim G_N$, Newton's constant, and hence has mass dimension $[\kappa] = -1$. As usual, when one expands the Einstein-Hilbert action and tries to quantise h, one finds infinities. Hence, we should think of this as an effective description with a cut-off, and we write all terms that are consistent with diffeomorphism symmetry. The natural cut-off is the Planck mass $M_p = \sqrt{\hbar c/G_N}$, and the action becomes

$$\mathcal{L} = \int_M \mathrm{d}^4x \sqrt{-g} \left(M_p^2 R + c_1 R^2 + c_2 R^{\mu\nu} R_{\mu\nu} + \frac{c_3 R^3}{M_p^2} + \ldots + \mathcal{L}_{\text{matter}} \right) \tag{1.2}$$

As we mentioned, this theory is valid for small energies and one can expand all scattering amplitudes in powers of $\kappa\Lambda_0 \sim \Lambda_0/M_p$ where Λ_0 is the energy of the scattering process. However, as the process reaches energy scales comparable to M_p, we see that this expansion breaks down and our theory becomes no longer valid[1]. Unfortunately, the Planck scale is precisely the energy we would require to probe quantum effects of gravity.

This opened the question of what a theory of quantum gravity would look like. The hope would be that GR has a UV fixed point[2] where the scattering amplitudes are finite, and the infinite number of couplings, plus any extra fields that disappear in the low energy dynamics, are controlled by a finite number of parameters. Unfortunately, finding such a theory is a complicated task. There is common lore that there is no decoupling regime of quantum gravity [15]. That is, the precise fixed point and its dynamics are heavily dependent on the matter content of the theory at all energies right up to the UV. Without a guiding principal on what the matter content should look like, the chances of stumbling across the correct UV theory

[1] Note that this is an irreconcilable problem. While some theories only have a finite number of divergences that need to be cancelled by counter terms, the mass dimension of κ in GR means that each loop order produces new divergences that must be cancelled. All of these counter terms must be fixed by experiment and hence one would have to do an infinite number experiments to render the theory predictive. One can show that the non-correctable divergences occur at 2nd loop order [11]. Such theories we call non-renormalisable and cannot be used as a high energy description of the physics.

[2] A UV fixed point is a point in theory space at which the couplings are independent of the cut-off Λ. That is, it is a fixed point of the renormalisation group flow. For it to be a UV fixed point of GR, there must be some RG flow line going from (a perturbation away from) the UV fixed point, through the point in theory space corresponding to (1.2).

seems highly unlikely.

One such guiding principal is supersymmetry. This fermionic symmetry collects bosons and fermions into multiplets with adjacent spin. The higher the amount of supersymmetry, the more spins are grouped into each multiplet. Imposing that $\mathcal{L}_{\text{matter}}$ is such that the gravitational action is supersymmetric produces a theory called supergravity [16]. Not only does supersymmetry put restrictions on the matter content of the theory, it also gives relations between the coupling constants. Since bosons and fermions contribute to scattering amplitudes with opposite signs, it was hoped that these relations would lead to cancellations that would remove the divergences and hence leave the theory finite, or at least renormalisable. Unfortunately, it is now known that supergravity in $D = 4$ dimensions[3] with all but the maximal $\mathcal{N} = 8$ supersymmetry is also divergent[4] and there are arguments that even this is divergent at 7-loop order[5] [18]. The problem of a UV complete theory of gravity therefore remained open. Fortunately, at around the same time, another theory was starting to be studied that seemed to be both UV finite, and completely determine the matter content and couplings of the theory.

String theory was initially created to try to describe the strong interaction. In the 1960's there were an abundance of new particles being discovered with an approximately linear relationship between their spin and mass squared $J = M^2\alpha'$, α' being called the Regge slope. An expression for the scattering amplitude that reproduced the Regge slope and was consistent with crossing symmetry was suggested by Veneziano in [19]. This model eventually became known as the dual resonance model and it was shown that it was consistent with the scattering of relativistic string states [20–24]. Unfortunately, this theory had many properties that made it unsuitable for describing hadronic physics. The double resonance model slowly faded in popularity as QCD proved to be more and more successful. However, one property that made it unusable as a model for hadrons also made it the perfect theory for a description of quantum gravity - the existence of a massless spin 2 state [25].

The existence of a graviton-like state in the spectrum of string theory was an appealing property. It was also known for some time that the infinite tower of massive higher spins in the spectrum meant that the theory was UV finite. At high energies, the higher spin states become approximately massless. When this occurs, the Coleman-Mandula theorem says that the S-matrix must be trivial and hence there are no divergences [26]. Given this promising behaviour, people started to study string theory as a potential *'theory of everything'*.

The key postulate of string theory is that objects in physics are not point-like particles, but instead extended objects like strings. Typically, string theory is described by an action on the 2 dimensional worldsheet - the generalisation of the worldline of a particle [27, 28]. Borrowing intuition from the action of a relativistic particle, people initially considered the Nambu-Goto action of the string which equates the equations of motion with volume minimisation of the worldsheet. This formulation was not well-suited for quantisation as it contains the square root of fields. A reformulation of the theory came in [29] and was used by Polyakov to quantise the string. The action, since called the Polyakov action, is a non-linear σ-model, taking values in

[3] Here and in the rest of the thesis, we will denote 'external' dimensions with D.

[4] In fact, maximal supergravity in $d \geq 6$ is known to be divergent.

[5] Even if it is not divergent at that order, there are non-perturbative arguments that suggest that this theory is inconsistent and requires a UV completion of M-theory on a 7-torus. [17]

the spacetime and takes the form

$$S = \frac{T}{2} \int \mathrm{d}^2\sigma \sqrt{-h} h^{\alpha\beta} g_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu \tag{1.3}$$

Here $h_{\alpha\beta}$ is the worldsheet metric and the X^μ are fields that can be thought of as coordinates on a target space M with metric g. This theory has diffeomorphism invariance and Weyl invariance which can be used to fix the worldsheet metric to a conformal Minkowski metric $\mathrm{e}^{\lambda\sigma}\eta_{\alpha\beta}$ [30]. Upon doing so, the residual gauge symmetries are a local conformal invariance and Weyl invariance, which gives the usual description of a string as a quantum CFT. As these are gauge symmetries, they must be anomaly-free in the quantum theory. Enforcing this has interesting consequences for the properties of string theory, as we will see.

A natural question arises when considering strings propagating through spacetime relating to whether the strings are open or closed. Both possibilities are valid and give rise to interesting consequences of string theory. Closed strings have a spectrum of states of increasing mass and spin. In the massless spectrum, one has a metric, an antisymmetric rank 2 B-field, and a scalar dilaton $\hat{\varphi}$. Both strings therefore describe gravitational interactions, but also prescribe some other matter content through the massive higher spin states. Since these strings have no boundary, one does not need to prescribe boundary conditions. Open strings on the other hand require boundary conditions of the form of Neumann or Dirichlet. These are respectively of the form

$$\partial_\sigma X^I = 0 \qquad X^J = c^J \quad \text{(const)} \tag{1.4}$$

It can be shown that Neumann boundary conditions imply that the end of the string moves at the speed of light. Dirichlet boundary conditions were largely ignored until Polchinski highlighted their importance in the 1990's [31]. Having the end-point of a string fixed in certain coordinates describes a p dimensional space-like surface which was called a D_p-brane. Looking at the massless spectrum of the open string [32], one finds perturbations along the direction of the brane which describe massless spin-1 fields on the brane. Hence, open strings describe gauge theories on the world-volume of the brane[6]. There are also fluctuations transverse to the direction of the brane which have been interpreted as fluctuations of the branes themselves. This suggests that the D_p-branes themselves should be thought of as dynamical objects in string theory. Hence, the full spectrum of string theory is described by gravitational-like interactions of closed strings, gauge-boson-like interactions of open strings, along with the higher mass and higher spin states of the strings and the branes.

This promising progress was hindered by a few key issues. Firstly, unitarity required bosonic string theory to exist in 26 dimensions [33]. Secondly, all states of the bosonic string have integer spin and hence only describe bosonic matter in the target space. Finally, the ground state of the string theory was always tachyonic and there was no consistent way to remove it. The latter two issues were solved by introducing worldsheet fermions to get a supersymmetric theory [34, 35]. While this allowed fermionic matter to be described, the ground state was still tachyonic. However, there was now a consistent way to remove this state through what has been called the GSO projection [36]. In total, there are five consistent string theories. There

[6] This is the generalisation of the worldsheet to higher dimensional objects.

is type I, the two heterotic string theories with SO(32) and $E_8 \times E_8$ gauge groups respectively, and the two type II theories - IIA and IIB [37].

Type IIA and type IIB [37] both have $\mathcal{N} = 2$ spacetime supersymmetry[7] but differ by the chirality of the generators. Type IIA is non-chiral so has $\mathcal{N} = (1,1)$ supersymmetry while IIB is chiral and has $\mathcal{N} = (2,0)$. They are obtained through different choices of GSO projection on the worldsheet. In the massless spectrum, both theories have the same NSNS sector - the metric, the B-field and dilaton $\hat{\varphi}$ - but differ in the RR sector. These contain antisymmetric tensors A_{p+1} of rank $(p+1) = 1, 3$ for IIA and $(p+1) = 0, 2, 4$ for IIB. These have field strengths F_{p+2}, where F_5 is further restricted to be self-dual for IIB. It turns out that the D_p-branes are electrically charged, and D_{10-p-4}-branes are magnetically charged, under A_{p+1}. Hence there is a restriction to the type of branes that can occur in each theory[8]. The open string spectrum then gives rise to a gauge theory on the brane much like it did in the bosonic case. The string itself is electrically charged under the B-field and there is another 5-brane called the NS5 brane that is magnetically charged.

Type I string theory [36] comes from an orientifold projection of type II in the presence of 32 D_9-branes for anomaly cancellation. The projection preserves only $\mathcal{N} = 1$ spacetimes supersymmetry, and the 32 D9-branes means that there is an SO(32) gauge symmetry. The massless spectrum of the closed string gives the metric, the dilaton and the RR 2-form. The RR 2-form couples electrically to the D_1-brane and magnetically to the D_5-brane. Hence we see that only these branes occur in type I string theory.

Finally, the heterotic string [38, 39] combines both the bosonic string and the superstring. Since the action is a 2-dimensional CFT, the operators factorise into so-called left and right moving sectors. In the heterotic string, the left moving modes are taken from the bosonic string and the right moving modes are taken from the superstring. The additional modes from the left moving sector form gauge potentials for the gauge groups SO(32) or $E_8 \times E_8$. These are enforced by the requirement of anomaly cancellation. Since the left and right moving sectors have to be independent in this theory, this forces the string to be closed and hence there are no D_p-branes in heterotic string theory.

These 5 seemingly distinct string theories were later shown to be related through a web of dualities that also relate strong and weak coupling phenomena [40–44]. It was further conjectured that all of these theories were in fact different limits of one overarching 11 dimensional theory called M-theory [45, 46]. This seemed tantalisingly close to a unique fundamental theory of the universe but there was still one requirement of string theory that had to be solved. While the superstring had reduced the required number of dimensions down from 26, all 5 string theories still required 10 spacetime dimensions [47]. The question became whether it was possible to put strings on a background that resembled something more physical.

A background is defined to be a target space geometry on which a quantum theory of strings

[7]Throughout this thesis, $\mathcal{N}$ will denote the number of supersymmetry generators the theory is symmetric under. The dimension of the generators and their flavour depends on the dimension of the spacetime and the signature of the metric. In even dimensions with a Lorentzian signature, the fundamental spin representations are Weyl spinors meaning that we can classify spinors with respect to their chirality (i.e. eigenvalue under some γ_d). We therefore write $\mathcal{N} = (p, q)$ to denote p even generators, and q odd.

[8]Note that a D_p-brane has a world-volume of dimension $p + 1$. Much like a particle is charged under a vector field, a p-brane can be charged under a $p + 1$-form.

is well-defined. The key requirement for a well-defined quantum theory of strings is that the conformal symmetry must not be anomalous. This translates into the requirement that the β-functions of the couplings, interpreted as target-space fields, must vanish. To make progress, it is important to work in a regime of string theory in which the string length scale l_s is much less than the typical length scale of the target space geometry defined by the metric l.

$$l_s \ll l \sim \frac{\partial g}{\partial x} \tag{1.5}$$

This defines an energy scale at which our theory is valid, as well as an expansion which is done in a parameter $\alpha' = l_s^2$. In this limit, the world sheet CFT is weakly coupled and we can use usual QFT methods to determine the β-functions. At lowest order, we recover the equations of motion of supergravity. We learn then that, at low energies and weak string coupling[9], string theory backgrounds are simply supergravity backgrounds [48]. This provided the important link between string theory, and the empirically verified theory of general relativity.

At low energies, the massless modes dominate the dynamics with stringy affects suppressed. At higher energies, we approach a scale closer to the string length scale l_s and hence the extended nature of the string becomes more important. We would therefore need to add α' corrections to these equations which would change the geometry [49, 50]. We would also have to be cognisant of effects arising from winding modes of the string[10], or branes wrapping cycles. Once we move into this regime, the rich structure of string theory comes into the foreground and one can use strong/weak coupling dualities such as T-duality, and the larger U-duality[11]. At low energies, branes also become non-dynamical objects and instead take the form of solitonic solutions of the supergravity equations [46, 59–64]. Importantly, they can still interact with the massless modes of the string through gauge fields defined at the end of an open string, and by sourcing flux.

The class of all solutions to supergravity is quite a large set. However, we can restrict ourselves to a more refined set depending on the application. Given the unphysical number of dimensions required in string theory/M-theory, a lot of focus has been put into backgrounds of the form

$$\mathbb{M} = M_{\text{ext}} \times M_{\text{int}} \tag{1.6}$$

where M_{ext} is some, usually maximally symmetric non-compact, space called the external space which is somehow deemed the 'physically observable' space. M_{int} is called the internal space and has properties that affect the physics on the external space. For physical applications, we would like the external space to resemble a vacuum of our universe, and hence we look for backgrounds with $M_{\text{ext}} = \mathbb{R}^{D-1,1}$, with the most physically relevant being of course $D = 4$. With the discovery of the AdS/CFT correspondence [65], a lot of interest has grown in backgrounds with $M_{\text{ext}} = \text{AdS}_D$.

[9] While α' controls, in a sense, the importance of the extended nature of the string, the string coupling g_s determines the strength of the quantum nature of the string and appears in string-loop expansions. It is in fact not a free parameter but is determined by the dilaton $\hat{\varphi}$.

[10] Winding modes are non-perturbative in α', but perturbative in the string coupling g_s.

[11] In recent years, people have considered backgrounds that may involve such exotic string states through double/exceptional field theory and non-geometric backgrounds [51–58]. While this work has interesting links to the generalised geometry we shall describe later, it will not be the focus of this thesis.

Different choices of M_{int} have interesting consequences for the dynamics of the effective theory on the external space. Choosing a space with isometries can lead to gauged matter in the effective theory [66]. Conversely, people have tried to break the large gauge groups of heterotic theories through flux induced potentials to groups that could resemble a GUT theory in $\mathbb{R}^{3,1}$ [67,68]. The geometry can even give us information on the massless scalar field content of the effective theory[12]. One particularly interesting class of backgrounds are those that lead to a supersymmetric effective theory - so called supersymmetric backgrounds. The archetypal example of such a compactification is where M_{int} is a Calabi-Yau manifold [69].

The desire to understand the precise role of supersymmetric backgrounds in physics led to an interesting interplay between geometry and string theory. While often mathematics is used to advance our physical understanding of a system, string theory started to guide our understanding of mathematics. Perhaps the most famous example of this interplay comes from the mirror symmetry conjecture [70, 71]. This postulates that, for every Calabi-Yau 3-fold X with Hodge numbers[13] $h^{1,1}, h^{1,2}$, there exists a 'mirror Calabi-Yau' $\tilde{X}$ with Hodge numbers $\tilde{h}^{1,1} = h^{1,2}, \tilde{h}^{1,2} = h^{1,1}$. This surprising prediction seems unlikely from the point of view of complex geometry, but arises naturally when examining the 2-dimensional CFT describing the string on a Calabi-Yau background. Other examples include a conjecture for an extension to Yau's theorem [72] to particular non-Kähler manifolds [73, 74] that arose from studying supersymmetric backgrounds of heterotic strings. The link between string theory and geometry grew deeper with the discovery of the topological strings [75] which are obtained through a 'topological twist' of the full theory. It was found that certain correlation functions defined new geometric invariants of the target space, called Gromov-Witten invariants [76], which could be used to distinguish between different symplectic manifolds. Partition functions of other topological models can be written in terms of holomorphic Ray-Singer torsions which are an invariant of complex geometry [77]. While much of the focus has been on Calabi-Yau manifolds and non-kähler analogues, the desire to understand supersymmetric backgrounds of various dimensions has led to increased interest in extending these results to other geometries, such as G_2 structures in 7 dimensions. In particular, there has been a desire to find an analogue of Yau's theorem [78], and to find the associated topological string/M-theory [79].

Supersymmetric backgrounds provide a vital playground for understanding the behaviour of string theory at low string coupling. Phenomenologically, these provide the most likely candidate for a stringy model of the universe that we inhabit[14]. While the Calabi-Yau manifold is one particular example of a supersymmetric background, there are more general cases. Unfortunately, little is known about generic backgrounds away from the Calabi-Yau case. This thesis tries to answer questions about generic supersymmetric backgrounds including what is their precise geometry, and what is the structure of the moduli space. It is important to get a proper picture of the full landscape of string backgrounds to understand what kind of super-

[12] The massless scalar fields parameterise the moduli space of possible compactifications.

[13] The Hodge numbers on a complex manifold are the dimensions of the associated Dolbeault cohomology group.

[14] Conversely, using strong/weak dualities of string theory such as the AdS/CFT correspondence, we can gain insight into the strong coupling regime of certain field theories by studying weakly coupled strings propagating on some fixed geometry. Supersymmetric backgrounds have been the setting of most tests of the AdS/CFT correspondence and are where the duality is best understood [80, 81].

symmetric theories can be induced from string theory. In doing so, we may learn something about string/M-theory itself, and possibly shed some light on the mathematical conjectures surrounding certain supersymmetric backgrounds.

1.2 Plan of Thesis

This thesis focuses on understanding generic features of supersymmetric flux backgrounds of string and M-theory with a Minkowski external space. We study the geometry of the internal space with arbitrary fluxes turned on and determine the properties that lead to supersymmetry. These problems are most naturally studied within the framework of generalised geometry. It turns out that supersymmetric backgrounds have a consistent structure within generalised geometry given by integrable G-structures. These are described in terms of a complex tensor ψ, and a complex subbundle $L \subset E_{\mathbb{C}}$ which we call exceptional complex structures. The supersymmetry conditions are given by involutivity and a moment map. This is true for both heterotic and maximal strings. Using these we can begin to answer questions about generic points in the moduli space, including the local dimension and the form of the Kähler potential. This work may also have interesting implications for geometry. The formulation of supersymmetry constraints in terms of a moment map gives a possible link with geometric invariant theory. Using these tools, we may be able to find an analogue of Yau's theorem for G_2 structures and the Hull-Strominger system. Quantising the Kähler potential a la Pestun and Witten [3] may also provide information on a geometric subsector of M-theory. This thesis is a structured as follows.

We begin in chapter 2 with a review of the background material necessary for the thesis. A lot of the discussion of supersymmetric backgrounds is done in the language of G-structures and so we review some of the key ideas and constructions in 2.1. We provide the definition of a G-structure, introduce compatible connections and define integrability. We look at intrinsic torsion as an obstruction to the integrability of a structure. Finally, we briefly look at how holonomy and G-structures are related. In section 2.2, we review some of the previous work done on supersymmetric backgrounds of string theory. In particular, we look at them within the context of G-structures. We initially look at backgrounds without flux and see how different G-structures arise for M-theory, type II, and heterotic backgrounds. We then reintroduce flux and look at some resulting no-go theorems. We look at how flux creates non-zero values for the intrinsic torsion of the G-structures we found previously, and how this can affect the geometry. Finally, we briefly review how one builds the effective theory on the lower dimensional space from the geometric data of the internal space. This is well known for Calabi-Yau compactifications, but the arguments start to break down once we turn on flux. Section 2.3 is used to give a broad review of generalised geometry - the mathematical framework used to describe generic flux backgrounds throughout this thesis. We review the geometries required to describe the NSNS sector, heterotic backgrounds, and type II/M-theory backgrounds. In particular, we look at the construction of the generalised tangent bundle, the Dorfman derivative, generalised connections, and G-structures within this framework. Finally, in 2.4 we look at how generalised geometry has already been used to answer generic questions of supersymmetric flux backgrounds of string theory. In particular, we look at GMPT backgrounds and exceptional Calabi-Yau spaces.

Chapter 3 describes how one can use the language of integrable G-structures in $\mathrm{O}(6,6+n)$ geometry to describe generic flux backgrounds of heterotic strings. In section 3.1 we give a brief introduction to the Hull–Strominger system as well as a review of what is known about the infinitesimal moduli problem in terms of deformations of a holomorphic Courant algebroid. In section 3.2 we describe the formulation of heterotic backgrounds in terms of $\mathrm{O}(6,6+n)\times\mathbb{R}^+$ generalised geometry. In particular, these are given in terms of a generalised tensor ψ, and a bundle L_{-1}. We do this first for the case with no gauge bundle in section 3.2.1, then we reintroduce the gauge bundle in 3.2.2. We also discuss the equivalence between supersymmetry and integrability for the structures. In section 3.3, we explore involutivity of L_{-1} and give the superpotential in terms of ψ. We also show how these are related to the F-term conditions of the Hull–Strominger system. In section 3.4, we give the Kähler potential on the space of structures and derive a moment map for the action of generalised diffeomorphisms. We compute both of these explicitly and show that the moment map reproduces the final supersymmetry conditions, now with a geometric interpretation. This reinterpretation of the supersymmetry conditions as the vanishing of some moment map provides some interesting links with geometric invariant theory which we highlight in section 3.4.3. In section 3.5 we find the infinitesimal moduli and show that they are related to the previously known $\bar{D}$ cohomology.

In chapter 4 we extend the ideas explored in the previous chapter to exceptional generalised geometry. We start with $\mathrm{E}_{6(6)}\times\mathbb{R}^+$ geometry and apply it to $\mathcal{N}=1$ backgrounds with an external $\mathbb{R}^{4,1}$. In section 4.1 we review previous work on the generalised structures of $\mathcal{N}=1$ backgrounds. These are described in terms of exceptional Calabi-Yau spaces. In section 4.2 we restrict our analysis to a subset of the structures. These define an $\mathrm{SU}^*(6)$ structure and have properties reminiscent of the exceptional complex structures defined in the previous chapter. While they do not fully define a supersymmetric background we argue that one can still find interesting information by studying these backgrounds. In particular, in section 4.3 we find an expression for the generic moduli of these structures provided they satisfy a type of $\partial\bar{\partial}$-lemma. We argue that one can use this expression to find the number massless hypermultiplets in the 5 dimensional effective theory.

Chapter 5 is the main chapter of this thesis and analyses the generic structure of supersymmetric backgrounds of type II and M-theory for $\mathbb{R}^{3,1}$ external spaces. We find that the geometry is entirely described by exceptional complex structures similar to those defined in the previous chapters. These give simple analogues for the general analysis of $\mathrm{SU}(7)$ structures we then give in section 5.1. Section 5.2 shows explicitly how G_2 manifolds and the solutions of GMPT fit into the general analysis. Section 5.3 first shows how the involutivity and moment map conditions can be viewed as F- and D-term supersymmetry conditions in a rewriting of the supergravity as an effective $D=4$, $\mathcal{N}=1$ theory. It then connects our analysis to the Geometrical Invariant Theory picture and the G_2 functional of Hitchin. In particular, we see that Hitchin's extremisation is equivalent to finding the stationary points of the norm functional, and we go on to outline the naive connection to stability. Section 5.4 addresses the general moduli problem, and calculates the moduli of generic "type-0" structures (including G_2) and the full set of moduli of GMPT solutions.

This work may provide a natural generalisation to Hitchin functionals with applications to quantum corrections of M-theory effective actions. We explore these ideas in chapter 6. In section 6.1 we review the work done by Nigel Hitchin on functionals defining integrable $SL(3,\mathbb{C})$, $SU(3,3)$, and G_2 structures. We then look at how these theories have been quantised and applied to topological string theory in section 6.2. In section 6.3 we describe how the Kähler potential is a natural candidate for the Hitchin functional of exceptional complex structures, and we describe the quantisation procedure. In section 6.4 we briefly describe some possible physical applications of this quantised theory.

Chapter 7 is reserved for discussion and future directions of work. The appendices contain additional details which may be useful for the reader but are not important for the main text. They are arranged as follows. Appendix A lays out the conventions and notation that we use throughout the thesis. It contains the $O(d,d)$, $O(6,6+n)$, and exceptional algebras for both M-theory and type IIB solutions. We also give an embedding of the $O(6,6)$ algebra into the $E_{7(7)}\times\mathbb{R}^+$ algebra for type IIB. Appendix B discusses some results on curvature in exceptional generalised geometry. We show that, unlike $O(d,d)$ geometry, there is no obvious projection of the generalised Riemann curvature to get something tensorial. Appendix C reviews (generalised) complex structures in terms of moment maps. This is the picture we will extend to the exceptional case throughout the thesis. Appendix D we provide explicit calculations of the superpotential, the Kähler potential, and the moment map referenced in chapter 3. In appendix E we prove the general form of exceptional complex structures in $E_{6(6)}\times\mathbb{R}^+$ geometry and in appendix F we provide detailed calculations of the moduli of these exceptional complex structures in various cases outlined in chapter 4. In appendix G we provide a detailed calculation of the moduli of the GMPT backgrounds of chapter 5. Appendix H is used to state some properties of determinants of Laplacians that are need for chapter 6. Finally, in appendices I and J we derive the metric on the moduli space of exceptional complex structures around the points corresponding to Calabi-Yau and G_2 manifolds respectively. These are needed for some of the quantisation calculations done in chapter 6.

Chapter 2

G-Structures, Flux Backgrounds and Generalised Geometry

In this chapter, we will review some of the technical background required for the rest of the thesis. A lot of the discussion of supersymmetric backgrounds of string theory will be framed in the language of G-structures and so we will review those first. Then we will look at some of the work that has been done in the past on backgrounds both with and without flux. We will then go onto describe the construction of generalised geometry which will be the language that most of the thesis is framed within. Finally, we will look at how generalised geometry has been utilised in the past to describe supersymmetric backgrounds.

2.1 G-Structures

The presence of supersymmetry usually implies the existence of a G-structure on the manifold, where the G refers to some Lie group. These geometries have been studied by mathematicians, initially in terms of what they called the *local equivalence problem* [82]. They provide a fairly general class of geometries whose structure is rich enough that many complex problems can be solved. In this section we will provide a brief mathematical review of G-structures and give some examples, focusing on those of importance for string theory. We will follow [82–85] amongst others.

2.1.1 Definition of a G-Structure

Let M be a closed n-dimensional manifold and take $T \to M$ to be its tangent bundle, $T^* \to M$ to be its cotangent bundle. A frame at a point $p \in M$ is a set of n linearly independent vectors $v_i \in T_p$. We will take the set of frames at a point $p \in M$ to be the set

$$F_p = \{\{v_1, ..., v_n\} \text{ a frame at } p \in M\} \tag{2.1}$$

We define the frame bundle $F \to M$ to be the disjoint union

$$F = \bigsqcup_{p \in M} \{p, F_p\} \tag{2.2}$$

It is easy to see that F defines a $\mathrm{GL}(n, \mathbb{R})$ principal bundle over M. Indeed, $\mathrm{GL}(n, \mathbb{R})$ acts on an element $\mathbf{v} = \{v_1, ..., v_n\} \in F_p$ by left multiplication.

$$g \cdot \mathbf{v} = \{gv_1, ..., gv_n\} \in F_p \tag{2.3}$$

Moreover, this is a transitive and free action on the fibre and hence we get a $\mathrm{GL}(n, \mathbb{R})$ principal bundle. We can then define a G-structure on a manifold to be [82, 86]

Definition 1. Take G to be a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$. A G-structure is a G principal subbundle $P_G \to M$, $P_G \subset F$.

A local frame can be viewed as a local section $\sigma : M \to F$. Any local frame that is also a section of P_G is said to be compatible with the G-structure. We will sometimes drop the G when the group is clear by context.

There are many topological obstructions that can exclude the existence of a particular G-structure on a manifold. Famously, the first Chern class $C_1(X)$ of a Kähler manifold is the obstruction to the existence of an SU(3) structure [72]. In general, however, determining whether such a reduction of the frame bundle exists can often be a challenging problem to solve. Fortunately, in many cases, and in particular the cases we will be interested in, a G-structure is equivalent to the existence of a set of global, non-vanishing tensors that are preserved by the action of G[1].

Example 1. A Riemann structure is an $\mathrm{O}(n)$ structure and is defined by a metric g. An $\mathrm{O}(n)$ structure always exists on a manifold.

Example 2. An orientation is an $\mathrm{SL}(n, \mathbb{R})$ structure and is defined by a globally non-vanishing top form τ

Example 3. A complex structure is a $\mathrm{GL}(n/2, \mathbb{C})$ structure and is defined by a map $J : T \to T$, such that $J^2 = -1$

Example 4. An $\mathrm{SL}(n/2, \mathbb{C})$ structure is defined by a complex, separable $\frac{n}{2}$-form Ω

Example 5. A symplectic structure is an $\mathrm{Sp}(n)$ structure and is defined by a non-degenerate 2-form ω

Example 6. A hermitian structure is a $\mathrm{U}(n/2)$ structure and is defined by a complex structure and symplectic structure ω, J

Example 7. A Calabi-Yau structure is an $\mathrm{SU}(n/2)$ structure and is defined by a holomorphic 3-form and a symplectic structure Ω, ω

Example 8. A G_2 structure is defined by a positive, stable 3-form ϕ on a 7-manifold

Example 9. A Spin(7) structure is a defined by a stable 4-form χ on an 8-manifold

Note that examples 7, 8 and 9 are of particular importance in string theory as each of the groups is a subgroup of $\mathrm{SO}(n)$ and hence can define what are called special holonomy groups. These are the groups of interest for supersymmetric backgrounds as we will see later.

[1] This action is induced by that of $\mathrm{GL}(n, \mathbb{R})$ on tensors.

2.1.2 Equivalence and Local Flatness

We would like to have a notion when two G-structures are equivalent in some sense. Consider a diffeomorphism $f : M \to N$. This defines a map $\mathrm{d}f : TM \to TN$ that is a linear isomorphism on the fibres $T_pM \xrightarrow{\sim} T_{f(p)}N$. This induces a well defined principal bundle morphism

$$\begin{aligned} (\mathrm{d}f)_* : FM &\longrightarrow FN \\ \mathbf{v} &\longmapsto (\mathrm{d}f)_*\mathbf{v} = \{(\mathrm{d}f)v_1, ..., (\mathrm{d}f)v_n\} \end{aligned} \tag{2.4}$$

This leads to the following definitions.

Definition 2. Let M and N be smooth manifolds with G-structures $P_G \to M$, $Q_G \to N$. A smooth map $f : M \to N$ is said to be G-structure preserving if $(\mathrm{d}f)_*(P_G) \subseteq Q_G$.

Definition 3. Let $P_G \to M$ and $P'_G \to M$ be two G-structures on a manifold M. We say that the G-structures are equivalent if there exists a diffeomorphism $f : M \to M$ that is G-structure preserving. That is, if $(\mathrm{d}f)_*(P_G) = P'_G$.

There is an important concept in the mathematical literature relating to the local equivalence problem. We would like to know if there is an atlas on our manifold that is compatible with the G-structure.

Definition 4. A G-structure $P_G \to M$ is said to be *locally flat* if in every patch $\mathcal{U} \subset M$, there exists a coordinate chart x^μ such that the coordinate frame $\{\partial_1, ..., \partial_n\}$ is a local section of P_G.

Often, a G-structures is defined to be integrable if is (equivalent to) a locally flat G-structure [82]. We will take a slightly weaker version of integrability which is sometimes called first-order integrability. G-structures with this property are such that, at each point $p \in M$, there exists local coordinates x^μ such that the coordinate frame $\{\partial_1, ..., \partial_n\}_{p\in M}$ is in $(P_G)_p$. One can show that this is equivalent to the existence of a torsion-free compatible connection [87].

Example 10. By the fundamental theorem of Riemannian geometry, every $\mathrm{O}(n)$ structure is first-order integrable. A locally flat $\mathrm{O}(n)$ structure is equivalent to a flat metric.

Example 11. By the Newlander-Nirenberg theorem [88], any first-order integrable $\mathrm{GL}(n/2, \mathbb{C})$ structure is locally flat.

2.1.3 Connections, Torsion and Integrability

Given a principal bundle $P_G \to M$, we can define the vertical subspace of the tangent space at a point T_pP to be the kernel of the derivative of the projection. That is

$$\mathrm{Ver}_p(P) = \ker(\mathrm{d}\pi)_p \qquad \mathrm{Ver}(P) = \bigsqcup_{p\in P} \mathrm{Ver}_p(P) \tag{2.5}$$

While the vertical space is well defined, there does not exist a canonical definition of a horizontal subspace such that

$$T_pP = \mathrm{Ver}_p(P) \oplus \mathrm{Hor}_p(P) \quad \forall\, p \in P \tag{2.6}$$

Any choice of a smooth distribution $\mathrm{Hor}(P)$ such that the fibre at p is compliment to $\mathrm{Ver}_p(P)$ $p \in P$ is called a connection. However, a general choice does not necessarily provide the nice properties that we usually associate with a connection. Hence, we will refine the definition to be compatible with the G action on P.

Definition 5. A principal connection on a principal bundle $P_G \to M$ is a choice of smooth distribution $\mathrm{Hor}(P)$ such that $\mathrm{Hor}_p(P) \oplus \mathrm{Ver}_p(P) = T_pP \; \forall \, p \in P$ and $\mathrm{Hor}_{g \cdot p}(P) = \mathrm{d}\gamma_p(\mathrm{Hor}_p(P))$ where $\gamma_p : P \to P$ is the diffeomorphism induced by the right action of G on P.

There is a canonical isomorphism between the vertical space at a point and the Lie algebra $\mathfrak{g}$ associated to the structure group of P. One can show that a choice of horizontal subspace is equivalent to a choice of surjective map $\omega_p : T_pP \to \mathfrak{g}$ such that $\omega_p(\mathrm{Ver}_p) = \mathfrak{g}$. We can then define the horizontal subspace to be the kernel of this map

$$\mathrm{Hor}_p(P) = \ker \omega_p \tag{2.7}$$

We can view ω_p as a $\mathfrak{g}$-valued 1-form at p and hence we can extend its definition to the whole of P, which would define the distribution $\mathrm{Hor}(P)$. One can show that $\mathrm{Hor}(P)$ is smooth if and only if ω is smooth. Moreover, ω defines a principle connection if it is pseudo $\mathrm{Ad}\, G$-invariant. That is,

$$\gamma_g^* \omega = \mathrm{Ad}_{g^{-1}} \circ \omega \tag{2.8}$$

where Ad denotes the adjoint action of G on $\mathfrak{g}$.

Given a choice of connection, we can define projection maps onto the horizontal and vertical subspaces, $\mathrm{proj}_{H/V}$. Given this, we can define the covariant derivative ∇_v along the direction of some vector $v \in T_xM$ of some (local) section $\sigma : M \to P$. The covariant derivative at $x \in M$ along v is defined to be

$$\nabla_v \sigma = \mathrm{proj}_V(\mathrm{d}\sigma(v)) \tag{2.9}$$

A choice of connection on P also defines a connection on all associated bundles. Indeed, take N to be some space with a G-action and define $\Pi : P \times N \to P \times_G N$ to be the projection onto the associated bundle. The horizontal distribution of $P \times_G N$ is defined by

$$\mathrm{Hor}(P \times_G N) = \mathrm{d}\Pi(\mathrm{Hor}(P) \oplus \{0\}) \tag{2.10}$$

In particular, take $P = F$ the frame bundle, and $N \simeq \mathbb{R}^n$ to be the fundamental $\mathrm{GL}(n, \mathbb{R})$ module. The associated bundle in this case is precisely the tangent bundle and hence (2.10) tells us that every choice of horizontal space on the frame bundle defines a horizontal space on the tangent bundle. This defines a linear connection on T via (2.9) and the identification of $\mathrm{Ver}_p(T) \simeq T_{\pi(p)}M$. Conversely, it is possible to show that every linear connection ∇ on T defines a unique horizontal distribution $\mathrm{Hor}(T)$. There is then a unique principal connection $\mathrm{Hor}(F)$ on F such that the associated connection is $\mathrm{Hor}(T)$. This shows that there is a bijection

$$\{\text{Principal connections on } F\} \quad \longleftrightarrow \quad \{\text{Linear connections on } T\} \tag{2.11}$$

Of course, choosing the G-space N to be another $\mathrm{GL}(n, \mathbb{R})$ module in (2.10) would define the associated connection on all $\mathrm{GL}(n, \mathbb{R})$ vector bundles over M.

Suppose we have a restricted G-structure $P_G \subset F$. We get the following definition.

Definition 6. A linear connection ∇ on T is said to be compatible with the G-structure if the corresponding principal connection on F is a principal connection on P_G. That is, it is compatible if $\mathrm{Hor}(F) \subset TP$.

It is always possible to find a compatible connection to a G-structure. Indeed, any principal connection on P_G defines a principal connection on F via $\mathrm{Hor}(P) \subset TP \subset TF$. Suppose ∇ is a G-compatible connection. The induced connection (also denoted by ∇) on other $\mathrm{GL}(n, \mathbb{R})$ vector bundles $E \to M$ will have the following property. If $E = E_1 \oplus E_2$, where the E_i are G-submodules, then $\forall\, v \in T_x M$ we have that

$$\nabla_v e_i \in \Gamma(E_i) \quad \forall\, e_i \in \Gamma(E_i) \tag{2.12}$$

That is, ∇ respects the G-representation structure. If the G-structure is defined by a set of invariant tensors τ_i, then compatibility is equivalent to [85].

$$\nabla \tau_i = 0 \tag{2.13}$$

Given a linear connection on T, we define the torsion to be given by

$$t(v, w) = \nabla_v w - \nabla_w v - [v, w] \qquad v, w \in \Gamma(T) \tag{2.14}$$

If this vanishes for all v, w then we say the connection is torsion-free. In line with the comments about integrability made in the last section, we have the following definition.

Definition 7. We say a G-structure $P_G \subset F$ is *integrable*, or *torsion-free*, if there exists a compatible principal connection $\mathrm{Hor}(P) \subset TP \subset TF$ whose associated linear connection on T is torsion free.

While the existence of a compatible connection is guaranteed, the existence of a torsion-free compatible connection depends strongly on the geometry of the G-structure P_G and hence is a much more interesting problem. We will review an obstruction to the existence of an integrable G-structure called *intrinsic torsion.*

2.1.4 Intrinsic Torsion of a G-Structure

Suppose we have a G-structure $P_G \subset F$ with compatible connections ∇, ∇' on T. Recall a linear connection is an $\mathbb{R}$-linear map obeying the Leibniz property.

$$\begin{aligned} \nabla : T &\longrightarrow T^* \otimes T \\ fv &\longmapsto \mathrm{d}f \otimes v + f \nabla v \end{aligned} \tag{2.15}$$

We can then view the difference $\nabla - \nabla'$ as a map : $T \to T^* \otimes T$ such that

$$\begin{aligned}(\nabla - \nabla')(fv) &= \nabla(fv) - \nabla'(fv) \\ &= \mathrm{d}f \otimes v + +f\nabla v - \mathrm{d}f \otimes v - f\nabla' v \\ &= f(\nabla - \nabla')v\end{aligned} \tag{2.16}$$

Hence, the difference between two connections is a tensor $\Sigma \in \Gamma(T^* \otimes \operatorname{End} T)$. Since ∇, ∇' are compatible, we know that they are associated to $\mathfrak{g}$-valued 1-forms ω, ω' on P_G. Given a local frame compatible with the G-structure $s : M \to P_G$, we find that

$$(\nabla - \nabla')_s = s^*(\omega - \omega') \in \Gamma(T^*) \otimes \mathfrak{g} \tag{2.17}$$

The frame-independent statement of this fact if that $\Sigma \in \Gamma(T^* \otimes \operatorname{ad} P_G)$.

One can also show the converse. Namely, if ∇ is a connection on T that is compatible with a G-structure P_G, then for any $\Sigma \in (T^* \otimes \operatorname{ad} P_G)$ the connection $\nabla' = \nabla + \Sigma$ is also a compatible connection. Hence, the affine space of G-compatible connections $\mathcal{A}_G$ is isomorphic to

$$\mathcal{A}_G \simeq \Gamma(T^* \otimes \operatorname{ad} P_G) \tag{2.18}$$

We would like to understand whether $\exists \nabla \in \mathcal{A}_G$ that is torsion free. Suppose we have a connection ∇_0 with torsion t_0, and a connection $\nabla' = \nabla_0 + \Sigma$ with torsion t'. Then we find

$$\begin{aligned}t'(v,w) - t_0(v,w) &= \nabla'_v w - \nabla'_w v - [v,w] - (\nabla_{0\,v} w - \nabla_{0\,w} v - [v,w]) \\ &= \Sigma(v) \cdot w - \Sigma(w) \cdot v\end{aligned} \tag{2.19}$$

Consider a map

$$\begin{aligned}\tau : \Gamma(T^* \otimes \operatorname{ad} P_G) &\longrightarrow \Gamma(\wedge^2 T^* \otimes T) \\ \tau(\Sigma)(v,w) &\longmapsto \Sigma(v) \cdot w - \Sigma(w) \cdot v\end{aligned} \tag{2.20}$$

$\operatorname{im} \tau \subset \Gamma(\wedge^2 T^* \otimes T)$ denotes the subset of the space of torsions that can be affected by changing ∇. Therefore, the affine space of torsions of G-compatible connections $\mathcal{T}_G$ is precisely

$$\mathcal{T}_G = t_0 + \operatorname{im} \tau \tag{2.21}$$

The statement that there is a torsion-free connection is equivalent to saying that $0 \in \mathcal{T}_G$, or equivalently, that $t_0 \in \operatorname{im} \tau$. This is perhaps a less appealing definition as it requires the existence of a reference connection ∇_0.

We define the space of intrinsic torsions to be the quotient space

$$W^{\text{int}} := \Gamma(\wedge^2 T^* \otimes T) / \operatorname{im} \tau \tag{2.22}$$

For any given G-structure and compatible connection ∇, the projection of t onto W^{int} is independent of the choice of connection by construction. Hence, this is a property of the G-structure itself, rather than any connection. We call the value a given G-structure takes its *intrinsic tor-*

sion. A torsion-free compatible connection exists if and only if the intrinsic torsion of the G-structure vanishes.

Example 12. W^{int} for an SU(3) structure can be shown to decompose into SU(3) representations as [89]

$$W^{\text{int}} \sim (\mathbf{1} \oplus \mathbf{1}) \oplus (\mathbf{8} \oplus \mathbf{8}) \oplus (\mathbf{6} \oplus \overline{\mathbf{6}}) \oplus 2(\mathbf{3} \oplus \overline{\mathbf{3}}) \tag{2.23}$$

So we can represent the intrinsic torsion for an SU(3) structure with 5 tensors $W_1, ..., W_5$. $W_1 \in \mathbf{1} \oplus \mathbf{1}$ is a complex scalar, $W_2 \in \mathbf{8} \oplus \mathbf{8}$ is a complex primitive (1,1)-form, $W_3 \in \mathbf{6} \oplus \overline{\mathbf{6}}$ is a real primitive $(2,1) + (1,2)$ form, and W_4, W_5 are real 1-forms. One can show that these obey the following

$$\mathrm{d}\omega = \frac{3}{2}\,\mathrm{im}(\bar{W}_1\Omega) + W_4 \wedge \omega + W_3 \qquad \mathrm{d}\Omega = W_1\omega \wedge \omega + W_2 \wedge \omega + \bar{W}_5 \wedge \Omega \tag{2.24}$$

Hence an SU(3) structure is integrable if and only if $\mathrm{d}\Omega = \mathrm{d}\omega = 0$.

Example 13. W^{int} for a G_2 structure can be shown to decompose into G_2 representations as [90]

$$W^{\text{int}} \sim \mathbf{1} \oplus \mathbf{7} \oplus \mathbf{14} \oplus \mathbf{27} \tag{2.25}$$

We can therefore represent the intrinsic torsion of a G_2 structure with 4 real tensors $X_1, ..., X_4$. X_1 is a scalar, X_2 is a 1-form, $X_3 \in \Omega^2_{14}(M)$, and $X_4 \in \Omega^3_{27}(M)$. One can show that they satisfy

$$\mathrm{d}\phi = X_1 * \phi + X_4 \wedge \phi + X_3 \qquad \mathrm{d} * \phi = \frac{4}{3} X_4 \wedge *\phi + X_2 \wedge \phi \tag{2.26}$$

Hence, a G_2 structure is integrable if and only if $\mathrm{d}\phi = \mathrm{d} * \phi = 0$.

2.1.5 Relation Between G-Structures and Holonomy

Often, the backgrounds relevant to string theory are said to be ‘special holonomy manifolds’ [91, 92]. These are backgrounds with restricted holonomy $\mathrm{Hol}(\nabla) \subset \mathrm{SO}(d)$. We have seen that every G-structure on a manifold has a compatible connection associated to it. One might expect that the holonomy of these compatible connections would also be restricted in some way. Here we will briefly describe the relation between G-structures and holonomy. First, we define holonomy of a connection on a principal bundle.

Definition 8. Let $P \to M$ be a principal bundle with fibre G, and $D = \mathrm{Hor}(P) \subset TP$ be a connection on P. We define the holonomy of P at $p \in P$ to be $\mathrm{Hol}_p(P, D) = \{g \in G \,|\, p \sim g \cdot p\}$, where $p \sim q$ if there exists a horizontal curve γ in P from p to q.

One can show that $\mathrm{Hol}_p(P, D)$ is a subgroup of G and depends on the choice of p only up to conjugation. Hence, we define $\mathrm{Hol}(P, D)$ to be the isomorphism class of $\mathrm{Hol}_p(P, D)$. If we take $P = F$ the frame bundle of M, then there exists an associated linear connection ∇ on T, whose holonomy group is defined in the usual way which we will denote by $\mathrm{Hol}(\nabla)$. Both of these groups are some subgroup of $\mathrm{GL}(n, \mathbb{R})$ and in fact, one can show that

$$\mathrm{Hol}(F, D) = \mathrm{Hol}(\nabla) \tag{2.27}$$

We will assume that $\text{Hol}(\nabla)$ is a connected, closed subgroup of $\text{GL}(n, \mathbb{R})$ as those will be the cases of most interest to us.

Consider a point $p \in F$ and define the set $Q = \{q \in F \,|\, q \sim p\}$. One can show that $Q \subset F$ defines a submanifold and that $H = \text{Hol}(F, D)$ acts freely on it. Hence, $Q \to M$ defines a principal H bundle. Moreover, one can show that the connection D is compatible with Q, i.e. $D \subset TQ$. This shows that the existence of a connection ∇ on T with holonomy H implies the existence of an H structure on M with compatible connection ∇. It is in fact possible to prove the converse [93]. Hence, there exists a connection ∇ with holonomy group H if and only if there exists an H-structure $P_H \subset F$ on M. Hence, the question of existence of holonomy groups becomes a topological question of the existence of principal subbundles of the frame bundle.

The question of which holonomy groups can appear via torsion free connections is a much more interesting problem. The problem for a general manifold is quite hard, however a list of connected, torsion-free, Riemannian holonomy groups that can exist on an open ball of $\mathbb{R}^n$ was originally put forward by Berger [94] and has since been extended [95]. The groups of interest for supersymmetric backgrounds are examples of what are called special holonomy groups. In 6, 7 and 8 dimensions respectively, these special holonomy groups are

$$\text{SU}(3) \qquad \text{G}_2 \qquad \text{Spin}(7) \tag{2.28}$$

While these are important in the study of supersymmetric backgrounds, manifolds with these holonomy groups do not provide a complete classification, as we will see.

2.2 Flux Backgrounds

Understanding the full landscape of string theory backgrounds has been a central area of research for a long time. Supersymmetric backgrounds in particular are attractive as an arena to study string theory for many reasons. Firstly, it has been shown that, at least in the context of M-theory and type II, if a background satisfies the supersymmetry conditions, and the fluxes satisfy the Bianchi identities, then the background will satisfy the equations of motion [96,97]. Hence, we can reduce a system of second order PDEs to a system of first order PDEs. This substantially simplifies the problem and makes finding solutions much easier. Secondly, if our universe is supersymmetric then it is likely to be broken to $\mathcal{N} = 1$ at the compactification scale, and then further broken by some four-dimensional affects, such as gaugino condensation [98–100]. Supersymmetry also restricts the form of the moduli space and ensures it has a Kähler structure. This simplification allows us to make progress in analysing its structure.

In this section we will review some of the work that has been done on supersymmetric backgrounds, both with and without flux. We will follow the work of [101] closely, and mostly look at compactifications down to $D = 4$ Minkowski. As we will see, most of the discussion of supersymmetric backgrounds has been done in the language of (non-)integrable G-structures. We will first review backgrounds in which the flux is turned off and determine the G-structures implied by supersymmetry. We will then reintroduce the fluxes and see how they affect the integrability. Finally, we will look at how the properties of the lower dimensional effective theory[2] are determined by the geometrical data of the internal manifold. For example, we will see how the massless scalar fields are determined by the moduli of M_{int}.

2.2.1 Compactifications without Flux

We shall take the following ansatz for the spacetime manifold and metric.

$$\mathbb{M} = M_{\text{ext}} \times M_{\text{int}} \qquad \mathrm{d}s^2 = \mathrm{e}^{2\Delta(y)} \tilde{g}_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + g_{mn} \mathrm{d}y^m \mathrm{d}y^n \tag{2.29}$$

where x^μ are coordinates on M_{ext}, y^m coordinates on M_{int}, and where Δ, the warp factor, is a function on M_{int}. We also take the configuration to be maximally symmetric on M_{ext} which implies that the external metric $\tilde{g}$ is AdS or Minkowski[3], and that all the background fermions vanish.

The condition that this background preserves some supersymmetry is the requirement that the supersymmetric variation of all the fields vanishes. In any background, we assume all the fermions are turned off. Here we further assume that the fluxes are turned off. By Grassmann parity, the supersymmetry variations of all the bosonic fields must always involve the background fermions. Since we have assumed these are zero, the supersymmetry conditions on the bosons are automatically satisfied.

$$\delta_\epsilon(\text{metric}) = 0 \qquad \delta_\epsilon(\text{flux}) = 0 \tag{2.30}$$

We therefore turn our attention to the variation of the fermion fields. In any supergravity theory there are gravitino(s) ψ, plus possible dilatinos λ, and gauginos χ. The precise combination of these depends on whether we are working in heterotic, type II or M-theory backgrounds. We will address each of these in turn.

M-Theory Backgrounds

For M-theory backgrounds, we only have one gravitino ψ. With all the fluxes turned off, the only relevant bosonic field is the 11 dimensional metric $\hat{g}$ and the variation of the fermion is simply [103–106]

$$\delta\psi = \hat{\nabla}\epsilon \tag{2.31}$$

where $\hat{\nabla}$ is the Levi-Civita connection on the 11 dimensional space $\mathbb{M}$. Hence, supersymmetry is equivalent to the existence of a globally defined parallel spinor ϵ. From the discussion in the previous section, we see that this implies a restricted holonomy group and hence the geometry of the spacetime is given by a reduced structure group P_G for some $G \subset \mathrm{Spin}(10,1)$. We would like to see what this implies for the internal manifold M_{int}.

Considering the parts of (2.31) along external and internal directions respectively, we find

$$\left(\tilde{\nabla}_\mu + \frac{1}{2}\tilde{\gamma}_\mu\tilde{\gamma}_5 \otimes \nabla\!\!\!/\Delta\right)\epsilon = 0 \tag{2.32}$$

$$\nabla_m \epsilon = 0 \tag{2.33}$$

[2] We will often refer to this simply as the effective theory.

[3] We will not consider de Sitter vacua here as they do not permit supersymmetric backgrounds. Moreover, it been suggested that there are no meta-stable non-supersymmetric de Sitter vacua of string theory at all [102]

where $\tilde{\nabla}$ is the Levi-Civita connection of $\tilde{g}$, ∇ is the Levi-Civita connection of g, and $\tilde{\gamma}$ are the γ-matrices of $\tilde{g}$ under the decomposition $\mathrm{Cliff}(10,1) \to \mathrm{Cliff}(3,1) \times \mathrm{Cliff}(7)$. From equation (2.32) we find that

$$[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu]\epsilon = -\frac{1}{2}\left(\tilde{\gamma}_{\mu\nu} \otimes (\nabla\!\!\!/ \Delta)^2\right)\epsilon \tag{2.34}$$

However, we also have

$$[\tilde{\nabla}_\mu, \tilde{\nabla}_\mu]\epsilon = \frac{1}{4}R_{\mu\nu\rho\sigma}\tilde{\gamma}^{\rho\sigma}\epsilon = \frac{\Lambda}{6}\tilde{\gamma}_{\mu\nu}\epsilon \tag{2.35}$$

where Λ is the cosmological constant associated to the maximally symmetric external metric $\tilde{g}$. We therefore find that

$$\frac{\Lambda}{3} + (\nabla\!\!\!/ A)^2 = 0 \tag{2.36}$$

Hence $(\nabla\!\!\!/ \Delta)^2$ is constant. However, on a compact manifold, the only constant value that $(\nabla\!\!\!/ \Delta)^2$ can take is 0. Hence, the warp factor Δ is constant and the cosmological constant must vanish. That is, the external space must be Minkowski. This provides our first no-go theorem.

Given the decomposition of the Clifford algebra, we can decompose ϵ into internal and external spinors

$$\epsilon = \eta_+(x) \otimes \xi(y) + \eta_- \otimes \xi^* \tag{2.37}$$

where $\eta_\pm$ are chiral spinors on the external space of opposite chirality, and ξ is a complex spinor on the internal space. Equations (2.32), (2.33) then become

$$\tilde{\nabla}_\mu \eta_\pm = 0 \qquad \nabla_m \xi = 0 \tag{2.38}$$

This translates to $\eta_\pm$ being some constant spinor on $\mathbb{R}^{3,1}$ and ξ being a parallel spinor on M_{int}. The constant external spinors $\eta_\pm$ generate the $\mathcal{N}=1$ rigid supersymmetry of the effective theory. The parallel internal spinor tells us we have a reduced holonomy group $\mathrm{Hol}(\nabla) \subset \mathrm{Spin}(7)$ and hence an integrable $\mathrm{Hol}(\nabla)$-structure.

To find the structure group, we need to consider the stabiliser of the internal spinor ξ. This is the complex combination of two real Spin(7) spinors $\xi = \zeta_1 + i\zeta_2$, which are each individually stabilised by some $G_2 \subset \mathrm{Spin}(7)$ wherever they are non-vanishing. This provides us with two interesting cases.

$$\begin{array}{ccc} \zeta_1 = \zeta_2 & \longleftrightarrow & G_2 \\ \zeta_1, \zeta_2 \text{ linearly independent} & \longleftrightarrow & SU(3) \end{array} \tag{2.39}$$

It is worth noting that, unlike in the other cases, the existence of these G-structures alone is no restriction to M_{int}. This is because any 7 dimensional spin manifold actually has an SU(2) structure [107]. However, the requirement that these G-structures are integrable, as is implied by the compatibility of the Levi-Civita connection, is a restriction. We should also note that the cases outlined above are not the most general cases for the spinor ξ. We can have more complicated geometries that are not so easily defined by conventional G-structures [108].

Type II Backgrounds

Here we find that, while some of the details are different, the story is largely the same as for the M-theory case. Now we have two gravitinos ψ^i and two dilatinos λ^i of opposite (the same) chirality for type IIA (IIB). The variation of these fields with all the fluxes turned off are [109–111]

$$\delta_\epsilon \psi = \hat{\nabla}\epsilon \qquad \delta_\epsilon \lambda = (\hat{\nabla}\!\!\!/\hat{\varphi})\epsilon \tag{2.40}$$

where $\hat{\varphi}$ is the dilaton. Here we have combined the gravitinos and dilatinos into one object $\psi = (\psi^1, \psi^2)$, $\lambda = (\lambda^1, \lambda^2)$. The supersymmetry parameter ϵ now consists of two chiral spinors $\epsilon = (\epsilon^1_+, \epsilon^2_\mp)$, where the upper sign corresponds to IIA and the lower to IIB.

We can run through the same argument as before and learn that the vanishing of the first equation in (2.40) implies that the warp factor must vanish and the external space must be Minkowski. We decompose the supersymmetry parameters under $\mathrm{Cliff}(9,1) \to \mathrm{Cliff}(3,1) \times \mathrm{Cliff}(6)$ as

$$\begin{aligned} \epsilon^1_+ &= \eta^1_+(x) \otimes \xi^1_+(y) + \eta^1_- \otimes \xi^1_- \\ \epsilon^2_\mp &= \eta^2_+(x) \otimes \xi^2_\mp(y) + \eta^2_- \otimes \xi^2_\pm \end{aligned} \tag{2.41}$$

where the $\xi^i_\pm$ are chiral spinors on 6 dimensions, and $\xi^{i*}_+ = \xi^i_-$. From this decomposition, we learn that the η^i must be constant spinors on $\mathbb{R}^{3,1}$ and the ξ^i must be parallel spinors on M_{int}. The second equation in (2.40) just implies that the dilaton is constant.

The existence of the parallel spinors implies a reduced holonomy group and hence a G-structure on the manifold. Again, we need to analyse the stabiliser group of the ξ^i of which there are two interesting cases.

$$\begin{array}{ccc} \xi^1 = \xi^2 & \longleftrightarrow & \mathrm{SU}(3) \\ \xi^1, \xi^2 \text{ linearly independent} & \longleftrightarrow & \mathrm{SU}(2) \end{array} \tag{2.42}$$

Interestingly, in both cases, the external spinors η^i are independent of one another. These individually generate supersymmetry in the effective theory and hence, when there is no flux, the background automatically preserves $\mathcal{N} = 2$. We therefore require flux for a genuine $\mathcal{N} = 1$ theory coming from type II.

Heterotic Backgrounds

Heterotic backgrounds have been the focus of many phenomenological efforts in string theory due to the ease of finding models that produce chiral Fermions and the Standard Model gauge group [67, 68, 112, 113]. They were first studied in [69] where they gave the supersymmetry variations of the single gravitino ψ, the dilatino λ, and the gauginos χ^a. We will work in the string frame where the warp factor is trivial. For vanishing fluxes, the supersymmetry variations become

$$\delta_\epsilon \psi = \hat{\nabla}\epsilon \qquad \delta_\epsilon \lambda = \sqrt{2}(\hat{\nabla}\!\!\!/\hat{\varphi})\epsilon \qquad \delta_\epsilon \chi^a = -\frac{1}{4}\mathrm{e}^{\hat{\varphi}}\gamma^{mn}F^a_{mn}\epsilon \tag{2.43}$$

Here we have required the vanishing of the gauge-invariant and Lorentz-invariant field strength [114, 115]

$$H = \mathrm{d}B + \frac{1}{4}\alpha'(\omega_3(A) - \omega_3(\Theta)) \tag{2.44}$$

where A and Θ are the connections on the gauge bundle and tangent bundle on M_{int} respectively, and ω_3 is the Chern-Simons term of the relevant connection.

Splitting the gravitino variation into internal and external pieces, we once again see that the external space must be flat[4] and that the internal space has a single parallel spinor ξ, where

$$\epsilon = \eta_+(x) \otimes \xi_+(y) + \eta_- \otimes \xi_- \tag{2.45}$$

Unlike in the previous cases, we only have one internal spinor and hence the geometry of the internal space is always given by an integrable SU(3) structure.

The vanishing of the dilatino term tells us that the dilaton must be constant. The gaugino variation can be re-expressed in the following way

$$0 = \delta_\epsilon \chi^a \qquad \Leftrightarrow \qquad 0 = \frac{1}{2}\xi^\dagger \gamma^{mn} F^a_{mn} \xi = \omega^\# \lrcorner F^a \tag{2.46}$$

where we have used the SU(3) structure of the internal space to write the spinor bilinear in terms of a symplectic form ω. We therefore require A to be an instanton [116,117] and to satisfy a constraint coming from the exterior derivative of (2.44) which states

$$\text{Tr}(F \wedge F) = \text{Tr}(R \wedge R) \tag{2.47}$$

where R is the curvature 2-form defined by Θ.

2.2.2 Compactifications with Flux

We have seen that, in many cases, the geometry of fluxless compactifications can be described in terms of some integrable G-structure. As described in section 2.1, these geometries have nice properties and are well understood. Despite the attraction of these simple geometries, people have looked for backgrounds that include non-trivial fluxes. While these break the integrability of the G-structures, they solve some of the problems that arise in the cases above. For example, in type II, fluxes were introduced to find genuine $\mathcal{N} = 1$ compactifications, while fluxes in other circumstances have been used to reduce the typically large number of moduli. We have also seen that flux is required to support a non-zero cosmological constant and hence find AdS solutions.

The key point in the conventional analysis of backgrounds with flux is that the geometry must be defined in terms of torsionful connections. That is, the flux generates a non-zero value of the intrinsic torsion of the G-structure. The precise value of the intrinsic torsion can have different implications of the geometry of the internal space.

The way fluxes enter the equations of motion and the supersymmetry conditions have led to many no-go theorems for fluxed compactifications [118–122]. For the moment I will just focus on the fairly general theorem[5] of [121] which states that there cannot be a reduction of type II or M-theory[6] down to Minkowski space where the internal space M_{int} is compact, non-singular, and there exists non-trivial n-form flux for $n > 1$. The statement of this theorem requires only the equations of motion, not supersymmetry and so is fairly broad. The proof of the statement

[4] In fact, in [69] they show that the external space must be flat even for non-vanishing flux H

[5] In fact, arguments that non-zero flux imposes a negative cosmological constant first appeared in [123,124].

[6] They even show this is true for massive IIA but we won't be considering that case here.

runs as follows. Consider the Einstein equations in the full supergravity

$$R_{MN} = T_{MN} - \frac{1}{d-2}\,\hat{g}_{MN}\,T^L{}_L \tag{2.48}$$

where $d = 10$ or 11 is the dimension of the full spacetime, and $\hat{g}$ is the metric on $\mathbb{M}$. We take the metric as in (2.30) but assume the external metric to be Minkowski. Considering the components of the Ricci curvature along external directions, we find

$$R_{\mu\nu} = -\eta_{\mu\nu}(\nabla^2 A + 2(\nabla A)^2) = T_{\mu\nu} - \frac{1}{d-2}\,\mathrm{e}^{2A}\eta_{\mu\nu}\,T^L{}_L \tag{2.49}$$

Contracting with η on both sides we find

$$-2\mathrm{e}^{-2A}\left(\nabla^2 \mathrm{e}^{2A}\right) = \mathrm{e}^{2A}\left(T^\mu{}_\mu - \frac{4}{d-2}T^L{}_L\right) \tag{2.50}$$

We proceed to show that the right hand side of (2.50) is always non-negative, and is 0 if and only if the only non-zero flux is a 1-form.

For non-singular geometries, the energy-momentum tensor only gets contributions from n-form fluxes. They contribute a term of the kind

$$T_{MN} = F_{ML_1\ldots L_{n-1}}F_N{}^{L_1\ldots L_{n-1}} - \frac{1}{2n}g_{MN}F^2 \tag{2.51}$$

Hence the contribution to the right hand side of (2.50) is

$$\tilde{T} = T^\mu{}_\mu - \frac{4}{d-2}T^L_L = -F_{\mu L_1\ldots L_{n-1}}F^{\mu L_1\ldots L_{n-1}} + \frac{4}{d-2}\left(1-\frac{1}{n}\right)F^2 \tag{2.52}$$

We want fluxes that do not break the isometry of the external Minkowski space and hence they must have entirely internal legs or 4 external legs. If F has only internal legs then the first term on the right hand side of (2.52) vanishes and $F^2 > 0$. Hence we get a strictly positive contribution to $\tilde{T}$ unless $n = 1$. In that case, the contribution vanishes identically. If F has 4 external legs then one can show that

$$F_{\mu L_1\ldots L_{n-1}}F^{\mu L_1\ldots L_{n-1}} = \frac{4}{n}F^2 \tag{2.53}$$

Hence, we get the contribution

$$\tilde{T} = -F^2\frac{4(d-n-1)}{n(d-2)} > 0 \tag{2.54}$$

where we have used the fact that F must have a temporal component and hence $F^2 < 0$.

Finally, we return to (2.50). We multiply both sides by e^{2A} and integrate over M. Since we have assumed M is compact with no singularities, the left hand side vanishes. However, the right hand side is strictly positive for any non-zero n-flux for $n > 1$. This is a contradiction and proves the result.

There are ways around the proof. For example, one can add higher derivative corrections to the equations of motion coming from string theory. If we include these, it has been shown that the positivity condition no-longer holds and you can have warped compactifications [125–127]. Another possibility is to add localised sources. These will produce singularities in the geometry which will contribute when we integrate (2.50) over the manifold. We need

$$\int_{M_{\text{int}}} e^{4A}(\tilde{T}_{\text{flux}} + \tilde{T}_{\text{loc}}) = 0 \tag{2.55}$$

and hence the local sources must contribute negatively to the energy-momentum tensor. We know that branes can be sources for flux, and so we can consider their contribution. For example, a D_p-brane along spacetime and wrapping an internal $p-3$ cycle Σ, we find the contribution to the energy momentum tensor (in the Einstein frame) is [128]

$$\tilde{T}_{\text{loc}} = \frac{7-p}{2} T_p\, \delta(\Sigma) \qquad T_p = \left(2\pi\sqrt{\alpha'}\right)^{-(p+1)} e^{\frac{p-3}{4}\hat{\varphi}} > 0 \tag{2.56}$$

Therefore, for $p < 7$, the brane also contributes positively[7]. We also get similar contributions from other branes [106, 120]. To cancel these contributions, we need objects of negative tension. Fortunately, orientifold planes provide such objects and can be inserted to avoid this no-go theorem[8] [129, 130].

While these no-go theorems are important in understanding string-theory compactifications, we shall not address them in this thesis. Instead, one should consider the type II or M-theory geometries we describe to either be non-compact, or describe the geometry away from sources. We now turn to how flux can affect the intrinsic torsion of the G-structures outlined previously. We will be working with the assumption that $M_{\text{int}} = \mathbb{R}^{3,1}$. Throughout this thesis, we will be using the democratic formalism for RR fluxes of type II [131]. Since we are also assuming that the fluxes do not break the isometry of $\mathbb{R}^{3,1}$, we will sometimes write a flux with 4 external legs as the dual of an entirely internal flux.

M-Theory Backgrounds

M-theory backgrounds with flux have been studied in many contexts [104, 106, 108, 122, 129, 130, 132–134]. We will focus on understanding how the introduction of flux affects the intrinsic torsion of the G-structures outlined previously. The supersymmetry variation of the gravitino with arbitrary 4-form flux $\hat{G}$ is [122]

$$\delta_\epsilon \psi_M = \left[\hat{\nabla}_M - \frac{1}{144}\left(\hat{\gamma}_M{}^{NPQR} - 8\delta^N{}_M \hat{\gamma}^{PQR}\right)\hat{G}_{NPQR}\right]\epsilon \tag{2.57}$$

where the $\hat{\gamma}$ are the full 11 dimensional γ-matrices. Taking the ansatz for the for the spinor as in (2.37), we find that the external spinors $\eta_\pm$ must be chiral and constant, as before. We then find the following conditions on the internal spinors [104]

$$0 = \left[\left(\mu i e^{-\Delta} + \frac{1}{2}\nabla\!\!\!\!/\,\Delta\right) + \frac{1}{144} G_{mnpq}\gamma^{mnpq}\right]\xi \tag{2.58}$$

[7] We see that $p \geq 7$ avoids this issue and hence F-theory is free from this no-go theorem.

[8] In fact there are further constraints on the number and type of branes and orientifold planes due to the tadpole cancellation condition [101].

$$0 = \left[\nabla_m + \frac{1}{144}\left(G_{pqrs}\gamma^{pqrs}{}_m - 8G_{mnpq}\gamma^{npq}\right)\right]\xi \tag{2.59}$$

Here, we have written the 4-form flux $\hat{G}$ as a piece along the external space and a piece along the internal space.

$$\hat{G} = \frac{3}{4!}\mu\epsilon_{\mu\nu\rho\sigma}(\mathrm{d}x)^{\mu\nu\rho\sigma} + \frac{1}{4!}G_{mnpq}(\mathrm{d}y)^{mnpq} \tag{2.60}$$

This is the most general form of $\hat{G}$ that does not break the isometry of the external Minkowski space.

From analysing equations (2.58), (2.59), [122] were able to show that, provided the 7-manifold only had a G_2 structure and nothing more refined, one could not solve these equations with non-trivial flux without curving the external space. However, as expressed earlier, any 7-manifold that is spin also has an SU(2) structure. This, in particular, implies the existence of a G_2 3-form ϕ, along with a vector field v. [104] were then able to use these structures to put constraints on the intrinsic torsion of the G_2 structure arising from the supersymmetry conditions.

From equation (2.59), we see that the internal Levi-Civita connection is no longer compatible with the internal spinor ξ. If we assume that the manifold has at least a G_2 structure generated by ξ, then this tells us that the compatible connection has some torsion piece that depends on the internal 4-form flux G. We can define the G_2 3-form by a bilinear in ξ

$$\phi_{abc} = -\mathrm{i}\xi^\dagger\gamma_{abc}\xi \qquad *\phi_{abcd} = \xi^\dagger\gamma_{abcd}\xi \tag{2.61}$$

As we saw in (2.26), the intrinsic torsion of the G_2 structure is precisely measured by $\mathrm{d}\phi$ and $\mathrm{d}*\phi$. The torsion classes take the form

$$\mathrm{d}\phi = X_1 * \phi + X_4 \wedge \phi + X_3 \qquad \mathrm{d}*\phi = \frac{4}{3}X_4 \wedge *\phi + X_2 \wedge \phi \tag{2.62}$$

Replacing the exterior derivative with the Levi-Civita connection and using (2.59), (2.61) we can find the values of the X_i in terms of G. This then puts constraints on the possible torsion classes that can arise. Moreover, one can put constraints on the possible values of G that lead to a solution of the supersymmetry equations. We will not explicitly give the answer here since it is messy and not particularly illuminating for the remainder of the thesis. The results are, however, given in [104].

We note that even with the introduction of the flux, this is not the most general form an M-theory internal manifold. This is because we have still had to make some assumptions about the internal spinor, such as it gives a well defined G_2 structure. There could be cases in which the real or imaginary part of ξ degenerate at some points on the manifold. The G_2 description would break down at those points.

Type II Backgrounds

A comprehensive review of flux backgrounds of type II theories with an SU(3) structure is given in [101]. The supersymmetry variations of the 2 gravitinos and the 2 dilatinos are given in the string frame by

$$\delta_\epsilon \psi_M = \hat{\nabla}_M \epsilon + \frac{1}{8} H_{MNP} \Gamma^{NP} \mathcal{P} \epsilon + \frac{1}{16} \mathrm{e}^{\hat{\varphi}} \sum_n \not{F}_n \Gamma_M \mathcal{P}_n \epsilon \tag{2.63}$$

$$\delta_\epsilon \lambda = \left(\not{\partial} \hat{\varphi} + \frac{1}{2} \not{H} \mathcal{P} \right) \epsilon + \frac{1}{8} \mathrm{e}^{\hat{\varphi}} \sum_n (-1)^n (5-n) \not{F}_n \mathcal{P}_n \epsilon \tag{2.64}$$

where H is the NS 3-form flux, the F_n are RR fluxes, $\hat{\varphi}$ is the dilaton and the $\mathcal{P}, \mathcal{P}_n$ are operators acting on the doublet $\epsilon = (\epsilon^1, \epsilon^2)$ and depend on whether we are in IIA or IIB. For IIA we have

$$\mathcal{P} = \Gamma_{11} \qquad \mathcal{P}_n = (\Gamma_{11})^{n/2} \sigma^1 \tag{2.65}$$

For IIB we have

$$\mathcal{P} = -\sigma^3 \qquad \mathcal{P}_n = \begin{cases} \sigma^1 & \frac{n+1}{2} \text{ even} \\ \mathrm{i}\sigma^2 & \frac{n+1}{2} \text{ odd} \end{cases} \tag{2.66}$$

To ensure we preserve only $\mathcal{N} = 1$ supersymmetry, we need to assume that the external spinors η^i in (2.41) are proportional to one another. We will consider this case here and examine more general cases later. We can absorb that proportionality into the definition of the internal spinor as follows.

$$\begin{aligned} \epsilon^1 &= \eta_+(x) \otimes (a\, \xi_+(y)) + \eta_- \otimes (\bar{a}\, \xi_-) \\ \epsilon^2 &= \eta_+(x) \otimes (b\, \xi_\mp(y)) + \eta_- \otimes (\bar{b}\, \xi_\pm) \end{aligned} \tag{2.67}$$

We also are assuming that the internal manifold only has an SU(3) structure and so the internal spinors are equal. We then split the supersymmetry conditions (2.63), (2.64) into internal and external pieces. We find that the external spinor has to be constant and we get a complicated set of conditions on the internal spinor. Much like in the G_2 case, we find that the Levi-Civita connection is no longer compatible with the internal spinor $\xi_\pm$. Instead, the compatible connection depends on the fluxes H, F_n which arrange themselves into the torsion classes of the SU(3) structure.

We define the SU(3) structure through the spinor bilinears

$$\omega_{mn} = \mp 2\mathrm{i}\, \xi_\pm^\dagger \gamma_{mn} \xi_\pm \qquad \Omega_{mnp} = -2\mathrm{i}\, \xi_- \gamma_{mnp} \xi_+ \tag{2.68}$$

As we saw in (2.24), the torsion classes of the SU(3) structure are defined by $\mathrm{d}\omega$ and $\mathrm{d}\Omega$ via

$$\mathrm{d}\omega = \frac{3}{2} \operatorname{im}(\bar{W}_1 \Omega) + W_4 \wedge \omega + W_3 \qquad \mathrm{d}\Omega = W_1 \omega \wedge \omega + W_2 \wedge \omega + \bar{W}_5 \wedge \Omega \tag{2.69}$$

where the W_i are differential forms falling into irreducible SU(3) representations. The different values that the W_i take depend on the fluxes and describe different geometries for the internal space. Moreover, there are strict constraints on the possible values of the flux that give solutions to the supersymmetry equations and the Bianchi identities.

Solutions to the supersymmetry constraints for various fluxes have been thoroughly studied [106, 135–141] and a summary of all possible $\mathcal{N} = 1$ Minkowski flux backgrounds with SU(3) structure was given in [142, Sec 4]. We won't give the full summary as it is again complicated and not that illuminating for the rest of the thesis. However, we will provide some details on certain cases that can arise. This, of course, depends on whether we are in IIA or IIB. We find the following possible constraints on the torsion classes

$$\begin{array}{cc} \textbf{IIA} & \textbf{IIB} \\ W_1 = W_2 = 0, \quad \bar{W}_5 = 2W_4 & W_1 = W_2 = 0, \quad \bar{W}_5 = 2W_4 \\ W_1 = W_3 = W_4 = 0 & W_1 = W_2 = W_3 = 0 \\ & W_1 = W_2 = W_4 = 0 \end{array} \tag{2.70}$$

The torsion classes that do not vanish depend on the fluxes. The following table describes the possible geometries that arise when various torsion classes vanish [101]

Geometry	**Torsion Classes**
Complex	$W_1 = W_2 = 0$
Symplectic	$W_1 = W_3 = W_4 = 0$
Half Flat	$\operatorname{im} W_1 = \operatorname{im} W_2 = W_4 = W_5 = 0$
Special hermitian	$W_1 = W_2 = W_4 = W_5 = 0$
Nearly Kähler	$W_2 = W_3 = W_4 = W_5 = 0$
Almost Kähler	$W_1 = W_3 = W_4 = W_5 = 0$
Kähler	$W_1 = W_2 = W_3 = W_4 = 0$
Calabi-Yau	$W_1 = W_2 = W_3 = W_4 = W_5 = 0$
Conformal Calabi-Yau	$W_1 = W_2 = W_3 = 3W_4 - 2W_5 = 0$

We observe that for type IIA, we either have a complex manifold, or a symplectic manifold. For type IIB however, the manifold is always complex but there may be a more refined structure depending on which other torsion classes vanish. We should note that the IIB cases are not exhaustive as there exist solutions that interpolate between the different cases [143]

Heterotic Backgrounds

Heterotic backgrounds with non-vanishing flux are free from the no-go theorems of type II and M-theory because of the presence of gauge fields and the higher α' corrections to the equations of motion. Therefore, these have been studied greatly over the years [69, 144–150]. The flux compactifications were first studied in [144, 146] and resulted in the Hull-Strominger system. This is a set of differential conditions on the geometry of the internal manifold that are equivalent to the supersymmetry constraints. We will leave a more complete review of the Hull-Strominger system to section 3.1 and will just comment on how the flux affects the torsion of the G-structure.

As noted, a heterotic background preserving $\mathcal{N} = 1$ supersymmetry always has an SU(3) structure. This simple structure implies the existence of a symplectic form ω and a complex 3-form Ω which we can build out of spinor bilinears as in (2.68). The supersymmetry conditions imply the following for the differential forms

$$\mathrm{d}(\mathrm{e}^{-2\hat{\varphi}}\Omega) = 0 \qquad \mathrm{i}(\partial - \bar{\partial})\omega = H \tag{2.71}$$

These equations describe the torsion classes of the SU(3) structure. Comparing with (2.24) we see that

$$W_1 = W_2 = 0 \quad W_5 = 2\mathrm{d}\phi \tag{2.72}$$

In particular, we see that we always have a complex structure J, and hence the Dolbeault differentials $\partial, \bar{\partial}$ are well-defined. The torsion classes W_3, W_4 will depend on H through the equation

$$\mathrm{d}\omega = J \cdot H \tag{2.73}$$

2.2.3 The Effective Theory

An important aspect of compactifications of string theory is understanding the effective theory in the lower dimensional space. This arises by doing a Kaluza-Klein-like reduction of all the fields on the full space $\mathbb{M}$ to the external $\mathbb{R}^{3,1}$ and truncating[9] to the massless modes. In the cases where the compactifications are described by integrable G-structures, the effective theory is well understood and can be described in terms of the moduli space of the G-structure.

The massless scalar fields in the effective theory should parameterise some space, often called the *scalar manifold* or *moduli space*. If all the scalar fields are constant on $\mathbb{R}^{3,1}$ then this should constitute a vacuum of the effective theory. Hence, the scalar manifold should be the moduli space of supersymmetric backgrounds of the theory. In the cases that the fluxes are turned off we know that the geometry can be described in terms of an integrable G-structure. Hence, the scalar manifold is related to the geometric moduli space of the G-structure. On the other hand, the massless fields of the effective theory should lift to fields of the full supergravity theory in 10 or 11 dimensions and can be written as perturbations around some fixed background. In practice, this means we can expand the fields on the whole space into internal and external pieces, where the internal pieces label the geometric moduli.

In the case of G_2 manifolds for M-theory backgrounds, the geometric moduli space is locally diffeomorphic to $H^3(M)$ [151]. Given a harmonic basis ρ_i of $H^3(M)$ we can expand the perturbations of the metric and 3-form A as[10]

$$A(x, y) = \alpha^i(x)\rho_i(y) \qquad \delta\phi(x, y) = \beta^i(x)\rho_i(y) \tag{2.74}$$

Since the G_2 3-form ϕ determines the internal metric, the second term induces the metric perturbation. The functions α^i, β^i form the scalar fields in the effective theory. We know this theory is supersymmetric and so they should combine into the complex scalars ξ^i of the chiral fields. From supersymmetry arguments, we know that the kinetic term should be

$$\int_{\mathbb{R}^{3,1}} K_{ij} \mathrm{d}\xi^i \wedge *\mathrm{d}\bar{\xi}^i \tag{2.75}$$

where K_{ij} is the Kähler metric on the scalar manifold. It is possible to show [152,153] that this

[9]In general this will not be a consistent truncation. More accurately, we want to find the effective theory by integrating out the massive modes.

[10]There can be other terms in the expansion of the metric and A, however these would not be scalars from the point of view of $\mathbb{R}^{3,1}$

Kähler metric has Kähler potential given by

$$\mathcal{K} \sim \ln \int_M \psi \wedge *\bar{\psi} \qquad \psi \in H^3(M)_{\mathbb{C}} \tag{2.76}$$

In the case of Calabi-Yau compactification of type II (or heterotic), the geometric moduli is locally split into Kähler and complex moduli [154]. This means that we can expand the perturbations of the supergravity fields in terms of a harmonic basis ω_a of $H^{1,1}(M)$ - the Kähler moduli - and a harmonic basis χ_k of $H^{2,1}(M)$. We will also use a real harmonic basis μ_K of $H^3(M)$. The NS sector decomposes as

$$\begin{aligned} \delta g_{mn} &= \mathrm{i}\bar{z}^k(x)\left(\tfrac{(\bar{\chi}_k)_{m\bar{p}\bar{q}}\Omega^{\bar{p}\bar{q}}{}_n}{|\Omega|^2}\right) \qquad & \delta g_{m\bar{n}} &= \mathrm{i}v^a(x)(\omega_a)_{m\bar{n}}(y) \\ \hat{\varphi}(x,y) &= \hat{\varphi}(x) & B(x,y) &= b^a(x)\omega_a(y) \end{aligned} \tag{2.77}$$

where we have used complex coordinates on the Calabi-Yau manifold. The RR sector in IIA decomposes as[11]

$$C_1(x,y) = 0 \qquad C_3(x,y) = \xi^K(x)\mu_K(y) \tag{2.78}$$

and the RR sector in IIB decomposes as

$$C_0(x,y) = C_0(x) \qquad C_2(x,y) = c^a(x)\omega_a(y) \qquad C_4(x,y) = \rho^a(x)(*\omega_a)(y) \tag{2.79}$$

Since Calabi-Yau compactifications define $\mathcal{N} = 2$ supersymmetry, these scalars will align into hypermultiplets, vector multiplets, and tensor multiplets. The scalars in the vector multiplets are given by

$$\text{IIA:} \quad t^a = b^a + \mathrm{i}v^a \qquad\qquad \text{IIB:} \quad z^k \tag{2.80}$$

The kinetic term in the action for these fields takes the exact same form as in (2.75) except that the Kähler potential in each case is given by [154–158]

$$\mathcal{K}_{\text{IIA}} \sim \ln \int_M \omega \wedge \omega \wedge \omega \qquad\qquad \mathcal{K}_{\text{IIB}} \sim \ln \int_M \Omega \wedge \bar{\Omega} \tag{2.81}$$

The scalars in the hypermultiplet have a similar kinetic term where the metric is hyper-Kähler and was found in [159].

As we have seen, to break the supersymmetry further one needs to include fluxes. Introducing fluxes has the benefit of inducing a superpotential [157, 158] and hence a potential for the moduli. Ensuring that this potential is minimised in the vacuum can fix some of the typically large number of moduli. Unfortunately, once we turn on fluxes for the gauge potentials, their expansion in terms of harmonic forms breaks down entirely. In fact, identifying the correct geometric moduli has been an extremely difficult problem in general. For small flux, one can try to argue that the moduli will be those of a Calabi-Yau, except some will attain a mass. This process is called moduli stabilisation but breaks down for larger fluxes [160, 161].

It is clear that understanding the geometric moduli space of supersymmetric backgrounds is key to understanding the dynamics of the effective theory on $\mathbb{R}^{3,1}$. However, it has been an

[11] C_1 does not provide any scalar fields in $\mathbb{R}^{3,1}$ since there are no harmonic 1-forms on a Calabi-Yau. It will contribute vector fields to the action but we will not discuss those here.

open problem for some time to know exactly what this is. The work in this thesis makes a significant step in solving that problem.

2.3 Generalised Geometry

Having a clear understanding of the geometry of the internal manifold of compactifications has been vital to the study of the effective physics of string theory in lower dimensions. Unfortunately, conventional G-structures give a far from complete picture. As we have seen, the introduction of flux breaks the integrability of the G-structure, and classifying the possible torsion classes can be an involved process. Moreover, as we have suggested above, the G-structures may not even be globally well-defined if the internal spinors degenerate at certain points on the manifold. On top of this, understanding the moduli space, the Kähler potential, and superpotential is a largely unknown problem except for some specific cases. Conventional geometry seems limited in attaining a complete picture of the geometry and moduli space of string backgrounds.

Fortunately, the last 20 years have seen the development of a new geometric formalism that is ideally suited to the full degrees of freedom of supergravity. This formalism, called *generalised geometry*, works by considering objects defined not on the tangent bundle $T \to M$, but on a Leibniz algebroid $E \to M$, that has some local decomposition into vectors and differential forms. There is some enlarged structure group $\mathrm{GL}(d, \mathbb{R})$[12] $\subset \mathcal{G} \subset \mathrm{GL}(\mathrm{rk}\, E, \mathbb{R})$ and the local decomposition of E is determined by the group $\mathcal{G}$. We shall call these structures $\mathcal{G}$-generalised geometry, or just $\mathcal{G}$-geometry.

All $\mathcal{G}$-generalised geometries are examples of transitive, local, locally split, closed-form Leibniz algebroids. These were studied and classified by Baraglia in [162]. However, their first appearance was in [163] for $\mathrm{O}(d, d)$ geometry, and developed in the work of Hitchin and Gualtieri [164, 165]. Other examples involving exceptional groups were later discovered [166], as well as others suitable for different aspects of string theory [167, 168]. In this section we will review the mathematical background of generalised geometries relevant in string theory.

2.3.1 $\mathrm{O}(d, d)$ Geometry

The first instance of generalised geometry was that defined in [164, 165]. There, they considered the bundle

$$E = T \oplus T^* \tag{2.82}$$

E is often called *the generalised tangent bundle* and it has a natural $\mathrm{O}(d, d)$ structure on it, where d is the dimension of the manifold, given by the inner product

$$\eta(x + \xi, y + \eta) = \frac{1}{2}\left(\xi(y) + \eta(x)\right) \tag{2.83}$$

[12]Here, as in the rest of the thesis, d will correspond to the dimension of M_{int}.

If we take a frame $\hat{e}_i$ of T, with dual frame e^i, then we can express the inner product with the matrix

$$\eta(\cdot,\cdot) = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \tag{2.84}$$

which is clearly preserved by $\mathrm{O}(d,d)$. The algebra of symmetries of this structure have an expression in terms of natural geometric bundles. This bundle is called the adjoint bundle[13]

$$\mathrm{ad}\,\tilde{F} = (T \otimes T^*) \oplus \wedge^2 T^* \oplus \wedge^2 T \tag{2.85}$$

This has a natural action on E given in appendix A.2

This bundle has a natural antisymmetric bracket $[\![\cdot,\cdot]\!]$ that acts on sections of E that gives the structure $(E, \eta, [\![\cdot,\cdot]\!], a)$ the structure of a Courant algebroid [163,169,170]. Here $a : E \to T$ is the natural projection and is called the *anchor map*. The Courant bracket is given by[14]

$$[\![x+\xi, y+\eta]\!] = [x,y] + \mathcal{L}_x\eta - \mathcal{L}_y\xi - \frac{1}{2}\mathrm{d}(x \lrcorner \eta - y \lrcorner \xi) \tag{2.86}$$

This is not a Lie bracket since it fails the Jacobi identity. One can check that

$$\begin{aligned} \mathrm{Jac}(X,Y,Z) &= [\![X,Y]\!]Z]\!] + \text{cyclic perms} \\ &= \frac{1}{3}\mathrm{d}\left(\eta([\![X,Y]\!],Z) + \text{cyclic perms}\right) \end{aligned} \tag{2.87}$$

One can define a derivative operator, called the *Dorfman derivative*[15] which acts as

$$L_X Y = \mathcal{L}_x y + \mathcal{L}_x \eta - y \lrcorner \mathrm{d}\xi \tag{2.88}$$

Then we have

$$[\![X,Y]\!] = \frac{1}{2}\left(L_X Y - L_Y X\right) \qquad \mathrm{d}\eta(X,Y) = \frac{1}{2}\left(L_X Y + L_Y X\right) \tag{2.89}$$

The collection (E, η, L, a) is a *Leibniz algebroid* [171–173] and turns out to be useful in the study of string theory backgrounds.

Symmetries and H-Twists

The Dorfman derivative is clearly covariant under usual diffeomorphisms of the manifold. Moreover, we can also consider transformations by 2-forms B given by

$$\mathrm{e}^B(x+\xi) = x + \xi + x \lrcorner B \tag{2.90}$$

[13] This is indeed the adjoint bundle defined by taking the $\mathrm{O}(d,d)$ frame bundle $\tilde{F}$, and the Lie algebra $\mathfrak{o}(d,d)$ and defining $\mathrm{ad}\,\tilde{F} = (\tilde{F} \times \mathfrak{o}(d,d))/\mathrm{O}(d,d)$

[14] Here, and in the rest of the thesis, $\lrcorner$ denotes the interior product of a multivector and form. The conventions we use are given in appendix A.1.

[15] This is sometimes called the generalised Lie derivative as it plays an analogous role to the Lie derivative in conventional differential geometry.

Putting this into the derivative, we find that

$$L_{e^B X} e^B Y = e^B L_X Y + y \lrcorner (x \lrcorner dB) \tag{2.91}$$

If $B \in \Omega^2_{cl}(M)$ is closed, then B is a symmetry of the Dorfman derivative. We call the symmetry group the set of *generalised diffeomorphisms* and it is given by the semi-direct product

$$\mathrm{GDiff} = \mathrm{Diff} \ltimes \Omega^2_{cl}(M) \sim \mathrm{Diff} \ltimes \Omega^2_{ex}(M) \tag{2.92}$$

The second term is taken in a local patch. We see from (2.88) that this is precisely the set of transformations generated by L_X. The statement that these are symmetries of the Leibniz structure is precisely the statement that L_X acts as a derivation on itself.

Notice, however, that if we choose any closed[16] 3-form $H \in \Omega^3(M)$ then we can define another differential operator via

$$L^H_X Y = \mathcal{L}_x y + \mathcal{L}_x \eta - y \lrcorner d\xi + y \lrcorner (x \lrcorner H) \tag{2.93}$$

We call this the *H-twisted Dorfman derivative* and it has the same symmetries as the untwisted case. We also see from (2.91) that

$$L^H_{(e^B X)}(e^B Y) = L^{H+dB}_X Y \tag{2.94}$$

Hence, by reparameterising the sections $X, Y \in \Gamma(E)$, we can move between different representatives of the cohomology class of H. Inequivalent twists are labelled by $H^3(M)$.

In a local patch $\mathcal{U}_i \subset M$, we can always find some $B_i \in \Omega^2(\mathcal{U}_i)$ such that $H = dB_i$. Hence, on this local patch we can calculate the twisted Dorfman derivative via the untwisted

$$L^H_X Y = L_{e^{B_i} X} e^{B_i} Y \tag{2.95}$$

However, this is just a local expression. We know that on an overlap $\mathcal{U}_i \cap \mathcal{U}_j$, the potentials B_i, B_j are related by an exact form $B_j = B_i + d\Lambda_{ji}$. For the sections to be globally well-defined we require

$$X = e^{B_i}(x_i + \xi_i) = e^{B_j}(x_j + \xi_j) \qquad \text{on } \mathcal{U}_i \cap \mathcal{U}_j \tag{2.96}$$

where the $x_i \in \Gamma(\mathcal{U}_i, T)$, $\xi_i \in \Omega^1(\mathcal{U}_i)$ are local vector fields and 1-forms. For this to be true, we require patching of the 1-forms and vector fields to mix into one another. That is, we require

$$x_i = x_j \tag{2.97}$$

$$\xi_i = \xi_j + x_j \lrcorner d\Lambda_{ji} \tag{2.98}$$

This implies that the $X_i = x_i + \xi_i$ are in fact local sections of a twisted bundle E_H which is

[16] Closure is required to ensure this still satisfies the axioms of a Leibniz algebroid. Namely we need $[L^H_X, L^H_Y] = L^H_{L^H_X Y}$

defined to be an extension of the tangent bundle by the cotangent bundle.

$$T^* \longrightarrow E_H \longrightarrow T \tag{2.99}$$

This provides an alternative description of twists by a 3-form H. One can either consider the untwisted bundle with a twisted Dorfman derivative, or consider the usual Dorfman derivative acting on a twisted bundle. The gauge potential $H = \mathrm{d}B$ defines an isomorphism between the Leibniz algebroids

$$(E_H, L) \qquad \longleftrightarrow \qquad (E, L^H) \tag{2.100}$$

In this thesis we will use these pictures interchangeably as they are equivalent. Because of this, we will drop the H subscript in future discussions.

In fact, the patching has to obey particular compatibility conditions on triple intersections implying the bundle E_H is 'twisted by a gerbe', and B the properties of a *connective structure on a gerbe* [174]

$$\Lambda_{ji} + \Lambda_{ik} + \Lambda_{kj} = \mathrm{d}\Lambda_{kji} \qquad \text{on } \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k \tag{2.101}$$

Generalised Spinors

Sections of E, along with the inner product on E, define a Clifford algebra Cliff(E) defined by

$$X \cdot X = \eta(X, X) \tag{2.102}$$

This has a natural representation representation on the exterior algebra $\wedge^\bullet T^*$ given by

$$\not{X}\Phi = (\not{x} + \not{\xi})\Phi = x \lrcorner \Phi + \xi \wedge \Phi \tag{2.103}$$

where $x + \xi \in E$, $\Phi \in \wedge^\bullet T^*$. We use the slashed notation to denote the Clifford action to mimic the usual notation of contraction with γ-matrices. It is easy to check that the Clifford condition (2.102) is met. One can also check that if $A \in \mathfrak{gl}(d, \mathbb{R})$ then the induced Clifford action is

$$\not{A}\Phi = A \cdot \Phi + \frac{1}{2}(\mathrm{Tr}\, A)\Phi \tag{2.104}$$

where $\cdot$ denotes the usual adjoint action of $\mathfrak{gl}(d)$ on polyforms. This implies that the spinor bundle S is actually isomorphic to $\wedge^\bullet T^* \otimes (\det T)^{1/2}$ as a representation of Spin(d, d). To account for this $\det T^*$ factor, we often include an $\mathbb{R}^+$ factor in the structure group. Physically this relates to the trombone symmetry of supergravity backgrounds. We can then embed the physical GL$(d, \mathbb{R})$ subgroup such that the spinor bundle takes the form

$$S = \wedge^\bullet T^* = \wedge^{\mathrm{ev}} T^* \oplus \wedge^{\mathrm{odd}} T^* = S_+ \oplus S_- \tag{2.105}$$

On the right hand side we have decomposed S into irreducible Spin(d, d) representations. We call sections of $S_\pm$ chiral spinors of positive/negative chirality respectively.

There is a natural O(d, d)-invariant pairing on spinors that takes values in $\det T^*$, called the

Mukai pairing. It is given by

$$(\Phi, \Psi) = \sum_n (-1)^{\lfloor n/2 \rfloor} \Phi_{d-n} \wedge \Psi_n \tag{2.106}$$

Generalised Metrics

A generalised metric is a reduction of the structure group to its maximally compact subgroup $O(d,d) \to O(d) \times O(d)$[17]. This is equivalent to defining a splitting of E into $C_+ \oplus C_-$ such that the inner product η is positive/negative definite on $C_\pm$ respectively, and they are orthogonal. This defines a conventional metric on E via

$$G(V,W) = \eta(V,W)_{C_+} - \eta(V,W)_{C_-} \tag{2.107}$$

The subscript denotes the restriction of those vectors to the relevant subspaces.

Any conventional metric g defines a generalised metric on the untwisted space via

$$C_\pm = \{v \pm g(v,\cdot) \,|\, v \in T\} \qquad \Rightarrow \qquad G = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \tag{2.108}$$

To generalise this to include the H twist we need to multiply $C_\pm$ by e^B. Then we find that the metric is given by

$$G = \mathrm{e}^B \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \mathrm{e}^{-B} = \begin{pmatrix} g - Bg^{-1}B & -Bg^{-1} \\ g^{-1}B & g^{-1} \end{pmatrix} \tag{2.109}$$

In fact, it is possible to show that any generalised metric takes this form, and hence uniquely defines a B and a metric via the formula above.

Generalised Complex Structures

Generalised complex structures are the analogue of complex structures in $O(d,d)$ geometry [164,165]. It is defined by a complex structure on E that preserves the $O(d,d)$ structure. That is

$$\mathcal{J} : E \longrightarrow E \qquad \mathcal{J}^2 = -1 \qquad \eta(\mathcal{J}X, \mathcal{J}Y) = \eta(X,Y) \tag{2.110}$$

This defines a reduction of the structure group $O(d,d) \to U(\frac{d}{2}, \frac{d}{2})$. We can decompose the complexified generalised tangent bundle into eigenspaces of $\mathcal{J}$

$$E_{\mathbb{C}} = L_1 \oplus L_{-1} \tag{2.111}$$

where L_n is the ni-eigenbundle of $\mathcal{J}$. Clearly $L_{-1} = \overline{L_1}$ and $L_1 \cap L_{-1} = \{0\}$. One can also show that

$$\eta(L_1, L_1) = 0 \tag{2.112}$$

[17] We can choose generalised metrics of different signature by taking $O(p,q) \times O(q,p)$.

These conditions are equivalent to $\mathcal{J}$ and so we get the alternative definition of generalised complex structure.

Definition 9. A *generalised complex structure* is a $\mathrm{U}(\frac{d}{2},\frac{d}{2})$ structure and is defined by a sub-bundle $L \subset E_{\mathbb{C}}$ satisfying the following.

i) L is maximally isotropic - $\dim_{\mathbb{C}} L = d$, and $(L, L) = 0$

ii) L has real index 0 - $L \cap \overline{L} = \{0\}$

An object satisfying only (i) is called a Dirac structure [163]

Given a generalised complex structure, we can define a unique line bundle $\mathcal{U} \subset S_{\mathbb{C}}$ that satisfies

$$\not{L}_1 \Phi = 0 \quad \forall\, \Phi \in \Gamma(\mathcal{U}) \tag{2.113}$$

given any non-vanishing local section $\Phi \in \Gamma(\mathcal{U})$, we have $\left(\Phi, \bar{\Phi}\right) \neq 0$. Φ is called the *pure spinor*[18] associated to the generalised complex structure. We can also decompose $S_{\mathbb{C}}$ into eigenbundles of $\mathcal{J}$ and we find

$$S_{\mathbb{C}} = \sum_{n=-3}^{3} S_n \qquad \mathcal{U} = S_3 \tag{2.114}$$

Definition 10. A generalised complex structure is integrable if L_1 is involutive under the Courant bracket[19]

$$[\![L_1, L_1]\!] \subseteq L_1 \tag{2.115}$$

A generalised complex structure is integrable if and only if any section $\Phi \in \Gamma(\mathcal{U})$ satisfies

$$\mathrm{d}\Phi = \not{X}\Phi \qquad \text{some } X \in \Gamma(E) \tag{2.116}$$

Moreover, if the generalised complex structure is integrable then one can show that the exterior derivative splits into generalised Dolbeault operators $\mathrm{d} = \partial + \bar{\partial}$ satisfying

$$\partial : S_n \longrightarrow S_{n+1} \qquad \bar{\partial} : S_n \longrightarrow S_{n-1} \tag{2.117}$$

Example 14. A complex structure J defines a generalised complex structure

$$\mathcal{J}_J = \begin{pmatrix} J & 0 \\ 0 & -J^{\dagger} \end{pmatrix} \qquad L_1 = T^{1,0} \oplus T^{*0,1} \qquad \mathcal{U} = \wedge^{0,3} T^* \tag{2.118}$$

This is integrable if and only if the complex structure is integrable.

Example 15. A symplectic structure ω defines a generalised complex structure

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & \omega^{-1} \\ -\omega & 0 \end{pmatrix} \qquad L_1 = \mathrm{e}^{\mathrm{i}\omega} T_{\mathbb{C}} \qquad \mathcal{U} = \left\langle \mathrm{e}^{-\mathrm{i}\omega} 1 \right\rangle \tag{2.119}$$

[18] Note that given any line bundle $\mathcal{U} \subset S_{\mathbb{C}}$, we can find a subbundle of $L \subset E_{\mathbb{C}}$ from (2.113). A section $\Phi \in \Gamma(\mathcal{U})$ is said to be a pure spinor if the subbundle L is a Dirac structure.

[19] The isotropy condition means we can equally define integrability as involutivity under the Dorfman derivative.

Generalised Calabi-Yau Structures

We can define a refined structure if the pure spinor bundle $\mathcal{U}$ is trivial.

Definition 11. A *generalised Calabi-Yau structure* [164] is an $\mathrm{SU}(\frac{d}{2},\frac{d}{2})$ structure and is defined by a global non-vanishing pure spinor $\Phi \in \Gamma(S_{\mathbb{C}})$ satisfying

$$\left(\Phi, \bar{\Phi}\right) \neq 0 \tag{2.120}$$

The generalised Calabi-Yau structure is integrable if $\mathrm{d}\Phi = 0$.

2.3.2 $\mathrm{O}(6, 6+n)$ Geometry

We can define a slight generalisation of the generalised geometry above to involve the geometry of gauge connections on some principle bundle $P_G \to M$. These are precisely the degrees of freedom relevant for Heterotic string theory and the geometry was first defined in [167]. We will also restrict ourselves to the case that $\dim M = 6$ as that will be the case of interest in chapter 3. The structure defined by $T \oplus T^*$ is often called an exact Courant algebroid [175] because the sequence (2.99) is an exact sequence on fibres. The isomorphism classes of such algebroids are labelled by $H^3(M)$ [176]. The geometry relevant for heterotic strings is a non-exact Courant algebroid and is defined by the following set of extensions[20]

$$\begin{array}{ccccc} T^* & \longrightarrow & E & \longrightarrow & E' \\ \mathrm{ad}\, P_G & \longrightarrow & E' & \longrightarrow & T \end{array} \tag{2.121}$$

This has a natural inner product of signature $\mathrm{O}(6, 6+n)$ where $n = \dim G$, and we have assumed G is compact. It is given by

$$\eta(v + \Sigma + \lambda, w + \Lambda + \sigma) = \frac{1}{2}(\lambda(y) + \sigma(x)) + \mathrm{Tr}(\Sigma\Lambda) \tag{2.122}$$

As before, there is an isomorphism between E and an untwisted bundle $T^* \oplus \mathrm{ad}\, P_G \oplus T$ given by a choice of gauge connections. These gauge connections are $A \in \Gamma(T^* \otimes \mathrm{ad}\, P_G)$, $B \in \Omega^2(M)$ and give globally well-defined field strengths

$$F = \mathrm{d}A + A \wedge A \qquad H = \mathrm{d}B + \omega_3(A) \tag{2.123}$$

$\omega_3(A)$ is the Chern-Simons term for the connection A. We can define a twisted Dorfman derivative via

$$\begin{aligned} L^{H+F}_V W = {} & \mathcal{L}_v w \\ & + \mathcal{L}_v \sigma - w \lrcorner \mathrm{d}\lambda + w \lrcorner (v \lrcorner H) + 2\,\mathrm{Tr}(\Lambda \mathrm{d}_A \Sigma) - 2\,\mathrm{Tr}(\Lambda(v \lrcorner F)) + 2\,\mathrm{Tr}(\Sigma(w \lrcorner F)) \\ & + [\Sigma, \Lambda] + v \lrcorner \mathrm{d}_A \Lambda - w \lrcorner \mathrm{d}_A \Sigma - v \lrcorner (w \lrcorner F) \end{aligned} \tag{2.124}$$

[20] There have been other Courant algebroids used for studying heterotic strings which have been called holomorphic string algebroids [74,177–179]. These are very closely related to the bundle defined in this section, being locally equivalent to $T^{1,0} \oplus \mathrm{ad}\, P_G \oplus T^{*1,0}$. These algebroids can be used to describe supersymmetric backgrounds and have a very close relation to the exceptional complex structures we define later in chapter 3. The generalised geometry defined here, however, can be applied to non-supersymmetric backgrounds as well.

where $\mathrm{d}_A\Lambda = \mathrm{d}\Lambda + [A, \Lambda]$. As before, we can equivalently work in either the twisted bundle, or the twisted derivative picture.

2.3.3 $\mathrm{E}_{d(d)} \times \mathbb{R}^+$ Geometry

We have seen that the NS sector and the gauge sector can be unified into one geometric formalism, a statement that will be made more clear in the following sections. In this section we will review generalised geometry which incorporates all the degrees of freedom of supergravity. The structure group is $\mathrm{E}_{d(d)} \times \mathbb{R}^+$ for M-theory, and $\mathrm{E}_{d+1(d+1)} \times \mathbb{R}^+$ for type II. This was first developed [166] and later studied in [180–182].

The generalised tangent bundle

The relevant vector bundle for generalised geometry is the generalised tangent bundle

$$E \simeq \begin{cases} T \oplus \wedge^2 T^* \oplus \wedge^5 T^* \oplus (T^* \otimes \wedge^7 T^*) & \text{M-theory} \quad (2.125a) \\ T \oplus (S \otimes T^*) \oplus \wedge^3 T^* \oplus (S \otimes \wedge^5 T^*) \oplus (T^* \otimes \wedge^6 T^*) & \text{Type IIB} \quad (2.125b) \end{cases}$$

where $S \simeq \mathbf{2}$ of $\mathrm{SL}(2, \mathbb{R})$. Note that some of the terms above will vanish for $D \geq 5$. While everything we say in this chapter applies to both M-theory and type IIB compactifications, we will mostly focus on M-theory for ease. We should note that (2.125a), (2.125b) are only isomorphisms and are not unique. This is because E is defined as an extension of the tangent bundle by the bundle of differential forms. More precisely, we define E (for M-theory compactifications[21]) via

$$\begin{array}{rcccc} \wedge^2 T^* & \longrightarrow & E'' & \longrightarrow & T \\ \wedge^5 T^* & \longrightarrow & E' & \longrightarrow & E'' \\ T^* \otimes \wedge^7 T^* & \longrightarrow & E & \longrightarrow & E' \end{array} \tag{2.126}$$

This means that for local patches $\mathcal{U}_i, \mathcal{U}_j \subset M$ with $\mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset$, and for local sections

$$v_i + \omega_i + \sigma_i + \tau_i \in \Gamma(\mathcal{U}_i, E) \tag{2.127}$$

the patching is given by

$$v_j = v_i \tag{2.128}$$

$$\omega_j = \omega_i + v_i \lrcorner \mathrm{d}\Lambda_{ji} \tag{2.129}$$

$$\sigma_j = \sigma_i + \omega_i \wedge \mathrm{d}\Lambda_{ji} + \frac{1}{2} v_i \lrcorner \mathrm{d}\Lambda_{ji} \wedge \mathrm{d}\Lambda_{ji} + v_i \lrcorner \mathrm{d}\tilde{\Lambda}_{ji} \tag{2.130}$$

$$\begin{aligned} \tau_j = \tau_i + &+ j\mathrm{d}\Lambda_{ji} \wedge \sigma_i - j\mathrm{d}\tilde{\Lambda}_{ji} \wedge \omega_i + j\mathrm{d}\Lambda_{ji} \wedge v_i \lrcorner \mathrm{d}\tilde{\Lambda}_{ji} + \frac{1}{2} j\mathrm{d}\Lambda_{ji} \wedge \mathrm{d}\Lambda_{ji} \wedge \omega_i \\ &+ \frac{1}{6} j\mathrm{d}\Lambda_{ji} \wedge v_i \lrcorner \mathrm{d}\Lambda_{ji} \wedge \mathrm{d}\Lambda_{ji} \end{aligned} \tag{2.131}$$

where $\Lambda_{ji}, \tilde{\Lambda}_{ji}$ are local two and five-forms respectively on $\mathcal{U}_i \cap \mathcal{U}_j$. We can write this in a more compact form by saying $V_j = \mathrm{e}^{\mathrm{d}\Lambda_{ji} + \mathrm{d}\tilde{\Lambda}_{ji}} \cdot V_i$ where we using the exponentiated form of the

[21] From here we will assume that the precise form of the expressions given will only apply to M-theory unless otherwise specified

adjoint action outlined in appendix A.4 We call secions of E generalised vectors.

We note that the Λ_{ji} define what is called a *connective structure on a gerbe* [174], meaning that there successive gauge transformations that obey various compatibility conditions on overlapping patches. Namely

$$\Lambda_{ji} + \Lambda_{ik} + \Lambda_{kj} = \mathrm{d}\Lambda_{kji} \qquad \text{on } \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k \tag{2.132}$$

$$\Lambda_{kji} + \Lambda_{lkj} + \Lambda_{ilk} + \Lambda_{jil} = \mathrm{d}\Lambda_{lkji} \qquad \text{on } \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k \cap \mathcal{U}_l \tag{2.133}$$

The five-forms $\tilde{\Lambda}_{ji}$ obey a similar set of compatibility conditions on up to 7-fold overlaps. The conditions are complicated by the fact that they depend on the Λ patching as well. We will not give the precise patching here as it is not important for the following.

Defining an isomorphism as in (2.125a), or equivalently a splitting of the extension (2.126), is equivalent to choosing a three and six-form gauge potential $A, \tilde{A}$ which are patched on overlaps via

$$A_j = A_i + \mathrm{d}\Lambda_{ji} \tag{2.134}$$

$$\tilde{A}_j = \tilde{A}_i + \mathrm{d}\tilde{\Lambda}_{ji} - \frac{1}{2}\mathrm{d}\Lambda_{ji} \wedge A_i \tag{2.135}$$

Then, given a section of the direct sum bundle $\hat{V} \in \Gamma(T \oplus \wedge^2 T^* \oplus \wedge^5 T^* \oplus (T^* \otimes \wedge^7 T^*))$, we can define a section of E, with the correct patching, by

$$V = \mathrm{e}^{A+\tilde{A}} \cdot \hat{V} \in \Gamma(E) \tag{2.136}$$

The precise form of the gauge potentials means that we can define the following global objects

$$F = \mathrm{d}A \qquad \tilde{F} = \mathrm{d}\tilde{A} - \frac{1}{2} A \wedge F \tag{2.137}$$

Note that the second term is a seven-form and physically corresponds to the dual of an external 4-form flux. While this trivially vanishes for $d = 6$, the gauge potential $\tilde{A}$ is still present. Different values of F and $\tilde{F}$ define different isomorphism classes of E and hence we often say that *the generalised tangent bundle is twisted by the flux.*

A priori, this bundle has structure group $\mathrm{GL}(\mathrm{rk}\, E, \mathbb{R})$. However, for supergravity applications, we would like the structure group to be $\mathrm{E}_{d(d)} \times \mathbb{R}^+$. To do so we need to introduce some invariant tensors. These depend on the dimension d of the internal space. For $d = 6$ there is a cubic invariant, and for $d = 7$ one needs a symplectic and quartic invariant.

$$\mathrm{E}_{6(6)} \; : \quad c : S^3 E \longrightarrow \det T^* \tag{2.138}$$

$$\mathrm{E}_{7(7)} \; : \quad s : \wedge^2 E \longrightarrow \det T^* \qquad q : S^4 E \longrightarrow (\det T^*)^2 \tag{2.139}$$

These are explicitly given in A.4. With this, we get the required structure group and E transforms in the fundamental representation. All of the relevant representations are summarised in table 2.1. The generalised tangent bundle is always chosen to have weight 1 under the $\mathbb{R}^+$.

The $\mathbb{R}^+$ factor is very important here since E does not form a representation of $E_{d(d)}$ alone[22]. The importance of the $\mathbb{R}^+$ factor, known as the trombone symmetry, was first noticed in [181] where they explain that physically it is related to the warp factor of the compactification.

The Dorfman Derivative

An important object within the framework of generalised geometry is the *Dorfman derivative*. This is an $\mathbb{R}$-bilinear map

$$L_\cdot\cdot\colon \Gamma(E)\times\Gamma(E)\longrightarrow\Gamma(E) \tag{2.140}$$

such that for all $V,W\in\Gamma(E)$, $f\in C^\infty(M)$, the following hold

$$L_U(L_V W) = L_{L_U V}W + L_V(L_U W) \tag{2.141}$$
$$L_V(fW) = a(V)(f)W + fL_V W \tag{2.142}$$

where $a : E\longrightarrow T$ is called *the anchor map.* (2.141) says that the Dorfman derivative is a derivation on itself, and (2.142) says that L_V is a covariant differential operator on E for all $V\in\Gamma(E)$. E endowed with such an operator is called a Leibniz algebroid.

The form of the Dorfman derivative was first found in [180]. In a local patch $\mathcal{U}_i\subset M$ we can write the Dorfman derivative as

$$\begin{aligned} L_{V_i}V_i' = \mathcal{L}_{v_i}v_i' &+ (\mathcal{L}_{v_i}\omega_i' - v_i'\lrcorner \mathrm{d}\omega_i) + (\mathcal{L}_{v_i}\sigma_i' - v_i'\lrcorner \mathrm{d}\sigma_i - \omega_i'\wedge \mathrm{d}\omega_i) \\ &+ (\mathcal{L}_{v_i}\tau_i' - j\sigma_i'\wedge\mathrm{d}\omega_i - j\omega_i'\wedge\mathrm{d}\sigma_i) \end{aligned} \tag{2.143}$$

where $V_i = v_i+\omega_i+\sigma_i+\tau_i\in\Gamma(\mathcal{U}_i,E)$, and similarly for V'. for this to be well-defined we need it to be consistent with the patching (2.128) - (2.131). A simple calculation shows that this works because of the closure of $\mathrm{d}\Lambda_{ji}$, $\mathrm{d}\tilde{\Lambda}_{ji}$. From (2.143) we can see that the action of the Dorfman derivative is generated by the Lie derivative along v, and the action of exact differential forms $\mathrm{d}\omega,\mathrm{d}\sigma$. These are precisely the diffeomorphism and $A,\tilde{A}$ gauge degrees of freedom of the M-theory backgrounds. Because of this, we say that the Dorfman derivative *generates the gauge transformations of the supergravity background.* We will give an example of this when we talk about the generalised metric.

Without choosing an isomorphism (2.125a), we cannot define a global expression for the Dorfman derivative as there is no global notion of a two-form or five-form. However, once we choose the isomorphism, or equivalently the gauge potentials $A,\tilde{A}$ with field strength $F,\tilde{F}$, we can define a global expression in terms of the untwisted generalised vectors. To do this we define the *twisted Dorfman derivative*

$$L^F_{\hat{V}}\hat{V}' = \mathrm{e}^{-A-\tilde{A}}L_{\mathrm{e}^{A+\tilde{A}}\hat{V}}\mathrm{e}^{A+\tilde{A}}\hat{V}' \tag{2.144}$$

$$\begin{aligned} =&\mathcal{L}_v v' + (\mathcal{L}_v\omega' - v'\lrcorner(\mathrm{d}\omega - v\lrcorner F)) \\ &+ (\mathcal{L}_v\sigma' - v'\lrcorner(\mathrm{d}\sigma - v\lrcorner\tilde{F} + \omega\wedge F) - \omega'\wedge(\mathrm{d}\omega - v\lrcorner F)) \\ &+ (\mathcal{L}_v\tau' - j\sigma'\wedge(\mathrm{d}\omega - v\lrcorner F) - j\omega'\wedge(\mathrm{d}\sigma - v\lrcorner\tilde{F} + \omega\wedge F)) \end{aligned} \tag{2.145}$$

[22] It is $(T\oplus\wedge^2T^*\oplus\wedge^5T^*\oplus(T^*\otimes\wedge^7T^*))\otimes(\det T^*)^{-1/(9-d)}$ that forms the $E_{d(d)}$ representation [162].

where $\hat{V} = v + \omega + \sigma + \tau \in \Gamma(T \oplus \wedge^2 T^* \oplus \wedge^5 T^* \oplus (T^* \otimes \wedge^7 T^*))$ is a global section, and similarly for $\hat{V}'$. Here we see that the Dorfman derivative is twisted by the flux while the generalised tangent bundle remains untwisted. This is an equivalent formulation of generalised geometry and we will often take this approach.

Finally, we observe that the Dorfman derivative is not antisymmetric. We can define an antisymmetric bracket, often called the *Courant bracket*[23], as the antisymmetric part of the Dorfman derivative

$$[\![V, V']\!] = \frac{1}{2}\left(L_V V' - L_{V'} V\right) \tag{2.146}$$

$$\begin{aligned} &= [v, v'] + (v \lrcorner \mathrm{d}\omega' - v' \lrcorner \mathrm{d}\omega + \frac{1}{2}\mathrm{d}(v \lrcorner \omega' - v' \lrcorner \omega)) \\ &\quad + (v \lrcorner \mathrm{d}\sigma' - v' \lrcorner \mathrm{d}\sigma + \frac{1}{2}\mathrm{d}(v \lrcorner \sigma' - v' \lrcorner \sigma) + \frac{1}{2}(\omega \wedge \mathrm{d}\omega' - \omega' \wedge \mathrm{d}\omega)) \\ &\quad + (\mathcal{L}_v \tau' - \mathcal{L}_{v'} \tau - j\sigma' \wedge \mathrm{d}\omega + j\sigma \wedge \mathrm{d}\omega' - j\omega' \wedge \mathrm{d}\sigma + j\omega \wedge \mathrm{d}\sigma') \end{aligned} \tag{2.147}$$

As with the Dorfman derivative, this is a local expression. To give a global expression we can define a twisted Courant bracket $[\![\cdot, \cdot]\!]^F$ by using L^F. One should note that this does not define a Lie bracket as it does not satisfy the Jacobi identity, as we shall see below once we have defined a few more objects.

Generalised Tensors

We are now in a position to start defining objects analogous to those defined in conventional generalised geometry. We naturally define generalised covectors as sections of the following bundle whose fibres transform in the antifundamental

$$E^* \simeq T^* \oplus \wedge^2 T \oplus \wedge^5 T \oplus (T \otimes \wedge^7 T) \tag{2.148}$$

Generalised tensors are given by sections of some $\mathcal{T} \subset E^{\otimes r} \otimes E^{* \otimes s}$ for some r and s. The fibres of $\mathcal{T}$ must transform in some representation of $\mathrm{E}_{d(d)} \times \mathbb{R}^+$. As before we can view these bundles as twisted by the flux, or we can keep them untwisted and take the twisted Dorfman derivative. We have listed some of the important tensor bundles and their representations in the table below. Note that the subscript denotes the $\mathbb{R}^+$ weight. We will go through each in turn.

$\mathcal{G}$	E	$\mathrm{ad}\,\tilde{F}$	N	K
$\mathrm{O}(6,6)$	$\mathbf{12}$	$\mathbf{66}$	$\mathbf{1}$	$\mathbf{220}$
$\mathrm{E}_{7(7)} \times \mathbb{R}^+$	$\mathbf{56_1}$	$\mathbf{1_0} + \mathbf{133_0}$	$\mathbf{133_2}$	$\mathbf{912_{-1}}$
$\mathrm{E}_{6(6)} \times \mathbb{R}^+$	$\mathbf{27_1}$	$\mathbf{1_0} + \mathbf{78_0}$	$\mathbf{27'_2}$	$\mathbf{351'_{-1}}$

Table 2.1: The $\mathrm{E}_{d(d)} \times \mathbb{R}^+$ representations that certain tensor bundles transform in. We include the corresponding $\mathrm{O}(6,6)$ representations for reference.

[23]This name is just because it mimics the Courant bracket of $\mathrm{O}(d,d)$ geometry. The structure it defines does not give a Courant algebroid.

The Adjoint Bundle

Formally we can define the adjoint bundle as an associated bundle. Let $\tilde{F} \to M$ be the *generalised frame bundle* which is an $\mathrm{E}_{d(d)} \times \mathbb{R}^+$ principle bundle [181, 182]. Analogously to conventional geometry, the frame bundle (at a point $p \in M$) is defined to be the set of frames of E that are also compatible with the invariant tensors (2.138), (2.139). Let $\mathfrak{e}_d \oplus \mathbb{R}$ be the split form of the Lie algebra of the structure group. Let Ad denote the adjoint action of $\mathrm{E}_{6(6)} \times \mathbb{R}^+$. Then the adjoint bundle is defined to be

$$\operatorname{ad}\tilde{F} := (\tilde{F} \times \mathfrak{e}_6 \oplus \mathbb{R})/\operatorname{Ad} \tag{2.149}$$

This is a vector bundle with fibres that transform as $\mathfrak{e}_d \oplus \mathbb{R}$. As with E, we can decompose this into natural geometric bundles and is given by

$$\operatorname{ad}\tilde{F} \simeq \mathbb{R} \oplus (T \otimes T^*) \oplus \wedge^3 T^* \oplus \wedge^6 T^* \oplus \wedge^3 T \oplus \wedge^6 T \tag{2.150}$$

Given that the fibres are isomorphic to the Lie algebra of the structure group, and section $R \in \Gamma(\operatorname{ad}\tilde{F})$ has a natural action on E which is given in appendix A.4.

An alternative description of the adjoint bundle is as a particular subbundle of $E \otimes E^*$. In fact, there is a unique $\mathrm{E}_{d(d)} \times \mathbb{R}^+$ covariant projection onto the adjoint bundle which we express the following way

$$\times_{\mathrm{ad}} : E \times E^* \longrightarrow \operatorname{ad}\tilde{F} \tag{2.151}$$

It is given explicitly in appendix A.4. With this, one can actually show that the Dorfman derivative (2.143) can be written in the more covariant form

$$L_V V' = \langle V, \mathrm{d}\rangle\, V' - (d \times_{\mathrm{ad}} V) \cdot V' \tag{2.152}$$

Here $\langle \cdot, \cdot \rangle$ denotes the natural pairing of E and E^*, and we are viewing $\mathrm{d} \in T^* \xhookrightarrow{a^*} E^*$. Finally, $\cdot$ denotes the adjoint action of $\operatorname{ad}\tilde{F}$ on E as described above. This expression then naturally generalises to give an action of the Dorfman derivative on any generalised tensor

$$L_V \alpha = \langle V, \mathrm{d}\rangle\, \alpha - (d \times_{\mathrm{ad}} V) \cdot \alpha \tag{2.153}$$

The expression for the Dorfman derivative acting on $R \in \Gamma(\operatorname{ad}\tilde{F})$ is given in A.4.

$\boldsymbol{N \subset S^2 E}$

There is a particular subbundle of the symmetric product of E which is important in exceptional generalised geometry. We denote it by N and it falls into a particular $\mathrm{E}_{d(d)}$ representation as shown in table 2.1. It is given by

$$N \simeq T^* \oplus \wedge^4 T^* \oplus (T^* \otimes \wedge^6 T^*) \oplus (\wedge^3 T^* \otimes \wedge^7 T^*) \oplus (\wedge^6 T^* \otimes \wedge^7 T^*) \tag{2.154}$$

Again there are unique covariant maps

$$\times_N : E \times E \longrightarrow N \tag{2.155}$$
$$\times_E : N \times E^* \longrightarrow E \tag{2.156}$$

These are defined in appendix A.4. These bundles are important in the structure of the Dorfman derivative and Courant bracket. One can show that the following identities hold.

$$\frac{1}{2}(L_V V' + L_{V'} V) = d \times_E (V \times_N V') \tag{2.157}$$
$$\mathrm{Jac}(V_1, V_2, V_3) = d \times_E ([\![V_1, V_2]\!] \times_N V_3 + (\text{cyclic perms.})) \tag{2.158}$$

where $\mathrm{Jac}(V_1, V_2, V_3) = [\![[\![V_1, V_2]\!], V_3]\!] + (\text{cyclic perms.})$ is the Jacobiator.

In fact, the bundle N also appears in an interesting context with exceptional field theory as noticed in [181]. In this formalism, one not only enlarges the tangent bundle to encompass the symmetries of supergravity, but one also enlarges the spacetime to try to capture the U-duality structure of string theory. One can then define the Dorfman derivative much like in (2.153), however now d is now replaced with the exterior derivative on the fully enlarged space $\tilde{\mathrm{d}}$. In doing so a number of problems arise. Firstly, the derivative no longer satisfies the Leibniz property (2.141), and secondly, the supergravity action is no longer invariant under the action generated. These are clearly not desired properties of an operator generating gauge symmetries of an action.

These issues are remedied by dictating that these formulae only hold once you have chosen a physical space $M \subset \tilde{M}$. This is called a section and is roughly equivalent to choosing the anchor map $a : E \to T$. However, one cannot choose any submanifold to get a well-defined Dorfman derivative L. It turns out that the physical space must have a tangent bundle T which satisfies $T \times_N T = 0$[24]. In fact, there is a slightly stronger constraint, aptly named the strong constraint, that says that if d is the derivative operator along the physical space then the operator

$$\mathrm{d} \times_{N^*} \mathrm{d} \equiv 0$$

on any generalised tensor. There are many choices of spaces that satisfy this strong constraint and they are related to each other via U-dualities. We note here that since the anchor map a is part of the definition of our structure, we already have a well-defined spacetime and we are only trying to study the symmetries of supergravity and not the dualities of string theory.

The Torsion Bundle

Another important bundle is the space of generalised torsions. This will be relevant when we introduce generalised connections in the next section. For now we will just state the properties of the bundle without motivation.

The torsion bundle $K \subset E^* \otimes \mathrm{ad}\, \tilde{F}$ has the following expression in terms of natural geometric

[24] It is easy to check using A.4 that T does in fact satisfy this in our formalism

bundles

$$\begin{aligned} K \simeq \wedge^7 T^* \oplus \wedge^4 T^* \oplus (\wedge^2 T^* \otimes T) \oplus S^2 T \oplus (T^* \otimes \wedge^3 T) \oplus (\wedge^4 T \otimes T) \\ \oplus (S^2 T^* \otimes \wedge^7 T) \oplus (\wedge^2 T \otimes \wedge^6 T) \oplus (\wedge^4 T \otimes \wedge^7 T) \oplus (\wedge^7 T)^2 \end{aligned} \tag{2.159}$$

This transforms in the representation given in table 2.1. While this will become more important later, we note now that the flux $F + \tilde{F} \in \Omega^4(M) \oplus \Omega^7(M) \subset \Gamma(K)$. Hence we can view the flux as an element of the torsion bundle. Then, combined with the map $K \times E \to \mathrm{ad}\,\tilde{F}$, we can write the twisted Dorfman derivative (2.145) in a more covariant way

$$L^F_{\hat{V}} \hat{V}' = \left\langle \hat{V}, \mathrm{d} \right\rangle \hat{V}' - (\mathrm{d} \times_{\mathrm{ad}} \hat{V} - F(\hat{V})) \cdot \hat{V}' \tag{2.160}$$

where $F(\hat{V}) = v \lrcorner F - \omega \wedge F + v \lrcorner \tilde{F} \in \Gamma(\mathrm{ad}\,\tilde{F})$. This again naturally generalises to give an expression for the Dorfman derivative on any generalised tensor.

2.3.4 Generalised Connections, Torsion and Integrability

An important object in any geometric theory is the connection. We can define a generalised connection for any $\mathcal{G}$-geometry [167, 180–186] in complete analogy. A generalised connection D is defined to be an $\mathbb{R}$-linear map

$$D : \Gamma(\mathcal{T}) \longrightarrow \Gamma(E^* \otimes \mathcal{T}) \tag{2.161}$$

such that

$$D(f\alpha) = \mathrm{d}f \otimes X + f D\alpha \tag{2.162}$$

for all $f \in C^\infty(M)$, $\alpha \in \Gamma(\mathcal{T})$ where $\mathcal{T}$ is some generalised tensor bundle. We have again used the identification $\mathrm{d}f \in \Omega^1(M) \xhookrightarrow{a^*} \Gamma(E^*)$.

Torsion

The torsion of a generalised connection is some tensor $T \in \Gamma(E^* \otimes \mathrm{ad}\,\tilde{F})$ which is defined by[25]

$$T(V) \cdot \alpha := L^D_V \alpha - L_V \alpha \tag{2.163}$$

where L^D_V is the Dorfman derivative (2.153), except with every instance of d replaced with D. That is

$$L^D_V \alpha = \langle V, D \rangle \alpha - (D \times_{\mathrm{ad}} V) \cdot \alpha \tag{2.164}$$

Note that $T(V) \in \Gamma(\mathrm{ad}\,\tilde{F})$ and so $T(V) \cdot \alpha$ denotes the adjoint action of $T(V)$ on the generalised tensor α. One may think that the generalised torsion can fill out the whole of $E^* \otimes \mathrm{ad}\,\tilde{F}$. However, due to the precise form of the Dorfman derivative, we find that the torsion can only live in $K \oplus E^* \subset E^* \otimes \mathrm{ad}\,\tilde{F}$. This highlights the importance of the bundle K defined in section 2.3.3.

[25] One can show that $t(v) \cdot a := \mathcal{L}^\nabla_v a - \mathcal{L}_v a$ gives an equivalent definition of the torsion of a conventional connection ∇

G-structures and Integrability

Much like in conventional geometry, one can define a G-structure to be a principle subbundle of the frame bundle $P_G \subset \tilde{F}$ with fibre $G \subset \mathcal{G}$. Often, the existence of a G-structure is equivalent to the existence of some globally non-vanishing generalised tensors that are preserved by G. We already saw some examples of generalised G-structures in section 2.3.1. As in conventional geometry, we define a G-structure to be integrable if there exists a torsion free compatible generalised connection. That is, if the generalised tensors α_i define the G-structure, then it is integrable if there exists a connection D such that

$$D\alpha_i = 0 \quad \forall\, i \qquad\qquad L^D_V = L_V \tag{2.165}$$

While a compatible connection always exists, the existence of a torsion free compatible connection depends on a geometric property of the G-structure called *the intrinsic torsion.* We will define this now and show that for a torsion free compatible connection to exist, the intrinsic torsion must vanish. We will follow section 2.1.4 and [187, section 5.1] closely.

Note that compatible (but not necessarily torsion free) connections will not, in general, be unique. Given any two compatible connections D, D', their difference will be a tensor $D' - D = \Sigma \in \Gamma(E^* \otimes \operatorname{ad} P_G) \subseteq \Gamma(E^* \otimes \operatorname{ad} \tilde{F})$, where $\operatorname{ad} P_G \to M$ is the adjoint bundle of P. That is, it is a vector bundle with fibres $\mathfrak{g}$, the Lie algebra of G. This has a natural action on generalised tensors, inherited from $\operatorname{ad} \tilde{F}$, and will annihilate the α_i. This gives the space of G-compatible connections an affine structure with space isomorphic to $\Gamma(E^* \otimes \operatorname{ad} P_G)$.

Given two generalised connections D, D', we can find their respective torsions T, T' via (5.11). Moreover, the difference in torsion is given by

$$\begin{aligned} T'(V) - T(V) &= L^{D'}_V - L^D_V && (2.166)\\ &= \big\langle V, D' - D\big\rangle - \big((D' - D) \times_{\mathrm{ad}} V\big) && (2.167) \end{aligned}$$

This clearly only depends linearly on $\Sigma = D' - D$ and hence we can define a vector bundle map

$$\begin{aligned} \tau : E^* \otimes \operatorname{ad} P_G &\longrightarrow W \simeq K \oplus E^* \\ \Sigma &\longmapsto T' - T \end{aligned} \tag{2.168}$$

We can think of the image of τ as the part of the torsion we can change by changing the compatible connection. We will define $W_G = \operatorname{im} \tau$. Note that, in general, τ is not surjective and so there exists a subspace of the torsion which cannot be affected by changing the compatible connection. This space is called the intrinsic torsion space and we shall denote it by $W^G_{\mathrm{int}} \simeq W/W_G$. The precise embedding of $W/W_G \hookrightarrow W$ can be given by decomposing W, W_G into irreducible G representations.

Given a compatible connection D with torsion T, we can consider the projection onto W^G_{int}, called the intrinsic torsion $T^{\mathrm{int}} \in \Gamma(W^G_{\mathrm{int}})$. As explained, this part of the torsion is unchanged by changing $D \to D' = D + \Sigma$. Hence, one can find a Σ such that the torsion T' of D' vanishes if and only if the intrinsic torsion T^{int} vanishes.

Generalised Metrics and Generalised Levi-Civita Connections

A *generalised metric* is defined to be a reduction of the structure group to its maximally compact subgroup, which we denote with H_d. It turns out that such a structure is always integrable, but the torsion free compatible connection is not unique. We call any such connection a *generalised Levi-Civita Connection*. This is the generalisation of the fundamental theorem of Riemannian geometry. We will outline a proof here. The aim is to show that the intrinsic torsion for H_d structures always vanishes. We will focus on the case of $d = 7$ for simplicity, but similar proofs hold for all geometries.

As mentioned above, the generalised metric defines an $H_7 = \mathrm{SU}(8)$ structure[26]. With such a structure we can decompose all of the relevant bundles into SU(8) representations

$$E^* \otimes \operatorname{ad} P_{\mathrm{SU}(8)} \sim \mathbf{28} \oplus \mathbf{36} \oplus \mathbf{420} \oplus \mathbf{1280} \oplus c.c. \tag{2.169}$$

$$W \sim \mathbf{28} \oplus \mathbf{36} \oplus \mathbf{420} \oplus c.c. \tag{2.170}$$

Here $\sim$ denotes the decomposition of the fibres into subrepresentations. We can see that all of the SU(8) representations that appear in W appear in $E^* \otimes \operatorname{ad} P_{\mathrm{SU}(8)}$. Hence, in this particular case, the map τ defined in (2.168) is surjective. This is shown more explicitly in [181]. This means that $W_G = W$ and hence $W^G_{\mathrm{int}} = 0$, i.e. the intrinsic torsion of any SU(8)-compatible connection vanishes. Hence, there always exists a $\Sigma \in \Gamma(E^* \otimes \operatorname{ad} P_{\mathrm{SU}(8)})$ such that $D + \Sigma$ is torsion free. This proves the first statement that the generalised Levi-Civita connection always exists. We can also see from (2.169) that the kernel of τ on each fibre is 2×1280 dimensional. Hence, if D is a torsion free compatible connection, then so is $D + \Sigma$ if $\Sigma_p \in \mathbf{1280} \oplus c.c.$ at each point $p \in M$. Hence, the space of generalised Levi-Civita connections is an affine space isomorphic to $\Gamma(\mathbf{1280} \oplus c.c.)$. This proves the second statement that such a connection is not unique.

A natural question to ask here is if we can define a notion of generalised curvature within generalised geometry. The natural candidate for the generalised Riemann curvature R would be

$$R(X,Y) \cdot Z := [D_X, D_Y]Z - D_{[\![X,Y]\!]}Z \tag{2.171}$$

where $X, Y, Z \in \Gamma(E)$ and $D_X = \langle X, D\rangle$. In fact, (2.171) is not a generalised tensor as it is not $C^\infty(M)$-linear. Indeed, if we take $X' = fX$, $Y' = gY$, $Z' = hZ$ for some $f, g, h \in C^\infty(M)$ then we find

$$R(X',Y') \cdot Z' = fghR(X,Y)Z + \frac{1}{2} hD_{(f\mathrm{d}g - g\mathrm{d}f)\times_E (X \times_N Y)}Z \tag{2.172}$$

While there has been some work defining a tensorial Riemann tensor for $\mathrm{O}(d,d)\times\mathbb{R}^+$ generalised geometry [188–190], there does not exist such an object in exceptional generalised geometry. We will go into more detail in appendix B.

[26] In fact the maximally compact subgroup is $\mathrm{SU}(8)/\mathbb{Z}_2$. However, as we note later, we always assume we can take the double-cover.

2.4 Generalised Geometry and Supersymmetric Backgrounds

Generalised geometries have found particular use in describing backgrounds of supergravity. This may seem unsurprising since the geometry and topology of both the manifold, and the gauge field structure, is built into the definition of the generalised tangent bundle. Hence, defining objects on E as opposed to T naturally includes the flux degrees of freedom of any background. We recall from section 2.2 that it was precisely these degrees of freedom that have been so hard to characterise through conventional geometric techniques. In this section we will review some of the key ways these generalised structures have been employed to study generic aspects of flux backgrounds.

2.4.1 Applications of $\mathrm{O}(d,d)$ Generalised Geometry

The $\mathrm{O}(d,d)$ group had been an important object in the study of string theory for some time. The discrete group $\mathrm{O}(d,d,\mathbb{Z})$ is the T-duality group of string theory on a d dimensional torus [191, 192]. It was also known that the NS sector of string theory compactified on such a torus has the continuous group as a global symmetry and the metric and B-field transform as the generalised metric (2.109) [192–195]. Moreover, it was shown that the moduli of toroidal compactifications of string theory parameterise the coset $\mathrm{O}(d,d)/\mathrm{O}(d)\times\mathrm{O}(d)$ [196]. With these observations, people used $\mathrm{O}(d,d)$ transformations to try to find new backgrounds of string theory [197–199].

The generalised geometry of Hitchin and Gualtieri gives a way of manifestly realising these symmetries through the background geometry. Originally with applications to mirror symmetry in mind, supersymmetric backgrounds of type II with an SU(3) structure were described in terms of pure spinors in [142]. They were able to rewrite the supersymmetry conditions on the internal spinors as differential constraints on the generalised spinors. To do so they used the Clifford map between polyforms and spinor bilinears, and then employed the constraints coming from the Killing spinor equations. In their work, they noticed that one pure spinor is always left unaffected by the RR fluxes. However, the NS flux H enters with a non-canonical action.

In [156] they noticed that an $\mathrm{SU}(3)\times\mathrm{SU}(3)$ structure defined by 2 pure generalised spinors is precisely the algebraic data defined by 2 non-vanishing internal spinors[27]. These encompass the cases in which the internal manifold has an SU(3) structure, an SU(2) structure, and intermediate cases that cannot be described by conventional G-structures[28]. In [202], the authors were once again able to find the differential constraints on 2 generalised spinors to define an $\mathcal{N}=1$ background. They found that

$$(\mathrm{d}+H\wedge)(\mathrm{e}^{2\Delta-\hat{\varphi}}\Phi_1)=0 \qquad (\mathrm{d}+H\wedge)(\mathrm{e}^{2\Delta-\hat{\varphi}}\Phi_2)=\mathrm{d}\Delta\wedge\bar{\Phi}_2+F \tag{2.173}$$

where Δ is the warp factor and $\hat{\varphi}$ is the dilaton. The Φ_i are even/odd polyforms depending on whether we are in type IIA or IIB. F is shorthand for the formal sum of fluxes, rescaled using the norm of the internal spinors. We see that Φ_1 defines a (twisted) generalised Calabi-Yau manifold, and Φ_2 depends on all the fluxes. These backgrounds have been called GMPT backgrounds after

[27] See [200, 201] for a similar investigation of $\mathrm{G}_2\times\mathrm{G}_2$ structures for compactifications on 7-manifolds.

[28] These are cases where the internal spinors become parallel at certain points. When this occurs, the SU(2) structure degenerates into an SU(3) structure.

the authors. The equations (2.173) were reformulated in [203] to make the generalised complex geometry of the background more manifest. We will look at these backgrounds in more detail in section 5.2. The description of supersymmetric backgrounds in terms of 2 pure spinors provides the background with a non-integrable $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure[29]. Specifically, the integrability is broken by the RR fluxes.

In [183] they showed that any non-supersymmetric background could be described in terms of an integrable $\mathrm{O}(1,9) \times \mathrm{O}(9,1)$ structure. They noted that one can describe the fields of the NS sector by a generalised metric provided one introduces an $\mathbb{R}^+$ factor into the structure group to properly describe the dilaton. A generalised metric is then equivalent to a choice of $g, B, \hat{\varphi}$. That is

$$\{g, B, \hat{\varphi}\}_{p \in M} \in \frac{\mathrm{O}(10,10) \times \mathbb{R}^+}{\mathrm{O}(1,9) \times \mathrm{O}(9,1)} \tag{2.174}$$

They introduced a generalised connection for $\mathrm{O}(d,d)$ generalised geometry, originally defined in [204], and found the conditions required for the generalised metric to be integrable. They found that one can rewrite type II supergravity in a manifestly $\mathrm{O}(1,9) \times \mathrm{O}(9,1)$ covariant form using a generalised Levi-Civita connection, i.e. a torsion free generalised connection that is compatible with the generalised metric. As we have seen, these connections always exist but are not unique. Despite this ambiguity, one can find certain projections that are unique. One can then use these projections to write the supersymmetry variations and the equations of motion. The equations of motion are given in terms of the generalised Ricci curvature and scalar, and the Bianchi identity.

$$R_{AB} + \frac{1}{16}\hat{\varphi}^{-1}\left(F, \Gamma_{AB} F\right) = 0 \qquad R = 0 \qquad \Gamma^A D_A F = 0 \tag{2.175}$$

Note that, in the absence of RR fluxes, the equations of motion are given by the vanishing of the generalised Ricci curvature.

These calculations show that generic supergravity backgrounds always have an integrable $\mathrm{O}(1,9) \times \mathrm{O}(9,1)$ structure on the generalised tangent bundle. Moving to supersymmetric backgrounds, one must work with a further reduced structure group, but the integrability is broken by the presence of RR fluxes. This is unsurprising given that only the B field is built into the structure of the geometry while the RR fluxes must be accounted for separately. As was noted in [205, 206], they behave as generalised spinors.

2.4.2 Applications of $\mathrm{O}(6, 6+n)$ Generalised Geometry

It has been shown that $\mathrm{O}(6, 6+n)$ is a symmetry of heterotic string theory reduced on a 6 dimensional torus [192, 196]. However, there has not been as much interest in how $\mathrm{O}(6, 6+n)$ generalised geometry can be applied to these backgrounds. Much of the mathematical literature has been interested in the holomorphic Courant (or string) algebroids [74, 177–179, 207]. Despite this, the generalised metric was defined in [186]. A generalised metric is a reduction of the structure group $\mathrm{O}(6, 6+n) \to \mathrm{O}(6) \times \mathrm{O}(6+n)$. This is equivalent to a choice of subbundle

[29] Integrability would be if both pure spinors are $\mathrm{d} + H\wedge$ closed. This would correspond to $\mathcal{N} = 2$ backgrounds.

$C_+ \subset E$ of rank 6 such that the restriction of the inner product η to C_+ is positive definite[30]. We denote by C_- the orthogonal compliment of C_+ under η.

A choice of metric is equivalent to a choice of bosonic fields

$$\{g, B, A\}_{p\in M} \in \frac{\mathrm{O}(6, 6+n)}{\mathrm{O}(6)\times \mathrm{O}(6+n)} \tag{2.176}$$

They show that one can always introduce a generalised connection that is compatible with this reduced structure and that is torsion-free. They go on to show that the equations of motion for the heterotic background are given by the vanishing of the generalised Ricci tensor

$$R_{AB} = 0 \tag{2.177}$$

This mirrors what we saw above with the NS sector of type II. In their work they do not fix the dilaton factor by considering an $\mathbb{R}^+$ term in the structure group. Instead, they relate the choice of generalised Levi-Civita connection to the choice of dilaton in the physical theory.

2.4.3 Applications of $\mathrm{E}_{d(d)} \times \mathbb{R}^+$ Generalised Geometry

Exceptional generalised geometry is the natural framework in which to study backgrounds of supergravity and string theory. We will see that all fluxes can be dealt with in a uniform way, and that arbitrary supersymmetric backgrounds will be described by integrable G-structures. For ease, we will again focus on the application to M-theory backgrounds. However, everything has an analogous story in type II backgrounds.

It has long been known [208–210] that the bosonic fields of supergravity backgrounds reduced on d dimensions form an element of the coset

$$\{\Delta, g, A, \tilde{A}\}_{p\in M} \in \frac{\mathrm{E}_{d(d)}\times \mathbb{R}^+}{H_d} \tag{2.178}$$

Also, Δ is the warp factor, g is the metric, and $A, \tilde{A}$ are the gauge potentials of the flux. We denote by H_d the maximally compact subgroup of $\mathrm{E}_{d(d)} \times \mathbb{R}^+$. (2.178) says that there is some $H_d \subset \mathrm{E}_{d(d)} \times \mathbb{R}^+$ which preserves the bosonic fields. In other words, a choice of bosonic fields reduces the structure group

$$\mathrm{E}_{d(d)} \times \mathbb{R}^+ \longrightarrow H_d \tag{2.179}$$

In fact, in order to describe fermions in this generalised setting, one must work with the double cover $\tilde{H}_d$ of the maximally compact subgroup [182]. We will always assume that we are working on a manifold in which there is a well defined lift of the structure group $H_d \to \tilde{H}_d$[31].

We can describe this H_d structure conveniently through a generalised metric

$$G : S^2E \longrightarrow \mathbb{R} \tag{2.180}$$

[30]The conventions here are slightly different to the conventions of [186] as their negative definite space would be rank 6. We use these conventions to match the conventions used in chapter 3. They also talk about when a particular C_+ is *admissible*. In our framework, with a prescribed inner product η on the space, all C_+ as described will be admissible.

[31]This is the analogue of a spin manifold in conventional geometry. There are some subtleties as to when this lift exists, as outlined in [166].

To find G, we must define what is called a *conformal split frame* of E. Given any frame $\{\hat{e}_a\}$ of T with $\{e^a\}$ the dual frame, and any choice of bosonic fields $\Delta, A, \tilde{A}$ we define the conformal split frame by

$$\begin{aligned} \hat{E}_a &= \mathrm{e}^{\Delta}\mathrm{e}^{A+\tilde{A}}\hat{e}_a & E^{ab} &= \mathrm{e}^{\Delta}\mathrm{e}^{A+\tilde{A}}e^{ab} \\ E^{abcde} &= \mathrm{e}^{\Delta}\mathrm{e}^{A+\tilde{A}}e^{abcde} & E^{a,a_1\dots a_7} &= \mathrm{e}^{\Delta}\mathrm{e}^{A+\tilde{A}}e^{a,a_1,,,a_7} \end{aligned} \tag{2.181}$$

where $e^{ab} = e^a \wedge e^b$, $e^{abcde} = e^a \wedge \dots \wedge e^e$ and $e^{a,a_1\dots a_7} = e^a \otimes e^{a_1} \wedge \dots \wedge e^{a_7}$. If we further impose that the $\hat{e}_a$ are orthonormal with respect to the conventional metric g then we can define the map G by

$$G(V,V) = v^a v_a + \frac{1}{2}\omega_{ab}\omega^{ab} + \frac{1}{5!}\sigma_{abcde}\sigma^{abcde} + \frac{1}{7!}\tau_{a,a_1\dots a_7}\tau^{a,a_1\dots a_7} \tag{2.182}$$

where these are the components of the generalised vectors in the conformal split frame (2.181), and where we have raised and lowered indices with the metric δ_{ab} (in this frame). One can show that this map is invariant under H_d transformations and hence the bosonic fields, $\{\Delta, g, A, \tilde{A}\}$, define an H_d structure as claimed above.

The action of the infinitesimal gauge transformations are generated by the Dorfman derivative [181,211]. That is, given a generalised metric G, if we take a small diffeomorphism generated by a vector $v \in \Gamma(T)$, and do a gauge transformation of the gauge potentials $A' = A + \mathrm{d}\omega$, $\tilde{A}' = \tilde{A} + \mathrm{d}\sigma - \frac{1}{2}\mathrm{d}\omega \wedge A$, then the gauge transformed G is given by

$$G' = G + L_V G \tag{2.183}$$

where $V = \mathrm{e}^{A+\tilde{A}}(v + \omega + \sigma + \tau)$. We call the combined action of diffeomorphisms and form field gauge transformations *generalised diffeomorphisms*.

As we saw for $\mathrm{O}(d,d)$ and $\mathrm{O}(6,6+n)$ geometries, an important object for the application of $\mathrm{E}_{d(d)} \times \mathbb{R}^+$ geometries is the *generalised Levi-Civita connections*. These were defined in [181,183] to be torsion free generalised connections that are compatible with the generalised metric. We already saw in section 2.3.4 that these objects always exist but are not unique. Moreover, the naive definition of the Riemann curvature is not tensorial. Despite this, it was shown in [181] that the generalised Ricci tensor $R_{MN} = R^P{}_{MPN}$ is tensorial. Moreover, it is independent of the choice of generalised Levi-Civita connection. It also has the property that the bosonic supergravity action and equations of motion on the internal space are given by

$$S = \int_M R\,\mathrm{vol}_G \qquad R_{MN} = 0 \tag{2.184}$$

respectively. Here vol_G is the $\mathrm{E}_{d(d)}$ invariant volume form given in the conformal split frame by $\mathrm{e}^{(9-d)\Delta}\sqrt{g}$. Hence we see that, much like in the previous sections, the equations of motion imply generalised Ricci flatness of the generalised Levi-Civita connections. However, unlike for $\mathrm{O}(d,d)$ geometries, the presence of RR fluxes does not break the Ricci flatness. This is because these fluxes are built into the structure of the generalised tangent bundle.

Supersymmetric Backgrounds

As we saw in section 2.2, geometries with $\mathcal{N}$ linearly independent solutions to the internal Killing spinor equations will preserve $\mathcal{N}$ supersymmetries in the effective theory[32]. The key observation in exceptional geometries is that the internal spinors transform in the fundamental representation of H_d. This allows us to define a clear reduction of the structure group for any required supersymmetry. This is in contrast to conventional geometry or $\mathrm{O}(d,d)$ geometry in which assumptions had to be made about the form of the internal spinors to properly define a reduction.

This observation was first made in [180] in which they analysed M-theory backgrounds with a 4 dimensional Minkowski space preserving $\mathcal{N}=1$ supersymmetry. Since the (double cover of the) maximally compact subgroup in this case is SU(8), the existence of a globally non-vanishing section of a bundle transforming in the **8** reduces the structure group $\mathrm{SU}(8) \to \mathrm{SU}(7)$. It was shown in [184] that the non-vanishing spinor satisfies the Killing spinor equation if and only if there exists a torsion free connection compatible with the reduced structure. This was shown using the reformulation of the Killing spinor equations in manifestly $\tilde{H}_d$ covariant way found in [181, 182]. It was also shown that these backgrounds are automatically generalised Ricci flat and hence solve the equations of motion. This provides a simple proof of the statement that the Killing spinor equations imply the equations of motion. In [180] it was also shown that an alternative description of such an SU(7) structure is given by a non-vanishing tensor ψ transforming in the **912** of $\mathrm{E}_{7(7)}$. They were able to use this to find a description of the effective superpotential on $\mathbb{R}^{3,1}$ in an $\mathrm{E}_{7(7)}$ covariant form in terms of ψ.

Supersymmetric backgrounds preserving 8 supercharges in $\mathbb{R}^{3,1}$, $\mathbb{R}^{4,1}$ and $\mathbb{R}^{5,1}$ were analysed in [187]. For $\mathbb{R}^{3,1}$, we get an $\mathcal{N}=2$ solution and hence we require two linearly independent solutions of the Killing spinor equation. This implies the existence of an SU(6) structure on the internal space[33]. They found a description of these backgrounds in terms of a triplet of adjoint-valued tensors and a non-vanishing generalised vector. They also found the differential conditions that these backgrounds must satisfy for the existence of a torsion free connection, given in terms of vanishing moment maps for the action of generalised diffeomorphisms on the space of structures. Using the results of [184] again one can show that this is equivalent to the Killing spinor equations. In analogy with fluxless compactifications preserving 8 supercharges, they labelled these backgrounds exceptional Calabi-Yau spaces. We will review these in more detail in section 4.1 for the $\mathbb{R}^{4,1}$ case.

The correspondence between integrable G-structures and supersymmetric backgrounds was originally proved for backgrounds preserving minimal supersymmetry[34], and was extended to all supersymmetric backgrounds in [185]. There, they showed that a background preserving any supersymmetry defines an integrable $G_{\mathcal{N},d}$ structure. The precise group depends on the amount of preserved supersymmetry and the dimension of the internal space. The groups are listed in the table below.

They found this by analysing the algebra of Killing vectors and Killing spinors. They showed

[32]We will ignore accidental additional supersymmetries, like is the case for Calabi-Yau compactifications.

[33]For compactifications down to $\mathbb{R}^{4,1}$ the structure group is USp(6), and to $\mathbb{R}^{5,1}$ it is $\mathrm{SU}(2) \times \mathrm{USp}(4)$.

[34]In fact, $D=4$ $\mathcal{N}=2$ backgrounds were also shown to be integrable SU(6) structures.

d	$\tilde{H}_d$	$G_{\mathcal{N},d} \subset \tilde{H}_d$
7	SU(8)	$\mathrm{SU}(8-\mathcal{N})$
6	USp(8)	$\mathrm{USp}(8-2\mathcal{N})$
5	$\mathrm{SU}(4) \times \mathrm{SU}(4)$	$\mathrm{SU}(4-2\mathcal{N}_+) \times \mathrm{SU}(4-2\mathcal{N}_-)$
4	USp(4)	$\mathrm{USp}(4-2\mathcal{N})$

Table 2.2: The G-structures in $\mathrm{E}_{d(d)} \times \mathbb{R}^+$ geometry that describe a background preserving $\mathcal{N}$ supersymmetries.

that the closure of the algebra is equivalent to the vanishing of the intrinsic torsion of a connection compatible with the G-structures listed above. This work shows that the geometry of supergravity backgrounds have a simple mathematical formulation which can be exploited to understand deep questions about such backgrounds, particularly about the moduli space of these geometries.

We also mention that a lot of work has been done on understanding the generalised geometry of supersymmetric backgrounds with an AdS external space. It was shown in [212] that such backgrounds are described by manifolds with G-structures of weak generalised holonomy, where the groups are the same as listed above. Weak holonomy is characterised by the existence of a compatible connection with non-vanishing singlet intrinsic torsion only. These were originally looked at in [213] where the authors found the AdS analogue of the exceptional Calabi-Yau spaces defined previously. They called these backgrounds exceptional Sasaki-Einstein spaces. They were defined by the same tensors as in the Minkowski space, just with different integrability conditions that depend on the cosmological constant. The AdS_5 geometries were further studied in [214] and related to particular CFTs through the AdS/CFT correspondence. in particular, they matched the marginal deformations of the CFT to deformations of the USp(6) structures on the gravity side. Similar results for $\mathcal{N}=2$ SCFT_3 were analysed in [215].

Chapter 3

Generalised Geometry for Heterotic Backgrounds

In this chapter, we analyse how $\mathrm{O}(6, 6+n) \times \mathbb{R}^+$ generalised geometry can be used to describe and study Heterotic string backgrounds preserving minimal supersymmetry. These have been studied in great detail in the past due to their ability to reproduce the gauge groups of the standard model with relative ease [67, 68, 112, 113, 216–220]. We will find that all of these backgrounds can be described uniformly through $\mathrm{SU}(3) \times \mathrm{Spin}(6+n)$ structures. These will have properties that are reminiscent of complex and generalised complex structures. Using these similarities, we will be able to provide direct links to geometric invariant theory [221] and we will be able to recover the results on moduli as found in [178, 179, 207]. The work in this chapter follows the work of [2] very closely. This chapter is structured as follows. First, we review the Hull-Strominger system, providing the equations for the F-terms and the D-terms, as well as summarising some of the previous results on moduli. We then introduce the generalised structures for supersymmetric heterotic backgrounds. This is done in terms of some holomorphic object ψ transforming in the $\mathbf{220_1}$, and some subbundle L_1 of the generalised tangent bundle. Next we show that the F-terms are (almost) given by involutivity of L_1 under the Dorfman derivative. We also provide an expression for the super potential in terms of the holomorphic object ψ. Next we show that the D-terms are given by the vanishing of a moment map. Taken together, the involutivity and the moment map imply that the $\mathrm{SU}(3) \times \mathrm{Spin}(6+n)$ structure is integrable. Next, we look at how the moment map provides a possible GIT picture of the moduli space and we outline how this may lead to a generalisation of Yau's theorem for non-Kähler manifolds. Finally, we study the moduli of these $\mathrm{SU}(3) \times \mathrm{Spin}(6+n)$ structures.

3.1 Review of the Hull-Strominger System

We begin with a review of the Hull–Strominger system [144, 146]. This is a set of equations describing the geometry of general $\mathcal{N} = 1$ backgrounds of the heterotic string on a ten-dimensional manifold M that is a product of a six-dimensional manifold X with four-dimensional Minkowski space $M = \mathbb{R}^{3,1} \times X$, with trivial warp factor in the string frame.

The condition of $\mathcal{N} = 1$ supersymmetry implies the existence of a global nowhere-vanishing spinor ϵ on X. This defines an SU(3) structure on X which can be equivalently described in terms of a complex three-form Ω (with a compatible almost complex structure I) and a real two-form ω satisfying

$$\Omega \wedge \omega = 0, \qquad \tfrac{1}{8}\mathrm{i}\,\Omega \wedge \bar{\Omega} = \tfrac{1}{6}\omega \wedge \omega \wedge \omega. \tag{3.1}$$

As usual, the forms are defined as bilinears in the spinor ϵ

$$\Omega_{mnp} = \epsilon^{\mathrm{T}}\gamma_{mnp}\epsilon, \qquad \omega_{mn} = -\mathrm{i}\,\epsilon^{\dagger}\gamma_{mn}\epsilon. \tag{3.2}$$

The supersymmetry conditions in the form of the Killing spinor equations imply that this SU(3) structure is not integrable but instead satisfies

$$\mathrm{d}(\mathrm{e}^{-2\varphi}\Omega) = 0, \qquad \mathrm{d}(\mathrm{e}^{-2\varphi}\omega \wedge \omega) = 0, \tag{3.3}$$

where φ is the dilaton. These conditions are known as "conformally holomorphic" and "conformally balanced" respectively. Note that the first condition implies that X has an integrable complex structure whose canonical bundle is holomorphically trivial.

Heterotic compactifications come with a connection A on a vector bundle $V \to X$ whose field strength F is valued in $\mathrm{End}(V)$, and a connection Θ on the tangent bundle T whose curvature R is valued in $\mathrm{End}(T)$. Supersymmetry implies that both connections are instantons [222, 223]

$$F_{0,2} = 0, \qquad \omega^{\sharp} \lrcorner F = 0, \qquad \text{and} \qquad R_{0,2} = 0, \qquad \omega^{\sharp} \lrcorner R = 0, \tag{3.4}$$

where $\omega^{\sharp}$ is ω with its indices raised using the metric on X and a subscript indicates the $(0,2)$-form part of the curvature with respect to the complex structure defined by the SU(3) structure. In other words, V and T must be holomorphic vector bundles with connections that solve the hermitian Yang–Mills equations with zero slope. A theorem due to Donaldson–Uhlenbeck–Yau then guarantees a unique solution provided V and T are polystable [224, 225].

The final supersymmetry condition is the anomaly cancellation condition. This couples the intrinsic torsion of the SU(3) structure with the B field and the connections. It is given by

$$\mathrm{i}(\partial - \bar{\partial})\omega = H \coloneqq \mathrm{d}B + \tfrac{1}{4}\alpha' \left(\omega_3(A) - \omega_3(\Theta)\right), \tag{3.5}$$

where ω_3 is the Chern–Simons three-form for the relevant connection, for example

$$\omega_3(A) = \mathrm{Tr}(A \wedge \mathrm{d}A + \tfrac{2}{3}A \wedge A \wedge A). \tag{3.6}$$

This implies a non-trivial Bianchi identity for the NSNS three-form flux H

$$\mathrm{d}H = \tfrac{1}{4}\alpha'(\mathrm{Tr}\,F \wedge F - \mathrm{Tr}\,R \wedge R). \tag{3.7}$$

For convenience, in what follows we will drop explicit reference to α', absorbing it into the definition of B and ω. Moreover we will mostly ignore the tangent bundle connection Θ with the understanding that it can be reintroduced afterwards by taking V to be a $G = \mathrm{G}_{\text{gauge}} \times \mathrm{GL}(6, \mathbb{R})$ vector bundle, where $\mathrm{G}_{\text{gauge}}$ is the gauge group for A, together with a suitable definition of the trace, as, for example, in [186].

It is useful to group these equations into so-called F-terms and D-terms. As was discussed in [207], the F-term conditions correspond to

$$\mathrm{d}(\mathrm{e}^{-2\varphi}\Omega) = 0, \qquad \mathrm{i}(\partial - \bar{\partial})\omega = H, \qquad F_{0,2} = 0. \tag{3.8}$$

The remainder are the D-terms

$$\mathrm{d}(\mathrm{e}^{-2\varphi}\omega \wedge \omega) = 0, \qquad \omega^{\sharp} \lrcorner F = 0. \tag{3.9}$$

One can view the F-terms as determining a holomorphic structure on a certain bundle $\mathcal{Q}$ [178]. The remaining D-term conditions – a conformally balanced metric and polystability of V – must then be imposed. More precisely one requires the bundle $\mathcal{Q}$ to be holomorphic, where $\mathcal{Q}$ is defined via a series of extensions as

$$\begin{array}{ccccc} T^{*1,0} & \longrightarrow & \mathcal{Q} & \longrightarrow & \mathcal{A}, \\ (\operatorname{ad} P_G)_{\mathbb{C}} & \longrightarrow & \mathcal{A} & \longrightarrow & T^{1,0}. \end{array} \tag{3.10}$$

where $\operatorname{ad} P_G$ is a vector bundle with fibre $\mathfrak{g}$, the Lie algebra of the gauge group. This is an example of a holomorphic Courant algebroid [74, 177]. Equivalently there exists a holomorphic differential $\bar{D}$ such that

$$\bar{D} \colon \Omega^{(p,q)}(X, \mathcal{Q}) \to \Omega^{(p,q+1)}(X, \mathcal{Q}), \qquad \bar{D}^2 = 0. \tag{3.11}$$

The condition $\bar{D}^2 = 0$ is equivalent to the integrability of the conventional complex structure, the holomorphicity of the gauge bundle and the Bianchi identities for F, R and H.

The moduli of the background appear in the massless spectrum of the four-dimensional theory and so a full knowledge of the moduli space is important for both phenomenology and more formal questions. Once one moves away from Calabi–Yau type solutions and allows non-zero fluxes, the moduli are much more difficult to understand. Fortunately, identifying the holomorphic structure $\bar{D}$ streamlines the analysis of the moduli space for heterotic compactifications [178, 179, 207, 226]. The moduli can be thought of as deformations of $\bar{D}$ that still satisfy $\bar{D}^2 = 0$ and the D-term conditions. Given some mild assumptions on the bundle V, it is known [178] that the hermitian Yang–Mills equations do not impose any extra conditions on the infinitesimal moduli of the system (and that the same result holds for T). It is also known that while deformations of the hermitian structure preserving the conformally balanced condition (3.3) may a priori be infinite dimensional, once you impose the anomaly cancellation condition you are reduced to a finite number of moduli. Up to $(0, 2)$ variations of the NSNS two-form B, the infinitesimal moduli of the Hull–Strominger system are then given by deformations of the holomorphic structure on $\mathcal{Q}$. That is they are counted by the cohomology

$$H^{0,1}_{\bar{D}}(X, \mathcal{Q}). \tag{3.12}$$

We should note that these actually include non-physical moduli which correspond to deformations of the connection Θ that do not change the physical fields, such as the metric.[1] These appear in this construction as one treats Θ as an independent field (and part of the gauge connection), whereas in reality it is determined by the other fields of the background. To find the physical moduli, one must remove this over counting – this has yet to be understood.

The story outlined above is valid for infinitesimal deformations. Using holomorphicity, one can also study finite deformations [179]. These are known to obey the Maurer–Cartan equation for an L_3 algebra (an L_∞ algebra up to degree 3). The deformations can be packaged into

$$y \in \Omega^{(0,1)}(X, \mathcal{Q}), \qquad b \in \Omega^{(0,2)}(X), \tag{3.13}$$

where y encodes deformations of the holomorphic structure – deformations of the complex structure, complexified hermitian structure and gauge connection – and b encodes the $(0,2)$ deformations of the B field. Note that the b modulus vanishes if $h^{0,2} = 0$ [227] – we will make no such assumption here and so shall keep explicit reference to it. To linear order the moduli are determined by the set of equations

$$\bar{D}y - \tfrac{1}{2}\partial b = 0, \tag{3.14}$$

$$\bar{\partial} b = 0, \tag{3.15}$$

$$\partial(\mathrm{e}^{-2\varphi}\imath_\mu\Omega) = 0, \tag{3.16}$$

where $\mu \in \Omega^{(0,1)}(X, T^{1,0})$ is a complex structure deformation. These are the equations we will recover in section 5.4.

3.2 Generalised Structures for $\mathcal{N} = 1$ Heterotic Backgrounds

Generalised geometry provides a useful framework for studying generic supersymmetric backgrounds of maximal supergravities in terms of integrable generalised G-structures. In particular, it gives a geometric interpretation of generic properties of type II and M-theory backgrounds, such as the superpotential and Kähler potential for $\mathcal{N} = 1$ solutions with four external dimensions, as well as tools to tackle questions about the moduli space [1]. Heterotic (and type I) theories can also be formulated in terms of generalised geometry, as we will now summarise briefly. We will then discuss how generalised geometry can be used to characterise $\mathcal{N} = 1$ heterotic backgrounds.

Ignoring the gauge bundle for now, the bosonic field content of the heterotic theory is the same as the NSNS sector of type II supergravity. Hence the relevant generalised geometry is that of $\mathrm{O}(6,6) \times \mathbb{R}^+$ generalised geometry on a generalised tangent bundle E defined as an extension of T by T^* [164,165]

$$T^* \longrightarrow E \longrightarrow T, \tag{3.17}$$

where E admits an $\mathrm{O}(6,6) \times \mathbb{R}^+$ structure. As usual, there is a natural differential operator known as the generalised Lie (or Dorfman) derivative on E. An (off-shell) configuration of the

[1] These are counted by $H^{(0,1)}_{\bar{\nabla}}(X, \operatorname{End} T)$, where $\bar{\nabla}$ is the antiholomorphic part of the covariant derivative defined by Θ.

bosonic fields defines a generalised metric that reduces the structure group of E to $SO(6) \times SO(6) \simeq SU(4) \times SU(4)$.

We can reintroduce the gauge connection and obtain full heterotic backgrounds as follows. Combining the connection Θ with the gauge connection A to give a single connection on the principal bundle P_G, where $G = G_{\text{gauge}} \times GL(6,\mathbb{R})$, the generalised tangent bundle E is defined as the extension

$$\begin{array}{ccccc} T^* & \longrightarrow & E' & \longrightarrow & E, \\ \operatorname{ad} P_G & \longrightarrow & E & \longrightarrow & T, \end{array} \tag{3.18}$$

where $\operatorname{ad} P_G$ is the vector bundle with fibre $\mathfrak{g}$, the Lie algebra of the extended gauge group G. This structure with its Dorfman derivative is known as a transitive Courant algebroid [228] – it has been used to describe heterotic supergravity in [167, 186] (see also [55] in the double field theory context). We reviewed some of the key points in section 2.3.2. In particular, given a generalised vector $V \in \Gamma(E)$, there is a Dorfman derivative L_V defined by (A.26). Locally we have a (non-canonical) isomorphism

$$E \simeq T \oplus \operatorname{ad} P_G \oplus T^*. \tag{3.19}$$

This has a natural $O(6, 6+n)$ structure on it defined by the inner product

$$\eta(v + \Lambda + \lambda, w + \Sigma + \sigma) = \tfrac{1}{2}\imath_v \sigma + \tfrac{1}{2}\imath_w \lambda + \operatorname{Tr}(\Lambda\Sigma), \tag{3.20}$$

where n is the dimension of $\mathfrak{g}$. While we will not give the exact form of the adjoint bundle $\operatorname{ad} \tilde{F}$ whose fibres are the Lie algebra $O(6, 6+n)$, we note that

$$T^* \otimes \mathfrak{g} \subseteq \operatorname{ad} \tilde{F} \simeq \wedge^2 E. \tag{3.21}$$

An (off-shell) configuration of the bosonic fields, that is a metric g, two-form B and one-form gauge field A, again define a generalised metric that in this case reduces the structure group to $SO(6) \times SO(6+n)$ [186]. Further requiring the fields to give a solution preserving $\mathcal{N} = 1$ supersymmetry is equivalent to a further reduction to an integrable $SU(3) \times SO(6+n)$ structure. As in previous work on $\mathcal{N} = 1$ structures [1], we will find it useful to also consider a weaker $\mathbb{R}^+ \times U(3) \times SO(6+n)$ structure. We will see how these are defined in terms of generalised structures in section 3.2.2 and how to define the conditions for integrability.

Note that in the formalism where one includes the Θ connection by extending the gauge bundle V to be a $G_{\text{gauge}} \times GL(6,\mathbb{R})$ bundle, there are non-physical degrees of freedom, since the connection Θ on the tangent space connection is thought of as independent of the metric and B. One can remove these by setting the value of Θ by hand. As was discussed in [167], one can get around this issue by identifying an $O(6)$ subbundle of the $GL(6,\mathbb{R})$ bundle, then identifying it with one of the $O(6)$ structures defined on the $T \oplus T^*$ part of the generalised tangent bundle. This gives a structure group $O(6) \times G_{\text{gauge}} \times O(6)$. The trade off is that the generalised connections relevant for this construction will not be torsion free, but instead appear with a particular non-vanishing intrinsic torsion. We will not take this approach in this chapter.

3.2.1 $\mathbb{R}^+ \times$ U(3) $\times$ SU(4) and SU(3) $\times$ SU(4) Structures

Let us start by considering the simple case where we ignore the gauge bundle, applicable to both the heterotic and type II theories. As discussed in [179, Appendix C], the existence of a nowhere-vanishing spinor that can parameterise $\mathcal{N} = 1$ supersymmetry transformations in four dimensions requires a reduction of the structure group from that defined by the generalised metric, namely $\mathrm{SU}(4) \times \mathrm{SU}(4)$, to $\mathrm{SU}(3) \times \mathrm{SU}(4) \subset \mathrm{O}(6,6) \times \mathbb{R}^+$. Following [1], it will be useful for us to also define a slightly weaker $\mathbb{R}^+ \times \mathrm{U}(3) \times \mathrm{SU}(4)$ structure. These will play roles analogous to $\mathrm{SL}(3,\mathbb{C})$ structures and $\mathrm{GL}(3,\mathbb{C})$ structures in conventional geometry.

Each structure is defined by a generalised tensor that is invariant under the reduced structure group[2]

$$\begin{aligned} \mathrm{SU}(3) \times \mathrm{SU}(4) \text{ structure}: &\quad \psi \in \Gamma(\det T^* \otimes \wedge^3 E_{\mathbb{C}}), \\ \mathbb{R}^+ \times \mathrm{U}(3) \times \mathrm{SU}(4) \text{ structure}: &\quad J \in \Gamma(\operatorname{ad} \tilde{F}). \end{aligned} \tag{3.22}$$

They are stabilised by the same $\mathrm{SU}(3) \times \mathrm{SU}(4)$, but J is also invariant under a $\mathbb{C}^*$ action. As discussed in detail in [1], one should think of this as generalising the relation between an $\mathrm{SL}(3,\mathbb{C})$ structure Ω and a $\mathrm{GL}(3,\mathbb{C})$ structure I. The differential conditions which ensure supersymmetry of the on-shell solution are then equivalent to the integrability of this structure, in line with the general discussion of [184]. In the next section we will see how we can reformulate the conditions for integrability of the $\mathbb{R}^+ \times \mathrm{U}(3) \times \mathrm{SU}(4)$ structure, and in the following section consider the extra conditions that make the $\mathrm{SU}(3) \times \mathrm{SU}(4)$ structure integrable.

Let us begin by defining the structure J. At a point on the manifold, the generalised metric defines an $\mathrm{SU}(4)\times\mathrm{SU}(4)$ subgroup of $\mathrm{O}(6,6)\times\mathbb{R}^+$, with the invariant spinor reducing this further to $\mathrm{SU}(3) \times \mathrm{SU}(4)$. There is a $\mathrm{U}(1) \subset \mathrm{SU}(4)$ that commutes with the $\mathrm{SU}(3)$. The commutant of this $\mathrm{U}(1)$ inside $\mathrm{O}(6,6) \times \mathbb{R}^+$ is an $\mathbb{R}^+ \times \mathrm{U}(3) \times \mathrm{SU}(4)$, where the $\mathrm{U}(1)$ is generated at each point of the internal manifold by a section $J \in \Gamma(\operatorname{ad} \tilde{F})$.[3] This leads us to define

Definition 12. A *generalised* $\mathbb{R}^+ \times U(3) \times SU(4)$ *structure* is a section $J \in \Gamma(\operatorname{ad} \tilde{F})$ that generates this U(1) subgroup at each point.

By construction, J defines a generic reduction of the structure group of the generalised tangent bundle E to $\mathbb{R}^+ \times \mathrm{U}(3) \times \mathrm{SU}(4)$.[4] Different choices of J are related by local $\mathrm{O}(6,6) \times \mathbb{R}^+$ transformations, giving an orbit of structures within the **66** representation space.

Decomposing $\mathrm{O}(6,6)$ using explicit $\mathrm{SU}(4) \times \mathrm{SU}(4)$ indices, we have

$$\mathbf{66} = (\mathbf{15},\mathbf{1}) \oplus (\mathbf{1},\mathbf{15}) \oplus (\mathbf{6},\mathbf{6}) \ni (\mu^{\alpha}{}_{\beta}, \mu^{\dot{\alpha}}{}_{\dot{\beta}}, \mu^{\alpha\beta\dot{\alpha}\dot{\beta}}), \tag{3.23}$$

where the nowhere-vanishing spinor ϵ is invariant under an SU(3) subgroup of the first SU(4) factor. Using this, we can write J as

$$J^{\alpha}{}_{\beta} = 4\,\epsilon^{\alpha}\bar{\epsilon}_{\beta} - (\bar{\epsilon}\epsilon)\delta^{\alpha}{}_{\beta}, \qquad J^{\dot{\alpha}}{}_{\dot{\beta}} = J^{\alpha\beta\dot{\alpha}\dot{\beta}} = 0, \tag{3.24}$$

[2] Note that, as we will argue below, the particular determinant weight of the ψ structure is required to make ψ a holomorphic function on the space of $\mathrm{SU}(3) \times \mathrm{SU}(4)$ structures.

[3] As in the type II and M-theory case [1], one can also define J at each point on the manifold as being conjugate to a certain element of $\mathfrak{su}(4) \times \mathfrak{su}(4)$ that commutes with the desired $\mathfrak{su}(3) \times \mathfrak{su}(4)$.

[4] Note that the standard *generalised complex structure* [164, 165] is also defined by choosing the generator of a U(1) subgroup but in that case the commutant would be U(3, 3).

where we have normalised $\bar{\epsilon}\epsilon = 1$. Decomposing further under the $\mathrm{SU}(3) \times \mathrm{U}(1)$ subgroup of the first $\mathrm{SU}(4)$ factor, we have

$$\mathbf{66} = (\mathbf{8},\mathbf{1})_0 \oplus (\mathbf{3},\mathbf{1})_{-2} \oplus (\bar{\mathbf{3}},\mathbf{1})_2 \oplus (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{15})_0 \oplus (\mathbf{3},\mathbf{6})_1 \oplus (\bar{\mathbf{3}},\mathbf{6})_{-1}, \tag{3.25}$$

where a non-bold subscript denotes the $\mathrm{U}(1)$ charge. J lies in the singlet $(\mathbf{1},\mathbf{1})_0$ representation.

From the expression (3.24) and the parameterisation of the generalised metric in terms of a conventional metric g and two-form field B, one finds that J generically takes the form

$$J = \tfrac{1}{2}\,\mathrm{e}^{-B} \cdot (I - \omega + \omega^\sharp), \tag{3.26}$$

where I is the almost complex structure on $T_{\mathbb{C}}$ defined by the three-form Ω, and ω is the compatible fundamental two-form. The B field acts by the exponentiated adjoint action, which is nilpotent at degree three. In analogy with a conventional complex structure, we can use J to decompose the generalised tangent space into eigenspaces. Under $\mathrm{SU}(3) \times \mathrm{U}(1) \times \mathrm{SU}(4)$, the adjoint action of J on the complexification of E splits as

$$\begin{aligned} E_{\mathbb{C}} &= L_1 \oplus L_{-1} \oplus L_0, \\ \mathbf{12}_{\mathbb{C}} &= (\mathbf{3},\mathbf{1})_1 \oplus (\bar{\mathbf{3}},\mathbf{1})_{-1} \oplus (\mathbf{1},\mathbf{6})_0. \end{aligned} \tag{3.27}$$

Given the form (3.26), it is then easy to see that L_{-1} takes the generic form

$$L_{-1} = \mathrm{e}^{-B-\mathrm{i}\omega} \cdot T^{0,1} = \{\bar{v} + \imath_{\bar{v}}(B + \mathrm{i}\omega) \mid \bar{v} \in \Gamma(T^{0,1})\}, \tag{3.28}$$

where as above $T^{0,1} \subset T_{\mathbb{C}}$ is the $-\mathrm{i}$ eigenbundle for the action of the almost complex structure I.[5] As with a conventional almost complex structure, we have an alternative definition purely in terms of the subbundle L_{-1}:

Definition 13. An $\mathbb{R}^+ \times \mathrm{U}(3) \times \mathrm{SU}(4)$ structure is a subbundle $L_{-1} \subset E_{\mathbb{C}}$ such that

i) $\dim_{\mathbb{C}} L_{-1} = 3$,

ii) $\eta(L_{-1}, L_{-1}) = 0$,

iii) $L_{-1} \cap \bar{L}_{-1} = \{0\}$,

iv) The map $h\colon L_{-1} \times L_{-1} \to \mathbb{C}$, defined by $h(V,W) = \eta(V, \bar{W})$, is a definite hermitian inner product.

Note that we could equally well define the structure in terms of L_1.

Turning to the $\mathrm{SU}(3) \times \mathrm{SU}(4)$ structure ψ, we note that the bundle

$$K = \det T^* \otimes \wedge^3 E, \tag{3.29}$$

transforms in the $\mathbf{220_1}$ representation of $\mathrm{O}(6,6) \times \mathbb{R}^+$ (where the bold subscript denotes the $\mathbb{R}^+$ weight [183]). Decomposing first under $\mathrm{SU}(4) \times \mathrm{SU}(4)$ and then under $\mathrm{SU}(3) \times \mathrm{U}(1) \times \mathrm{SU}(4)$,

[5] We will denote $(0,1)$-vectors with a bar. Unbarred objects will denote either generic vectors or $(1,0)$-vectors depending on context. The complex conjugate of a vector or one-form will be indicated with a superscript *.

we have

$$\begin{aligned}\mathbf{220} &= (\mathbf{10},\mathbf{1}) \oplus (\overline{\mathbf{10}},\mathbf{1}) \oplus (\mathbf{1},\mathbf{10}) \oplus (\mathbf{1},\overline{\mathbf{10}}) \oplus (\mathbf{15},\mathbf{6}) \oplus (\mathbf{6},\mathbf{15}) \\ &= (\mathbf{1},\mathbf{1})_3 \oplus (\bar{\mathbf{6}},\mathbf{1})_{-1} \oplus (\mathbf{3},\mathbf{1})_1 \oplus (\mathbf{1},\mathbf{1})_{-3} \oplus (\mathbf{6},\mathbf{1})_1 \oplus (\bar{\mathbf{3}},\mathbf{1})_{-1} \oplus \dots .\end{aligned} \tag{3.30}$$

where the subscripts now denote the U(1) charge. In particular, we see that the $\mathrm{SU}(3) \times \mathrm{SU}(4)$ singlet in the decomposition implies that each choice of J defines a unique line bundle $\mathcal{U}_J \subset K_{\mathbb{C}}$, satisfying

$$V \bullet \psi = 0 \quad \forall\, V \in \Gamma(L_{-1}), \qquad \eta(\psi,\bar{\psi}) \neq 0, \tag{3.31}$$

where ψ is a local section of $\mathcal{U}_J$, η is the pairing on sections of K induced from the symmetric pairing η on E, and the product $V \bullet \psi$ is the projection map $E \otimes K \to H$, where H is the generalised tensor bundle transforming in the $\mathbf{495_1}$ representation of $\mathrm{O}(6,6) \times \mathbb{R}^+$. Equivalently, a local section ψ is defined by $J\psi = -3\mathrm{i}\psi$ under the adjoint action of J.[6] Mirroring the definition of a nowhere-vanishing three-form for an almost complex structure, we then have

Definition 14. Given a choice of J with trivial line bundle $\mathcal{U}_J$, a generalised $\mathrm{SU}(3) \times \mathrm{SU}(4)$ structure is a global nowhere-vanishing section $\psi \in \Gamma(\mathcal{U}_J)$.

Note that two different choices of ψ that are related by multiplication by a nowhere-vanishing complex function define the same structure J. Decomposing with explicit $\mathrm{SU}(4) \times \mathrm{SU}(4)$ indices we have

$$\begin{aligned}\mathbf{220} &= (\mathbf{10},\mathbf{1}) \oplus (\overline{\mathbf{10}},\mathbf{1}) \oplus (\mathbf{15},\mathbf{6}) \oplus (\mathbf{6},\mathbf{15}) \oplus (\mathbf{1},\mathbf{10}) \oplus (\mathbf{1},\overline{\mathbf{10}}) \\ &\ni (\kappa^{\alpha\beta}, \kappa_{\alpha\beta}, \kappa^{\alpha}{}_{\beta}{}^{\dot\alpha\dot\beta}, \kappa^{\dot\alpha}{}_{\dot\beta}{}^{\alpha\beta}, \kappa^{\dot\alpha\dot\beta}, \kappa_{\dot\alpha\dot\beta}).\end{aligned} \tag{3.32}$$

In terms of the spinor ϵ we then have

$$\psi^{\alpha\beta} = \sqrt{g}\,\mathrm{e}^{-2\varphi}\,\epsilon^{\alpha}\epsilon^{\beta}, \tag{3.33}$$

with all the other components vanishing. Recall that ψ is defined up to a complex function. We fixed the normalisation $\bar{\epsilon}\epsilon = 1$, so that the phase of ϵ encodes the phase freedom in ψ, while the overall scale of ψ is parameterised by the dilaton $\mathrm{e}^{-2\varphi}$, in line with the fact that the combination $\sqrt{g}\,\mathrm{e}^{-2\varphi}$ is the $\mathrm{O}(6,6)$ invariant volume defined by the generalised metric [183].

Again we can use the generalised metric to translate this into a tensor expression following [179]. As we have mentioned a generalised metric gives a reduction of the structure group of E to $\mathrm{SO}(6)_+ \times \mathrm{SO}(6)_- \simeq \mathrm{SU}(4)_+ \times \mathrm{SU}(4)_-$. The $\mathrm{O}(6,6) \times \mathbb{R}^+$ generalised tangent bundle E then decomposes under $\mathrm{SO}(6)_+ \times \mathrm{SO}(6)_-$ as $E = C_+ \oplus C_-$, giving a corresponding decomposition of $\wedge^3 E$ as

$$\wedge^3 E = \wedge^3 C_+ \oplus (\wedge^2 C_+ \otimes C_-) \oplus (C_+ \otimes \wedge^2 C_-) \oplus \wedge^3 C_-, \tag{3.34}$$

as in (3.32), where the $\wedge^3 C_\pm$ spaces decompose into complex self-dual and anti-self-dual components transforming in the $\mathbf{10}$ and $\overline{\mathbf{10}}$ representations. Note that, in terms of the splitting

[6] This corresponds to taking $\psi \in (\mathbf{1},\mathbf{1})_{-3}$. We make this choice to match with the usual conventions of Ω being the holomorphic object on the space of structures.

defined by the generalised metric we have

$$(C_+)_{\mathbb{C}} = L_1 \oplus L_{-1}, \qquad (C_-)_{\mathbb{C}} = L_0. \tag{3.35}$$

Let $\hat{E}^+_a = \hat{e}_a + e_a - \imath_{\hat{e}_a} B$ be an explicit basis for C_+, where $\hat{e}_a$ is an orthonormal basis for T defined by the metric g, and e_a is the dual basis. The expression (3.33) defines the tensor

$$\begin{aligned}\psi &= \sqrt{g}\,\mathrm{e}^{-2\varphi}\,\tfrac{1}{3!}(\epsilon^{\mathrm{T}}\gamma^{abc}\epsilon)\,\hat{E}^+_a \wedge \hat{E}^+_b \wedge \hat{E}^+_c \\ &= \mathrm{e}^{-2\varphi}\,\mathrm{e}^{-B-\mathrm{i}\omega}\cdot\Omega,\end{aligned} \tag{3.36}$$

where the exponential $\mathrm{e}^{-B-\mathrm{i}\omega}$ acts via the adjoint action and in going to the second line we use the isomorphism $\wedge^3 T \otimes \wedge^6 T^* \simeq \wedge^3 T^*$. This expression ensures ψ is stabilised by the correct $\mathrm{SU}(3)\times\mathrm{SU}(4)$ subgroup. We note that given an $\mathcal{N}=2$ structure encoded by a pair of pure spinors $\Phi_\pm$, one can construct ψ as

$$\psi^{MNP} = (\bar{\Phi}_+, \Gamma^{MNP}\Phi_-), \tag{3.37}$$

where Γ^M are the $\mathrm{O}(6,6)$ gamma matrices and $(\cdot,\cdot)$ is the Mukai pairing.

3.2.2 $\mathbb{R}^+ \times \mathrm{U}(3) \times \mathrm{Spin}(6+n)$ and $\mathrm{SU}(3)\times\mathrm{Spin}(6+n)$ Structures

It is straightforward to extend this story to include the gauge bundle. Since many of the results are analogous to the previous section, we will sketch the key points. As noted in (3.19), the generalised tangent bundle is locally given by

$$E \simeq T \oplus \mathrm{ad}\,P_G \oplus T^*, \tag{3.38}$$

where $\mathrm{ad}\,P_G$ is the adjoint bundle with fibres given by the Lie algebra $\mathfrak{g}$ of the gauge group G. Sections of E thus encode diffeomorphisms and gauge transformations of both the gauge field A and the two-form B. Again, there are two generalised structures each defined by a generalised tensor that is invariant under the reduced structure group

$$\begin{aligned}\mathrm{SU}(3)\times\mathrm{Spin}(6+n)\text{ structure}: &\quad \psi \in \Gamma(\det T^* \otimes \wedge^3 E_{\mathbb{C}}), \\ \mathbb{R}^+\times\mathrm{U}(3)\times\mathrm{Spin}(6+n)\text{ structure}: &\quad J \in \Gamma(\mathrm{ad}\,\tilde{F}).\end{aligned} \tag{3.39}$$

These are stabilised by the same $\mathrm{SU}(3)\times\mathrm{Spin}(6+n)$, but J is also invariant under a $\mathbb{C}^*$ action.

We begin with the weaker $\mathbb{R}^+\times\mathrm{U}(3)\times\mathrm{Spin}(6+n)$ structure defined by J. Mirroring the discussion in the previous subsection, one finds that J generically takes the form

$$J = \tfrac{1}{2}\,\mathrm{e}^{-B}\mathrm{e}^{-A}\cdot(I - \omega + \omega^\sharp), \tag{3.40}$$

where now we include a twisting by the one-form gauge field A. Again, we can use J to decompose the generalised tangent space E into eigenspaces. Noting that the fibres of E transform in the $(\mathbf{12}+\boldsymbol{n})$ representation of $\mathrm{O}(6,6+n)$ and decomposing under $\mathrm{U}(1)\times\mathrm{SU}(3)\times\mathrm{Spin}(6+n)$

we find that

$$\begin{aligned} E_{\mathbb{C}} &= L_1 \oplus L_{-1} \oplus L_0, \\ \mathbf{12}+\boldsymbol{n} &= (\mathbf{3},\mathbf{1})_1 + (\overline{\mathbf{3}},\mathbf{1})_{-1} + (\mathbf{1},\mathbf{6}+\boldsymbol{n})_0, \end{aligned} \tag{3.41}$$

where $(\mathbf{6}+\boldsymbol{n})$ is the fundamental representation of $\mathrm{Spin}(6+n)$. Identifying L_{-1} as the subbundle transforming as $(\overline{\mathbf{3}},\mathbf{1})_{-1}$, given the form of J in (3.40), one can check that L_{-1} takes the generic form

$$L_{-1} = \mathrm{e}^{-B-\mathrm{i}\,\omega}\mathrm{e}^{-A}T^{0,1} = \{\bar{v} + \imath_{\bar{v}}A + \imath_{\bar{v}}(B+\mathrm{i}\,\omega) - \mathrm{Tr}(\imath_{\bar{v}}A\,A) \mid \bar{v} \in \Gamma(T^{0,1})\}, \tag{3.42}$$

where $T^{0,1} \subset T_{\mathbb{C}}$ is the $-\mathrm{i}$ eigenbundle for the almost complex structure I. As before, one can use L_{-1}, subject to some algebraic conditions, as a definition of the $\mathbb{R}^+ \times \mathrm{U}(3) \times \mathrm{Spin}(6+n)$ structure.

As in the case without the gauge bundle, an $\mathrm{SU}(3) \times \mathrm{Spin}(6+n)$ structure ψ is a nowhere-vanishing section of

$$\psi \in \Gamma(\det T^* \otimes \wedge^3 E_{\mathbb{C}}). \tag{3.43}$$

Again, ψ is not a generic element but needs to lie in a particular orbit of $\mathrm{Spin}(6,6+n)$ so that its stabiliser is $\mathrm{SU}(3)\times\mathrm{Spin}(6+n)$. Using a generalised metric, we can write $E = C_+ \oplus C_-$, where C_+ is a six-dimensional subbundle on which η is positive definite, defined in (3.20). Letting $\hat{E}^+_m$ be a basis for C_+, we can write

$$\begin{aligned} \psi &= \sqrt{g}\mathrm{e}^{-2\varphi}\tfrac{1}{3!}(\epsilon^{\mathrm{T}}\gamma^{mnp}\epsilon)\hat{E}^+_m \wedge \hat{E}^+_n \wedge \hat{E}^+_p \\ &= \mathrm{e}^{-2\varphi}\mathrm{e}^{-B-\mathrm{i}\omega}\mathrm{e}^{-A}\cdot\Omega. \end{aligned} \tag{3.44}$$

This expression guarantees that ψ is stabilised by the correct $\mathrm{SU}(3)\times\mathrm{Spin}(6+n)$ group.

3.2.3 Supersymmetry and Intrinsic Torsion

The existence of the ψ structure is just the algebraic part of the supersymmetry conditions for an $\mathcal{N}=1$ background (namely the requirement that one has a non-vanishing spinor). There are also differential conditions given by the Killing spinor equations, which can be translated into the F- and D-term conditions in (3.8) and (3.9) respectively. As we will discuss, these are equivalent to the structure being torsion-free or "integrable" [167, 184, 186]. As in [1], it will be useful to consider the intrinsic torsion for both J and ψ as one can view an integrable ψ in terms of an integrable J together with a further differential condition in the form of a moment map for generalised diffeomorphisms.

We call a structure torsion-free or integrable if there exists a generalised connection that is compatible with the structure and is torsion-free. For example, a torsion-free $\mathrm{SU}(3)\times\mathrm{Spin}(6+n)$ structure is equivalent to the existence of ψ and a connection D such that

$$D\psi = 0, \qquad L^D_V - L_V = T(V) = 0, \tag{3.45}$$

where L_V is the Dorfman derivative defined in (A.26) with the gauge sector turned off, L^D_V is the Dorfman derivative with ∂ replaced by D, and the generalised torsion is a map $T\colon \Gamma(E) \to \Gamma(\operatorname{ad}\tilde{F})$. The obstruction to the existence of such a torsion-free connection is a non-vanishing intrinsic torsion.

Starting with the simpler case where we ignore the gauge bundle, following the standard analysis [181, 183–185, 187], we find that the intrinsic torsion for ψ and J live in subbundles of $\wedge^3 E \oplus E^*$ transforming as $\mathbf{220} \oplus \mathbf{12}$ and decomposing under the structure group via

$$\begin{aligned} W^{\text{int}}_{\text{SU}(3)\times\text{SU}(4)} :&(\mathbf{3},\mathbf{6})_{-2} \oplus (\bar{\mathbf{3}},\mathbf{6})_{2} \oplus (\mathbf{1},\mathbf{1})_{-3} \oplus (\mathbf{1},\mathbf{1})_{3} \\ &\oplus (\mathbf{3},\mathbf{1})_{1} \oplus (\bar{\mathbf{3}},\mathbf{1})_{-1} \oplus (\mathbf{1},\mathbf{6})_{0}, \end{aligned} \tag{3.46}$$

$$W^{\text{int}}_{\mathbb{R}^+\times\text{U}(3)\times\text{SU}(4)} :(\mathbf{3},\mathbf{6})_{-2} \oplus (\bar{\mathbf{3}},\mathbf{6})_{2} \oplus (\mathbf{1},\mathbf{1})_{-3} \oplus (\mathbf{1},\mathbf{1})_{3}, \tag{3.47}$$

where the subscript denotes the U(1) charge under J.

When we include the gauge bundle the representations in which the intrinsic torsion for each structure lives are given by

$$\begin{aligned} W^{\text{int}}_{\text{SU}(3)\times\text{Spin}(6+n)} :&(\mathbf{3},\mathbf{6}+\boldsymbol{n})_{-2} \oplus (\bar{\mathbf{3}},\mathbf{6}+\boldsymbol{n})_{2} \oplus (\mathbf{1},\mathbf{1})_{-3} \oplus (\mathbf{1},\mathbf{1})_{3} \\ &\oplus (\mathbf{3},\mathbf{1})_{1} \oplus (\bar{\mathbf{3}},\mathbf{1})_{-1} \oplus (\mathbf{1},\mathbf{6}+\boldsymbol{n})_{0}, \end{aligned} \tag{3.48}$$

$$W^{\text{int}}_{\mathbb{R}^+\times\text{U}(3)\times\text{Spin}(6+n)} :(\mathbf{3},\mathbf{6}+\boldsymbol{n})_{-2} \oplus (\bar{\mathbf{3}},\mathbf{6}+\boldsymbol{n})_{2} \oplus (\mathbf{1},\mathbf{1})_{-3} \oplus (\mathbf{1},\mathbf{1})_{3}, \tag{3.49}$$

where a subscript denotes the U(1) charge with respect to J and $(\mathbf{6}+\boldsymbol{n})$ is the fundamental representation of Spin$(6+n)$.

Since $\mathcal{N}=1$ supersymmetry in four dimensions follows from integrability of the SU(3) $\times$ SU(4) structure, and integrability is equivalent to the vanishing of the intrinsic torsion of the structure, we need some natural differential conditions which enforce the vanishing of the above components of the intrinsic torsion. These differential conditions should then be thought of as the supersymmetry conditions for the background, but now with a geometric interpretation. The form of these conditions will be the subject of the next two sections.

3.3 Involutivity, the Superpotential and F-Terms

In this section we will consider the integrability of the weaker $\mathbb{R}^+ \times \text{U}(3) \times \text{SU}(4)$ and $\mathbb{R}^+ \times \text{U}(3) \times \text{Spin}(6+n)$ structures, defined by J, and the show how these conditions can be defined as an involutivity condition of a subbundle or equally as coming from varying a superpotential. This matches an earlier observation, in the case of pure $\text{O}(d,d)$ generalised geometry, relating supersymmetry of the underlying sigma model to integrability of a subbundle [229]. We will also briefly discuss the connection to the holomorphic Courant algebroid [74, 177, 178] given in (3.10). We will turn to the extra conditions that one must impose on ψ to guarantee an honest $\mathcal{N}=1$ background in the next section.

3.3.1 Involutivity Conditions

As with conventional complex structures and the $\mathcal{N}=1$ structures defined in [1], it turns out that integrability of the J structure is equivalent to involutivity of a subbundle of the generalised tangent bundle. For the $\mathbb{R}^+ \times \mathrm{U}(3) \times \mathrm{SU}(3)$ structure we define

Definition 15. A torsion-free $\mathbb{R}^+ \times \mathrm{U}(3) \times \mathrm{SU}(4)$ structure J is one for which L_{-1} is involutive under the Dorfman derivative

$$L_V W \in \Gamma(L_{-1}) \qquad \forall\, V, W \in \Gamma(L_{-1}). \tag{3.50}$$

Note that one can replace the Dorfman derivative with the Courant bracket in this condition: the difference between the two is a term of the form $\mathrm{d}(\eta(V,W))$, but $\eta(V,W)$ vanishes for $V, W \in \Gamma(L_{-1})$ from definition 13. We also note that since $\bar{L}_{-1} \simeq L_1$, involutivity of L_{-1} is equivalent to involutivity of L_1.

It is straightforward to see that involutivity of L_{-1} is equivalent to vanishing intrinsic torsion for the $\mathbb{R}^+ \times \mathrm{U}(3) \times \mathrm{SU}(4)$ structure. Recall first that we can always find a generalised connection D that is compatible with the structure, so that $DJ = 0$, but this is not necessarily torsion-free. Now consider the definition (3.45) of the torsion of a connection where we restrict to $V, W \in \Gamma(L_{-1})$

$$L_V W = L^D_V W - T(V) \cdot W. \tag{3.51}$$

Compatibility of the connection guarantees $L^D_V W \in \Gamma(L_{-1})$, so involutivity reduces to checking that $T(V) \cdot W$ lies only in L_{-1}. Note also that since the left-hand side does not depend on the choice of connection and $L^D_V W$ lies in $\Gamma(L_{-1})$ for any choice of D, only the intrinsic torsion can contribute to the components of $T(V) \cdot W$ that lie outside of L_{-1}. The intrinsic torsion representations that appear in $T(V) \cdot W \in \Gamma(E)$ are

$$\begin{aligned} &(\bar{\mathbf{3}}, \mathbf{6})_2 \otimes (\bar{\mathbf{3}}, \mathbf{1})_{-1} \otimes (\bar{\mathbf{3}}, \mathbf{1})_{-1} \supset (\mathbf{1}, \mathbf{6})_0, \\ &(\mathbf{1}, \mathbf{1})_3 \otimes (\bar{\mathbf{3}}, \mathbf{1})_{-1} \otimes (\bar{\mathbf{3}}, \mathbf{1})_{-1} \supset (\mathbf{3}, \mathbf{1})_1. \end{aligned} \tag{3.52}$$

A non-zero $(\bar{\mathbf{3}}, \mathbf{6})_2$ component of the intrinsic torsion would generate a $(\mathbf{1}, \mathbf{6})_0 \simeq L_0$ term in $L_V W$, while a non-zero $(\mathbf{1}, \mathbf{1})_3$ component would generate a $(\mathbf{3}, \mathbf{1})_1 \simeq L_1$ part. Requiring both of these to be absent so that $L_V W \in \Gamma(L_{-1})$ sets both of these components of the intrinsic torsion to zero. Complex conjugation then implies that the whole of the intrinsic torsion vanishes. This shows that the $\mathbb{R}^+ \times \mathrm{U}(3) \times \mathrm{SU}(4)$ structure defined by J, or equivalently L_{-1}, is integrable if and only if L_{-1} is involutive with respect to the Dorfman derivative.

The discussion up to this point has been rather abstract. One might wonder how integrability for J translates into concrete equations for the SU(3) structure that underlies the Hull–Strominger system discussed in section 3.1. Given that we have an explicit description of the subbundle L_{-1}, given in (3.28), we can check how involutivity constrains the SU(3) structure. Taking any $v, w \in \Gamma(T)$ one finds

$$L_{\mathrm{e}^{-B-\mathrm{i}\,\omega} v}(\mathrm{e}^{-B-\mathrm{i}\,\omega} w) = \mathrm{e}^{-B-\mathrm{i}\,\omega} L^{H+\mathrm{i}\,\mathrm{d}\omega}_v w = \mathrm{e}^{-B-\mathrm{i}\,\omega}\big([v,w] - \imath_v \imath_w (H + \mathrm{i}\,\mathrm{d}\omega)\big), \tag{3.53}$$

where $H = \mathrm{d}B$ and we have used the expression for the Dorfman derivative in (A.26) after setting the gauge field to zero. If in particular we choose the vectors to be $\bar{v}, \bar{w} \in \Gamma(T^{0,1})$ so that $\mathrm{e}^{-B-\mathrm{i}\,\omega}\bar{v} \in \Gamma(L_{-1})$, then for L_{-1} to be involutive (so that the right-hand side lies only in L_{-1}), we require that $[\bar{v}, \bar{w}] - \imath_{\bar{v}} \imath_{\bar{w}} (H + \mathrm{i}\,\mathrm{d}\omega)$ is a section of $\Gamma(T^{0,1})$ alone. Splitting into vector and one-form equations, this gives the conditions

$$[\bar{v}, \bar{w}] \in \Gamma(T^{0,1}), \qquad \imath_{\bar{v}}\imath_{\bar{w}}(H + \mathrm{i}\,\mathrm{d}\omega) = 0, \tag{3.54}$$

which must hold for all choices of $\bar{v}, \bar{w} \in \Gamma(T^{0,1})$. The first of these is simply the requirement that the almost complex structure I is integrable, so that it is an honest complex structure. This also implies that the corresponding complex three-form Ω satisfies $\mathrm{d}\Omega = \bar{a} \wedge \Omega$ for some $\bar{a} \in \Omega^{0,1}(X)$. The second condition can be understood by decomposing according to complex type as $H = H_{3,0} + H_{2,1} + H_{1,2} + H_{0,3}$ and $\omega = \omega_{1,1}$. Since $\bar{v}$ and $\bar{w}$ are $(0,1)$-vectors, the second of the conditions gives $H_{0,3} = 0$ and $H_{1,2} + \mathrm{i}\,\bar{\partial}\omega = 0$. As both H and ω are real, these imply $H_{3,0} = H_{0,3} = 0$ and $H_{2,1} + H_{1,2} + \mathrm{i}(\bar{\partial} - \partial)\omega = 0$. Putting this together, we have

$$L_{-1} \text{ is involutive} \quad \Leftrightarrow \quad \begin{array}{c} [\bar{v}, \bar{w}] \in \Gamma(T^{0,1}) \\ H = \mathrm{i}(\partial - \bar{\partial})\omega \end{array} \tag{3.55}$$

Note that these are (almost) the equations coming from the F-term conditions (3.8) with the gauge bundle turned off. The F-term equations are slightly stronger since they imply that Ω is conformally holomorphic, fixing $\bar{a}$ in terms of the dilaton φ, whereas the above conditions leave $\bar{a}$ undetermined. We will come back to this point when we discuss the superpotential in section 3.3.2. Note also that these are the same set of conditions as the integrability of a "half generalised complex structure" [230], which appear from a worldsheet analysis of $(2,0)$ non-linear sigma model geometry.

The involutivity condition naturally extends to the $\mathbb{R}^+ \times \mathrm{U}(3) \times \mathrm{Spin}(6+n)$ case. Given the explicit description of L_{-1} in (3.42) and the expression for the Dorfman derivative in (A.26), we can relate integrability for the $\mathbb{R}^+ \times \mathrm{U}(3) \times \mathrm{Spin}(6+n)$ structure, in the form of involutivity of L_{-1}, to the data of the Hull–Strominger system, namely the SU(3) structure and the connection on V. Taking generic vectors $v, w \in \Gamma(T)$ one now finds

$$L_{\mathrm{e}^{-B-\mathrm{i}\,\omega}\mathrm{e}^{-A}v}(\mathrm{e}^{-B-\mathrm{i}\,\omega}\mathrm{e}^{-A}w) = \mathrm{e}^{-B-\mathrm{i}\,\omega}\mathrm{e}^{-A}\big([v,w] - \imath_v\imath_w(H + \mathrm{i}\,\mathrm{d}\omega) - \imath_v\imath_w F\big), \tag{3.56}$$

where

$$H = \mathrm{d}B + \omega_3(A), \qquad \omega_3(A) = \mathrm{Tr}(A \wedge \mathrm{d}A + \tfrac{2}{3}A \wedge A \wedge A), \tag{3.57}$$

$$F = \mathrm{d}A + A \wedge A, \qquad \mathrm{d}H = \mathrm{Tr}(F \wedge F). \tag{3.58}$$

As before, specialising to $\bar{v}, \bar{w} \in \Gamma(T^{0,1})$ so that $\mathrm{e}^{-B-\mathrm{i}\,\omega}\mathrm{e}^{-A}\bar{v} \in \Gamma(L_{-1})$, for involutivity of L_{-1} we require that the expression in the parentheses in (3.56) lies only in $\Gamma(T^{0,1})$. This implies

$$L_{-1} \text{ is involutive} \quad \Leftrightarrow \quad \begin{array}{c} [\bar{v}, \bar{w}] \in \Gamma(T^{0,1}) \\ H = \mathrm{i}(\partial - \bar{\partial})\omega \\ F_{0,2} = 0 \end{array} \tag{3.59}$$

As before, we have an integrable complex structure on the manifold, implying $\mathrm{d}\Omega = \bar{a} \wedge \Omega$ for some $\bar{a} \in \Omega^{0,1}(X)$, and the three-form flux H is fixed by d^I of the hermitian form ω. In addition, the $(0,2)$ component of the curvature F must vanish, implying that the gauge bundle is holomorphic. Again, these are the F-term equations (3.8), up to the conformal holomorphicity condition for Ω.

In order to describe the heterotic theory, as mentioned, we can include the tangent bundle connection within the gauge sector, as discussed in [74,167,186]. This has the effect of redefining H to be its full heterotic form and adds a holomorphicity condition for the tangent bundle connection so that

$$H = \mathrm{d}B + \omega_3(A) - \omega_3(\Theta), \qquad R_{0,2} = 0, \tag{3.60}$$

where Θ is the ∇^- connection and R is the corresponding curvature two-form.

It is interesting to compare how the involutivity condition on L_{-1} defines the holomorphic structure of the geometry to the holomorphic Courant algebroid $\mathcal{Q}$ given in (3.10) and used in the papers [74,177]. Defining the perpendicular subbundle $L_{-1}^{\perp}$, such that, on a patch U_i,

$$V \in \Gamma(L_{-1}^{\perp}) \qquad \Leftrightarrow \qquad \eta(V,W) = 0 \quad \forall\, W \in \Gamma(L_{-1}), \tag{3.61}$$

we have

$$\begin{aligned} L_{-1}^{\perp}/L_{-1} &\simeq \mathrm{e}^{-B-\mathrm{i}\omega}\mathrm{e}^{-A}\cdot\left(T^{1,0}\oplus T^{0,1}\oplus T^{*1,0}\oplus(\operatorname{ad}P_G)_{\mathbb{C}}\right)/\mathrm{e}^{-B-\mathrm{i}\omega}\mathrm{e}^{-A}\cdot T^{0,1}, \\ &\simeq \mathrm{e}^{-B-\mathrm{i}\omega}\mathrm{e}^{-A}\cdot\left(T^{1,0}\oplus T^{*1,0}\oplus(\operatorname{ad}P_G)_{\mathbb{C}}\right), \\ &\simeq T^{1,0}\oplus T^{*1,0}\oplus(\operatorname{ad}P_G)_{\mathbb{C}} \simeq \mathcal{Q}. \end{aligned} \tag{3.62}$$

Hence we see that L_{-1} indeed determines $\mathcal{Q}$ and furthermore the involutivity of L_{-1} implies that $\mathcal{Q}$ is holomorphic.[7] As a bundle, all $\mathcal{Q}$ are isomorphic to $T^{1,0}\oplus T^{*1,0}\oplus(\operatorname{ad}P_G)_{\mathbb{C}}$. However the corresponding holomorphic Courant algebroids (or more precisely "Bott–Chern algebroids" in the language of [74]) are distinguished by the choice of ω and A, such that inequivalent algebroids are distinguished by the Aeppli class defined in [74].

3.3.2 The Superpotential

It is known that the F-term conditions in (3.8) can be derived starting from a heterotic superpotential [207,231–233]

$$\mathcal{W} = \int_X \mathrm{e}^{-2\varphi}\Omega\wedge(H + \mathrm{i}\,\mathrm{d}\omega), \tag{3.63}$$

and requiring $\mathcal{W} = \delta\mathcal{W} = 0$ under variations of the structures Ω, ω and fields B and φ [207,233]. Building on work on flux superpotentials [234,235] and their description in generalised geometry [180], we conjectured in [1] that the superpotential is given by the singlet part of the intrinsic torsion of the ψ structure and explicitly showed this was true for the examples of G_2 in M-theory and generic $\mathcal{N}=1$ backgrounds of type II theories. Here we will show that the singlet torsion does indeed give the superpotential in the case of heterotic backgrounds and that it is a holomorphic function of ψ. We also discuss how the superpotential conditions imply involutivity of L_{-1}. Not only does this provide a covariant expression for the superpotential for

[7] Note that it is the adjoint bundle for the complexified group, $G_{\mathbb{C}}$, that appears here. If L_{-1} is involutive, so that we have $F_{0,2}=0$, the transition functions that define $(\operatorname{ad}P_G)_{\mathbb{C}}$ can be taken to be holomorphic, so that $\mathcal{Q}$ is also holomorphic.

generic heterotic backgrounds, it also provides further justification for the claim made in [1].

Given that an infinitesimal change in ψ can be parameterised by an element of the $\mathrm{O}(6,6+n)\times\mathbb{R}^+$ Lie algebra and ψ transforms in the $(\mathbf{1},\mathbf{1})_{-3}$, the variations of the $\mathrm{SU}(3)\times\mathrm{Spin}(6+n)$ structure ψ transform as $(\mathbf{1},\mathbf{1})_{-3}$, $(\bar{\mathbf{3}},\mathbf{1})_{-1}$ and $(\mathbf{3},\mathbf{6}+\boldsymbol{n})_{-2}$. Thus $\delta\mathcal{W}/\delta\psi=0$ constrains the dual $(\mathbf{1},\mathbf{1})_{3}$, $(\mathbf{3},\mathbf{1})_{1}$ and $(\bar{\mathbf{3}},\mathbf{6}+\boldsymbol{n})_{2}$ components of the intrinsic torsion. Note that this means the vanishing of the variation of $\mathcal{W}$ implies $\mathcal{W}=0$, as $\mathcal{W}$ is the singlet component of the intrinsic torsion. We also note that the superpotential condition is slightly stronger than involutivity of L_{-1}, which constrained only the $(\mathbf{1},\mathbf{1})_{3}$ and $(\bar{\mathbf{3}},\mathbf{6}+\boldsymbol{n})_{2}$ components, leaving $(\mathbf{3},\mathbf{1})_{1}$ undetermined. The involutivity condition implies there is an integrable complex structure and hence $\mathrm{d}\Omega=\bar{a}\wedge\Omega$. The extra superpotential constraint is precisely what is needed to fix the $(0,1)$-form $\bar{a}$.

As for $\mathrm{E}_{7(7)}\times\mathbb{R}^+$ backgrounds with $\mathcal{N}=1$ supersymmetry, one can rephrase involutivity as a holomorphic condition on ψ itself. Let $V\in\Gamma(L_{-1})$ and D be a compatible connection, such that $D\psi=0$. From the definition of the torsion of D in (3.45), we have

$$L_V\psi=-T(V)\cdot\psi \qquad \text{for } V\in\Gamma(L_{-1}). \tag{3.64}$$

Naively one would expect $L^D_V\psi$ to appear on the right-hand side. This would contain terms of the form $D_V\psi$, $(D\times_{\mathrm{ad}}V)\cdot\psi$ and $(D\cdot V)\psi$ (where the final term appears as ψ has a non-zero weight under the $\mathbb{R}^+$ action). However, using the fact that ψ is a singlet and that it has weight one under $\mathbb{R}^+$, one finds that the terms which involve D acting on V cancel identically, leaving only $D_V\psi$ which vanishes due to the compatibility of the connection. The remaining torsion term is linear in V and, since $L_V\psi$ is independent of D, only the intrinsic torsion can appear in $T(V)\cdot\psi$. Using the $\mathrm{U}(1)\times\mathrm{SU}(3)\times\mathrm{SU}(4)$ decomposition, one can check that the $(\bar{\mathbf{3}},\mathbf{6}+\boldsymbol{n})_{2}$, $(\mathbf{1},\mathbf{1})_{3}$ and $(\mathbf{3},\mathbf{1})_{1}$ parts of the intrinsic torsion (3.46) appear, which are the same components that appear in $\delta\mathcal{W}/\delta\psi$. This gives us an alternative description of the involutivity condition as

$$\text{involutive } L_{-1} \qquad\Leftrightarrow\qquad L_V\psi=U(V)\,\psi \quad \forall\, V\in\Gamma(L_{-1}), \tag{3.65}$$

where $U\in\Gamma(L^*_{-1})$ is the $(\mathbf{3},\mathbf{1})_{1}$ component of the $\mathrm{SU}(3)\times\mathrm{SU}(4)$ intrinsic torsion, and $U(V)=U_MV^M$ is a pairing between sections of E^* and E so that $U(V)$ is a scalar function. If we further require that U vanishes, we have

$$\frac{\delta\mathcal{W}}{\delta\psi}=0 \qquad\Leftrightarrow\qquad L_V\psi=0 \quad \forall\, V\in\Gamma(L_{-1}), \tag{3.66}$$

so that we have an alternative description of the superpotential condition (recall that $\delta\mathcal{W}/\delta\psi=0$ implies $\mathcal{W}=0$). As discussed in [1], we expect that one can take a given ψ that satisfies the involutivity condition and rescale it by an appropriate complex function so that the stronger superpotential condition is satisfied. Note that these expressions show that involutivity and the superpotential itself are holomorphic in ψ. Since L_{-1} is fixed by $V\bullet\psi=0$ (see (3.31)), L_{-1} depends holomorphically on ψ. The conditions that $L_V\psi=U(V)\psi$ and $L_V\psi=0$ for all $V\in\Gamma(L_{-1})$ are then also holomorphic in ψ (since $\bar{\psi}$ does not appear).

Our conjecture that the superpotential is given by the singlet of the intrinsic torsion can be translated to the statement that

$$\mathcal{W} = \int_X W \sim \int_X \eta(\psi, T), \tag{3.67}$$

where T is the intrinsic torsion of the structure. The pairing of T with ψ projects onto the $(\mathbf{1}, \mathbf{1})_3$ component. Note also that ψ is weight one and T is weight zero under the $\mathbb{R}^+$ action, so that their pairing is a weight-one scalar. A weight-one scalar is a section of $\det T^*$ and so gives a volume form that can be integrated over the manifold. From the previous discussion, the $(\mathbf{1}, \mathbf{1})_3$ component of the torsion can be obtained from ψ alone, and so the superpotential itself is a holomorphic function of ψ.

There are alternative ways to write $\mathcal{W}$ to make the dependence on ψ more obvious. One can always find a torsion-free connection D that is compatible with the generalised metric structure discussed in section 3.2.2. Using this one can write the integrand of the superpotential as

$$W \sim \operatorname{Tr}(J, D \times_{\text{ad}} \psi), \tag{3.68}$$

where J is the $\mathbb{R}^+ \times \mathrm{U}(3) \times \mathrm{Spin}(6+n)$ structure defined in section 3.2.2.[8] Note that since neither J nor the generalised connection are weighted under $\mathbb{R}^+$, the right-hand side of (3.68) is a section of $\det T^*$ and hence we can integrate it over the manifold to give

$$\mathcal{W} \sim \int_X \operatorname{Tr}(J, D \times_{\text{ad}} \psi). \tag{3.69}$$

This expression is the easiest to use for direct calculations. Naively it does not appear to be holomorphic in ψ as J is a function of ψ and $\bar{\psi}$. However, we can rewrite it as

$$\mathcal{W} \sim \int_X \frac{\eta(\bar{\psi}, (D \times_{\text{ad}} \psi) \cdot \psi)}{\eta(\bar{\psi}, \psi)}, \tag{3.70}$$

where, as in [1], the weight of ψ is such that the dependence on $\bar{\psi}$ drops out. That is, under an infinitesimal antiholomorphic variation of $\bar{\psi}$, only the terms that are proportional to $\bar{\psi}$ contribute to the variation of $\eta(\bar{\psi}, (D \times_{\text{ad}} \psi) \cdot \psi)$, while the other components are projected out. This leaves a trivial scaling transformation $\bar{\psi} \to \mathrm{e}^{\bar{c}} \bar{\psi}$, under which our expression is clearly invariant thanks to $\eta(\bar{\psi}, \psi)$ in the denominator. Hence $\mathcal{W}$ does not vary under deformations of $\bar{\psi}$ and so it is indeed holomorphic in ψ, as we claimed.

As we show in appendix D, using the explicit expressions for J and ψ in terms of the underlying SU(3) structure, the superpotential reduces to

$$\mathcal{W} \sim \int_X \mathrm{e}^{-2\varphi} \Omega \wedge (H + \mathrm{i}\, \mathrm{d}\omega). \tag{3.71}$$

This is precisely the form of the superpotential in (3.63) and used in [207, 232, 233]. Hence

[8] As for the case of $\mathrm{E}_{7(7)} \times \mathbb{R}^+$ generalised geometry [1], it is easy to see that this expression does not depend on the choice of connection (such torsion-free compatible connections are not unique). In particular, there are no singlets in the undetermined parts of D when one decomposes under the $\mathcal{N} = 1$ structure group. This means that any expression that is an $\mathrm{SU}(3) \times \mathrm{O}(6+n)$ singlet, is linear in D and involves only $\mathrm{SU}(3) \times \mathrm{O}(6+n)$ invariant tensors, will depend only on the singlet part of the $\mathrm{SU}(3) \times \mathrm{O}(6+n)$ intrinsic torsion.

our expression (3.69) is the covariant form of the superpotential for a generic four-dimensional $\mathcal{N} = 1$ heterotic background determined by ψ.

Having seen how the F-term conditions of the Hull–Strominger system can be understood as involutivity for a subbundle defined by a generalised structure or the vanishing of the superpotential, in the next section we will discuss how the remaining D-term equations can be imposed by requiring the vanishing of a moment map for generalised diffeomorphisms. This moment map will be defined using ψ, and its vanishing will be equivalent to the vanishing of the remaining components of the intrinsic torsion for the $\mathrm{SU}(3) \times \mathrm{Spin}(6+n)$ structure, confirming the claim that a four-dimensional $\mathcal{N} = 1$ heterotic background is equivalent to an integrable $\mathrm{SU}(3) \times \mathrm{Spin}(6+n)$ structure.

3.4 The Kähler Potential, Moment Map and D-Terms

As we have seen, integrability of the $\mathrm{U}(3) \times \mathrm{Spin}(6+n) \times \mathbb{R}^+$ structure – in the form of involutivity of L_{-1} – gives a subset of the supersymmetry conditions required of an $\mathcal{N} = 1$, $D = 4$ heterotic background. As we have mentioned, the remaining conditions come from the vanishing of a moment map for the action of diffeomorphisms and gauge transformations (generalised diffeomorphisms). Much of what follows is analogous to the story for $\mathrm{E}_{7(7)} \times \mathbb{R}^+$ backgrounds. For this reason, we shall be brief and refer the interested reader to the longer discussion in [1].

3.4.1 The Kähler Potential

We know that the moduli space of a generic four-dimensional $\mathcal{N} = 1$ theory admits a Kähler metric which will be related to the Kähler potential on the space of $\mathrm{SU}(3)\times\mathrm{Spin}(6+n)$ structures. Here we will give an expression for this potential in terms of the object ψ.

At each point $p \in X$, ψ is stabilised by some $\mathrm{SU}(3) \times \mathrm{Spin}(6+n) \subset \mathrm{O}(6, 6+n) \times \mathbb{R}^+$ subgroup. Hence at each point, ψ is an element of the coset

$$\psi|_p \in C = \frac{\mathrm{O}(6, 6+n) \times \mathbb{R}^+}{\mathrm{SU}(3) \times \mathrm{Spin}(6+n)}. \tag{3.72}$$

An $\mathrm{SU}(3) \times \mathrm{Spin}(6+n)$ structure is then a section of the fibre bundle

$$C \longrightarrow \mathcal{C} \longrightarrow X. \tag{3.73}$$

Hence we can define the space of $\mathrm{SU}(3) \times \mathrm{Spin}(6+n)$ structures to be the set of sections of $\mathcal{C}$:

$$\mathcal{Z} \simeq \Gamma(\mathcal{C}). \tag{3.74}$$

There is a natural Kähler structure on this space, determined by supersymmetry. First, note that the homogeneous space $\mathrm{O}(6, 6+n)/\mathrm{U}(3) \times \mathrm{Spin}(6+n)$ admits a pseudo-Kähler structure [236]. The space $\mathcal{C}$ can be viewed as a complex line bundle over this homogeneous space with the zero section removed. This reflects the fact that we only have an $\mathbb{R}^+$ action, and hence we have a cone over a Kähler base. This complex cone over a Kähler base has a natural Kähler structure which then induces one on the space of sections. In this case, the Kähler potential $\mathcal{K}$ on $\mathcal{Z}$ is given by

$$\mathcal{K} = \int_X \eta(\psi, \bar{\psi})^{\frac{1}{2}}, \tag{3.75}$$

where ψ is viewed as a complex coordinate on the space of structures. Note that the weight of ψ ensures that $\eta(\psi, \bar{\psi})^{1/2}$ is a top-form and hence can be integrated. Different choices of weight would correspond to different Kähler metrics, with the weight we have chosen corresponding to the metric picked out by supersymmetry (as we saw with holomorphy of the superpotential).

As was shown in [179], the object ψ does indeed give a complex coordinate on $\mathcal{Z}$. The particular form of ψ and its $\mathbb{R}^+$ weight turns out to be very natural. Consider the anchor map

$$\pi\colon E \to T, \tag{3.76}$$

which simply projects on the vector component of a generalised vector. This induces a map $\pi\colon \wedge^3 E \to \wedge^3 T$ which, together with $\wedge^3 T \otimes \wedge^6 T^* \simeq \wedge^3 T^*$, gives

$$\pi(\psi) \sim \mathrm{e}^{-2\varphi}\,\Omega. \tag{3.77}$$

Thus, via the anchor map, the object ψ defines an ordinary complex three-form $\pi(\psi)$ on the manifold. This three-form is Ω up to a dilaton factor, and is precisely the form that is holomorphic (closed under $\bar{\partial}$) in the Hull–Strominger system (3.3). Note that, so long as we consider only deformations fixing the cohomology of the H flux, we are fixing the underlying Courant algebroid and thus the anchor map π. The induced map is therefore complex linear and has no moduli dependence. This means that if ψ is holomorphic on the coset C then so is the three-form $\mathrm{e}^{-2\varphi}\,\Omega$.

We can define a *non-holomorphic* coordinate on $\mathcal{Z}$ as

$$\chi = \eta(\psi, \bar{\psi})^{-1/4}\psi. \tag{3.78}$$

This is a complex section of $\wedge^3 E \otimes (\det T^*)^{1/2} \sim \mathbf{220_{1/2}}$ and gives the Kähler potential (3.75) as

$$\mathcal{K} = \int_X \eta(\chi, \bar{\chi}). \tag{3.79}$$

We will see that this non-holomorphic parameterisation is useful for writing the symplectic structure on $\mathcal{Z}$. The symplectic structure on $\mathcal{Z}$ is given by $\varpi = \mathrm{i}\,\partial'\bar{\partial}'\mathcal{K}$, where $\delta = \partial' + \bar{\partial}'$ is the functional derivative on $\mathcal{Z}$. Contracting two vectors $\alpha, \beta \in \Gamma(T\mathcal{Z})$ into ϖ, one has

$$\begin{aligned} \imath_\beta \imath_\alpha \varpi = \frac{\mathrm{i}}{2} \int_X \eta(\psi, \bar{\psi})^{-1/2} \Big(& \eta(\imath_\alpha \delta\psi, \imath_\beta \delta\bar{\psi}) - \eta(\imath_\beta \delta\psi, \imath_\alpha \delta\bar{\psi}) \\ & - \tfrac{1}{2}\eta(\psi, \bar{\psi})^{-1} \eta(\imath_\alpha \delta\psi, \bar{\psi}) \eta(\psi, \imath_\beta \delta\bar{\psi}) + \tfrac{1}{2}\eta(\psi, \bar{\psi})^{-1} \eta(\imath_\beta \delta\psi, \bar{\psi}) \eta(\psi, \imath_\alpha \delta\bar{\psi}) \Big). \end{aligned} \tag{3.80}$$

Rewriting this in terms of χ gives

$$\imath_\beta \imath_\alpha \varpi = \frac{\mathrm{i}}{2} \int_X \left(\eta(\imath_\alpha \delta\chi, \imath_\beta \delta\bar{\chi}) - \eta(\imath_\beta \delta\chi, \imath_\alpha \delta\bar{\chi}) \right). \tag{3.81}$$

While we leave the full calculation to appendix D, one can show that the Kähler potential takes the form

$$\mathcal{K} = \int_X \mathrm{i}\, \mathrm{e}^{-2\varphi} \Omega \wedge \bar{\Omega}. \tag{3.82}$$

In fact, it takes this form up to an overall constant which can be removed by rescaling ψ. With this rescaling χ is given by

$$\chi = \frac{1}{3!} g^{1/4} \mathrm{e}^{-\varphi} \Omega^{mnp} \hat{E}^{+}_{mnp}, \tag{3.83}$$

where $\hat{E}^{+}_{mnp} = \hat{E}^{+}_{m} \wedge \hat{E}^{+}_{n} \wedge \hat{E}^{+}_{p}$, and the $\hat{E}^{+}_{m}$ are defined as in (3.44). We will see later that, while (3.82) appears to only depend on the complex structure parameters (which vary Ω), it does in fact capture all possible deformations of the structure.

3.4.2 The Moment Map

One can then restrict to the subspace of ψ structures for which L_{-1} is involutive, that is

$$\hat{\mathcal{Z}} = \{\psi \in \mathcal{Z} \mid J \text{ is integrable}\}. \tag{3.84}$$

As we showed in (3.65) in the discussion of the superpotential, this condition is holomorphic in ψ. Hence $\hat{\mathcal{Z}}$ inherits its Kähler metric $\mathcal{Z}$, which is defined by the same Kähler potential. Following the discussion in [1], one can then define a moment map for the action of generalised diffeomorphisms on $\hat{\mathcal{Z}}$ as follows. Infinitesimally, generalised diffeomorphisms are generated by the Dorfman derivative along a generalised vector $V \in \Gamma(E)$. A generalised diffeomorphism defines a deformation of χ as

$$\imath_{\rho_V} \delta\chi = L_V \chi, \tag{3.85}$$

where $\rho_V \in \Gamma(T\hat{\mathcal{Z}})$ is the induced vector field. The corresponding moment map is defined by

$$\imath_{\rho_V} \imath_\alpha \varpi = \imath_\alpha \delta\mu(V), \tag{3.86}$$

from which we deduce

$$\mu(V) = -\frac{\mathrm{i}}{2} \int_X \eta(\psi, \bar{\psi})^{-1/2} \eta(L_V \psi, \bar{\psi}) = -\frac{\mathrm{i}}{2} \int_X \eta(L_V \chi, \bar{\chi}), \tag{3.87}$$

where $\mu \colon \hat{\mathcal{Z}} \to \mathfrak{gdiff}^*$ is the moment map. We will use the form of the moment map in terms of both ψ and χ in the following, so we give them both above.

How does the moment map constrain the structure? In other words, which components of the intrinsic torsion can appear in μ? Recall that we can always find a compatible connection ($D\psi = D\chi = 0$) that is not necessarily torsion free. Using this we can rewrite the moment map as

$$\mu(V) = -\frac{\mathrm{i}}{2} \int_X \eta(L^D_V \chi, \bar{\chi}) + \frac{\mathrm{i}}{2} \int_X \eta(T^{\text{int}}(V) \cdot \chi, \bar{\chi}). \tag{3.88}$$

The first term vanishes by the compatibility of D. Assuming that the associated weaker $\mathbb{R}^+ \times \mathrm{U}(3) \times \mathrm{Spin}(6+n)$ structure is integrable, and hence its intrinsic torsion (3.49) vanishes, the final term is zero for all $V \in \Gamma(E)$ if and only if the $(\mathbf{3}, \mathbf{1})_1 + (\bar{\mathbf{3}}, \mathbf{1})_{-1} + (\mathbf{1}, \mathbf{6}+\boldsymbol{n})_0$ part of the intrinsic torsion in (3.48) vanishes.[9] That is, imposing that the moment map vanishes, $\mu = 0$, gives the final condition for the $\mathrm{SU}(3) \times \mathrm{Spin}(6+n)$ structure to be integrable. We then have

Definition 16. A torsion-free generalised $\mathrm{SU}(3) \times \mathrm{Spin}(6+n)$ structure is one where the associated subbundle L_{-1} is involutive and the moment map (3.87) vanishes.

We now check that the vanishing of the moment map imposes the remaining equations of the Hull–Strominger system that do not appear in the involutivity conditions found in the previous section. Taking a generic generalised vector $V = \mathrm{e}^{-B}\mathrm{e}^{-A}(v+\lambda+\Lambda)$ where $v \in \Gamma(T)$, $\lambda \in \Gamma(T^*)$ and $\Lambda \in \Gamma(\operatorname{ad} P_G)$, a long calculation in appendix D shows that

$$\begin{aligned}\mu(V) = \tfrac{1}{2}\int_X \imath_v(2\partial\varphi - 2\bar{\partial}\varphi + \bar{a} - a)\mathrm{e}^{-2\varphi}\Omega\wedge\bar{\Omega} - 4\,\mathrm{e}^{-2\varphi}\operatorname{Tr}(\Lambda F)\wedge\omega\wedge\omega \\ + 2\lambda\wedge\mathrm{d}(\mathrm{e}^{-2\varphi}\omega\wedge\omega),\end{aligned} \tag{3.89}$$

where we have used the fact that the complex structure is integrable (which comes from involutivity) and so $\mathrm{d}\Omega = \bar{a}\wedge\Omega$ for some $\bar{a} \in \Omega^{0,1}(X)$. It is clear that imposing the vanishing of the moment map for all $V = \mathrm{e}^{-B}\mathrm{e}^{-A}(v+\lambda+\Lambda)$ gives

$$\bar{a} = 2\bar{\partial}\varphi, \qquad F\wedge\omega\wedge\omega = 0, \qquad \mathrm{d}(\mathrm{e}^{-2\varphi}\omega\wedge\omega) = 0, \tag{3.90}$$

which are equivalent to

$$\mathrm{d}(\mathrm{e}^{-2\varphi}\Omega) = 0, \qquad \omega^\sharp \lrcorner F = 0, \qquad \mathrm{d}(\mathrm{e}^{-2\varphi}\omega\wedge\omega) = 0. \tag{3.91}$$

These are precisely the missing supersymmetry equations. Hence the Hull–Strominger system is equivalent to an integrable $\mathrm{SU}(3) \times \mathrm{Spin}(6+n)$ structure.

Physically, $\mathrm{SU}(3) \times \mathrm{Spin}(6+n)$ structures that are related by diffeomorphisms and gauge transformations (GDiff) give equivalent backgrounds, so the moduli space of structures $\mathcal{M}_\psi$ should be viewed as the space of torsion-free $\mathrm{SU}(3) \times \mathrm{Spin}(6+n)$ structures quotiented by the action of these transformations. Since $\hat{\mathcal{Z}}$ admits both a symplectic structure and a Kähler structure, there are two ways to view this quotient, namely as a symplectic quotient by GDiff or as a standard quotient by the complexified group $\mathrm{GDiff}_{\mathbb{C}}$:

$$\mathcal{M}_\psi = \{\psi \in \hat{\mathcal{Z}} \mid \mu = 0\}/\mathrm{GDiff} \equiv \hat{\mathcal{Z}}/\!\!/\mathrm{GDiff} \simeq \hat{\mathcal{Z}}/\mathrm{GDiff}_{\mathbb{C}}. \tag{3.92}$$

How is $\mathcal{M}_\psi$ related to the moduli space of $D = 4$, $\mathcal{N} = 1$ heterotic backgrounds? First note that even within $\mathcal{M}_\psi$, different choices of ψ can lead to the same background, that is, the same set of physical fields.[10] Instead, it is the generalised metric that determines the physical fields, so we should take the moduli space of the background to be choices of $\psi \in \mathcal{M}_\psi$ that lead to different generalised metrics. Said differently, while deformations of ψ at a point take values in $\mathrm{O}(6,6+n)\times\mathbb{R}^+/(\mathrm{SU}(3)\times\mathrm{O}(6+n))$, only those that are also in $\mathrm{O}(6,6+n)\times\mathbb{R}^+/(\mathrm{O}(6)\times\mathrm{O}(6+n)$ change the physical fields. Fortunately, it is easy to take this into account. First note that

[9] Checking that $\mu(V) = 0$ for all V is equivalent to showing μ itself vanishes.

[10] Without the gauge sector, this is the statement that there is a family of ψ's that give the same $\mathrm{O}(6)\times\mathrm{O}(6)$ structure.

constant shifts of the dilaton can be absorbed in the definition of the four-dimensional metric (recall that we are working in string frame). Second, note that a deformation of ψ that lives in $(\mathrm{Spin}(6) \times \mathrm{Spin}(6+n)/(\mathrm{SU}(3) \times \mathrm{Spin}(6+n))$ would correspond to a change of the Killing spinor ϵ that leaves the physical background unchanged. Such deformations are possible only if there is a second Killing spinor to rotate into, and so the background would secretly preserve $\mathcal{N} = 2$ supersymmetry. Notice however that changes of ϵ by a constant phase do not lead to extra Killing spinors and such a phase can be absorbed into the four-dimensional spinors appearing in the split of the ten-dimensional spinor. This constant phase corresponds to the U(1) generated by J. Putting this together, assuming we do have an honest $\mathcal{N} = 1$ background, the unphysical deformations of ψ come from constant shifts of the dilaton and constant phase rotations. Given the form of ψ in (3.44), a constant shift of the dilaton by $\varphi \to \varphi - c/2$ simply rescales by the exponentiated $\mathbb{R}^+$ action of c on a weight-one object. The physical moduli space $\mathcal{M}$ of the background is then

$$\text{Moduli space of } \mathcal{N} = 1 \text{ background, } \mathcal{M} = \mathcal{M}_\psi /\!\!/ \mathrm{U}(1) \simeq \mathcal{M}_\psi / \mathbb{C}^*, \tag{3.93}$$

where $\lambda \in \mathbb{C}^*$ acts as $\psi \to \lambda\psi$. Note that this implies the Kähler potential scales as $\mathcal{K} \to |\lambda|\mathcal{K}$. The Kähler potential $\tilde{\mathcal{K}}$ on the physical moduli space is then

$$\tilde{\mathcal{K}} = -3 \log \mathcal{K}. \tag{3.94}$$

We can compare this expression with those found in the literature. The generic form of the Kähler potential, given an arbitrary (conventional) SU(3) structure, in the heterotic theory was given in [232] following [231, 237, 238] and for generic heterotic vacua in [233, 239] (matching the original expressions in the case of Calabi–Yau compactifications [154, 240, 241]). One finds

$$\tilde{\mathcal{K}} = -\log \mathcal{V} - \log(S + \bar{S}) - \log \int_X \mathrm{i}\, \Psi \wedge \bar{\Psi}, \tag{3.95}$$

where $\mathcal{V}$ is the volume calculated from ω, $\operatorname{re} S \propto \mathrm{e}^{-2\varphi}\mathcal{V}$ and $\Psi \propto \mathrm{e}^{-2\varphi}\Omega$. Using the SU(3) structure relations and that the dilaton is independent of the internal manifold, one can rewrite the above expression as

$$\begin{aligned} \tilde{\mathcal{K}} &= -\log \mathcal{V} - \log(\mathrm{e}^{-2\varphi}\mathcal{V}) - \log \mathrm{e}^{-4\varphi}\mathcal{V} \\ &= -\log(\mathrm{e}^{-6\varphi}\mathcal{V}^3) \\ &= -3 \log \int_X \mathrm{i}\, \mathrm{e}^{-2\varphi}\Omega \wedge \bar{\Omega}. \end{aligned} \tag{3.96}$$

This matches both the form of $\mathcal{K}$ that we give above and confirms the coefficient of -3 in moving from the Kähler potential $\mathcal{K}$ on the moduli space of $\mathrm{SU}(3) \times \mathrm{Spin}(6+n)$ structure to the Kähler potential $\tilde{\mathcal{K}}$ on the physical moduli space, as mentioned around (5.43).

When one has an honest Calabi–Yau background, the Kähler potential can be separated into terms that give the metric for complex structure, Kähler and bundle moduli, plus a universal term for the dilaton. On a general $\mathcal{N} = 1$ background, such a split is not possible and one simply has (3.75). This also explains another possible point of confusion. Looking at (3.82), one might be tempted to think that it depends only on complex structure parameters (which vary Ω). However, this is an artifact of expressing the general form of the Kähler potential (3.75) at a chosen point on the parameter space. Variations of the Kähler potential should be written in terms of variations of the full structure ψ, and not simply Ω, and then one will capture all of the possible deformations. Put another way, in writing (3.75) we have not picked out the holomorphic parameterisation of ψ.[11]

3.4.3 Extremisation of the Kähler Potential and GIT

As we have seen, the Hull–Strominger system is equivalent to the existence of an involutive subbundle and the vanishing of a moment map for generalised diffeomorphisms. However, as for the $E_{7(7)}\times\mathbb{R}^+$ backgrounds discussed in [1], the vanishing of the moment map is equivalent to extremising the Kähler potential over complexified generalised diffeomorphisms simply because $\hat{\mathcal{Z}}$ is Kähler [243]. This reformulation allows us to make a direct connection to the work of [74].

If we take $\mathcal{I}$ to be the complex structure on $\hat{\mathcal{Z}}$, then the action of complexified generalised diffeomorphisms are generated by $\rho_V \in \Gamma(T\hat{\mathcal{Z}})$ and $\mathcal{I}\rho_W \in \Gamma(T\hat{\mathcal{Z}})$. Since ψ is a holomorphic coordinate on the space of structures, we have

$$\mathcal{L}_{\mathcal{I}\rho_V}\psi = \imath_{\mathcal{I}\rho_V}\partial'\psi = \mathrm{i}\,\imath_{\rho_V}\partial'\psi = \mathrm{i}\,L_V\psi, \tag{3.97}$$

where $\mathcal{L}$ is the Lie derivative on $\hat{\mathcal{Z}}$, and we have split the exterior (functional) derivative into holomorphic and antiholomorphic parts $\delta = \partial' + \bar{\partial}'$. Varying the Kähler potential along the orbit of an imaginary GDiff, we have

$$\begin{aligned}
\mathcal{L}_{\mathcal{I}\rho_V}\mathcal{K} &= \frac{1}{2}\int_X \eta(\psi,\bar{\psi})^{-1/2}\big[\eta(\imath_{\mathcal{I}\rho_V}\delta\psi,\bar{\psi}) + \eta(\psi,\imath_{\mathcal{I}\rho_V}\delta\bar{\psi})\big] \\
&= \frac{\mathrm{i}}{2}\int_X \eta(\psi,\bar{\psi})^{-1/2}\big[\eta(L_V\psi,\bar{\psi}) - \eta(\psi,L_V\bar{\psi})\big] \\
&= \mathrm{i}\int_X \eta(\psi,\bar{\psi})^{-1/2}\eta(L_V\psi,\bar{\psi}) \\
&= -2\,\mu(V).
\end{aligned} \tag{3.98}$$

Thus we can think of the D-terms as coming from the vanishing of a moment map, or, since $\mathcal{K}$ is invariant under the real group GDiff, the extremisation of the Kähler potential with respect to $\mathrm{GDiff}_{\mathbb{C}}$.

In the work of [74], the Hull–Strominger system is viewed as extremising a "dilaton functional" over variations of the holomorphic Courant algebroid (3.10) with fixed Aeppli class. We note first that the dilaton functional is precisely the Kähler potential defined above. Moreover, as discussed around (3.62), the involutive bundle L_{-1} defines the holomorphic Courant algebroid $\mathcal{Q}$ with a hermitian metric (ω, A)[12] defining a given Aeppli class. The authors of [74]

[11] Note that even in the Calabi–Yau case, the Kähler potential is naively independent of the gauge field moduli. However, the holomorphic Kähler moduli are shifted relative to the naive ones, and once these are picked out the dependence on the gauge moduli becomes explicit [242].

[12] This is labelled (ω,θ^h) in the language of [74].

show that the variations within a fixed Aeppli class are given by[13]

$$\delta\omega = 2\,\mathrm{Tr}(\theta F) + \partial\xi^* + \bar{\partial}\xi, \qquad \delta A_{0,1} = -\bar{\partial}_A\theta. \tag{3.99}$$

Examining equations (3.138)–(3.141), one sees that these are precisely the transformations generated by $\mathrm{e}^{-B-\mathrm{i}\omega}\mathrm{e}^{-A}(-\mathrm{i}\xi + \mathrm{i}\xi^* + \theta) \in \mathfrak{gdiff}_{\mathbb{C}}$. Hence, extremising the dilaton functional follows directly from our picture of extremising the Kähler potential over complex generalised diffeomorphisms. Interestingly, we have a larger set of variations which are not included in those considered in [74], namely variations parameterised by some complex vector field $v \in \Gamma(T_{\mathbb{C}}) \simeq \mathfrak{diff}_{\mathbb{C}}$. As shown in (3.89), it is these variations that ensure $\mathrm{e}^{-2\varphi}\Omega$ is a holomorphic section (closed under $\bar{\partial}$). As shown in [74], provided such a section exists the variational problem of the dilaton functional is equivalent to the Hull–Strominger system. In our formulation however, the existence of a holomorphic volume form becomes part of the variational problem and does not need to be implemented by hand.

The present work also answers a question posed in [74], namely whether there exists a moment map interpretation of the Hull–Strominger system. Furthermore, this interpretation provides a fascinating link with geometric invariant theory (GIT).[14] As in many other classic problems (including the hermitian Yang–Mills equations [224, 225, 244, 245] and the equations of Kähler–Einstein geometry [246–248]), we can view the space of integrable $\mathrm{SU}(3) \times \mathrm{Spin}(6+n)$ structures as a quotient by a complexified group of some infinite-dimensional space of structures. Geometric invariant theory then tells us that we should identify

$$\hat{\mathcal{Z}} /\!\!/ \mathrm{GDiff} \simeq \hat{\mathcal{Z}}^{\mathrm{ps}} / \mathrm{GDiff}_{\mathbb{C}}, \tag{3.100}$$

where $\hat{\mathcal{Z}}^{\mathrm{ps}}$ is the subspace of $\hat{\mathcal{Z}}$ of "polystable points". This arises as it is not guaranteed that all $\mathrm{GDiff}_{\mathbb{C}}$ orbits will intersect with the surface $\mu^{-1}(0)$. If an orbit does not intersect this surface, we call the points along it unstable and these are not included in $\hat{\mathcal{Z}}^{\mathrm{ps}}$. By understanding which points are polystable, one would be able to relate the existence of solutions to a differential equation, namely $\mu = 0$, to the algebraic data of the complex orbits. In (3.92) we skipped over this subtlety of having to restrict to a subspace of $\hat{\mathcal{Z}}$ as it turns out that it is not be relevant for the infinitesimal moduli problem in section 5.4.

The standard procedure for identifying which points in $\hat{\mathcal{Z}}$ are polystable runs as follows. One considers $\mathrm{U}(1) \subset \mathrm{GDiff}$ actions generated by some $\rho_V \in \Gamma(T\hat{\mathcal{Z}})$. Under complexification we get some $\mathbb{C}^* \subset \mathrm{GDiff}_{\mathbb{C}}$ action, $\psi \to \psi(\nu)$, $\nu \in \mathbb{C}^*$, and we consider the limit $\nu \to 0$. If there is a limiting point in $\hat{\mathcal{Z}}/\mathbb{C}^*$ (for example if the latter space was compact, which however is not that case here) then in the limit the $\mathbb{C}^*$ action should coincide with the rescaling action

$$\lim_{\nu \to 0} \psi(\nu) = \nu^{w(\psi, V)} \psi_0 \tag{3.101}$$

for some $\psi_0 \in \hat{\mathcal{Z}}$. Here $w(\psi, V) \in \mathbb{Z}$ is called the weight, and is quantised because we have a

[13] This is given by $\delta\omega = \mathrm{i}\, c(h^{-1}\delta h, F_h) + \partial\xi^{0,1} + \overline{\partial\xi^{0,1}}$ in the language of [74].

[14] See [221] and references therein for a review of GIT.

U(1) action. In this limit we also find that

$$\lim_{\nu\to 0}\mathcal{K}(\nu) = |\nu|^{w(\psi,V)}\mathcal{K}_0. \tag{3.102}$$

By considering all possible $\mathrm{U}(1) \subset \mathrm{GDiff}$ subgroups, or one-parameter-subgroups, one then defines

$$\begin{aligned}
&\text{if } w(\psi,V) < 0 \text{ for all 1-PS then } \psi \text{ is stable,}\\
&\text{if } w(\psi,V) \le 0 \text{ for all 1-PS then } \psi \text{ is semistable,}\\
&\text{if } w(\psi,V) > 0 \text{ for some 1-PS then } \psi \text{ is unstable.}
\end{aligned} \tag{3.103}$$

The usual argument for the correspondence (3.100) relies on the "norm functional" (in this case the Kähler potential) being convex over the action of $\mathrm{GDiff}_{\mathbb{C}}$. This then ensures that there is a unique minimum of the functional, i.e. a point where $\mu = 0$, within the complex orbit of the stable points. However, as is pointed out in [74], there are concave orbits given by primitive deformations of ω. Therefore, there may be multiple points along a given $\mathrm{GDiff}_{\mathbb{C}}$ orbit for which $\mu = 0$ and so the correspondence (3.100) may be more subtle. Despite this, understanding polystability should give us conditions for the existence of solutions to the Hull–Strominger system, if not uniqueness.

It is interesting to consider this constraint for U(1) subgroups of the gauge group G, generated by some $\theta \in \Gamma(\operatorname{ad} P_G)$.[15] First note that we can express the weight as follows

$$w(\psi,V)\mathcal{K}_0 = \mathcal{L}_{\mathcal{I}\rho V}\mathcal{K}_0 = -2\,\mu_0(V), \tag{3.104}$$

where $\mu_0(V)$ is the moment map evaluated on ψ_0. Hence we can define ψ to be semistable if $\mu_0(V) \ge 0$. In order to lift the generator of the U(1) action into a generalised vector we take, as usual, $V = \mathrm{e}^{-B}\mathrm{e}^{-A}\theta = \mathrm{e}^{-A}\theta$, then from (3.89), we have

$$\mu(\theta) \sim \int_X \mathrm{e}^{-2\varphi}\,\mathrm{Tr}(\theta F)\wedge\omega\wedge\omega. \tag{3.105}$$

For $\varphi = 0$, this is precisely the expression for the weight for the GIT problem associated to the hermitian Yang–Mills equations. The requirement that (in an appropriate limit) (3.105) is greater than or equal to zero for all possible θ has been shown to be equivalent to the slope stability of the gauge bundle $P \to M$. (See, for example, [249] for a review.) More generally, for conformally balanced hermitian metrics, in our case when $\mathrm{d}(\mathrm{e}^{-2\varphi}\omega\wedge\omega) = 0$, a theorem of Buchdahl and Li–Yau [73, 250] states that solutions of the hermitian Yang–Mills equations require slope stability with respect to $\mathrm{e}^{-2\varphi}\omega\wedge\omega$, precisely the combination that appears in our weight expression. Note that here the balanced condition actually comes from extremising the Kähler potential under the action of complex one-form gauge transformations of B, so it would be a consequence of our more general stability condition.

This, of course, requires further investigation. For the moment, we content ourselves with pointing out that gauge conditions resembling slope stability appear naturally in the GIT picture, and that by understanding the constraints coming from all possible U(1) subgroups, one might be able to characterise polystability for the full Hull–Strominger system. Note for exam-

[15] Note that here θ is an honest gauge parameter and not a section of the generalised tangent space.

ple, we could consider circle actions on the manifold generated by some vector field $\xi \in \Gamma(T)$. One might expect those coming from Hamiltonian symplectomorphisms to be related to the picture of Calabi–Yau stability developed in [246, 247].

3.5 Moduli of Heterotic Backgrounds

We will now analyse the massless moduli of a generic heterotic background in terms of some cohomological structure. We have seen that the conditions for a $D = 4$, $\mathcal{N} = 1$ Minkowski background can be rephrased in terms of integrable $\mathrm{SU}(3)\times\mathrm{Spin}(6+n)$ structures. By using this language we will be able to give a new interpretation to previous results found on infinitesimal moduli [178, 179]. We will follow the methods of [1] closely.

As discussed around (3.93), the physical moduli space is given by

$$\mathcal{M} = \mathcal{M}_\psi/\mathbb{C}^* \qquad \mathcal{M}_\psi = \{\psi \mid J \text{ is integrable}\}/\!\!/\mathrm{GDiff} \simeq \hat{\mathcal{Z}}/\mathrm{GDiff}_{\mathbb{C}}. \tag{3.106}$$

Writing the moduli space in this way greatly simplifies the deformation theory. First, relating the symplectic quotient to a complex quotient means that we do not need to solve the moment map condition. Instead, we need only consider deformations of ψ that preserve the involutivity of L_{-1}, up the action of complexified generalised diffeomorphisms. Second, those elements of $\mathrm{GDiff}_{\mathbb{C}}$ that preserve J simply rescale ψ by a function. The moment map fixes this factor, up to an overall constant $\mathbb{C}^*$ rescaling. Thus we can actually identify the moduli space simply as a quotient of the space of integrable J structures

$$\mathcal{M} = \{J \mid J \text{ is integrable}\}/\mathrm{GDiff}_{\mathbb{C}} \tag{3.107}$$

Hence, to understand the local structure of the physical moduli space, we need to consider only deformations of L_{-1} up to complex generalised diffeomorphisms.

Infinitesimally this can be reinterpreted as the cohomology of the following complex

$$\Gamma(E_{\mathbb{C}}) \xrightarrow{\mathrm{d}_1} \Gamma(\mathfrak{C}) \xrightarrow{\mathrm{d}_2} \Gamma(W^{\text{int}}_{\mathbb{R}^+\times\mathrm{U}(3)\times\mathrm{Spin}(6+n)}), \tag{3.108}$$

where $\mathfrak{C}$ is a vector subbundle of $\operatorname{ad}\tilde{F}_{\mathbb{C}}$ such that $\Xi\cdot L_{-1} \not\subseteq L_{-1}$ for all non-zero sections $\Xi \in \Gamma(\mathfrak{C})$. We consider deformed bundles

$$L'_{-1} := (1 - \Xi)\cdot L_{-1} \qquad \Xi \in \Gamma(\mathfrak{C}), \tag{3.109}$$

such that the new L'_{-1} is involutive with respect to the Dorfman derivative to linear order in Ξ. Since L'_{-1} is involutive if and only if the intrinsic torsion of the corresponding $\mathbb{R}^+ \times \mathrm{U}(3) \times \mathrm{Spin}(6 + n)$ structure vanishes, this defines a linear map, denoted by d_2 above. The deformation is integrable if and only if $\Xi \in \ker \mathrm{d}_2$. There is also a notion of trivial deformations given by the action of complex generalised diffeomorphisms acting on L_{-1}. Infinitesimally this is just the Dorfman derivative along a complexified generalised vector. That is, a deformation

$L'_{-1} = (1+\Xi)\cdot L_{-1}$ is trivial if there is some $V \in \Gamma(E_{\mathbb{C}})$ such that

$$L'_{-1} = (1 + L_V)L_{-1}. \tag{3.110}$$

Again we can define a linear map d_1 such that a deformation generated by $\Xi \in \Gamma(\mathfrak{C})$ is trivial if and only if $\Xi \in \operatorname{im} \mathrm{d}_1$. It is simple to show using (A.30) that any trivial deformation is integrable and hence $\mathrm{d}_2 \circ \mathrm{d}_1 = 0$. This means (3.108) is a complex whose cohomology counts the physical moduli.

We will now find explicit expressions for the maps d_1 and d_2 using the parametrisation of L_{-1} given in (3.42), and show that we recover the cohomology of [178,179]. Note that the choice of $\mathfrak{C}$ is not unique for a given L_{-1} and different choices change the form of the linear maps. A canonical choice comes from thinking of the fibres of $\mathfrak{C}$ as quotient spaces $(\mathfrak{o}_{6,6+n} \oplus \mathbb{R})/\mathfrak{p}$, where $\mathfrak{p}$ is the parabolic subalgebra preserving L_{-1}. Since we are only interested in the cohomology, which is independent of the exact choice of $\mathfrak{C}$, we will choose convenient a representative.

Recall the form of L_{-1}

$$L_{-1} = \mathrm{e}^{-B-\mathrm{i}\,\omega}\mathrm{e}^{-A}T^{0,1}. \tag{3.111}$$

We take $\mathfrak{C}$ to be

$$\mathfrak{C} \simeq \mathrm{e}^{-B-\mathrm{i}\,\omega}\mathrm{e}^{-A}\cdot\big[(T^{1,0}\otimes T^{*0,1}) \oplus \wedge^{1,1}T^*_{\mathbb{C}} \oplus \wedge^{0,2}T^* \oplus (T^{*0,1}\otimes \operatorname{ad} P_G)\big]. \tag{3.112}$$

We note that these bundles should be taken to be complexified as above, which we assume from this point forward. For any non-zero section Ξ of this bundle we see that

$$\begin{aligned} &\Xi\colon L_{-1} \to \mathrm{e}^{-B-\mathrm{i}\,\omega}\mathrm{e}^{-A}(T^{1,0}\oplus T^* \oplus \operatorname{ad} P_G) \simeq E_{\mathbb{C}}/L_{-1},\\ &\Xi = \mathrm{e}^{-B-\mathrm{i}\,\omega}\mathrm{e}^{-A}\cdot(-\mu + x + b + \alpha) \in \Gamma(\mathfrak{C}), \end{aligned} \tag{3.113}$$

where $\mu \in \Gamma(T^{1,0}\otimes T^{*0,1})$, $x \in \Gamma(T^{*1,1})$, $b \in \Gamma(T^{*0,2})$, and $\alpha \in \Gamma(T^{*0,1}\otimes \operatorname{ad} P_G)$ – these are what one might call complex structure, hermitian, and bundle moduli. (Again note that we are taking all of the bundles above to be complexified.) This shows that (3.112) is a good choice of $\mathfrak{C}$. We can then define our deformed bundle

$$L'_{-1} = (1 - \Xi)L_{-1}. \tag{3.114}$$

To linear order in the deformation, we can rewrite this in a more convenient form as

$$L'_{-1} = \mathrm{e}^{-\Theta}(1+\mu)T^{0,1}, \tag{3.115}$$

where $\Theta = B + \mathrm{i}\omega + x + b + \operatorname{Tr}(A\wedge\alpha) + A + \alpha$.[16] It is worth stressing that by deforming within the space of structures we are including deformations that do not change the generalised metric, that is do not change the physical supergravity fields. In terms of the ψ structure, the additional degrees of freedom parameterise Spin(6)/SU(3) and transform in the **3** of SU(3) – these correspond to deforming the putative Killing spinor, while keeping the supergravity fields

[16] Here we note that to linear order $1 + x + b + \alpha = \mathrm{e}^{b+x}\mathrm{e}^{\alpha}$ and then used the Baker–Campbell–Hausdorff formula together with (A.23).

fixed. If there are any such integrable deformations they would imply that the background actually defined an $\mathcal{N}=2$ rather than $\mathcal{N}=1$ solution. We will return to this point below.

We now want to examine the conditions on Ξ (or equivalently Θ) for L'_{-1} to be involutive, that is, for the deformation to be integrable. From (3.114), two general sections $V, W \in \Gamma(L'_{-1})$ can be parametrised by Θ, μ and two vectors $\bar{v}, \bar{w} \in \Gamma(T^{0,1})$. The Dorfman derivative of W along V can then be written in terms of a twisted derivative as

$$L_{\mathrm{e}^{-\Theta}(1+\mu)\bar{v}}\left(\mathrm{e}^{-\Theta}(1+\mu)\bar{w}\right) = \mathrm{e}^{-\Theta} L^{\tilde{H}+\tilde{F}}_{\bar{v}+\mu\cdot\bar{v}}(\bar{w}+\mu\cdot\bar{w}), \tag{3.116}$$

where $\tilde{H}$ and $\tilde{F}$ are given to first order in the deformation by

$$\begin{aligned}\tilde{H} &= \mathrm{d}B + \omega_3(A+\alpha) + \mathrm{i}\,\mathrm{d}\omega + \mathrm{d}x + \mathrm{d}b + \mathrm{d}\,\mathrm{Tr}(A\wedge\alpha)\\ &= 2\mathrm{i}\,\partial\omega + 2\,\mathrm{Tr}(\alpha\wedge F) + \mathrm{d}x + \mathrm{d}b,\end{aligned} \tag{3.117}$$

$$\begin{aligned}\tilde{F} &= \mathrm{d}(A+\alpha) + (A+\alpha)\wedge(A+\alpha)\\ &= F + \mathrm{d}_A\alpha,\end{aligned} \tag{3.118}$$

where $\mathrm{d}_A = \mathrm{d} + [A,\cdot]$. Involutivity of L'_{-1} is then equivalent to

$$L^{\tilde{H}+\tilde{F}}_{\bar{v}+\mu\cdot\bar{v}}(\bar{w}+\mu\cdot\bar{w}) = \bar{u} + \mu\cdot\bar{u}, \tag{3.119}$$

for some $\bar{u} \in \Gamma(T^{0,1})$. Using the expression for the twisted Dorfman derivative from (A.26), to first-order in the deformation we have

$$\begin{aligned}L^{\tilde{H}+\tilde{F}}_{\bar{v}+\mu\cdot\bar{v}}(\bar{w}+\mu\cdot\bar{w}) =& [\bar{v},\bar{w}] + [\mu\cdot\bar{v},\bar{w}] + [\bar{v},\mu\cdot\bar{w}]\\ &- \imath_{\bar{v}}\imath_{\bar{w}}\big(2\,\mathrm{Tr}(\alpha\wedge F) + \mathrm{d}x + \mathrm{d}b\big) - 2\mathrm{i}\,\imath_{\mu\cdot\bar{v}}\imath_{\bar{w}}\partial\omega - 2\mathrm{i}\,\imath_{\bar{v}}\imath_{\mu\cdot\bar{w}}\partial\omega\\ &- \imath_{\bar{v}}\imath_{\bar{w}}\bar{\partial}_A\alpha - \imath_{\mu\cdot\bar{v}}\imath_{\bar{w}}F - \imath_{\bar{v}}\imath_{\mu\cdot\bar{w}}F\\ \equiv& \bar{u} + \mu\cdot\bar{u}.\end{aligned} \tag{3.120}$$

Decomposing according to complex type, we require

$$[\bar{v},\bar{w}] + [\mu\cdot\bar{v},\bar{w}]^{0,1} + [\bar{v},\mu\cdot\bar{w}]^{0,1} = \bar{u} \tag{3.121}$$

$$[\mu\cdot\bar{v},\bar{w}]^{1,0} + [\bar{v},\mu\cdot\bar{w}]^{1,0} = \mu\cdot\bar{u}, \tag{3.122}$$

$$\imath_{\bar{v}}\imath_{\bar{w}}\bar{\partial}_A\alpha + \imath_{\mu\cdot\bar{v}}\imath_{\bar{w}}F - \imath_{\mu\cdot\bar{w}}\imath_{\bar{v}}F = 0, \tag{3.123}$$

$$\imath_{\bar{v}}\imath_{\bar{w}}\big(2\,\mathrm{Tr}(\alpha\wedge F) + \bar{\partial}x + \partial b\big) + 2\mathrm{i}\,\imath_{\mu\cdot\bar{v}}\imath_{\bar{w}}\partial\omega + 2\mathrm{i}\,\imath_{\bar{v}}\imath_{\mu\cdot\bar{w}}\partial\omega = 0, \tag{3.124}$$

$$\imath_{\bar{v}}\imath_{\bar{w}}\bar{\partial}b = 0. \tag{3.125}$$

Let us consider each of these conditions in turn. As we are working to first order in the deformations, dotting (3.121) with μ and substituting into (3.122) gives

$$\mu\cdot[\bar{v},\bar{w}] = [\mu\cdot\bar{v},\bar{w}]^{1,0} + [\bar{v},\mu\cdot\bar{w}]^{1,0}. \tag{3.126}$$

Expanding out in components and using a torsion-free compatible $\mathrm{GL}(3,\mathbb{C})$ connection,[17] one

[17]This exists as the undeformed solution admits an honest complex structure, I.

can show this condition is equivalent to $\imath_{\bar{w}}\imath_{\bar{v}}\bar{\partial}\mu = 0$, where μ is treated as a $(0,1)$-form with a holomorphic vector index. As this must vanish for all $\bar{v}$ and $\bar{w}$, we find

$$\bar{\partial}\mu = 0. \tag{3.127}$$

This is the expected condition on first-order deformations of a complex structure.

The third condition (3.123) can be rewritten using $\imath_{\bar{v}}\imath_{\bar{w}}\imath_{\mu}F = \imath_{\mu\cdot\bar{v}}\imath_{\bar{w}}F - \imath_{\mu\cdot\bar{w}}\imath_{\bar{v}}F$, where $\imath_{\mu}F = e^{a} \wedge \imath_{\mu_a}F$, to give

$$\bar{\partial}_A\alpha + \imath_\mu F = 0. \tag{3.128}$$

The fourth condition (3.124) can be rewritten using $\imath_{\bar{v}}\imath_{\mu\cdot\bar{w}}\partial\omega - \imath_{\bar{w}}\imath_{\mu\cdot\bar{v}}\partial\omega = -\imath_{\bar{w}}\imath_{\bar{v}}\imath_{\mu}\partial\omega$ to give

$$2\,\mathrm{Tr}(\alpha \wedge F) + \bar{\partial}x + \partial b + 2\mathrm{i}\,\imath_\mu\partial\omega = 0. \tag{3.129}$$

The final condition (3.125) is simply

$$\bar{\partial}b = 0. \tag{3.130}$$

Taken together, the conditions are

$$\begin{aligned} \bar{\partial}\mu &= 0, & (3.131)\\ \bar{\partial}b &= 0, & (3.132)\\ \bar{\partial}x + 2\mathrm{i}\,\imath_\mu\partial\omega + 2\,\mathrm{Tr}(\alpha \wedge F) + \partial b &= 0, & (3.133)\\ \bar{\partial}_A\alpha + \imath_\mu F &= 0. & (3.134) \end{aligned}$$

These equations give the map d_2 on the different components of Ξ. It is comforting to note that these equations agree with those that have appeared before in work on heterotic moduli. To be precise, our equations match those in [178,179], which we reproduce in (3.14) and (3.15), after noting that $x_{\text{here}} = 2x_{\text{there}}$, $\mu_{\text{here}} = -\mu_{\text{there}}$ and $b_{\text{here}} = \mathcal{B}_{\text{there}}$.[18] The only equation we are missing is (3.16) which is equivalent to the deformed complex three-form being conformally holomorphic. However, as we saw in section 3.4.2, this condition is imposed by the moment map, not involutivity. (Alternatively, one can see it as the extra condition that is imposed by the superpotential.) The particular missing equation is associated to the moment map condition that fixes ψ (up to an overall constant) as a section of $\mathcal{U}_J$ once J is determined. Since we have shown that we can describe the moduli space in terms of deformations of J alone it does not appear. Note however, even if we had been using ψ to parameterise the moduli space, we would still not have had to impose this relation. The point is that, as we have argued, at the level of the cohomology imposing moment map conditions is equivalent to quotienting by complex generalised diffeomorphisms. In other words, there will be representatives in the cohomology class for which this missing condition is satisfied and hence we do not need to impose it as an extra condition here. (This was actually the reason we could parameterise the moduli space using J alone.) This illustrates the usefulness of this approach as it reduces the complexity of the equations governing the moduli. As a separate point, note also that the integrability

[18]The factor of two in x is down to a choice of conventions. The minus sign that appears in μ is due to our μ deforming $T^{0,1}$ while the μ in [178,179] is a deformation of $T^{*1,0}$.

conditions above are holomorphic in the complex parameters Ξ, as we would expect from our general discussion around (3.65).

We now examine the conditions for a deformation to be trivial. This will tell us what an "exact" deformation is and thus give the resulting cohomology that counts the inequivalent, non-trivial deformations. A deformation is to be regarded as trivial if the resulting L'_{-1} is related to the undeformed subbundle by the action of the Dorfman derivative. In other words, if L'_{-1} is simply a $\mathrm{GDiff}_{\mathbb{C}}$ rotation of L_{-1}, the deformation is trivial. Let V be a section of L_{-1} and W be a section of $E_{\mathbb{C}}$ such that

$$\begin{aligned} V &= e^{-B-\mathrm{i}\,\omega}e^{-A}\bar{v}, \\ W &= e^{-B-\mathrm{i}\,\omega}e^{-A}(w+\bar{w}+\xi+\bar{\xi}+\theta) = \mathrm{e}^{-B-\mathrm{i}\,\omega}\mathrm{e}^{-A}W', \end{aligned} \tag{3.135}$$

where w is a $(1,0)$-vector, $\bar{v}$ and $\bar{w}$ are $(0,1)$-vectors, ξ and $\bar{\xi}$ are $(1,0)$- and $(0,1)$-forms, and θ is a complex gauge parameter. Note that w and $\bar{w}$ (and ξ and $\bar{\xi}$) are independent degrees of freedom and not related by complex conjugation, $\bar{w}\neq w^*$. Peeling off the twisting by $-B-\mathrm{i}\,\omega$ and $-A$, the action of $\mathrm{GDiff}_{\mathbb{C}}$ by W on a section of L_{-1} is

$$\begin{aligned} (1+L^{H+\mathrm{i}\,\mathrm{d}\omega+F}_{W'})\bar{v} &= \bar{v}+[w+\bar{w},\bar{v}]-\imath_{\bar{v}}\mathrm{d}(\xi+\bar{\xi})-\imath_{w+\bar{w}}\imath_{\bar{v}}(H+\mathrm{i}\,\mathrm{d}\omega) \\ &\quad +2\,\mathrm{Tr}(\theta\,\imath_{\bar{v}}F)-\imath_{\bar{v}}\mathrm{d}_A\theta-\imath_{w+\bar{w}}\imath_{\bar{v}}F \\ &= \bar{v}'-\imath_{\bar{v}'}\bar{\partial}w-\imath_{\bar{v}'}\bar{\partial}\xi-\imath_{\bar{v}'}\partial\bar{\xi}-\imath_{\bar{v}'}\bar{\partial}\bar{\xi}-2\mathrm{i}\,\imath_w\imath_{\bar{v}'}\partial\omega \\ &\quad +2\,\mathrm{Tr}(\theta\,\imath_{\bar{v}'}F)-\imath_{\bar{v}'}\bar{\partial}_A\theta-\imath_w\imath_{\bar{v}'}F, \end{aligned} \tag{3.136}$$

where $\bar{v}'=\bar{v}+[\bar{w},\bar{v}]+[w,\bar{v}]^{0,1}$ is a trivial rotation of $\bar{v}$ and we are working to first order in the components of W.

We want to compare this with the expression for a linear deformation of L_{-1}. Using the $\mathrm{O}(6,6+n)$ algebra [167] given in (A.23) and the Baker–Campbell–Hausdorff formula, L'_{-1} can be rewritten as

$$\begin{aligned} L'_{-1} &= \mathrm{e}^{-B-\mathrm{i}\,\omega-x-b-\mathrm{Tr}(A\wedge\alpha)}\mathrm{e}^{-A-\alpha}(1+\mu)\bar{v} \\ &= \mathrm{e}^{-B-\mathrm{i}\,\omega}\mathrm{e}^{-A}(\bar{v}+\mu\cdot\bar{v}+\imath_{\bar{v}}x+\imath_{\bar{v}}b+\imath_{\bar{v}}\alpha). \end{aligned} \tag{3.137}$$

Comparing (3.136) with the components in the parenthesis in (3.137), one sees that a deformation of L'_{-1} is actually the action of $\mathrm{GDiff}_{\mathbb{C}}$, and so trivial, if

$$\mu = -\bar{\partial}w, \tag{3.138}$$

$$x = -\bar{\partial}\xi-\partial\bar{\xi}+2\mathrm{i}\,\imath_w\partial\omega+2\,\mathrm{Tr}(\theta\,F), \tag{3.139}$$

$$b = -\bar{\partial}\bar{\xi}, \tag{3.140}$$

$$\alpha = -\bar{\partial}_A\theta+\imath_wF. \tag{3.141}$$

Combined, these derivatives form the operator d_1. One can check that these satisfy (3.131)–(3.134) (so that exact deformations are automatically closed) provided $\{\partial,\bar{\partial}\}=0$, $\bar{\partial}^2=\bar{\partial}_A^2=0$, implying the original solution has a complex structure and a holomorphic gauge bundle, and F and H satisfy the appropriate Bianchi identities. These will each hold as we are assuming we are deforming around an $\mathcal{N} = 1$ solution. Combining (3.138)–(3.141) with (3.131)–(3.134), we recover precisely the cohomology of [178] up to the b term which is not present in their analysis. This is included in the linear terms in the same calculation in [179] and is related to deformations of $B_{0,2}$.

It is worth analysing this b modulus further. As we mentioned above, our parameterisation of the deformation includes not only deformations of the physical fields preserving $\mathcal{N} = 1$ supersymmetry but also potential deformations of the Killing spinors, with the same background geometry. The latter type of deformations correspond to the background admitting additional supersymmetries. Specifically one can show that a particular combination of b and μ will leave the generalised metric invariant and hence correspond to such additional supersymmetries. From the form of the equations (3.132) and (3.139) we see that if $h^{0,2}$ vanishes then there are no moduli for deformations of b and hence all the deformations correspond to physical deformations of the background – in other words this is sufficient for the background not to admit additional supersymmetries. A counter example is the solution on $\mathrm{K3} \times \mathrm{T}^2$ with trivial gauge group. In this case $h^{0,2} \neq 0$ and the b modulus survives. The additional degree of freedom corresponds to rotating the choice of $\mathcal{N} = 1$ subalgebra picked out by ψ within the $\mathcal{N} = 2$ supersymmetry algebra.

Chapter 4

Generalised Geometry for 5 Dimensional External Spaces

In this chapter, we will look at backgrounds of M-theory which have a 5 dimensional Minkowski external space. These were originally studied in the context of generalised geometry in [187] and were shown to have a $\mathrm{USp}(6) \subset \mathrm{E}_{6(6)} \times \mathbb{R}^+$ structure defined via a so-called V and H structure. Here we will restrict our analysis to only the H-structure. We will find that this can also be described in terms of an exceptional complex structure in $\mathrm{E}_{6(6)} \times \mathbb{R}^+$. That is, there is an involutive bundle and a moment map structure which define an integrable H-structure. Using these structures, we will be able to draw some interesting conclusions about such backgrounds. We will be able to fully classify the possible exceptional complex structures which will put restrictions on the geometry of the internal space. We will also be able to analyse the moduli space in many cases and determine the exact moduli of all backgrounds satisfying a particular generalised $\partial\bar{\partial}$-lemma. These will be related to the number of massless hypermultiplets in the effective theory. This chapter is structured as follows. First, we review the work of [187] and recall the definition of H and V structures. Then we restrict to the H-structures and define the $\mathrm{SU}^*(6)$ structure, and the weaker $\mathrm{U}^*(6) \times \mathbb{R}^+$ structure, in terms of a holomorphic object χ, and a subbundle $L_1 \subset E_{\mathbb{C}}$ respectively. We show that integrability of these structures is given by involutivity of L_1 and the vanishing of a moment map. Next, we classify the possible forms of L_1 and then use this classification to determine the moduli of these structures in all backgrounds of constant type. Finally, we finish with some possible applications to AdS_5 backgrounds, and finding the spectrum of the associated CFT_4.

4.1 Review of Exceptional Calabi-Yau Structures

We will be interested in backgrounds with an external $\mathbb{R}^{4,1}$ that preserve minimal supersymmetry. Recall in section 2.4.3 we stated that the geometry of backgrounds preserving minimal supersymmetry (8 supercharges) with a $\mathbb{R}^{4,1}$ external space was given by an integrable $\mathrm{USp}(6)$ structure [184]. This is because the supersymmetry parameter transforms in the fundamental of the (double cover) of the maximally compact subgroup $\mathrm{USp}(8)$ [182,187]. More generally, the existence of $\mathcal{N}$ globally linearly independent spinors defines a $\mathrm{USp}(8-2\mathcal{N})$ structure. Moreover, as was shown in [185, 187], these spinors satisfy the Killing spinor equations if and only if the $\mathrm{USp}(8-2\mathcal{N})$ structure is integrable[1]. In this chapter, we will be interested in the geometric structures associated to the existence of a torsion free USp(6) structure. These were extensively studied in [187] and we will briefly review their work here.

As with conventional geometry, it is more convenient to define G-structures in terms of generalised tensors, rather than spinorial objects. We would also like to find a condition for integrability in terms of some differential conditions on those generalised tensors. This would be the equivalent of the closure conditions of ω and Ω for an SU(3) structure. In [187] they showed that a USp(6) structure is defined by the combination of what they call an H-structure and a V-structure, satisfying some compatibility conditions and differential conditions. In the effective 5 dimensional theory, the H-structure will be related to the scalars in the hypermultiplets, while the V-structure will be related to the scalars in the vector multiplets, hence the nomenclature[2].

The H-structure is defined by a triplet of weighted adjoint valued tensors $J_\alpha \in \Gamma((\det T^*)^{1/2} \otimes \operatorname{ad}\tilde{F})$, $\alpha = 1, 2, 3$. These have to form a highest weight $\mathfrak{su}(2)$ algebra of $\mathfrak{e}_6$. In particular we require

$$[J_\alpha, J_\beta] = 2\kappa\epsilon_{\alpha\beta\gamma}J_\gamma \qquad \operatorname{Tr}(J_\alpha J_\beta) = -\kappa^2\delta_{\alpha\beta} \tag{4.1}$$

Where κ is some section of $(\det T^*)^{1/2}$. Alone, these tensors define an $\mathrm{SU}^*(6)$ structure, where $\mathrm{SU}^*(6)$ is a particular non-compact real form of $\mathrm{SL}(6,\mathbb{C})$[3] [252, 253]. This G-structure is integrable if and only if the following generalised 1-forms vanish.

$$\mu_\alpha(V) := -\frac{1}{2}\epsilon_{\alpha\beta\gamma}\int_M \operatorname{Tr}(J_\beta L_V J_\gamma) \stackrel{!}{=} 0 \tag{4.2}$$

The maps, μ_α can be viewed as moment maps for the action of generalised diffeomorphisms on the hyper-kähler moduli space of H-structures $\mathcal{M}_H$. We will expand a little more in the next section on how this works.

The V-structure is defined by a single generalised vector $K \in \Gamma(E)$ that satisfies

$$c(K, K, K) = \kappa^2 \tag{4.3}$$

This describes an $F_{4(4)}$ structure which is integrable if

$$L_K K = 0 \tag{4.4}$$

Again, this can be interpreted as the vanishing of a different moment map for the action of generalised diffeomorphisms on the Kähler moduli space of V-structures $\mathcal{M}_V$.

Finally, the USp(6) structure is defined by an H-structure and a V-structure obeying an

[1] It was later shown in [212] that for compactifications down to AdS spaces, the generalised geometry is described by the same G-structures, now satisfying a property called weak generalised holonomy. They define this to be a G-structure where the intrinsic torsion lies in a singlet representation of the group G. The value it takes depends on the value of the cosmological constant.

[2] In fact, in [187], they introduce these structures for compactifications down to 4, 5, and 6 dimensional Minkowski space preserving 8 supercharges. The H and V-structures for compactifications down to 4 dimensions were first introduced in [251].

[3] It can be identified as the following subgroup of $\mathrm{SL}(6,\mathbb{C})$. If J is an antisymmetric 6×6 matrix such that $J^2 = -\mathrm{Id}$, then $U \in \mathrm{SU}^*(6)$ with U^* its complex conjugate if and only if $UJ = JU^*$.

additional compatibility condition and integrability condition. These are

$$J_+ \cdot K = 0 \qquad L_K J_\alpha = 0 \tag{4.5}$$

where $J_\pm = J_1 \pm \mathrm{i} J_2$. The additional compatibility condition ensures that the stabiliser group of the two structures is $F_{4(4)} \cap \mathrm{SU}^*(6) = \mathrm{USp}(6)$, and the extra differential condition is required to ensure the intrinsic torsion completely vanishes. These completely describe the geometry of $\mathcal{N} = 1$ backgrounds with $\mathbb{R}^{4,1}$ external space.

4.2 $\mathrm{SU}^*(6)$ and $\mathbb{R}^+ \times \mathrm{U}^*(6)$ Structures

It turns out that we can reinterpret the H-structure in a way that is similar to exceptional complex structures of heterotic backgrounds described in chapter 3. In particular, we will find that we can define an involutive subbundle of the generalised tangent bundle. While this does not describe the full geometry of the supersymmetric background, we will be able to find useful information from the analysis of this subsector. In this section, we will give a definition of exceptional complex structures of $\mathrm{E}_{6(6)} \times \mathbb{R}^+$ geometry and see how we can reinterpret the $\mathrm{SU}^*(6)$ structures as such an object. To do so, we will break the explicit $\mathrm{SU}(2)$ symmetry of the H-structure. We will return to this point later in the chapter.

First we note that to define an $\mathrm{SU}^*(6)$ structure, it is sufficient to just define J_+. Indeed we can then obtain the full J_α via

$$J_- = \bar{J}_+ \qquad J_3 = (-8\,\mathrm{Tr}(J_+ J_-))^{-1/2}\,\mathrm{i}\,[J_+, J_-]$$

In fact, it turns out that $\chi := \kappa J_+ \in \Gamma(\det T^* \otimes \mathrm{ad}\,\tilde{F})$ is a holomorphic coordinate on the space of $\mathrm{SU}^*(6)$ structures[4]. This will be useful later when understanding the Kähler structure on $\mathcal{M}_H$. From these tensors we can define two reductions of the structure group

$$\begin{aligned} \mathrm{SU}^*(6) \text{ structure :} \quad & \chi = \kappa J_+ \in \Gamma(\det T^* \otimes \mathrm{ad}\,\tilde{F}) \\ \mathbb{R}^+ \times \mathrm{U}^*(6) \text{ structure :} \quad & \tilde{J} \in \Gamma(\mathrm{ad}\,\tilde{F}) \end{aligned} \tag{4.6}$$

where $\tilde{J} = \kappa^{-1} J_3$ is the unweighted J_3. Note that since $\tilde{J}$ is unweighted, it is left invariant by the $\mathbb{R}^+ \subset \mathrm{E}_{6(6)} \times \mathbb{R}^+$. The additional $\mathrm{U}(1)$ symmetry comes from the action generated by $\tilde{J}$ itself. Hence $\tilde{J}$ does define a $\mathbb{R}^+ \times \mathrm{U}^*(6)$ structure as claimed. Alternatively, we can define the $\mathbb{R}^+ \times \mathrm{U}^*(6)$ structure more directly from the supersymmetry.

Definition 17. Supersymmetry selects a $\mathrm{USp}(6) \subset \mathrm{USp}(8)$ structure, and hence a well defined $\mathrm{SU}(2)$ commutant within $\mathrm{USp}(8)$. An $\mathbb{R}^+ \times \mathrm{U}^*(6)$ structure is given by any $\tilde{J} \in \Gamma(\mathrm{ad}\,\tilde{F})$ that generates some $\mathrm{U}(1) \subset \mathrm{SU}(2)$ commutant.

Given a $\tilde{J}$, one can use it to decompose the generalised tangent bundle into eigenbundles.

[4] This was first noticed by Danial Waldram and Edward Tasker.

We find that

$$\begin{aligned} E_{\mathbb{C}} &= L_1 \oplus L_{-1} \oplus L_0 \\ \mathbf{27} &\to \mathbf{6}_1 \oplus \mathbf{6}_{-1} \oplus \mathbf{15}_0 \end{aligned} \tag{4.7}$$

In the second line we have expressed this decomposition in terms of $\mathrm{U}^*(6)$ representations. Here the subscripts denote the charge under the $\mathrm{U}(1) \subset \mathrm{U}^*(6)$ generated by the $\tilde{J}$. Much like for conventional almost complex structures and almost generalised complex structures, we have a definition of $\mathrm{U}^*(6)$ structures purely in terms of L_1.

Definition 18. An $\mathbb{R}^+ \times \mathrm{U}^*(6)$ structure is defined by a subbundle $L_1 \subset E_{\mathbb{C}}$ such that

i) $\dim_{\mathbb{C}} L_1 = 6$

ii) $L_1 \times_N L_1 = 0$

iii) $L_1 \cap \bar{L}_1 = \{0\} \qquad L_1 \cap L_0 = \{0\}$

iv) The map $\zeta : L_1 \times (L_{-1})^* :\to \mathbb{R}$ defined by

$$\zeta(V, Z) = \mathrm{Tr}\left((V \times_{\mathrm{ad}} Z)(\bar{V} \times_{\mathrm{ad}} \bar{Z})\right) \tag{4.8}$$

is negative $\forall\, V \in L_1$, $Z \in (L_{-1})^*$

We call such structures *almost exceptional complex structures.* Any bundle obeying the first two conditions is called an *almost exceptional Dirac structure* [165].

Note that we could equally well define the structure in terms of L_{-1}. While the third and fourth conditions appear to depend on the full decomposition (4.7), one can define L_0 by L_1 via the following. Let $A = \{Z \in E^* \mid \langle V, Z\rangle = 0\ \forall\, V \in L_1 \oplus L_{-1}\} \subset E^*$, where $\langle\cdot,\cdot\rangle$ is the natural pairing between E and E^*. That is, A is the null space of $L_1 \oplus L_{-1}$. Then we define

$$L_0 = (L_1 \times_{\mathrm{ad}} A) \cdot L_{-1} \tag{4.9}$$

Once we have found L_0 such that (iii) holds, we have a well-defined splitting of the dual space E^* into $(L_{\pm 1})^*$ and $(L_0)^*$.

We can decompose the weighted adjoint bundle into eigenbundles of $\tilde{J}$. We find

$$\mathbf{78} \oplus \mathbf{1} \to \mathbf{1}_{+2} \oplus \mathbf{1}_{-2} \oplus \mathbf{20}_{+1} \oplus \mathbf{20}_{-1} \oplus \mathrm{ad}\, P_{\mathbb{R}^+ \times \mathrm{U}^*(6)} \tag{4.10}$$

The singlets imply that an $\mathbb{R}^+ \times \mathrm{U}^*(6)$ structure defines a line bundle $\mathcal{U}_{\tilde{J}} \subset (\det T^*) \otimes \mathrm{ad}\, \tilde{F}$. One can show that it is defined by

$$V \bullet \chi = 0 \quad \forall\, V \in \Gamma(L_1), \qquad \mathrm{Tr}(\chi\bar{\chi}) \neq 0 \tag{4.11}$$

where χ is a local section of $\mathcal{U}_{\tilde{J}}$. The product $V \bullet \chi$ is defined by the projection $E \otimes (\det T^*) \otimes \mathrm{ad}\, \tilde{F} \to C$ where C is the generalised tensor bundle transforming in the $\mathbf{351_4}$ of $\mathrm{E}_{6(6)} \times \mathbb{R}^+$[5].

[5] Note that this is R_4 in the tensor hierarchy of $\mathrm{E}_{6(6)}$ [254]

One can equally define a local section χ by the condition $\tilde{J} \cdot \chi = [\tilde{J}, \chi] = 2\mathrm{i}\chi$. We can then give the definition

Definition 19. Given an almost exceptional complex structure $\tilde{J}$ with trivial line bundle $\mathcal{U}_{\tilde{J}}$, an $\mathrm{SU}^*(6)$ structure is a global non-vanishing section of $\mathcal{U}_{\tilde{J}}$.

We expect that all $\mathrm{SU}^*(6)$ structures will arise in this way and any two will be related by some $\mathrm{E}_{6(6)} \times \mathbb{R}^+$ transformation. Note that any two $\mathrm{SU}^*(6)$ structures χ, χ' which define the same $\mathbb{R}^+ \times \mathrm{U}^*(6)$ structure will be related by some non-vanishing function f

$$\chi' = f\chi \tag{4.12}$$

Much like for complex structures and generalised complex structures, suitable values for χ will not fill out the whole of $\mathbf{78} \oplus \mathbf{1}$. Instead, they will exist in some particular $\mathrm{E}_{6(6)} \times \mathbb{R}^+$ orbit.

It is important to reiterate that since $\mathrm{SU}^*(6) \not\subset \mathrm{USp}(8)$, a choice of χ does not define a generalised metric and hence does not fully define a supergravity background. Despite this, we can still find some useful information about the hypermultiplets in the effective theory by studying such structures.

4.2.1 Involutivity, Moment Maps, and Integrability

We will now look at the conditions imposed on χ by integrability. Recall from section 4.1 that the integrability of the $\mathrm{SU}^*(6)$ structure is a subset of the supersymmetry conditions. This was given as the vanishing of a triplet of moment maps for the action of generalised diffeomorphisms on the space of H-structures. We will see that we will be able to recast this as an involutivity condition on L_1, which gives the integrability of the exceptional complex structure, along with the vanishing of just a single moment map. In taking this description for integrable H-structures, one loses the explicit hyper-Kähler structure that is guaranteed by supersymmetry. Instead, we break the structure to just a Kähler structure. We will explore the implications of this hidden SU(2) symmetry later in this chapter.

The intrinsic torsion for the generalised structures lies in a subbundle of the torsion bundle $K \sim \mathbf{351_{-1}}$. Decomposing into $\mathrm{U}^*(6)$ structures we find that they transform as [187]

$$W_{\text{int}}^{\mathrm{SU}^*(6)} : \quad \mathbf{15}_2 \oplus \mathbf{15}_{-2} \oplus \mathbf{15}_0 \oplus \mathbf{6}_1 \oplus \mathbf{6}_{-1} \tag{4.13}$$

$$W_{\text{int}}^{\mathbb{R}^+ \times \mathrm{U}^*(6)} : \quad \mathbf{15}_2 \oplus \mathbf{15}_{-2} \tag{4.14}$$

where again the subscripts denote the U(1) charge. We saw earlier that the integrability of a complex structure can be described in terms of involutivity of eigenbundles. Again we find that we can define

Definition 20. An integrable $\mathbb{R}^+ \times \mathrm{U}^*(6)$ structure, or *exceptional complex structure*, is an almost exceptional complex structure that is involutive under the Dorfman derivative. That is

$$L_V W \in \Gamma(L_1) \qquad \forall\, V, W \in \Gamma(L_1) \tag{4.15}$$

In analogy with generalised complex geometry, an involutive structure that does not satisfy $L_1 \cap L_{-1} = \{0\}$ is called an *exceptional Dirac structure.*

In general, $L_V W \neq [\![V, W]\!]$. However, given the definition of an exceptional complex structure and (2.157), we find that the two agree on $\Gamma(L_1)$. Hence, it is equivalent to determine integrability with respect to the Courant bracket.

To see that this definition is correct, one can introduce a compatible connection D that is not necessarily torsion free. Using (5.11) we find

$$L_V W = L^D_V W - T(V) \cdot W \tag{4.16}$$

Because of the compatibility of D, the first term must be a section of L_1. Moreover, since the left hand side does not depend on the choice compatible connection, the projection of $L_V W$ onto $L_0 \oplus L_{-1}$ can only depend on the intrinsic torsion T^{int}. Given (4.14) we can see that

$$T^{\text{int}}(V) \cdot W \in \Gamma(L_0) \tag{4.17}$$

Hence, for $L_V W$ to be a section of L_1 for all $V, W \in \Gamma(L_1)$, we need that $T^{\text{int}}|_{\mathbf{15}_{-2}} = 0$. The complex conjugate of this condition then sets the whole of $T^{\text{int}} = 0$. Hence we see that definition 20 is correct.

From (4.13) we can see that the integrability conditions for $\mathbb{R}^+ \times \mathrm{U}^*(6)$ structures is a subset of the conditions for an integrable $\mathrm{SU}^*(6)$ structure. This makes sense as the $\mathbb{R}^+ \times \mathrm{U}^*(6)$ structure is a strictly weaker structure. Given χ defining an $\mathbb{R}^+ \times \mathrm{U}^*(6)$ structure that is integrable, we want to know what additional conditions are required so that the $\mathrm{SU}^*(6)$ structure is integrable. Let's consider just the map μ_3 from (4.2). In [187], they show that this is equal to

$$\mu_3(V) \propto \int_M \mathrm{Tr}(\kappa\, J_3 T^{\text{int}}(V)) + \frac{1}{2}\int_M T^{\text{int}}(J_3 \cdot V) \cdot \kappa^2 \tag{4.18}$$

From this we can see that $\mu_3 \equiv 0$ if and only if the singlet part of $T^{\text{int}}(V)$ is 0 for all $V \in \Gamma(E)$. From the decomposition of E given in (4.7), we see that this is equivalent to the $\mathbf{15}_0 \oplus \mathbf{6}_1 \oplus \mathbf{6}_{-1}$ part of T^{int} vanishing. This is precisely the remaining part of the intrinsic torsion of the $\mathrm{SU}^*(6)$ structure given in (4.13). This motivates the following definition.

Definition 21. An integrable $\mathrm{SU}^*(6)$ structure χ, or equivalently J_+, is an $\mathrm{SU}^*(6)$ structure with an integrable $\mathrm{U}^*(6)$ structure, along with the vanishing of the moment map μ_3.

As we mentioned, μ_3 is a moment map for the action of generalised diffeomorphisms on the space of $\mathrm{SU}^*(6)$ structures. Indeed, following [187], the space of H-structures $\mathcal{A}_H$ has a Kähler structure[6] with Kähler potential given by

$$\mathcal{K} = \int_M (\mathrm{Tr}(\chi\bar{\chi}))^{1/2} \tag{4.19}$$

Here χ is a holomorphic coordinate on $\mathcal{A}_H$. Splitting the functional derivative into holomorphic and antiholomorphic parts $\delta = \partial' + \bar{\partial}'$, and using $\varpi = \mathrm{i}\partial'\bar{\partial}'\mathcal{K}$ we can find the Kähler form on

[6] In fact, it is a hyper-kähler structure picked out by supersymmetry but we shall focus on just one of the Kähler structures for now, returning to the full structure later.

the space as

$$\imath_\beta \imath_\alpha \varpi = \int_M \frac{\mathrm{i}}{(\mathrm{Tr}(\chi\bar{\chi}))^{1/2}} \left[-\frac{1}{4} \frac{\mathrm{Tr}(\imath_\alpha \delta\chi\bar{\chi})\,\mathrm{Tr}(\chi\delta\imath_\beta\bar{\chi})}{\mathrm{Tr}(\chi\bar{\chi})} + \frac{1}{4} \frac{\mathrm{Tr}(\imath_\beta \delta\chi\bar{\chi})\,\mathrm{Tr}(\chi\delta\imath_\alpha\bar{\chi})}{\mathrm{Tr}(\chi\bar{\chi})} \right. \\ \left. + \frac{1}{2}\,\mathrm{Tr}(\imath_\alpha\delta\chi\imath_\beta\delta\bar{\chi}) - \frac{1}{2}\,\mathrm{Tr}(\imath_\beta\delta\chi\imath_\alpha\delta\bar{\chi}) \right] \tag{4.20}$$

Using the non-holomorphic coordinate J_+, this takes the much simpler form

$$\imath_\beta \imath_\alpha \varpi = \frac{\mathrm{i}}{2} \int_M \left[\mathrm{Tr}(\imath_\alpha \delta J_+ \imath_\beta \delta J_-) - \mathrm{Tr}(\imath_\beta \delta J_+ \imath_\alpha \delta J_-)\right] \tag{4.21}$$

Here $\alpha, \beta \in \Gamma(T\mathcal{A}_H)$. Taking $\beta = \rho_V$ to be the vector generated by an infinitesimal generalised diffeomorphism, i.e. $\imath_{\rho_V} \delta J_+ = L_V J_+$, then we see that

$$\begin{aligned}
\imath_{\rho_V} \imath_\alpha \varpi &= \frac{\mathrm{i}}{2} \int_M \left[\mathrm{Tr}(\imath_\alpha \delta J_+ \imath_{\rho_V} \delta J_-) - \mathrm{Tr}(\imath_{\rho_V} \delta J_+ \imath_\alpha \delta J_-)\right] &(4.22)\\
&= \frac{\mathrm{i}}{2} \int_M \left[\mathrm{Tr}(\imath_\alpha \delta J_+ L_V J_-) - \mathrm{Tr}(L_V J_+ \imath_\alpha \delta J_-)\right] &(4.23)\\
&= -\frac{\mathrm{i}}{2} \int_M \left[\mathrm{Tr}(J_- L_V \imath_\alpha \delta J_+) + \mathrm{Tr}(\imath_\alpha \delta J_- L_V J_+)\right] &(4.24)\\
&= -\imath_\alpha \delta \left(\frac{\mathrm{i}}{2} \int_M \mathrm{Tr}(J_- L_V J_+) \right) &(4.25)\\
&= -\imath_\alpha \delta \mu_3(V) &(4.26)
\end{aligned}$$

Hence we see that, up to an overall sign which doesn't matter, μ_3 is precisely the moment map defined.

4.2.2 Example - Calabi-Yau Manifolds

We know that Calabi-Yau three-folds provide a supersymmetric background when all the fluxes vanish. We should then be able to embed some of the data of the Calabi-Yau into the formalism of exceptional complex structures. It turns out that exceptional complex structures describe the conventional complex structure of the Calabi-Yau. That is, embedding the Calabi-Yau structure into the language of exceptional complex structures takes the following pattern of inclusions.

$$\begin{array}{ccccc}
\mathrm{USp}(6) & \subset & \mathrm{SU}^*(6) & \subset & \mathbb{R}^+ \times \mathrm{U}^*(6) \\
\cup & & \cup & & \cup \\
\mathrm{SU}(3) & \subset & \mathrm{SL}(3,\mathbb{C}) & \subset & \mathrm{GL}(3,\mathbb{C})
\end{array} \tag{4.27}$$

From [187] we find that the relevant structures are

$$\mathrm{SU}^*(6)\ \text{structure} : \qquad \chi = \tfrac{1}{2}\kappa^2(-\Omega + \Omega^\#) \tag{4.28}$$

$$\mathbb{R}^+ \times \mathrm{U}^*(6)\ \text{structure} : \begin{cases} \tilde{J} = \tfrac{1}{2}(I - \mathrm{vol} - \mathrm{vol}^\#) \\ L_1 = \mathrm{e}^{\mathrm{i}\,\mathrm{vol}} \cdot (T^{1,0} \oplus \wedge^{0,2} T^*) \end{cases} \tag{4.29}$$

Here, Ω is the $\mathrm{SL}(3,\mathbb{C})$ structure and I is the associated $\mathrm{GL}(3,\mathbb{C})$ structure. That is, I is the conventional complex structure of the Calabi-Yau. $\mathrm{vol} = \frac{\mathrm{i}}{8}\Omega \wedge \bar{\Omega}$ is the volume form picked out by the $\mathrm{SL}(3,\mathbb{C})$ structure, and the musical isomorphisms are defined by the $\mathrm{SU}(3)$ metric of the Calabi-Yau, g. κ^2 is some section of the bundle $\det T^*$. It is a simple check using the formulae for the adjoint action in appendix A that L_1 is the $+i$ eigenbundle of $\tilde{J}$, and χ lives in the $-2\mathrm{i}$ eigenbundle of $\tilde{J}$.

Now we look at what the integrability conditions given by definitions 20 and 21 imply for the $\mathrm{GL}(3,\mathbb{C})$ and $\mathrm{SL}(3,\mathbb{C})$ structures. First, let's consider the integrability of the $\mathbb{R}^+ \times \mathrm{U}^*(6)$ structure. Taking $V = \mathrm{e}^{\mathrm{i}\,\mathrm{vol}}(v+\omega), V' = \mathrm{e}^{\mathrm{i}\,\mathrm{vol}}(v'+\omega') \in \Gamma(L_1)$, we have

$$\begin{aligned} L_V V' &= L_{\mathrm{e}^{\mathrm{i}\,\mathrm{vol}}(v+\omega)} \mathrm{e}^{\mathrm{i}\,\mathrm{vol}}(v'+\omega') \\ &= \mathrm{e}^{\mathrm{i}\,\mathrm{vol}} L_{v+\omega}(v'+\omega') \\ &= \mathrm{e}^{\mathrm{i}\,\mathrm{vol}} \left[\mathcal{L}_v v' + (\mathcal{L}_v \omega' - v' \lrcorner \mathrm{d}\omega) - \omega' \wedge \mathrm{d}\omega\right] \end{aligned} \tag{4.30}$$

For this to be a section of L_1 also, we require that $\mathcal{L}_v v' \in \Gamma(T^{1,0})$ for any $v, v' \in \Gamma(T^{1,0})$. This is precisely the statement that the $\mathrm{GL}(3,\mathbb{C})$ structure is integrable. If this is the case then the exterior derivative decomposes into the Dolbeault operators $\mathrm{d} = \partial + \bar{\partial}$. With this, we see that the last term vanishes, and the middle term in the parentheses is equal to $v \lrcorner \partial\omega' - v' \lrcorner \partial\omega \in \Gamma(\wedge^{0,2}T^*)$. Hence we see that

$$\mathbb{R}^+ \times \mathrm{U}^*(6) \text{ structure integrable} \quad \Leftrightarrow \quad \mathrm{GL}(3,\mathbb{C}) \text{ structure integrable} \tag{4.31}$$

Now let's consider the vanishing of the moment map μ_3. Using the algebra in appendix A, one can show that

$$\mu_3(V) = \frac{\mathrm{i}}{8}\int_M -\frac{1}{2}L_V\kappa^2 + \frac{1}{2}\kappa^2\left[-\bar{\Omega}^\# \lrcorner \mathcal{L}_v\Omega + \Omega^\# \lrcorner \mathcal{L}_v\bar{\Omega} - (\Omega^\# \wedge \bar{\Omega}^\#) \lrcorner \mathrm{d}\sigma\right] \tag{4.32}$$

This first term vanishes since $L_V\kappa^2$ is a total derivative[7]. The final term gives

$$-\frac{1}{2}\int_M \kappa^2\, \mathrm{vol}^\# \lrcorner \mathrm{d}\sigma \propto \int_M \mathrm{vol}^\# \lrcorner \kappa^2 \mathrm{d}\sigma \propto \int_M \mathrm{d}(\mathrm{vol}^\# \lrcorner \kappa^2) \wedge \sigma \overset{!}{=} 0 \quad \forall\, \sigma \in \Gamma(\wedge^5 T^*) \tag{4.33}$$

This is true if and only if $\kappa^2 = c\,\mathrm{vol}$ for some constant $c \in \mathbb{R}$, which we can set to 1 without loss of generality. This just says that the volume form picked out by the $\mathrm{SU}^*(6)$ structure is the same as that picked out by the $\mathrm{SL}(3,\mathbb{C})$ structure. With this, the rest of (4.32) is proportional to

$$\begin{aligned} \int_M (\mathcal{L}_v\Omega \wedge \bar{\Omega} - \Omega \wedge \mathcal{L}_v\bar{\Omega}) &= \int_M (v \lrcorner \mathrm{d}\Omega \wedge \bar{\Omega} + \mathrm{d}(v \lrcorner \Omega) \wedge \bar{\Omega} - \Omega \wedge v \lrcorner \mathrm{d}\bar{\Omega} - \Omega \wedge \mathrm{d}(v \lrcorner \bar{\Omega}) \\ &= 2\int_M v \lrcorner (\bar{a} - a)\, \Omega \wedge \bar{\Omega} \end{aligned} \tag{4.34}$$

In moving to the second line we have used integration by parts to put the derivatives on the $\Omega, \bar{\Omega}$, and have used the integrability of I to write $\mathrm{d}\Omega = \bar{a} \wedge \Omega$ for some $\bar{a} \in \Gamma(T^{*0,1})$. This

[7] Here we use the fact that $\int_M \mathrm{d}(...) = 0$ which means we are taking M to be compact without boundary, or that the fields die off sufficiently quickly at infinity.

vanishes for all $v \in \Gamma(T)$ if and only if $\bar{a} = 0$, or equivalently, if $\mathrm{d}\Omega = 0$. Therefore, we have

$$\mathrm{SU}^*(6) \text{ structure integrable} \quad \Leftrightarrow \quad \mathrm{SL}(3, \mathbb{C}) \text{ structure integrable} \tag{4.35}$$

4.2.3 A Closer Analysis of Exceptional Complex Structures

In this section we will examine what definition 20 implies for the structure of L_1. We will find that the isotropy and reality conditions place strong restrictions on the possible form of the ECS in terms of natural bundles. First, we define a notion of *type* similar to that of generalised complex structures defined in [165], and also defined in exceptional geometry in [1]. We make this definition only for compactifications of of M-theory.

Definition 22. The *type* of an almost exceptional complex structure $L_1 \subset E_{\mathbb{C}}$ is the (complex) codimension of its image under the anchor map. That is, if $a : E \to T$ is the anchor map, then

$$\operatorname{type} L_1 = \operatorname{codim}_{\mathbb{C}} a(L_1) = 6 - \dim_{\mathbb{C}} a(L_1) \tag{4.36}$$

We will find that the only allowed types of an ECS are 0 and 3.

We would like to classify the possible forms of L_1 based on the criteria set out in definition 20. We will only state the results here and leave the proofs for appendix E. Let us first focus on condition (ii), the isotropy condition that states

$$L_1 \times_N L_1 = 0 \tag{4.37}$$

If we write $V_i = v_i + \omega_i + \sigma_i \in L_1$, then using the formula for the projection onto N around (A.56) we find that the elements of L_1 must satisfy

$$v_1 \lrcorner \omega_2 + v_2 \lrcorner \omega_1 = 0 \tag{4.38}$$

$$j\omega_1 \wedge \omega_2 + j\omega_2 \wedge \omega_1 = 0 \tag{4.39}$$

$$\omega_1 \wedge \omega_2 - v_1 \lrcorner \sigma_2 - v_2 \lrcorner \sigma_3 = 0 \tag{4.40}$$

Careful consideration of these equations shows that any exceptional Dirac structure must be of the following form.

Proposition 1. Any isotropic subbundle $L \subset E_{\mathbb{C}}$ has the form

$$\mathrm{e}^{\alpha+\beta} \cdot (\Delta \oplus S_2 \oplus S_5) \tag{4.41}$$

where $\alpha \in \Omega^3(M)$ and $\beta \in \Omega^6(M)$ are arbitrary but fixed, and where $\Delta \subset T$, $S_2 \subset \wedge^2 T^*$, $S_5 \subset \wedge^5 T^*$ satisfy the following. For all $v \in \Delta$, $\omega, \omega' \in S_2$ and $\sigma \in S_5$ we have

$$\begin{aligned} v \lrcorner \omega &= 0 \qquad & v \lrcorner \sigma &= 0 \\ \omega \wedge \omega' &= 0 \qquad & j\omega \wedge \sigma &= 0 \end{aligned} \tag{4.42}$$

We now turn our attention to conditions (i) and (iii) in definition 20. One finds that imposing $\dim_{\mathbb{C}} L_1 = 6$ restricts us to type 0, 3 and 6. Then imposing $L_1 \cap L_0 = \{0\}$ excludes the type 6 case. We can therefore summarise the general form of an ECS in proposition 2. First, we will introduce some notation which will be useful both here and later when we discuss the moduli of these structures.

Let $\Delta \subset T$ be some subbundle. We define by $\mathcal{F}^k_p(\Delta) \subset \wedge^k T^*$ to be the bundle of differential k-forms ϕ satisfying

$$\phi(x_1, ..., x_p, v_1, ..., v_{k-p}) = 0 \quad \forall\, x_1, ..., x_p \in \Gamma(T),\ v_1, ..., v_{k-p} \in \Gamma(\Delta) \tag{4.43}$$

Note that this defines a filtration of the fibres of $\wedge^k T^*$ with

$$0 = \mathcal{F}^k_k(\Delta) \subseteq \mathcal{F}^k_{k-1}(\Delta) \subseteq ... \subseteq \mathcal{F}^k_0(\Delta) \subseteq \mathcal{F}^k_{-1}(\Delta) := \wedge^k T^* \tag{4.44}$$

where we have defined $\mathcal{F}^k_{-1}(\Delta) = \wedge^k T^*$ for ease later. We can now summarise the results stated above as follows.

Proposition 2. An exceptional complex structure can only be of type 0 or type 3, and their general form is given by

$$\begin{aligned} &\text{type 0:} && \mathrm{e}^{\alpha+\beta} \cdot T_{\mathbb{C}} \\ &\text{type 3:} && \mathrm{e}^{\alpha+\beta} \cdot (\Delta \oplus \mathcal{F}^2_1(\Delta)) \end{aligned} \tag{4.45}$$

where $\Delta \subset T_{\mathbb{C}}$ is rank 3.

α, β, Δ are not generic and are constrained by condition (iv) of definition 20. In principle, one could find the non-linear conditions imposed by (iv), however we will not do that here. Instead, we note that one can show that the following choice of isotropic bundle satisfies (i)-(iii) of the definition of ECS.

$$L_1 = \mathrm{e}^{\mathrm{i}\,\mathrm{vol}} \cdot T_{\mathbb{C}} \tag{4.46}$$

where vol is any non-zero volume form. However, this defines an $\mathrm{SL}(6, \mathbb{R}) \times \mathbb{R}^+$ structure, and one finds that $\zeta(V, Z) > 0$. One can also show that if we choose some real $\rho \in \Omega^3(M)$ then

$$L_1 = \mathrm{e}^{\mathrm{i}\rho} \cdot T_{\mathbb{C}} \qquad \Rightarrow \qquad \zeta(\mathrm{e}^{\mathrm{i}\rho} \cdot v, \mathrm{e}^{\mathrm{i}\rho} \cdot (\mathrm{vol}^\# \lrcorner \nu)) \propto |\nu(v)|^2 \,\mathrm{Tr}(K_\rho^2) \tag{4.47}$$

where $K_\rho : T \to T$ is the map introduced by Hitchin in [255]. The statement that $\zeta < 0$ is equivalent to the statement that $\mathrm{Tr}(K_\rho^2) < 0$, and hence $\rho = \mathrm{re}\,\Omega$ for some $\mathrm{SL}(3, \mathbb{C})$ structure Ω.

We will now turn to the conditions for integrability of the $\mathrm{U}^*(6) \times \mathbb{R}^+$ structure. Recall from (4.15) that an ECS is integrable if and only if it is involutive with respect to the Dorfman derivative. We will write $L_1 = \mathrm{e}^{\alpha+\beta} \cdot (v + \omega) \in \Gamma(L_1)$, and similarly for V', where $v \in \Gamma(\Delta)$ and $\omega \in \Gamma(\mathcal{F}^2_1)$. Note that in the case that $\Delta = T_{\mathbb{C}}$, $\mathcal{F}^2_1(\Delta) = 0$ and hence this expression covers both type 0 and type 3. Involutivity then becomes

$$L_V V' = \mathrm{e}^{\alpha+\beta} \cdot \big([v, v'] + (\mathcal{L}_v \omega' - v' \lrcorner \mathrm{d}\omega + v' \lrcorner (v \lrcorner \mathrm{d}\alpha)) - \omega' \wedge \mathrm{d}\omega\big) \in \Gamma(L_1) \tag{4.48}$$

For this to be true we require $[\Delta, \Delta] \subseteq \Delta$. In the type 0 case this is trivial, but in the type 3 case this implies the existence of a 3d foliation. The 2-form piece implies that for all $v, v' \in \Gamma(\Delta)$

$$v \lrcorner \mathrm{d}\omega' - v' \lrcorner \mathrm{d}\omega + v' \lrcorner (v \lrcorner \mathrm{d}\alpha) \in \Gamma(\mathcal{F}_1^k) \tag{4.49}$$

A short calculation shows that, provided Δ is integrable in the sense of Frobenius, the de Rham differential restricts to $\mathrm{d} : \Gamma(\mathcal{F}_p^k) \to \Gamma(\mathcal{F}_p^{k+1})$. Hence, $\Gamma(\mathcal{F}_p^\star)$ defines a filtration of the de Rham complex. It is then clear that for all $v \in \Gamma(\Delta)$, $\omega' \in \Gamma(\mathcal{F}_1^2)$ we have $v \lrcorner \mathrm{d}\omega' \in \Gamma(\mathcal{F}_1^2)$. We further require that $v' \lrcorner (v \lrcorner \mathrm{d}\alpha) \in \Gamma(\mathcal{F}_1^2)$ which we can restate as

$$v \lrcorner (v' \lrcorner (v'' \lrcorner \mathrm{d}\alpha)) = 0 \quad \forall\, v, v', v'' \in \Gamma(\Delta) \tag{4.50}$$

Finally, we need the 5-form term to vanish in (4.48). Since $\omega' \in \Gamma(\mathcal{F}_1^2)$ and $\mathrm{d}\omega \in \Gamma(\mathcal{F}_1^3)$, you can show that $\omega' \wedge \mathrm{d}\omega \in \Gamma(\mathcal{F}_3^5)$. This space trivially vanishes because Δ is of rank (at least) 3. Therefore, the final term vanishes trivially. We can summarise the results as follows.

Proposition 3. An integrable ECS is of the form $L_1 = \mathrm{e}^{\alpha+\beta} \cdot (\Delta \oplus \mathcal{F}_1^2(\Delta))$ where

$$[\Delta, \Delta] \subseteq \Delta \qquad v \lrcorner (v' \lrcorner (v'' \lrcorner \mathrm{d}\alpha)) = 0 \quad \forall\, v, v', v'' \in \Gamma(\Delta) \tag{4.51}$$

In the case that the ECS is of type 0, the second condition just implies $\mathrm{d}\alpha = 0$. This is just the statement that the (complex) flux vanishes. This may not be the case for type 3 solutions.

4.3 Moduli of H-structures

As previously mentioned, an $\mathrm{SU}^*(6)$ structure does not define a generalised metric and hence does not define a supergravity background. However, much like the moduli space of a Calabi-Yau locally splitting into Kähler and complex moduli, the moduli space of a USp(6) structure splits locally into H-structure and V-structure moduli. Therefore, by studying the moduli of the H-structure, we will be able to retrieve some information about the spectrum of the effective theory on $\mathbb{R}^{4,1}$. The moduli space $\mathcal{M}_H$ was described in [187] in terms of a hyper-Kähler quotient of the space of $\mathrm{SU}^*(6)$ structure by generalised diffeomorphisms. In the analysis above, we have broken the SU(2) symmetry of the structure and in doing so, we will be able to make explicit statements about the moduli of the structure in terms of natural cohomology groups. This comes at the cost of losing the explicit hyper-Kähler construction, with only one of the Kähler structures manifest. In trying to reinstate the SU(2) symmetry, we will find some interesting structure of the cohomology arising.

The moduli space of $\mathrm{SU}^*(6)$ structures is defined to be the space of $\mathrm{SU}^*(6)$ structures that are integrable, up to diffeomorphisms and form field gauge transformations. That is, the moduli space is given by

$$\mathcal{M}_H = \{\chi \in \mathcal{A}_H \,|\, L_1 \text{ involutive, } \mu_3 = 0\}/\mathrm{GDiff} =: \hat{\mathcal{A}}_H /\!\!/ \mathrm{GDiff} \tag{4.52}$$

The second inequality arises because μ_3 is precisely the moment map for the action of GDiff on $\mathcal{A}_H$. Hence, $\mathcal{M}_H$ is precisely the Kähler quotient of $\hat{\mathcal{A}}_H := \{\chi \in \mathcal{A}_H \,|\, L_1 \text{ involutive}\}$. From this description, we are able to exploit a general result about group actions that preserve a Kähler structure. That is, the space can be viewed as either a Kähler quotient, or a quotient by the complexified group $\mathrm{GDiff}_{\mathbb{C}}$[8] (see for example [256]). We can therefore write the moduli space of H-structures in the convenient form[9]

$$\mathcal{M}_H = \hat{\mathcal{A}}_H / \mathrm{GDiff}_{\mathbb{C}} \tag{4.53}$$

We can make a further simplification by considering $\mathrm{SU}^*(6)$ structures up to overall scaling $\mathcal{M}_H/\mathbb{C}^*$. Physically, this is motivated by the fact that rescaling χ by an overall constant can be viewed as rescaling the internal Killing spinors by an overall constant. This can then be absorbed into the definition of the external components of the decomposition of the Killing spinors. With this in mind, the scale of χ is unphysical in a full supergravity background and won't appear as moduli in the effective theory on $\mathbb{R}^{4,1}$. Given this simplification, we find

$$\mathcal{M}_H/\mathbb{C}^* = \{L_1 \,|\, L_1 \text{ involutive}\}/\mathrm{GDiff}_{\mathbb{C}} =: \hat{\mathcal{Z}}_{\mathbb{R}^+\times \mathrm{U}^*(6)}/\mathrm{GDiff}_{\mathbb{C}} \tag{4.54}$$

where $\hat{\mathcal{Z}}_{\mathbb{R}^+\times \mathrm{U}^*(6)}$ is the space of involutive $\mathbb{R}^+ \times \mathrm{U}^*(6)$ structures. This holds because a choice of L_1 determines χ up to some scaling, as is described around (4.10). This problem is now completely analogous to the problem of finding moduli of conventional complex structures [257]. We will follow their work very closely.

4.3.1 Deformation Theory and Moduli of SU*(6) Structures

At a point $p \in M$, the space of almost exceptional complex structures is given by the coset

$$Q_{\mathbb{R}^+\times \mathrm{U}^*(6)} = E_{6(6)} \cdot \tilde{J}_0 = \mathrm{E}_{6(6)}/\mathrm{U}^*(6) = \mathrm{E}_{6,\mathbb{C}}/P \tag{4.55}$$

where $\tilde{J}_0$ is some fixed exceptional complex structure and P is the parabolic subgroup that stabilises L_1

$$P = \mathrm{Stab}\, L_1 = \mathrm{GL}(6,\mathbb{C}) \ltimes \mathbb{C}^{21} \tag{4.56}$$

By considering all possible $p \in M$ we find that $\tilde{J}$ must be a section of the bundle

$$Q_{\mathbb{R}^+\times \mathrm{U}^*(6)} \longrightarrow \mathcal{Q}_{\mathbb{R}^+\times \mathrm{U}^*(6)} \longrightarrow M \tag{4.57}$$

Infinitesimally, the deformations are given by sections of the bundle

$$\mathfrak{e}_{6,\mathbb{C}}/\mathfrak{p} \longrightarrow \mathfrak{Q}_{\mathbb{R}^+\times \mathrm{U}^*(6)} \longrightarrow M \tag{4.58}$$

[8] One has to be careful in defining this complexified group since the natural complexification is not well defined. What we mean by $\mathrm{GDiff}_{\mathbb{C}}$ is the group generated by $\rho_V, \mathcal{I}\rho_V \in \Gamma(T\mathcal{A}_H)$, where $\mathcal{I}$ is the complex structure on $\mathcal{A}_H$.

[9] In fact, as is described in chapters 3 and 5, one really needs to consider the space $\hat{\mathcal{A}}_H^{\mathrm{ps}}$ of 'polystable' points in $\hat{\mathcal{A}}_H$. This has interesting links to geometric invariant theory of which we go into more detail in those chapters. Here, we are just interested in the infinitesimal structure of the moduli space for which this technicality is not important. Hence, we will ignore this detail in this section.

In practice, we choose an embedding $\mathfrak{e}_{6,\mathbb{C}}/\mathfrak{p} \hookrightarrow \mathfrak{e}_{6,\mathbb{C}}$. Then, given some section $A \in \Gamma(\mathfrak{Q}_{\mathbb{R}^+\times \mathrm{U}^*(6)})$, we can define the deformed L_1 bundle L_1' by

$$L_1' = (1+\epsilon A)\cdot L_1 \tag{4.59}$$

for some small parameter $\epsilon << 1$ and we view A as a map : $L_1 \to E_{\mathbb{C}}/L_1$. Through the embedding $\mathfrak{e}_{6,\mathbb{C}}/\mathfrak{p} \hookrightarrow \mathfrak{e}_{6,\mathbb{C}}$, we get an embedding $E_{\mathbb{C}}/L_1 \hookrightarrow E_{\mathbb{C}}$.

By assumption, the original bundle L_1 is involutive and hence the intrinsic torsion vanishes. For a generic deformation A, L_1' will have some non-zero intrinsic torsion that appears as an obstruction to the involutivity of the bundle with respect to the Dorfman derivative. By expanding the involutivity condition to first order in ϵ, we find a map

$$\mathrm{d}_2 : \Gamma(\mathfrak{Q}_{\mathbb{R}^+\times \mathrm{U}^*(6)}) \longrightarrow \Gamma(W_{\mathrm{int}}^{\mathbb{R}^+\times \mathrm{U}^*(6)}) \tag{4.60}$$

The integrable deformations are determined by the kernel of this map. That is, L_1' is integrable if and only if $A \in \ker \mathrm{d}_2$.

We also have a notion of trivial deformation given by complexified generalised diffeomorphisms. To linear order, these are given by the action of the Dorfman derivative along some complexified vector $V \in \Gamma(E_{\mathbb{C}})$. That is, L_1' is said to be a trivial deformation if

$$L_1' = (1+\epsilon L_V)L_1 \quad \text{some } V \in \Gamma(E_{\mathbb{C}}) \tag{4.61}$$

This defines a second map

$$\mathrm{d}_1 : \Gamma(E_{\mathbb{C}}) \longrightarrow \Gamma(\mathfrak{Q}_{\mathbb{R}^+\times \mathrm{U}^*(6)}) \tag{4.62}$$

where a deformation A is trivial if an only if $A \in \operatorname{im} \mathrm{d}_1$. It is an easy check that any trivial deformation is involutive to linear order in ϵ. Indeed,

$$\begin{aligned} L_{W+\epsilon L_V W}(W' + \epsilon L_V W') &= L_W W' + \epsilon(L_{L_V W}W' + L_W L_V W') + O(\epsilon^2) \\ &= (1+\epsilon L_V)L_W W' + O(\epsilon^2) \end{aligned} \tag{4.63}$$

This implies that $\mathrm{d}_2 \circ \mathrm{d}_1 = 0$, and hence we have a three-term complex

$$\Gamma(E_{\mathbb{C}}) \xrightarrow{\ \mathrm{d}_1\ } \Gamma(\mathfrak{Q}_{\mathbb{R}^+\times \mathrm{U}^*(6)}) \xrightarrow{\ \mathrm{d}_2\ } \Gamma(W_{\mathrm{int}}^{\mathbb{R}^+\times \mathrm{U}^*(6)}) \tag{4.64}$$

where the cohomology of (4.64) counts the moduli of the $\mathrm{SU}^*(6)$ structure. In the rest of this section we will provide different ways of analysing this cohomology for different cases.

4.3.2 Type 0 Structures

A generic[10] $\mathbb{R}^+ \times \mathrm{U}^*(6)$ structure is of type 0 and these are of the form

$$L_1 = \mathrm{e}^{\alpha+\beta}\cdot T_{\mathbb{C}} \qquad \alpha \in \Omega^3(M)_{\mathbb{C}},\ \beta \in \Omega^6(M)_{\mathbb{C}} \tag{4.65}$$

[10] Generic in the sense that any the space of L_1 with non-surjective projection onto T are measure 0 in the Grassmannian of all L_1.

The algebraic conditions arising from definition 20 put algebraic conditions on α, β. In the case that $\beta = 0$, $\alpha = \mathrm{i}\rho$ for some $\rho \in \Omega^3(M)_{\mathbb{R}}$, these conditions ensure that ρ defines an $\mathrm{SL}(3,\mathbb{C})$ structure as in [255]. It is easy to see from (2.145) that this is an integrable $\mathbb{R}^+ \times \mathrm{U}^*(6)$ structure if $\mathrm{d}\alpha = 0$.

To find the moduli of this structure we can choose the following embeddings.

$$E_{\mathbb{C}}/L_1 = \wedge^2 T^* \oplus \wedge^5 T^* \tag{4.66}$$

$$\mathfrak{Q}_{\mathbb{R}^+ \times \mathrm{U}^*(6)} = \wedge^3 T^* \oplus \wedge^6 T^* \tag{4.67}$$

Then a generic deformation of L_1 of the form (4.65) will be

$$L_1' = (1 + \epsilon(a+b))L_1 = \mathrm{e}^{\alpha+\beta+\epsilon(a+\tilde{b})}T_{\mathbb{C}} \tag{4.68}$$

where the formula on the right hand side is to be taken to first order in a, b, and where $\tilde{b} = b - \frac{1}{2}a \wedge \alpha$. From this it is clear that

$$L_1' \text{ integrable} \quad \Leftrightarrow \quad \mathrm{d}a = \mathrm{d}b = 0 \tag{4.69}$$

Of course the condition on b is trivial. We then want to consider when a deformation is trivial. That is, when we can write it in the form

$$L_1' = (1 + \epsilon L_V)L_1 \qquad \text{some } V \in E_{\mathbb{C}} \tag{4.70}$$

Writing $V = \mathrm{e}^{\alpha+\beta}(V + \omega + \sigma)$, we find that the trivial L_1' can be written as

$$L_1' = \mathrm{e}^{\alpha+\beta-\mathrm{d}\omega-\mathrm{d}\tilde{\sigma}}T_{\mathbb{C}} \tag{4.71}$$

where $\tilde{\sigma} = \sigma + \frac{1}{2}\alpha \wedge \omega$. Hence, the deformation is trivial if and only if a, b are exact. From this it is clear to see that the moduli are counted by the de Rham cohomology groups

$$\mathcal{H} = H^3 \oplus H^6 \tag{4.72}$$

4.3.3 Type 3 Structures

As we have seen, the only possibility other than type 0 is type 3. In this case we have

$$L_1 = \mathrm{e}^{\alpha+\beta} \cdot (\Delta \oplus \mathcal{F}_1^2(\Delta)) \tag{4.73}$$

For convenience, we will define a dual filtration of multivectors $\mathfrak{F}_p^k(\Delta) \subset \wedge^k T$ given by $\xi \lrcorner \phi = 0$ for all $\xi \in \mathfrak{F}_p^k(\Delta)$, and for all $\phi \in \mathcal{F}_p^k(\Delta)$. It is possible to show that one can choose the following for the quotient spaces.

$$E_{\mathbb{C}}/L_1 = \left(T/\mathfrak{F}_0^1\right) \oplus \left(\wedge^2 T^*/\mathcal{F}_1^2\right) \oplus \wedge^5 T^* \tag{4.74}$$

$$\mathfrak{Q}_{\mathbb{R}^+ \times \mathrm{U}^*(6)} = \left[\left(T/\mathfrak{F}_0^1\right) \otimes \left(T^*/\mathcal{F}_0^1\right)\right] \oplus \left(\wedge^3 T/\mathfrak{F}_2^3\right) \oplus \left(\wedge^3 T^*/\mathcal{F}_1^3\right) \oplus \wedge^6 T^* \tag{4.75}$$

While these spaces may seem confusing at first, things are made easier by choosing some space $\Sigma \subset T_{\mathbb{C}}$ that is complement to Δ. In the case that $\Delta \sim T^{1,0}$, there is a canonical choice of Σ. However, when $\Delta \cap \bar{\Delta} \neq 0$, there may not be a well-defined choice. We expect that, since $a(E) = T$, a unique choice should be selected by L_0 somehow but we have not checked this. Alternatively, we will always assume that these ECS are part of a supergravity background and hence there will be a metric. We can always choose $\Sigma = \Delta^\perp$ with respect to this metric. In any case, to calculate the following one may need to make such a choice. In that case, these quotients simplify to

$$E_{\mathbb{C}}/L_1 = \Sigma \oplus (\Sigma^* \otimes \Delta^*) \oplus \wedge^2\Delta^* \oplus \wedge^5 T^* \tag{4.76}$$

$$\mathfrak{Q}_{\mathbb{R}^+\times \mathrm{U}^*(6)} = [\Sigma \otimes \Delta^*] \oplus \wedge^3\Sigma \oplus \wedge^3\Delta^* \oplus (\wedge^2\Delta^* \otimes \Sigma^*) \oplus \wedge^6 T^* \tag{4.77}$$

The final result should be independent of this choice of embedding and so we will work with the general form (4.74), (4.75).

An important consideration to make in the type 3 case is the possibility of non-trivial flux. As we saw in proposition 3, the complex flux locally defined by $\mathrm{d}\alpha$ does not need to vanish. This means that the physical flux $F = \mathrm{d}A$, does not need to vanish either. Moreover, while it is locally expressed as $\mathrm{d}A$, the gauge potential A may not be globally defined and hence F may be in a non-trivial cohomology class. In fact, as we shall see in the AdS case later, the cohomology class of F represents something physical, such as the number of M5 branes wrapping a cycle. This becomes an issue if we try to calculate the integrable deformations of a bundle L_1 twisted by the local gauge potentials A, since all of the deformation parameters will not be globally well-defined, as they will also be twisted by A. Instead, we shall assume that we are working with the flux-twisted Dorfman derivative[11]. This has the trade off of having the moduli counted by the cohomology of a 'flux-twisted' differential. To find such a differential that squares to 0, it will be convenient to work with the complex-flux twisted Dorfman derivative $L_V^{F_{\mathbb{C}}}$ and consider deformations of the untwisted bundle $\tilde{L}_1 = \Delta \oplus \mathcal{F}_1^2(\Delta)$. This has the same quotient bundles as (4.74), (4.75).

Let's consider a general deformation element $R = r + X + \theta + \mu \in \Gamma(\mathfrak{Q}_{\mathbb{R}^+\times \mathrm{U}^*(6)})$ where $r \in \Gamma\left[\left(T/\mathfrak{F}_0^1\right) \otimes \left(T^*/\mathcal{F}_0^1\right)\right]$, $X \in \Gamma(\wedge^3 T/\mathfrak{F}_2^3)$, etc. Then we consider the deformed bundle

$$\tilde{L}_1' = (1+R)\tilde{L}_1 = \mathrm{e}^{\theta+\mu}(1+r+X)\cdot(\Delta + \mathcal{F}_1^2(\Delta)) \tag{4.78}$$

We then consider the conditions for $\tilde{L}_1'$ to be involutive under $L_V^{F_{\mathbb{C}}}$. We will leave the detailed calculation to the appendix and for now just note that the moduli are controlled by two cohomology groups related to Δ. Firstly, since Δ is involutive with respect to the Lie bracket, this defines a Lie algebroid and has an associated differential

$$\mathrm{d}_\Delta : \wedge^p\left(T^*/\mathcal{F}_1^1\right) \longrightarrow \wedge^{p+1}\left(T^*/\mathcal{F}_1^1\right) \qquad \mathrm{d}_\Delta^2 = 0 \tag{4.79}$$

If we take $i : \Delta \hookrightarrow T$ to be the natural inclusion, then $i^* : T^* \to (T^*/\mathcal{F}_1^1)$. We can define the

[11] The equivalence between the twisted Dorfman derivative and twisted generalised tangent bundle pictures was highlighted in chapter 2.3

differential above via $i^* \circ \mathrm{d} = \mathrm{d}_\Delta \circ i^*$, where we take the natural extension of i^* to $\wedge^p T^*$. This will define cohomology groups which we will denote by H^p_Δ.

The second cohomology group of interest is defined in terms of the filtration $\mathcal{F}^p_k(\Delta)$. Recall that $\mathrm{d} : \mathcal{F}^p_k(\Delta) \to \mathcal{F}^{p+1}_k(\Delta)$ if Δ is an integrable distribution. Hence, the de Rham differential descends to the following complex.

$$\left(\wedge^1 T^*/\mathcal{F}^1_k\right) \xrightarrow{\mathrm{d}} \left(\wedge^2 T^*/\mathcal{F}^2_k\right) \xrightarrow{\mathrm{d}} \ldots \xrightarrow{\mathrm{d}} \left(\wedge^6 T^*/\mathcal{F}^6_k\right) \tag{4.80}$$

We then denote the cohomology groups associated to this complex as $H^p_{\mathcal{F}_k}$. We note that neither of these cohomologies are the basic cohomology of foliated spaces defined in e.g. [258].

After a lengthy calculation, one finds that the moduli are counted by the cohomology of a differential that we will label $\mathrm{d}_{\Delta,F}$ which creates the following complex

$$\begin{aligned} &\Gamma\left(\left(T/\mathfrak{F}^1_0\right) \oplus \left(\wedge^2 T^*/\mathcal{F}^2_1\right) \oplus \wedge^5 T^*\right) \\ \xrightarrow{\mathrm{d}_{\Delta,F}} &\Gamma\left(\left(\wedge^3 T/\mathfrak{F}^3_2\right) \oplus \left[\left(T/\mathfrak{F}^1_0\right) \otimes \left(T^*/\mathcal{F}^1_0\right)\right] \oplus \left(\wedge^3 T^*/\mathcal{F}^3_1\right) \oplus \wedge^6 T^*\right) \\ \xrightarrow{\mathrm{d}_{\Delta,F}} &\Gamma\left(\left[\left(\wedge^3 T/\mathfrak{F}^3_2\right) \otimes \left(T^*/\mathcal{F}^1_0\right)\right] \oplus \left[\left(T/\mathfrak{F}^1_0\right) \otimes \wedge^2\left(T^*/\mathcal{F}^1_0\right)\right] \oplus \left(\wedge^4 T^*/\mathcal{F}^4_1\right)\right) \end{aligned} \tag{4.81}$$

If we take $R = X + r + \theta + \mu \in \Gamma(\mathfrak{Q}_{\mathbb{R}^+ \times \mathrm{U}^*(6)})$, and $V = v + \omega + \sigma \in \Gamma(E_\mathbb{C}/L_1)$, then the closure conditions are[12]

$$0 = \mathrm{d}_\Delta X \tag{4.82}$$

$$0 = \mathrm{d}_\Delta r - jX \lrcorner j^2 F_\mathbb{C} \tag{4.83}$$

$$0 = \mathrm{d}\theta - r \cdot F_\mathbb{C} \tag{4.84}$$

and the exactness conditions are

$$r = \mathrm{d}_\Delta v \tag{4.85}$$

$$\theta = \mathrm{d}\omega - v \lrcorner F_\mathbb{C} \tag{4.86}$$

$$\mu = \mathrm{d}\sigma + \omega \wedge F_\mathbb{C} \tag{4.87}$$

We are implicitly taking projections onto relevant quotient spaces where needed above. It is an easy check to see that $\mathrm{d}^2_{\Delta,F} = 0$. If $F_\mathbb{C}$ happens to be exact then one can show, assuming constraints on α that are slightly stronger than those implied by involutivity[13], that the cohomology is counted by

$$H^0_\Delta(M, \wedge^3 T/\mathfrak{F}^3_2) \oplus H^1_\Delta(M, T/\mathfrak{F}^1_0) \oplus H^3_{\mathcal{F}_1}(M) \oplus H^6_\mathrm{d}(M) \tag{4.88}$$

4.3.4 A Generic Result on the Moduli

The moduli found in the previous sections determine the moduli of all structures of constant type. However, there are cases where the type changes over the manifold. As we will see later

[12] The definition of $jX \lrcorner j^2 F_\mathbb{C}$ can be found in appendix A

[13] This extra condition is detailed in appendix F.

around (4.103), a change of type 0 → type 3 can be interpreted as the 3-form twist α becoming singular. In the case that we transition from type 0 → real type 3, this may indicate the presence of a brane and hence understanding these type changing solutions may indicate how to incorporate branes into the generalised picture. We would therefore like a general result on the moduli that does not appeal to the specific type of the structure.

Being able to analyse the cohomology requires a good choice of embeddings $\mathfrak{e}_{6,\mathbb{C}}/\mathfrak{p} \hookrightarrow \mathfrak{e}_{6,\mathbb{C}}$ and $E_{\mathbb{C}}/L_1 \hookrightarrow E_{\mathbb{C}}$. While the final result will not depend on this choice, certain embeddings will simplify the problem greatly. One particularly convenient choice is picked out by the $\mathrm{U}^*(6)$ structure itself. Decomposing under this group we find

$$E_{\mathbb{C}} = \mathfrak{X}_1 \oplus \mathfrak{X}_{-1} \oplus \wedge^2 \mathfrak{X}^* \tag{4.89}$$

$$\operatorname{ad} \tilde{F}_{\mathbb{C}} = \operatorname{ad} P_{\mathbb{R}^+ \times \mathrm{U}^*(6)} \oplus \wedge^3 \mathfrak{X}^*_{-1} \oplus \wedge^6 \mathfrak{X}^*_{-2} \oplus \wedge^3 \mathfrak{X}^*_1 \oplus \wedge^6 \mathfrak{X}^*_{+2} \tag{4.90}$$

$$W_{\text{int}}^{\mathbb{R}^+ \times \mathrm{U}^*(6)} = \wedge^4 \mathfrak{X}^*_{-2} \oplus \wedge^4 \mathfrak{X}^*_2 \tag{4.91}$$

where $\mathfrak{X}$ is a bundle that transforms in the **6** of $\mathrm{SU}^*(6)$. A natural choice of embeddings is then

$$E_{\mathbb{C}}/L_1 = \wedge^2 \mathfrak{X}^* \oplus \wedge^5 \mathfrak{X}^*_{-1} \qquad \mathfrak{Q}_{\mathbb{R}^+ \times \mathrm{U}^*(6)} = \wedge^3 \mathfrak{X}^*_{-1} \oplus \wedge^6 \mathfrak{X}^*_{-2} \tag{4.92}$$

Since L_1 defines an integrable structure, by assumption, there exists a torsion free connection D that is compatible with the exceptional complex structure $\tilde{J}$. From (5.11) we know that we can replace the definitions of $\mathrm{d}_1, \mathrm{d}_2$ in terms of the Dorfman derivative with definitions involving L^D_V. This means that we can write the maps $\mathrm{d}_1, \mathrm{d}_2$ in terms of the connection D. Moreover, viewing $D : \Gamma(\mathcal{T}) \to \Gamma(E^* \otimes \mathcal{T})$, we can decompose E^* into $\mathrm{U}^*(6)$ representations. The compatibility of D implies that it is consistent to define a decomposition of D as

$$D = D_1 + D_{-1} + D_0 \tag{4.93}$$

where $D_n = \pi_n D$ where π_n is the projection of E^* to the subspace with $\tilde{J}$ charge n.

With these decompositions, we find that the complex (4.64) can be written

$$\begin{array}{ccccc} \Gamma(\wedge^2 \mathfrak{X}^*) & \xrightarrow{D_{-1}} & \Gamma(\wedge^3 \mathfrak{X}^*_{-1}) & \xrightarrow{D_{-1}} & \Gamma(\wedge^4 \mathfrak{X}^*_{-2}) \\ & \nearrow^{D_0} & & \nearrow^{D_0} & \\ \Gamma(\wedge^5 \mathfrak{X}^*_{-1}) & \xrightarrow{D_{-1}} & \Gamma(\wedge^6 \mathfrak{X}^*_{-2}) & & \end{array} \tag{4.94}$$

Note that the involutivity of L_1 implies that $D_{-1}^2 = 0$. In fact, L_1 defines a Lie algebroid and D_{-1} is the associated differential

$$D_{-1} : \wedge^p \mathfrak{X}^*_q \longrightarrow \wedge^{p+1} \mathfrak{X}^*_{q-1} \tag{4.95}$$

In full generality, not much can be said about the cohomology of (4.94) without more knowledge of the maps D_{-1}, D_0. However, if we make the following assumption, we can give a generic result about the moduli of the $\mathrm{SU}^*(6)$ structures.

Definition 23. D_0, D_{-1} are said to satisfy the *generalised $\partial\bar{\partial}$-lemma* if they satisfy the following

$$\operatorname{im} D_0 \cap \ker D_{-1} \subseteq \operatorname{im} D_{-1} D_0 \tag{4.96}$$

We show in appendix E that provided the generalised $\partial\bar{\partial}$-lemma holds, and D_0 is a cochain homomorphism, then the cohomology $\mathcal{H}$ of the complex (4.64) is given by the cohomology of D_{-1}. More precisely we have

$$\mathcal{H} = H^3_{D_-} \oplus H^6_{D_-} \tag{4.97}$$

We will see later that while these are the geometric moduli, not all of them may be physical.

Example - Calabi-Yau

We return to the explicit example of compactification on a Calabi-Yau. Following the method set out above, we decompose $E_{\mathbb{C}}$, $\operatorname{ad}\tilde{F}_{\mathbb{C}}$ into eigenspaces of $\tilde{J}$. This is outlined in appendix E but for now, we just note that there is an isomorphism between this complex and the following, using the holomorphic three-form Ω.

$$\begin{array}{ccccc} \Omega^2(M)_{\mathbb{C}} & \xrightarrow{\ \partial\ } & \Omega^3(M)_{\mathbb{C}} & \xrightarrow{\ \partial\ } & \Omega^4(M)_{\mathbb{C}} \\ & \overset{D_0}{\nearrow} & & \overset{D_0}{\nearrow} & \\ \Omega^5(M)_{\mathbb{C}} & \xrightarrow{\ \partial\ } & \Omega^6(M)_{\mathbb{C}} & & \end{array} \tag{4.98}$$

where $D_0 = \Omega^{\#} \lrcorner \bar{\partial} + \bar{\Omega}^{\#} \lrcorner \partial$. One can show that this satisfies the generalised $\partial\bar{\partial}$-lemma and hence the moduli are counted by

$$H^3_{\partial} \oplus H^6_{\partial} \tag{4.99}$$

We can see some of the known moduli of Calabi-Yau manifolds in there. For example, $H^{1,2}_{\partial}$ are the complex structure moduli.

4.3.5 Reintroducing the SU(2) Symmetry

We return now to the question of the broken SU(2) symmetry of the SU*(6) structure. Recall that we could define an SU*(6) structure as a triplet of weighted adjoint elements J_α that form a highest weight $\mathfrak{su}(2)$ algebra under the adjoint bracket. We then formed a $\mathbb{R}^+ \times \mathrm{U}^*(6)$ structure by considering just the unweighted J_3. However, we could equally have chosen any of the J_α, or indeed any $\mathfrak{u}(1) \subset \mathfrak{su}(2)$. This means that in any supergravity background we have a $\mathrm{SU}(2)/\mathrm{U}(1) \simeq \mathbb{CP}^1$ of equivalent $\mathbb{R}^+ \times \mathrm{U}^*(6)$ structures which are given by

$$\tilde{J}_u = u^\alpha \tilde{J}_\alpha \qquad |u|^2 = 1 \tag{4.100}$$

where the $\tilde{J}_\alpha$ are the unweighted J_α. Moreover, an integrable SU*(6) structure implies there exists a torsion free connection which is compatible with all the J_α. Any such a connection will also be compatible with the $\tilde{J}_u$ and hence all $\mathbb{R}^+ \times \mathrm{U}^*(6)$ structures will be integrable in a supergravity background.

We can equivalently think of this in terms of involutive bundles

$$\tilde{J}_u \quad \Leftrightarrow \quad L_u = v^i L_i \tag{4.101}$$

where the $i = \pm$, and the $v^i \in \mathbb{CP}^1$ are functions on the u^α. For all possible u, the L_u define a Lie algebroid and hence have an associated differential D_u which will count the moduli. Moreover, since they are physically counting the moduli of the same structure, the cohomology groups should be isomorphic. We therefore see that we have a set of quasi-isomorphic derivations labelled by $\mathbb{CP}^1$.

Example - Calabi-Yau

For a Calabi-Yau manifold, the triplet of unweighted adjoint tensors are given by

$$\tilde{J}_1 = -\rho + \rho^\# \qquad \tilde{J}_2 = -\hat{\rho} + \hat{\rho}^\# \qquad \tilde{J}_3 = \frac{1}{2}(I - \mathrm{vol} - \mathrm{vol}^\#) \tag{4.102}$$

where ρ, $\hat{\rho}$ are the real and imaginary parts of Ω respectively. One finds that for $\tilde{J}_u$, we have

$$L_u = \begin{cases} \mathrm{e}^{\frac{\mathrm{i}}{2}\left(\frac{u^1+\mathrm{i}u^2}{1+u^3}\Omega + \frac{u^1-\mathrm{i}u^2}{1-u^3}\bar{\Omega}\right)} T_{\mathbb{C}} & u^3 \neq \pm 1 \\ L_{\pm 1} & u^3 = \pm 1 \end{cases} \tag{4.103}$$

Hence, at all points except the north and south pole of the $\mathbb{CP}^1$, the $\mathbb{R}^+ \times \mathrm{U}^*(6)$ structure is of type 0. At the north and south pole the structure is of type 3. Moreover, each bundle has an associated differential and we find that

$$D_u \sim \begin{cases} \mathrm{d} & u^3 \neq \pm 1 \\ \partial & u^3 = 1 \\ \bar{\partial} & u^3 = -1 \end{cases} \tag{4.104}$$

These differentials do not seem isomorphic. However, we saw that the moduli for the Calabi-Yau defined by L_1 were given by the Dolbeault cohomology groups $H^3_\partial \oplus H^6_\partial$, whereas the type 0 moduli were counted by the de Rham cohomology groups $H^3 \oplus H^6$. The statement that these are isomorphic is equivalent to Hodge's Theorem on Calabi-Yau manifolds.

4.4 Applications to $\mathrm{AdS}_5 \times M_6$ Backgrounds

In the case of $\mathrm{E}_{6(6)} \times \mathbb{R}^+$ geometry, we find that supersymmetric AdS solutions also have an involutive bundle structure. As was discussed briefly in chapter 2, AdS solutions are characterised in generalised geometry through weak holonomy [212]. Specifically, these solutions have a USp(6) structure which has a non-zero intrinsic torsion that is constrained to live in a singlet representation of W^{int}. As was shown in [213], this singlet lies only in the μ_3 moment map given in (4.2). We can therefore characterise AdS_5 solutions as those with an $\mathrm{SU}^*(6)$ structure satisfying

$$L_1 \text{ involutive}, \qquad \mu_3(V) = 3m \int_M c(V, K, K) \tag{4.105}$$

where m is the inverse AdS radius, and K is a suitable $F_{4(4)}$ structure satisfying other compatibility and integrability conditions. Given this, we can apply some of the tools we have found above to analysing CFTs through the AdS/CFT correspondence.

The general local geometry of AdS_5 backgrounds was analysed in [259] in terms of a local SU(2) structure. This means that there are local 1-forms ζ_1, ζ_2 such that the metric looks like

$$\mathrm{d}s^2(M_6) = \mathrm{d}s^2_{\mathrm{SU}(2)} + \zeta_1^2 + \zeta_2^2 \tag{4.106}$$

There is also a complex 2-form Ω and a real fundamental 2-form ω which capture the geometry of $\mathrm{d}s^2_{\mathrm{SU}(2)}$. These, and other useful objects, can be defined in terms of spinor bilinears[14].

$$\begin{aligned} \sin\theta &= \bar{\epsilon}^+\epsilon^- & Y &= \omega - \sin\theta\,\zeta_1\wedge\zeta_2 = -\mathrm{i}\bar{\epsilon}^+\gamma_{(2)}\epsilon^+ \\ \tilde{\zeta}_1 &= \cos\theta\,\zeta_1 = \bar{\epsilon}^+\gamma_{(1)}\epsilon^+ & Y' &= \zeta_1\wedge\zeta_2 - \sin\,\theta\omega = \mathrm{i}\bar{\epsilon}^+\gamma_{(2)}\epsilon^- \\ \tilde{\zeta}_2 &= \cos\theta\,\zeta_2 = \mathrm{i}\bar{\epsilon}^+\gamma_{(1)}\epsilon^- & X &= -\Omega\wedge(\sin\theta\,\zeta_1 - \mathrm{i}\zeta_2) = \epsilon^{+\mathrm{T}}\gamma_{(3)}\epsilon^+ \\ \tilde{\Omega} &= \cos\theta\,\Omega = \epsilon^{-\mathrm{T}}\gamma_{(2)}\epsilon^+ & V &= \cos\theta\,\omega\wedge\zeta_2 = \bar{\epsilon}^+\gamma_{(3)}\epsilon^- \end{aligned} \tag{4.107}$$

Here the γ are the gamma matrices for Cliff(6) in an orthonormal frame for M. There are some other useful spinor bilinear identities we can define

$$\begin{aligned} \bar{\epsilon}^+\epsilon^+ = \bar{\epsilon}^-\epsilon^- &= 1 & \epsilon^{+\mathrm{T}}\epsilon^- &= 0 \\ \mathrm{i}*X &= \epsilon^{-\mathrm{T}}\gamma_{(3)}\epsilon^+ & -\tilde{\zeta}_1\wedge Y &= \mathrm{i}\bar{\epsilon}^+\gamma_{(3)}\epsilon^+ \\ \tfrac{1}{3!}Y\wedge Y\wedge Y &= \mathrm{i}\bar{\epsilon}^+\gamma_{(6)}\epsilon^+ & Z &= *\tilde{\zeta}_1 = \mathrm{i}\bar{\epsilon}^+\gamma_{(5)}\epsilon^- \end{aligned} \tag{4.108}$$

The Killing spinor equations put constraints on these tensors. In [259] it was shown that the set of necessary and sufficient local conditions are (for $m \neq 0$)

$$\begin{aligned} \mathrm{d}(\mathrm{e}^{3\Delta}\sin\theta) &= 2m\mathrm{e}^{2\Delta}\tilde{\zeta}_1 & \mathrm{d}(\mathrm{e}^{4\Delta}\tilde{\Omega}) &= 3m\mathrm{e}^{3\Delta}X \\ \mathrm{d}(\mathrm{e}^{5\Delta}\tilde{\zeta}_2) &= *F + 4m\mathrm{e}^{4\Delta}Y & \mathrm{d}(\mathrm{e}^{3\Delta}V) &= \mathrm{e}^{3\Delta}\sin\theta F + 2m\mathrm{e}^{2\Delta}*Y' \end{aligned} \tag{4.109}$$

One can also deduce the following important relations

$$\mathrm{d}(\mathrm{e}^{3\Delta}X) = 0 \qquad \mathrm{d}(\mathrm{e}^{\Delta}Y') = -\xi\lrcorner F \qquad \mathrm{d}(\mathrm{e}^{\Delta}Z) = \mathrm{e}^{\Delta}Y'\wedge F \tag{4.110}$$

where ξ is a Killing vector, which can be shown using the Killing spinor equations, given by

$$\xi = \mathrm{e}^{\Delta}\tilde{\zeta}_2^{\#} \tag{4.111}$$

These results were translated into $E_{6(6)} \times \mathbb{R}^+$ geometry in [213]. They found that the appropriate $\mathrm{SU}^*(6)$ and $F_{4(4)}$ structures are given by

$$J_3 = -\frac{1}{2}\kappa Y_R + \frac{1}{2}\kappa(\tilde{\zeta}_1\wedge Y - \tilde{\zeta}_1^{\#}\wedge Y^{\#}) - \frac{1}{2}\kappa(\frac{1}{3!}Y\wedge Y\wedge Y + \frac{1}{3!}Y^{\#}\wedge Y^{\#}\wedge Y^{\#}) \tag{4.112}$$

$$J_+ = \frac{1}{2}\kappa\left(\tilde{\Omega}_R - \mathrm{i}*X + \mathrm{i}*X^{\#}\right) \tag{4.113}$$

$$K = \xi - \mathrm{e}^{\Delta}Y' + \mathrm{e}^{\Delta}Z \tag{4.114}$$

[14] Note that we use slightly different notation to [259], instead following the notation of [213].

where $\kappa^2 = \mathrm{e}^{3\Delta}\,\mathrm{vol}_6$ and the subscript R denotes raising one index to creating a GL(6) adjoint element. For us, we will only be interested in the $\mathrm{U}^*(6) \times \mathbb{R}^+$ defined by the unweighted J_3, which we have labelled $\tilde{J}$. We can find the decomposition of the generalised tangent bundle under $\tilde{J}$ and we find

$$L_1 = \mathrm{e}^{-\mathrm{i}\cos\theta\,\zeta_1\wedge\omega - \mathrm{i}\sin\theta\,\mathrm{vol}_6} \cdot \left[\Delta \oplus \mathcal{F}_1^2(\Delta)\right] \tag{4.115}$$

$$\Delta = T^{1,0}M_4 \oplus \mathbb{C}(\sin\theta\,\zeta_2^\# - \mathrm{i}\zeta_1^\#) \tag{4.116}$$

Here, $T^{1,0}M_4$ is the complex structure on the 4-dimensional subspace of T orthogonal to ζ_1, ζ_2, defined by Ω. In their formulation, they have untwisted by the physical flux $F_\mathbb{R}$, which will be in some non-trivial cohomology class counting the number of M5 branes wrapping the internal cycles. We can combine this with the complex twist given above to form the complex flux $F_\mathbb{C}$ which is relevant for the deformation problem.

This bundle is globally type 3, even away from the locus $\sin\theta = 0$. This means that we can use some of the tools of the previous section to understand the moduli of of these backgrounds, and therefore the spectrum of the associated CFT. Understanding the moduli of the $\mathrm{SU}^*(6)$ structure should be all that is needed to understand the spectrum of the associated CFT since it was argued in [214] that there are no exactly marginal deformations coming from deformations of the $F_{4(4)}$ structure K. This matches the field theory result that there are no exactly marginal deformations of just the Kähler potential. The difficulty in applying the results of the previous section directly to this case is that the moduli calculated there were for integrable $\mathrm{SU}^*(6)$ structures, whereas these are not since the μ_3 moment map does not vanish. Fortunately, as is argued in [260], the spectrum only depends on the holomorphic object χ, which defines the $\mathrm{SU}^*(6)$ structure, and not at all on K. Hence the moduli are exactly counted by the cohomology of $\mathrm{d}_{\Delta,F}$, with $F_\mathbb{C}$ in a non-trivial cohomology class, as laid out in section 4.3.3.

We have therefore shown that calculating the entire spectrum of any given CFT_4 dual to some $\mathrm{AdS}_5 \times M_6$ is equivalent to calculating particular cohomology groups associated to a distribution Δ. Calculating the spectrum is a challenging problem related to calculating certain homology groups in some a priori non-commutative chiral ring. These would be some 7-dimensional analogue of the cyclic homologies in Calabi-Yau algebras [261]. Instead, we have shown that these are equivalent to certain cohomologies in a smooth commutative geometry, graded by the action of $\mathcal{L}_\xi$[15]. This is the M-theory analogue of the work done in the type IIB case, where the cyclic homologies have been related to the Kohn-Rossi cohomology groups of the associated Sasaki-Einstein manifold in [262], and extended to non-Sasaki-Einstein solutions in [260].

Looking at (4.116), we see that away from $\sin\theta = 0$, the distribution Δ defines a complex structure. However, on that locus, the complex structure degenerates and we find $\Delta\cap\bar{\Delta} \neq 0$. We could therefore call Δ an '*almost everywhere complex structure*'. There is not much literature on calculating the graded cohomology groups for these distributions. However, one may hope that standard techniques could be applied, and the exact dimensions could be calculated for the simple cases of the Maldacena-Nunez solutions [121].

[15] This is the geometric equivalent of the R-charge

Chapter 5

Generalised Geometry for 4 Dimensional External Spaces

In this chapter we will analyse properties of generic 4 dimensional Minkowski compactifications in terms of integrable SU(7) structures. We will start by defining these structures in terms of a holomorphic object ψ transforming in the $\mathbf{912_3}$ of $E_{7(7)} \times \mathbb{R}^+$. We will also introduce a slightly weaker $U(7) \times \mathbb{R}^+$ structure in terms of a subbundle $L_3 \subset E_{\mathbb{C}}$. This bundle has properties reminiscent of complex and generalised complex structures and so we have called it an exceptional complex structure. Next, we show that supersymmetry is equivalent to the involutivity of L_3 plus the vanishing of a moment map. Taken together, these imply that the SU(7) structure is integrable. We are able to give an expression for the superpotential and the Kähler potential of the effective theory in terms of the holomorphic object ψ. The moment map interpretation also provides possible links to geometric invariant theory which we briefly discuss next. Finally, we discuss the moduli of SU(7) structures, recovering the results of G_2 manifolds, and finding the exact set of moduli for GMPT backgrounds.

5.1 Generalised $\mathcal{N} = 1$ Structures

Our goal is to analyse generic Minkowski $\mathcal{N} = 1$, $D = 4$ flux compactifications of M-theory and type II supergravity. In this section, we will show that they define two closely related generalised G-structures, analogous to the $GL(3, \mathbb{C})$ and $SL(3, \mathbb{C})$ structures in conventional geometry. Remarkably, we will find that the supersymmetry conditions can be rephrased similarly as an involution condition and the vanishing of a moment map. Conventional G_2 structures are of course a special case, corresponding to a compactification of M-theory with vanishing flux, as are the general type II solutions of GMPT [101] and both will provide useful examples of generalised $\mathcal{N} = 1$ structures in the following sections.

Generic $\mathcal{N} = 1$, $D = 4$ Minkowski flux compactifications of M-theory have been analysed using conventional geometrical techniques several years ago [104–106, 108]. The metric takes a warped form

$$ds^2 = e^{2\Delta} ds^2(\mathbb{R}^{3,1}) + ds^2(M), \tag{5.1}$$

where M is the compactification manifold, the internal four-form flux is non-trivial and the eleven-dimensional Killing spinors take the form

$$\epsilon = \eta_+ \otimes \mathrm{e}^{\Delta/2}\zeta^{\mathrm{c}} + \eta_- \otimes \mathrm{e}^{\Delta/2}\zeta, \tag{5.2}$$

where $\eta_\pm$ are chiral spinors of Spin(3, 1) and ζ is a complex Spin(7) spinor. Supersymmetry implies $\bar{\zeta}\zeta$ is constant and there is vanishing four-form flux on the non-compact Minkowski space. In the G_2 case ζ is real. The analogous type II backgrounds were analysed by GMPT [101]. In this case the two type II Killing spinors take the form

$$\begin{aligned} \epsilon_1 &= \eta_+ \otimes \zeta_1^+ + \eta_- \otimes \zeta_1^-, \\ \epsilon_2 &= \eta_+ \otimes \zeta_2^\mp + \eta_- \otimes \zeta_2^\pm, \end{aligned} \tag{5.3}$$

where $\zeta_i^\pm$ are chiral Spin(6) spinors, and the upper and lower choices of sign refer to type IIA and IIB respectively. One can again construct a constant-norm, eight-component spinor

$$\zeta = \mathrm{e}^{\Delta/2}\mathrm{e}^{-\hat{\varphi}/6}\begin{pmatrix}\zeta_1^+ \\ \zeta_2^-\end{pmatrix}, \tag{5.4}$$

where $\hat{\varphi}$ is the dilaton. Note that in both the M-theory and type II compactifications, although ζ is nowhere vanishing, the individual Spin(7) components (the real and imaginary parts of ζ) or Spin(6) components (the $\zeta_i^\pm$) may vanish, and hence do not define conventional (global) G-structures.

However, these backgrounds do make sense globally as generalised G-structures [180,184]. To specify the background one needs the bosonic fields on M together with the Killing spinor ζ. In exceptional generalised geometry the bosonic fields define a generalised metric G. For example in M-theory G is equivalent to the set $\{\Delta, g, A, \tilde{A}\}$ where g is the seven-dimensional metric, A is the three-form potential on M and $\tilde{A}$ is the six-form potential encoding the dual of the four-form field strength on the Minkowski space. Geometrically G defines an $\mathrm{SU}(8)/\mathbb{Z}_2 \subset \mathrm{E}_{7(7)} \times \mathbb{R}^+$ generalised structure. The spinor ζ then transforms as the **8** representation of the double cover, SU(8). The stabiliser of such a nowhere-vanishing constant-norm element is SU(7).[1] In this way, we see that a supersymmetric $\mathcal{N} = 1$ background defines a generalised SU(7) structure. The differential conditions on the Killing spinor are then equivalent to the vanishing of the generalised intrinsic torsion of the SU(7) structure [184].

5.1.1 SU(7) and $\mathbb{R}^+ \times$ U(7) Structures

Rather than defining the SU(7) structure using the pair (G, ζ) one can also define it directly in terms of generalised tensors. In fact there will be two kinds of structure in $\mathrm{E}_{7(7)} \times \mathbb{R}^+$ that will interest us [180]:

$$\begin{aligned} J &: \text{ stabilised by } G = \mathbb{C}^* \times \mathrm{SU}(7) = \mathbb{R}^+ \times \mathrm{U}(7), \\ \psi &: \text{ stabilised by } G = \mathrm{SU}(7). \end{aligned} \tag{5.5}$$

We will refer to J as an exceptional complex structure and ψ as a generalised SU(7) structure. They are stabilised by the same SU(7), but J is also invariant under an extra $\mathbb{C}^*$ action.

[1] This is analogous to a nowhere-vanishing Spin(6) $\simeq$ SU(4) spinor being stabilised by SU(3).

This is directly analogous to the relation between an almost complex structure I in six dimensions (a $\mathrm{GL}(3,\mathbb{C})$ structure) and a complex three-form Ω (an $\mathrm{SL}(3,\mathbb{C})$ structure), or an almost generalised complex structure $\mathcal{J}$ and an almost generalised Calabi–Yau structure Φ.

To see how these structures are defined, for definiteness consider the M-theory case. Recall that the generalised tangent space is given by

$$\begin{aligned} E &\simeq T \oplus \wedge^2 T^* \oplus \wedge^5 T^* \oplus (T^* \otimes \wedge^7 T^*), \\ V &= v + \omega + \sigma + \tau, \end{aligned} \tag{5.6}$$

where $V \in \Gamma(E)$ and E transforms in the $\mathbf{56_1}$ of $\mathrm{E}_{7(7)} \times \mathbb{R}^+$. Here the bold subscript denotes the $\mathbb{R}^+$ weight, normalised so that the determinant bundle $\det T^*$ has weight 2. We will occasionally denote the components of a generalised vector explicitly as V^M, where $M = 1, \ldots 56$. One can then define [187] two $\mathrm{E}_{7(7)}$-invariant maps

$$s \colon \wedge^2 E \to \det T^*, \qquad q \colon S^4 E \to (\det T^*)^2, \tag{5.7}$$

namely the symplectic invariant s and symmetric quartic invariant q. We will also need the adjoint bundle

$$\operatorname{ad} \tilde{F} \simeq \mathbb{R} \oplus (T \otimes T^*) \oplus \wedge^3 T \oplus \wedge^3 T^* \oplus \wedge^6 T \oplus \wedge^6 T^*, \tag{5.8}$$

transforming in Lie algebra representation $\mathbf{133_0} \oplus \mathbf{1_0}$, as well as a bundle K, given for example in [181], which contains the torsion of a generalised connection and transforms in the $\mathbf{912_{-1}}$ representation. We also recall that the generalised Lie derivative [180,181], or Dorfman bracket, generates infinitesimal diffeomorphisms and gauge transformations and takes the form

$$L_V \alpha = \mathcal{L}_v \alpha - (\mathrm{d}\omega + \mathrm{d}\sigma) \cdot \alpha, \tag{5.9}$$

when acting on a arbitrary generalised tensor α, where $\mathcal{L}$ is the conventional Lie derivative, $\mathrm{d}\omega$ and $\mathrm{d}\sigma$ are regarded as sections of $\operatorname{ad}\tilde{F}$ and $\cdot$ denotes the adjoint action. In the following it will also be useful to use the "twisted" generalised Lie derivative defined for $A \in \Gamma(\wedge^3 T^* M)$ and $\tilde{A} \in \Gamma(\wedge^6 T^* M)$ via (see for example [187, Appendix D])

$$\begin{aligned} L_V^{F+\tilde{F}} \alpha &:= \mathrm{e}^{-A-\tilde{A}} L_{\mathrm{e}^{A+\tilde{A}} \cdot V} \left(\mathrm{e}^{A+\tilde{A}} \cdot \alpha \right) \\ &= \mathcal{L}_v \alpha - \left(\mathrm{d}\omega - \imath_v F + \mathrm{d}\sigma - \imath_v \tilde{F} + \omega \wedge F \right) \cdot \alpha, \end{aligned} \tag{5.10}$$

where $F = \mathrm{d}A$ and $\tilde{F} = \mathrm{d}\tilde{A} - \frac{1}{2} A \wedge F$. Given a generalised connection D we can define the generalised torsion $T \colon \Gamma(E) \to \Gamma(\operatorname{ad}\tilde{F})$ via [183]

$$L_V \alpha = L_V^D \alpha - T(V) \cdot \alpha, \tag{5.11}$$

where

$$L_V^D \alpha = D_V \alpha - (D \times_{\mathrm{ad}} V) \cdot \alpha, \tag{5.12}$$

where $\times_{\mathrm{ad}}$ is a projection $\times_{\mathrm{ad}} \colon E^* \otimes E \to \operatorname{ad}\tilde{F}$ and $D_V = V^M D_M$ is the generalised derivative along V. One finds that this definition implies the torsion actually lies in $K \oplus E^* \subset E^* \otimes \operatorname{ad}\tilde{F}$.

Let us turn first to defining the structure J. Recall that, at a point on the manifold, the generalised metric defines an $\mathrm{SU}(8)/\mathbb{Z}_2$ subgroup of $\mathrm{E}_{7(7)} \times \mathbb{R}^+$, and the spinor ζ defines an SU(7) subgroup of SU(8). There is a $\mathrm{U}(1) \subset \mathrm{SU}(8)/\mathbb{Z}_2$ that commutes with this SU(7) subgroup. It is generated by an element of the SU(8) Lie algebra conjugate to the diagonal matrix

$$\alpha = (-1/2, -1/2, \ldots, 7/2) \in \mathrm{SU}(8) \subset \mathrm{E}_{7(7)} \oplus \mathbb{R}. \tag{5.13}$$

The normalisation is chosen so that $\exp(\mathrm{i}\theta J)$ with $0 \le \theta < 2\pi$ generates a U(1) subgroup of $\mathrm{SU}(8)/\mathbb{Z}_2$. Note that the commutant of this U(1) is an $\mathbb{R}^+ \times \mathrm{U}(7)$ subgroup of $\mathrm{E}_{7(7)} \times \mathbb{R}^+$. Globally the U(1) at each point will be generated by a section of the adjoint bundle $J \in \Gamma(\mathrm{ad}\,\tilde{F})$ that is conjugate to α at each point. This leads us to the definition:

Definition 24. A *generalised* $\mathbb{R}^+ \times U(7)$ *structure* or *almost exceptional complex structure* is a section $J \in \Gamma(\mathrm{ad}\,\tilde{F})$ that is conjugate at each point $p \in M$ to the element $\alpha \in \mathrm{SU}(8) \subset \mathrm{E}_{7(7)} \oplus \mathbb{R}$ given in (5.13).

Since the maximal compact subgroup $\mathrm{SU}(8)/\mathbb{Z}_2 \subset \mathrm{E}_{7(7)}$ and the maximal torus of SU(8) are each unique up to conjugation, every reduction of the structure group of E to $\mathbb{R}^+ \times \mathrm{U}(7)$ should be included in the definition. Furthermore all such structures will be related by local $\mathrm{E}_{7(7)} \times \mathbb{R}^+$ transformations. Hence, as discussed in [180], the choice of J does not fill out all of the **133** representation space but instead lies within a particular orbit. Concretely, decomposing $\mathrm{E}_{7(7)}$ using explicit SU(8) indices (see [180] or [182]) we have

$$\mathbf{133} = \mathbf{63} \oplus \mathbf{70} \ni (\mu^{\alpha}{}_{\beta}, \mu_{\alpha\beta\gamma\delta}), \tag{5.14}$$

and we can write J using the spinor ζ as

$$J^{\alpha}{}_{\beta} = 4\zeta^{\alpha}\bar{\zeta}_{\beta} - \tfrac{1}{2}(\bar{\zeta}\zeta)\delta^{\alpha}{}_{\beta}, \qquad J_{\alpha\beta\gamma\delta} = 0, \tag{5.15}$$

where we have normalised $\bar{\zeta}\zeta = 1$. For completeness, we note that further decomposing under $\mathrm{SU}(7) \times \mathrm{U}(1)$ we have

$$\mathbf{133} = \mathbf{1}_0 \oplus \mathbf{48}_0 \oplus (\mathbf{7}_{-4} \oplus \overline{\mathbf{7}}_4) \oplus (\mathbf{35}_2 \oplus \overline{\mathbf{35}}_{-2}) \tag{5.16}$$

where now the subscripts denote the U(1) charge, and J lies in the singlet $\mathbf{1}_0$ representation.

Given J, in analogy with a conventional almost complex structure, we can use it to decompose the complexified generalised tangent space. Under the adjoint action of J on sections of the generalised tangent bundle, decomposing under $\mathrm{SU}(7) \times \mathrm{U}(1)$, we find

$$\begin{aligned} E_{\mathbb{C}} &= L_3 \oplus L_{-1} \oplus L_1 \oplus L_{-3}, \\ \mathbf{56}_{\mathbb{C}} &= \mathbf{7}_3 + \mathbf{21}_{-1} + \overline{\mathbf{21}}_1 + \overline{\mathbf{7}}_{-3}. \end{aligned} \tag{5.17}$$

Thus we get four rather than two subbundles, with $L_{-3} \simeq \bar{L}_3$ and $L_{-1} \simeq \bar{L}_1$. As we will see, L_3 will play the analogue of the role of $T^{1,0}$ in conventional complex geometry. As such, this leads to the alternative definition

Definition 25. An almost exceptional complex structure is a subbundle $L_3 \subset E_{\mathbb{C}}$ such that

i) $\dim_{\mathbb{C}} L_3 = 7$,

ii) $L_3 \times_N L_3 = 0$,

iii) $L_3 \cap \bar{L}_3 = \{0\}$,

iv) The map $h\colon L_3 \times L_3 \to (\det T^*)_{\mathbb{C}}$ defined by $h(V,W) = \mathrm{i}\, s(V,\bar{W})$ is a definite hermitian inner product valued in $\det T^*$,

where $\times_N \colon E \times E \to N$, with N the generalised tensor bundle transforming in the $\mathbf{133_2}$ representation, is an $\mathrm{E}_{7(7)} \times \mathbb{R}^+$ covariant map given in appendix A. In analogy with the generalised complex structure case we call a subbundle L_3 satisfying the first two conditions a *(complex) exceptional polarisation.*

Note that the (complex) stabiliser groups in $\mathrm{E}_{7,\mathbb{C}}$ of all exceptional polarisations are isomorphic. However the corresponding real stabiliser groups in $\mathrm{E}_{7(7)} \times \mathbb{R}^+$ can differ. In particular, only almost exceptional complex structures are stabilised by a subgroup $\mathrm{U}(7) \times \mathbb{R}^+ \subset \mathrm{E}_{7(7)} \times \mathbb{R}^+$.

We now turn to the SU(7) structure ψ. Decomposing the **912** representation under $\mathrm{SU}(7) \times \mathrm{U}(1) \subset \mathrm{SU}(8)/\mathbb{Z}_2 \subset \mathrm{E}_{7(7)}$, we find

$$\begin{aligned} \mathbf{912} &= \mathbf{36} \oplus \mathbf{420} \oplus \text{c.c}, \\ &= \mathbf{1}_7 \oplus \mathbf{7}_3 \oplus \mathbf{28}_{-1} \oplus \mathbf{21}_{-1} \oplus \mathbf{35}_{-5} \oplus \mathbf{140}_3 \oplus \mathbf{224}_{-1} \oplus \text{c.c.} \end{aligned} \tag{5.18}$$

where the subscript denotes the U(1) charge. Consider the generalised tensor bundle transforming in the $\mathbf{912_3}$ representation of $\mathrm{E}_{7(7)} \times \mathbb{R}^+$ (where the bold subscript denotes the $\mathbb{R}^+$ weight; the reason for this particular choice will be discussed below)

$$\begin{aligned} \tilde{K} = (\det T^*)^2 \otimes K \simeq \mathbb{R} \oplus \Lambda^3 T^* \oplus (T^* \otimes \Lambda^5 T^*) \oplus (S^2 T^* \otimes \Lambda^7 T^*) \\ \oplus (\Lambda^3 T^* \otimes \Lambda^6 T^*) \oplus (\Lambda^3 T \otimes (\Lambda^7 T^*)^3) \oplus \dots, \end{aligned} \tag{5.19}$$

where $K \subset E^* \otimes \operatorname{ad} \tilde{F}$ is the torsion bundle [181]. The SU(7) singlet in the decomposition (5.18) implies that each almost exceptional complex structure J defines a unique line bundle $\mathcal{U}_J \subset \tilde{K}_{\mathbb{C}}$, satisfying

$$V \bullet \psi = 0 \quad \forall\, V \in \Gamma(L_3), \qquad s(\psi, \bar{\psi}) \neq 0, \tag{5.20}$$

where ψ is a local section of $\mathcal{U}_J$, the product $V \bullet \psi$ is defined by the projection map $E \otimes \tilde{K} \to C$ where C is the generalised tensor bundle transforming in the $\mathbf{8645_4}$ representation[2] of $\mathrm{E}_{7(7)} \times \mathbb{R}^+$, and s is the symplectic invariant on the **912** bundle $\tilde{K} \subset E \otimes E \otimes E$ induced from the symplectic invariant on the **56** bundle E. One can equivalently define a local section ψ by the condition $J\psi = 7\mathrm{i}\psi$ under the adjoint action of J. In complete analogy with the almost complex and almost generalised complex cases we are then led to define

Definition 26. Given an almost exceptional complex structure J with trivial line bundle $\mathcal{U}_J$, a *generalised SU(7) structure* is a global nowhere-vanishing section $\psi \in \Gamma(\mathcal{U}_J)$.

[2]Note that this representation is just the next step in the tensor hierarchy [263, 264] above **912**.

Again we expect all generalised geometries with SU(7) structure group will arise this way, and furthermore any two such structures will be related by a local $E_{7(7)} \times \mathbb{R}^+$ transformation. In particular, any two generalised SU(7) structures with the same almost exceptional complex structure J will be related by a nowhere vanishing complex function f

$$\psi' = f\psi. \tag{5.21}$$

Again [180], ψ parameterises a particular orbit in the **912** representation rather than filling out the whole representation space. One could always write down the (non-linear) conditions on ψ (and J for that matter) which define the relevant orbits, but we have not attempted to do so. This would give conditions that are the analogue of stability for a three-form Ω and non-degeneracy for a two-form ω. Instead, we can always write ψ concretely using the spinor ζ and generalised metric G. Under the decomposition in (5.18) we can write a section of $\tilde{K}$ in explicit SU(8) indices [180, 182] as

$$\kappa = (\kappa^{\alpha\beta}, \kappa^{\alpha\beta\gamma}{}_{\delta}, \bar{\kappa}_{\alpha\beta}, \bar{\kappa}_{\alpha\beta\gamma}{}^{\delta}) \in \Gamma(\tilde{K}_{\mathbb{C}}). \tag{5.22}$$

The SU(7) structure can then be written as

$$\psi^{\alpha\beta} = \lambda(\mathrm{vol}_G)^{3/2}\zeta^{\alpha}\zeta^{\beta}, \qquad \bar{\psi}_{\alpha\beta} = \psi^{\alpha\beta\gamma}{}_{\delta} = \bar{\psi}_{\alpha\beta\gamma}{}^{\delta} = 0, \tag{5.23}$$

where $\mathrm{vol}_G = \mathrm{e}^{2\Delta}\sqrt{g}$ is the $E_{7(7)}$-invariant volume defined by the generalised metric [182, 183] and λ is a non-zero complex number.

Recall that, since $SU(7) \subset SU(8)$, the generalised structure ψ also defines a generalised metric and so completely specifies the supergravity background. This is analogous to a G_2 structure in conventional geometry, where the invariant three-form φ defines a metric. In this way, our construction gives what one might call a "generalised G_2 structure". However this obscures the fact that the stabiliser group is actually SU(7) and not G_2 or $G_2 \times G_2$ as might be expected, so we do not follow this convention. Later we will see that for the example of a conventional G_2 structure, the invariant three-form φ does indeed define both ψ and J.

5.1.2 Supersymmetry and Integrability

We now turn to the conditions imposed on the generalised structures ψ and J by supersymmetry. As shown in [184, 185, 265], the vanishing of the generalised intrinsic torsion for the SU(7) structure is equivalent to $\mathcal{N} = 1$ supersymmetry for the Minkowski space solution. In what follows it will be useful to consider the intrinsic torsion for both ψ and J as the conditions for a torsion-free J are a subset of those for ψ. This will allow us to see that integrability for ψ follows from integrability for J, phrased in terms of an involution condition plus the vanishing of a moment map for generalised diffeomorphisms, that is the group of diffeomorphisms and form-field gauge transformations.

Following the analysis in [184, 185, 265], it is easy to show that intrinsic torsion for each generalised structure lies in a sub-bundle of **912** torsion bundle K transforming as

$$W^{\text{int}}_{\text{SU}(7)}: \quad \mathbf{1_{-7}} \oplus \bar{\mathbf{7}}_{\mathbf{-3}} \oplus \mathbf{21_{-1}} \oplus \mathbf{35_{-5}} \oplus \text{c.c.} \tag{5.24}$$

$$W^{\text{int}}_{\mathbb{R}^+\times\text{U}(7)}: \quad \mathbf{1_{-7}} \oplus \mathbf{35_{-5}} \oplus \text{c.c.} \tag{5.25}$$

where again, the subscript denotes that U(1) charge under the action of J. We saw earlier how integrability of a complex structure can be recast as involutivity of eigenspaces of the complex structure under the Lie bracket. It is thus natural to define:

Definition 27. A torsion-free $\mathbb{R}^+ \times \text{U}(7)$ structure J or *exceptional complex structure* is one satisfying involutivity of L_3 under the generalised Lie derivative

$$L_V W \in \Gamma(L_3), \qquad V, W \in \Gamma(L_3). \tag{5.26}$$

Again in analogy with the generalised geometry case, we call the weaker case of an involutive exceptional polarisation, an *exceptional Dirac structure*.

In general $L_V W \neq -L_W V$, however the definition of an exceptional polarisation implies

$$L_V W = [\![V, W]\!] \qquad V, W \in \Gamma(L_3), \tag{5.27}$$

where $[\![V, W]\!] = \frac{1}{2}(L_V W - L_W V)$ is the antisymmetric Courant bracket, and in fact the involution condition could be equally well defined using the Courant bracket as the generalised Lie derivative.

To prove that involutivity is equivalent to vanishing intrinsic torsion of the $\mathbb{R}^+ \times \text{U}(7)$ structure, we first recall that we can always find a generalised connection D that is compatible with the $\mathbb{R}^+ \times \text{U}(7)$ structure, so that $DJ = 0$, but it will not necessarily be torsion free. Consider the definition of the torsion (5.11) with $V, W = \alpha \in \Gamma(L_3)$. Compatibility of the connection with J ensures $L^D_V W \in \Gamma(L_3)$, so involutivity amounts to checking that $T(V) \cdot W$ is in L_3 only. Since the left-hand side of (5.11) does not depend on the choice of compatible connection, only the intrinsic torsion contributes to the components of $T(V) \cdot W$ not in L_3. Explicitly, the intrinsic torsion representations contribute to $T(V) \cdot W \in \Gamma(E)$ as

$$\begin{aligned} \mathbf{1}_{-7} \otimes \mathbf{7}_3 \otimes \mathbf{7}_3 &\supset \mathbf{21}_{-1}, \\ \mathbf{35}_{-5} \otimes \mathbf{7}_3 \otimes \mathbf{7}_3 &\supset \overline{\mathbf{21}}_1. \end{aligned} \tag{5.28}$$

In other words, a non-zero $\mathbf{1}_{-7}$ component of the torsion would generate a $\mathbf{21}_{-1} = L_{-1}$ term in $L_V W$. Requiring $L_V W \cap L_{-1} = \{0\}\ \forall V, W \in \Gamma(L_3)$ thus sets the $\mathbf{1}_{-7}$ component of the torsion to zero. In a similar way, one sees that the $\mathbf{35}_{-5}$ component is set to zero by $L_V W \cap L_1 = \{0\}$. One has $L_V W \cap L_{-3} = \{0\}$ identically just by counting the U(1) charges.

We now need to consider the remaining conditions that imply we have a torsion-free SU(7) structure and hence an $\mathcal{N} = 1$, $D = 4$ background. Comparing the representations that appear in the intrinsic torsion for the $\mathbb{R}^+ \times \text{U}(7)$ and SU(7) structures (5.24) and (5.25), we see there must then be an additional condition that sets the **7** and **21** components of the SU(7) intrinsic torsion to zero. As we will now show these appear as the vanishing of a moment map μ for the action of generalised diffeomorphisms on the space of ψ structures.

One first notes that the space of SU(7) structures on M admits a natural pseudo-Kähler metric. This is a consequence of viewing the theory as a rewriting of the ten- or eleven-dimensional theory so that only four supercharges are manifest (the analogous situation for $\mathcal{N}=2$ theories was described in [156, 187, 251, 266]). In analogy to [267], the local SO(9, 1) or SO(10, 1) Lorentz symmetry is broken and the four-dimensional scalar degrees of freedom, that is the space of generalised SU(7) structures, can be packaged into $\mathcal{N}=1$, $D=4$ chiral multiplets [180]. As such they must admit a Kähler metric, albeit infinite-dimensional. The explicit construction is as follows. At a point $p \in M$, a choice of ψ is equivalent to picking a point in the coset

$$\psi|_p \in Q_{\mathrm{SU}(7)} = \frac{\mathrm{E}_{7(7)} \times \mathbb{R}^+}{\mathrm{SU}(7)}, \tag{5.29}$$

so that an SU(7) structure on M corresponds to a section of the fibre bundle

$$Q_{\mathrm{SU}(7)} \to \mathcal{Q}_{\mathrm{SU}(7)} \to M. \tag{5.30}$$

We can then identify

$$\text{space of SU(7) structures, } \mathcal{Z} \simeq \Gamma(\mathcal{Q}_{\mathrm{SU}(7)}). \tag{5.31}$$

The key point is that the space $Q_{\mathrm{SU}(7)}$ admits a homogeneous pseudo-Kähler metric of signature (70, 16), picked out by supersymmetry. One first notes that the related space $\mathrm{E}_{7(7)}/\mathrm{U}(7)$ admits a homogeneous pseudo-Kähler metric by a classic result of Borel [236, Proposition 2]. The metric is unique up to an overall scale [268]. The space $Q_{\mathrm{SU}(7)}$ can be viewed as a complex line bundle over $\mathrm{E}_{7(7)}/\mathrm{U}(7)$, with the zero section removed, since we only have an $\mathbb{R}^+$ action. There is then a natural one-parameter family of conical, homogeneous Kähler metrics on $Q_{\mathrm{SU}(7)}$, distinguished by the relative size of the U(1) circle relative to the $\mathrm{E}_{7(7)}/\mathrm{U}(7)$ base. The infinite-dimensional space of structures $\mathcal{Z}$ then inherits a pseudo-Kähler structure from the pseudo-Kähler structure on $Q_{\mathrm{SU}(7)}$. Our choice of $\mathbb{R}^+$ weight for ψ picks out a particular Kähler metric within the one-parameter family with the explicit Kähler potential given by

$$\mathcal{K} = \int_M \left(\mathrm{i}\, s(\psi, \bar{\psi})\right)^{1/3}, \tag{5.32}$$

where ψ can be viewed as a complex coordinate on the space of structures, or more precisely as a holomorphic embedding $\psi\colon \mathcal{Z} \hookrightarrow \Gamma(\tilde{K}_{\mathbb{C}})$. Given the $\mathbb{R}^+$ weight of the $\tilde{K}$ bundle, we need to take the 1/3-power so that the integrand in (5.32) is a section of $\det T^*$ and hence can be integrated over M. A different choice of weight would have led to a different power in $\mathcal{K}$ and hence a different Kähler metric.

In analogy to the $\mathcal{N}=2$ case described in [156, 187, 251, 266], the existence of the Kähler structure follows from supersymmetry. As we mentioned above, one can consider rewriting the ten- or eleven-dimensional theory so that only four supercharges are manifest. Similar to [267], the local SO(9, 1) or SO(10, 1) Lorentz symmetry is broken and the internal degrees of freedom can be packaged into $\mathcal{N}=1$, $D=4$ chiral multiplets [180] coupled to four-dimensional supergravity. Note that there are an infinite number of fields as no Kaluza–Klein truncation is performed: one keeps all modes on the internal space. The scalar degrees of freedom should hence parameterise an infinite-dimensional Kähler space, but from our discussion, this is just the space of generalised SU(7) structures.[3] In this context, the particular weight of ψ, and hence Kähler metric, is fixed by the four-dimensional supersymmetry. In particular, as we will see in section 5.3.1, the power of 1/3 is required for the $D = 4$, $\mathcal{N} = 1$ superpotential on the space of chiral fields parameterising $\mathcal{Z}$ to be a holomorphic function of ψ.

We can write the symplectic structure corresponding to (5.32) very explicitly as follows. Using $\varpi = \mathrm{i}\,\partial'\bar{\partial}'\mathcal{K}$, we have, contracting two vectors $\alpha, \beta \in \Gamma(T\mathcal{Z})$ into the symplectic form,

$$\begin{aligned}\imath_\beta \imath_\alpha \varpi = \mathrm{i} \int_M \frac{1}{3} \frac{1}{\big(\mathrm{i}\, s(\psi, \bar{\psi})\big)^{2/3}} \Big(&\mathrm{i}\, s(\imath_\alpha \delta\psi, \imath_\beta \delta\bar{\psi}) - \mathrm{i}\, s(\imath_\beta \delta\psi, \imath_\alpha \delta\bar{\psi}) \\ &- \frac{2}{3} \frac{\mathrm{i}\, s(\imath_\alpha \delta\psi, \bar{\psi})\, \mathrm{i}\, s(\psi, \imath_\beta \delta\bar{\psi})}{\mathrm{i}\, s(\psi, \bar{\psi})} + \frac{2}{3} \frac{\mathrm{i}\, s(\imath_\beta \delta\psi, \bar{\psi})\, \mathrm{i}\, s(\psi, \imath_\alpha \delta\bar{\psi})}{\mathrm{i}\, s(\psi, \bar{\psi})} \Big).\end{aligned} \tag{5.33}$$

Note that if we define a new *non-holomorphic* parameterisation

$$\phi = \big(\mathrm{i} s(\psi, \bar{\psi})\big)^{-1/3} \psi, \tag{5.34}$$

which transforms in the $\mathbf{912_1}$ representation, the symplectic structure takes the simple form

$$\imath_\beta \imath_\alpha \varpi = -\frac{1}{3} \int_M \big(s(\imath_\alpha \delta\phi, \imath_\beta \delta\bar{\phi}) - s(\imath_\beta \delta\phi, \imath_\alpha \delta\bar{\phi}) \big), \tag{5.35}$$

that is, it is just the pull-back $\varpi = \frac{1}{3}\phi^* s$ of the symplectic form on the space of ϕ.

One can also restrict to the subspace of structures that define an (integrable) exceptional complex structure, so that L_3 is involutive,

$$\hat{\mathcal{Z}} = \{\psi \in \mathcal{Z} \mid J \text{ is integrable}\}. \tag{5.36}$$

As we will show in section 5.3.1, the integrability condition is holomorphic as a function of ψ and so $\hat{\mathcal{Z}}$ inherits a Kähler metric from the one on $\mathcal{Z}$, with the same Kähler potential.

Finally we can turn to the remaining integrability conditions for the SU(7) structure. As in our previous examples, the Kähler structure on $\hat{\mathcal{Z}}$ is invariant under generalised diffeomorphisms. Infinitesimally these are generated by the generalised Lie derivative, and parameterised by generalised vectors $V \in \Gamma(E)$. As deformations in the space of structures, this defines a vector field $\rho_V \in \Gamma(T\hat{\mathcal{Z}})$

$$\imath_{\rho_V} \delta\phi = L_V \phi, \tag{5.37}$$

[3] As we discuss below, the chiral multiplet space is strictly a $\mathbb{C}^*$ quotient of the space of structures.

where for convenience we are using the non-holomorphic structure ϕ. We then have

$$\begin{aligned} \imath_{\rho_V}\imath_\alpha\varpi &= \frac{\mathrm{i}}{3}\int_M \big(\mathrm{i}\, s(\imath_\alpha\delta\phi, L_V\bar{\phi}) - \mathrm{i}\, s(L_V\phi, \imath_\alpha\delta\bar{\phi})\big) \\ &= -\frac{\mathrm{i}}{3}\int_M \big(\mathrm{i}\, s(L_V\imath_\alpha\delta\phi, \bar{\phi}) + \mathrm{i}\, s(L_V\phi, \imath_\alpha\delta\bar{\phi})\big) \\ &= \imath_\alpha\delta\left(\tfrac{1}{3}\int_M s(L_V\phi, \bar{\phi})\right) \\ &= \imath_\alpha\delta\mu(V), \end{aligned} \tag{5.38}$$

where we have used compactness to integrate by parts and have defined

$$\begin{aligned} \mu(V) &= \tfrac{1}{3}\int_M s(L_V\phi, \bar{\phi}) \\ &= \tfrac{1}{3}\int_M s(L_V\psi, \bar{\psi})(\mathrm{i}\, s(\psi, \bar{\psi}))^{-2/3}, \end{aligned} \tag{5.39}$$

where in going to the second line we use $\int_M L_V(\cdots) = 0$. This gives a moment map $\mu\colon \hat{\mathcal{Z}} \to \mathfrak{gdiff}^*$, where, as before, $\mathfrak{gdiff}^*$ is the dual of the Lie algebra of generalised diffeomorphisms.

We now want to prove that integrability of ψ is equivalent to the vanishing of the moment map (5.39). Let D be a (torsionful) generalised connection compatible with the SU(7) structure, that is $D\psi = 0$ (and hence $D\phi = 0$). Using the definition of torsion (5.11), we have

$$\begin{aligned} \mu(V) &= \tfrac{1}{3}\int_M s((L^D_V\phi, \bar{\phi}) - s(T(V)\phi, \bar{\phi}) \\ &= \tfrac{1}{3}\int_M s(D_V\phi, \bar{\phi}) - s((D\times_{\mathrm{ad}} V)\phi, \bar{\phi}) - s(T(V)\phi, \bar{\phi}) \\ &= -\tfrac{1}{3}\int_M s(T(V)\phi, \bar{\phi}), \end{aligned} \tag{5.40}$$

where in moving to the last line we integrate the middle term by parts and use $D\phi = D\bar{\phi} = 0$. Since the definition of μ is independent of any choice of connection, only the SU(7) intrinsic torsion can contribute in the last expression. Given that the generalised vector $V \in \Gamma(E)$ transforms in the $\mathbf{7} + \mathbf{21} + \text{c.c.}$ representation, and ϕ is an SU(7) singlet, only the $\mathbf{7} + \mathbf{21} + \text{c.c.}$ representations of the SU(7) intrinsic torsion can appear[4] in μ. However, from (5.25) and (5.24), we see these are precisely the additional components that must be set to zero for an exceptional complex structure to be an integrable SU(7) structure. Thus we have shown that the following definition is consistent:

Definition 28. A torsion-free generalised SU(7) structure is one where L_3 is involutive and the moment map (5.39) vanishes.

Since two SU(7) structures that are related by diffeomorphisms and gauge transformations give physically equivalent backgrounds, the moduli space of SU(7) structures is naturally a symplectic quotient by generalised diffeomorphisms GDiff, or equivalently a quotient by the

[4]Note that there could in principle be a further kernel in the map from the intrinsic torsion to μ so that only one of the **7** and **21** representations appeared. However it is easy to show that both representations are in fact present.

complexified group $\mathrm{GDiff}_{\mathbb{C}}$:

$$\mathcal{M}_\psi = \hat{\mathcal{Z}} /\!/ \mathrm{GDiff} \simeq \hat{\mathcal{Z}} / \mathrm{GDiff}_{\mathbb{C}}. \tag{5.41}$$

Recall that the moduli space of G_2 holonomy manifolds in M-theory is associated with $\mathcal{N} = 1$, $D = 4$ chiral superfields [269–272]. For a generic $\mathcal{N} = 1$, $D = 4$ background, supersymmetry implies that the moduli space of integrable (torsion-free) generalised SU(7) structures should again define fields in chiral multiplets. However, note that not all deformations of ψ deform the physical fields on the internal space. In particular, only those within the coset $E_{7(7)} \times \mathbb{R}^+/(\mathrm{SU}(8)/\mathbb{Z}_2)$ are physical (deformations that change the generalised metric). First note that, from the warped product form (5.1), shifts of the warp factor $\Delta \to \Delta + c$ for some constant c can be absorbed in the four-dimensional metric. Second note that any modulus that lies in SU(8)/SU(7) would correspond to a change of Killing spinor ζ for the same physical background. However this just implies that the background admits a second Killing spinor and so really preserves $\mathcal{N} = 2$ supersymmetry. The exception to this is the change of ζ by a constant phase, that is by the U(1) generated by J, since this too can always be reabsorbed into the four-dimensional spinors in the ansatzë (5.2) and (5.3). Thus for honest $\mathcal{N} = 1$ backgrounds we only need to consider the action of this U(1) and the shift in Δ. As we note from the form of ψ in (5.23), shifting Δ simply rescales ψ, in fact via the $\mathbb{R}^+$ action. Put together we see that the physical moduli space is given by

$$\text{Moduli space of } \mathcal{N} = 1 \text{ background} = \mathcal{M}_\psi /\!/ \mathrm{U}(1) \simeq \mathcal{M}_\psi / \mathbb{C}^*,$$

where the $\mathbb{C}^*$ action acts as

$$\psi \to \lambda^3 \psi, \tag{5.42}$$

where we have normalised the $\mathbb{C}^*$ action to match the $\mathbb{R}^+$ action on ψ which implies $\mathcal{K} \to |\lambda|^2 \mathcal{K}$. Under the symplectic quotient, the physical moduli space has a Kähler potential $\tilde{\mathcal{K}}$ given by

$$\tilde{\mathcal{K}} = -3 \log \mathcal{K}. \tag{5.43}$$

This is the Kähler potential that determines the metric on the moduli space of the supergravity background. For example, in the G_2 case that we will discuss in section 5.2.1, this reduces to the known result that the Kähler potential $\tilde{\mathcal{K}} = -3 \log \int_M \mathrm{vol}$ describes the coupling of moduli in the four-dimensional effective theory, where vol is determined by the G_2 structure [270–272].

Note that, strictly, one should check that the kinetic terms and potentials in the $D = 4$ effective theory are given by $\tilde{\mathcal{K}}$ (specifically checking that the coefficient of -3 is correct). One check is to compare with the G_2 holonomy case, as we do in the next section. Alternatively, we can note that the quotient is simply the standard relation between the Kähler geometry in superconformal supergravity [273–275], using a compensator field, and the standard supergravity formalism where a gauge for the compensator is chosen. This fixes the $\mathbb{C}^*$ scaling of $\mathcal{K}$ and the factor of -3 comes from the standard normalisation of the gravitational coupling constant (as reviewed for example in [276]).

5.2 Examples of Integrable Generalised SU(7) Structures

We now present two classic examples of $\mathcal{N} = 1$, $D = 4$ backgrounds and describe how they can be understood as integrable generalised SU(7) structures. We discuss G_2 backgrounds in M-theory and $\mathcal{N} = 1$ GMPT backgrounds in type II theories. In both of these cases, we will see that involutivity of the L_3 subbundle reproduces a subset of the known differential conditions these backgrounds must satisfy. The final differential conditions come from the vanishing of the moment map. In particular, this gives a completely new way of viewing G_2 manifolds that, intriguingly, closely mirrors the discussion of complex structures.

5.2.1 G_2 Structures in M-theory

Recall that a G_2 structure is defined by a nowhere-vanishing three-form $\varphi \in \Gamma(\wedge^3 T^*M)$, which can be written in a local frame as

$$\varphi = e^{246} - e^{235} - e^{145} - e^{136} + e^{127} + e^{347} + e^{567}. \tag{5.44}$$

This defines a metric $g = e_a \otimes e_a$ and an orientation $\mathrm{vol} = e^{1\cdots 7} = \star 1$. The Hodge dual of φ is

$$\star\,\varphi = e^{1234} + e^{1256} + e^{3456} + e^{1357} - e^{1467} - e^{2367} - e^{2457}, \tag{5.45}$$

so that $\varphi \wedge \star\varphi = 7\,\mathrm{vol}$. The structure is integrable, that is we have a G_2 holonomy manifold, if and only if

$$\mathrm{d}\varphi = \mathrm{d}\star\varphi = 0. \tag{5.46}$$

Compactifying M-theory on a G_2 holonomy manifold with $\Delta = 0$ gives a $\mathcal{N} = 1$, $D = 4$ background. One can also include non-trivial three-form potential A such that $\mathrm{d}A = 0$.

We would like to first identify how a G_2 structure defines a generalised SU(7) structure. Before doing so it is useful to define the notion of "type" for almost exceptional complex structures in M-theory in an analogous way to the type of generalised complex structures given in [165]:

Definition 29. The *type* of an almost exceptional complex structure $L_3 \subseteq E_{\mathbb{C}}$ is the (complex) codimension of its projection onto the tangent bundle $T_{\mathbb{C}}$. That is, if $\pi\colon E \to T$ is the anchor map then

$$\operatorname{type} L_3 \coloneqq \operatorname{codim}_{\mathbb{C}} \pi(L_3) = 7 - \dim_{\mathbb{C}} \pi(L_3). \tag{5.47}$$

A generic[5] seven-dimensional subspace of a fibre of E will have a surjective projection onto the tangent space T, and hence a generic exceptional complex structure is type-0. We can always write such a space as

$$L_3 = \mathrm{e}^{\alpha+\beta} T_{\mathbb{C}} \qquad \text{for } \alpha \in \Gamma(\wedge^3 T^*_{\mathbb{C}}),\ \beta \in \Gamma(\wedge^6 T^*_{\mathbb{C}}). \tag{5.48}$$

This identically satisfies the first two conditions for an almost exceptional complex structure, and one gets simple constraints on the polyform $\alpha + \beta$ from $L_3 \cap \bar{L}_3 = \{0\}$ and the requirement

[5] Generic in the sense that the set of all seven-dimensional subspaces not of this type is measure zero in the Grassmannian.

that $\mathrm{i}\, s(V, \bar{W})$ for $V, W \in \Gamma(L_3)$ defines a definite hermitian inner product. Note that $T_{\mathbb{C}}$ is the simplest example of an exceptional Dirac structure (following the definition given in section 5.1.2), but is not an exceptional complex structure since, for example, $T_{\mathbb{C}} \cap \overline{T_{\mathbb{C}}} \neq 0$. In terms of the Killing spinor ζ, viewed as a complexified Spin(7) spinor, the requirement that the structure is type-0 is that the scalar $\zeta^{\mathrm{T}}\zeta$ is nowhere vanishing.[6] This is precisely the case discussed in [108] where the real and imaginary parts of ζ have different normalisations (and/or are non-orthogonal). The analyses in [104] and [106], on the other hand, fix equal norms and orthogonal real spinors and so define a structure that is strictly not type-0.

A G_2 structure, embedded in generalised geometry, defines the simplest example of a type-0 almost exceptional complex structure. Taking $\alpha = \mathrm{i}\varphi$ and $\beta = 0$, we have

$$L_3 = \mathrm{e}^{\mathrm{i}\varphi} T_{\mathbb{C}}, \tag{5.49}$$

so that a section of L_3 takes the form (using the "$\jmath$-notation" of [181])

$$\begin{aligned} \Gamma(L_3) \ni v &+ \mathrm{i}\, \imath_v\varphi - \tfrac{1}{2}\varphi \wedge \imath_v\varphi - \tfrac{1}{6}\mathrm{i}\, \jmath\varphi \wedge \varphi \wedge \imath_v\varphi \\ &= v + \mathrm{i}\, \imath_v\varphi - \star\imath_v\varphi - \mathrm{i}\, v^\flat \otimes \mathrm{vol}, \end{aligned} \tag{5.50}$$

for some $v \in \Gamma(T_{\mathbb{C}})$. The condition on $\mathrm{i}\, s(V, \bar{W})$ for $V, W \in \Gamma(L_3)$ is equivalent to the weighted metric

$$\tilde{g}(v, w) = \imath_v\varphi \wedge \imath_w\varphi \wedge \varphi \tag{5.51}$$

being positive definite for $v, w \in \Gamma(T)$. However, this is just the condition that φ is a (positive) stable form in the sense of Hitchin [255]. (It also implies $L_3 \cap \bar{L}_3 = \{0\}$.) The $\mathbb{R}^+ \times \mathrm{U}(7)$ structure J is given by

$$J = \varphi^\sharp - \varphi, \tag{5.52}$$

where $\varphi^\sharp$ is obtained from φ by raising its indices using the inverse metric g^{-1} defined by φ. One can check that this satisfies $JL_3 = 3\mathrm{i}L_3$ using the action of the **133** on the **56** given in appendix A.

We now turn to the integrability condition on J. Involutivity of L_3 is simple to show using the properties of the generalised Lie derivative. Writing generic sections of L_3 as $V = \mathrm{e}^{\mathrm{i}\varphi} v$ and $W = \mathrm{e}^{\mathrm{i}\varphi} w$, given two vectors $v, w \in \Gamma(T_{\mathbb{C}})$, we then have

$$L_V W = L_{\mathrm{e}^{\mathrm{i}\varphi} v}(\mathrm{e}^{\mathrm{i}\varphi} w) = \mathrm{e}^{\mathrm{i}\varphi} L_v^{\mathrm{i}\,\mathrm{d}\varphi + \frac{1}{2}\varphi\wedge\mathrm{d}\varphi} w = \mathrm{e}^{\mathrm{i}\varphi}\big([v, w] + \imath_w\imath_v(\mathrm{i}\,\mathrm{d}\varphi + \tfrac{1}{2}\varphi \wedge \mathrm{d}\varphi)\big), \tag{5.53}$$

where we have used the twisted generalised Lie derivative (5.10). The second term must vanish for the right-hand side to be a section of L_3 only. As this is defined for arbitrary v and w, the bundle L_3 is involutive on L_3 if and only if we have a *closed* G_2 structure

$$\text{involutivity of } L_3: \qquad \mathrm{d}\varphi = 0. \tag{5.54}$$

[6] Note that this condition involves $\zeta^{\mathrm{T}}\zeta$ and not $\bar{\zeta}\zeta$, which is what defines the SU(7) structure (see below (5.2)). Given that ψ is of the form (5.23), this condition amounts to requiring that the $1 \in \mathbb{R}$ component of ψ is non-vanishing.

This condition is weaker than a torsion-free G_2 structure (which requires $d \star \varphi = 0$ as well). A theorem due to Bryant states that, like symplectic structures, all closed G_2 structures are equivalent, taking a standard form in a local patch [277].

Now we examine the moment map for this example. To do so, we first need to define the SU(7) structure ψ. Recall that ψ is a section of

$$\begin{aligned}\tilde{K} \simeq \mathbb{R} \oplus \wedge^3 T^* \oplus (T^* \otimes \wedge^5 T^*) \oplus (S^2 T^* \otimes \wedge^7 T^*) \\ \oplus (\wedge^3 T^* \otimes \wedge^6 T^*) \oplus (\wedge^3 T \otimes (\wedge^7 T^*)^3) \oplus \ldots,\end{aligned} \tag{5.55}$$

and also that $V \bullet \psi = 0$ for all $V \in \Gamma(L_3)$. Since $\mathbb{R}$ is the lowest degree term in $\tilde{K}$, we note that, taking $1 \in \Gamma(\tilde{K})$, we must have $v \bullet 1 = 0$ for any vector $v \in \Gamma(T_{\mathbb{C}})$ viewed as a section of $\Gamma(E_{\mathbb{C}})$. Since $L_3 = e^{i\varphi} T_{\mathbb{C}}$ we see this means we can construct ψ as

$$\psi = e^{i\varphi} \cdot 1, \tag{5.56}$$

where the exponential acts on $1 \in \mathbb{R}$ via the adjoint action. The components of ψ have the form

$$\psi \sim (1, \varphi, j\varphi \wedge \varphi, \tilde{g}, \ldots). \tag{5.57}$$

Recall that $s(\psi, \bar{\psi}) \in \Gamma((\det T^*)^3)$, so it has $3 \times 7 = 21$ indices. Given that $\varphi \in \Gamma(\wedge^3 T^*)$, it hence must be degree 7 in φ, meaning the Kähler potential (5.32) is degree 7/3. This is precisely the same scaling as the G_2 Hitchin functional [151, 278] so that, up to an overall constant, we must have

$$\mathcal{K} = \int_M \left(i\, s(\psi, \bar{\psi})\right)^{1/3} \propto \int_M \varphi \wedge \star\varphi. \tag{5.58}$$

One can check this is indeed the case by an explicit calculation. Using the the twisted generalised Lie derivative and invariance of the symplectic form under a complexified $E_{7(7)}$ transformation, we can then calculate the moment map (5.39)

$$\begin{aligned}\mu(V) &= \tfrac{1}{3} \int_M s\left(L_V(e^{i\varphi} \cdot 1), e^{-i\varphi} \cdot 1\right) (i\, s(\psi, \bar{\psi}))^{-2/3} \\ &= \tfrac{1}{3} \int_M s(e^{i\varphi} L_{e^{-i\varphi} V}^{i\,d\varphi + \frac{1}{2}\varphi\wedge d\varphi} 1, e^{-i\varphi} \cdot 1)(i\, s(\psi, \bar{\psi}))^{-2/3} \\ &= \tfrac{1}{3} \int_M s(L_{e^{-i\varphi} V}^{i\,d\varphi + \frac{1}{2}\varphi\wedge d\varphi} 1, e^{-2i\varphi} \cdot 1)(i\, s(\psi, \bar{\psi}))^{-2/3}.\end{aligned} \tag{5.59}$$

As $e^{-i\varphi}$ has no kernel, we can relabel $e^{-i\varphi} V \to V$ to give

$$\mu(e^{i\varphi} V) = \tfrac{1}{3} \int_M s(L_V^{i\,d\varphi + \frac{1}{2}\varphi\wedge d\varphi} 1, e^{-2i\varphi} \cdot 1)(i\, s(\psi, \bar{\psi}))^{-2/3}. \tag{5.60}$$

Given $V = v + \omega + \sigma + \tau$ and $d\varphi = 0$

$$L_V^{i\,d\varphi + \frac{1}{2}\varphi\wedge d\varphi} 1 = L_V 1 = -(d\omega + d\sigma) \cdot 1 = -d\omega - j d\sigma. \tag{5.61}$$

For general $\gamma_{mnp} \in \Gamma(\wedge^3 T^*)$ and $\pi_{m,n_1\dots n_5} \in \Gamma(T^* \otimes \wedge^5 T^*)$ we have

$$\begin{aligned} s(\gamma + \pi, &\mathrm{e}^{-2\mathrm{i}\varphi} \cdot 1)(\mathrm{i}\, s(\psi, \bar{\psi}))^{-2/3}_{m_1\dots m_7} \\ &= \text{const} \times \gamma_{[m_1 m_2 m_3}(\star\varphi)_{m_4 m_5 m_6 m_7]} + \text{const} \times g^{np}\pi_{n,p[m_1 m_2 m_3 m_4}\varphi_{m_5 m_6 m_7]}, \end{aligned} \tag{5.62}$$

where, rather than evaluate the expression directly, we have used the facts that it must be linear in γ and π and a top form, and that the only G_2-invariant tensors are φ, $\star\varphi$, the metric g and its inverse. However for $\pi = j\mathrm{d}\sigma$ the second term vanishes. We thus have[7]

$$\begin{aligned} \mu(\mathrm{e}^{\mathrm{i}\varphi} V) &\propto \int_M \mathrm{d}\omega \wedge \star\varphi \\ &\propto \int_M \omega \wedge \mathrm{d} \star \varphi, \end{aligned} \tag{5.63}$$

where we have assumed that M is compact to integrate by parts.

$$\text{vanishing of moment map:} \quad \mathrm{d} \star \varphi = 0, \tag{5.64}$$

so that the G_2 structure must be torsion free.

We can extend this example to include fluxes by including them in the complexified transformation as

$$L_3 = \mathrm{e}^{\tilde{A}+A}\mathrm{e}^{\mathrm{i}\varphi}T = \mathrm{e}^{\tilde{A}-\frac{1}{2}\mathrm{i}A\wedge\varphi+A+\mathrm{i}\varphi}T, \tag{5.65}$$

where A and $\tilde{A}$ are three- and six-form potentials. The real $E_{7(7)}$ transformation by $A + \tilde{A}$ amounts to turning on four-form and seven-form fluxes, given by

$$F = \mathrm{d}A, \qquad \tilde{F} = \mathrm{d}\tilde{A} - \tfrac{1}{2}A \wedge F. \tag{5.66}$$

The involutivity condition is now

$$[v, w] + \imath_w \imath_v (F + \mathrm{i}\,\mathrm{d}\varphi + \tilde{F} + \tfrac{1}{2}\varphi \wedge \mathrm{d}\varphi - \mathrm{i}F \wedge \varphi) \in \Gamma(TM), \tag{5.67}$$

which holds if and only if

$$\mathrm{d}\varphi = F = \tilde{F} = 0. \tag{5.68}$$

In other words, involutivity of L_3 forces the G_2 structure to be closed and the fluxes to vanish. Note that one could include a warp factor by including e^{Δ} in the definition of L_3 – one would then find that involutivity also forces the warp factor to be constant. Since all the fluxes vanish, the twisted generalised Lie derivative is equal to the ordinary Lie derivative and the analysis of the $\mu = 0$ condition is exactly as before, that is, it simply implies $\mathrm{d} \star \varphi = 0$, and the G_2 structure is integrable. We have thus reproduced the standard conditions for a supersymmetric compactification of M-theory on a G_2 manifold. For the SU(7) structure there is strictly one extra degree of freedom, since we can always rescale ψ by a complex constant. As we discussed

[7] Note that an analogous argument gives the same expression for the variation of the Kähler potential for $\delta\varphi = \mathrm{d}\omega$. (This gives a reason for why the coefficient of the first term in (5.62) cannot vanish; one knows the generic variation of the Hitchin function is non-zero.) As we will discuss in section 5.3.2, this reflects the fact that the vanishing of the moment map is the same as the extremisation of the Kähler potential.

at the end of section 5.1.2, this rescaling is not physical.

Recall that SU(7) structures are equivalent if they differ by generalised diffeomorphisms. The gauge symmetries will simply shift

$$A \to A + \mathrm{d}\omega, \qquad \tilde{A} \to \tilde{A} + \mathrm{d}\sigma, \tag{5.69}$$

thus the physical gauge degrees of freedom parameterise the de Rham cohomology classes $H^3_{\mathrm{d}}(M, \mathbb{R})$ and $H^6_{\mathrm{d}}(M, \mathbb{R})$. The conventional diffeomorphisms on the other hand simply relate diffeomorphic G_2 structures. Locally the moduli space of integrable G_2 structures is diffeomorphic to a open set of $H^3_{\mathrm{d}}(M, \mathbb{R})$ (see for example [85]). Furthermore $H^6_{\mathrm{d}}(M, \mathbb{R}) = 0$. As we will show in section 5.4, all deformations of the SU(7) structure either deform the G_2 structure φ, deform A in $H^3_{\mathrm{d}}(M)$ or correspond to rescaling ψ. Thus, dropping the non-physical rescaling, the physical moduli space is

$$\mathcal{M}_\psi / \mathbb{C}^* \simeq H^3_{\mathrm{d}}(M, \mathbb{C}) \qquad \text{(locally)}, \tag{5.70}$$

with the Kähler metric given by (5.43), as in [270–272]. We will comment more on a formal way to treat this moduli problem in section 5.4.

In summary, we have shown that by embedding the problem in $\mathrm{E}_{7(7)} \times \mathbb{R}^+$ generalised geometry, the G_2 manifold has an intriguing reinterpretation, as a sort of generalised complex structure. There is an involutive complex subbundle whenever $\mathrm{d}\varphi = 0$, and the final condition $\mathrm{d} \star \varphi = 0$ comes from a moment map.

5.2.2 GMPT Structures in Type II

The GMPT solutions give $\mathcal{N} = 1$ compactifications of type II supergravities and were first analysed in [202] and further studied in [203]. While the solutions are not completely general,[8] they do cover a large class of compactifications in which the internal manifold has an SU(3) structure, an SU(2) structure, or an intermediate case where the two SU(3) structures can degenerate. The key observation of [202] was that these three cases are examples of SU(3) $\times$ SU(3) structures on the generalised tangent bundle $E_{\mathrm{O}(6,6)} = T \oplus T^*$ and can all be described as generalised Calabi–Yau manifolds admitting two pure spinors [164, 165]. We begin with a brief review of the key aspects of the GMPT solutions before embedding them into the SU(7) structures we have described above. We will use this formulation of the solutions to find their moduli in section 5.4.4.

The GMPT solutions admit two non-vanishing, compatible pure spinors $\{\Phi_+, \Phi_-\}$ with associated generalised complex structures $\{\mathcal{J}_+, \mathcal{J}_-\}$ satisfying

$$(\Phi_+, \slashed{V}\Phi_-) = (\Phi_+, \slashed{V}\bar{\Phi}_-) = 0 \quad \forall\, V \in E_{\mathrm{O}(6,6)} \quad \Leftrightarrow \quad [\mathcal{J}_+, \mathcal{J}_-] = 0, \tag{5.71}$$

where $[\cdot, \cdot]$ is the usual commutator and the slash denotes the Clifford action, as it will for the remainder of this section. This is a special case of the generalised Kähler structures defined

[8] The construction requires that the two internal spinors $\{\eta^1, \eta^2\}$ in (5.3) are nowhere vanishing. An example that falls outside of this classification is an NS5-brane wrapping a Calabi–Yau. As shown in [187], this class of solution can be embedded within exceptional generalised geometry.

in [279] and gives an $\mathrm{SU}(3)\times\mathrm{SU}(3)$ structure. The two pure spinors are constructed as bilinears of the Killing spinors $\{\zeta_1^\pm,\zeta_2^\pm\}$ given in (5.3). The Killing spinor equations in terms of $\Phi^\pm$ were first given in [202]. Here it will be useful to use an equivalent form derived by Tomasiello [203]

$$\mathrm{d}\Phi_\pm = 0, \qquad F = -8\,\mathrm{d}^{\mathcal{J}_\pm}(\mathrm{e}^{-3\Delta}\,\mathrm{im}\,\Phi_\mp), \tag{5.72}$$

$$\mathrm{d}(\mathrm{e}^{-\Delta}\,\mathrm{re}\,\Phi_\mp) = 0, \tag{5.73}$$

where $\mathrm{d}^{\mathcal{J}} = [\mathrm{d},\mathcal{J}]$, the upper/lower sign is for type IIA/B, Δ is the warp factor in the string frame and F is the Ramond–Ramond flux. It is easy to show that, if one assumes the generalised $\partial\bar\partial$-lemma (G.5) [280], then (5.72) implies that F is in the trivial cohomology class. The spinors are normalised so that

$$(\Phi_+,\bar\Phi_+) = (\Phi_-,\bar\Phi_-) = \tfrac{1}{8}\mathrm{e}^{6\Delta-2\hat\varphi}\,\mathrm{vol}, \tag{5.74}$$

where $(\cdot,\cdot)$ is the Mukai pairing (A.21), $\hat\varphi$ is the dilaton and vol is the volume form defined by the string-frame metric. Note that in [203] the twisted differential $\mathrm{d}_H = \mathrm{d} - H\wedge$ is used. Here we will use the convention that the B-field is included in the definition of the spinors and RR flux (that is they are twisted by e^{-B} relative to those in [203]) and hence the usual differential d appears.

We now show how to embed these solutions into the framework of generalised SU(7) structures. We start by defining L_3 as[9]

$$L_3 = \mathrm{e}^{C+8\,\mathrm{i}\,\mathrm{e}^{-3\Delta}\,\mathrm{im}\,\Phi_\mp}(L_1^{\mathcal{J}_\pm}\oplus\mathcal{U}_{\mathcal{J}_\pm}). \tag{5.75}$$

Here the upper/lower signs correspond to type IIA/B respectively, $L_1^{\mathcal{J}_\pm}\subset E_{\mathrm{O}(6,6)\mathbb{C}}\simeq(T\oplus T^*)_\mathbb{C}$ is the $+\mathrm{i}$-eigenspace of $\mathcal{J}_\pm$, $\mathcal{U}_{\mathcal{J}_\pm}$ is the pure spinor line bundle defined by $\mathcal{J}_\pm$ and C is the (global) polyform potential for the RR flux F. Note that in writing (5.75), we are implicitly using an embedding of the $\mathrm{O}(6,6)$ structures into the $\mathrm{E}_{7(7)}$ generalised tangent bundle and adjoint bundle: this is given in appendix A.6 for type IIB.[10] We will focus on type IIB for definiteness but analogous results hold in IIA with the appropriate embedding. It is relatively straightforward to check that L_3 satisfies the necessary and sufficient conditions to define an almost exceptional complex structure.

Now we turn to the involutivity condition. We will show first that the untwisted bundle $L_1^{\mathcal{J}_-}\oplus\mathcal{U}_{\mathcal{J}_-}$ is involutive if and only if $\mathcal{J}_-$ is integrable. One can check that $\mathrm{i}\,s(V,\bar W)$ is not positive definite, thus it defines only an exceptional Dirac structure, but not an exceptional complex structure. We find that the modified bundle L_3 is involutive provided an extra condition on the twisting factor $C+8\,\mathrm{i}\,\mathrm{e}^{-3\Delta}\,\mathrm{im}\,\Phi_-$ is satisfied. Let

$$V = W + \alpha\,\Phi_- \in \Gamma(L_1^{\mathcal{J}_-}\oplus\mathcal{U}_{\mathcal{J}_-}), \tag{5.76}$$

[9] Such a procedure for going from generalised to exceptional complex structures was originally formulated in an $\mathrm{E}_{6(6)}$ context by two of the current authors (AA and DW) with Michela Petrini and Edward Tasker [260].

[10] The powers of Δ in the normalisation (5.74) imply that $\Phi_\pm$ are sections of a weight-three bundle under the $\mathbb{R}^+$ action. The adjoint bundle is weight-zero, hence the $\mathrm{e}^{-3\Delta}$ factor in (5.75).

where $W \in \Gamma(L_1^{\mathcal{J}_-})$ and $\alpha \in C^\infty(M,\mathbb{C})$, and similarly for V'. Requiring

$$L_V V' = L_{W+\alpha\Phi_-}(W' + \alpha'\Phi) \in \Gamma(L_1^{\mathcal{J}_-} \oplus \mathcal{U}_{\mathcal{J}_-}), \tag{5.77}$$

implies first that

$$L_W W' \in L_1^{\mathcal{J}_-}, \qquad \forall\, W, W' \in L_1^{\mathcal{J}_-}, \tag{5.78}$$

that is, the generalised complex structure $\mathcal{J}_-$ associated to Φ_- must be integrable. From (C.32) and $\not{W}\Phi_- = \not{W}'\Phi_- = 0$ we then immediately have

$$\begin{aligned} L_W(\alpha'\Phi_-) &= (\mathcal{L}_v\alpha')\Phi_- + \alpha' L_W\Phi_- = \left\langle W, \mathrm{d}\alpha' + 2A\right\rangle \Phi_- \in \mathcal{U}_{\mathcal{J}_-}, \\ L_{\alpha\Phi_-}W' &= -\mathrm{d}(\alpha\Phi_-)\cdot W' = -\left\langle W', \mathrm{d}\alpha + 2A\right\rangle \Phi_- \in \mathcal{U}_{\mathcal{J}_-}, \end{aligned} \tag{5.79}$$

as required (in the second line $\mathrm{d}(\alpha\Phi_-)$ acts via the $\mathrm{E}_{7(7)}$ adjoint action). For the final term we have

$$L_{\alpha\Phi_-}(\alpha'\Phi_-) = -\mathrm{d}(\alpha\Phi_-)\cdot(\alpha'\Phi_-) = -\alpha'[(\not{\mathrm{d}}\alpha + \not{A})\Phi_-]\cdot\Phi_- = 0 \tag{5.80}$$

identically, as can be seen simply by counting the $\mathcal{J}_-$ charge. Hence we see

$$\text{involutive } L_1^{\mathcal{J}_-} \oplus \mathcal{U}_{\mathcal{J}_-} \qquad \Leftrightarrow \qquad \text{integrable } \mathcal{J}_-. \tag{5.81}$$

We now define

$$\Sigma = C + 8\,\mathrm{i}\,\mathrm{e}^{-3\Delta}\,\mathrm{im}\,\Phi_+, \tag{5.82}$$

so that $\mathrm{e}^\Sigma V \in \Gamma(L_3)$ if $V \in \Gamma(L_1^{\mathcal{J}_-} \oplus \mathcal{U}_{\mathcal{J}_-})$. We then have[11]

$$L_{\mathrm{e}^\Sigma V}(\mathrm{e}^\Sigma V') = \mathrm{e}^\Sigma L_V^{\mathrm{d}\Sigma} V' = \mathrm{e}^\Sigma\left[L_V V' - (\not{W}\mathrm{d}\Sigma)\cdot V'\right], \tag{5.83}$$

where $L^{\mathrm{d}\Sigma}$ is the twisted generalised Lie derivative (for type IIB, see [187]) and $(\not{W}\mathrm{d}\Sigma)$ acts on V' via the $\mathrm{E}_{7(7)} \times \mathbb{R}^+$ adjoint action. To be involutive we need the term in brackets to be an element of $L_1^{\mathcal{J}_-} \oplus \mathcal{U}_{\mathcal{J}_-}$. Since the first term is differential and the second algebraic in V and V' this can only be true if each term is separately a section of $L_1^{\mathcal{J}_-} \oplus \mathcal{U}_{\mathcal{J}_-}$. We have already analysed the first term. For the second term it means $\not{W}\mathrm{d}\Sigma \in \Gamma(\mathrm{ad}\,\tilde{F})$ must stabilise $L_1^{\mathcal{J}_-} \oplus \mathcal{U}_{\mathcal{J}_-}$. For the W component we have

$$-(\not{W}\mathrm{d}\Sigma)\cdot W' = \not{W}'\not{W}\mathrm{d}\Sigma. \tag{5.84}$$

If we will split the spinor bundle S^- into its $\mathcal{J}_-$ ni-eigenspaces, S_n, where $n = -3, -1, 1, 3$, and denote by a subscript n the projection of a polyform to S_n, this implies

$$\not{W}'\not{W}\mathrm{d}\Sigma \in S_3 \quad \Leftrightarrow \quad (\mathrm{d}\Sigma)_{-1} = (\mathrm{d}\Sigma)_{-3} = 0. \tag{5.85}$$

Combining these conditions with their complex conjugates we find

$$F = -8\,\mathrm{d}^{\mathcal{J}_-}(\mathrm{e}^{-3\Delta}\,\mathrm{im}\,\Phi_+). \tag{5.86}$$

[11] Note that the generalised Lie derivative is antisymmetric when L_3 is involutive, so checking involutivity with the generalised Lie derivative is equivalent to checking it with the Courant bracket. The condition that $L_3 \times_N L_3 = 0$ ensures this.

Moreover, just counting the $\mathcal{J}_-$ charges, we see that this condition is enough to imply $(\not{W} \mathrm{d}\Sigma) \cdot \Phi_- = 0$. Taken together, we see that the first two equations in (5.72) are necessary and sufficient conditions for involutivity of L_3:

$$\text{involutivity of } L_3 : \qquad \mathrm{d}\Phi_- = \not{A}\Phi_-, \qquad F = -8\, \mathrm{d}^{\mathcal{J}_-}(\mathrm{e}^{-3\Delta} \operatorname{im} \Phi_+). \tag{5.87}$$

As expected, involutivity does not provide a full solution to the supersymmetry equations. Instead we find that it implies essentially the first two (5.72) of the three conditions found in [203]. From section 5.1.2 we know that these equations only imply the vanishing of part of the intrinsic torsion, and that the vanishing of the rest of the intrinsic torsion, here given by the final equation $\mathrm{d}(\mathrm{e}^{-\Delta} \operatorname{re} \Phi_+) = 0$, is implied by the vanishing of the moment map (5.39). In other words we have

$$\text{vanishing of moment map :} \qquad A = 0, \qquad \mathrm{d}(\mathrm{e}^{-\Delta} \operatorname{re} \Phi_+) = 0, \tag{5.88}$$

that is, the generalised complex structure $\mathcal{J}_-$ is promoted to a generalised Calabi–Yau structure, and in addition the third condition of [203] is satisfied. Since the full set of equations (5.72) and (5.73) are equivalent to supersymmetry, the proof in [184, 185] that supersymmetry is equivalent to vanishing intrinsic torsion is sufficient for these last conditions to to indeed be equivalent to the vanishing of the the moment map. Thus, rather than give all the details, let us simply sketch below how the relevant conditions arise.

Since it defines an exceptional polarisation, the $L_1^{\mathcal{J}_-} \oplus \mathcal{U}_{\mathcal{J}_-}$ subbundle will have an associated singlet in the $\tilde{K}_{\mathbb{C}}$ bundle, just as for an almost exceptional complex structure. Given the decomposition under $\mathrm{O}(6,6) \times \mathrm{SL}(2,\mathbb{R}) \subset \mathrm{E}_{7(7)}$

$$\mathbf{912} = (\mathbf{352}', \mathbf{1}) + (\mathbf{220}, \mathbf{2}) + (\mathbf{12}, \mathbf{2}) + (\mathbf{32}, \mathbf{3}), \tag{5.89}$$

the only $\mathrm{SU}(3,3) \subset \mathrm{O}(6,6)$ singlet appears in the $\mathbf{32}$ representation, given by, up to $\det T^*$ factors, Φ_- itself. In fact, the $\mathbb{R}^+$ weight of $\tilde{K}$ is such that singlet is simply $\Phi_- \in \Gamma(\tilde{K}_{\mathbb{C}})$. It has the property that $V \bullet \Phi_- = 0$ for all $V \in \Gamma(L_1^{\mathcal{J}_-} \oplus \mathcal{U}_{\mathcal{J}_-})$. Given the twisting of L_3 it is then easy to see that the corresponding $\mathrm{SU}(7)$ structure is simply

$$\psi = \mathrm{e}^{\Sigma} \cdot \Phi_- = \mathrm{e}^{C + 8\,\mathrm{i}\, \mathrm{e}^{-3\Delta} \operatorname{im} \Phi_+} \cdot \Phi_-, \tag{5.90}$$

where, since Φ_- is already naturally the section of a weight-three bundle under the $\mathbb{R}^+$ action, we do not expect any additional powers of e^{Δ}. Turning to the moment map, we can repeat the same steps in the analysis for G_2 structures in section 5.2.1 to derive

$$\begin{aligned} \mu(\mathrm{e}^{\Sigma} V) &= \tfrac{1}{3} \int_M s(L_V^{\mathrm{d}\Sigma} \Phi_-, \mathrm{e}^{-\Sigma} \mathrm{e}^{\bar{\Sigma}} \cdot \bar{\Phi}_-)(\mathrm{i}\, s(\psi, \bar{\psi}))^{-2/3} \\ &= \tfrac{1}{3} \int_M s(L_V^{\mathrm{d}\Sigma} \Phi_-, \mathrm{e}^{-2\mathrm{i} \operatorname{im} \Sigma} \cdot \bar{\Phi}_-)(\mathrm{i}\, s(\psi, \bar{\psi}))^{-2/3}, \end{aligned} \tag{5.91}$$

where in the second line we have use the property that we can always choose a gauge for C such that Σ and $\bar{\Sigma}$ commute. We also have

$$L_V^{\mathrm{d}\Sigma}\Phi_- = L_V\Phi_- - (V\!\!\!/\,\mathrm{d}\Sigma)\cdot\Phi_- = L_V\Phi_- = L_Z\Phi_- - (\mathrm{d}\Lambda_- + \mathrm{d}\tilde{\Lambda})\cdot\Phi_-, \tag{5.92}$$

where we have used the fact, derived from the involutivity condition, that $\mathrm{d}\Sigma$ stabilises $L_1^{\mathcal{J}_-}\oplus\mathcal{U}_{\mathcal{J}_-}$ and hence the singlet $\Phi_-\in\Gamma(\tilde{K})$, and in the last expression have split $V = Z+\Lambda_+ +(\tilde{\Lambda}+\tau)$ using the decomposition (A.85). One can argue that the terms that survive in the moment map are of the form

$$\mu(V)\sim \text{const}\int(L_Z\Phi_-,\bar{\Phi}_-) + \text{const}\int(\mathrm{d}\Lambda_-,\mathrm{e}^{-\Delta}\,\mathrm{re}\,\Phi_+). \tag{5.93}$$

This form follows from keeping track of the $\mathrm{U}(1)\subset\mathrm{SL}(2,\mathbb{R})$ charge in the $\mathrm{O}(6,6)\times\mathrm{SL}(2,\mathbb{R})\subset\mathrm{E}_{7(7)}$ decomposition, noting the $\mathbb{R}^+$ weight to get the correct e^{Δ} factor, and recalling the algebraic relations between $\mathcal{J}_\pm$ and $\Phi_\pm$. In particular, the U(1) charge implies that the second term arises from the third-order term in $\mathrm{im}\,\Phi_+$ exponential. As was first noticed in [164], one can determine the real part of a pure spinor from the imaginary part[12] as a third-order expression in $\mathrm{im}\,\Phi_+$, hence the appearance of $\mathrm{re}\,\Phi_+$. The first term vanishes if and only if $A=0$, while, integrating by parts on the second term gives $\mathrm{d}(\mathrm{e}^{-\Delta}\,\mathrm{re}\,\Phi_+)=0$, the final condition in (5.73).

Calabi–Yau as $\mathcal{N}=1$

It is straightforward to describe the usual Calabi–Yau compactifications in our formalism. While these actually give $\mathcal{N}=2$ compactifications, we can still write them in our $\mathcal{N}=1$ language. In this case, the internal spinors are equal, $\zeta_1=\zeta_2$, and can be used to construct a complex three-form Ω and a real two-form ω. Given vanishing flux one finds that the dilaton and warp factor must be constant. The Killing spinor equations then imply

$$\mathrm{d}\Omega = 0,\qquad \mathrm{d}\omega = 0. \tag{5.94}$$

These objects can be embedded as generalised complex structures as

$$\Phi_- = \tfrac{1}{8}\mathrm{e}^{3\Delta-\hat{\varphi}}\Omega,\qquad L_1^{\mathcal{J}_-} = T^{0,1}\oplus T^{*1,0}, \tag{5.95}$$

$$\Phi_+ = \tfrac{1}{8}\mathrm{e}^{3\Delta-\hat{\varphi}}\mathrm{e}^{\mathrm{i}\,\omega},\qquad L_1^{\mathcal{J}_+} = T\lrcorner(1-\mathrm{i}\,\omega). \tag{5.96}$$

where these are chosen such that they have the correct normalisation according to (5.72) and (5.73).

Focusing on type IIB, we take

$$L_3 = \mathrm{e}^{\mathrm{i}\,\mathrm{e}^{-\hat{\varphi}}(\omega-\frac{1}{6}\omega\wedge\omega\wedge\omega)}(T^{0,1}\oplus T^{*1,0}\oplus\mathbb{C}\,\mathrm{e}^{3\Delta-\varphi}\Omega). \tag{5.97}$$

Integrability of L_3 then implies

$$\mathrm{d}\Omega = A\wedge\Omega,\qquad \mathrm{d}^I\omega = \mathrm{d}^I\hat{\varphi}\wedge\omega, \tag{5.98}$$

[12]In fact, in [164] they show that $\mathrm{im}\,\Phi$ can be obtained from $\mathrm{re}\,\Phi$. However the converse statement is also true.

where I is the (integrable) complex structure associated to Ω and $\mathrm{d}^I = [\mathrm{d}, I]$. Clearly these are not the full set of integrability conditions for Calabi–Yau. Imposing the vanishing of the moment map, we find that $A = \mathrm{d}\Delta = \mathrm{d}\hat{\varphi} = 0$ and hence the above become

$$\mathrm{d}\Omega = 0, \qquad \mathrm{d}^I\omega = 0 \quad \Leftrightarrow \quad \mathrm{d}\omega = 0. \tag{5.99}$$

Finally note that we could have instead taken the pure spinors to be

$$\Phi_- = \tfrac{1}{8}\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{3\Delta-\hat{\varphi}}\Omega, \qquad \Phi_+ = \tfrac{1}{8}\mathrm{e}^{\mathrm{i}\beta}\mathrm{e}^{3\Delta-\hat{\varphi}}\mathrm{e}^{\mathrm{i}\omega}, \tag{5.100}$$

where α, β are two real constants. This would not change the normalisation condition or the generalised metric, but would affect what we mean by the real and imaginary parts of $\Phi_\pm$ and hence would rearrange which terms appear in the involutivity and moment map conditions. This amounts to choosing which $\mathcal{N} = 1 \subset \mathcal{N} = 2$ we want to make manifest.

5.3 The Superpotential, the Kähler Potential and Extremisation

As we discussed in section 5.1.2, the existence of the Kähler metric on the space $\mathcal{Z}$ of generalised SU(7) structures is really just a reflection of fact that one can rewrite the full ten- or eleven-dimensional supergravity in $D = 4$, $\mathcal{N} = 1$ language, in line with the $\mathcal{N} = 2$ discussion of [156,187,251,266]. The internal degrees of freedom parameterised by ψ lie in chiral multiplets and hence parameterise a Kähler manifold. By including the unphysical constant overall scaling and phase of ψ we are in the superconformal formulation of the supergravity. The D-term (or more strictly the Killing prepotential $\mathcal{P}$) is just the moment map μ for the action of the GDiff gauge symmetry, with $V \in \Gamma(E)$ giving a parameterisation of $\mathfrak{gdiff}$:

$$\begin{aligned} &\text{Kähler potential :} && \mathcal{K} = \int_M \left(\mathrm{i}\, s(\psi, \bar{\psi})\right)^{1/3}, \\ &D\text{-term :} && \mathcal{P} = \tfrac{1}{3}\int_M s(L_V\psi, \bar{\psi})(\mathrm{i}\, s(\psi, \bar{\psi}))^{-2/3}. \end{aligned} \tag{5.101}$$

To complete the description of the chiral multiplet sector we need the generic superpotential $\mathcal{W}$ in terms of ψ. This was first discussed in [180]. The $D = 4$, $\mathcal{N} = 1$ supersymmetry conditions are the vanishing of the D-term, namely $\mathcal{P} = 0$, and the superpotential conditions $\delta\mathcal{W}/\delta\psi = \mathcal{W} = 0$. In terms of our previous discussion this means that the superpotential conditions should imply the involutivity of L_3. A missing ingredient thus far in our discussion is to show that involutivity is a holomorphic condition in terms ψ. In this section, we will extend the analysis of [180] to give the expression for $\mathcal{W}$ for a generic $\mathcal{N} = 1$ background. We will see that it is indeed a holomorphic function of ψ and furthermore show that, in the special cases of a G_2 structure and GMPT, it matches the standard expressions in the literature.

Recall also that the moment map picture implies that formally the moduli space of integrable SU(7) structures can be viewed as as a quotient by the complexification $\mathrm{GDiff}_{\mathbb{C}}$ of the generalised diffeomorphism group. As for the complex and generalised complex structure cases, the complexification does not really exist as a group, and instead what is really meant is modding out by the complexification of the orbits generated by the action of GDiff. The other focus of this section is to investigate this action and show that it gives a (generalised) reinterpretation of Hitchin's picture of integrable G_2 structures as extremising a particular functional. We will also comment very briefly on how this might suggest notions of stability for G_2 manifolds and their generalisations.

5.3.1 The Superpotential

In this section we will derive a general form for the superpotential $\mathcal{W}$, building on work on superpotentials in the presence of flux first proposed in [234,235], and the generalised geometry expressions given in [3]. A natural conjecture is that $\mathcal{W}$ is given by the singlet part of the intrinsic torsion for the SU(7) structure integrated over the internal manifold. As we will see, one can pick out this singlet by a projection that is holomorphic in terms of ψ, meaning that the superpotential is a holomorphic function of ψ, justifying ψ as the holomorphic coordinate on $\mathcal{Z}$.

As mentioned above we expect the supersymmetry conditions $\delta\mathcal{W}/\delta\psi = \mathcal{W} = 0$ to imply the involution condition on L_3. We note that the variations of the SU(7) structure ψ transform as $\mathbf{1}_7$, $\mathbf{7}_3$ and $\overline{\mathbf{35}}_5$, and so $\delta\mathcal{W}/\delta\psi = 0$ will constrain the dual $\mathbf{1}_{-7}$, $\overline{\mathbf{7}}_{-3}$ and $\mathbf{35}_{-5}$ components of the intrinsic torsion. This means $\delta\mathcal{W}/\delta\psi = 0$ implies $\mathcal{W} = 0$ (as $\mathcal{W}$ itself is the singlet) and furthermore is a slightly stronger condition than L_3 being involutive, which only constrained the $\mathbf{1}_{-7}$ and $\mathbf{35}_{-5}$ components.

Before turning to the superpotential itself, it is useful to show that one can rephrase involutivity as a holomorphic condition on ψ. Suppose $V \in \Gamma(L_3)$ and D is a compatible generalised connection, that is $D\psi = 0$. From the definition (5.11) we find

$$L_V\psi = -T(V)\cdot\psi \qquad \text{for } V \in \Gamma(L_3). \tag{5.102}$$

Note that this expression is linear in V. For any other $\mathbb{R}^+$ weight we would have gotten an additional factor of the form $(D\cdot V)\psi$ where $D\cdot V = D_M V^M$, and hence a non-linear expression. Since $L_V\psi$ is independent of D, only the intrinsic torsion contributes to $T(V)\cdot\psi$. From the $U(1)\times SU(7)$ representations it is easy to check that the $\mathbf{1}_{-7}$, $\overline{\mathbf{7}}_{-3}$, and $\mathbf{35}_{-5}$ components of the intrinsic torsion (5.24) appear, precisely the components in $\delta\mathcal{W}/\delta\psi$. This gives us an alternative formulation of the involutivity condition[13] (i.e. the vanishing of the $\mathbf{1}_{-7}$ and $\mathbf{35}_{-5}$ components)[14]:

$$\text{involutive } L_3 \qquad \Leftrightarrow \qquad L_V\psi = A(V)\,\psi \quad \forall\, V\in L_3, \tag{5.103}$$

where $A \in \Gamma(L_3^*)$ is the $\overline{\mathbf{7}}_{-3}$ component of the SU(7) intrinsic torsion, and $A(V) = A_M V^M$ is just the natural pairing between sections of E^* and E. We also see that we expect

$$\frac{\delta\mathcal{W}}{\delta\psi} = 0 \qquad \Leftrightarrow \qquad L_V\psi = 0 \quad \forall\, V\in L_3. \tag{5.104}$$

[13] Note that in the conventional and generalised complex structure cases we could equally well have formulated the conditions (C.5) and (C.32) as $\mathcal{L}_V\Omega = (\imath_V A)\Omega$ and $L_V\Phi = 2\langle V, A\rangle\,\Phi$ for all $V\in\Gamma(L_1)$.

[14] Note that relations of this form were first noted in the context of integrable structures in $E_{6(6)}$ generalised geometry by Edward Tasker (private communication).

In analogy with the complex structure and generalised complex structure cases, we expect that we can always take a ψ satisfying the involutive condition and rescale by a complex function $\psi' = f\psi$ so that the stronger superpotential condition is satisfied.

Crucially both of these conditions are linear in V and so can be viewed as a holomorphic expressions in ψ. (Note from (5.20) that L_3 is fixed by $V \bullet \psi = 0$ and so also only depends holomorphically on ψ.) If we had chosen a structure ψ' with a different $\mathbb{R}^+$-weight we would have had an additional $(D \cdot V)\psi'$ term. For the involutivity condition we could still have phrased the condition in the holomorphic form $L_V \psi' \propto \psi'$, however the $\delta\mathcal{W}/\delta\psi' = 0$ condition would not be holomorphic because it would have to be written as $L_V[(\mathrm{i}s(\psi', \bar{\psi}')^p \psi'] = 0$ for some suitable power p. Thus we anticipate that the superpotential $\mathcal{W}$ is a holomorphic function only if we take ψ transforming in $\mathbf{912_3}$.

Returning to the definition of the superpotential, why is it natural to conjecture that it is the singlet torsion $\mathbf{1}_{-7}$? Consider the AdS case for a moment. We know from [184] that the cosmological constant appears as a singlet of the intrinsic torsion when decomposed under SU(8) and this descends to the singlet for the SU(7) structure (since there is only one singlet). The supersymmetry conditions for an AdS background include the vanishing of derivatives of the superpotential (the F-terms) but the superpotential itself does not vanish. Instead, requiring the superpotential to vanish is the final condition for a Minkowski solution. Thus it is reasonable that the superpotential itself is simply the singlet of the torsion.

To see this more concretely, we conjecture

$$\mathcal{W} := \int_M W \sim \int_M \mathrm{i}\, s(\psi, T), \tag{5.105}$$

where T is the intrinsic torsion of the structure. The symplectic product with ψ projects onto the singlet component (specifically the $\mathbf{1}_{-7}$ component). We also note that ψ is weight-3 and T is weight-(-1) with respect to the $\mathbb{R}^+$ action. This means $\mathrm{i}\, s(\psi, T)$ is weight-2 and hence is a volume form which can be integrated over the manifold. From (5.102) we know that the $\mathbf{1}_{-7}$ component of the torsion is a holomorphic function of ψ, and hence the superpotential is holomorphic.

We can make the ψ dependence more manifest as follows. It was shown in [180], using the Killing spinor equations, that W can be written as[15]

$$(D \times_{\mathrm{ad}} \psi) \cdot \psi \sim W\psi, \tag{5.106}$$

where D is now a torsion-free SU(8) connection (not SU(7)), $\times_{\mathrm{ad}}$ is a projection to the adjoint representation $\mathbf{133}$, so that $D \times_{\mathrm{ad}} \psi$ transforms in the $\mathbf{133_2}$ representation, and W is the desired singlet component of the intrinsic torsion of the structure defined by ψ. Clearly we can project onto W by calculating

$$\frac{s(\bar{\psi}, (D \times_{\mathrm{ad}} \psi) \cdot \psi)}{s(\bar{\psi}, \psi)} \sim W. \tag{5.107}$$

At first sight, this appears to depend on $\bar{\psi}$ and so will not be holomorphic on $\mathcal{Z}$. However, this

[15] Technically, in [180] a specific choice of the connection D was taken. We show that the operator appearing here is independent of this choice at the end of this section.

apparent dependence factors out. Consider an infinitesimal variation of the structure

$$\begin{aligned} \delta\psi &\sim c\,\psi + a\cdot\psi + \tilde{a}\cdot\psi, \\ \delta\bar{\psi} &\sim \bar{c}\,\bar{\psi} + \bar{a}\cdot\bar{\psi} + \bar{\tilde{a}}\cdot\bar{\psi}, \end{aligned} \tag{5.108}$$

where we are acting with the Lie algebra $E_{7(7)}\oplus\mathbb{R}$. Decomposing under SU(7) as in (5.16), c is a complex singlet coming from $\mathbf{1}_0$ and the $\mathbb{R}$ action, while a and $\tilde{a}$ transform in $\mathbf{7}_{-4}$ and $\overline{\mathbf{35}}_{-2}$ representations respectively. Of the antiholomorphic parameters $(\bar{c}, \bar{a}, \bar{\tilde{a}})$, only $\bar{c}$ can appear in the relevant projection

$$s(\delta\bar{\psi}, (D\times_{\text{ad}}\psi)\cdot\psi) \sim W\, s(\delta\bar{\psi}, \psi) = \bar{c}\, W\, s(\bar{\psi}, \psi) \tag{5.109}$$

as the parts involving $\bar{a}$ and $\bar{\tilde{a}}$ are non-singlet and thus projected out. Thus we are left with only a $\mathbb{C}^*$ scaling of $\bar{\psi}$ by the antiholomorphic factor $e^{\bar{c}}$. However this scaling clearly factors out of (5.107) and hence W is indeed a holomorphic function of ψ.

In conclusion, the general expression for the superpotential of a generic $D=4$, $\mathcal{N}=1$ background up to an overall constant is

$$\mathcal{W} = \int_M W \sim \int_M \frac{s(\bar{\psi}, (D\times_{\text{ad}}\psi)\cdot\psi)}{s(\bar{\psi},\psi)} \sim \int_M \operatorname{Tr}\left(J, (D\times_{\text{ad}}\psi)\right). \tag{5.110}$$

We have included an alternative expression in a slightly simpler form that has the benefit of being easier to calculate explicitly. However, it is less obvious to see that it does not depend on antiholomorphic variations of the structure.

For completeness we should check that our expressions for W are well defined, in the sense that they do not depend on the parts of the torsion-free SU(8) connection D which are not determined by the generalised metric G. These undetermined components form the $\mathbf{1280}+\overline{\mathbf{1280}}$ parts of the connection, and they *do* appear in the unprojected operator $D\times_{\text{ad}}\psi$, which thus depends on the choice of the connection D. To see that they do *not* appear in our expressions for the superpotential above, note that J, ψ and the operators $\operatorname{Tr}(J(D\times_{\text{ad}}\psi))$ and $s(\bar{\psi}, (D\times_{\text{ad}}\psi)\cdot\psi)$ are all SU(7) singlets. This means that only SU(7) singlet parts of the connection can appear in them. A routine decomposition reveals that there are no singlets in the SU(7) decomposition of the $\mathbf{1280}+\overline{\mathbf{1280}}$ representation of SU(8), and thus these parts of the connection cannot appear in our expressions. As such, these operators represent a complex SU(7) singlet part of the intrinsic torsion, as claimed.

G_2 in M-theory

In the G_2 case, it is straightforward to calculate the superpotential directly and compare with the existing literature. As discussed in section 5.2.1, the SU(7) structure corresponding to a G_2 structure with flux has the form

$$\begin{aligned} \psi &= e^{\tilde{A}+A}e^{i\varphi}\cdot 1 = e^{\tilde{A}-\frac{1}{2}iA\wedge\varphi+A+i\varphi}\cdot 1 = e^{\gamma}\cdot 1, \\ L_3 &= e^{\tilde{A}-\frac{1}{2}iA\wedge\varphi+A+i\varphi}\cdot T = e^{\gamma}\cdot T_{\mathbb{C}}, \end{aligned} \tag{5.111}$$

where we have defined $\gamma = \tilde{A} - \frac{1}{2}\mathrm{i}A\wedge\varphi + A + \mathrm{i}\varphi$ as a sum of six- and three-forms. The Dorfman derivative of ψ along $V = \mathrm{e}^{\gamma}v \in \Gamma(L_3)$ satisfies

$$L_V\psi = L_{\mathrm{e}^{\gamma}\cdot v}(\mathrm{e}^{\gamma}\cdot 1) = \mathrm{e}^{\gamma}_{v}{}^{\Gamma}1 = \mathrm{e}^{\gamma}\cdot(\mathcal{L}_v 1 - \imath_v\Gamma\cdot 1) = -\mathrm{e}^{\gamma}\cdot\imath_v\Gamma\cdot 1, \tag{5.112}$$

where the complex flux

$$\Gamma = F + \mathrm{i}\,\mathrm{d}\varphi + \tilde{F} + \tfrac{1}{2}\varphi\wedge\mathrm{d}\varphi - \mathrm{i}F\wedge\varphi \in \Gamma(\wedge^4T^*\oplus\wedge^7T^*) \tag{5.113}$$

can be viewed as a section of the torsion bundle K. Using the various actions of γ as an adjoint element, we also have

$$T(V)\cdot\psi = T(\mathrm{e}^{\gamma}v)\cdot\mathrm{e}^{\gamma}\cdot 1 = \mathrm{e}^{\gamma}\cdot(\mathrm{e}^{-\gamma}\cdot T)(v)\cdot 1 = \mathrm{e}^{\gamma}\cdot\imath_v(\mathrm{e}^{-\gamma}\cdot T)\cdot 1. \tag{5.114}$$

Finally we note that

$$s(\psi,T) = s(\mathrm{e}^{\gamma}\cdot 1, T) = s(1,\mathrm{e}^{-\gamma}\cdot T) \sim (\mathrm{e}^{-\gamma}\cdot T)_{(7)}, \tag{5.115}$$

where $(\mathrm{e}^{-\gamma}\cdot T)_{(7)}$ is the seven-form component of $(\mathrm{e}^{-\gamma}\cdot T)$. However, using (5.102) and comparing (5.112) and (5.114), we see that $(\mathrm{e}^{-\gamma}\cdot T)_{(7)} = \Gamma_{(7)}$ and hence

$$\mathcal{W} \propto \int_M \mathrm{i}\,s(\psi,T) \propto \int_M \left(\tilde{F} + \tfrac{1}{2}\varphi\wedge\mathrm{d}\varphi - \mathrm{i}F\wedge\varphi\right). \tag{5.116}$$

The superpotential is simply the integral of the seven-form component of the complex flux.

We can compare this expression to those that have already appeared the literature. Beasley and Witten considered the M-theory superpotential on manifolds of G_2 holonomy [272] – this means we should assume $\mathrm{d}\varphi = 0$ to match their results. In addition, they take $\int_M \tilde{F} = -\frac{1}{2}\int_M A\wedge F$.[16] Using these assumptions, the above superpotential can be rewritten as

$$\mathcal{W} \propto \int_M \left(\tfrac{1}{2}A + \mathrm{i}\varphi\right)\wedge F, \tag{5.117}$$

which matches that given in [272]. More generally, the M-theory superpotential on manifolds with G_2 structure with flux has been discussed in a number of places [103, 132, 152, 282]. Following [282], we define

$$P_0 = \int_M (\tilde{F} + \tfrac{1}{2}A\wedge F) \quad \in (2\pi)^2\,\mathbb{Z}, \tag{5.118}$$

which allows us to rewrite our superpotential as

$$\begin{aligned}\mathcal{W} &\propto P_0 + \int_M \left(-\left(\tfrac{1}{2}A + \mathrm{i}\varphi\right)\wedge F + \tfrac{1}{2}\varphi\wedge\mathrm{d}\varphi\right)\\ &\propto P_0 - \tfrac{1}{2}\int_M (A + \mathrm{i}\varphi)\wedge\mathrm{d}(A+\mathrm{i}\varphi).\end{aligned} \tag{5.119}$$

[16] As discussed by Beasley and Witten, this comes about as the Page charge (the integral of $\frac{1}{(2\pi)^2}\mathrm{d}\tilde{A}$) is quantised. Since $\frac{1}{(2\pi)^2}\frac{1}{2}\int_M A\wedge F$ is only defined modulo an integer [281], one can take $\int_M(\tilde{F} + \frac{1}{2}A\wedge F) = 0$ without introducing extra ambiguities.

This matches the expression found in [282] up to an overall multiplicative constant.

Let us make one further comment. Recall that involutivity for a G_2 structure implied $d\varphi = dA = d\tilde{A} = 0$ and so $d\gamma = 0$. From (5.112) this means $L_V\psi = 0$ for all $V \in \Gamma(L_3)$ – in other words $\delta\mathcal{W}/d\psi = 0$. This is a result of our choice of normalisation of ψ. If we had scaled by a complex function f so that $\psi' = e^\gamma \cdot f$, we would have had an additional one-form contribution to the intrinsic torsion T and $L_V\psi$ would not vanish, consistent with the comments below (5.104).

GMPT

We can repeat the same analysis to give the superpotential in the GMPT case. The SU(7) structure has the form given in (5.90) and (5.75)

$$\psi = e^{\Sigma} \cdot \Phi_-, \qquad L_3 = e^{\Sigma}(L_1^{\mathcal{J}_-} \oplus \mathcal{U}_{\mathcal{J}_-}), \tag{5.120}$$

where $\Sigma = C + 8\,i\,e^{-3\Delta}\,\mathrm{im}\,\Phi_+$. Using (5.92), we then have

$$\begin{aligned} L_V\psi &= e^{\Sigma} \cdot \left[L_{Z+\alpha\Phi_-}\Phi_- - (\not{Z}d\Sigma) \cdot \Phi_-\right) \\ &= e^{\Sigma} \cdot \left[\not{Z}d\Phi_- - \alpha(d\Phi_-) \cdot \Phi_- - (\not{Z}d\Sigma) \cdot \Phi_-\right], \end{aligned} \tag{5.121}$$

where we take $Z \in \Gamma(L_1^{\mathcal{J}_-})$ so that $V = e^{\Sigma}(Z + \alpha\Phi_-) \in \Gamma(L_3)$ and have used the algebraic property $(Z + \alpha\Phi_-) \bullet \Phi_- = 0$. As in (5.114) we have for the torsion

$$T(V) \cdot \psi = e^{\Sigma} \cdot \left(e^{-\Sigma} \cdot T\right)(Z + \alpha\Phi_-) \cdot \Phi_-. \tag{5.122}$$

Finally we have

$$s(\psi, T) = s(e^{\Sigma} \cdot \Psi, T) = s(\Psi_-, e^{-\Sigma} \cdot T) \sim \left(\Phi_-, (e^{-\Sigma} \cdot T)_-\right), \tag{5.123}$$

where in the last expression we have the Mukai pairing of Φ_- and the odd-polyform component $(e^{-\Sigma} \cdot T)_-$ of the torsion. However, using (5.102) and comparing (5.121) and (5.122), we see that $(e^{-\Sigma} \cdot T)_- = d\Sigma$ and hence

$$\mathcal{W} \propto \int_M i\, s(\psi, T) \propto \int_M \left(\Phi_-, F + 8\,i\,d(e^{-3\Delta}\,\mathrm{im}\,\Phi_+)\right). \tag{5.124}$$

Taking into account the normalisations (5.74), we see that this is in precise agreement with the O(6, 6) generalised geometry expressions given in [156, 283–285].

5.3.2 The Kähler Potential, the Moment Map and Extremisation

Almost twenty years ago Hitchin [151] gave an intriguing reformulation of integrable G_2 structures as corresponding to stationary points of a suitable functional on the space of closed structures, that is those satisfying $d\varphi = 0$, taking the variation within the cohomology class of φ. In this section we will show that the Kähler potential $\mathcal{K}$ gives a natural generalised geometry extension of Hitchin's functional for SU(7) structures. In particular, we show that the moment map conditions $\mu = 0$ can be rephrased as stationary points of $\mathcal{K}$ when varying over the space of complexified generalised diffeomorphisms $\mathrm{GDiff}_{\mathbb{C}}$. In the case of G_2 structures we show that this is identical to Hitchin's variational problem.

We start by recalling that an infinitesimal generalised diffeomorphism defines a vector field $\rho_V \in \Gamma(T\mathcal{Z})$ on the space $\mathcal{Z}$ of generalised SU(7) structures given by[17]

$$\mathcal{L}_{\rho_V}\psi = \imath_{\rho_V}\delta\psi = L_V\psi. \tag{5.125}$$

The symplectic form ϖ on $\mathcal{Z}$ given in (5.33) is invariant under the action of GDiff, that is $\mathcal{L}_{\rho_V}\varpi = 0$, and μ in (5.39) is the corresponding moment map defined by $\imath_{\rho_V}\varpi = -\delta\mu(V)$. Note that it is straightforward to check that $\imath_{\rho_W}\delta\mu(V) = \mu([\![V, W]\!])$, where $[\![V, W]\!]$ is the Courant bracket, and hence the moment map is equivariant. We also immediately note $\mathcal{L}_{\rho_V}\psi = L_V\psi$ is holomorphic in ψ hence the GDiff action also preserves the complex structure on $\mathcal{Z}$.

It is a standard result from the supergravity literature that the moment map (or D-term) can be solved in terms of the Kähler potential [243]. Explicitly, if ρ_V generates the symmetry, one has, by definition,

$$\delta\mu(V) = -\imath_{\rho_V}\varpi = -\imath_{\rho_V}\left(\tfrac{1}{2}\delta\delta^{\mathcal{I}}\mathcal{K}\right) = -\tfrac{1}{2}\mathcal{L}_{\rho_V}\left(\mathcal{I}\delta\mathcal{K}\right) + \tfrac{1}{2}\delta\left(\imath_{\rho_V}\mathcal{I}\delta\mathcal{K}\right), \tag{5.126}$$

where $\delta^{\mathcal{I}} = [\mathcal{I}, \delta]$ and $\mathcal{I}$ is the complex structure on $\mathcal{Z}$. But we have $\mathcal{L}_{\rho_V}\mathcal{I} = 0$, so, assuming we choose the Kähler potential such that it is also invariant, that is $\mathcal{L}_{\rho_V}\mathcal{K} = 0$, the first term vanishes. Using $\imath_{\mathcal{I}\rho_V}\delta\mathcal{K} = -\imath_{\rho_V}\mathcal{I}\delta\mathcal{K}$, one then has (up to closed terms which are fixed to vanish by the requirement of equivariance)

$$\mu(V) = -\tfrac{1}{2}\imath_{\mathcal{I}\rho_V}\delta\mathcal{K} = -\tfrac{1}{2}\mathcal{L}_{\mathcal{I}\rho_V}\mathcal{K}. \tag{5.127}$$

To check this relation explicitly in our case, we first calculate $\mathcal{I}\rho_V$. Since ψ is holomorphic, splitting the exterior (functional) derivative on $\mathcal{Z}$ into holomorphic and antiholomorphic parts $\delta = \partial' + \bar{\partial}'$, we have

$$\mathcal{L}_{\mathcal{I}\rho_V}\psi = \imath_{\mathcal{I}\rho_V}\partial'\psi = \mathrm{i}\imath_{\rho_V}\partial'\psi = \mathrm{i}\mathcal{L}_{\rho_V}\psi = \mathrm{i}L_V\psi. \tag{5.128}$$

We then have

$$\begin{aligned}
\mathcal{L}_{\mathcal{I}\rho_V}\mathcal{K} &= \int_M \tfrac{1}{3}\left(\mathrm{i}\,s(\psi,\bar{\psi})\right)^{-2/3}\left(\mathrm{i}\,s(\imath_{\mathcal{I}\rho_V}\delta\psi,\bar{\psi}) + \mathrm{i}\,s(\psi,\imath_{\mathcal{I}\rho_V}\delta\bar{\psi})\right) \\
&= -\int_M \tfrac{1}{3}\left(\mathrm{i}\,s(\psi,\bar{\psi})\right)^{-2/3}\left(s(L_V\psi,\bar{\psi}) - s(\psi, L_V\bar{\psi})\right) \\
&= -\int_M \tfrac{2}{3}\left(s(\psi,\bar{\psi})\right)^{-2/3}\ s(L_V\psi,\bar{\psi}) \\
&= -2\,\mu(V),
\end{aligned} \tag{5.129}$$

where we used an integration by parts and compactness to reach the final line. This is in complete agreement with (5.127). For completeness, using the non-holomorphic structure ϕ, we

[17] Note that here $\mathcal{L}_{\rho_V}$ is the Lie derivative along ρ_V in the space of structures $\mathcal{Z}$, whereas L_V is the generalised Lie derivative on the manifold M.

can also check the invariance of $\mathcal{K}$:

$$\begin{aligned}\mathcal{L}_{\rho_V}\mathcal{K} &= \mathrm{i}\int_M s(\imath_{\rho_V}\delta\phi,\bar{\phi}) + s(\phi,\imath_{\rho_V}\delta\bar{\phi}) = \mathrm{i}\int_M s(L_V\phi,\bar{\phi}) + s(\phi, L_V\bar{\phi}) \\ &= \mathrm{i}\int_M L_V s(\phi,\bar{\phi}) = 0,\end{aligned} \tag{5.130}$$

where the action of L_V on a top-form reduces to the Lie derivative, which then vanishes due to compactness of M.

The relation (5.127) is striking because it shows that the zeros of the moment map can be equally well thought of as critical points of $\mathcal{K}$

$$\mu = 0 \qquad \Leftrightarrow \qquad \text{critical point of } \mathcal{K} \text{ under } \mathrm{GDiff}_{\mathbb{C}} \text{ action.} \tag{5.131}$$

The group GDiff does not really complexify, so what is really meant here is motion on the orbits generated by ρ_V and $\mathcal{I}\rho_V$. Since $\mathcal{K}$ is invariant under the former, the extremisation is really over iGDiff generate by $\mathcal{I}\rho_V$. For the set of critical points to form a nice moduli space after quotienting by GDiff, as in the symplectic quotient, strictly one needs to show that a critical point of the Kähler potential is non-degenerate transverse to the orbit of GDiff [151]. It is a general result that the Hessian for the imaginary transformations is given by

$$\mathcal{L}_{\mathcal{I}\rho_V}\mathcal{L}_{\mathcal{I}\rho_W}\mathcal{K} = -2\,\mathcal{L}_{\mathcal{I}\rho_V}\mu(W) = -2\,\imath_{\mathcal{I}\rho_V}\delta\mu(W) = 2\,\imath_{\mathcal{I}\rho_V}\imath_{\rho_W}\varpi = 2\,\tilde{g}(\rho_V,\rho_W), \tag{5.132}$$

where $\tilde{g}$ is the pseudo-Kähler metric on $\mathcal{Z}$. Because the metric is pseudo-Kähler, it is possible that $\tilde{g}(\rho_V,\rho_W)$ could vanish for all ρ_W and this not imply that $\rho_V = 0$. Since we want to mod out by real generalised diffeomorphisms, the non-degeneracy condition we require is that, at the extremum,

$$\tilde{g}(\rho_V,\rho_W) = 0 \quad \forall\, W \in \Gamma(E) \qquad \to \qquad \exists\, U \in \Gamma(E): \quad \mathrm{i}L_V\psi = L_U\psi. \tag{5.133}$$

In other words, any degeneracy in the direction of an imaginary GDiff transformation is always equivalent to a real GDiff transformation. One can rephrase this condition in terms of the operators discussed in section 5.4.5. However, at this point, we do not understand them well enough to check if the non-degeneracy is generically true. That said, from a physical perspective, since the equations of motion of supergravity are elliptic and supersymmetry implies the equations of motion, we would expect there to be a sensible finite-dimensional moduli space.

The extremisation of $\mathcal{K}$ is a generalised geometry extension of Hitchin's extremisation of a G_2 functional [151] as we will now see. We saw in section 5.2.1 that for G_2 structures, the Kähler potential is proportional to the G_2 Hitchin functional $V(\varphi)$

$$\mathcal{K}(\psi) \propto V(\varphi) = \int_M \varphi\wedge\star\varphi \qquad \text{for } \psi = \mathrm{e}^{\tilde{A}+A}\mathrm{e}^{\mathrm{i}\varphi}\cdot 1. \tag{5.134}$$

Furthermore, under an imaginary GDiff transformation it is straightforward to calculate

$$\imath_{\mathcal{I}\rho_V}\delta\psi = \mathrm{i}L_V\psi = \mathrm{i}\mathcal{L}_v\psi - \mathrm{i}(\mathrm{d}\omega+\mathrm{d}\sigma)\cdot\psi, = -\mathrm{d}(\imath_v\varphi)\cdot\psi - \mathrm{i}\left(\mathrm{d}\omega'+\mathrm{d}\sigma'\right)\cdot\psi. \tag{5.135}$$

where $\omega' = \omega - \imath_v A$ and $\sigma' = \sigma - \imath_v \tilde{A} - \frac{1}{2} A \wedge \imath_v A + \frac{1}{2}\varphi \wedge \imath_v \varphi$ and we have used the involutivity conditions $\mathrm{d}\varphi = \mathrm{d}A = \mathrm{d}\tilde{A} = 0$. We see that, up to real generalised diffeomorphisms, an imaginary GDiff is equivalent to an imaginary gauge transformation. Exponentiating, again up to real gauge transformations, we get

$$\psi \mapsto \psi' = \mathrm{e}^{\tilde{A}+A}\mathrm{e}^{\mathrm{i}\mathrm{d}\sigma'}\mathrm{e}^{\mathrm{i}(\varphi+\mathrm{d}\omega')} \cdot 1 = \mathrm{e}^{\tilde{A}+A}\mathrm{e}^{\mathrm{i}(\varphi+\mathrm{d}\omega')} \cdot (1 + \text{const} \times j\mathrm{d}\sigma' + \dots), \tag{5.136}$$

where $j\mathrm{d}\sigma'$ denotes $\mathrm{d}\sigma_{m,n_1\dots n_5} \in \Gamma(T^* \otimes \wedge^5 T^*)$ and the dots denote higher-order terms in $\mathrm{d}\sigma'$. In particular, we see the G_2 three-form is shifted within its cohomology class. We now want to extremise $\mathcal{K}$ with respect to the σ' and ω' variations. First note that it is independent of A and $\tilde{A}$ since it is a $\mathrm{E}_{7(7)} \times \mathbb{R}^+$-invariant. Next, we first show that $\mathrm{d}\sigma' = 0$ is an extremum with respect to the σ' variations. Writing the modified G_2 structure as $\varphi' = \varphi + \mathrm{d}\omega'$, linearising in $\pi = j\mathrm{d}\sigma'$ we then have, using the same arguments that led to (5.62),

$$\delta\mathcal{K} = \int_M \kappa \qquad \text{where} \qquad \kappa_{m_1\dots m_7} = \text{const} \times g'^{np}\pi_{n,p[m_1\dots m_4}\varphi'_{m_5 m_6 m_7]}. \tag{5.137}$$

However, the antisymmetry of $\mathrm{d}\sigma'$ implies κ vanishes and hence $\delta\mathcal{K} = 0$. This means we are back to extremising $\mathcal{K}(\psi')$ in (5.134) with φ replaced with $\varphi' = \varphi + \mathrm{d}\omega'$. But this is exactly the extremisation introduced by Hitchin [151]. For a variation $\delta\varphi' = \mathrm{d}\omega'$ it gives

$$\delta V(\varphi') \propto \int_M \delta\varphi' \wedge \star\varphi' = \int_M \mathrm{d}\omega' \wedge \star\varphi'. \tag{5.138}$$

Integrating by parts shows that V has a critical point for $\mathrm{d}\star\varphi' = 0$, recovering the condition from the vanishing of the moment map as we expected.

5.3.3 Moduli Spaces, GIT and Stability

The fact that the moduli space can be viewed either as a symplectic quotient or a quotient by the complexified group is a general result for group actions that preserve a Kähler structure (see for example the discussion in [256]). For the case in hand, we have

$$\mathcal{M}_\psi = \hat{\mathcal{Z}} /\!\!/ \mathrm{GDiff} \simeq \hat{\mathcal{Z}}^{\mathrm{ps}}/\mathrm{GDiff}_{\mathbb{C}}. \tag{5.139}$$

There is a subtlety we have glossed over previously which is that for the complex quotient one needs to consider not the full space of structures but a subset $\hat{\mathcal{Z}}^{\mathrm{ps}} \subset \hat{\mathcal{Z}}$ of "polystable" points. The equivalence of quotients in (5.139) is the Kempf–Ness theorem. This is part of "Geometric Invariant Theory" or GIT, as reviewed for example in [221]. The point is that not all complex orbits will intersect the space of zeros of the moment map $\mu^{-1}(0)$. If ψ lies on an orbit that fails to meet $\mu^{-1}(0)$ it is called unstable and is excluded from $\hat{\mathcal{Z}}^{\mathrm{ps}}$. Our setup is typical of a number of classic geometric problems: one has an infinite-dimensional Kähler manifold with a group action such that the vanishing of a moment map corresponds to the solution of a differential equation. For example, it appears in Atiyah and Bott's work on flat connections on Riemann surfaces [286], in the "hermitian Yang–Mills" equations of Donaldson–Uhlenbeck–Yau [224, 225, 245], Fine's formulation of the Calabi conjecture [287], and the equations of

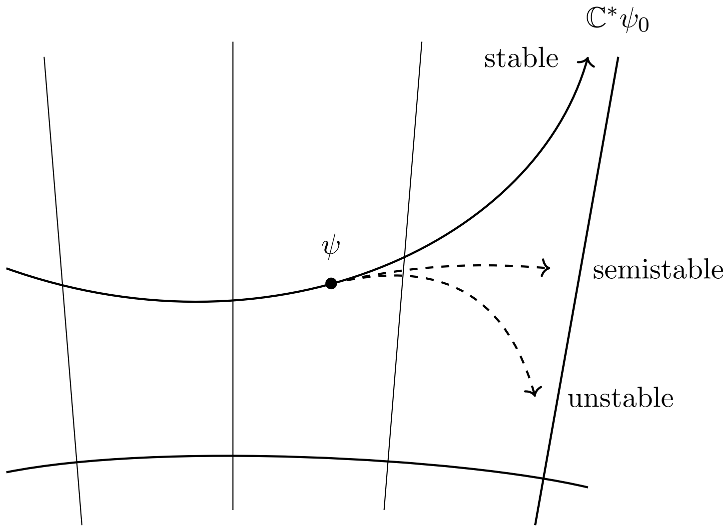


Figure 5.1: Stability for a 1-PS orbit of ψ.

Kähler–Einstein geometry [246, 247, 288]. Famously, in each case, developing the correct GIT notion of stability allows one to translate the question of existence of solutions to the differential equation into algebraic conditions arising from the analysis of the complex orbits.

In this section, we will sketch how our description of integrable SU(7) structures might translate into the GIT picture, and discuss the form of the moduli space. In general, stability can be understood in the following way. Consider a U(1) subgroup of the group action. For us this is some $\mathrm{U}(1) \subset \mathrm{GDiff}$ generated by some vector field $\rho_V \in \Gamma(T\hat{\mathcal{Z}})$. Under complexification this gives a $\mathbb{C}^*$ action on the space of involutive structures $\hat{\mathcal{Z}}$. Starting at some point $\psi \in \hat{\mathcal{Z}}$ the $\mathbb{C}^*$ action generates an orbit of structures $\psi(\nu)$ parameterised by $\nu \in \mathbb{C}^*$. If the space of structures were compact up to overall scalings of the SU(7) structure of the form $\psi \to \lambda^3\psi$ with $\lambda \in \mathbb{C}^*$, then in the limit $\nu \to 0$, the two $\mathbb{C}^*$ actions must coincide, giving a fixed line of structures (see figure 5.1)

$$\lim_{\nu\to 0} \psi(\nu) = \nu^{3w(\psi)}\psi_0 \quad \Rightarrow \quad \lim_{\nu\to 0} \mathcal{K} = |\nu|^{2w(\psi)}\mathcal{K}_0, \qquad \text{where } w(\psi) \in \mathbb{Z}, \tag{5.140}$$

where the weight $w(\psi)$ depends on the orbit (and hence the original structure ψ) and is necessarily quantised since we have a $\mathrm{U}(1) \subset \mathbb{C}^*$ action.[18] Considering all such U(1) subgroups, or "one-parameter subgroups" (1-PS), one then defines[19]

$$\begin{aligned} &\text{if } w(\psi) < 0 \text{ for all 1-PS then } \psi \text{ is stable,} \\ &\text{if } w(\psi) \leq 0 \text{ for all 1-PS then } \psi \text{ is semistable,} \\ &\text{if } w(\psi) > 0 \text{ for some 1-PS then } \psi \text{ is unstable.} \end{aligned} \tag{5.141}$$

The beautiful observation is then that if the function $\mathcal{K}$ is convex with respect to varying $|\nu|$, and is stable in both directions (that is for 1-PS generated by ρ_V and the inverse 1-PS generated by $-\rho_V$), then it must have a (unique) minimum. But we have already seen from (5.127) that

[18] We have normalised the U(1) charges relative to the $\mathbb{R}^+$ action, hence the factor of three in (5.140).

[19] More generally one can define stability for the action of the whole of the complexified group (in our case $\mathrm{GDiff}_{\mathbb{C}}$) but the Hilbert–Mumford criterion implies that stability for all the 1-PS is an equivalent condition.

a minimum of $\mathcal{K}$ is equivalent to the vanishing of the moment map $\mu(V) = 0$ for this particular V. Since stability is for all 1-PS it implies there is a unique minimum where $\mu(V) = 0$ for all V. Hence if ψ is stable[20] then there is a unique solution of the moment map in the orbit of ψ generated by $\mathrm{GDiff}_{\mathbb{C}}$. In the language of GIT we are identifying

$$\text{norm functional} = \text{Kähler potential } \mathcal{K} \tag{5.142}$$

which as we saw above is the $E_{7(7)} \times \mathbb{R}^+$ extension of Hitchin's G_2-functional.

In the Kähler–Einstein context, Yau [72] originally introduced the notion of a functional that is the integral of the square of the scalar curvature, and in the moment map picture is the integral of the square of the moment map. Critical points of the Yau functional are called "extremal metrics". In our context, the $\mathcal{N} = 1$, $D = 4$ supergravity picture gives a simple interpretation of the analogous object. Recall that the potential of the supergravity is given by

$$\mathcal{V} = \mathrm{e}^{\mathcal{K}} \left(\hat{g}^{i\bar{j}} D_i \mathcal{W} D_{\bar{j}} \bar{\mathcal{W}} - 3\mathcal{W}\bar{\mathcal{W}} \right) + \tfrac{1}{2} (\operatorname{re}\tau)^{ab} \mathcal{P}_a \mathcal{P}_b, \tag{5.143}$$

where $\hat{g}_{i\bar{j}}$ is the Kähler metric on the space of chiral fields Φ^i, $D_i\mathcal{W} = \partial_i\mathcal{W} - (\partial_i\mathcal{K})\mathcal{W}$, and $\operatorname{re}\tau_{ab}$ is an invariant metric on the Lie algebra of the moment map symmetry. If we consider SU(7) structures that are involutive (or strictly the slightly stronger condition that the superpotential is extremised (5.104)) the term in parentheses vanishes. The metric on the Lie algebra is fixed by the generalised metric G_{MN} (see for example [58]) and we are left with

$$\mathcal{V} \sim \int_M \mathrm{vol}_G^{-1} G^{MN} \mathcal{P}_N \mathcal{P}_M \sim \int_M \mathrm{vol}_G \mathcal{R}, \tag{5.144}$$

where vol_G is the $E_{7(7)}$-invariant volume form defined by the generalised metric. (Note that the factor of vol_G^{-1} in the first term comes from the fact that $\mathcal{P} \in \Gamma(\det T^* \otimes E^*)$.) We see that the potential is the square of the moment map. Furthermore, from the reformulation of supergravity in terms of $E_{7(7)} \times \mathbb{R}^+$ generalised geometry [181, 182], the potential is the supergravity action on M which is just the integral of the generalised Ricci scalar $\mathcal{R}$ as we write in the second term. Thus we have

$$\text{Yau functional} \sim \int_M \mathrm{vol}_G \mathcal{R}. \tag{5.145}$$

We see that extremising the Yau functional corresponds to generalised Ricci-flat solutions, that is generic solutions of the supergravity equations.

Central to the equivalence of stability and the vanishing of the moment map is the condition that the norm functional is convex. This is usually a consequence of the general result (5.132) that the second derivative is given by the Kähler metric

$$\mathcal{L}_{\mathcal{I}\rho_V} \mathcal{L}_{\mathcal{I}\rho_V} \mathcal{K} = 2\,\tilde{g}(\rho_V, \rho_V). \tag{5.146}$$

A positive-definite metric then implies convexity. As we have already mentioned, a key difference for SU(7) structures is that we have a pseudo-Kähler metric and so we can no longer guarantee

[20] The actual condition is the slightly more subtle notion of "polystability" which includes equivalence classes of semistable orbits, at the boundary between stable and unstable orbits.

that the norm functional is convex under the action of iGDiff. Thus a stable orbit may have more than one solution of the moment map, and unstable orbits may still include solutions, implying stability is only a sufficient condition for the existence of solutions. This problem is closely related to the degeneracy question, mentioned above, as to whether critical points of $\mathcal{K}$ form a nice moduli space.

The pseudo-Kähler structure raises other potential subtleties with the description of the moduli space of integrable SU(7) structures as we have presented it. First, the holomorphic involutivity condition might define a null subspace within the space of structures $\mathcal{Z}$, meaning there is no guarantee that the subspace $\hat{\mathcal{Z}}$ inherits a Kähler metric (since the pullback of the metric can be degenerate). Secondly, if the group action defining the moment map is null, there is similarly no guarantee that there is a Kähler metric on the symplectic quotient. Although we have not checked directly, physically we might expect that neither problem arises, the point being that supersymmetry implies that there must be a Kähler metric on the final moduli space, since it is a space of chiral superfields. Furthermore, unless the background secretly admits more supersymmetries, this metric must be positive definite (since it gives the kinetic terms of the four-dimensional fields). If there are extra supersymmetries these appear as deformations which change the SU(7) structure but not the generalised metric, and hence are unphysical.

This makes one wonder if there could be a more standard GIT picture underlying the conditions. Recall that at a point $p \in M$ the tangent space $TQ_{\mathrm{SU}(7)}$ to the $\mathrm{E}_{7(7)} \times \mathbb{R}^+/\mathrm{SU}(7)$ coset space (5.29) decomposes under $\mathrm{SU}(7) \times \mathrm{U}(1)$ as

$$TQ_{\mathrm{SU}(7)} \;:\quad (\mathbf{1}_0 \oplus \mathbf{1}_0) \oplus (\mathbf{7}_{-4} \oplus \overline{\mathbf{7}}_4) \oplus (\mathbf{35}_2 \oplus \overline{\mathbf{35}}_{-2}), \tag{5.147}$$

where the first two terms are generated by the action of J and the $\mathbb{R}^+$ scaling. The complex structure on $TQ_{\mathrm{SU}(7)}$ pairs the representations in parentheses, with a positive definite metric on $\mathbf{35}_2 \oplus \overline{\mathbf{35}}_2$ and a negative definite metric on the remaining directions giving a signature $(70, 16)$, which is then inherited by the full space of structures $\mathcal{Z}$. Focusing on the G_2 case, or perhaps more generally the type-0 case, we will now discuss how the negative deformations can potentially be removed. First one considers the space of exceptional complex structures J (rather than $\mathcal{Z}$), which removes the two singlet components in (5.147). Then one takes the symplectic quotient by the normal subgroup of gauge transformations generated by five-forms which removes the remaining $\mathbf{7} \oplus \overline{\mathbf{7}}$ components.

An exceptional complex structure J determines ψ up to rescaling by a function $\psi \to f\psi$. Thus we can define the space of exceptional complex structures as a symplectic quotient

$$\hat{\mathcal{X}}, \text{ space of exceptional complex structures} = \hat{\mathcal{Z}} /\!\!/ H. \tag{5.148}$$

The Lie algebra of H is given by $\mathfrak{h} \simeq C^\infty(M)$ and $\alpha \in \mathfrak{h}$ acts via $\rho_\alpha(\psi) = \mathrm{i}\alpha\psi$, giving the moment map

$$\mu_H(\alpha) = \int_M \alpha \left(\mathrm{i}\, s(\psi, \bar{\psi})\right)^{1/3}, \tag{5.149}$$

and in the quotient we set $\mu_H = \mathrm{vol}_0$ for some fixed reference volume form. Since the action preserves the Kähler structure on $\hat{\mathcal{Z}}$ there is then also a Kähler metric on $\hat{\mathcal{X}}$ though now based on the coset space $E_{7(7)}/U(7)$ with signature $(70, 14)$. The corresponding Kähler potential is given by choosing an arbitrary section $\psi \in \Gamma(\mathcal{U}_J)$ and calculating

$$\tilde{K} = \int_M \log\left(\mathrm{i}s(\psi,\bar{\psi})/\operatorname{vol}_0^3\right)\operatorname{vol}_0 . \tag{5.150}$$

The action of GDiff descends to $\hat{\mathcal{X}}$ (strictly we need to restrict to the subgroup $\mathrm{GDiff}_0 \subset \mathrm{GDiff}$ that preserves vol_0, that is in the Lie algebra $L_V \operatorname{vol}_0 = 0$, but we will ignore this subtlety). Hence one can define a corresponding moment map $\tilde{\mu}$ on $\hat{\mathcal{X}}$ given by

$$\tilde{\mu}(V) = \int_M \frac{s(L_V\psi,\bar{\psi})}{\mathrm{i}\,s(\psi,\bar{\psi})}\operatorname{vol}_0, \tag{5.151}$$

and define the quotient moduli space

$$\mathcal{M}_{\text{phys}} = \hat{\mathcal{X}}/\!\!/\mathrm{GDiff}. \tag{5.152}$$

We claim that this is isomorphic the physical moduli space $\mathcal{M}_\psi/\mathbb{C}^*$, where the $\mathbb{C}^*$ action is the constant rescaling $\psi \to \lambda^3\psi$. The point is that the vanishing of the moment map $\tilde{\mu}(V) = 0$ on $\hat{\mathcal{X}}$ implies the vanishing of the moment map $\mu(V) = 0$ on $\hat{\mathcal{Z}}$ except for those transformations that preserve J, that is $L_V J = 0$. However, such transformations simply rescale ψ. The effect is that for each J satisfying $\tilde{\mu} = 0$ the additional conditions from $\mu = 0$ simply fix the particular section $\psi \in \Gamma(\mathcal{U}_J)$. Up to an overall $\mathbb{C}^*$ rescaling $\psi \to \lambda^3\psi$, we expect one such solution for each J, and hence $\mathcal{M}_J$ is isomorphic to the physical moduli space $\mathcal{M}_\psi/\mathbb{C}^*$. (This is completely analogous to the $\mathrm{SL}(3,\mathbb{C})$ structure case.)

If we focus on G_2 structures, fixing an integrable J, the compatible ψ can be written as

$$\psi = \mathrm{e}^{\tilde{A}+A}\mathrm{e}^{\mathrm{i}\varphi}\cdot f, \tag{5.153}$$

for some function f and with $\mathrm{d}\varphi = \mathrm{d}A = \mathrm{d}\tilde{A} = 0$. We note that the group $G_\sigma \subset \mathrm{GDiff}$ of five-form gauge transformations forms a normal subgroup. Thus we can do the symplectic reduction by stages, first reducing by G_σ and then by the quotient group $\mathrm{GDiff}' = \mathrm{GDiff}/G_\sigma$. As we saw in section 5.3.2, the form of ψ we have written already satisfies $\mu(\sigma) = 0$ for all five-forms σ. Hence the symplectic quotient just identifies $\tilde{A} \sim \tilde{A} + \mathrm{d}\sigma$. Taking $H^6_{\mathrm{d}}(M,\mathbb{R}) = 0$, we have $\tilde{A} \sim 0$. By moving to the quotient space

$$\hat{\mathcal{X}}_\sigma = \hat{\mathcal{X}}/\!\!/G_\sigma, \tag{5.154}$$

we have effectively removed 14 of the allowed deformations. Direct calculation in the G_2 case implies that this removes precisely the negative directions in the metric, so that the Kähler metric on $\hat{\mathcal{X}}_\sigma$ is positive definite. Thus we have a conventional picture of stability with

$$\mathcal{M}_{\text{phys}} \simeq \hat{\mathcal{X}}_\sigma/\!\!/\mathrm{GDiff}' \simeq \hat{\mathcal{X}}_\sigma^{\text{ps}}/\mathrm{GDiff}'_{\mathbb{C}}. \tag{5.155}$$

This suggests that, at least formally, the space of integrable G_2 structures, complexified by including the closed three-form potential A, can be viewed as a GIT quotient of the space of closed G_2 structures. The $E_{7(7)}$ extension of Hitchin's G_2 functional $\mathcal{K}$ plays the role of the norm functional.

A choice of 1-PS in this case should be a diffeomorphism corresponding to circle actions on M since the gauge transformations in GDiff are always non-compact. If the diffeomorphism is generated by $\xi \in \Gamma(T)$, fixed points of the 1-PS amount to solutions to

$$L_\xi J = 0 \quad \text{where} \quad J = \mathrm{e}^A\big(\varphi^\sharp - \varphi\big), \tag{5.156}$$

where we have allowed for a non-trivial three-form potential. The value of the moment map at the fixed point, suitably normalised, should give an integer invariant. This will be the analogue of the Futaki invariant in Kähler–Einstein geometry [289]. Furthermore, these should be obstructions to the existence of solutions to the moment map. The simplest solution to (5.156), is to take $\mathcal{L}_\xi\varphi = \mathcal{L}_\xi A = 0$. In this case, the SU(7) structure $\psi \in \Gamma(\mathcal{U}_J)$ can only depend on the circle action through the function f. One would expect that the integer invariants would thus encode the topology of the line bundle $\mathcal{U}_J$, since the moment map is independent of the choice of section. The obstruction is thus that the bundle must be trivial, as we expect for the existence of a globally defined ψ. More interestingly however, the 1-PS motion may lead to other types of solution to (5.156), most notably exceptional complex structures with type-changes, perhaps associated to circle actions with fixed points. These are structures J which are no longer type-0 in the whole of M. This is possible since although the $\mathbb{C}^*$ action generated by ξ preserves the cohomology class of φ and A, the forms themselves may vanish or become singular at points in M. The moment map evaluated on such solutions should again give some integral invariant of the closed G_2 structure. Naively, understanding such configurations would be key to formulating any notion of stability.

5.4 Moduli of $\mathcal{N} = 1$ Backgrounds

The generalised SU(7) structure we have described characterises generic $\mathcal{N} = 1$ flux backgrounds with a four-dimensional Minkowski factor. A natural question to ask is what is the moduli space of these backgrounds? If the background is to be used for phenomenology, this will tell us about the massless chiral superfields in the four-dimensional effective theory (ignoring extra M-theory or stringy massless excitations localised at singularities, since we are in the supergravity limit). Although the answer is well-known for G_2 compactifications, very little is known about generic supersymmetric flux compactifications. In this section we will use the generalised geometrical description to show how the moduli are related to particular cohomologies. For G_2 this reproduces the well-known result that the number of chiral fields is counted by the third de Rham cohomology $H^3_{\mathrm{d}}(M,\mathbb{C})$. The analysis trivially extends to generic type-0 SU(7) structures giving the local moduli space as $H^3_{\mathrm{d}}(M,\mathbb{C}) \oplus H^6_{\mathrm{d}}(M,\mathbb{C})$. Remarkably it also gives a complete description of the moduli for the GMPT solutions, completing an analysis first considered in [203].

As we have seen, the moduli space of a $D = 4$, $\mathcal{N} = 1$ background is given by $\mathcal{M}_{\text{phys}} = \mathcal{M}_\psi/\mathbb{C}^*$, where $\mathcal{M}_\psi$ is the space of torsion-free SU(7) structures modulo generalised diffeomorphisms. In section 5.3.3 we argued that if the infinite-dimensional GIT picture is valid this is equivalent to to $\hat{\mathcal{X}}/\mathrm{GDiff}_{\mathbb{C}}$, where $\hat{\mathcal{X}}$ is the space of exceptional complex structures. If we have a solution J, the local moduli space thus corresponds to a finding the integrable deformations of J modulo complexified generalised diffeomorphisms. As we noted, strictly the GIT picture is not necessarily equivalent because the the metric on $\hat{\mathcal{X}}$ is not positive definite. However, assuming the critical points of $\mathcal{K}$ are non-degenerate transverse to the orbit of GDiff, infinitesimally this will produce the correct moduli space. A generic deformation then defines an element of the intrinsic torsion that must vanish for the deformation to be integrable. The complexified generalised diffeomorphisms will be generated by the Dorfman derivative and are necessarily integrable. This sets up a problem in cohomology and it is this that we aim to understand better. We will start with a quick review of the moduli of conventional complex structures as this will illustrate many of the key ideas that we will use in analysing the deformations of SU(7) structures.

5.4.1 Review of the Moduli Space of Complex Structures

Let us recall how the moduli space of integrable $\mathrm{SL}(3,\mathbb{C})$ structures arises. One starts by considering deformations of an integrable $\mathrm{GL}(3,\mathbb{C})$ structure. Define $Q_{\mathrm{GL}(3,\mathbb{C})} = \mathrm{GL}(6,\mathbb{R})/\mathrm{GL}(3,\mathbb{C})$ as the space of (almost) complex structures at a point $p \in M$. This can be viewed as

$$Q_{\mathrm{GL}(3,\mathbb{C})} = \mathrm{GL}(6,\mathbb{R})/\mathrm{GL}(3,\mathbb{C}) = \mathrm{GL}(6,\mathbb{R}) \cdot I_0 = \mathrm{GL}(6,\mathbb{C})/P, \tag{5.157}$$

where $\mathrm{GL}(6,\mathbb{R}) \cdot I_0$ is the orbit of a fixed complex structure I_0 under $g \in \mathrm{GL}(6,\mathbb{R})$, and P is the parabolic subgroup of $\mathrm{GL}(6,\mathbb{C})$ that stabilises L_1

$$P = \operatorname{Stab} L_1 = (\mathrm{GL}(3,\mathbb{C}) \times \mathrm{GL}(3,\mathbb{C})) \ltimes \mathbb{C}^9. \tag{5.158}$$

The orbit picture means that deformations of the complex structure are parameterised by a choice of element of $\mathfrak{gl}_{6,\mathbb{C}}/\mathfrak{p}$ at each point in the manifold. In other words, one takes a section of the vector bundle

$$\mathfrak{gl}_{6,\mathbb{C}}/\mathfrak{p} \to \mathfrak{Q}_{\mathrm{GL}(3,\mathbb{C})} \to M. \tag{5.159}$$

In practice one can view $\mathfrak{Q} \subset \operatorname{ad} \tilde{F}_{\mathbb{C}}$ by choosing an embedding $\mathfrak{gl}_{6,\mathbb{C}}/\mathfrak{p} \hookrightarrow \mathfrak{gl}_{6,\mathbb{C}}$. In particular, using the real structure one can decompose

$$\begin{aligned} \mathfrak{gl}_{6,\mathbb{C}} &= \mathfrak{gl}_{3,\mathbb{C}} \oplus \mathfrak{gl}_{3,\mathbb{C}} \oplus \mathfrak{q} \oplus \bar{\mathfrak{q}}, \\ \mathfrak{p} &= \mathfrak{gl}_{3,\mathbb{C}} \oplus \mathfrak{gl}_{3,\mathbb{C}} \oplus \mathfrak{q}, \end{aligned} \tag{5.160}$$

where we identify the (nilpotent) subalgebra $\mathfrak{q} \simeq \Gamma(T^{1,0}_p \otimes T^{*0,1}_p)$. The pair of $\mathfrak{gl}_{3,\mathbb{C}}$ algebras and $\mathfrak{q}$ preserve $L_1 = T^{1,0} \subset T_{\mathbb{C}}$. This means a deformation of L_1 at a point $p \in M$ can formally be parameterised by $\bar{\alpha}_p \in \bar{\mathfrak{q}}$ alone, that so one can identify $\mathfrak{Q}_{\mathrm{GL}(3,\mathbb{C})} \simeq T^{0,1} \otimes T^{*1,0}$. The deformed subbundle is then

$$L_1' = \mathrm{e}^{\bar{\alpha}} L_1 = (1 + \bar{\alpha}) L_1. \tag{5.161}$$

L_1' can then be used to define $L_{-1}' \subset T_{\mathbb{C}}$ via $L_{-1}' = \bar{L}_1'$ provided $L_1' \cap L_{-1}' = 0$. Note that nilpotency of $\mathfrak{q}$ implies $\exp \bar{\alpha} = 1 + \bar{\alpha}$.

As before, this new subbundle is integrable if and only if

$$[L'_1, L'_1] \subset L'_1. \tag{5.162}$$

One can check that for an arbitrary deformation parameterised by $\bar{\alpha}$ we have

$$\left(\mathrm{e}^{-\bar{\alpha}}[\mathrm{e}^{\bar{\alpha}}V, \mathrm{e}^{\bar{\alpha}}W]\right)^m = \underbrace{(1+\bar{\alpha})^p{}_q(V^q\mathrm{d}_pW^m - W^q\mathrm{d}_pV^m)}_{\in L_1} + \underbrace{V^qW^r(\partial\bar{\alpha} + [\bar{\alpha},\bar{\alpha}])^m{}_{qr}}_{\in L_{-1}}. \tag{5.163}$$

This gives the well-known result that a complex structure deformation is integrable if and only if $\bar{\alpha}$ satisfies the Maurer–Cartan equation.

$$\partial\bar{\alpha} + [\bar{\alpha},\bar{\alpha}] = 0. \tag{5.164}$$

If one is just interested in the infinitesimal moduli at this point, taking $\bar{\alpha} = \epsilon\bar{\beta}$, in the limit $\epsilon \to 0$ the condition is simply $\partial\bar{\beta} = 0$. In general there may be an obstruction to extending this solution for finite ϵ, although the Kodaira–Nirenberg–Spencer theorem states there is no obstruction if the cohomology class $H^{2,0}_\partial(M, T^{0,1})$ vanishes. For the moduli space one should mod out by deformations generated by diffeomorphisms. Infinitesimally, that is of the form

$$L'_1 = (1 + \epsilon\mathcal{L}_v)L_1 \qquad v \in \Gamma(T). \tag{5.165}$$

Writing $v = x + \bar{x}$ for a unique $x \in \Gamma(T^{1,0})$, one finds

$$L'_1 = (1 + \epsilon\partial\bar{x})L_1, \tag{5.166}$$

where one views $\partial\bar{x} \in \Gamma(\mathfrak{Q}_{\mathrm{GL}(3,\mathbb{C})})$. A deformation is then trivial if $\bar{\beta} = \partial\bar{x}$ for some $\bar{x} \in \Gamma(T^{0,1})$. Hence we get the result that the infinitesimal moduli of $\mathrm{GL}(3,\mathbb{C})$ structures is given by

$$H^{1,0}_\partial(M, T^{0,1}). \tag{5.167}$$

Finally we note that it is simple to connect this picture to the moduli space of $\mathrm{SL}(3,\mathbb{C})$ structures. An integrable complex structure I defines a line of $\mathrm{SL}(3,\mathbb{C})$ structures $\mathcal{U}_I$. Up to a constant $\mathbb{C}^*$ rescaling $\Omega \to \lambda\Omega$ there is a unique integrable structure $\Omega \in \Gamma(\mathcal{U}_I)$ (that is one satisfying $\mathrm{d}\Omega = 0$) for each complex structure I. Hence, we get the standard result that the moduli space of integrable $\mathrm{SL}(3,\mathbb{C})$ structures is just $H^{1,0}_\partial(M, T^{0,1}) \oplus \mathbb{C} \simeq H^{1,2}_\partial(M) \oplus H^{0,3}_\partial(M)$.

5.4.2 Moduli Space of SU(7) Structures

Now let us turn to the moduli space $\mathcal{M}_\psi$ of SU(7) structures ψ. As we have discussed, locally the physical moduli space $\mathcal{M}_\psi/\mathbb{C}^*$ can be identified with the space of deformations of J that remain integrable, modulo complex diffeomorphisms.

First, let us introduce some notation as we did in the previous section. We consider the space $Q_{\mathrm{U}(7)\times\mathbb{R}^+}$ of almost exceptional complex structures at a point $p \in M$. This can be viewed as

$$Q_{\mathrm{U}(7)\times\mathbb{R}^+} = \mathrm{E}_{7(7)}/\mathrm{U}(7) = \mathrm{E}_{7(7)} \cdot J_0 = \mathrm{E}_{7,\mathbb{C}}/P, \tag{5.168}$$

where $\mathrm{E}_{7(7)} \cdot J_0$ is the orbit of a fixed almost exceptional complex structure J_0 under $\mathrm{E}_{7(7)}$ at some fixed point on the manifold, and P is the parabolic subgroup that stabilises L_3

$$P = \mathrm{Stab}\, L_3 = \mathrm{GL}(7, \mathbb{C}) \ltimes \mathbb{C}^{42}. \tag{5.169}$$

By considering the orbit of J_0 at all points on the manifold, we see that infinitesimal deformations of the structure can be viewed as a sections of the vector bundle

$$\mathrm{E}_{7,\mathbb{C}}/\mathfrak{p} \to \mathfrak{Q}_{\mathrm{U}(7)\times\mathbb{R}^+} \to M. \tag{5.170}$$

Again, in practice we will embed $\mathfrak{Q}_{\mathrm{U}(7)\times\mathbb{R}^+} \hookrightarrow \mathrm{ad}\, \tilde{F}_{\mathbb{C}}$ by choosing an embedding $\mathrm{E}_{7,\mathbb{C}}/\mathfrak{p} \hookrightarrow \mathrm{E}_{7,\mathbb{C}}$. Explicitly, we write a generic infinitesimal deformation of L_3 as

$$L_3 \to L_3' = (1 + \epsilon A) \cdot L_3, \tag{5.171}$$

where we view $A \in \Gamma(\mathfrak{Q}_{\mathrm{U}(7)\times\mathbb{R}^+})$ as a map

$$A\colon L_3 \to E_{\mathbb{C}}/L_3, \tag{5.172}$$

and then make a choice of embedding $E_{\mathbb{C}}/L_3 \hookrightarrow E_{\mathbb{C}}$. As the original subbundle L_3 is involutive, the intrinsic torsion vanishes. For a generic deformation A, L_3' will have some non-zero intrinsic torsion that appears as an obstruction to the involutivity of L_3' with respect to the generalised Lie derivative (or equivalently the Courant bracket). Expanding to first order in ϵ we get a differential map d_2

$$\mathrm{d}_2\colon \Gamma(\mathfrak{Q}_{\mathrm{U}(7)\times\mathbb{R}^+}) \to \Gamma(W^{\mathrm{int}}_{\mathrm{U}(7)\times\mathbb{R}^+}), \tag{5.173}$$

where sections of $W^{\mathrm{int}}_{\mathrm{U}(7)\times\mathbb{R}^+}$ are the intrinsic torsion for the deformed almost exceptional complex structure.[21] The L_3' subbundle will be involutive if the intrinsic torsion vanishes, and so the deformed structure will be integrable if and only if $A \in \ker \mathrm{d}_2$.

We also have the notion of a trivial deformation. As we have discussed, this corresponds to the action of the complexified generalised diffeomorphism group $\mathrm{GDiff}_{\mathbb{C}}$. To linear order, such deformations are given by the action of the Dorfman derivative. That is, we consider L_3' to be equivalent to L_3 if

$$L_3' = (1 + \epsilon L_V)L_3 \quad \text{for some } V \in \Gamma(E_{\mathbb{C}}). \tag{5.174}$$

This defines a second differential map d_1

$$\mathrm{d}_1\colon \Gamma(E_{\mathbb{C}}) \to \Gamma(\mathfrak{Q}_{\mathrm{U}(7)\times\mathbb{R}^+}). \tag{5.175}$$

A trivial deformation should automatically be torsion free – by the Leibniz property of the

[21] From the discussion around (5.28), note that here $W^{\mathrm{int}}_{\mathrm{U}(7)\times\mathbb{R}^+}$ is strictly a complex bundle transforming in the $\mathbf{1}_{-7} \oplus \mathbf{35}_{-5}$ representation of $\mathrm{U}(7) \times \mathbb{R}^+$.

generalised Lie derivative we have

$$
\begin{aligned}
L_{W+_V W}(W' +_V W') &= L_W W' + \epsilon \left(L_{L_V W} W' + L_W(L_V W')\right) + O(\epsilon^2) \\
&= (1 + \epsilon L_V)\, L_W W' + O(\epsilon^2)
\end{aligned} \tag{5.176}
$$

and hence any trivial deformation is indeed integrable. This is precisely the statement that $\mathrm{d}_2 \circ \mathrm{d}_1 = 0$ and so we have a three-term complex

$$
\Gamma(E_{\mathbb{C}}) \xrightarrow{\mathrm{d}_1} \Gamma(\mathfrak{Q}_{\mathrm{U}(7)\times\mathbb{R}^+}) \xrightarrow{\mathrm{d}_2} \Gamma(W^{\text{int}}_{\mathrm{U}(7)\times\mathbb{R}^+}). \tag{5.177}
$$

Assuming there are no obstructions, the local moduli space of the SU(7) structure is modelled on the cohomology of this complex.

In the rest of this section we will calculate this cohomology for both the G_2, generic type-zero and GMPT structure examples. In the G_2 case we recover the known result that the moduli are counted by the third de Rham cohomology of the underlying manifold. In the GMPT case, we find new results – the full set of moduli were previously unknown. We will see that in both cases the ability to calculate the cohomology of (5.177) relies on finding a nice parameterisation of the embeddings $E_{\mathbb{C}}/L_3 \hookrightarrow E_{\mathbb{C}}$ and $\mathfrak{Q}_{\mathrm{U}(7)\times\mathbb{R}^+} \hookrightarrow \operatorname{ad}\tilde{F}_{\mathbb{C}}$. This then leads to a description of the moduli in terms of cohomologies defined by differentials that are naturally associated to the problem. For the G_2 (and general type-zero) case this is the de Rham differential, while for the GMPT solutions it is the generalised Dolbeault operator associated to the integrable generalised complex structure. One may hope that the general case could be solved in terms of some natural differential associated to the L_3 bundle – we make some comments on this at the end of this section in 5.4.5, noting some of the complications that arise.

5.4.3 Example 1: G_2 and Type-0 Geometries

Recall that we can embed G_2 structures into the language of exceptional complex structures via the definition

$$
L_3 = \mathrm{e}^{\mathrm{i}\varphi} \cdot T_{\mathbb{C}}. \tag{5.178}
$$

The involutivity of this bundle then gives $\mathrm{d}\varphi = 0$. A useful parameterisation of the quotient spaces as subspace of $E_{\mathbb{C}}$ and $\operatorname{ad}\tilde{F}_{\mathbb{C}}$ is given by

$$
\begin{aligned}
E_{\mathbb{C}}/L_3 &\simeq \wedge^2 T^*_{\mathbb{C}} \oplus \wedge^5 T^*_{\mathbb{C}} \oplus (T^*_{\mathbb{C}} \otimes \wedge^7 T^*_{\mathbb{C}}), \\
\mathfrak{Q}_{\mathrm{U}(7)\times\mathbb{R}^+} &\simeq \wedge^3 T^*_{\mathbb{C}} \oplus \wedge^6 T^*_{\mathbb{C}}.
\end{aligned} \tag{5.179}
$$

It is worth noting that these are not eigenspaces of the exceptional complex structure J and hence this is a different parameterisation to that given in (5.204) below. They instead come from the natural deformations of the underlying exceptional Dirac structure defined by T. They are invariant under the map $\mathrm{e}^{\mathrm{i}\varphi}$, meaning they can equally well be viewed as defining deformations of L_3. In the same way, we can also identify the space of the intrinsic torsion as

$$
W^{\text{int}}_{\mathrm{U}(7)\times\mathbb{R}^+} \simeq \wedge^4 T^*_{\mathbb{C}} \oplus \wedge^7 T^*_{\mathbb{C}}, \tag{5.180}
$$

that is the space of intrinsic torsion of the Dirac structure.

If we take $\alpha \in \Gamma(\wedge^3 T^*_{\mathbb{C}})$ and $\beta \in \Gamma(\wedge^6 T^*_{\mathbb{C}})$, the infinitesimal deformation is given by

$$L_3' = \big(1 + \epsilon(\alpha + \beta)\big) \cdot \mathrm{e}^{\mathrm{i}\varphi} \cdot T_{\mathbb{C}} = \mathrm{e}^{\mathrm{i}\varphi + \epsilon(\alpha + \tilde{\beta})} \cdot T_{\mathbb{C}} + O(\epsilon^2), \tag{5.181}$$

where $\tilde{\beta} = \beta - \frac{1}{2}\varphi \wedge \alpha$. Repeating the calculation in (5.53), we can use the twisted Dorfman derivative and $\mathrm{d}\varphi = 0$ to find

$$\text{involutive } L_3' \quad \Leftrightarrow \quad \mathrm{d}\alpha = \mathrm{d}\beta = 0. \tag{5.182}$$

Hence integrable deformations are given by *closed* three-forms and six-forms. For the trivial deformations, writing $V = v + \omega + \sigma + \tau \in \Gamma(E_{\mathbb{C}})$ we have, since $\mathcal{L}_v T = 0$,

$$\begin{aligned} L_3' &= (1 + \epsilon L_V)\, \mathrm{e}^{\mathrm{i}\varphi} \cdot T_{\mathbb{C}} \\ &= \big(1 - \epsilon(\mathrm{d}\omega - \mathrm{d}\sigma - \mathrm{e}^{\mathrm{i}\varphi}\mathcal{L}_v \mathrm{e}^{-\mathrm{i}\varphi})\big) \cdot \mathrm{e}^{\mathrm{i}\varphi} \cdot T_{\mathbb{C}} \\ &= (1 + \epsilon(\mathrm{d}\tilde{\omega} + \mathrm{d}\tilde{\sigma}))\, L_3, \end{aligned} \tag{5.183}$$

where $\tilde{\omega} = -\omega + \mathrm{i}\, \imath_v \varphi$ and $\tilde{\sigma} = -\sigma - \frac{1}{2}\varphi \wedge \imath_v \varphi$. Hence the complex (5.177) becomes

$$\Gamma(\wedge^2 T^*_{\mathbb{C}} \oplus \wedge^5 T^*_{\mathbb{C}}) \xrightarrow{\ \mathrm{d}\ } \Gamma(\wedge^3 T^*_{\mathbb{C}} \oplus \wedge^6 T^*_{\mathbb{C}}) \xrightarrow{\ \mathrm{d}\ } \Gamma(\wedge^4 T^*_{\mathbb{C}} \oplus \wedge^7 T^*_{\mathbb{C}}). \tag{5.184}$$

where d is the exterior derivative, and the inequivalent deformations are counted by

$$\frac{\{\alpha \in \Gamma(\wedge^3 T^*_{\mathbb{C}}), \beta \in \Gamma(\wedge^6 T^*_{\mathbb{C}}) \,|\, \mathrm{d}\alpha = \mathrm{d}\beta = 0\}}{\{\alpha = \mathrm{d}\tilde{\omega}, \beta = \mathrm{d}\tilde{\sigma}\}} = H^3_{\mathrm{d}}(M, \mathbb{C}) \oplus H^6_{\mathrm{d}}(M, \mathbb{C}). \tag{5.185}$$

That is, the inequivalent deformations are counted by the third and sixth de Rham cohomologies. For a G_2 manifold, the sixth de Rham cohomology is trivial and hence the cohomology of (5.177) is counted by $H^3_{\mathrm{d}}(M, \mathbb{C})$ alone. The imaginary elements are deformations of the G_2 structure while the real elements shift the gauge potential such that the flux remains zero. This is in complete agreement with standard analysis of the moduli space of G_2 compactifications of M-theory [270–272].

It is also clear from the way we have written these deformations that they are unobstructed. The action of complex gauge potentials $\alpha + \beta$ can be exponentiated for finite ϵ as in the final term of (5.181), such that the linearised closure condition is enough to imply the deformation is integrable. Thus the moduli space looks like $H^3_{\mathrm{d}}(M, \mathbb{C})$ in a finite patch. Formally this is the statement that there is an open subset of the moduli space $\mathcal{V} \subseteq \mathcal{M}_{\text{phys}}$ containing this exceptional complex structure, an open subset $\mathcal{U} \subseteq H^3_{\mathrm{d}}(M, \mathbb{C})$ containing $\mathbf{0}$, and a diffeomorphism $\mathcal{V} \to \mathcal{U}$.

Finally we note that the G_2-structure calculation extends straightforwardly to a generic type-0 structure. Recall these take the form

$$L_3 = \mathrm{e}^{\alpha + \beta} \cdot T_{\mathbb{C}} \tag{5.186}$$

where $\alpha \in \Gamma(\wedge^3 T^*_{\mathbb{C}})$, $\beta \in \Gamma(\wedge^6 T^*_{\mathbb{C}})$ and involutivity implies $\mathrm{d}\alpha = \mathrm{d}\beta = 0$. By following the same analysis as above, one sees that the deformations of this structure will again be given by

$$H^3_{\mathrm{d}}(M,\mathbb{C}) \oplus H^6_{\mathrm{d}}(M,\mathbb{C}). \tag{5.187}$$

This gives the moduli space of the class of supersymmetric backgrounds discussed in [108], complementary to those analysed in [104, 106]. It would be interesting to analyse further the conventional geometry of these solutions.

5.4.4 Example 2: GMPT Geometries

As we saw in section 5.2.2, we can write the GMPT solutions as

$$L_3 = \mathrm{e}^{\Sigma}[L_1^{\mathcal{J}_\pm} \oplus \mathcal{U}_{\mathcal{J}_\pm}], \qquad \Sigma = C + 8\,\mathrm{i}\,\mathrm{e}^{-3A}\,\mathrm{im}\,\Phi_{\mp}, \tag{5.188}$$

where the upper/lower signs correspond to type IIA/B respectively and the $\mathrm{O}(6,6)$ bundles are appropriately embedded into $\mathrm{E}_{7(7)}\times\mathbb{R}^+$. As before, we will work in type IIB for concreteness but similar results hold for type IIA. We will use the notation set out in section 5.2.2. In particular, recall that the generalised complex structure $\mathcal{J}_-$ defines a decomposition of the generalised spinor bundles into $\mathrm{i}n$-eigenspaces $S^+ = S_2 \oplus S_0 \oplus S_{-2}$ and $S^- = S_3 \oplus S_1 \oplus S_{-1} \oplus S_{-3}$ where $S_3 \simeq \mathcal{U}_{\mathcal{J}_-}$. We can always choose C such that the twisting Σ lies in $S_0 \oplus S_2$ since any component in S_{-2} acts trivially on L_3.

We take the parameterisation

$$\begin{aligned} E_{\mathbb{C}}/L_3 &= L_{-1}^{\mathcal{J}_-} \oplus (S_1 \oplus S_{-1} \oplus S_{-3}) \oplus \wedge^5 T^*_{\mathbb{C}}, \\ \mathfrak{Q}_{\mathrm{U}(7)\times\mathbb{R}^+} &= \wedge^2 (L_{-1}^{\mathcal{J}_-})^* \oplus (S_0 \oplus S_{-2}) \oplus \wedge^6 T^*_{\mathbb{C}}. \end{aligned} \tag{5.189}$$

As before, these are not eigenspaces of J. Instead the are the spaces of natural deformations of the underlying exceptional Dirac structure defined by $L_1^{\mathcal{J}_\pm} \oplus \mathcal{U}_{\mathcal{J}_-} \subset E_{\mathbb{C}}$. Since $\Sigma \in \Gamma(S_0 \oplus S_2)$, these spaces are invariant under the action of e^{Σ} and hence can be used to describe deformations of the twisted bundle (5.188). One can similarly identify the intrinsic torsion

$$W^{\mathrm{int}}_{\mathrm{U}(7)\times\mathbb{R}^+} \simeq \wedge^3 (L_{-1}^{\mathcal{J}_-})^* \oplus (S_{-1} \oplus S_{-3}) \tag{5.190}$$

as a subbundle of K.

We leave the details of the calculation to appendix G but to summarise, we note that we deform the L_3 bundle by $\varepsilon \in \Gamma(\wedge^2 (L_{-1}^{\mathcal{J}_-})^*)$, $\chi = \chi_0 + \chi_{-2} \in \Gamma(S_0 \oplus S_{-2})$ and $\Theta \in \Gamma(\wedge^6 T^*)$, then assuming the $\mathrm{d}\mathrm{d}^{\mathcal{J}}$-lemma (G.5) [280], one can show that the integrable moduli are counted by

$$[\varepsilon] \in H^2_{\mathrm{d}_L}(M), \qquad [\chi] \in H^0_{\bar\partial}(M) \oplus H^{-2}_{\bar\partial}(M), \qquad [\Theta] \in H^6_{\mathrm{d}}(M,\mathbb{C}). \tag{5.191}$$

The differentials d_L and $\bar\partial$ are operators associated to the generalised complex structure given by Φ_- in the IIB case, and are defined in [165]. The operator d_L is the differential associated to the Lie algebroid structure $L_{-1}^{\mathcal{J}_-}$. The operators $\bar\partial$ are not the Dolbeault operators but are the generalised Dolbeault operators defined on the spinor bundles by the decomposition of $\mathrm{d} = \partial + \bar\partial$.

Hence we have

$$\mathrm{d}_L\colon \wedge^p(L_{-1}^{\mathcal{J}_-})^* \to \wedge^{p+1}(L_{-1}^{\mathcal{J}_-})^*, \qquad \bar{\partial}\colon S_n \to S_{n-1}. \tag{5.192}$$

We see that the operators in the complex (5.177) are both given by $\mathrm{d}_L + \bar{\partial} + \mathrm{d}$ acting on the appropriate bundles. The second cohomology group of d_L counts the deformations of the $\mathcal{J}_-$ generalised complex structure [165]. The $\bar{\partial}$ cohomology groups count the deformations of F and $\operatorname{im}\Phi_+$. Since M is a generalised Calabi–Yau manifold, the cohomologies of d_L and $\bar{\partial}$ are actually isomorphic. We see that apart from the top form (which just measures the Wilson line for the dual NSNS six-form potential $\tilde{B}$), all of the moduli are counted by natural differentials associated to the integrable $\mathrm{SU}(3,3)$ structure of the GMPT solutions.

This includes and extends the results of [203], where the moduli of Φ_+ keeping Φ_- fixed (and vice versa) were examined. It was also suggested that one might be able to find the full moduli space by varying Φ_- and $\operatorname{re}\Phi_+$ independently while satisfying their closure conditions. It was hoped that one could then find a solution to the $\operatorname{im}\Phi_+$ equation by examining critical points of a modified Hitchin functional by varying over a fixed cohomology class. This allows an estimate of an upper bound for the number of moduli in this case. In contrast, we are able to find the exact number of moduli by finding variations of Φ_- and $\operatorname{im}\Phi_+$ such that

$$\mathrm{d}\Phi_- = 0, \qquad F = -8\,\mathrm{d}^{\mathcal{J}_-}(\mathrm{e}^{-3A}\operatorname{im}\Phi_+). \tag{5.193}$$

The final condition $\mathrm{d}(\mathrm{e}^{-A}\operatorname{re}\Phi_+) = 0$ is imposed by the vanishing of the moment map. However, as we have mentioned, imposing this is equivalent to quotienting by $\mathrm{GDiff}_{\mathbb{C}}$ and hence we get it without imposing a further differential condition. As we have noted several times, this construction works only away from sources and hence these deformations do not account for deformations of branes or orientifolds.

We can see how each of these deformations affects the form of L_3:

$$\Phi'_- = (1 + \not{\varepsilon})\Phi_-, \tag{5.194}$$

$$F' = F + \tfrac{1}{2}\mathrm{d}\big(\operatorname{re}(\not{\varepsilon}\mu + \chi)\big), \tag{5.195}$$

$$\operatorname{im}\Phi'_+ = \operatorname{im}\Phi_+ + \tfrac{1}{8}\mathrm{e}^{3A}\operatorname{im}(\not{\varepsilon}\mu + \chi). \tag{5.196}$$

Here μ is a polyform in $\Gamma(S_2)$, related to Σ and defined in appendix G.7. As noted by Hitchin [164], $\operatorname{re}\Phi_+$ is determined by $\operatorname{im}\Phi_+$, and hence these deformations determine the full solution $\{\Phi_+, \Phi_-, F\}$. Note that a small deformation of a GMPT solution remains within the GMPT class. GMPT describes all $\mathcal{N} = 1$ solutions for which the two internal spinors are nowhere vanishing – this is an open condition and hence will not be changed by small deformations [202, 203].

Finally we consider the existence of obstructions to the linear deformations described above. We begin with the observation that a polyform deformation can be lifted to a finite deformation simply by promoting it to an exponential. Indeed this is precisely what we have done in the derivation above. The real question then is whether there are any obstructions to the generalised complex structure deformation $\varepsilon \in \Gamma(\wedge^2(L_{-1}^{\mathcal{J}_-})^*)$. A result due to Hitchin [164] states that all deformations of generalised Calabi–Yau structures are unobstructed. Since we have a global

Φ_- that satisfies $\mathrm{d}\Phi_- = 0$, we have a generalised Calabi–Yau structure defined by $\mathcal{J}_-$. Taken together, this would seem to imply that the moduli are unobstructed, much like in the previous G_2 case.

Calabi–Yau as $\mathcal{N} = 1$

As we saw in section 5.2.2, we can embed a Calabi–Yau compactification in type IIB via

$$L_3 = \mathrm{e}^{\mathrm{i}\,\mathrm{e}^{-\varphi}(\omega - \frac{1}{6}\omega\wedge\omega\wedge\omega)}[T^{0,1} \oplus T^{*1,0} \oplus \mathbb{C}\,\mathrm{e}^{3A-\varphi}\Omega]. \tag{5.197}$$

As is shown in [165], for $\Phi_- \propto \Omega$ the generalised Dolbeault operator $\bar{\partial}$ reduces to the usual Dolbeault operator associated to the complex structure defined by Ω. It is also shown that

$$H^2_{\mathrm{d}_L}(M) = H^2_{\bar{\partial}}(M,\mathbb{C}) \oplus H^1_{\bar{\partial}}(M,T^{1,0}_{\mathbb{C}}) \oplus H^0_{\bar{\partial}}(M,\wedge^2 T^{1,0}_{\mathbb{C}}), \tag{5.198}$$

$$H^0_{\bar{\partial}}(M) = \bigoplus_{i=0}^{3} H^{i,i}_{\bar{\partial}}(M,\mathbb{C}), \tag{5.199}$$

$$H^{-2}_{\bar{\partial}}(M) = H^{0,2}_{\bar{\partial}}(M,\mathbb{C}) \oplus H^{1,3}_{\bar{\partial}}(M,\mathbb{C}), \tag{5.200}$$

where the cohomologies on the left-hand side are with respect to the generalised Dolbeault operators and those on the right-hand side are with respect to the usual Dolbeault operators. Using the isomorphism provided by the three-form Ω, we see that the moduli of such a solution are counted by the Hodge numbers

$$h^{2,1} + (h^{0,0} + h^{1,1} + h^{2,2} + h^{3,3}) + h^{3,3}. \tag{5.201}$$

Note that these are the complex dimensions. Here $h^{2,1}$ corresponds to the deformations of the complex structure associated to Ω. The real part of the Dolbeault groups in the parentheses corresponds to shifts in the RR polyform potential C. The imaginary part corresponds to shifts in $\operatorname{im}\Phi_+$, which count deformations of the Kähler potential ω, and the NSNS fields ϕ and B. Notice that we have one extra, non-physical modulus here. Finally the real part of the final $H^{3,3}_{\bar{\partial}}$ gives deformations of $\tilde{B} \in \Gamma(\wedge^6 T^*)$, the six-form potential dual to B. Again we have an extra, non-physical modulus given by the imaginary part of $H^{3,3}_{\bar{\partial}}$.

The two extra, non-physical moduli correspond to changing the $\mathcal{N} = 1 \subset \mathcal{N} = 2$ that is picked out by our formalism. These moduli do not change the SU(8) structure (which gives us the physical fields in the theory), though they do rotate the $\mathrm{SU}(7) \subset \mathrm{SU}(8)$. Indeed, we note that choosing an $\mathcal{N} = 1 \subset \mathcal{N} = 2$ is equivalent to choosing a $\mathrm{U}(1) \subset \mathrm{SU}(2)$. Hence there are 2 real or 1 complex parameters that encode this choice, precisely the counting we have. Note that these extra moduli appear only for Calabi–Yau compactifications as they are really $\mathcal{N} = 2$ – a generic GMPT solution is a genuine $\mathcal{N} = 1$ solution and hence all the moduli are physical.

5.4.5 Comments on the Generic Moduli Problem

We would like to calculate the cohomology of the following complex for a generic integrable $L_3 \subset E_{\mathbb{C}}$:

$$\Gamma(E_{\mathbb{C}}) \xrightarrow{\mathrm{d}_1} \Gamma(\mathfrak{Q}_{\mathrm{U}(7)\times\mathbb{R}^+}) \xrightarrow{\mathrm{d}_2} \Gamma(W^{\mathrm{int}}_{\mathrm{U}(7)\times\mathbb{R}^+}). \tag{5.202}$$

We can use the SU(7) structure to decompose the bundles as J eigenspaces following (5.17), (5.16) and (5.25)

$$\begin{aligned} E_{\mathbb{C}} &= \mathfrak{X}_3 \oplus (\wedge^2\mathfrak{X}^*)_1 \oplus (\wedge^5\mathfrak{X}^*)_{-1} \oplus \mathfrak{X}^*_{-3} \\ \mathrm{ad}\,\tilde{F}_{\mathbb{C}} &= \mathrm{ad}\,P_{\mathrm{U}(7)\times\mathbb{R}^+} \oplus (\wedge^3\mathfrak{X})_2 \oplus (\wedge^6\mathfrak{X})_4 \oplus (\wedge^3\mathfrak{X}^*)_{-2} \oplus (\wedge^6\mathfrak{X}^*)_{-4} \\ W^{\mathrm{int}}_{\mathrm{U}(7)\times\mathbb{R}^+} &= (\wedge^4\mathfrak{X}^*)_{-5} \oplus (\wedge^7\mathfrak{X}^*)_{-7} \end{aligned} \tag{5.203}$$

where $\mathfrak{X}$ transforms in the **7** of SU(7). A natural parametrisation of embeddings is then

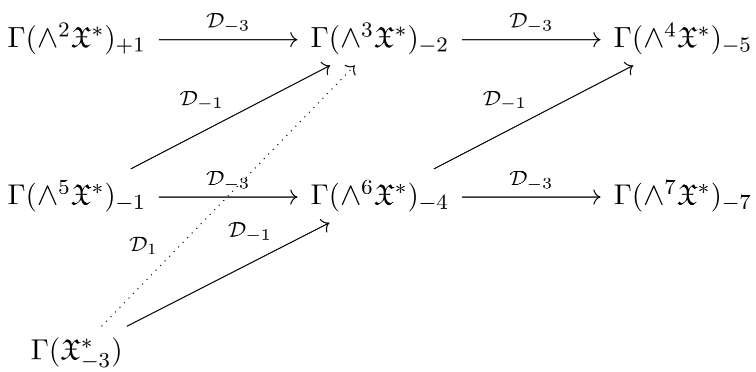
$$E_{\mathbb{C}}/L_3 = (\wedge^5\mathfrak{X}^*)_1 \oplus (\wedge^2\mathfrak{X}^*)_{-1} \oplus \mathfrak{X}^*_{-3}, \qquad \mathfrak{Q}_{\mathrm{U}(7)\times\mathbb{R}^+} = (\wedge^3\mathfrak{X}^*)_{-2} \oplus (\wedge^6\mathfrak{X}^*)_{-4}. \tag{5.204}$$

As L_3 defines an integrable $\mathrm{U}(7)\times\mathbb{R}^+$ structure, we have a torsion-free compatible connection $\mathcal{D}$. Since d_1 and d_2 are defined in terms of the Dorfman derivative L_V and $\mathcal{D}$ is torsion free, we can replace all Dorfman derivatives with $L^{\mathcal{D}}_V$, as in (5.12). This implies the maps d_1 and d_2 can be written in terms of $\mathcal{D}$. Moreover, viewing the derivative as a map $\mathcal{D}\colon R \to E^* \otimes R$, for any given generalised tensor bundle R, we can decompose E^* and hence $\mathcal{D}$ into operators

$$\mathcal{D} = \mathcal{D}_3 + \mathcal{D}_{-1} + \mathcal{D}_1 + \mathcal{D}_{-3}. \tag{5.205}$$

The compatibility of the generalised connection ensures that these operators map U(7) representations into U(7) representations in a way that will be clear in a moment. We can think of these operators as the generalisation of the Dolbeault operators to SU(7) structures.

Describing the operators $\mathrm{d}_1, \mathrm{d}_2$ in this parametrisation, one finds that the complex (5.202) decomposes as



(5.206)

Note that the involutivity of L_3 implies that $(\mathcal{D}_{-3})^2 = 0$. In fact L_3 defines a Lie algebroid and $\mathcal{D}_{-3}$ is the associated differential

$$\mathcal{D}_{-3}\colon \wedge^p\mathfrak{X}^* \to \wedge^{p+1}\mathfrak{X}^*, \tag{5.207}$$

similarly to the situation for a Dirac structure in [165]. It seems likely that under certain assumptions – notably some generalised version of the $\partial\bar{\partial}$-lemma – it is possible to write the cohomology of (5.202) in terms of the cohomology groups $H^{\bullet}_{\mathcal{D}_{-3}}(M)$ of $\mathcal{D}_{-3}$. This would be in line with the theory of deformations of complex structures [290], generalised complex structures [165], or more generally Dirac structures [291]. However the existence of the $\mathcal{D}_1$ action between $\mathfrak{X}^*_{-3}$ and $\wedge^3\mathfrak{X}^*_{-2}$ makes the analysis considerably more subtle than that for the G_2 and GMPT examples.

Chapter 6

Quantising the Exceptional Hitchin Functional

Hitchin functionals were first introduced by Nigel Hitchin in [255] as an interesting way to study the properties of $\mathrm{SL}(3,\mathbb{C})$ structures. They have since been extended to G_2 structures [151], and generalised complex structures [164]. Not only do these have interesting mathematical applications, but they have been shown to have applications to topological strings, both in the B-model [3] and in the G_2 string [79, 292, 293]. In this chapter, we will review the work done on Hitchin functionals of various types, as well as their applications to topological strings. We will then indicate how the Kähler potentials of chapters 4 and 5 define exceptional Hitchin functionals for the exceptional complex structures in each case. Moreover, we will quantise these functionals, perturbing around a Calabi-Yau and a G2 manifold respectively. This is the exceptional geometry analogue of the calculation Pestun and Witten did for generalised complex structures [3]. The partition function for the $\mathrm{SU}^*(6)$ structures may provide a geometric interpretation for the 1-loop correction to the universal hypermultiplet in 5 dimensions. The partition function for the $\mathrm{SU}(7)$ structures could give an indication of loop corrections to certain terms in the effective actions of M-theory.

6.1 The Hitchin Functionals

6.1.1 The Hitchin Functional for $\mathrm{SL}(3,\mathbb{C})$ Structures

Understanding $\mathrm{SL}(3,\mathbb{C})$ structures as the critical points of certain functionals varied over cohomology classes was first found in [255], and later studied in [294]. The key point is that, while the $\mathrm{SL}(3,\mathbb{C})$ structure is often described in terms of some holomorphic 3-form, it can in fact be determined from the real part only. In fact, provided a real 3-form lies in a particular $\mathrm{GL}(6,\mathbb{R})$ orbit of $\wedge^3 T^*$, one can find the usual complex structure and holomorphic 3-form describing the $\mathrm{SL}(3,\mathbb{C})$ structure. One can then understand integrability as critical points of a certain functional $H(\rho)$. We will review the work of [255, 294] here.

SL(3, ℂ) Structures at a Point

We will start by just considering the structures on a real 6 dimensional vector space W before extending this to 6-manifolds. Given W, there is a natural isomorphism

$$\wedge^5 W^* \cong W \otimes \wedge^6 W^* \tag{6.1}$$

We are considering the geometry of 3-forms on this space so consider any $\rho \in \wedge^3 W^*$. Using the isomorphism above, we can define the following maps

$$\begin{aligned} K_\rho : W &\longrightarrow W \otimes \wedge^6 W^* \\ w &\longmapsto (w \lrcorner \rho) \wedge \rho \end{aligned} \tag{6.2}$$

$$\begin{aligned} \lambda : \wedge^3 W^* &\longrightarrow (\wedge^6 W^*)^2 \\ \rho &\longmapsto \frac{1}{6} \operatorname{Tr} K_\rho^2 \end{aligned} \tag{6.3}$$

These satisfy the following properties

$$\operatorname{Tr} K_\rho = 0 \qquad K_\rho^2 = \lambda(\rho) 1 \tag{6.4}$$

We will be considering the space of 3-forms given by[1]

$$U = \{\rho \in \wedge^3 W^* \mid \lambda(\rho) < 0\} \tag{6.5}$$

It was shown in [255] that $\rho \in U$ if and only if the stabiliser of ρ is $\mathrm{SL}(3, \mathbb{C}) \subset \mathrm{GL}(6, \mathbb{R})$. Hence, ρ should define a complex structure on W and a complex 3-form, Ω. From (6.4) we can immediately see that we can define a complex structure by

$$I_\rho = \frac{1}{\sqrt{-\lambda(\rho)}} K_\rho \tag{6.6}$$

To uniquely define the complex 3-form, one needs to define an orientation on W. An alternative characterisation for U is $\rho \in U$ if and only if $\rho = \alpha + \bar{\alpha}$ where $\alpha \in \wedge^3 W^* \otimes \mathbb{C}$, $\alpha \wedge \bar{\alpha} \neq 0$. It turns out that α is unique up to complex conjugation. We order $\alpha, \bar{\alpha}$ so that $\alpha \wedge \bar{\alpha}$ is positive with respect to the orientation picked. We can then define a new $\hat{\rho} \in U$ by $\hat{\rho} = \mathrm{i}(\bar{\alpha} - \alpha)$. This uniquely defines a complex 3-form $\Omega = \rho + \mathrm{i}\hat{\rho}$ which is the usual definition of the $\mathrm{SL}(3, \mathbb{C})$ structure. Indeed, one can show that Ω is consistent with the complex structure I_ρ in that, when decomposed into I_ρ eigenspaces, $\Omega \in \wedge^{3,0} W^*$. Finally, we note that

$$\mathrm{i}\Omega \wedge \bar{\Omega} = 2\rho \wedge \hat{\rho} = 4\sqrt{-\lambda(\rho)} \tag{6.7}$$

The space U has a Kähler structure on it. Indeed, given some orientation $\epsilon \in \wedge^6 W^*$, we

[1] We say an element $\tau \in (\wedge^6 W^*)^2$ is positive if $\exists\, s \in \wedge^6 W^*$ such that $\tau = s \otimes s$. We say $\tau < 0$ if $-\tau$ is positive.

can define a symplectic structure by

$$\omega(\alpha, \beta)\epsilon = \alpha \wedge \beta \tag{6.8}$$

Using the function $\varphi(\rho)\epsilon = \sqrt{-\lambda(\rho)}$ we can define a Hamiltonian function vector field X_φ which has the property

$$X_\varphi(\rho) = -\hat{\rho} \tag{6.9}$$

Since $U \subset \wedge^3 W^*$, we can take $T_\rho U \cong \wedge^3 W^*$ and so we can view X_φ instead as a map $X_\varphi : U \longrightarrow \wedge^3 W^*$. It is clear from the construction of $\hat{\rho}$ that $\hat{\hat{\rho}} = -\rho$. Therefore, $X_\varphi \circ X_\varphi(\rho) = -\rho$ and so the derivative DX_φ satisfies

$$(DX_\varphi)^2 = -1 \tag{6.10}$$

We take this to be the complex structure $\mathcal{I}$ on U.

In fact, there is a more intuitive picture of the complex structure on U related to the complex structure on W. Take $\rho \in U$, which defines a complex structure I_ρ on W. Then consider the tangent space to U at ρ. As mentioned, $T_\rho U \cong \wedge^3 W^*$. We then decompose this into I_ρ eigenspaces. Then the eigenspaces of $\mathcal{I}$ are given by

$$T^{1,0}_\rho U = \wedge^{3,0} W^* \oplus \wedge^{2,1} W^* \qquad T^{0,1}_\rho U = \wedge^{1,2} W^* \oplus \wedge^{0,3} W^* \tag{6.11}$$

U therefore has a Kähler structure[2] with a Kähler metric of complex signature (4,6)[3] defined by

$$g(\alpha, \beta) = \omega(\mathcal{I}\alpha, \beta) \tag{6.12}$$

Indeed, if $\theta_1, \theta_2, \theta_3$ is a complex basis for $\wedge^{1,0} W^*$ then we find that the following must have opposite signs under the metric g.

$$\left\{ \begin{array}{c} \theta_1 \wedge \theta_2 \wedge \theta_3 \\ \theta_1 \wedge \theta_2 \wedge \bar{\theta}_2 + \theta_1 \wedge \theta_3 \wedge \bar{\theta}_3 \\ \text{cyclic perms of 1,2,3} \end{array} \right\} \qquad \left\{ \begin{array}{c} \theta_1 \wedge \theta_2 \wedge \bar{\theta}_3 \\ \theta_1 \wedge \theta_2 \wedge \bar{\theta}_2 - \theta_1 \wedge \theta_3 \wedge \bar{\theta}_3 \\ \text{cyclic perms of 1,2,3} \end{array} \right\} \tag{6.13}$$

Hence, we just need to find the sign of $\Omega = \rho + \mathrm{i}\hat{\rho}$. In fact

$$g(\Omega, \bar{\Omega})\epsilon = \omega(\mathcal{I}\Omega, \bar{\Omega})\epsilon = \mathrm{i}\Omega \wedge \bar{\Omega} = 2\varphi(\rho)\epsilon \tag{6.14}$$

But φ is a positive homogeneous function of degree 2 and so the metric is positive on $\wedge^{3,0} W^*$, and has complex signature (4,6) overall as stated.

$\mathbf{SL(3, \mathbb{C})}$ Structures on 6-Manifolds

We now extend the analysis above to 3-forms on a closed, oriented 6-manifold M. We take a globally non-vanishing 3-form $\rho \in \Omega^3(M)$. This will define a section $\lambda(\rho) \in \Gamma((\wedge^6 T^*)^2)$ which we will take to be globally negative. Hence, ρ defines an $\mathrm{SL}(3, \mathbb{C})$ structure on M via the

[2] In fact, U has a special Kähler structure.

[3] Note that in [255], Hitchin finds the signature to be $(1, 9)$. However, there appears to be a slight flaw in his argument which we correct here. This calculation does not affect the rest of his paper.

mechanism above extended pointwise to the whole of M.

We define a functional, which we call the Hitchin functional[4]

$$H_\Omega(\rho) = \int_M \varphi(\rho)\epsilon = \int_M \sqrt{-\lambda(\rho)} \tag{6.15}$$

The key observation of [255] is that ρ is a critical point of $H(\rho)$ within a given cohomology class $[\rho] \in H^3(M)$ if and only if it is the real part of a non-vanishing holomorphic 3-form Ω defining the $\mathrm{SL}(3,\mathbb{C})$ structure. To see this, we note that

$$\delta H = \int_M \delta\varphi\epsilon \tag{6.16}$$

From the symplectic interpretation in the previous section, we know that at a point $\delta\varphi(\delta\rho) = \omega(X_\varphi, \delta\rho)$, and that $X_\varphi = -\hat{\rho}$. Hence

$$\delta H = -\int_M \hat{\rho} \wedge \delta\rho \tag{6.17}$$

We are restricting the variational problem to a given cohomology class $[\rho] \in H^3(M)$ and hence we can take $\delta\rho = \mathrm{d}b$ for some $b \in \Omega^2(M)$. Therefore we see that

$$\delta H = -\int_M \hat{\rho} \wedge \mathrm{d}b \equiv 0 \qquad \Leftrightarrow \qquad d\hat{\rho} = 0 \tag{6.18}$$

Hence, the complex 3-form $\Omega = \rho + \mathrm{i}\hat{\rho}$ is closed, i.e. a holomorphic section, if and only if ρ is a critical point of the functional.

We can go further and show that, provided the $\partial\bar{\partial}$-lemma holds, these critical points are non-degenerate transverse to the action of (orientation preserving) diffeomorphisms. Hence, on any $\partial\bar{\partial}$ manifold, there is a diffeomorphism between an open patch $[\rho] \in \mathcal{U} \subset H^3(M)$, and an open patch in the moduli space $\Omega \in \mathcal{V} \subset \mathcal{M}_\Omega$. This is shown by considering the second variation of the Hitchin functional to get the Hessian and showing that if it is degenerate along some $\delta\rho = \mathrm{d}b$, then one can write $\mathrm{d}b = \mathcal{L}_v\rho$ for some vector field v.

While we won't go through the full argument laid out in [255], we will go through the derivation of the second variation. This will be important for the quantisation procedures later.

[4]The subscript Ω is used here to distinguish the Hitchin functional for $\mathrm{SL}(3,\mathbb{C})$ structures from the Hitchin functional of other structures that we will define later. It should not be taken to mean the complex object associated to ρ. Where the meaning is clear we will drop the subscript.

The second variation is essentially given by the integral of the metric $g(\delta_1\rho, \delta_2\rho)$. We have

$$\begin{aligned} \delta^2 H &= \delta_2 \left(-\int_M \hat{\rho} \wedge \delta_1 \rho \right) \\ &= \delta_2 \left(\int_M X_\varphi \wedge \delta_1 \rho \right) \\ &= \int_M \delta_2 X_\varphi \wedge \delta_1 \rho \\ &= \int_M (DX_\varphi)(\delta_2\rho) \wedge \delta_1 \rho \\ &= \int_M \mathcal{I}\delta_2\rho \wedge \delta_1\rho \end{aligned} \tag{6.19}$$

Now let us take the variation $\delta_1\rho = \delta_2\rho = \mathrm{d}b$, and expand $b = b_{20} + b_{11} + b_{02}$. b is a real 2-form and so we must have $b_{02} = \bar{b}_{20}$, and b_{11} is a real (1,1)-form. It is possible to show that the terms involving b_{02}, b_{20} appear as total derivatives under the integral and hence vanish. We are left with

$$\delta^2 H_\Omega = \int_M 2\mathrm{i}\partial b_{11} \wedge \bar{\partial} b_{11} \tag{6.20}$$

This is the action quantised in [3], as we will review later.

6.1.2 The Hitchin Functional for SU(3, 3) Structures

Hitchin later showed in [164] that a completely analogous structure holds for $\mathrm{SU}(3,3)$ structures in $\mathrm{O}(6,6)$ generalised geometry. We will review this work here and we will find that, once again, the complex pure spinor $\Phi \in \wedge^{\mathrm{ev/odd}} T^*$ can be described by the real part only, provided it lies in the correct open orbit of $\mathrm{O}(6,6)$. One can also then describe the integrability of these structures in terms of the extremisation of a particular functional over a given cohomology class.

SU(3, 3) Structures at a Point

On the space of generalised spinors $S \simeq \wedge^{\mathrm{ev/odd}} W^* \otimes (\wedge^6 W)^{1/2}$, the Mukai pairing is skew-symmetric and invariant under $\mathrm{O}(6,6)$. Hence there is a moment map

$$\begin{aligned} \mu : S &\longrightarrow \mathfrak{g}^* \\ \mu(\rho)(a) &= \frac{1}{2} \langle \sigma(a)\rho, \rho \rangle \end{aligned} \tag{6.21}$$

Here $\mathfrak{g} = \mathfrak{o}(6,6)$, $a \in \mathfrak{g}$, and $\sigma : \mathfrak{g} \longrightarrow \mathrm{End}\, S$ is the representation of $\mathfrak{g}$ on spinors. We will use the inner product $\mathrm{Tr}(XY)$ to identify $\mathfrak{g}^* \simeq \mathfrak{g}$, and so $\mu(\rho) \in \mathfrak{g}$. We define an invariant quartic function by

$$\begin{aligned} q : S &\longrightarrow \mathbb{R} \\ \rho &\longmapsto \mathrm{Tr}\, \mu(\rho)^2 \end{aligned} \tag{6.22}$$

One can show that $\mu(\rho)^2 = \frac{1}{48} q(\rho) 1$ and that, if $q(\rho) < 0$, then ρ is the real part of a pure spinor Φ that defines an $\mathrm{SU}(3,3)$ structure, i.e. a generalised Calabi-Yau structure. That is,

$\rho = \Phi + \bar{\Phi}$ where $\langle \Phi, \bar{\Phi} \rangle \neq 0$ and Φ is unique up to complex conjugation. We will again focus on the geometry of the space

$$U = \{\rho \in S \,|\, q(\rho) < 0\} \subset S \tag{6.23}$$

Using the Mukai pairing as the symplectic form, we can find a Kähler geometry on U via the following. We define a function $\varphi = \sqrt{-q(\rho)/3}$ and take the corresponding Hamiltonian vector field X_φ. Then one finds that $X_\varphi = \hat{\rho}$ where $2\Phi = \rho + \mathrm{i}\hat{\rho}$. Viewing X_φ as a function $U \longrightarrow S$, we can take the derivative DX_φ. Then, the observation that $\hat{\hat{\rho}} = -\rho$ implies that $(DX_\varphi)^2 = -1$, and hence DX_φ determines a complex structure $\mathcal{I}$ on U. This again has a convenient interpretation when the generalised complex structure defines either a conventional complex structure ($\Phi = \Omega$) or a symplectic structure ($\Phi = \mathrm{e}^{\mathrm{i}\omega}$). In each case we find that

$$T^{0,1}_{\rho_\Omega} U = \wedge^{3,2} W^* \oplus \wedge^{3,0} W^* \oplus \wedge^{2,1} W^* \oplus \wedge^{1,0} W^* \qquad T^{0,1}_{\rho_\omega} U = \mathrm{e}^{\mathrm{i}\omega} \oplus \mathrm{e}^{\mathrm{i}\omega} \wedge^2 W^* \tag{6.24}$$

We can define a Kähler metric of complex signature $(7, 9)$ via

$$g(\rho, \sigma) = \langle \mathcal{I}\rho, \sigma \rangle \tag{6.25}$$

SU(3, 3) Structures on 6-Manifolds

To lift this problem to SU(3, 3) structures defined on manifolds, we take $S \cong \wedge^{\mathrm{ev/odd}} T^*$. To do this, we must lift the structure group to $\mathrm{O}(6, 6) \times \mathbb{R}^+$, the structure group relevant for type I theories [183]. Then, given some non-vanishing section $\rho \in \Gamma(S)$, we get a global section $q(\rho) \in \Gamma((\wedge^6 T^*)^2)$ which we will assume to be globally negative. We then have a top-form $\varphi(\rho)$.

We define a functional, which we will call the extended Hitchin functional[5]

$$H_\Phi(\rho) = \int_M \varphi(\rho) = \int_M \sqrt{-(\rho)/3} \tag{6.26}$$

The first variation can be calculated by using the symplectic interpretation above. At a point $\delta\varphi = \langle X_\varphi, \delta\rho \rangle = \langle \hat{\rho}, \delta\rho \rangle$. Hence

$$\delta H = \int_M \langle \hat{\rho}, \delta\rho \rangle \tag{6.27}$$

Hitchin was then able to show that a closed, stable form $\rho \in \Omega^{\mathrm{ev/odd}}(M)$ extremises the extended Hitchin functional if and only if $\rho + \mathrm{i}\hat{\rho} = 2\Phi$ defines an integrable generalised Calabi-Yau structure. to see this, we take the first variation with $\delta\rho = \mathrm{d}b$ for some $b \in \Omega^{\mathrm{odd/ev}}(M)$ and find

$$\delta H = \int_M \langle \hat{\rho}, \mathrm{d}b \rangle \equiv 0 \qquad \Leftrightarrow \qquad \mathrm{d}\hat{\rho} = 0 \tag{6.28}$$

Hitchin goes further to say that these extrema are non-degenerate transverse to the orbit of generalised diffeomorphisms[6] provided the generalised $\partial\bar{\partial}$-lemma holds. He then uses this to

[5]The subscript Φ is used to distinguish the Hitchin functional for SU(3, 3) structures from other Hitchin functionals defined in this section. It should not be taken to be the complex object associated to ρ. Where the meaning is clear we will drop the subscript.

[6]In Hitchin's language, generalised diffeomorphisms are described by the extension of (orientation preserving)

understand the local structure of the moduli space in terms of generalised cohomology groups. We will not go through the full argument but we will provide the second variation, or the Hessian, as that will be important for understanding the topological B-model later [3]. The second variation is again given by

$$\delta^2 H = \delta_1 \left(\int_M \langle \hat{\rho}, \delta_2 \rangle \right) \tag{6.29}$$

$$= \int_M \langle \delta_1 X_\varphi, \delta_2 \rho \rangle \tag{6.30}$$

$$= \int_M \langle (DX_\varphi)\delta_1 \rho, \delta_2 \rho \rangle \tag{6.31}$$

$$= \int_M \langle \mathcal{I} \delta_1 \rho, \delta_2 \rho \rangle \tag{6.32}$$

If we take the complex structure case, so $\Phi = \Omega$, we can take $\delta_1 \rho = \delta_2 \rho = \mathrm{d}b$ and write $b = b_{00} + b_{20} + b_{11} + b_{20} + b_{22}$, then up to exact terms under the differential, and overall constants, the second variation is given by

$$\delta^2 H_\Phi = \int_M b_{11} \wedge \partial\bar{\partial} b_{11} + b_{00} \wedge \partial\bar{\partial} b_{22} \tag{6.33}$$

6.1.3 The Hitchin Functional for $\mathbf{G_2}$ Structures

The final functional we will review is that of G_2 structures, also discovered and studied by Hitchin in [151, 294]. While the overall picture of G_2 is very similar to that of the (generalised) complex structures above, in that an integrable G_2 structure can be described by an extremised functional over a particular cohomology class, some of the details are different. In particular, there is no natural Kähler structure on the space of G_2 structures. Interestingly, this Kähler structure is restored when you consider G_2 as a subset of SU(7) structures, as outlined in chapter 5.

$\mathbf{G_2}$ Structures at a Point

Take W a real 7 dimensional vector space and $\phi \in \wedge^3 W^*$. We get a map

$$\begin{aligned} B_\phi : S^2 W &\longrightarrow \wedge^7 W^* \\ (v, w) &\longmapsto -\frac{1}{6} (v \lrcorner \phi) \wedge (w \lrcorner \phi) \wedge \phi \end{aligned} \tag{6.34}$$

Alternatively, we can view this as a map $K_\phi : W \longrightarrow W^* \otimes \wedge^7 W^*$ and so we have

$$(\det K_\phi)^{1/9} \in \wedge^7 W^* \tag{6.35}$$

diffeomorphisms by exact 2-forms

$$\Omega^{\text{exact}}(M) \longrightarrow \mathcal{G} \longrightarrow \text{Diff}_0$$

which, provided $\det K_\phi \neq 0$, defines a natural orientation on W. This gives an inner product on W given by

$$g_\phi(v,w) = B_\phi(v,w)(\det K_\phi)^{-1/9} \tag{6.36}$$

One can show that the stabiliser of ϕ such that g_ϕ is positive definite is $\mathrm{G}_2 \subset \mathrm{GL}(7,\mathbb{R})$ and that φ is in the orbit of the standard G_2 form

$$\tilde{\phi} = e^{123} - e^{145} - e^{167} - e^{246} + e^{257} - e^{347} - e^{356} \qquad g_{\tilde{\phi}} = \sum_i e^i \otimes e^i \tag{6.37}$$

We call such 3-forms positive and stable, and we will take the subspace $U \subset \wedge^3 W^*$ of positive, stable 3-forms.

Fixing some orientation $\epsilon \in \wedge^7 W^*$, we can define a positive, homogeneous function of degree 7/3 on U by

$$\varphi(\phi)\epsilon = (\det K_\phi)^{1/9} \tag{6.38}$$

Taking $*_\phi$ to be the Hodge operator defined by g_ϕ, one can show that

$$\varphi(\phi)\epsilon = \frac{1}{6}\phi \wedge *_\phi\phi \qquad \delta\varphi\epsilon = \frac{7}{18} *_\phi \phi \wedge \delta\phi$$

This expresses $*_\phi\phi$ in terms of the derivative of some function. From now on we will refer to $*_\phi$ as $*$, knowing that it depends non-linearly on ϕ. We can equally define the G_2 structure in terms of the 4-form $*\phi$, although we won't consider that here.

G_2 Structures on 7-Manifolds

To consider G_2 structures on a 7 dimensional manifold M, we take a global non-vanishing $\phi \in \Omega^3(M)$ which is everywhere positive and stable. $(\det K_\phi)^{1/9}$ defines a global non-vanishing section of $\wedge^7 T^*$. We can then define a functional, we will call the G_2 Hitchin functional, given by[7]

$$H_\phi(\phi) = \int_M \varphi(\phi)\epsilon = \int_M \frac{1}{6}\phi \wedge *\phi \tag{6.39}$$

The first variation is then

$$\delta H = \int_M \delta\varphi\epsilon = \int_M \frac{7}{18} * \phi \wedge \delta\phi \tag{6.40}$$

Hitchin then found that a closed, positive, stable $\phi \in \Omega^3(M)$ defines an integrable[8] G_2 structure if and only if ϕ extremises the G_2 Hitchin functional restricted to a given cohomology class $[\phi] \in H^3(M)$. Indeed, taking the variation $\delta\phi = \mathrm{d}b$ for some $b \in \Omega^2(M)$ in the first variation we find

$$\delta H \sim \int_M *\phi \wedge \mathrm{d}b \equiv 0 \qquad \Leftrightarrow \qquad \mathrm{d} * \phi = 0 \tag{6.41}$$

Hitchin again goes further to say that these critical points are non-degenerate transverse to

[7]The subscript ϕ is used to distinguish the G_2 Hitchin functional from other Hitchin functionals defined in this section. It should not be thought of as the variable ϕ over which the variational problem is taken. Where the meaning is clear, we will drop the subscript.

[8]Technically, this is just a torsion-free G_2 structure, which does not guarantee local flatness. We will not consider this subtlety here.

the orbit of diffeomorphisms. This means that locally there is a diffeomorphism between an open patch $[\phi] \in \mathcal{U} \subset H^3(M)$ and an open subset $\mathcal{V} \subset \mathcal{M}_\phi$. He showed this by considering the second variation, or Hessian, which is given by the following expression.

$$\delta^2 H = \int_M \frac{4}{3} * \pi_1(\delta_1\phi) \wedge \pi_1(\delta_2\phi) + *\pi_7(\delta_1\phi) \wedge \pi_7(\delta_2\phi) - *\pi_{27}(\delta_1\phi) \wedge \pi_{27}(\delta_2\phi) \tag{6.42}$$

where the π_n are projections of the space of three forms onto their irreducible G_2 subrepresentations, $\wedge^3 T^* = \wedge^3_1 T^* \oplus \wedge^3_7 T^* \oplus \wedge^3_{27} T^*$.

If we take $\delta\phi = \mathrm{d}b$ and write $b = B + v\lrcorner\phi$, with $B \in \wedge^2_{14} T^*$, then [292] showed that, up to total derivative terms, the second variation is given by

$$\delta^2 H_\phi = \int_M *\pi_7(\mathrm{d}B) \wedge \pi_7(\mathrm{d}B) - *\pi_{27}(\mathrm{d}B) \wedge \pi_{27}(\mathrm{d}B) \tag{6.43}$$

6.2 Quantisation of the Hitchin Functionals

Beyond being interesting mathematically for understanding the moduli space of various G-structures, Hitchin functionals have found use in understanding the quantum nature of string theory. It was first hoped that the quantisation of the Hitchin functional for $\mathrm{SL}(3,\mathbb{C})$ structures would match the 1-loop correction to the topological B-model, given the correspondence between the observables in that model and properties of the complex structure on the target manifold. However, it was shown in [3] that it was the quantisation of the Hitchin functionals of $\mathrm{SU}(3,3)$ structures of $\mathrm{O}(6,6)$ geometry that gave the correct 1-loop calculation. The link between topological strings and generalised complex structures was further examined in [295]. The G_2 Hitchin functional was quantised in [292] and compared to the topological G_2 string of [79, 293] but was found to not quite agree. We will briefly review their work here.

6.2.1 Quantisation of H_Ω and H_Φ

We will first review the work of [3]. There, they tried to understand the relation between the Hitchin functionals, and the topological B-model at 1-loop. The topological B-model [75,77,296] is given by a twist of the $\mathcal{N} = 2$ σ-model from a 2 dimensional Riemann surface to, in our case, a Calabi-Yau manifold X. The correlations functions are only dependent on the complex structure moduli of the target space X. There is another twist of the σ-model that gives the A-model, whose observables are related to the symplectic structure on X. These two theories are related by mirror symmetry. Given the geometrical nature of the B-model, and the ability of the Hitchin functional to classically describe integrable complex structures, it was conjectured in [297–299] that these two theories should somehow be related.

They performed the quantisation of the Hitchin functional at quadratic order and compared it to the partition function of the topological B-model. It was found in [77], that the 1-loop contribution to the B-model partition function can be written in terms of the holomorphic Ray-Singer torsions [300]. To define these, we consider the $\bar{\partial}$ complex on the bundles of holomorphic p-forms $\Omega^{p,0}(X)$ and take the alternating product of determinants of Laplacians on the space of (p,q)-forms

$$I^{\mathrm{RS}}_{\bar{\partial},p}(\Omega^{p,0}) = \left(\prod_{q=0}^{3} \left(\det{}'\Delta_{pq}\right)^{q(-1)^{q+1}} \right) \tag{6.44}$$

Here, $\det'$ means the ζ regularised product of non-zero eigenvalues. The Laplacian Δ_{pq} is just the restriction of $\Delta_{\bar{\partial}} = \frac{1}{2}\Delta_{\mathrm{d}}$ to $\Omega^{p,q}(X)$. It is known that, while these do depend on the complex structure, they do not depend on the Hermitian metric chosen to define Δ. The 1-loop partition function is then given by

$$Z_{B,1-loop} = \prod_{p=0}^{3} \left(I^{\mathrm{RS}}_{\bar{\partial},p} \right)^{p(-1)^p} = \frac{I_1}{I_0^3} \tag{6.45}$$

where for short-hand we have written $I^{\mathrm{RS}}_{\bar{\partial},p} = I_p$ and used the Hodge duality to get $I_{3-p} = I_p$.

To compare this to the Hitchin functional, they took the real part of the $\mathrm{SL}(3,\mathbb{C})$ structure on X, ρ_0, which they assumed to be closed and a minimum of the Hitchin functional. They then considered the partition function of the functional over exact variations $\rho = \rho_0 + \mathrm{d}b$.

$$Z_\Omega([\rho_0]) = \int Db\, \mathrm{e}^{-H_\Omega(\rho)} \tag{6.46}$$

The 1-loop contribution to this is given at the quadratic order, and hence the action we want to quantise is given by (6.20). This has a gauge structure given by the following

$$\delta b_{11} = \partial b_{01} + \bar{\partial} b_{10} \qquad \delta b_{10} = \partial b_{00} \qquad \delta b_{01} = \bar{\partial} b_{00} \tag{6.47}$$

where the reality conditions enforce $\bar{b}_{10} = b_{01}$, b_{00} a real function.

BV quantisation allows us to take account of the gauge structure[9]. We start by introducing new fields for each of the gauge parameters (called 'ghosts', and 'ghosts for ghosts' in the BV language). These are the fields b_{pq} with $p,q \leq 1$ and with statistics $(-1)^{p+q}$. We introduce a nilpotent BRST operator Q which generates the gauge transformations above. We then need to introduce antifields to each of the fields with opposite statistics. These are conjugate variables to each of the fields above with respect to the odd-symplectic bracket given by $\langle f, g\rangle = \int f \wedge g$. We can therefore take the conjugate fields to be b_{pq} with $p,q \geq 2$ and with statistics $(-1)^{p+q+1}$.

The BV action is defined to be

$$\begin{aligned} S &= S_0 + \sum_i \langle \Phi_i^*, Q\Phi_i \rangle \\ &= \int_M b_{11} \wedge \partial\bar{\partial} b_{11} + b_{22} \wedge (\partial b_{01} + \bar{\partial} b_{10}) + b_{32} \wedge \bar{\partial} b_{00} + b_{23} \wedge \partial b_{00} \end{aligned} \tag{6.48}$$

To quantise this, we need to gauge fix. This requires picking a gauge-fixing fermion or choosing a Lagrangian submanifold of the space of fields. We choose the Lagrangian submanifold given

[9] A review of BV quantisation is given in [301]

by

$$\begin{aligned} b_{11} &= \partial^\dagger\bar{\partial}^\dagger\alpha_{22} & b_{22} &= \partial^\dagger\alpha_{32} + \bar{\partial}^\dagger\alpha_{23} \\ b_{10} &= \partial^\dagger\alpha_{20} + \mathrm{i}\partial\alpha_{00} & b_{23} &= \partial^\dagger\alpha_{33} \\ b_{01} &= \bar{\partial}^\dagger\alpha_{02} + \mathrm{i}\bar{\partial}\alpha_{00} & b_{32} &= \bar{\partial}^\dagger\alpha_{33} \\ & & b_{33} &= 0 \end{aligned} \tag{6.49}$$

After performing the path integral over this Lagrangian submanifold, and careful consideration of the operators that appear in the determinants, [3] showed that at 1-loop, the Hitchin functional gives

$$Z_\Omega^{1-loop} = \frac{I_1}{I_0} \tag{6.50}$$

This clearly does not agree with the 1-loop calculation of the B-model (6.45).

This would seem to suggest that the topological B-model is not a theory of simply complex structures, at least not in the sense of the $\mathrm{SL}(3,\mathbb{C})$ Hitchin functional. In [3], they decided to look at the extended Hitchin functional for $\mathrm{SU}(3,3)$ structures instead. At the classical level, these functionals describe the same objects. Indeed, [164, Prop 12] shows that on a Calabi-Yau manifold with $b_1 = 0$, $\rho \in \Omega^{\mathrm{odd}}(M)$ is an extremum of the extended Hitchin functional if and only if $\rho = \mathrm{re}\,\Omega \in \Omega^3(M)$ is the real part of an integrable $\mathrm{SL}(3,\mathbb{C})$ structure[10]. However, the off-shell degrees of freedom are given by all even polyforms, and hence they will provide additional contributions to the 1-loop partition function.

The 1-loop contribution is given by the path integral of (6.33) over b_{11}, b_{22}, b_{00}. The integral over the b_{11} fields will be the same as for the $\mathrm{SL}(3,\mathbb{C})$ case and hence it was just left to consider the path integral of

$$\int_M b_{00} \wedge \partial\bar{\partial} b_{22} \tag{6.51}$$

This has a much more complicated gauge structure than previously and hence the BV quantisation is more involved. However, [3] gives details of the process and finds, after much calculation, that the 1-loop contribution to the extended Hitchin functional is

$$Z_\Phi^{1-loop} = \frac{I_1}{I_0^3} = Z_B^{1-loop} \tag{6.52}$$

This tells us that, at least to 1-loop, the B-model is in fact a theory of generalised complex structures.

6.2.2 Quantisation of H_ϕ

We will now look at the work of [292]. Here, they aimed to do the analogous thing [3] but to the Hitchin functional for G_2 structures H_ϕ. Their hope was to compare it to the topological G_2 string defined in an earlier piece of work [79]. While they didn't find full agreement, we will outline their calculation here as it will be useful for us when we come to look at the exceptional Hitchin functionals below.

[10] In fact, ρ can be an exact B-field transformation of $\mathrm{re}\,\Omega$, however these are trivial deformations from the point of view of generalised geometry.

They wanted to calculate the path integral over $b \in \Omega^2(M)$ of

$$\delta^2 H_\phi = \int_M \frac{4}{3} * \pi_1(\mathrm{d}b) \wedge \pi_1(\mathrm{d}b) + *\pi_7(\mathrm{d}b) \wedge \pi_7(\mathrm{d}b) - *\pi_{27}(\mathrm{d}b) \wedge \pi_{27}(\mathrm{d}b) \tag{6.53}$$

The gauge symmetry of this action is diffeomorphisms which act on ϕ via $\delta\phi = \mathrm{d}(v \lrcorner \phi)$ for some arbitrary vector field $v \in \chi(M)$. We note that the 2-form b can be decomposed into G_2 representations $b_7 \in \Omega^2_7(M)$ and $b_{14} \in \Omega^2_{14}(M)$. We can therefore use the diffeomorphism freedom to fix $b_7 = 0$. This can be done through an algebraic redefinition of variables $b \to b - v \lrcorner \phi$ and hence will not introduce any determinants into the calculation of the partition function[11].

With this we are left with the path integral over $b_{14} = B$ of the functional (6.43)

$$\int_M *\pi_7(\mathrm{d}B) \wedge \pi_7(\mathrm{d}B) - *\pi_{27}(\mathrm{d}B) \wedge \pi_{27}(\mathrm{d}B) = \int_M \mathrm{d}B \wedge *(2\pi_7 - 1)\mathrm{d}B \tag{6.54}$$

The gauge structure of this action is given by

$$QB = \pi_{14}(\mathrm{d}A) \qquad QA = \mathrm{d}C \qquad QC = 0 \tag{6.55}$$

We then introduce antifields $\tilde{B} \in \Omega^5_{14}(M)$, $\tilde{A} \in \Omega^6(M)$, $\tilde{C} \in \Omega^7(M)$ with the action

$$S = \int_M \mathrm{d}B \wedge *(2\pi_7 - 1)\mathrm{d}B + \tilde{B} \wedge \mathrm{d}A + \tilde{A} \wedge \mathrm{d}C \tag{6.56}$$

The π_{14} in QB is absorbed into the definition of $\tilde{B}$ as an element of the $\Omega^5_{14}(M)$. This gives the gauge structure for the antifields

$$Q\tilde{B} = 3\mathrm{d} * (2\pi_7 - 1)\mathrm{d}B \qquad Q\tilde{A} = \mathrm{d}\tilde{B} \qquad Q\tilde{C} = \mathrm{d}A \tag{6.57}$$

We again need to pick a Lagrangian submanifold. The details of choosing such a space are complicated by the fact that B is not an arbitrary 2-form, but one living in a subrepresentation. However, they show in [292, App C] that one can choose the following representatives for the Lagrangian space.

$$\begin{aligned} B &= \mathrm{d}^\dagger(2\pi_7 - 1)\mathrm{d}\beta & \qquad \tilde{B} &= *\pi_{14}(\mathrm{d}\mathrm{d}^\dagger\tilde{\beta}) \\ A &= \mathrm{d}^\dagger\alpha & \tilde{A} &= \mathrm{d}^\dagger\tilde{\alpha} \\ & & \tilde{C} &= 0 \end{aligned} \tag{6.58}$$

where $\beta \in \Omega^2_{14}(M)$, $\alpha, \tilde{\beta} \in \Omega^2(M)$, $\tilde{\alpha} \in \Omega^7(M)$.

[11] A similar argument explains why only b_{11} contributes to the quantisation of H_Ω, and why only b_{00}, b_{11}, b_{22} contribute to the quantisation of H_Φ

Performing the path integral then gives[12]

$$\begin{aligned} Z_\phi &= \left(\det{}'\left(\Delta_{14}^2 - \frac{3}{2}\pi_{14}\mathrm{d}\mathrm{d}^\dagger\right)\right)^{-1/2} |\det\,\pi_{14}\mathrm{d}_1||\det \mathrm{d}_0|^{-1} \\ &= \left(\det{}'\Delta_{14}^2\right)^{-1/2} |\det \pi_{14} d_1|^2 \left(\det{}'\Delta_1^0\right)^{-1/2} \\ &= \left(\det{}'\Delta_{14}^2\right)^{-1/2} \left(\det{}'\Delta^1\right) \left(\det{}'\Delta^0\right)^{-3/2} \\ &= \left(\det{}'\Delta_{14}\right)^{-1/2} \left(\det{}'\Delta_7\right) \left(\det{}'\Delta_1\right)^{-3/2} \end{aligned} \tag{6.59}$$

We have used properties of determinants written in appendix H and in the last line we have used the fact that the Laplacian operator has the same spectrum on isomorphic G_2 representations. Any G_2 manifold has three independent topological invariants defined by products of determinants of Laplacians. These are given by

$$I_0 = \left(\det{}'\Delta_7\right)^{-1/2} \left(\det{}'\Delta_1\right)^{7/2} \tag{6.60}$$

$$I_1 = \left(\det{}'\Delta_{14}\right)^{-1/2} \left(\det{}'\Delta_7\right) \tag{6.61}$$

$$\begin{aligned} I^{\mathrm{RS}} &= \left(\det{}'\Delta^3\right)^{-1/2} \left(\det{}'\Delta^2\right)^{3/2} \left(\det{}'\Delta\right)^{-5/2} \left(\det{}'\Delta^0\right)^{7/2} \\ &= \left(\det{}'\Delta_{27}\right)^{-1/2} \left(\det{}'\Delta_{14}\right)^{3/2} \left(\det{}'\Delta_7\right)^{-3/2} \left(\det{}'\Delta_1\right)^{3} \end{aligned} \tag{6.62}$$

The final expression is in fact the Ray-Singer, or analytic, torsion of the manifold M [302]. There is one other independent expression we can build out of determinants of Laplacians that is related to the G_2 structure on the manifold. Unlike the expressions above, which are independent of the G_2 metric chosen, the following will depend on the metric and so is not topological. It is given by the analytic torsion of the following complex that exists on any G_2 manifold.

$$\begin{aligned} &\check{D} : 0 \longrightarrow \wedge_1^0 T^* \xrightarrow{\mathrm{d}} \wedge_7^1 T^* \xrightarrow{\pi_7 \mathrm{d}} \wedge_7^2 T^* \xrightarrow{\pi_1 \mathrm{d}} \wedge_1^3 T^* \longrightarrow 0 \\ &\mathrm{Tor}(\check{D}) = \left(\det{}'\Delta_7\right)^{-1/2} \left(\det{}'\Delta_1\right)^{3/2} \end{aligned} \tag{6.63}$$

We can then express Z_ϕ in terms of the topological invariants and $\mathrm{Tor}(\check{D})$. We find

$$Z_\phi = I_0^{-3/4}\, I_1\, \mathrm{Tor}(\check{D})^{3/4} \tag{6.64}$$

Clearly, this is not independent of the choice of G_2 metric, unlike the case for H_Ω and H_Φ. This may not be so surprising since, unlike for (generalised) complex structures, the existence of a G_2 structure always defines a metric. Hence, any dependence on a G_2 structure will naturally lead to dependence on a metric.

This quantisation of this functional did not match the 1-loop contribution to the topological G_2 string. [292] also looked at a structure defined in $O(7,7)$ geometry defining a $\mathrm{G}_2 \times \mathrm{G}_2$ structure that were defined in [201]. Unfortunately, that did not match the topological string partition function either. Despite H_ϕ not providing the geometric interpretation for the topological string they were hoping for, the work of [292] will be useful in the following sections when we come

[12]To see why this is the kinetic operator requires understanding how the de Rham operator acts on different G_2 representations. These are outlined explicitly in [277].

to quantise the exceptional Hitchin functionals, and provide possible loop corrections to some terms in the effective actions of M-theory.

6.3 The Exceptional Hitchin Functionals

We have already seen a candidate for a Hitchin-like functional for the exceptional complex structures described in the previous chapters - the Kähler potential. To understand why this is the natural choice, we need to reframe some of the discussion of the other Hitchin functionals into complex objects. This only exists for the H_Ω and H_Φ Hitchin functionals as there is no natural complex structure on the space of G_2 structures. One could view the Hitchin functional for SU(7) structures as the natural extension of H_ϕ to a complex parameter space.

For both $SL(3,\mathbb{C})$ and $SU(3,3)$ structures, the Hitchin functionals were defined in terms of a real stable (poly)form ρ. This (poly)form defined a positive homogeneous function $\varphi(\rho)$ and a complex object

$$\rho + \mathrm{i}\hat{\rho} \qquad \hat{\rho} = \omega^{-1} \lrcorner \mathrm{d}\varphi \tag{6.65}$$

where ω is the symplectic structure on the space of ρ given by either the exterior product on 3-forms, or the Mukai pairing on polyforms. The complex object is then the usual definition of the relevant G-structure. Using this, we can rewrite the Hitchin functionals in the following way

$$H_\Omega = \int_M \mathrm{i}\Omega \wedge \bar{\Omega} \sim \mathcal{K}_{SL(3,\mathbb{C})} \qquad H_\Phi = \int_M -\mathrm{i}\left\langle \Phi, \bar{\Phi} \right\rangle \sim \mathcal{K}_{SU(3,3)} \tag{6.66}$$

We see that, in each case, the Hitchin functional is just the Kähler potential on the space of structures. These are discussed in appendix C. In that appendix, we also saw that integrability could be rephrased as involutivity and a vanishing moment map. In addition, the vanishing of the moment map is equivalent to extremising these potentials over complexified (generalised) diffeomorphisms.

These properties of the conventional Hitchin functionals suggest that the natural choice for the exceptional Hitchin functionals are the Kähler potentials in each case. For example, the SU(7) Hitchin functional would be[13]

$$H_\psi = \mathcal{K}_{SU(7)} = \int_M \left(\mathrm{i}\, s(\psi, \bar{\psi})\right)^{1/3} \tag{6.67}$$

In principle, one could determine what this functional is only in terms of $\mathrm{re}\,\psi$ and consider the real problem, however we have not done so here[14]. We would like to understand what role H_ψ plays in the quantum effective action of M-theory or string theory, if any. This involves

[13] The subscript ψ is only used to distinguish this Hitchin functional from the other Hitchin functionals described. It should not be thought of as the argument ψ over which we take the variational problem. Where the meaning is clear, we will drop the subscript.

[14] Following the SU(3,3) case and following [180], it seems like the Hitchin functional is given by the trace of the moment map $\zeta(a) = s(a \cdot \lambda, \lambda)$ which gives

$$H_\psi = \int_M q(\lambda)^{1/6}$$

Here, $\lambda = \mathrm{re}\,\psi$, $a \in \Gamma(\mathrm{ad}\,\tilde{F})$ and ζ is the moment map induced because s is an $E_{7(7)}$ invariant symplectic structure. Then varying with respect to λ gives a term which looks like $\mathrm{i}(\bar{\psi} - \psi)$ if we take λ as in [180] quantising the quadratic contribution to this functional from a perturbation around some fixed background

$$\psi = \psi_0 + \delta\psi \qquad \longrightarrow \qquad S_0 = \delta^2 H(\delta\psi, \delta\bar{\psi}) \tag{6.68}$$

The first problem we run into is determining what is the space of variations we should take. Recall from the other Hitchin functionals, we don't take an arbitrary variation, but a variation just within some cohomology class. This made sense when the structure was a polyform, however ψ has an expansion in terms of natural bundles other than exterior powers of T^*. Therefore, it is not clear what the analogue of de Rham cohomology would be. Moreover, the real objects defined in the previous Hitchin functionals existed in an open orbit of $\mathrm{GL}(d,\mathbb{R})$ and hence an arbitrary exact deformation still defined a suitable G-structure. Here, neither the complex object ψ, nor its real part $\mathrm{re}\,\psi$, live in an open orbit of $\mathrm{E}_{d(d)} \times \mathbb{R}^+$ and hence we must be careful to define appropriate variations.

One option take the same variations that we took for the classic equations of motion. That is, the vanishing of the moment map for the integrability was equivalent to extremising over complexified generalised diffeomorphisms. We might therefore expect that the variations we should take are therefore $\delta\psi = L_V\psi$ for some complex $V \in \Gamma(E_{\mathbb{C}})$. However, running the same argument for $\mathrm{SU}(3,3)$ structures would not give the same quadratic term and hence the partition function would be different[15].

To determine which point of view to take, we look at the example of a compactification of type IIA on a Calabi-Yau manifold. This is an example of a GMPT background which were analysed in chapter 5. The SU(7) structure in this case is given by

$$\psi \sim \mathrm{e}^{8\mathrm{i}\hat{\rho}} \cdot \Phi_+ \qquad \Phi_+ = \mathrm{e}^{\mathrm{i}\omega} \tag{6.70}$$

where $\hat{\rho} = \mathrm{im}\,\Omega$ is the imaginary part of the of the holomorphic 3-form Ω, and ω is the Kähler form. We can label an arbitrary generalised vector as $V = v + (b_0 + b_2 + b_4 + b_6) + \mathrm{e}^{8\mathrm{i}\hat{\rho}} B_1 \mathrm{e}^{-8\mathrm{i}\hat{\rho}} + \tilde{B}_5 + \tau \in \Gamma(E_{\mathbb{C}})$, where the subscript denotes the degree of the differential form, and where $\tau \in \Gamma(T^* \otimes \wedge^6 T^*)$[16]. Then, the generalised diffeomorphism along V gives

$$\psi' = (1 + L_V)\psi = \mathrm{e}^{8\mathrm{i}\hat{\rho} + 8\mathrm{i}\mathrm{d}(v\lrcorner\hat{\rho}) - \mathrm{d}(b_0+b_2+b_4) - \mathrm{d}\tilde{B}_5 - 32\mathrm{d}(v\lrcorner\hat{\rho}\wedge\hat{\rho})} \cdot \left(\mathrm{e}^{\mathrm{i}\omega + \mathrm{i}\mathrm{d}(v\lrcorner\omega) + \mathrm{d}B_1} \cdot 1\right) \tag{6.72}$$

We also have real diffeomorphisms as a gauge symmetry. Examining (6.72) and using the freedom to change the real parts of $v, b_n, B_1, \tilde{B}_5$, we see that we can give the deformations just

[15] It is easy to check that generalised diffeomorphisms can only vary ψ within the same *type* of generalised complex structure. Hence, the variation around a Calabi-Yau 3-fold would give $\delta\rho = \mathrm{d}b_2 + \mathrm{d}b_4$ which would give a 1-loop action of

$$\int_M b_{11} \wedge \partial\bar{\partial} b_{11} \tag{6.69}$$

The partition function of this, we have seen, is not the same as the 1-loop B-model.

[16] For type IIA the generalised tangent bundle can be calculated through a dimensional reduction of the M-theory generalised tangent bundle and gives

$$E = T \oplus \mathbb{R} \oplus \wedge^2 T^* \oplus \wedge^4 T^* \oplus \wedge^6 T^* \oplus T^* \oplus \wedge^5 T^* \oplus T^* \otimes \wedge^6 T^* \tag{6.71}$$

as

$$\psi' = \mathrm{e}^{8\mathrm{i}\hat{\rho}+\mathrm{id}(b_0+b_2+b_4)+\mathrm{id}\tilde{B}_5} \cdot \left(\mathrm{e}^{\mathrm{i}\omega+\mathrm{id}B_1} \cdot 1\right) \tag{6.73}$$

where $b_n, B_1, \tilde{B}_5$ are real forms, and where we have used the freedom in v to set $b_2 \in \Omega^{1,1}(M)$. We have then made some redefinitions to absorb other terms involving v.

We see that complexified generalised diffeomorphisms recover the perturbations of [3] for generalised complex structures[17]

$$\delta\hat{\rho} = \mathrm{d}b_0 + \mathrm{d}b_{11} + \mathrm{d}b_4 \tag{6.74}$$

However, we also get more perturbations coming from B (which deforms the Kähler form), and $\tilde{B}$. Complexified generalised diffeomorphisms therefore seem to give a minimal extension of the Pestun and Witten model that includes both the NSNS and the RR degrees of freedom of string theory. More generally, we would find that complexified generalised diffeomorphisms give variations of both spinors $\Phi_\pm$ simultaneously, and hence give variations of the full geometry of the background. These variations are therefore the natural candidate for the off-shell degrees of freedom.

To find the quadratic action to quantise, we first need to write $\delta\psi = R \cdot \psi$, for some $R \in \Gamma(\mathrm{ad}\,\tilde{F})$. The 1-loop contribution is then given by

$$\delta^2 H = \int_M g(R, \bar{R}) \tag{6.75}$$

where g is the hermitian metric on the cone $\mathrm{E}_{7(7)} \times \mathbb{R}^+/\mathrm{SU}(7)$, calculated around the point $\psi \in \mathcal{Z}$. While calculating the full NSNS and RR contributions to the partition function on the Calabi-Yau would be an interesting problem on its own, we will instead focus on two other cases relating to compactifications of M-theory which have been studied previously in this thesis. Firstly, we will look at the case of the Hitchin functional for $\mathrm{SU}^*(6)$ structures in $\mathrm{E}_{6(6)} \times \mathbb{R}^+$ geometry perturbed around a Calabi-Yau manifold. Then we will look at perturbations of $\mathrm{SU}(7)$ structures around a G_2 manifold.

6.3.1 The Hitchin Functional for SU*(6) Structures

The Hitchin functional for $\mathrm{SU}^*(6)$ can be written as

$$H_\chi = \int_M \left(\mathrm{Tr}(\chi\bar{\chi})\right)^{1/2} \tag{6.76}$$

We would like to understand the 1-loop partition function when perturbing around a Calabi-Yau background. For $\mathrm{SU}^*(6)$ structures, there is an ambiguity in the choice of χ but we will choose[18]

$$\chi = \mathrm{e}^{\mathrm{i}\rho} \cdot 1 \tag{6.77}$$

[17] In that paper they perturb $\rho = \mathrm{re}\,\Omega$ but it is equivalent to perturb $\hat{\rho} = \mathrm{im}\,\Omega$.

[18] It would be an interesting consistency check to see if the partition function calculated for $\chi = \frac{1}{2}(I - \mathrm{vol} - \mathrm{vol}^\#$ is the same as the one calculated in this section.

where $\rho = \operatorname{re}\Omega$. Taking the set of variations to be given by complexified generalised diffeomorphisms, we find

$$\begin{aligned} L_V \chi &= L_V(\mathrm{e}^{\mathrm{i}\rho} \cdot 1) \\ &= (\mathcal{L}_v - \mathrm{d}\omega - \mathrm{d}\sigma)\, \mathrm{e}^{\mathrm{i}\rho} \cdot 1 \\ &= (\mathrm{i}\mathcal{L}_v \rho + \frac{1}{2}[\mathcal{L}_v \rho, \rho]) \cdot \mathrm{e}^{\mathrm{i}\rho} \cdot 1 - (\mathrm{d} + \mathrm{d}\sigma) \cdot \mathrm{e}^{\mathrm{i}\rho} \cdot 1 \\ &= (\mathrm{i}\mathrm{d}(v \lrcorner \rho) - \frac{1}{2}\mathrm{d}(v \lrcorner \rho) \wedge \rho - \mathrm{d}\omega - \mathrm{d}\sigma) \cdot \mathrm{e}^{\mathrm{i}\rho} \cdot 1 \\ &= R \cdot \chi \end{aligned} \tag{6.78}$$

In the final line we have expressed the variation in terms of an adjoint action of $R \in \Gamma(\operatorname{ad} \tilde{F}_{\mathbb{C}}$. We see that R is given by the action of exact 3-forms and 6-forms.

We also have the gauge symmetry of real generalised diffeomorphisms. Examining the expression above, we see that we can fix the gauge symmetry by setting $\operatorname{re}\omega$, $\operatorname{re}\sigma$, and the $\Omega^{0,2}(M)$ part of $\operatorname{im}\omega$ to 0. With this simplification, we see that we can choose R to be

$$R = \mathrm{i}\mathrm{d}b + \mathrm{i}\mathrm{d}c \tag{6.79}$$

where $b \in \Omega^{1,1}(M)_{\mathbb{R}}$ and $c \in \Omega^5(M)_{\mathbb{R}}$. The quadratic action is then given by the metric on the cone $\mathrm{E}_{6(6)} \times \mathbb{R}^+/\mathrm{SU}^*(6)$ which is calculated in detail in appendix I. The final result is

$$S_0 = \int_M \partial b \wedge \bar{\partial} b + \mathrm{d}c \wedge * \mathrm{d}c \tag{6.80}$$

This action is decoupled and hence the partition function factorises into Z_b and Z_c. The partition function for b is the same as that for $\mathrm{SL}(3,\mathbb{C})$ structures that we reviewed in section 6.2.1. The partition function for c is that of a free 5-form can be calculated using BV quantisation. One finds, using the identities of [3], that it is trivial on a Calabi-Yau manifold. Hence, the 1-loop contribution is

$$Z_\chi = \frac{I_1}{I_0} \tag{6.81}$$

where I_n is the n^{th} holomorphic Ray-Singer torsion described in section 6.2.1. Despite the need to introduce a metric to write the action, it turns out to be independent of this choice, and depends only on the complex structure moduli of the Calabi-Yau.

6.3.2 The Hitchin Functional for SU(7) Structures

As mentioned, the Hitchin functional for SU(7) structures can be written

$$H_\psi = \int_M \left(\mathrm{i}\, s(\psi, \bar{\psi})\right)^{1/3} \tag{6.82}$$

We would like to understand the 1-loop contribution to the path integral when perturbed around a G_2 background. That is, we would like to quantise the quadratic piece of the functional arising from variations

$$\psi = \mathrm{e}^{\mathrm{i}\phi} \cdot 1 + \delta\psi \tag{6.83}$$

It is important to understand which variations to take. Unlike in the previous Hitchin functionals, the real part of ψ is not in an open orbit of the **912**. Hence, we cannot take an arbitrary variation and still expect to have an SU(7) structure. Instead, we argued that we should take variations generated by complexified generalised diffeomorphisms. We write

$$\delta\psi = L_V\psi = R \cdot \psi \tag{6.84}$$

where in the final equality, we have used the fact that ψ defines an integrable SU(7) structure to express the Dorfman derivative along some $V \in \Gamma(E_{\mathbb{C}})$ as the adjoint action of some $R \in \Gamma(\operatorname{ad}\tilde{F}_{\mathbb{C}})$. We find that

$$\begin{aligned} L_V\psi &= L_V(\mathrm{e}^{\mathrm{i}\phi}\cdot 1) \\ &= (\mathcal{L}_v - \mathrm{d}\omega - \mathrm{d}\sigma)\mathrm{e}^{\mathrm{i}\phi}\cdot 1 \\ &= (\mathcal{L}_v\mathrm{e}^{\mathrm{i}\phi})\cdot 1 - (\mathrm{d}\omega + \mathrm{d}\sigma)\cdot\mathrm{e}^{\mathrm{i}\phi}\cdot 1 \\ &= (\mathrm{i}\mathcal{L}_v\phi + \frac{1}{2}[\mathcal{L}_v\phi,\phi])\cdot\mathrm{e}^{\mathrm{i}\phi}\cdot 1 - (\mathrm{d}\omega + \mathrm{d}\sigma)\cdot\mathrm{e}^{\mathrm{i}\phi}\cdot 1 \\ &= (\mathrm{i}\mathrm{d}(v\lrcorner\phi) + \frac{1}{2}\mathrm{d}(v\lrcorner\phi)\wedge\phi - \mathrm{d}\omega - \mathrm{d}\sigma)\cdot\mathrm{e}^{\mathrm{i}\phi}\cdot 1 \end{aligned} \tag{6.85}$$

We can see that R is given in terms of exact 3-forms and 6-forms. However, using the gauge symmetry of real diffeomorphisms we can make some simplifications. Examining the expression above we can see that we can use the gauge symmetry to set $\operatorname{re}\omega$, $\operatorname{re}\sigma$, and $\pi_7(\operatorname{im}\omega)$ to 0. Hence, we can choose the variation to be

$$R = \mathrm{i}\mathrm{d}a + \mathrm{i}\mathrm{d}b \tag{6.86}$$

where $a \in \Omega^2_{\mathbf{14}}(M)_{\mathbb{R}}$, $b \in \Omega^5(M)_{\mathbb{R}}$. The quadratic action is then given by the metric on the cone $(\mathrm{E}_{7(7)} \times \mathbb{R}^+)/\mathrm{SU}(7)$

$$\delta^2 H(\delta\psi, \delta\bar{\psi}) = \int_M g(R, \bar{R}) \tag{6.87}$$

A detailed calculation of this metric is given in appendix J but we shall just give the final result here.

$$S_0 = \int_M \pi_7(\mathrm{d}a)\wedge *\pi_7(\mathrm{d}a) - \pi_{27}(\mathrm{d}a)\wedge *\pi_{27}(\mathrm{d}a) - \mathrm{d}b\wedge *\mathrm{d}b \tag{6.88}$$

We have absorbed real constants into the definition of a and b in the action above.

Fortunately, a and b are decoupled in the action and hence we can find the partition function relatively easily. We notice that the term involving a is precisely the quadratic term in the G_2 Hitchin functional (6.43). We already saw the contribution of this term to the path integral in (6.59). The term involving b is just the action of a free 5-form on a 7-manifold. The contribution to the path integral is easily calculated using BV quantisation, and is also given in [292, App D] to be

$$Z_b = (\det\Delta_1)^{-1/2} I^{\mathrm{RS}} \tag{6.89}$$

Altogether, the 1-loop contribution to the partition function is then

$$\begin{aligned} Z_\psi &= Z_a Z_b \\ &= (\det \Delta_{14})^{-1/2} (\det \Delta_7) (\det \Delta_1)^{-2} I^{\text{RS}} \\ &= I_1 I^{\text{RS}} I_0^{-1} \text{Tor}(\check{D}) \end{aligned} \tag{6.90}$$

Again, we find that the partition function is not independent of the specific G_2 structure as it has a $\text{Tor}(\check{D})$ term. It would be interesting to see how this explicitly depends on the G_2 moduli, much like is done for the complex moduli of the topological B-model.

6.4 Possible Physical Applications

It has been known for some time that the genus g partition functions of the topological A and B model provide certain amplitudes in the 4 dimensional effective theory of type IIA and B string theory compactified on a Calabi-Yau. Indeed, in any Calabi-Yau compactification scheme, there are 2 universal multiplets - the 'graviphoton' multiplet which contains a graviton and a vector whose field strengths are denoted R, T respectively, and the 'universal hypermultiplet' which has scalars (S, Z). It was shown in [303] that the higher order corrections to the graviphoton in type IIA (resp. B) are given schematically by

$$S_{\text{graviphoton}} = S_{\text{tree}} + \int_M \sum_{g=1}^{\infty} F_g \left(g R^2 T^{2g-2} + 2g(g-1)(RT)^2 T^{2g-4} \right) \tag{6.91}$$

where, it turns out, F_g is the genus g partition function of the topological A (resp. B) model, initially calculated in [77]. They also showed in [303] that the corrections to the universal hypermultiplet are given by $\tilde{F}_g$, the genus g partition function of the topological B (resp. A) model. The action is given by

$$S_{\text{universal hyper}} = S_{\text{tree}} + \int_M \sum_{g=1}^{\infty} \tilde{F}_g \left(g(\partial\partial S)^2 (\partial Z)^{2g-2} + 2g(g-1)(\partial\partial S\, \partial Z)^2 (\partial Z)^{2g-4} \right) \tag{6.92}$$

The 1-loop partition function F_1 for the B model is precisely what is calculated in [3] from the Hitchin functional of generalised complex structures, reviewed above. These arguments can be lifted to the 5 dimensional universal hypermultiplet [304–306].

One may conjecture that the partition function Z_χ calculated for the $SU^*(6)$ structures in 6.3.1 gives the 1-loop corrections to the universal hypermultiplet in 5 dimensions. Indeed, we saw that the partition function is topological in the sense that it is independent of the precise Calabi-Yau metric used. This would match the analysis in 4 dimensions, as well as the work of [306] in which they calculate the corrections to the universal hypermultiplet in terms of the Euler characteristic of the Calabi-Yau. It would be interesting to see if we can match precisely the corrections they get from our expression for Z_χ given in terms of holomorphic Ray-Singer torsions. Similar arguments were used in [77] to relate the holomorphic Ray-Singer torsions to the graviphoton corrections.

A more challenging question is the application of the partition function Z_ψ calculated for the SU(7) structures in 6.3.2. In a similar argument, one may guess that the partition function corresponds to 1-loop corrections to the action of the chiral fields in the 4 dimensional effective theory. There is not much in the literature to compare this with and so we may leave it as a prediction. To check, one would need to do the full loop calculation in the supergravity theory. One may expect a certain amount of cancellation due to supersymmetry and so there may be a localisation argument that reproduces the path integral calculated. From the geometric point of view, it seems at least plausible that the 1-loop corrections to the Kähler metric on the chiral cone come from a suitable path integral of the Kähler potential. Alternatively, there has been some work on loop corrections of M-theory using the M-theory/F-theory duality and then compactifying on a Calabi-Yau 4-fold [307–310]. It may be interesting to see if their work provides an indication of how Z_ψ precisely plays a role in the quantum corrections to the effective 4 dimensional theory.

Chapter 7

Discussion

In this thesis we set out to understand better the geometry of generic supersymmetric flux background of string theory and M-theory. We focused on Minkowski backgrounds in $D = 4, 5$ and studied them within the framework of generalised geometry. This is a mathematical formalism that combines the structure of manifolds with diffeomorphisms, and gauge fields with gauge transformations, into a Leibniz algebroid. The precise construction of this algebroid naturally encodes the geometry of the manifold, along with the gerbe structure of the gauge fields. Moreover, the structure group of the algebroid corresponds with the symmetry group of the respective reduced supergravity being studied. It has been known for some time that, when lifted to generalised geometry, the supersymmetric backgrounds of string and M-theory are described by integrable G-structures. Integrability is, much like in conventional geometry, defined to be the existence of a torsion-free compatible connection. Using this knowledge, we set out to build the geometry of Minkowski backgrounds of type II and M-theory in $D = 4, 5$, as well as the geometry of Minkowski backgrounds of heterotic theories in $D = 4$. Moreover, using the integrable G-structure, we hoped to learn more about the moduli space of these structures.

Interestingly, despite the difference in groups for each of the cases, we found a common structure within the geometry. We found that the G-structures for all supersymmetric backgrounds could be described in terms of some global non-vanishing tensor ψ, which defined a particular subbundle of the generalised tangent bundle. The subbundle defined a strictly weaker $\mathbb{C}^* \times G$-structure, as it only defined ψ up to some complex scaling. The integrability of the G-structure was equivalent to the involutivity of this subbundle under the Dorfman derivative, as well as the vanishing of a particular moment map. This moment map was that associated to the action of generalised diffeomorphisms on the space of G-structures. We highlighted how this was analogous to the story of integrable $\mathrm{SL}(3, \mathbb{C})$ structures on a 6-manifold. Given the correspondence, we called these structures *exceptional complex structures*.

Using the properties of the integrable G-structures, we were able to propose a procedure for finding the moduli of any given supersymmetric background. The vanishing moment map provides the moduli space the structure of a Kähler quotient of the space of involutive structures, by generalised diffeomorphisms. However, the Kähler quotient is equivalent to the quotient by the complexified group. Moreover, given rescalings of ψ correspond to a non-physical modulus, we we were able to express the physical moduli space as the space of involutive subbundles quotiented by the action of complexified generalised diffeomorphisms. The physical moduli space should give the space of massless modes in the lower dimensional effective theory defined on the external Minkowski space. A subtlety arises if the background preserves more supersymmetry, but we were able to account for this in particular examples. We could therefore express the physical moduli of a particular background as the cohomology of a complex of deformations of the involutive bundle.

In the heterotic case, we were able to express the generic subbundle $L_1 \subset T \oplus \operatorname{ad} P_G \oplus T^*$ in terms of the tensors defining the SU(3) structure of the background. We saw that involutivity and the vanishing of the moment map recreated precisely the equations of the Hull-Strominger system. Moreover, we were able to match the complex controlling the deformations of L_1, to the complex describing the moduli of Heterotic backgrounds previously found.

For $D = 5$ backgrounds of type II and M-theory, the exceptional complex structure described only part of the full G-structure defining supersymmetry. Despite this, we were able to classify all the possible exceptional complex structures on compactifications of M-theory. Moreover, we were able to exploit the additional structure in $D = 5$ that L_1 should be part of a triplet of equivalent structures, to show that compactifications of M-theory should either be type 0, have a complex structure, or have a 3 dimensional foliation. Moreover, we found the exact moduli of generic backgrounds satisfying a certain property we called the *generalised* $\partial\bar{\partial}$*-lemma*. These moduli are expressed in terms of 'natural' cohomology groups associated to the Lie algebroid structure of L_1. We showed that this reproduced what we expected for Calabi-Yau compactifications of M-theory.

For $D = 4$ backgrounds, we studied the cases of G_2 manifolds in M-theory, and GMPT backgrounds in type II. We should that involutivity and the vanishing moment reproduce the supersymmetry equations in each case as we expected. While we couldn't find a generic result for the moduli of generic backgrounds, we were able to find the exact moduli in the two cases studied. For G_2 manifolds this recreated the expected results. The moduli of GMPT backgrounds were not previously known and we were able to express them in terms of cohomology groups associated to the generalised complex structure with which they are defined.

We found that there were two additional interesting outcomes from this description of supersymmetric backgrounds. The first came from looking at the correspondence between Kähler quotients and complex quotients more carefully. In fact, to ensure the quotient space has a nice topology, one needs to restrict themselves to a set of so-called 'polystable points' within the space of G-structures. This idea came from geometric invariant theory which is the study of group quotients of projective spaces. While this theory mostly applies to finite dimensional spaces, similar techniques have been used to study infinite dimensional problems, like the existence of Kähler-Einstein metrics. The possibility of using geometric invariant theory to prove a generalisation of Yau's theorem to the Hull-Strominger system have been proposed in multiple places. Our work on heterotic backgrounds provided a direct interpretation in terms of moment maps, a key feature of many GIT problems, and hence provided a more direct avenue for studying this problem. Moreover, we were able to show that our moment map theory extends work done previously by others on the so-called dilaton functional. For the $D = 4$ backgrounds in M-theory, we described how GIT may give a way to find a generalisation of Yau's theorem to G_2 manifolds. We noted that the Hilbert-Mumford criterion would not directly apply as the Kähler potential is not convex. However, we also showed that a slight adaption to the problem gave a convex norm functional and hence gave a promising candidate for approaching this problem.

The second interesting feature is the correspondence of the Kähler potentials on the space of exceptional complex structures, and Hitchin functionals. Hitchin functionals have been studied in great detail for complex structures, generalised complex structures, and G_2 structures. Moreover, the 1-loop contribution to the path integral of the Hitchin functional of generalised complex structures have been linked to the topological B-model, and hence can be used to calculate coefficients in the 4d effective action of string theory. We were able to study the 1-loop contribution to the path integral of the Kähler potential of $SU^*(6)$ structures perturbed around a Calabi-Yau, and SU(7) structures perturbed around a G_2 manifold. This provides a clear candidate for certain 1-loop corrections to effective actions coming from M-theory.

7.1 Future Directions

There are many possibilities for future research coming from the work in this thesis. An obvious extension is to continue the work on quantising the exceptional Hitchin functionals and understanding their role in the quantum description of M-theory, if they have any. The relation between the topological string and the Hitchin functional of $SU(3,3)$ structures has been well studied [3, 75, 296–298]. Not only are these theories interesting mathematically, but they describe a physical subsector of the full string theory, relating to certain graviphoton terms in the effective field theory in 4 dimensions [77, 303]. These describe strings on backgrounds with H-flux, and their connection to observables in M-theory have been examined in [311, 312]. The exceptional Hitchin functionals provide a candidate theory that includes the RR fluxes as well, or alternatively a subsector of M-theory. Much work has been done on understanding the effective action coming from M-theory in lower dimensions [283, 305, 307, 313]. It would be instructive to see if we could match the calculation of the 1-loop of the exceptional Hitchin functional to certain coefficients in the effective actions in $D = 4, 5$. In particular, we expect the quantisation of the exceptional complex structure in $D = 5$ to give corrections to the universal hypermultiplet in the effective theory. If this is true, it would provide evidence that the analogous structure in 7 dimensions would provide corrections in 4 dimensions, and we may be able to gain predictions from studying it further. We also have an analogous Hitchin functional for the heterotic string. If we ignore the gauge bundle, then the quantisation would be very similar to that of the conventional Hitchin functional, except now including information about the symplectic structure ω. This would likely give a geometric interpretation of a topological subsector of heterotic string theory [314]. We would like to see if this quantisation matches the results of that paper.

Another avenue of future research would be to understand the condition for polystability of the exceptional complex structures in various settings. We say in chapters 3 and 5 that understanding the moduli space of supersymmetric backgrounds requires a better understanding of the complex orbits of the GDiff group on the space of structures. Mathematicians have been particularly interested in the heterotic case as it provides a natural extension to Yau's theorem to a non-Kähler space [74, 177]. Similar approaches have been used understand Kähler-

Einstein equations [248], flat connections on Riemann surfaces [244], and slope-stability of gauge bundles [73, 245, 250]. Mathematically, converting the infinite dimensional problem to a form suitable for the techniques of GIT can be quite involved - for the Kähler-Einstein case it involves taking a particular limit of certain powers of line bundles over the space. However, we can follow certain ideas to gain intuition for what features may be important in the study of stability. One object that is often studied is an invariant to the $\mathrm{GDiff}_{\mathbb{C}}$ flow called the Futaki invariant. This appears as an obstruction to stability and we can define an analogue for exceptional complex structures in terms of the complex object ψ. Naively, this Futaki invariant seems to measure the Chern class of the line bundle that ψ lives in. Non-triviality of this bundle is a clear obstruction to the existence of an SU(7) structure so it does not seem to tell us anything interesting. However, there may be cases where the limit of the $\mathrm{GDiff}_{\mathbb{C}}$ flow may produce a structure of a different type. In these cases, the Futaki invariant seems to give much more information on possible obstructions. Therefore, studying and classifying the phenomenon of type change would be a very important step towards understanding stability.

The work on $D = 4$ compactifications and SU(7) structures has provided a procedure for finding the moduli of any given flux background. We were able to use this to find the exact moduli of GMPT backgrounds which describe all type II backgrounds with non-vanishing internal Killing spinors. These describe very broad classes of backgrounds but they are not exhaustive. In particular, the NS5 brane solution is not contained within this set of solutions [120, 150]. We could apply the same techniques to find the moduli of these backgrounds. We also found the moduli of so-called type 0 solutions. We would like to get a better understanding of which backgrounds these correspond to. They likely describe backgrounds in which the internal Killing spinor satisfies $\epsilon^T \epsilon \neq 0$. Hence they may link to the work of [108]. We could also look at the moduli of the triple M5-brane intersection in M-theory backgrounds [315]. It is likely that this falls outside of the type 0 class, but we could still apply the same techniques to find the moduli. The work of [104] provides even more examples to test this work.

It is worth noting that the majority of this work applies only to smooth backgrounds away from sources. Hence, the moduli we have found do not include brane moduli. To understand these better, we would like to understand calibrations in flux backgrounds in more detail [235, 316, 317]. Some work has already been done in this direction [318–321]. The exceptional geometry of SU(7) structures may provide insight into how to provide a unified description for supersymmetric flux calibrations. Similar work was done in [322] for AdS_5 backgrounds.

Extending the exceptional geometry described in this thesis to AdS backgrounds would also be of great use to understanding the AdS/CFT correspondence. For AdS_5 backgrounds, we saw that a non-zero cosmological constant does not break the involutivity of the subbundle L_1. Instead, the cosmological constant appears as a non-vanishing moment map. Kähler quotients with non-vanishing moment maps are well understood and hence we were able to find the spectrum of some dual CFT4 in terms of cohomology groups associated to some distribution $\Delta \subset T_{\mathbb{C}}$[1]. It would be interesting to see if these cohomology groups, graded by the R-charge, could be calculated explicitly for simple cases such as the Maldacena-Nunez solutions [121]. In

[1]There is on going work on this problem in the IIB case by Anthony Ashmore, Mariana Graña, Michela Petrini, Edward Tasker, and Daniel Waldram.

the $D = 4$ case, the issue is reversed. The cosmological constant does not appear in the moment map, but instead as an obstruction to the involutivity of the subbundle L_3. This breaks the Lie algebroid structure of the subbundle and hence it is not clear if the moduli will be counted by some natural cohomology groups. However, there may be enough structure to still find some information of the moduli and, as a consequence, the conformal manifold of CFT_3 which are not well understood.

Bibliography

[1] A. Ashmore, C. Strickland-Constable, D. Tennyson, and D. Waldram, "Generalising G_2 geometry: involutivity, moment maps and moduli", arXiv:1910.04795 [hep-th].

[2] A. Ashmore, C. Strickland-Constable, D. Tennyson, and D. Waldram, "Heterotic backgrounds via generalised geometry: moment maps and moduli", arXiv:1912.09981 [hep-th].

[3] V. Pestun and E. Witten, "The Hitchin functionals and the topological B-model at one loop", *Lett. Math. Phys.* **74** (2005)21–51, arXiv:hep-th/0503083 [hep-th].

[4] P. A. M. Dirac, "The quantum theory of the emission and absorption of radiation", *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character* **114** 767, (1927)243–265.

[5] M. Gell-Mann and F. E. Low, "Quantum electrodynamics at small distances", *Phys. Rev.* **95** (Sep, 1954)1300–1312.

[6] K. G. Wilson, "The Renormalization Group and Strong Interactions", *Phys. Rev. D* **3** (1971)1818.

[7] K. G. Wilson, "Renormalization group and critical phenomena. 1. Renormalization group and the Kadanoff scaling picture", *Phys. Rev. B* **4** (1971)3174–3183.

[8] K. Wilson and J. B. Kogut, "The Renormalization group and the epsilon expansion", *Phys. Rept.* **12** (1974)75–199.

[9] G. 't Hooft and M. Veltman, "One loop divergencies in the theory of gravitation", *Ann. Inst. H. Poincare Phys. Theor. A* **20** (1974)69–94.

[10] D. M. Capper, G. Leibbrandt, and M. R. Medrano, "Calculation of the graviton self-energy using dimensional regularization", *Phys. Rev. D* **8** (Dec, 1973)4320–4331.

[11] M. H. Goroff and A. Sagnotti, "The Ultraviolet Behavior of Einstein Gravity", *Nucl. Phys. B* **266** (1986)709–736.

[12] S. Deser and P. van Nieuwenhuizen, "One Loop Divergences of Quantized Einstein-Maxwell Fields", *Phys. Rev. D* **10** (1974)401.

[13] S. Deser, H.-S. Tsao, and P. van Nieuwenhuizen, "One Loop Divergences of the Einstein Yang-Mills System", *Phys. Rev. D* **10** (1974)3337.

[14] S. Deser and P. van Nieuwenhuizen, “Nonrenormalizability of the Quantized Dirac-Einstein System”, *Phys. Rev. D* **10** (1974)411.

[15] N. Arkani-Hamed, L. Motl, A. Nicolis, and C. Vafa, “The String landscape, black holes and gravity as the weakest force”, *JHEP* **06** (2007)060, `arXiv:hep-th/0601001`.

[16] D. Z. Freedman, P. van Nieuwenhuizen, and S. Ferrara, “Progress Toward a Theory of Supergravity”, *Phys. Rev. D* **13** (1976)3214–3218.

[17] M. B. Green, H. Ooguri, and J. H. Schwarz, “Nondecoupling of Maximal Supergravity from the Superstring”, *Phys. Rev. Lett.* **99** (2007)041601, `arXiv:0704.0777 [hep-th]`.

[18] K. Stelle, “Non-renormalization theorems in maximal supergravity”, *Proc. Steklov Inst. Math.* **272** 1, (2011)246–255.

[19] G. Veneziano, “Construction of a crossing-simmetric, regge-behaved amplitude for linearly rising trajectories”, *Il Nuovo Cimento A (1965-1970)* **57** 1, (1968)190–197.

[20] S. Fubini and G. Veneziano, “Level structure of dual-resonance models”, *Nuovo Cim. A* **64** (1969)811–840.

[21] S. Fubini, D. Gordon, and G. Veneziano, “A general treatment of factorization in dual resonance models”, *Phys. Lett. B* **29** (1969)679–682.

[22] K. Bardakci and S. Mandelstam, “Analytic solution of the linear-trajectory bootstrap”, *Phys. Rev.* **184** (1969)1640–1644.

[23] Y. Nambu, “Quark model and the factorization of the Veneziano amplitude”,.

[24] L. Susskind, “Harmonic-oscillator analogy for the veneziano model”, *Phys. Rev. Lett.* **23** (1969)545–547.

[25] T. Yoneya, “Connection of Dual Models to Electrodynamics and Gravidynamics”, *Prog. Theor. Phys.* **51** (1974)1907–1920.

[26] S. R. Coleman and J. Mandula, “All Possible Symmetries of the S Matrix”, *Phys. Rev.* **159** (1967)1251–1256.

[27] J. Polchinski, *String theory: Volume 1, an introduction to the bosonic string*. Cambridge university press, 1998.

[28] J. Polchinski, *String theory: Volume 2, superstring theory and beyond*. Cambridge university press, 1998.

[29] L. Brink, P. Di Vecchia, and P. S. Howe, “A Locally Supersymmetric and Reparametrization Invariant Action for the Spinning String”, *Phys. Lett. B* **65** (1976)471–474.

[30] D. Friedan, E. J. Martinec, and S. H. Shenker, “Conformal Invariance, Supersymmetry and String Theory”, *Nucl. Phys. B* **271** (1986)93–165.

[31] J. Polchinski, "Dirichlet Branes and Ramond-Ramond charges", *Phys. Rev. Lett.* **75** (1995)4724–4727, arXiv:hep-th/9510017.

[32] C. V. Johnson, *D-branes.* Cambridge university press, 2002.

[33] C. Lovelace, "Pomeron form-factors and dual Regge cuts", *Phys. Lett. B* **34** (1971)500–506.

[34] P. Ramond, "Dual Theory for Free Fermions", *Phys. Rev. D* **3** (1971)2415–2418.

[35] A. Neveu and J. Schwarz, "Tachyon-free dual model with a positive-intercept trajectory", *Phys. Lett. B* **34** (1971)517–518.

[36] F. Gliozzi, J. Scherk, and D. I. Olive, "Supersymmetry, Supergravity Theories and the Dual Spinor Model", *Nucl. Phys. B* **122** (1977)253–290.

[37] M. B. Green and J. H. Schwarz, "Supersymmetrical String Theories", *Phys. Lett. B* **109** (1982)444–448.

[38] M. B. Green and J. H. Schwarz, "Anomaly Cancellation in Supersymmetric D=10 Gauge Theory and Superstring Theory", *Phys. Lett. B* **149** (1984)117–122.

[39] D. J. Gross, J. A. Harvey, E. J. Martinec, and R. Rohm, "The Heterotic String", *Phys. Rev. Lett.* **54** (1985)502–505.

[40] J. H. Schwarz and A. Sen, "Duality symmetric actions", *Nucl. Phys. B* **411** (1994)35–63, arXiv:hep-th/9304154.

[41] J. H. Schwarz, "An SL(2,Z) multiplet of type IIB superstrings", *Phys. Lett. B* **360** (1995)13–18, arXiv:hep-th/9508143. [Erratum: Phys.Lett.B 364, 252 (1995)].

[42] P. Hořava and E. Witten, "Heterotic and type i string dynamics from eleven dimensions", *Nuclear Physics B* **460** 3, (1996)506 – 524.

[43] J. Polchinski and E. Witten, "Evidence for heterotic - type I string duality", *Nucl. Phys. B* **460** (1996)525–540, arXiv:hep-th/9510169.

[44] M. Dine, P. Y. Huet, and N. Seiberg, "Large and Small Radius in String Theory", *Nucl. Phys. B* **322** (1989)301–316.

[45] E. Witten, "String theory dynamics in various dimensions", *Nucl. Phys. B* **443** (1995)85–126, arXiv:hep-th/9503124.

[46] E. Bergshoeff, E. Sezgin, and P. Townsend, "Supermembranes and Eleven-Dimensional Supergravity", *Phys. Lett. B* **189** (1987)75–78.

[47] J. Schwarz, "Physical States and Pomeron Poles in the Dual Pion Model", *Nucl. Phys. B* **46** (1972)61–74.

[48] J. Callan, Curtis G., E. Martinec, M. Perry, and D. Friedan, "Strings in Background Fields", *Nucl. Phys. B* **262** (1985)593–609.

[49] J. Callan, Curtis G., I. R. Klebanov, and M. Perry, "String Theory Effective Actions", *Nucl. Phys. B* **278** (1986)78–90.

[50] I. Jack and D. Jones, "String Theory Effective Actions and σ Model Beta Functions", *Phys. Lett. B* **193** (1987)449–454.

[51] C. Hull and R. Reid-Edwards, "Non-geometric backgrounds, doubled geometry and generalised T-duality", *JHEP* **09** (2009)014, `arXiv:0902.4032 [hep-th]`.

[52] C. Hull and B. Zwiebach, "Double Field Theory", *JHEP* **09** (2009)099, `arXiv:0904.4664 [hep-th]`.

[53] E. Plauschinn, "Non-geometric backgrounds in string theory", *Phys. Rept.* **798** (2019)1–122, `arXiv:1811.11203 [hep-th]`.

[54] O. Hohm, C. Hull, and B. Zwiebach, "Generalized metric formulation of double field theory", *JHEP* **08** (2010)008, `arXiv:1006.4823 [hep-th]`.

[55] O. Hohm and S. K. Kwak, "Double Field Theory Formulation of Heterotic Strings", *JHEP* **06** (2011)096, `arXiv:1103.2136 [hep-th]`.

[56] P. du Bosque, "Advancements in Double & Exceptional Field Theory on Group Manifolds", other thesis, 10, 2017.

[57] O. Hohm and H. Samtleben, "Exceptional Field Theory I: $E_{6(6)}$ covariant Form of M-Theory and Type IIB", *Phys. Rev. D* **89** 6, (2014)066016, `arXiv:1312.0614 [hep-th]`.

[58] O. Hohm and H. Samtleben, "Exceptional field theory. II. $\mathrm{E}_{7(7)}$", *Phys. Rev.* **D89** (2014)066017, `arXiv:1312.4542 [hep-th]`.

[59] J. Hughes, J. Liu, and J. Polchinski, "Supermembranes", *Phys. Lett. B* **180** (1986)370–374.

[60] P. K. Townsend, "Supersymmetric extended solitons", *Physics Letters B* **202** 1, (1988)53–57.

[61] M. Duff and K. Stelle, "Multimembrane solutions of D = 11 supergravity", *Phys. Lett. B* **253** (1991)113–118.

[62] A. Dabholkar, G. W. Gibbons, J. A. Harvey, and F. Ruiz Ruiz, "Superstrings and Solitons", *Nucl. Phys. B* **340** (1990)33–55.

[63] M. Duff and J. Lu, "Elementary five-brane solutions of D = 10 supergravity", *Nucl. Phys. B* **354** (1991)141–153.

[64] M. Duff and J. Lu, "Type II p-branes: The Brane scan revisited", *Nucl. Phys. B* **390** (1993)276–290, `arXiv:hep-th/9207060`.

[65] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity", *Int. J. Theor. Phys.* **38** (1999)1113–1133, arXiv:hep-th/9711200.

[66] M. Trigiante, "Gauged Supergravities", *Phys. Rept.* **680** (2017)1–175, arXiv:1609.09745 [hep-th].

[67] V. Braun, Y.-H. He, B. A. Ovrut, and T. Pantev, "A Standard model from the E(8) x E(8) heterotic superstring", *JHEP* **06** (2005)039, arXiv:hep-th/0502155.

[68] V. Braun, Y.-H. He, B. A. Ovrut, and T. Pantev, "A Heterotic standard model", *Phys. Lett. B* **618** (2005)252–258, arXiv:hep-th/0501070.

[69] P. Candelas, G. T. Horowitz, A. Strominger, and E. Witten, "Vacuum Configurations for Superstrings", *Nucl. Phys. B* **258** (1985)46–74.

[70] L. J. Dixon, "SOME WORLD SHEET PROPERTIES OF SUPERSTRING COMPACTIFICATIONS, ON ORBIFOLDS AND OTHERWISE", in *Summer Workshop in High-energy Physics and Cosmology*. 10, 1987.

[71] W. Lerche, C. Vafa, and N. P. Warner, "Chiral Rings in N=2 Superconformal Theories", *Nucl. Phys. B* **324** (1989)427–474.

[72] S.-T. Yau, "On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampére equation", *Comm. Pure Appl. Math.* **31** 3, (1978)339–411.

[73] S.-T. Yau and J. Li, "Hermitian–Yang–Mills connections on non-Kähler manifolds", *World Scient. Publ. London* (1987)560–573.

[74] M. Garcia-Fernandez, R. Rubio, C. Shahbazi, and C. Tipler, "Canonical metrics on holomorphic Courant algebroids", arXiv:1803.01873 [math.DG].

[75] E. Witten, "Topological Sigma Models", *Commun. Math. Phys.* **118** (1988)411.

[76] M. Gromov, "Pseudo holomorphic curves in symplectic manifolds", *Inventiones mathematicae* **82** 2, (1985)307–347.

[77] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, "Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes", *Commun. Math. Phys.* **165** (1994)311–428, arXiv:hep-th/9309140.

[78] S. Donaldson, "Some recent developments in Kähler geometry and exceptional holonomy", *arXiv e-prints* (Aug., 2018)arXiv:1808.03995, arXiv:1808.03995 [math.DG].

[79] J. de Boer, A. Naqvi, and A. Shomer, "The Topological G(2) string", *Adv. Theor. Math. Phys.* **12** 2, (2008)243–318, arXiv:hep-th/0506211.

[80] E. Witten, "Anti-de Sitter space and holography", *Adv. Theor. Math. Phys.* **2** (1998)253–291, arXiv:hep-th/9802150.

[81] S. Gubser, I. R. Klebanov, and A. M. Polyakov, “Gauge theory correlators from noncritical string theory”, *Phys. Lett. B* **428** (1998)105–114, arXiv:hep-th/9802109.

[82] S. Sternberg, *Lectures on differential geometry*, vol. 316. American Mathematical Soc., 1999.

[83] M. Nakahara, *Geometry, topology and physics.* CRC Press, 2003.

[84] P. Piccione and D. V. Tausk, *The theory of connections and G-structures: applications to affine and isometric immersions.* IMPA, 2006.

[85] D. Joyce, *Compact manifolds with special holonomy.* Oxford University Press, 2000.

[86] S. Kobayashi, *Transformation groups in differential geometry.* Springer Science & Business Media, 2012.

[87] V. Guillemin, “The integrability problem for g-structures”, *Transactions of the American Mathematical Society* **116** (1965)544–560.

[88] A. Newlander and L. Nirenberg, “Complex analytic coordinates in almost complex manifolds”, *Annals of Mathematics* **65** 3, (1957)391–404.

[89] S. Chiossi and S. Salamon, “The intrinsic torsion of su (3) and g 2 structures”, in *Differential Geometry, Valencia 2001*, pp. 115–133. World Scientific, 2002.

[90] M. Fernández and A. Gray, “Riemannian manifolds with structure groupg 2”, *Annali di matematica pura ed applicata* **132** 1, (1982)19–45.

[91] S. S. Gubser, “TASI lectures: Special holonomy in string theory and M theory”, in *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 2001): Strings, Branes and EXTRA Dimensions*, pp. 197–233. 1, 2002. arXiv:hep-th/0201114.

[92] B. R. Greene, “String theory on Calabi-Yau manifolds”, in *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 96): Fields, Strings, and Duality*, pp. 543–726. 6, 1996. arXiv:hep-th/9702155.

[93] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, vol. 1. New York, London, 1963.

[94] M. Berger, “Sur les groupes d’holonomie homogènes de variétés à connexion affine et des variétés riemanniennes”, *Bulletin de la Société Mathématique de France* **83** (1955)279–330.

[95] R. Bryant, “Classical, exceptional, and exotic holonomies: a status report”, in *Actes de la table ronde de géométrie différentielle en l’honneur de Marcel Berger. Collection SMF Séminaires and congres*, vol. 1, pp. 93–166. 1996.

[96] J. P. Gauntlett, J. B. Gutowski, and S. Pakis, “The Geometry of D = 11 null Killing spinors”, *JHEP* **12** (2003)049, arXiv:hep-th/0311112.

[97] J. P. Gauntlett, D. Martelli, J. Sparks, and D. Waldram, "Supersymmetric AdS(5) solutions of type IIB supergravity", *Class. Quant. Grav.* **23** (2006)4693–4718, arXiv:hep-th/0510125.

[98] F. Quevedo, "Lectures on superstring phenomenology", *AIP Conf. Proc.* **359** (1996)202–242, arXiv:hep-th/9603074.

[99] J. Derendinger, L. E. Ibanez, and H. P. Nilles, "On the Low-Energy d = 4, N=1 Supergravity Theory Extracted from the d = 10, N=1 Superstring", *Phys. Lett. B* **155** (1985)65–70.

[100] M. Dine, R. Rohm, N. Seiberg, and E. Witten, "Gluino Condensation in Superstring Models", *Phys. Lett. B* **156** (1985)55–60.

[101] M. Grana, "Flux compactifications in string theory: A Comprehensive review", *Phys. Rept.* **423** (2006)91–158, arXiv:hep-th/0509003 [hep-th].

[102] G. Obied, H. Ooguri, L. Spodyneiko, and C. Vafa, "De Sitter Space and the Swampland", arXiv:1806.08362 [hep-th].

[103] B. S. Acharya and B. J. Spence, "Flux, supersymmetry and M theory on seven manifolds", arXiv:hep-th/0007213 [hep-th].

[104] P. Kaste, R. Minasian, and A. Tomasiello, "Supersymmetric M theory compactifications with fluxes on seven-manifolds and G structures", *JHEP* **07** (2003)004, arXiv:hep-th/0303127 [hep-th].

[105] K. Behrndt and C. Jeschek, "Fluxes in M theory on seven manifolds: G structures and superpotential", *Nucl. Phys.* **B694** (2004)99–114, arXiv:hep-th/0311119 [hep-th].

[106] G. Dall'Agata and N. Prezas, "N = 1 geometries for M theory and type IIA strings with fluxes", *Phys. Rev.* **D69** (2004)066004, arXiv:hep-th/0311146 [hep-th].

[107] T. Friedrich, I. Kath, A. Moroianu, and U. Semmelmann, "On nearly parallel g2-structures", *Journal of Geometry and Physics* **23** 3-4, (1997)259–286.

[108] A. Lukas and P. M. Saffin, "M theory compactification, fluxes and AdS_4", *Phys. Rev.* **D71** (2005)046005, arXiv:hep-th/0403235 [hep-th].

[109] J. H. Schwarz and P. C. West, "Symmetries and Transformations of Chiral N=2 D=10 Supergravity", *Phys. Lett. B* **126** (1983)301–304.

[110] J. H. Schwarz, "Covariant Field Equations of Chiral N=2 D=10 Supergravity", *Nucl. Phys. B* **226** (1983)269.

[111] E. Bergshoeff, M. de Roo, B. Janssen, and T. Ortin, "The Super D9-brane and its truncations", *Nucl. Phys. B* **550** (1999)289–302, arXiv:hep-th/9901055.

[112] V. Braun, P. Candelas, R. Davies, and R. Donagi, "The MSSM Spectrum from (0,2)-Deformations of the Heterotic Standard Embedding", *JHEP* **05** (2012)127, arXiv:1112.1097 [hep-th].

[113] V. Braun, Y.-H. He, B. A. Ovrut, and T. Pantev, "The Exact MSSM spectrum from string theory", *JHEP* **05** (2006)043, arXiv:hep-th/0512177.

[114] E. Bergshoeff, M. de Roo, B. de Wit, and P. van Nieuwenhuizen, "Ten-Dimensional Maxwell-Einstein Supergravity, Its Currents, and the Issue of Its Auxiliary Fields", *Nucl. Phys. B* **195** (1982)97–136.

[115] G. Chapline and N. Manton, "Unification of Yang-Mills Theory and Supergravity in Ten-Dimensions", *Phys. Lett. B* **120** (1983)105–109.

[116] M. Atiyah, N. J. Hitchin, V. Drinfeld, and Y. Manin, "Construction of Instantons", *Phys. Lett. A* **65** (1978)185–187.

[117] S. K. Donaldson and R. P. Thomas, "Gauge theory in higher dimensions",.

[118] S. Ivanov and G. Papadopoulos, "A No go theorem for string warped compactifications", *Phys. Lett. B* **497** (2001)309–316, arXiv:hep-th/0008232.

[119] G. Curio and A. Krause, "Four flux and warped heterotic M theory compactifications", *Nucl. Phys. B* **602** (2001)172–200, arXiv:hep-th/0012152.

[120] J. P. Gauntlett, D. Martelli, S. Pakis, and D. Waldram, "G structures and wrapped NS5-branes", *Commun. Math. Phys.* **247** (2004)421–445, arXiv:hep-th/0205050.

[121] J. M. Maldacena and C. Nunez, "Supergravity description of field theories on curved manifolds and a no go theorem", *Int. J. Mod. Phys. A* **16** (2001)822–855, arXiv:hep-th/0007018.

[122] K. Behrndt and C. Jeschek, "Fluxes in M theory on seven manifolds and G structures", *JHEP* **04** (2003)002, arXiv:hep-th/0302047.

[123] G. Gibbons, "ASPECTS OF SUPERGRAVITY THEORIES", in *XV GIFT Seminar on Supersymmetry and Supergravity*. 6, 1984.

[124] B. de Wit, D. Smit, and N. Hari Dass, "Residual Supersymmetry of Compactified D=10 Supergravity", *Nucl. Phys. B* **283** (1987)165.

[125] C. S. Chan, P. L. Paul, and H. L. Verlinde, "A Note on warped string compactification", *Nucl. Phys. B* **581** (2000)156–164, arXiv:hep-th/0003236.

[126] K. Behrndt and S. Gukov, "Domain walls and superpotentials from M theory on Calabi-Yau three folds", *Nucl. Phys. B* **580** (2000)225–242, arXiv:hep-th/0001082.

[127] P. Mayr, "Stringy world branes and exponential hierarchies", *JHEP* **11** (2000)013, arXiv:hep-th/0006204.

[128] S. B. Giddings, S. Kachru, and J. Polchinski, "Hierarchies from fluxes in string compactifications", *Phys. Rev. D* **66** (2002)106006, arXiv:hep-th/0105097.

[129] K. Dasgupta, G. Rajesh, and S. Sethi, "M theory, orientifolds and G - flux", *JHEP* **08** (1999)023, arXiv:hep-th/9908088.

[130] S. Kachru and J. McGreevy, "M theory on manifolds of G(2) holonomy and type IIA orientifolds", *JHEP* **06** (2001)027, arXiv:hep-th/0103223.

[131] E. Bergshoeff, R. Kallosh, T. Ortin, D. Roest, and A. Van Proeyen, "New formulations of D = 10 supersymmetry and D8 - O8 domain walls", *Class. Quant. Grav.* **18** (2001)3359–3382, arXiv:hep-th/0103233.

[132] G. Dall'Agata and N. Prezas, "Scherk–Schwarz reduction of M-theory on G_2-manifolds with fluxes", *JHEP* **10** (2005)103, arXiv:hep-th/0509052 [hep-th].

[133] B. S. Acharya, "A Moduli fixing mechanism in M theory", arXiv:hep-th/0212294.

[134] P. S. Aspinwall and R. Kallosh, "Fixing all moduli for M-theory on K3xK3", *JHEP* **10** (2005)001, arXiv:hep-th/0506014.

[135] G. Dall'Agata, "On supersymmetric solutions of type IIB supergravity with general fluxes", *Nucl. Phys. B* **695** (2004)243–266, arXiv:hep-th/0403220.

[136] A. R. Frey, "Notes on SU(3) structures in type IIB supergravity", *JHEP* **06** (2004)027, arXiv:hep-th/0404107.

[137] K. Behrndt, M. Cvetic, and P. Gao, "General type IIB fluxes with SU(3) structures", *Nucl. Phys. B* **721** (2005)287–308, arXiv:hep-th/0502154.

[138] K. Pilch and N. P. Warner, "A New supersymmetric compactification of chiral IIB supergravity", *Phys. Lett. B* **487** (2000)22–29, arXiv:hep-th/0002192.

[139] K. Pilch and N. P. Warner, "N = 1 supersymmetric solutions of IIB supergravity from Killing spinors", arXiv:hep-th/0403005.

[140] C. N. Gowdigere, D. Nemeschansky, and N. P. Warner, "Supersymmetric solutions with fluxes from algebraic Killing spinors", *Adv. Theor. Math. Phys.* **7** 5, (2003)787–806, arXiv:hep-th/0306097.

[141] J. Bovy, D. Lust, and D. Tsimpis, "N = 1,2 supersymmetric vacua of IIA supergravity and SU(2) structures", *JHEP* **08** (2005)056, arXiv:hep-th/0506160.

[142] M. Grana, R. Minasian, M. Petrini, and A. Tomasiello, "Supersymmetric backgrounds from generalized Calabi-Yau manifolds", *JHEP* **08** (2004)046, arXiv:hep-th/0406137.

[143] A. R. Frey and M. Grana, "Type IIB solutions with interpolating supersymmetries", *Phys. Rev. D* **68** (2003)106002, arXiv:hep-th/0307142.

[144] C. M. Hull, “Compactifications of the Heterotic Superstring”, *Phys. Lett.* **B178** (1986)357–364.

[145] C. Hull, “σ Model Beta Functions and String Compactifications”, *Nucl. Phys. B* **267** (1986)266–276.

[146] A. Strominger, “Superstrings with Torsion”, *Nucl. Phys.* **B274** (1986)253.

[147] K. Becker, M. Becker, K. Dasgupta, and P. S. Green, “Compactifications of heterotic theory on nonKahler complex manifolds. 1.”, *JHEP* **04** (2003)007, arXiv:hep-th/0301161.

[148] K. Becker, M. Becker, P. S. Green, K. Dasgupta, and E. Sharpe, “Compactifications of heterotic strings on nonKahler complex manifolds. 2.”, *Nucl. Phys. B* **678** (2004)19–100, arXiv:hep-th/0310058.

[149] K. Becker and L.-S. Tseng, “Heterotic flux compactifications and their moduli”, *Nucl. Phys. B* **741** (2006)162–179, arXiv:hep-th/0509131.

[150] J. P. Gauntlett, D. Martelli, and D. Waldram, “Superstrings with intrinsic torsion”, *Phys. Rev. D* **69** (2004)086002, arXiv:hep-th/0302158.

[151] N. Hitchin, “The geometry of three-forms in six and seven dimensions”, arXiv:math/0010054 [math.DG].

[152] T. House and A. Micu, “M-theory compactifications on manifolds with G_2 structure”, *Class. Quant. Grav.* **22** (2005)1709–1738, arXiv:hep-th/0412006 [hep-th].

[153] A. Kennon, “G_2-Manifolds and M-Theory Compactifications”, arXiv:1810.12659 [hep-th].

[154] P. Candelas and X. de la Ossa, “Moduli Space of Calabi-Yau Manifolds”, *Nucl. Phys.* **B355** (1991)455–481.

[155] S. Ferrara and S. Sabharwal, “Quaternionic Manifolds for Type II Superstring Vacua of Calabi-Yau Spaces”, *Nucl. Phys. B* **332** (1990)317–332.

[156] M. Grana, J. Louis, and D. Waldram, “Hitchin functionals in N=2 supergravity”, *JHEP* **01** (2006)008, arXiv:hep-th/0505264 [hep-th].

[157] T. W. Grimm and J. Louis, “The Effective action of type IIA Calabi-Yau orientifolds”, *Nucl. Phys. B* **718** (2005)153–202, arXiv:hep-th/0412277.

[158] T. W. Grimm and J. Louis, “The Effective action of $N = 1$ Calabi-Yau orientifolds”, *Nucl. Phys. B* **699** (2004)387–426, arXiv:hep-th/0403067.

[159] S. Ferrara and S. Sabharwal, “Dimensional Reduction of Type II Superstrings”, *Class. Quant. Grav.* **6** (1989)L77.

[160] V. Balasubramanian, P. Berglund, J. P. Conlon, and F. Quevedo, "Systematics of moduli stabilisation in Calabi-Yau flux compactifications", *JHEP* **03** (2005)007, arXiv:hep-th/0502058.

[161] B. de Carlos, S. Gurrieri, A. Lukas, and A. Micu, "Moduli stabilisation in heterotic string compactifications", *JHEP* **03** (2006)005, arXiv:hep-th/0507173.

[162] D. Baraglia, "Leibniz algebroids, twistings and exceptional generalized geometry", *J. Geom. Phys.* **62** (2012)903–934, arXiv:1101.0856 [math.DG].

[163] T. J. Courant, "Dirac manifolds", *Transactions of the American Mathematical Society* **319** 2, (1990)631–661.

[164] N. Hitchin, "Generalized Calabi–Yau manifolds", *Quart. J. Math.* **54** (2003)281–308, arXiv:math/0209099 [math-dg].

[165] M. Gualtieri, "Generalized complex geometry", arXiv:math/0401221 [math.DG].

[166] C. Hull, "Generalised Geometry for M-Theory", *JHEP* **07** (2007)079, arXiv:hep-th/0701203.

[167] A. Coimbra, R. Minasian, H. Triendl, and D. Waldram, "Generalised geometry for string corrections", *JHEP* **11** (2014)160, arXiv:1407.7542 [hep-th].

[168] C. Strickland-Constable, "Subsectors, Dynkin Diagrams and New Generalised Geometries", *JHEP* **08** (2017)144, arXiv:1310.4196 [hep-th].

[169] Z.-J. Liu, A. Weinstein, and P. Xu, "Manin triples for lie bialgebroids", *J. Differential Geom.* **45** 3, (1997)547–574.

[170] D. Roytenberg and A. Weinstein, "Courant algebroids and strongly and strongly homotopy lie algebras", *Letters in Mathematical Physics* **46** 1, (1998)81–93.

[171] J.-L. Loday and T. Pirashvili, "Universal enveloping algebras of leibniz algebras and (co) homology", *Mathematische Annalen* **296** 1, (1993)139–158.

[172] Y. Kosmann-Schwarzbach, "Derived brackets", *Letters in Mathematical Physics* **69** 1, (2004)61–87.

[173] R. Ibáñez, M. de León, J. C. Marrero, and E. Padrón, "Leibniz algebroid associated with a nambu-poisson structure", *Journal of Physics A: Mathematical and General* **32** 46, (1999)8129.

[174] N. J. Hitchin, "Lectures on special Lagrangian submanifolds", *AMS/IP Stud. Adv. Math.* **23** (2001)151–182, arXiv:math/9907034.

[175] Z. Chen, M. Stiénon, and P. Xu, "On regular courant algebroids", *J. Symplectic Geom.* **11** 1, (03, 2013)1–24.

[176] I. Vaisman, "Transitive courant algebroids", *International Journal of Mathematics and Mathematical Sciences* **2005** 11, (2005)1737–1758.

[177] M. Garcia-Fernandez, R. Rubio, and C. Tipler, "Holomorphic string algebroids", arXiv:1807.10329 [math.AG].

[178] X. de la Ossa and E. E. Svanes, "Holomorphic Bundles and the Moduli Space of N=1 Supersymmetric Heterotic Compactifications", *JHEP* **10** (2014)123, arXiv:1402.1725 [hep-th].

[179] A. Ashmore, X. De La Ossa, R. Minasian, C. Strickland-Constable, and E. E. Svanes, "Finite deformations from a heterotic superpotential: holomorphic Chern-Simons and an L_∞ algebra", *JHEP* **10** (2018)179, arXiv:1806.08367 [hep-th].

[180] P. Pires Pacheco and D. Waldram, "M-theory, exceptional generalised geometry and superpotentials", *JHEP* **09** (2008)123, arXiv:0804.1362 [hep-th].

[181] A. Coimbra, C. Strickland-Constable, and D. Waldram, "$E_{d(d)} \times \mathbb{R}^+$ generalised geometry, connections and M theory", *JHEP* **02** (2014)054, arXiv:1112.3989 [hep-th].

[182] A. Coimbra, C. Strickland-Constable, and D. Waldram, "Supergravity as Generalised Geometry II: $E_{d(d)} \times \mathbb{R}^+$ and M theory", *JHEP* **03** (2014)019, arXiv:1212.1586 [hep-th].

[183] A. Coimbra, C. Strickland-Constable, and D. Waldram, "Supergravity as Generalised Geometry I: Type II Theories", *JHEP* **11** (2011)091, arXiv:1107.1733 [hep-th].

[184] A. Coimbra, C. Strickland-Constable, and D. Waldram, "Supersymmetric Backgrounds and Generalised Special Holonomy", *Class. Quant. Grav.* **33** 12, (2016)125026, arXiv:1411.5721 [hep-th].

[185] A. Coimbra and C. Strickland-Constable, "Supersymmetric Backgrounds, the Killing Superalgebra, and Generalised Special Holonomy", *JHEP* **11** (2016)063, arXiv:1606.09304 [hep-th].

[186] M. Garcia-Fernandez, "Torsion-free generalized connections and Heterotic Supergravity", *Commun. Math. Phys.* **332** 1, (2014)89–115, arXiv:1304.4294 [math.DG].

[187] A. Ashmore and D. Waldram, "Exceptional Calabi-Yau spaces: the geometry of $\mathcal{N} = 2$ backgrounds with flux", *Fortsch. Phys.* **65** 1, (2017)1600109, arXiv:1510.00022 [hep-th].

[188] P. Aschieri, F. Bonechi, and A. Deser, "On Curvature and Torsion in Courant Algebroids", arXiv:1910.11273 [math.DG].

[189] O. Hohm and B. Zwiebach, "On the Riemann Tensor in Double Field Theory", *JHEP* **05** (2012)126, arXiv:1112.5296 [hep-th].

[190] D. S. Berman, C. D. Blair, E. Malek, and M. J. Perry, “The $O_{D,D}$ geometry of string theory”, *Int. J. Mod. Phys. A* **29** (2014)1450080, arXiv:1303.6727 [hep-th].

[191] B. Sathiapalan, “Duality in statistical mechanics and string theory”, *Phys. Rev. Lett.* **58** (Apr, 1987)1597–1599.

[192] J. Maharana and J. H. Schwarz, “Noncompact symmetries in string theory”, *Nucl. Phys. B* **390** (1993)3–32, arXiv:hep-th/9207016.

[193] A. H. Chamseddine, “N=4 Supergravity Coupled to N=4 Matter”, *Nucl. Phys. B* **185** (1981)403.

[194] K. Meissner and G. Veneziano, “Symmetries of cosmological superstring vacua”, *Phys. Lett. B* **267** (1991)33–36.

[195] K. Meissner and G. Veneziano, “Manifestly O(d,d) invariant approach to space-time dependent string vacua”, *Mod. Phys. Lett. A* **6** (1991)3397–3404, arXiv:hep-th/9110004.

[196] K. Narain, M. Sarmadi, and E. Witten, “A note on toroidal compactification of heterotic string theory”, *Nuclear Physics B* **279** 3, (1987)369 – 379.

[197] A. Sen, “Twisted black p-brane solutions in string theory”, *Phys. Lett. B* **274** (1992)34–40, arXiv:hep-th/9108011.

[198] S. Hassan and A. Sen, “Twisting classical solutions in heterotic string theory”, *Nucl. Phys. B* **375** (1992)103–118, arXiv:hep-th/9109038.

[199] M. Gasperini, J. Maharana, and G. Veneziano, “From trivial to nontrivial conformal string backgrounds via O(d,d) transformations”, *Phys. Lett. B* **272** (1991)277–284.

[200] C. Jeschek and F. Witt, “Generalised G(2) - structures and type IIb superstrings”, *JHEP* **03** (2005)053, arXiv:hep-th/0412280.

[201] F. Witt, “Generalised G(2) manifolds”, *Commun. Math. Phys.* **265** (2006)275–303, arXiv:math/0411642.

[202] M. Grana, R. Minasian, M. Petrini, and A. Tomasiello, “Generalized structures of N=1 vacua”, *JHEP* **11** (2005)020, arXiv:hep-th/0505212 [hep-th].

[203] A. Tomasiello, “Reformulating supersymmetry with a generalized Dolbeault operator”, *JHEP* **02** (2008)010, arXiv:0704.2613 [hep-th].

[204] M. Gualtieri, *Branes on Poisson varieties.* 10, 2007. arXiv:0710.2719 [math.DG].

[205] M. Fukuma, T. Oota, and H. Tanaka, “Comments on T dualities of Ramond-Ramond potentials on tori”, *Prog. Theor. Phys.* **103** (2000)425–446, arXiv:hep-th/9907132.

[206] S. Hassan, “SO(d,d) transformations of Ramond-Ramond fields and space-time spinors”, *Nucl. Phys. B* **583** (2000)431–453, arXiv:hep-th/9912236.

[207] X. de la Ossa, E. Hardy, and E. E. Svanes, "The Heterotic Superpotential and Moduli", *JHEP* **01** (2016)049, arXiv:1509.08724 [hep-th].

[208] E. Cremmer and B. Julia, "The so(8) supergravity", *Nuclear Physics B* **159** 1, (1979)141 – 212.

[209] "Spontaneously broken n = 8 supergravity", *Physics Letters B* **84** 1, (1979)83 – 86.

[210] "d = 11 supergravity with local su(8) invariance", *Nuclear Physics B* **274** 2, (1986)363 – 400.

[211] M. Grana, R. Minasian, M. Petrini, and D. Waldram, "T-duality, Generalized Geometry and Non-Geometric Backgrounds", *JHEP* **04** (2009)075, arXiv:0807.4527 [hep-th].

[212] A. Coimbra and C. Strickland-Constable, "Supersymmetric AdS backgrounds and weak generalised holonomy", arXiv:1710.04156 [hep-th].

[213] A. Ashmore, M. Petrini, and D. Waldram, "The exceptional generalised geometry of supersymmetric AdS flux backgrounds", *JHEP* **12** (2016)146, arXiv:1602.02158 [hep-th].

[214] A. Ashmore, M. Gabella, M. Graña, M. Petrini, and D. Waldram, "Exactly marginal deformations from exceptional generalised geometry", *JHEP* **01** (2017)124, arXiv:1605.05730 [hep-th].

[215] A. Ashmore, "Marginal deformations of 3d $\mathcal{N} = 2$ CFTs from AdS_4 backgrounds in generalised geometry", *JHEP* **12** (2018)060, arXiv:1809.03503 [hep-th].

[216] V. Braun, Y.-H. He, B. A. Ovrut, and T. Pantev, "A heterotic standard model", *Phys.Lett.* **B618** (2005)252–258, arXiv:hep-th/0501070 [hep-th].

[217] L. B. Anderson, J. Gray, A. Lukas, and B. Ovrut, "Stabilizing the complex structure in heterotic Calabi–Yau vacua", *JHEP* **02** (2011)088, arXiv:1010.0255 [hep-th].

[218] L. B. Anderson, J. Gray, A. Lukas, and E. Palti, "Two hundred heterotic Standard Models on smooth Calabi–Yau threefolds", *Phys. Rev.* **D84** (2011)106005, arXiv:1106.4804 [hep-th].

[219] L. B. Anderson, J. Gray, A. Lukas, and E. Palti, "Heterotic line bundle Standard Models", *JHEP* **06** (2012)113, arXiv:1202.1757 [hep-th].

[220] L. B. Anderson, J. Gray, A. Lukas, and B. Ovrut, "Stabilizing all geometric moduli in heterotic Calabi-Yau vacua", *Phys. Rev.* **D83** (2011)106011, arXiv:1102.0011 [hep-th].

[221] R. Thomas, "Notes on GIT and symplectic reduction for bundles and varieties", arXiv:math/0512411 [math.DG].

[222] S. Ivanov, "Heterotic supersymmetry, anomaly cancellation and equations of motion", *Phys. Lett.* **B685** (2010)190–196, arXiv:0908.2927 [hep-th].

[223] D. Martelli and J. Sparks, "Non-Kahler heterotic rotations", *Adv. Theor. Math. Phys.* **15** 1, (2011)131–174, `arXiv:1010.4031 [hep-th]`.

[224] S. K. Donaldson, "ANTI SELF-DUAL YANG-MILLS CONNECTIONS OVER COMPLEX ALGEBRAIC SURFACES AND STABLE VECTOR BUNDLES", *Proc. Lond. Math. Soc.* **50** (1985)1–26.

[225] K. Uhlenbeck and S.-T. Yau, "On the existence of Hermitian-Yang–Mills connections in stable vector bundles", *Comm. Pure Appl. Math.* **39** S1, (1986)257–293.

[226] L. B. Anderson, J. Gray, A. Lukas, and B. Ovrut, "The Atiyah Class and Complex Structure Stabilization in Heterotic Calabi-Yau Compactifications", *JHEP* **10** (2011)032, `arXiv:1107.5076 [hep-th]`.

[227] J. McOrist and R. Sisca, "Small gauge transformations and universal geometry in heterotic theories", `arXiv:1904.07578 [hep-th]`.

[228] P. Ševera, "Letters to Alan Weinstein about Courant algebroids", `arXiv:1707.00265 [math.DG]`.

[229] C. Hull and U. Lindström, "The Generalised Complex Geometry of (p, q) Hermitian Geometries", *Commun. Math. Phys.* (2019)1–16, `arXiv:1810.06489 [hep-th]`.

[230] C. Hull and U. Lindström, "Strong Kähler with Torsion as Generalised Geometry", *PoS* **CORFU2018** (2019)121, `arXiv:1904.03412 [hep-th]`.

[231] S. Gurrieri, A. Lukas, and A. Micu, "Heterotic on half-flat", *Phys. Rev.* **D70** (2004)126009, `arXiv:hep-th/0408121 [hep-th]`.

[232] I. Benmachiche, J. Louis, and D. Martinez-Pedrera, "The Effective action of the heterotic string compactified on manifolds with SU(3) structure", *Class. Quant. Grav.* **25** (2008)135006, `arXiv:0802.0410 [hep-th]`.

[233] J. McOrist, "On the Effective Field Theory of Heterotic Vacua", *Lett. Math. Phys.* **108** 4, (2018)1031–1081, `arXiv:1606.05221 [hep-th]`.

[234] S. Gukov, C. Vafa, and E. Witten, "CFT's from Calabi-Yau four folds", *Nucl. Phys.* **B584** (2000)69–108, `arXiv:hep-th/9906070 [hep-th]`. [Erratum: Nucl. Phys.B608,477(2001)].

[235] S. Gukov, "Solitons, superpotentials and calibrations", *Nucl. Phys.* **B574** (2000)169–188, `arXiv:hep-th/9911011 [hep-th]`.

[236] A. Borel, "Kählerian coset spaces of semisimple lie groups", *Proceedings of the National Academy of Sciences of the United States of America* **40** 12, (1954)1147.

[237] S. Gurrieri, J. Louis, A. Micu, and D. Waldram, "Mirror symmetry in generalized Calabi-Yau compactifications", *Nucl. Phys.* **B654** (2003)61–113, `arXiv:hep-th/0211102 [hep-th]`.

[238] S. Gurrieri, A. Lukas, and A. Micu, "Heterotic String Compactifications on Half-flat Manifolds. II.", *JHEP* **12** (2007)081, arXiv:0709.1932 [hep-th].

[239] P. Candelas, X. de la Ossa, and J. McOrist, "A Metric for Heterotic Moduli", *Commun. Math. Phys.* **356** 2, (2017)567–612, arXiv:1605.05256 [hep-th].

[240] A. Strominger, "Yukawa Couplings in Superstring Compactification", *Phys. Rev. Lett.* **55** (1985)2547.

[241] L. J. Dixon, V. Kaplunovsky, and J. Louis, "On Effective Field Theories Describing (2,2) Vacua of the Heterotic String", *Nucl. Phys.* **B329** (1990)27–82.

[242] J. P. Derendinger, L. E. Ibanez, and H. P. Nilles, "On the Low-Energy Limit of Superstring Theories", *Nucl. Phys.* **B267** (1986)365–414.

[243] J. Wess and J. Bagger, *Supersymmetry and supergravity.* Princeton University Press, Princeton, NJ, USA, 1992.

[244] M. F. Atiyah, "Complex analytic connections in fibre bundles", *Transactions of the American Mathematical Society* **85** 1, (1957)181–207.

[245] K. Uhlenbeck and S.-T. Yau, "A note on our previous paper: On the existence of Hermitian Yang–Mills connections in stable vector bundles", *Comm. Pure Appl. Math.* **42** 5, (1989)703–707.

[246] A. Fujiki, "Moduli space of polarized algebraic manifolds and Kähler metrics", *Sugaku Expositions* **5** 2, (1992)173–191. Sugaku Expositions.

[247] S. Donaldson, *Remarks on gauge theory, complex geometry and 4-manifold topology*, vol. 5 of *World Sci. Ser. 20th Century Math.*, ch. 42, pp. 384–403. 1997.

[248] X.-X. Chen, S. Donaldson, and S. Sun, "Kähler–Einstein metrics and stability", arXiv:1210.7494 [math.DG].

[249] S. Trautwein, "A survey of the GIT picture for the Yang-Mills equation over Riemann surfaces", arXiv:1511.08122.

[250] N. P. Buchdahl, "Hermitian-einstein connections and stable vector bundles over compact complex surfaces", *Mathematische Annalen* **280** 4, (Dec, 1988)625–648.

[251] M. Grana, J. Louis, A. Sim, and D. Waldram, "E7(7) formulation of N=2 backgrounds", *JHEP* **07** (2009)104, arXiv:0904.2333 [hep-th].

[252] M. Günaydin, L. Romans, and N. Warner, "Iib, or not iib: That is the question", *Physics Letters B* **164** 4, (1985)309 – 314.

[253] R. Gilmore, *Lie groups, physics, and geometry: an introduction for physicists, engineers and chemists.* Cambridge University Press, 2008.

[254] D. S. Berman, M. Cederwall, A. Kleinschmidt, and D. C. Thompson, "The gauge structure of generalised diffeomorphisms", *JHEP* **01** (2013)064, arXiv:1208.5884 [hep-th].

[255] N. Hitchin, "The geometry of three-forms in six dimensions", *J. Diff. Geom.* **55** (2000)547–576.

[256] N. J. Hitchin, A. Karlhede, U. Lindstrom, and M. Rocek, "Hyperkahler Metrics and Supersymmetry", *Commun. Math. Phys.* **108** (1987)535.

[257] K. Kodaira, *Complex manifolds and deformation of complex structures*. Springer, 2006.

[258] G. Habib and K. Richardson, "Modified differentials and basic cohomology for riemannian foliations", *Journal of Geometric Analysis* **23** 3, (2013)1314–1342.

[259] J. P. Gauntlett, D. Martelli, J. Sparks, and D. Waldram, "Supersymmetric AdS(5) solutions of M theory", *Class. Quant. Grav.* **21** (2004)4335–4366, arXiv:hep-th/0402153.

[260] A. Ashmore, M. Petrini, E. Tasker, and D. Waldram. To appear.

[261] V. Ginzburg, "Calabi-Yau algebras", arXiv:math/0612139.

[262] R. Eager, J. Schmude, and Y. Tachikawa, "Superconformal Indices, Sasaki-Einstein Manifolds, and Cyclic Homologies", *Adv. Theor. Math. Phys.* **18** 1, (2014)129–175, arXiv:1207.0573 [hep-th].

[263] E. A. Bergshoeff, I. De Baetselier, and T. A. Nutma, "E_{11} and the embedding tensor", *JHEP* **09** (2007)047, arXiv:0705.1304 [hep-th].

[264] B. de Wit, H. Nicolai, and H. Samtleben, "Gauged supergravities, tensor hierarchies, and M-theory", *JHEP* **02** (2008)044, arXiv:0801.1294 [hep-th].

[265] A. Coimbra and C. Strickland-Constable, "Generalised Structures for $\mathcal{N} = 1$ AdS Backgrounds", *JHEP* **11** (2016)092, arXiv:1504.02465 [hep-th].

[266] M. Grana, J. Louis, and D. Waldram, "SU(3) x SU(3) compactification and mirror duals of magnetic fluxes", *JHEP* **04** (2007)101, arXiv:hep-th/0612237 [hep-th].

[267] B. de Wit and H. Nicolai, "$d = 11$ Supergravity With Local SU(8) Invariance", *Nucl. Phys.* **B274** (1986)363–400.

[268] J. Dorfmeister and Z.-D. Guan, "Fine structure of reductive pseudo-Kählerian spaces", *Geometriae Dedicata* **39** 3, (1991).

[269] G. Papadopoulos and P. K. Townsend, "Compactification of $D = 11$ supergravity on spaces of exceptional holonomy", *Phys. Lett.* **B357** (1995)300–306, arXiv:hep-th/9506150 [hep-th].

[270] J. A. Harvey and G. W. Moore, "Superpotentials and membrane instantons", arXiv:hep-th/9907026 [hep-th].

[271] J. Gutowski and G. Papadopoulos, "Moduli spaces and brane solitons for M theory compactifications on holonomy G_2 manifolds", *Nucl. Phys.* **B615** (2001)237–265, arXiv:hep-th/0104105 [hep-th].

[272] C. Beasley and E. Witten, "A note on fluxes and superpotentials in M theory compactifications on manifolds of G_2 holonomy", *JHEP* **07** (2002)046, arXiv:hep-th/0203061 [hep-th].

[273] S. Ferrara, M. Kaku, P. K. Townsend, and P. van Nieuwenhuizen, "Gauging the graded conformal group with unitary internal symmetries", *Nucl. Phys.* **B129** (1977)125–134.

[274] M. Kaku, P. K. Townsend, and P. van Nieuwenhuizen, "Properties of conformal supergravity", *Phys. Rev.* **D17** (1978)3179. [,853(1978)].

[275] M. Kaku and P. K. Townsend, "Poincaré supergravity as broken superconformal gravity", *Phys. Lett.* **76B** (1978)54–58.

[276] R. Kallosh, L. Kofman, A. D. Linde, and A. Van Proeyen, "Superconformal symmetry, supergravity and cosmology", *Class. Quant. Grav.* **17** (2000)4269–4338, arXiv:hep-th/0006179 [hep-th]. [Erratum: Class. Quant. Grav.21,5017(2004)].

[277] R. L. Bryant, "Some remarks on G_2 structures", arXiv:math/0305124 [math.DG].

[278] N. Hitchin, "Stable forms and special metrics", arXiv:math/0107101 [math.DG].

[279] M. Gualtieri, "Generalized Kähler geometry", arXiv:1007.3485 [math.DG].

[280] G. R. Cavalcanti, "New aspects of the ddc-lemma", *arXiv Mathematics e-prints* (Jan., 2005)math/0501406, arXiv:math/0501406 [math.DG].

[281] E. Witten, "Five-brane effective action in M theory", *J. Geom. Phys.* **22** (1997)103–133, arXiv:hep-th/9610234 [hep-th].

[282] N. Lambert, "Flux and Freund–Rubin superpotentials in M-theory", *Phys. Rev.* **D71** (2005)126001, arXiv:hep-th/0502200 [hep-th].

[283] I. Benmachiche and T. W. Grimm, "Generalized $N = 1$ orientifold compactifications and the Hitchin functionals", *Nucl. Phys.* **B748** (2006)200–252, arXiv:hep-th/0602241 [hep-th].

[284] P. Koerber and L. Martucci, "From ten to four and back again: How to generalize the geometry", *JHEP* **08** (2007)059, arXiv:0707.1038 [hep-th].

[285] P. Koerber and L. Martucci, "Warped generalized geometry compactifications, effective theories and non-perturbative effects", *Fortsch. Phys.* **56** (2008)862–868, arXiv:0803.3149 [hep-th].

[286] M. Atiyah and R. Bott, "The Yang–Mills equations over Riemann surfaces", *Phil. Trans. R. Soc. A* **308** 1505, (1983)523–615.

[287] J. Fine, "The hamiltonian geometry of the space of unitary connections with symplectic curvature", 2011.

[288] X. Chen, S. Donaldson, and S. Sun, "Kähler-Einstein metrics on Fano manifolds. I–III", *J. Am. Math. Soc.* **28** 1, (2015)183–278.

[289] A. Futaki, "An obstruction to the existence of einstein kähler metrics.", *Inventiones mathematicae* **73** (1983)437–444.

[290] M. Manetti, "Lectures on deformations of complex manifolds (deformations from differential graded viewpoint)", *Rend. Mat. Appl. (7)* **24** 1, (2004)1–183.

[291] M. Gualtieri, M. Matviichuk, and G. Scott, "Deformation of Dirac structures via L_∞ algebras", arXiv:1702.08837 [math.DG].

[292] J. de Boer, P. de Medeiros, S. El-Showk, and A. Sinkovics, "G(2) Hitchin functionals at one loop", *Class. Quant. Grav.* **25** (2008)075006, arXiv:0706.3119 [hep-th].

[293] J. de Boer, P. de Medeiros, S. El-Showk, and A. Sinkovics, "Open G(2) strings", *JHEP* **02** (2008)012, arXiv:hep-th/0611080.

[294] N. J. Hitchin, "Stable forms and special metrics", arXiv:math/0107101.

[295] A. Kapustin, "Topological strings on noncommutative manifolds", *Int. J. Geom. Meth. Mod. Phys.* **1** (2004)49–81, arXiv:hep-th/0310057.

[296] E. Witten, *Mirror manifolds and topological field theory*, vol. 9, pp. 121–160. 1998. arXiv:hep-th/9112056.

[297] R. Dijkgraaf, S. Gukov, A. Neitzke, and C. Vafa, "Topological M-theory as unification of form theories of gravity", *Adv. Theor. Math. Phys.* **9** 4, (2005)603–665, arXiv:hep-th/0411073.

[298] A. A. Gerasimov and S. L. Shatashvili, "Towards integrability of topological strings. I. Three-forms on Calabi-Yau manifolds", *JHEP* **11** (2004)074, arXiv:hep-th/0409238.

[299] N. Nekrasov, "A la recherche de la M-theorie perdue Z theory: Chasing M / f theory", in *Annual International Conference on Strings, Theory and Applications*. 12, 2004. arXiv:hep-th/0412021.

[300] D. B. Ray and I. M. Singer, "Analytic torsion for complex manifolds", *Annals of Mathematics* **98** 1, (1973)154–177.

[301] J. Gomis, J. Paris, and S. Samuel, "Antibracket, antifields and gauge theory quantization", *Phys. Rept.* **259** (1995)1–145, arXiv:hep-th/9412228.

[302] J. Cheeger, "Analytic torsion and reidemeister torsion", *Proceedings of the National Academy of Sciences* **74** 7, (1977)2651–2654.

[303] I. Antoniadis, E. Gava, K. Narain, and T. Taylor, "Topological amplitudes in string theory", *Nucl. Phys. B* **413** (1994)162–184, arXiv:hep-th/9307158.

[304] A. Strominger, "Loop corrections to the universal hypermultiplet", *Phys. Lett. B* **421** (1998)139–148, arXiv:hep-th/9706195.

[305] L. Anguelova, M. Rocek, and S. Vandoren, "Quantum corrections to the universal hypermultiplet and superspace", *Phys. Rev. D* **70** (2004)066001, arXiv:hep-th/0402132.

[306] I. Antoniadis, R. Minasian, S. Theisen, and P. Vanhove, "String loop corrections to the universal hypermultiplet", *Class. Quant. Grav.* **20** (2003)5079–5102, arXiv:hep-th/0307268.

[307] T. W. Grimm, K. Mayer, and M. Weissenbacher, "Higher derivatives in Type II and M-theory on Calabi-Yau threefolds", *JHEP* **02** (2018)127, arXiv:1702.08404 [hep-th].

[308] T. W. Grimm, T. G. Pugh, and M. Weissenbacher, "The effective action of warped M-theory reductions with higher derivative terms — part I", *JHEP* **01** (2016)142, arXiv:1412.5073 [hep-th].

[309] T. W. Grimm, T. G. Pugh, and M. Weissenbacher, "The effective action of warped M-theory reductions with higher-derivative terms - Part II", *JHEP* **12** (2015)117, arXiv:1507.00343 [hep-th].

[310] T. W. Grimm, K. Mayer, and M. Weissenbacher, "One-modulus Calabi-Yau fourfold reductions with higher-derivative terms", *JHEP* **04** (2018)021, arXiv:1712.07074 [hep-th].

[311] R. Gopakumar and C. Vafa, "M theory and topological strings. 1.", arXiv:hep-th/9809187.

[312] R. Gopakumar and C. Vafa, "M theory and topological strings. 2.", arXiv:hep-th/9812127.

[313] T. W. Grimm, "The N=1 effective action of F-theory compactifications", *Nucl. Phys. B* **845** (2011)48–92, arXiv:1008.4133 [hep-th].

[314] I. Antoniadis, E. Gava, K. Narain, and T. Taylor, "Topological amplitudes in heterotic superstring theory", *Nucl. Phys. B* **476** (1996)133–174, arXiv:hep-th/9604077.

[315] J. P. Gauntlett, D. A. Kastor, and J. H. Traschen, "Overlapping branes in M theory", *Nucl. Phys. B* **478** (1996)544–560, arXiv:hep-th/9604179.

[316] J. P. Gauntlett, "Branes, calibrations and supergravity", *Clay Math. Proc.* **3** (2004)79–126, arXiv:hep-th/0305074.

[317] M. H. Emam, "Calibrated brane solutions of M-theory", other thesis, 9, 2004.

[318] T. Z. Husain, "Generalized Calibrations & the Characterization of M2-Brane Backgrounds", arXiv:0907.1665 [hep-th].

[319] G. Papadopoulos and P. Sloane, "Kappa symmetry, generalized calibrations and spinorial geometry", *JHEP* **05** (2006)050, arXiv:hep-th/0601164.

[320] J. F. Cascales and A. M. Uranga, "Branes on generalized calibrated submanifolds", *JHEP* **11** (2004)083, arXiv:hep-th/0407132.

[321] D. Martelli and J. Sparks, "G structures, fluxes and calibrations in M theory", *Phys. Rev. D* **68** (2003)085014, arXiv:hep-th/0306225.

[322] O. de Felice and J. Geipel, "Generalised Calibrations in AdS backgrounds from Exceptional Sasaki-Einstein Structures", arXiv:1704.05949 [hep-th].

Appendix A

Conventions

A.1 Exterior and Interior Products

We use the following conventions for the exterior and interior products of differential forms and multivectors. Here $u \in \Gamma(\wedge^p T)$, $v \in \Gamma(\wedge^q T)$, $\lambda \in \Omega^p(M)$, $\rho \in \Omega^q(M)$, and we take $p \geq q$ without loss of generality.

$$(u \wedge v)^{a_1 \dots a_{p+q}} = \frac{(p+q)!}{(p!q!} v^{[a_1 \dots a_p} u^{-p+1 \dots a_{p+q}]} \tag{A.1}$$

$$(\lambda \wedge \rho)_{a_1 \dots a_{p+q}} = \frac{(p+q)!}{p!q!} \lambda_{[a_1 \dots a_p} \rho_{a_{p+1} \dots a_{p+q}]} \tag{A.2}$$

$$(v \lrcorner \lambda)_{a_1 \dots a_{p-q}} = \frac{1}{q!} v^{b_1 \dots b_q} \lambda_{b_1 \dots b_q a_1 \dots a_{p-q}} \tag{A.3}$$

$$(u \lrcorner \rho)^{a_1 \dots a_{p-q}} = \frac{1}{q!} u^{a_1 \dots a_{p-q} b_1 \dots b_q} \rho_{b_1 \dots b_q} \tag{A.4}$$

$$(jv \lrcorner j\rho)^a{}_b = \frac{1}{(q-1)!} v^{ac_1 \dots c_{q-1}} \rho_{bc_1 \dots c_{q-1}} \tag{A.5}$$

$$(j\lambda \wedge \rho)_{a,a_1 \dots a_{p+q-1}} = \frac{(p+q-1)!}{(p-1)!q!} \lambda_{a[a_1 \dots a_{p-1}} \rho_{a_p \dots a_{p+q-1}]} \tag{A.6}$$

The following is notation used in chapter 4 which we clarify now. For $X \in \Gamma(\wedge^3 T)$, $F \in \Omega^4(M)$, we define

$$(jX \lrcorner j^2 F)^a{}_{bc} = \frac{1}{2} X^{apq} F_{bcpq} \tag{A.7}$$

The following are important in the projection maps defined in the next sections. We take $\tau \in \Gamma(T^* \otimes \wedge^7 T^*)$, $t \in \Gamma(T \otimes \wedge^7 T)$, $\sigma \in \Omega^5(M)$, $x \in \Gamma(T)$.

$$(jt \lrcorner j\tau)^a{}_b = \frac{1}{7!} t^{a,c_1 \dots c_7} \tau_{b,c_1 \dots c_7} \tag{A.8}$$

$$(j^{p+1}\lambda \wedge \tau)_{a_1 \dots a_{p+1}, b_1 \dots b_7} = (p+1) \lambda_{[a_1 \dots a_p} \tau_{a_{p+1}] b_1 \dots b_7} \tag{A.9}$$

$$(j^3 \sigma \wedge \sigma')_{a_1 a_2 a_3, b_1 \dots b_7} = \frac{7!}{5!2!} \sigma_{a_1 a_2 a_3 [b_1 b_2} \sigma'_{b_3 \dots b_7]} \tag{A.10}$$

$$(v \lrcorner j\tau)_{a,a_1 \dots a_6} = v^b \tau_{a,ba_1 \dots a_6} \tag{A.11}$$

A.2 O(d,d) Algebra

The generalised tangent bundle and adjoint bundle for $\mathrm{O}(d,d)\times\mathbb{R}^+$ geometry are as follows

$$E = T \oplus T^* \tag{A.12}$$

$$\mathrm{ad}\,\tilde{F} = \mathbb{R} \oplus (T\otimes T^*) \oplus \wedge^2 T^* \oplus \wedge^2 T \tag{A.13}$$

We take the following sections of these bundles, where each term matches with the expressions above in the obvious way.

$$X = x + \xi \qquad R = l + r + B + \beta \tag{A.14}$$

The following gives the adjoint action $R\cdot X = X'$

$$x' = lx + r\cdot x - \beta \lrcorner \xi \tag{A.15}$$

$$\xi' = l\xi + r\cdot\xi + x\lrcorner B \tag{A.16}$$

The following gives the Lie algebra bracket $[R,R'] = R''$.

$$l'' = 0 \tag{A.17}$$

$$r'' = [r,r'] - (j\beta\lrcorner jB' - j\beta'\lrcorner jB) \tag{A.18}$$

$$B'' = r\cdot B' - r'\cdot B \tag{A.19}$$

$$\beta'' = r\cdot\beta' - r'\cdot\beta \tag{A.20}$$

The Mukai pairing for two pure spinors $\Phi,\Psi\in S$ is given by

$$(\Phi,\Psi) = \sum_n (-1)^{\lfloor n/2\rfloor}\Phi_{d-n}\wedge\Psi_n \tag{A.21}$$

A.3 O$(6,6+n)$ Algebra

Here we collect a number of useful formula for the $6,6+n\times\mathbb{R}^+$ generalised geometry relevant for type I and heterotic backgrounds. A more detailed discussion can be found in [167].

The adjoint action of a two-form B, a two-vector β and a one-form gauge field A on a generalised vector $V = v+\lambda+\Lambda$ are given by

$$\begin{aligned}
\mathrm{e}^B V &= v + \lambda - \imath_v B + \Lambda, \\
\mathrm{e}^\beta V &= v + \lambda - \beta\lrcorner\lambda + \Lambda, \\
\mathrm{e}^A V &= v + \lambda + 2\,\mathrm{Tr}(\Lambda A) - \mathrm{Tr}(\imath_v A\, A) + \Lambda - \imath_v A.
\end{aligned} \tag{A.22}$$

Note that B commutes with itself, while A has a non-trivial commutator with itself

$$[A,A'] = -2\,\mathrm{Tr}(A\wedge A'). \tag{A.23}$$

One can check that the natural inner product

$$\eta(v+\Lambda+\lambda, w+\Sigma+\sigma) = \tfrac{1}{2}\imath_v\sigma + \tfrac{1}{2}\imath_w\lambda + \operatorname{Tr}(\Lambda\Sigma), \tag{A.24}$$

is preserved by the above action.

The twisted Dorfman derivative is defined by

$$L_{\mathrm{e}^{-B}\mathrm{e}^{-A}V}(\mathrm{e}^{-B}\mathrm{e}^{-A}W) = \mathrm{e}^{-B}\mathrm{e}^{-A}L_V^{H+F}W, \tag{A.25}$$

where for $V = v+\lambda+\Lambda$ and $W = w+\rho+\Sigma$, we have

$$\begin{aligned} L_V^{H+F}W = [v,w] \\ &+ \mathcal{L}_v\rho - \imath_w\mathrm{d}\lambda - \imath_v\imath_w H + 2\operatorname{Tr}(\Sigma\,\mathrm{d}_A\Lambda) - 2\operatorname{Tr}(\Sigma\,\imath_v F) + 2\operatorname{Tr}(\Lambda\,\imath_w F) \\ &+ [\Lambda,\Sigma] + \imath_v\mathrm{d}_A\Sigma - \imath_w\mathrm{d}_A\Lambda - \imath_v\imath_w F, \end{aligned} \tag{A.26}$$

where we have defined

$$\mathrm{d}_A\Lambda = \mathrm{d}\Lambda + [A,\Lambda], \tag{A.27}$$

$$F = \mathrm{d}A + A\wedge A, \tag{A.28}$$

$$H = \mathrm{d}B + \operatorname{Tr}(A\wedge\mathrm{d}A + \tfrac{2}{3}A\wedge A\wedge A). \tag{A.29}$$

We also have the usual rule for the commutator of two Dorfman derivatives

$$[L_U, L_V]W = L_{L_U V}W = L_{[\![U,V]\!]}W, \tag{A.30}$$

where $[\![\cdot,\cdot]\!]$ is the Courant bracket, the antisymmetrisation of the Dorfman derivative.

A.4 $\mathbf{E}_{d(d)}\times\mathbb{R}^+$ Algebra for M-theory

The generalised tangent bundle and adjoint bundle for $\mathrm{E}_{d(d)}\times\mathbb{R}^+$ geometry are as follows

$$E = T\oplus\wedge^2T^*\oplus\wedge^5T^*\oplus(T^*\otimes\wedge^7T^*) \tag{A.31}$$

$$\operatorname{ad}\tilde{F} = \mathbb{R}\oplus(T\otimes T^*)\oplus\wedge^3T^*\oplus\wedge^6T^*\oplus\wedge^3T\oplus\wedge^6T \tag{A.32}$$

We take the following sections of these bundles, where each term matches with the expressions above in the obvious way.

$$V = v+\omega+\sigma+\tau \qquad R = l+r+a+\tilde{a}+\alpha+\tilde{\alpha} \tag{A.33}$$

The following gives the adjoint action $R\cdot V = V'$

$$v' = lv + r\cdot v + \alpha\lrcorner\omega - \tilde{\alpha}\lrcorner\sigma \tag{A.34}$$

$$\omega' = l\omega + r\cdot\omega + v\lrcorner a + \alpha\lrcorner\sigma + \tilde{\alpha}\lrcorner\tau \tag{A.35}$$

$$\sigma' = l\sigma + r\cdot\sigma + v\lrcorner\tilde{a} + a\wedge\omega + \alpha\lrcorner\tau \tag{A.36}$$

$$\tau' = l\tau + r \cdot \tau + ja \wedge \sigma - j\tilde{a} \wedge \omega \tag{A.37}$$

The following gives the Lie algebra bracket $[R, R'] = R''$

$$l'' = \tfrac{1}{3}(\alpha \lrcorner a' - \alpha' \lrcorner a) + \tfrac{2}{3}(\tilde{\alpha}' \lrcorner \tilde{a} - \tilde{\alpha} \lrcorner \tilde{a}') \tag{A.38}$$

$$\begin{aligned} r'' = {} & [r, r'] + j\alpha \lrcorner ja' - j\alpha' \lrcorner ja - \tfrac{1}{3}\mathbb{I}(\alpha \lrcorner a' - \alpha' \lrcorner a) \\ & + j\tilde{\alpha}' \lrcorner j\tilde{a} - j\tilde{\alpha} \lrcorner j\tilde{a}' - \tfrac{2}{3}\mathbb{I}(\tilde{\alpha}' \lrcorner \tilde{a} - \tilde{\alpha} \lrcorner \tilde{a}') \end{aligned} \tag{A.39}$$

$$a'' = r \cdot a' - r' \cdot a + \alpha' \lrcorner \tilde{a} - \alpha \lrcorner \tilde{a}' \tag{A.40}$$

$$\tilde{a}'' = r \cdot \tilde{a}' - r' \cdot \tilde{a} - a \wedge a' \tag{A.41}$$

$$\alpha'' = r \cdot \alpha' - r' \cdot \alpha + \tilde{\alpha}' \lrcorner a - \tilde{\alpha} \lrcorner a' \tag{A.42}$$

$$\tilde{\alpha}'' = r \cdot \tilde{\alpha}' - r' \cdot \tilde{\alpha} - \alpha \wedge \alpha' \tag{A.43}$$

Dorfman Derivative

The following is the Dorfman derivative on vectors.

$$\begin{aligned} L_V V' = {} & \mathcal{L}_v v' + (\mathcal{L}_v \omega' - v' \lrcorner \mathrm{d}\omega) + (\mathcal{L}_v \sigma' - v' \lrcorner \mathrm{d}\sigma - \omega' \wedge \mathrm{d}\omega) \\ & + (\mathcal{L}_v \tau' - j\mathrm{d}\omega \wedge \sigma' - j\mathrm{d}\sigma \wedge \omega') \end{aligned} \tag{A.44}$$

The following is the Dorfman derivative on adjoint elements.

$$\begin{aligned} L_V R = {} & \mathcal{L}_v l + (\mathcal{L}_v r + j\alpha \lrcorner j\mathrm{d}\omega - \tfrac{1}{3}\mathbb{I}\alpha \lrcorner \mathrm{d}\omega - j\tilde{\alpha} \lrcorner j\mathrm{d}\sigma + \tfrac{2}{3}\mathbb{I}\tilde{\alpha} \lrcorner \mathrm{d}\sigma) \\ & + (\mathcal{L}_v a + r \cdot \mathrm{d}\omega - \alpha \lrcorner \mathrm{d}\sigma) + (\mathcal{L}_v \tilde{a} + r \cdot \mathrm{d}\sigma + \mathrm{d}\omega \wedge a) \\ & + (\mathcal{L}_v \alpha - \tilde{\alpha} \lrcorner \mathrm{d}\omega) + \mathcal{L}_v \tilde{\alpha} \end{aligned} \tag{A.45}$$

To obtain the twisted Dorfman derivative we make the following substitutions.

$$\mathrm{d}\omega \to \mathrm{d}\omega - v \lrcorner F \qquad \mathrm{d}\sigma \to \mathrm{d}\sigma - v \lrcorner \tilde{F} + \omega \wedge F \tag{A.46}$$

Invariant Tensors

The following is the cubic invariant for $\mathrm{E}_{6(6)}$

$$c(V, V, V) = -\left(v \lrcorner \omega \wedge \sigma + \frac{1}{3!}\omega \wedge \omega \wedge \omega\right) \tag{A.47}$$

The following is the symplectic invariant for $\mathrm{E}_{7(7)}$

$$s(V, V') = -\frac{1}{4}\left(v \lrcorner \tau' - v' \lrcorner \tau + \sigma \wedge \omega' - \sigma' \wedge \omega\right) \tag{A.48}$$

The Killing Form

The Killing form for $\mathrm{E}_{d(d)}$ is

$$\mathrm{Tr}(R, R') = \frac{1}{2}\left(\frac{1}{9-d}\,\mathrm{Tr}\, r\, \mathrm{Tr}\, r' + \mathrm{Tr}\, rr' + \alpha \lrcorner a' + \alpha' \lrcorner a - \tilde{\alpha} \lrcorner \tilde{a}' - \tilde{\alpha}' \lrcorner \tilde{a}\right) \tag{A.49}$$

Projections

Let $Z = \zeta + u + s + t \in \Gamma(E^*)$. Then the projection $E \times E^* \to \mathrm{ad}\,\tilde{F}$ is given by

$$l = -\tfrac{1}{3}u \lrcorner \omega - \tfrac{2}{3}s \lrcorner \sigma - t \lrcorner \tau \tag{A.50}$$

$$r = v \otimes \zeta - ju \lrcorner j\omega + \tfrac{1}{3}u \lrcorner \omega \mathbb{I} - js \lrcorner j\sigma + \tfrac{2}{3}s \lrcorner \sigma \mathbb{I} - jt \lrcorner j\tau \tag{A.51}$$

$$a = \zeta \wedge \omega + u \lrcorner \sigma + s \lrcorner \tau \tag{A.52}$$

$$\tilde{a} = \zeta \wedge \sigma + u \lrcorner \tau \tag{A.53}$$

$$\alpha = v \wedge u + s \lrcorner \omega + t \lrcorner \sigma \tag{A.54}$$

$$\tilde{\alpha} = -v \wedge s - t \lrcorner \omega \tag{A.55}$$

If $Y = \lambda + \kappa + \mu + \nu + \pi \in \Gamma(N)$ then the projection $E \times E \to N$ is given by

$$\lambda = v \lrcorner \omega' + v' \lrcorner \omega \tag{A.56}$$

$$\kappa = v \lrcorner \sigma' + v' \lrcorner \sigma - \omega \wedge \omega' \tag{A.57}$$

$$\begin{aligned} \mu = {} & (j\omega \wedge \sigma' + j\omega' \wedge \sigma) - \tfrac{1}{4}(\sigma \wedge \omega' + \sigma' \wedge \omega) \\ & + (v \lrcorner j\tau' + v' \lrcorner j\tau) - \frac{1}{4}(v \lrcorner \tau' + v' \lrcorner \tau) \end{aligned} \tag{A.58}$$

$$\nu = j^3\omega \wedge \tau' + j^3\omega' \wedge \tau - j^3\sigma \wedge \sigma' \tag{A.59}$$

$$\pi = j^6\sigma' \wedge \tau + j^6\sigma \wedge \tau' \tag{A.60}$$

A.5 $\mathrm{E}_{d+1(d+1)} \times \mathbb{R}^+$ Algebra for Type IIB

The generalised tangent bundle and adjoint bundle for $\mathrm{E}_{d+1(d+1)} \times \mathbb{R}^+$ geometry in type IIB are as follows

$$E = T \oplus (S \otimes T^*) \oplus \wedge^3 T^* \oplus (S \otimes \wedge^5 T^*) \oplus (T^* \otimes \wedge^6 T^*) \tag{A.61}$$

$$\begin{aligned} \mathrm{ad}\,\tilde{F} = {} & \mathbb{R} \oplus (T \otimes T^*) \oplus (S \otimes S^*)_0 \oplus (S \otimes T) \oplus (S \otimes \wedge^2 T^*) \\ & \oplus \wedge^4 T \oplus \wedge^4 T^* \oplus (S \otimes \wedge^6 T) \oplus (S \otimes \wedge^6 T^*) \end{aligned} \tag{A.62}$$

where S transforms in the doublet of $\mathrm{SL}(2,\mathbb{R})$ and the subscript 0 denotes the traceless parts. We take the following sections of these bundles, where each term matches the expressions above in the obvious way.

$$V = v + \lambda^i + \rho + \sigma^i + \tau \qquad R = l + r + a + \beta^i + B^i + \gamma + C + \tilde{\alpha}^i + \tilde{a}^i \tag{A.63}$$

The following gives the adjoint action $R \cdot V = V'$

$$v' = lv + r \cdot v + \epsilon_{ij}\beta^i \lrcorner \lambda^j + \gamma \lrcorner \rho + \epsilon_{ij}\tilde{\alpha}^i \sigma^j \tag{A.64}$$

$$\lambda'^i = l\lambda^i + r \cdot \lambda^i + a^i{}_j \lambda^j + v \lrcorner B^i + \beta^i \lrcorner \rho - \gamma \lrcorner \sigma^i - \tilde{\alpha}^i \tau \tag{A.65}$$

$$\rho' = l\rho + r \cdot \rho + v \lrcorner + C\epsilon_{ij}\lambda^i \wedge B^j + \epsilon_{ij}\beta^i \lrcorner \sigma^j + \gamma \lrcorner \tau \tag{A.66}$$

$$\sigma'^i = l\sigma^i + r \cdot \sigma^i + a^i{}_j \sigma^j + v \lrcorner \tilde{a}^i + \rho \wedge B^i - C \wedge \lambda^i + \beta^i \lrcorner \tau \tag{A.67}$$

$$\tau' = l\tau + r \cdot \tau - \epsilon_{ij} j\sigma^i \wedge B^j - j\rho \wedge C + \epsilon_{ij} j\lambda^i \wedge \tilde{a}^j \tag{A.68}$$

The following gives the Lie algebra bracket $[R, R'] = R''$

$$l'' = +\tfrac{1}{4}\epsilon_{ij}(\beta^i \lrcorner B'^j - \beta'^i \lrcorner B^j) + \tfrac{1}{2}(\gamma \lrcorner C' - \gamma' \lrcorner C) + \tfrac{3}{4}\epsilon_{ij}(\tilde{\alpha}^i \lrcorner \tilde{a}'^j - \tilde{\alpha}'^i \lrcorner \tilde{a}^j) \tag{A.69}$$

$$\begin{aligned} r'' &= [r, r'] + \epsilon_{ij}(j\beta^i \lrcorner jB'^j - j\beta'^i \lrcorner jB^j) - \tfrac{1}{4}\mathbb{I}\epsilon_{ij}(\beta^i \lrcorner B'^j - \beta'^i \lrcorner B^j) \\ &\quad + (j\gamma \lrcorner jC' - j\gamma' \lrcorner jC) - \tfrac{1}{2}\mathbb{I}(\gamma \lrcorner C' - \gamma' \lrcorner C) \\ &\quad + \epsilon_{ij}(j\tilde{\alpha}^i \lrcorner \tilde{a}'^j - \tilde{\alpha}'^i \lrcorner \tilde{a}^j) - \tfrac{3}{4}\mathbb{I}(\tilde{\alpha}^i \lrcorner \tilde{a}'^j - \tilde{\alpha}'^i \lrcorner \tilde{a}^j) \end{aligned} \tag{A.70}$$

$$\begin{aligned} a''^i{}_j &= (a \cdot a' - a' \cdot a)^i{}_j + \epsilon_{jk}(\beta^i \lrcorner B'^k - \beta'^i \lrcorner B^k) - \tfrac{1}{2}\delta^i{}_j \epsilon_{kl}(\beta^k \lrcorner B'^l - \beta'^k \lrcorner B^l) \\ &\quad + \epsilon_{jk}(\tilde{\alpha}^j \lrcorner \tilde{a}'^k - \tilde{\alpha}'^j \lrcorner \tilde{a}^k_{-} \tfrac{1}{2}\delta^i{}_j \epsilon_{kl}(\tilde{\alpha}^k \lrcorner \tilde{a}'^l - \tilde{\alpha}'^k \lrcorner \tilde{a}^l) \end{aligned} \tag{A.71}$$

$$\beta''^i = (r \cdot \beta'^i - r' \cdot \beta^i) + (a \cdot \beta' - a' \cdot \beta)^i - (\gamma \lrcorner B'^i - \gamma' \lrcorner B^i) - (\tilde{\alpha}^i \lrcorner C' - \tilde{\alpha}'^i \lrcorner C) \tag{A.72}$$

$$B''^i = (r \cdot B'^i - r' \cdot B^i) + (a \cdot B' - a' \cdot B)^i + (\beta^i \lrcorner C' - \beta'^i \lrcorner C) - (\gamma \lrcorner \tilde{a}'^i - \gamma' \lrcorner \tilde{a}^i) \tag{A.73}$$

$$\gamma'' = (r \cdot \gamma' - r' \cdot \gamma) + \epsilon_{ij}\beta^i \wedge \beta'^j + \epsilon_{ij}(\tilde{\alpha}^i \lrcorner B'^j - \tilde{\alpha}'^i \lrcorner B^j) \tag{A.74}$$

$$C'' = (r \cdot C' - r' \cdot C) - \epsilon_{ij}B^i \wedge B'^j + \epsilon_{ij}(\beta^i \lrcorner \tilde{a}'^j - \beta'^i \lrcorner \tilde{a}^j) \tag{A.75}$$

$$\tilde{\alpha}''^i = (r \cdot \tilde{\alpha}' - r' \cdot \tilde{\alpha})^i + (a \cdot \tilde{\alpha}' - a' \cdot \tilde{\alpha})^i - (\beta^i \wedge \gamma' - \beta'^i \wedge \gamma) \tag{A.76}$$

$$\tilde{a}''^i = (r \cdot \tilde{a}' - r' \cdot \tilde{a})^i + (a \cdot \tilde{a}' - a' \cdot \tilde{a})^i + (B^i \wedge C' - B'^i \wedge C) \tag{A.77}$$

Here ϵ_{ij} is the $\mathrm{SL}(2,\mathbb{R})$ invariant antisymmetric tensor with $\epsilon_{12} = -1$.

The Dorfman Derivative

The following is the Dorfman derivative on vectors

$$\begin{aligned} L_V V' = \mathcal{L}_v v' &+ (\mathcal{L}_v \lambda'^i - v' \lrcorner \mathrm{d}\lambda^i) + (\mathcal{L}_v \rho' - v' \lrcorner \mathrm{d}\rho + \epsilon_{ij}\mathrm{d}\lambda^i \wedge \lambda'^j) \\ &+ (\mathcal{L}_v \sigma'^i - v' \lrcorner \mathrm{d}\sigma^i + \mathrm{d}\rho \wedge \lambda'^i - \mathrm{d}\lambda^i \wedge \rho') \\ &+ (\mathcal{L}_v \tau' - \epsilon_{ij} j\lambda'^i \wedge \mathrm{d}\sigma^j + j\rho' \wedge \mathrm{d}\rho + \epsilon_{ij} j\sigma'^i \wedge \mathrm{d}\lambda^j) \end{aligned} \tag{A.78}$$

The following is the Dorfman derivative on adjoint elements

$$\begin{aligned} L_V R = (&\mathcal{L}_v l + \tfrac{1}{2}\gamma \lrcorner \mathrm{d}\rho + \tfrac{1}{4}\epsilon_{kl}\beta^k \lrcorner \mathrm{d}\lambda^l + \tfrac{3}{4}\epsilon_{kl}\tilde{\alpha}^k \lrcorner \mathrm{d}\sigma^l) \\ &+ (\mathcal{L}_v r + j\gamma \lrcorner j\mathrm{d}\rho - \tfrac{1}{2}\mathbb{I}\gamma \lrcorner \mathrm{d}\rho + \epsilon_{ij} j\beta^i \lrcorner \mathrm{d}\lambda^j - \tfrac{1}{4}\mathbb{I}\epsilon_{ij}\beta^i \lrcorner \mathrm{d}\lambda^j \\ &\qquad + \epsilon_{ij} j\tilde{\alpha}^i \lrcorner \mathrm{d}\sigma^j - \tfrac{3}{4}\mathbb{I}\epsilon_{ij}\tilde{\alpha}^i \lrcorner \mathrm{d}\sigma^j) \\ &+ (\mathcal{L}_v a^i{}_j + \epsilon_{jk}\beta^i \lrcorner \mathrm{d}\lambda^k - \tfrac{1}{2}\delta^i{}_j \epsilon_{kl}\beta^k \lrcorner \mathrm{d}\lambda^l + \epsilon_{jk}\tilde{\alpha}^i \lrcorner \mathrm{d}\sigma^k - \tfrac{1}{2}\delta^i{}_j \epsilon_{kl}\tilde{\alpha}^k \lrcorner \mathrm{d}\sigma^l) \\ &+ (\mathcal{L}_v \beta^i - \gamma \lrcorner \mathrm{d}\lambda^i - \tilde{\alpha}^i \lrcorner \mathrm{d}\rho) \\ &+ (\mathcal{L}_v B^i + r \cdot \mathrm{d}\lambda^i + a^i{}_j \mathrm{d}\lambda^j + \beta^i \lrcorner \mathrm{d}\rho - \gamma \lrcorner \mathrm{d}\sigma^i) \\ &+ (\mathcal{L}_v \gamma + \epsilon_{ij}\tilde{\alpha}^i \lrcorner \mathrm{d}\lambda^j) \\ &+ (\mathcal{L}_v C + r \cdot \mathrm{d}\rho + \epsilon_{ij}\mathrm{d}\lambda^i \wedge B^j + \epsilon_{ij}\beta^i \lrcorner \mathrm{d}\sigma^j) \\ &+ \mathcal{L}_v \tilde{\alpha}^i \\ &+ (\mathcal{L}_v \tilde{a}^i + r \cdot \mathrm{d}\sigma^i + a^i{}_j \mathrm{d}\sigma^j - \mathrm{d}\lambda^i \wedge C + B^i \wedge \mathrm{d}\rho^i) \end{aligned} \tag{A.79}$$

To obtain the twisted Dorfman derivative we make the following substitutions.

$$\mathrm{d}\lambda^i \to \mathrm{d}\lambda^i - v\lrcorner F^i \qquad \mathrm{d}\rho \to \mathrm{d}\rho - v\lrcorner F - \epsilon_{ij}\lambda^i \wedge F^j \qquad \mathrm{d}\sigma^i \to \mathrm{d}\sigma^i + \lambda^i \wedge F - \rho \wedge F^i \tag{A.80}$$

where

$$F^i = \mathrm{d}B^i \qquad F = \mathrm{d}C - \tfrac{1}{2}\epsilon_{ij}F^i \wedge B^j \tag{A.81}$$

Invariant Tensors

The following is the cubic invariant for $\mathrm{E}_{6(6)}$

$$c(V,V,V) = -\frac{1}{2}\left(v\lrcorner\rho \wedge \rho + \epsilon_{ij}\rho \wedge \lambda^i \wedge \lambda^j - 2\epsilon_{ij} v\lrcorner\lambda^i \sigma^j\right) \tag{A.82}$$

The following is the symplectic invariant for $\mathrm{E}_{7(7)}$

$$s(V,V') = -\frac{1}{4}\left(v\lrcorner\tau' - v'\lrcorner\tau + \epsilon_{ij}\lambda^i \wedge \sigma'^j - \epsilon_{ij}\lambda'^i \wedge \sigma^j - \rho \wedge \rho'\right) \tag{A.83}$$

The Killing Form

The Killing form for $\mathrm{E}_{d+1(d+1)}$ is

$$\begin{aligned}\mathrm{Tr}(R,R') = \frac{1}{2}\bigg(&\frac{1}{8-d}\,\mathrm{Tr}\,r\,\mathrm{Tr}\,r' + \mathrm{Tr}\,rr' + \mathrm{Tr}\,aa' + \gamma\lrcorner C' + \gamma'\lrcorner C \\ &+ \epsilon_{ij}\beta^i\lrcorner B^j + \epsilon_{ij}\beta'^i\lrcorner B^j + \epsilon_{ij}\tilde{\alpha}^i\lrcorner\tilde{a}'^j + \epsilon_{ij}\tilde{\alpha}'^i\lrcorner\tilde{a}^j\bigg)\end{aligned} \tag{A.84}$$

A.6 Embedding of $\mathrm{O}(6,6) \subset \mathrm{E}_{7(7)} \times \mathbb{R}^+$ for type IIB

We will follow the conventions and notation of [213] for $\mathrm{E}_{7(7)} \times \mathbb{R}^+$ generalised geometry applied to type IIB. Recall that the generalised tangent and adjoint spaces and their decompositions into $\mathrm{O}(6,6)$ generalised bundles take the form

$$\begin{aligned} E &\simeq T \oplus 2T^* \oplus \wedge^3 T^* \oplus 2\wedge^5 T^* \oplus (T^* \otimes \wedge^6 T^*) \\ &\simeq E_{\mathrm{O}(6,6)} \oplus S^- \oplus (\wedge^6 T^* \otimes E_{\mathrm{O}(6,6)}), \\ \mathrm{ad}\,\tilde{F} &\simeq 4\mathbb{R} \oplus (T \otimes T^*) \oplus 2\wedge^2 T^* \oplus 2\wedge^2 T \oplus \wedge^4 T^* \oplus 2\wedge^6 T^* \oplus 2\wedge^6 T \\ &\simeq 4\mathbb{R} \oplus \mathrm{ad}\,\tilde{F}_{\mathrm{O}(6,6)} \oplus S^+ \oplus (\wedge^6 T \otimes S^+) \oplus \wedge^6 T^* \oplus \wedge^6 T. \end{aligned} \tag{A.85}$$

We use the following rules for embedding the O(6,6) structures into the $E_{7(7)} \times \mathbb{R}^+$ structures for type IIB.

$$\begin{aligned} E_{O(6,6)} &\to E \\ v + \lambda &\mapsto v - s^i \lambda \end{aligned} \tag{A.86}$$

$$\begin{aligned} \operatorname{ad} \tilde{F}_{O(6,6)} &\to \operatorname{ad} \tilde{F} \\ r + \beta + B &\mapsto \tfrac{1}{8} \operatorname{Tr}(r) + \big(r - \tfrac{1}{8} \mathbb{I} \operatorname{Tr}(r)\big) + \tfrac{1}{4} \operatorname{Tr}(r)(r^i \epsilon_{jk} s^k + s^i \epsilon_{jk} r^k) - s^i B + r^i \beta, \end{aligned} \tag{A.87}$$

$$\begin{aligned} S^+ &\to \operatorname{ad} \tilde{F} \\ \Sigma &\mapsto r^i \epsilon_{jk} r^k \Sigma_{(0)} + r^i \Sigma_{(2)} + \Sigma_{(4)} + s^i \Sigma_{(6)}, \end{aligned} \tag{A.88}$$

$$\begin{aligned} S^- &\to E \\ \Sigma &\mapsto r^i \Sigma_{(1)} + \Sigma_{(3)} + s^i \Sigma_{(5)}, \end{aligned} \tag{A.89}$$

where $\Sigma_{(k)} \in \Gamma(\wedge^k T^*)$ are the components of the polyform Λ and r^i, s^i are real constant $SL(2, \mathbb{R})$ doublets such that $\epsilon_{ij} r^i s^j = 1$.

Appendix B

A Note on Curvature in Exceptional Generalised Geometry

In conventional Riemannian geometry (or even the study of manifolds with connection) one can define a curvature. This is central to many theorems in mathematics and many areas of physics, including supergravity. Given that the only requirement to define curvature is the choice of some connection, it seems natural to ask what the generalisation of this to generalised geometry would be. As we will see however, the naive choice for generalised Riemann curvature does not work and there does not appear to be a simple way to adapt the formula to fix the problems that arise.

Recall the formula for the Riemann curvature tensor $R \in \wedge^2 T^* \otimes \mathrm{ad}\tilde{P}_{GL(d,\mathbb{R})}$ is given by

$$R(u,v) = [\nabla_u, \nabla_v] - \nabla_{[u,v]} \qquad \forall u, v \in T \tag{B.1}$$

Hence, given a generalised connection D it seems natural to try to define a generalised Riemann curvature tensor in any generalised geometry as

$$R(U,V) = [D_U, D_V] - D_{[\![U,V]\!]} \qquad \forall U, V \in E \tag{B.2}$$

However, as is stated in [181], $\mathcal{R}$ as defined is not a tensor. Indeed making the changes $U \longrightarrow fU$, $V \longrightarrow gV$ and then act on some hX for $f, g, h \in C^\infty(M)$ and $X \in E$ we get

$$R(fU, gV)hX = R(U,V)X - \frac{1}{2} h D_{(gdf - fdg)\times_E (U \times_N V)} X \tag{B.3}$$

This result is completely generic and holds in all $O(d,d) \times \mathbb{R}^+$ and $E_{d(d)} \times \mathbb{R}^+$ generalised geometries.

B.1 Generalised Riemann Tensor in $\mathrm{O}(d,d) \times \mathbb{R}^+$ Generalised Geometry

It was shown in [189] that one can adapt the definition of Riemann curvature in ordinary geometry to define a genuine tensor in $O(d,d) \times \mathbb{R}^+$ generalised geometry. Their work is primarily done from the point of view of double field theory and enlarged spacetimes, however their results are applicable to generalised geometry. We will briefly review now what they did.

They work in some frame of the generalised tangent bundle. Using capital Latin letters to denote this frame, they take the connection

$$D_M A^N = d_M A^N + \Gamma_{MP}{}^N A^P \qquad D_M A_N = d_M A_N - \Gamma_{MN}{}^P A_P \tag{B.4}$$

Then using (B.2) we find that we get the usual formula for $R_{MNK}{}^L$

$$R_{MNK}{}^L = 2d_{[M}\Gamma_{N]K}{}^L + 2\Gamma_{[M|Q}{}^L\Gamma_{|N]K}{}^Q \tag{B.5}$$

However, as seen in (B.3) this does not define a tensor. However they noticed that one can define a generalised tensor as

$$\mathcal{R}_{MNKL} = R_{MNKL} + R_{KLMN} + \Gamma_{QMN}\Gamma^Q{}_{KL} \tag{B.6}$$

This has a lot of properties one would want from a generalised Riemann curvature. for a start it is an $O(d,d)$-tensor and it has the correct transformation properties under a generalised diffeomorphism. Moreover, under suitable contractions it provides the generalised Ricci tensor and scalar $\mathcal{R}_{MN}$, $\mathcal{R}$ that are used in the Lagrangian formulation of the NS-NS sector of supergravity [182, 183, 189].

Conversely, this does not immediately have the symmetry properties of the conventional Riemann curvature. While it is symmetric under the exchange of the first and last pairs of indices, it is not antisymmetric under swapping the first or last pair. They show in [189] that these properties can be obtained if we make further assumptions about the connection.

We can find a frame independent definition of the generalised Riemann curvature in terms of the $O(d,d)$ inner product η. For $U, V, W, X \in E$ we have

$$\begin{aligned}\eta(\mathcal{R}(U,V)W,X) = \eta(R(U,V)W,X) + \eta(R(W,X)U,V)\\ - \frac{1}{4}\eta(\eta(U,DV) - \eta(V,DU), \eta(W,DX) - \eta(X,DW))\end{aligned} \tag{B.7}$$

In the final term, the first inner product is done between the generalised vector parts, and the second is done between the generalised covector parts generated by the covariant derivative D. Note that the inner product on E^* is generated by $\langle\cdot,\cdot\rangle$ and the identification $E^* \cong E$.

B.2 Generalised Riemann Tensors in $\mathbf{E}_{d(d)} \times \mathbb{R}^+$ Generalised Geometry

We notice that the formula (B.7) can be rewritten in a form that also works for $E_{d(d)}$ generalised geometry. We use the projections $\times_N : E \times E \to N$, $\times_E : E^* \times N \to E$ and write

$$\begin{aligned}\mathcal{R}(U,V)W \times_N X =& R(U,V)W \times_N X + R(W,X)U \times_N V\\ &- \frac{1}{4}\left(\mu(U,DV) - \mu(V,DU)\right) \times_N \left(\mu(W,DX) - \mu(X,DW)\right)\end{aligned} \tag{B.8}$$

Here we have defined a map $\mu : E \times E^* \times E \to E$. If we use the letters $M, N, P, \ldots$ to denote generalised tangent bundle indices and $A, B, C, \ldots$ to denote N-bundle indices, and if η^A_{MN}, ζ^{MN}_A are the projectors $\times_N$, $\times_E$ respectively, then we define μ as

$$\mu^{PQ}_{MN} = \eta^A_{MN}\zeta^{PQ}_A \tag{B.9}$$

We will use these indices and tensors to show how the formula (B.8) is not tensorial in general.

Firstly we note that we want the projectors $\eta^A_{MN}, \zeta^{MN}_A$ to be covariant with respect to the action of the structure group, G. Hence if $R \in G$ then we require

$$-R^P{}_M\eta^A_{PN} - R^Q{}_N\eta^A_{MQ} + R^A{}_B\eta^B_{MN} = 0 \qquad R^M{}_P\zeta^{PN}_A + R^N{}_Q\zeta^{MQ}_A - R^B{}_A\zeta^{MN}_B = 0 \tag{B.10}$$

where the index structure denotes the representation that R falls into. We also want our connection to be compatible with the G-structure. This gives the following conditions on the connection

$$-\Gamma^Q_{MN}\eta^A_{QP} - \Gamma^Q_{MP}\eta^A_{NQ} + \Gamma^A_{MB}\eta^B_{NP} = 0 \qquad \Gamma^N_{MQ}\zeta^{QP}_A + \Gamma^P_{MQ}\zeta^{NQ}_A - \Gamma^B_{MA}\eta^{NP}_B = 0 \tag{B.11}$$

Here we have assumed that we can choose a frame in which η, ζ are locally constant maps and that the action of the connection can be extended to the N bundle. The form of the maps given in appendix A supports the first assumption, while the second assumption should be possible by the fact that $N \subset S^2E$ which has a natural action of the connection on it.

Working in this frame we can now rescale the vectors U, V, W, X by the functions f, g, h, k respectively. In doing so we find

$$\begin{aligned}[\mathcal{R}(U', V')W' \times_N X']^A =& fghk[\mathcal{R}(U, V)W \times_N X]^N \\ &+ \frac{1}{4}hk\Gamma^A_{MB}[(fdg - gdf) \times_E (U \times_N V)]^M[W \times_N X]^B \\ &+ \frac{1}{4}fg\Gamma^A_{MB}[(hdk - kdh) \times_E (W \times_N X)]^M[U \times_N V]^B\end{aligned} \tag{B.12}$$

It is not clear how one would cancel these additional terms given that they live in the N bundle. The only case where this works is for $O(d, d)$ generalised geometry in which $N \sim \mathbf{1}$ and hence the $\Gamma^A_{MB} = 0$.

So we see that for generic generalised geometry we still do not have a good definition of Riemann curvature. This is interesting since if we would like to perhaps extend generalised geometry to try to describe either α' corrections or higher derivative gravity then we would like to have some kind of Riemann curvature.

We should note that this work does not rule out the possibility of finding some tensor which behaves like a generalised Riemann tensor, just that the two most obvious candidates for one do not work in general. It may be possible to define a curvature tensor on a case-by-case basis. For example, in $E_{6(6)}$ generalised geometry, $N \cong E^*$ and so we can think of $\eta^A{}_{MN} \sim c_{MNP}$. The symmetry of indices may mean that there is some formula involving generalised vectors which exactly cancels any non-tensorial terms. Ideally however, one would like a formula that works in all cases of generalised geometry.

Appendix C

Integrability and Moment Maps for (Generalised) Complex Structures

We review two examples of familiar geometric structures that appear when describing supersymmetric backgrounds: conventional complex structures in six dimensions and their generalised geometry extensions first introduced by Hitchin and Gualtieri [164, 165]. In each case, involutivity of an appropriate vector bundle under a bracket is equivalent to the integrability of the structure.[1] We will then also discuss how the extra differential conditions that promote these structures to integrable $\mathrm{SL}(3,\mathbb{C})$ and generalised Calabi–Yau structures come from a moment map for the action of diffeomorphisms and, in the latter case, gauge symmetries. These two examples will provide the model for how we analyse generic four-dimensional $\mathcal{N}=1$ flux backgrounds.

C.1 Complex Structures

Let M be a six-dimensional manifold with tangent bundle T. Recall that an almost complex structure on M is a conventional G-structure with $G=\mathrm{GL}(3,\mathbb{C})$. It is defined by a nowhere-vanishing tensor $I\in\Gamma(\operatorname{End}T)$, with $I^2=-\mathbb{I}$, that allows one to decompose the complexified tangent bundle into subbundles

$$T\otimes\mathbb{C}\coloneqq T_{\mathbb{C}}=L_1\oplus L_{-1}, \tag{C.1}$$

where sections of L_1 have charge $+\mathrm{i}$ under the action of I, and $\bar{L}_1\simeq L_{-1}$. Typically, L_1 is written $T^{1,0}$ but we will use this notation to highlight the similarities to the work in the later sections. Consider two vectors $V,W\in\Gamma(L_1)$. A standard way to define an integrable structure is to require that the Lie bracket of two $(1,0)$-vector fields gives another $(1,0)$-vector field. In other words, L_1 is involutive under the Lie bracket

$$[V,W]\in\Gamma(L_1)\qquad\forall\, V,W\in\Gamma(L_1). \tag{C.2}$$

[1] Note that we use "integrable" and "torsion-free" interchangeably. For a conventional G-structure, integrable is a stronger condition: torsion-free implies the G-structure is flat to first-order, while integrable implies the G-structure is locally equivalent to the flat model. See [277] for some remarks on this nomenclature.

Using I to project onto L_1 it is then straightforward to show that involutivity of the bracket is equivalent to the vanishing of the Nijenhuis tensor, or equivalently, in the language of G-structures, the vanishing of the intrinsic torsion.

Every almost complex structure I defines a unique "canonical" line bundle $\mathcal{U}_I \subset \wedge^3 T^*_{\mathbb{C}}$ satisfying

$$\imath_V \Omega = 0 \quad \forall\, V \in \Gamma(L_{-1}), \qquad \Omega \wedge \bar{\Omega} \neq 0, \tag{C.3}$$

where Ω is a local section of $\mathcal{U}_I$. If this bundle is trivial, one can introduce a refinement of the almost complex structure by considering $G = \mathrm{SL}(3, \mathbb{C})$ structures. Each such structure is defined by a nowhere-vanishing section $\Omega \in \Gamma(\mathcal{U}_I)$ so that any two such structures defining the same complex structure differ by nowhere-vanishing complex function f

$$\Omega' = f\Omega. \tag{C.4}$$

Note that, as $\mathrm{SL}(3, \mathbb{C}) \subset \mathrm{GL}(3, \mathbb{C})$, given a suitable complex three-form Ω (one stabilised by $\mathrm{SL}(3, \mathbb{C})$) one can construct an almost complex structure I, as described by Hitchin [255]. It is natural then to ask the question, if we have a torsion-free $\mathrm{GL}(3, \mathbb{C})$ structure (a complex structure), what extra condition do we need to impose to have a torsion-free $\mathrm{SL}(3, \mathbb{C})$ structure? From the intrinsic torsion in each case, it is straightforward to see that the $\mathrm{GL}(3, \mathbb{C})$ structure is torsion-free if

$$\mathrm{d}\Omega = A \wedge \Omega, \tag{C.5}$$

for some $(0, 1)$-form A, while for a torsion-free $\mathrm{SL}(3, \mathbb{C})$ structure we should have

$$\mathrm{d}\Omega = 0. \tag{C.6}$$

Thus A encodes the extra intrinsic torsion components of the $\mathrm{SL}(3, \mathbb{C})$ structure.

This additional integrability condition can be viewed as the vanishing of a moment map. One first notes that the space of $\mathrm{SL}(3, \mathbb{C})$ structures admits a natural pseudo-Kähler metric [255]. At a point $p \in M$, a choice of Ω is equivalent to picking a point in the coset

$$\Omega|_p \in Q_{\mathrm{SL}(3,\mathbb{C})} = \frac{\mathrm{GL}(6, \mathbb{R})}{\mathrm{SL}(3, \mathbb{C})}. \tag{C.7}$$

The choice of $\mathrm{SL}(3, \mathbb{C})$ structure on M thus corresponds to a section of the fibre bundle

$$Q_{\mathrm{SL}(3,\mathbb{C})} \to \mathcal{Q}_{\mathrm{SL}(3,\mathbb{C})} \to M, \tag{C.8}$$

that is, we can identify

$$\text{space of SL}(3, \mathbb{C}) \text{ structures, } \mathrm{SL}(3, \mathbb{C}) \simeq \Gamma(\mathcal{Q}_{\mathrm{SL}(3,\mathbb{C})}). \tag{C.9}$$

This infinite-dimensional space then inherits a pseudo-Kähler structure from the pseudo-Kähler structure[2] on the coset space $Q_{\mathrm{SL}(3,\mathbb{C})}$, with a Kähler potential given by

$$\mathcal{K} = \mathrm{i} \int_M \Omega \wedge \bar{\Omega}, \tag{C.10}$$

where Ω can be viewed as a complex coordinate on the space of structures (or more precisely as a holomorphic embedding $\Omega\colon \mathrm{SL}(3,\mathbb{C}) \hookrightarrow \Gamma(\wedge^3 T^*_{\mathbb{C}})$). One can also restrict to the subspace of structures that define an (integrable) complex structure, so that L_1 is involutive,

$$\mathrm{SL}(3,\mathbb{C}) = \{\Omega \in \mathrm{SL}(3,\mathbb{C}) \mid I \text{ is integrable}\}. \tag{C.11}$$

Given that the integrability condition (C.5) is holomorphic – it is independent of $\bar{\Omega}$ – this space inherits a pseudo-Kähler metric from $\mathrm{SL}(3,\mathbb{C})$ with the same Kähler potential.

Diffeomorphisms act on $\mathrm{SL}(3,\mathbb{C})$ since the integrability conditions on I are diffeomorphism invariant. Infinitesimally they define a vector field $\rho_V \in \Gamma(T\mathrm{SL}(3,\mathbb{C}))$ such that

$$\imath_{\rho_V} \delta\Omega = \mathcal{L}_V \Omega, \tag{C.12}$$

where δ is the exterior (functional) derivative on $\mathrm{SL}(3,\mathbb{C})$ and $V \in \Gamma(T)$ generates the diffeomorphism. Clearly the Kähler potential (C.10) is diffeomorphism invariant. Furthermore, since $\mathcal{L}_V\Omega$ is independent of $\bar{\Omega}$, we see that diffeomorphisms also preserve the complex structure on $\mathrm{SL}(3,\mathbb{C})$. Together this implies they preserve the Kähler form.[3] Explicitly this is given by

$$\varpi = \mathrm{i}\, \partial' \bar{\partial}' \mathcal{K}, \tag{C.13}$$

where we have decomposed $\delta = \partial' + \bar{\partial}'$ into holomorphic and antiholomorphic derivatives. For an arbitrary vector $\alpha \in \Gamma(T\mathrm{SL}(3,\mathbb{C}))$ we then have

$$\begin{aligned} \imath_{\rho_V} \imath_\alpha \varpi &= -\int_M \big(\imath_\alpha \delta\Omega \wedge \mathcal{L}_V \bar{\Omega} - \mathcal{L}_V \Omega \wedge \imath_\alpha \delta\bar{\Omega}\big) = \int_M \big(\mathcal{L}_V \imath_\alpha \delta\Omega \wedge \bar{\Omega} + \mathcal{L}_V \Omega \wedge \imath_\alpha \delta\bar{\Omega}\big) \\ &= \int_M \imath_\alpha \big(\mathcal{L}_V \delta\Omega \wedge \bar{\Omega} + \mathcal{L}_V \Omega \wedge \delta\bar{\Omega}\big) = \imath_\alpha \delta\mu(v), \end{aligned} \tag{C.14}$$

where

$$\mu(V) = \int_M \mathcal{L}_V \Omega \wedge \bar{\Omega}. \tag{C.15}$$

defines a moment map $\mu\colon \mathrm{SL}(3,\mathbb{C}) \to \mathfrak{diff}^*$, where $\mathfrak{diff}$ is the Lie algebra of diffeomorphisms. It is straightforward to check that μ is equivariant.

Given the integrability condition (C.5), we can integrate by parts, to write

$$\begin{aligned} \mu(V) &= \int_M \Big(-\imath_V \Omega \wedge \bar{A} \wedge \bar{\Omega} + \imath_V (A \wedge \Omega) \wedge \bar{\Omega}\Big) \\ &= \int_M (\imath_V A - \imath_V \bar{A})\, \Omega \wedge \bar{\Omega}, \end{aligned} \tag{C.16}$$

[2] This metric has signature $(18,2)$ [255].

[3] Note that there may be further subtleties if the integrability condition defines a null subspace within $\mathrm{SL}(3,\mathbb{C})$ or if the group action defining the moment map is null. We comment on this for the case of SU(7) structures in section 5.3.3.

where we have used $A \wedge \bar{\Omega} = \bar{A} \wedge \Omega = 0$. The moment map vanishes for all $V \in \Gamma(T)$ if and only if

$$A = \bar{A} = 0. \tag{C.17}$$

In other words, we see that the vanishing of the moment map imposes the final condition (C.6) that promotes a complex structure to a torsion-free $\mathrm{SL}(3,\mathbb{C})$ structure.

Since two $\mathrm{SL}(3,\mathbb{C})$ structures that are related by a diffeomorphism are equivalent, the moduli space of $\mathrm{SL}(3,\mathbb{C})$ structures is naturally a quotient, defined as

$$\mathcal{M}_\Omega = \{\Omega \in \mathrm{SL}(3,\mathbb{C}) \mid \mu = 0\}/\mathrm{Diff}. \tag{C.18}$$

As we have seen, the Kähler geometry on the space of structures $\mathrm{SL}(3,\mathbb{C})$ is preserved by the action of the diffeomorphism group, thus we can view the moduli space either as a symplectic quotient by Diff or as a quotient by the complexified group

$$\mathcal{M}_\Omega = \mathrm{SL}(3,\mathbb{C})/\!\!/\mathrm{Diff} \simeq \mathrm{SL}(3,\mathbb{C})/\mathrm{Diff}_\mathbb{C}. \tag{C.19}$$

Note that the complexification of the diffeomorphism group $\mathrm{Diff}_\mathbb{C}$ is not really well defined. What is really meant is the complexification of the orbits, that is, if the vector field $\rho_V \in \Gamma(T\mathrm{SL}(3,\mathbb{C}))$ generates the action of diffeomorphisms on the spaces of structures, we can complexify this to also include the orbits generated by $\mathcal{I}\rho_V$, where $\mathcal{I}$ is the complex structure on $\mathrm{SL}(3,\mathbb{C})$. Since Ω is a holomorphic function on $\mathrm{SL}(3,\mathbb{C})$ we have

$$\imath_{\mathcal{I}\rho_V}\delta\Omega = -\imath_{\rho_V}(\mathcal{I}\delta\Omega) = \mathrm{i}\,\imath_{\rho_V}\delta\Omega = \mathrm{i}\,\mathcal{L}_V\Omega = \mathcal{L}_{IV}\Omega + 2\mathrm{i}(\imath_V A)\Omega, \tag{C.20}$$

where in the last expression we have used (C.5) and the fact that $\imath_{IV}\Omega = \mathrm{i}\,\imath_V\Omega$ and $\imath_{IV}A = -\mathrm{i}\,\imath_V A$. Thus in (C.19), up to diffeomorphisms, for each fixed complex structure, the action of $\mathrm{Diff}_\mathbb{C}$ simply rescales Ω until (C.6) is satisfied and the moment map vanishes.

C.2 Generalised Complex Structures

Let us now review the analogous story for the generalised complex structures (GCS) of Hitchin and Gualtieri [164,165]. We will see again that involutivity and a moment map characterise the integrable structures and lead to a local description of the moduli space as a Kähler quotient.

Consider a six-dimensional manifold M with a generalised tangent bundle $E_{\mathrm{O}(6,6)} = T \oplus T^*$. This admits a natural $\mathrm{O}(6,6)$ structure given by the inner product

$$\langle x+\xi, y+\eta \rangle = \eta(x) + \xi(y). \tag{C.21}$$

As was noted in [183], the relevant structure group for supergravity is actually $\mathrm{O}(6,6)\times\mathbb{R}^+$ to account for the dilaton. We take all generalised vectors to be weight zero under the $\mathbb{R}^+$ action. Given a generalised vector $V = x + \xi \in \Gamma(E)$, there is a natural generalised Lie derivative L_V such that, acting on a generalised vector $W = y + \eta$,

$$L_{x+\xi}(y+\eta) = [x,y] + \mathcal{L}_x\eta - \imath_y \mathrm{d}\xi. \tag{C.22}$$

This generates conventional diffeomorphisms and one-form gauge transformations, parameterised by x and ξ respectively. Its antisymmetrisation $[\![V,W]\!] := \frac{1}{2}(L_V W - L_W V)$ is the Courant bracket. $E_{\mathrm{O}(6,6)}$ generates a Clifford algebra Cliff$(6,6)$ via the inner product above, which has a natural representation on sections Ψ of the spinor bundle $S := \wedge^\bullet T^*$ via

$$\not{V}\Psi = \imath_x \Psi + \xi \wedge \Psi. \tag{C.23}$$

The slash notation signifies the Clifford action and can be viewed as contraction with the $\mathrm{O}(6,6)$ gamma matrices Γ^M. There is an invariant antisymmetric pairing (Ψ, Σ) on spinors given by the Mukai pairing (A.21), with the property that

$$(\Psi, \not{V}\Sigma) = (-\not{V}\Psi, \Sigma). \tag{C.24}$$

As a representation of $\mathrm{Spin}(6,6) \times \mathbb{R}^+$ the spinor bundle is reducible as one can define the analogue of Majorana–Weyl spinors[4]

$$S^+ = \wedge^{\text{even}} T^*, \qquad S^- = \wedge^{\text{odd}} T^*. \tag{C.25}$$

The exterior derivative gives a map $\mathrm{d}\colon S^\pm \to S^\mp$ such that the action of the generalised Lie derivative can be written as

$$L_V \Psi = \mathrm{d}(\not{V}\Psi) + \not{V}\mathrm{d}\Psi, \tag{C.26}$$

for any $\Psi \in \Gamma(S)$.

In analogy to a conventional almost complex structure, a generalised almost complex structure $\mathcal{J}$ is a endomorphism $\mathcal{J}\colon \Gamma(E_{\mathrm{O}(6,6)}) \to \Gamma(E_{\mathrm{O}(6,6)})$ such that

$$\mathcal{J}^2 = -\mathbb{I}, \qquad \langle \mathcal{J}V, \mathcal{J}V\rangle = \langle V, V\rangle \quad \forall\, V \in \Gamma(E). \tag{C.27}$$

As a generalised tensor, $\mathcal{J}$ is nowhere vanishing so defines reduction of the structure group of $E_{\mathrm{O}(6,6)}$ from $\mathrm{O}(6,6) \times \mathbb{R}^+$ to $\mathrm{U}(3,3) \times \mathbb{R}^+$. It gives a decomposition of the complexified generalised tangent bundle

$$E_{\mathrm{O}(6,6)\mathbb{C}} = L_1 \oplus L_{-1}, \tag{C.28}$$

where $L_{\pm 1}$ has charge $\pm\mathrm{i}$ under $\mathcal{J}$. Note that L_1 is maximally isotropic: $\langle L_1, L_1\rangle = 0$. This defines an isomorphism $L_1^* \simeq \bar{L}_1 = L_{-1}$. A generalised almost complex structure is integrable if L_1 is involutive with respect to the generalised Lie derivative

$$L_V W \in \Gamma(L_1) \qquad \forall\, V, W \in \Gamma(L_1), \tag{C.29}$$

which also implies $L_V W = [\![V,W]\!]$. Using the notion of generalised intrinsic torsion introduced in [184], one can show that this involution condition is equivalent to the vanishing of the generalised intrinsic torsion of the $\mathrm{U}(3,3) \times \mathbb{R}^+$ structure defined by $\mathcal{J}$.

Each generalised almost complex structure defines a unique pure spinor line bundle $\mathcal{U}_{\mathcal{J}} \subset S$

[4] It was important that we take the structure group to be $\mathrm{O}(6,6) \times \mathbb{R}^+$, or its double cover $\mathrm{Spin}(6,6) \times \mathbb{R}^+$, here since polyforms do not form a representation of $\mathrm{Spin}(6,6)$ alone. It also implies the antisymmetric pairing gives a top-form rather than a scalar. Without the $\mathbb{R}^+$ factor, one would have to take $S \simeq \bigwedge^\bullet T^* \otimes (\det T)^{1/2}$.

satisfying

$$\not{V}\Phi = 0 \quad \forall\, V \in \Gamma(L_1), \qquad (\Phi, \bar{\Phi}) \neq 0, \tag{C.30}$$

where Φ is a local section of $\mathcal{U}_{\mathcal{J}}$ and $(\cdot,\cdot)$ is the Mukai pairing defined in (A.21). If the pure spinor line bundle is trivial, one can choose a global nowhere-vanishing section. This defines an SU(3,3) or generalised Calabi–Yau (GCY) structure [164].[5] Two such structures defining the same GCY structure differ by nowhere vanishing complex function f

$$\Phi' = f\Phi. \tag{C.31}$$

From the generalised intrinsic torsion it is straightforward to see that the corresponding generalised complex structure is integrable if

$$\mathrm{d}\Phi = \not{A}\Phi, \tag{C.32}$$

where $A \in \Gamma(L_{-1})$ acts on Φ via the Clifford action. The generalised Calabi–Yau structure is integrable if

$$\mathrm{d}\Phi = 0, \tag{C.33}$$

and hence A encodes the extra components of the intrinsic torsion of the SU(3,3) structure.

As in the previous example of a complex structure, one can view the additional integrability condition as the vanishing of a moment map. One first notes that the space of SU(3,3) structures on M admits a natural pseudo-Kähler metric [151, 164] – the construction follows that of the almost complex structure case. At a point $p \in M$, a choice of Φ is equivalent to picking a point in the coset

$$\Phi|_p \in Q_{\mathrm{SU}(3,3)} = \frac{\mathrm{O}(6,6)\times\mathbb{R}^+}{\mathrm{SU}(3,3)}, \tag{C.34}$$

so that an SU(3,3) structure on M corresponds to a section of the fibre bundle

$$Q_{\mathrm{SU}(3,3)} \to \mathcal{Q}_{\mathrm{SU}(3,3)} \to M. \tag{C.35}$$

We can then identify

$$\text{space of SU}(3,3)\text{ structures, } \mathrm{SU}(3,3) \simeq \Gamma(\mathcal{Q}_{\mathrm{SU}(3,3)}). \tag{C.36}$$

This infinite-dimensional space inherits a pseudo-Kähler structure from the pseudo-Kähler structure[6] on the coset space $Q_{\mathrm{SU}(3,3)}$, with a Kähler potential given by

$$\mathcal{K} = \mathrm{i}\int_M (\Phi, \bar{\Phi}). \tag{C.37}$$

Again Φ can be viewed as a complex coordinate on the space of structures (or more precisely as a holomorphic embedding $\Phi\colon \mathrm{SU}(3,3) \hookrightarrow \Gamma(S_{\mathbb{C}})$) and one can also restrict to the subspace

[5] Given a GCY structure, one can recover the generalised almost complex structure by identifying L_1 as the null space of Φ.

[6] This metric has signature (30, 2) [156, 164].

of structures that define an (integrable) generalised complex structure, so that L_1 is involutive,

$$\mathrm{SU}(3,3) = \{\Phi \in \mathrm{SU}(3,3) \mid \mathcal{J} \text{ is integrable}\}. \tag{C.38}$$

The condition (C.32) is holomorphic and so $\mathrm{SU}(3,3)$ inherits a pseudo-Kähler metric from $\mathrm{SU}(3,3)$, with the same Kähler potential.

The group of generalised diffeomorphisms GDiff, that is diffeomorphisms and gauge transformations, acts on $\mathrm{SU}(3,3)$ and preserves the Kähler structure. The action is generated by vector fields $\rho_V \in \Gamma(T\mathrm{SU}(3,3))$ defined via the generalised Lie derivative

$$\imath_{\rho_V}\delta\Phi = L_V\Phi. \tag{C.39}$$

Given the Kähler form as defined in (C.13) and an arbitrary vector $\alpha \in \Gamma(T\mathrm{SU}(3,3))$, one finds

$$\begin{aligned}\imath_{\rho_V}\imath_\alpha\varpi &= -\int_M (\imath_\alpha\delta\Phi, L_V\bar{\Phi}) - (L_V\Phi, \imath_\alpha\delta\bar{\Phi}) = \int_M (L_V\imath_\alpha\delta\Phi, \bar{\Phi}) + (L_V\Phi, \imath_\alpha\delta\bar{\Phi}) \\ &= \imath_\alpha\delta\int_M (L_V\Phi, \bar{\Phi}) = \imath_\alpha\delta\mu(V)\end{aligned} \tag{C.40}$$

where

$$\mu(V) = \int_M (L_V\Phi, \bar{\Phi}), \tag{C.41}$$

defines a moment map $\mu\colon \mathrm{SU}(3,3) \to \mathfrak{gdiff}^*$. Here $\mathfrak{gdiff}$ is the Lie algebra of generalised diffeomorphisms generated by the generalised Lie derivative.

From (C.26), the integrability condition (C.32) and (C.24) we have

$$\begin{aligned}\mu(V) &= \int_M \big(\slashed{V}\mathrm{d}\Phi + \slashed{V}\Phi, \mathrm{d}\bar{\Phi}\big) = \int_M (\slashed{V}\slashed{A}\Phi, \Phi) + (\slashed{V}\Phi, \bar{\slashed{A}}\bar{\Phi}) \\ &= \int_M (\slashed{V}(\slashed{A} - \bar{\slashed{A}})\Phi, \bar{\Phi}) + ((\slashed{A} - \bar{\slashed{A}})\slashed{V}\Phi, \bar{\Phi}) = 2\int_M \langle V, A - \bar{A}\rangle(\Phi, \bar{\Phi}),\end{aligned} \tag{C.42}$$

where in going to the second line we have used $\slashed{A}\bar{\Phi} = \bar{\slashed{A}}\Phi = 0$. Thus we see the moment map vanishes for all V if and only if $A = \bar{A} = 0$, that is, if the $\mathrm{SU}(3,3)$ structure is integrable.

Again, we consider two $\mathrm{SU}(3,3)$ structures that are related by a generalised diffeomorphism as equivalent and so the moduli space of $\mathrm{SU}(3,3)$ structures is a symplectic quotient.[7] Since the group action preserves the Kähler structure, we can view also view the moduli space as a quotient by the complexified group $\mathrm{GDiff}_{\mathbb{C}}$

$$\mathcal{M}_\Phi = \mathrm{SU}(3,3)/\!\!/\mathrm{GDiff} \simeq \mathrm{SU}(3,3)/\mathrm{GDiff}_{\mathbb{C}}. \tag{C.43}$$

As before, if $\mathcal{I}$ is the complex structure on $\mathrm{SU}(3,3)$, we have

$$\imath_{\mathcal{I}\rho_V}\delta\Phi = -\imath_{\rho_V}(\mathcal{I}\delta\Phi) = \mathrm{i}(\imath_{\rho_V}\delta\Phi) = \mathrm{i}\, L_V\Phi = -L_{\mathcal{J}V}\Phi + 2\mathrm{i}\,\langle V, A\rangle\,\Phi, \tag{C.44}$$

where in the last expression we have used (C.32) and the fact that $\mathcal{J}V \circ \Phi = -\mathrm{i}\,\imath_V\Phi$ and

[7] As with the previous $\mathrm{SL}(3,\mathbb{C})$ structures, this can be more nuanced. We refer the reader to section 5.3.3 for a discussion of this for $\mathrm{SU}(7)$ structures.

$\langle \mathcal{J}V, A\rangle = \mathrm{i}\,\langle V, A\rangle$. Thus, up to generalised diffeomorphisms, for each fixed complex structure, the action of $\mathrm{GDiff}_{\mathbb{C}}$ simply rescales Φ until $\mathrm{d}\Phi = 0$ and the moment map vanishes.

Appendix D

Explicit Calculations of the Heterotic Superpotential, Kähler Potential and Moment Map

In this appendix, we lay out in detail how one calculates the superpotential, Kähler potential and the moment map using the explicit form of ψ and J given in the main text.

D.1 The Superpotential

To see that our expression for the superpotential (3.69) matches the conventional expression given in (3.63), we expand in $\mathrm{O}(6, 6+n)$ indices:

$$\begin{aligned} W &\sim \int_X J^A{}_B D_C \psi^{CB}{}_A \\ &\sim \int_X D_C (J^A{}_B \psi^{CB}{}_A) - \psi^{CBA} D_{[C} J_{AB]} \\ &\sim \int_X \psi^{ABC} D_{[A} J_{BC]} \\ &\sim \int_X \sqrt{g}\, \mathrm{e}^{-2\varphi} \Omega^{\bar\mu\bar\nu\bar\rho} D_{[\bar\mu} J_{\bar\nu\bar\rho]}, \end{aligned} \tag{D.1}$$

where we have used the fact that the boundary term vanishes identically, and have raised/lowered indices with η. To reach the final lines we have used results from the previous section on the contraction of ψ with a section of $\wedge^3 E$. Hence all that remains is to determine the form of $D_{[\bar\mu} J_{\bar\nu\bar\rho]}$. Using the components of the connection from [186], we have that

$$\begin{aligned} D_{[\bar\mu} J_{\bar\nu\bar\rho]} &= \nabla_{[\bar\mu} J_{\bar\nu\bar\rho]} - \tfrac{1}{3} H_{[\bar\mu}{}^{\sigma}{}_{\bar\nu|} J_{|\sigma|\bar\rho]} \\ &= \tfrac{1}{3} (-\mathrm{d}\omega)_{\bar\mu\bar\nu\bar\rho} + \tfrac{\mathrm{i}}{3} H_{[\bar\mu}{}^{\sigma}{}_{\bar\nu} g_{|\sigma|\bar\rho]} \\ &\sim (H + \mathrm{i}\mathrm{d}\omega)_{\bar\mu\bar\nu\bar\rho}, \end{aligned} \tag{D.2}$$

where we have used $g_{\mu\bar{\nu}} = -\mathrm{i}\omega_{\mu\bar{\nu}}$ for an SU(3) structure. Hence

$$\begin{aligned}\mathcal{W} &\sim \int_X \sqrt{g}\, \mathrm{e}^{-2\varphi}\Omega^{\bar{\mu}\bar{\nu}\bar{\rho}}(H + \mathrm{i}\,\mathrm{d}\omega)_{\bar{\mu}\bar{\nu}\bar{\rho}} \\ &\sim \int_X \mathrm{e}^{-2\varphi}\Omega \wedge (H + \mathrm{i}\,\mathrm{d}\omega).\end{aligned} \tag{D.3}$$

This is precisely the form of the superpotential in (3.63) and used in [207, 233]. Hence our expression (3.69) is the covariant form of the superpotential for a generic four-dimensional $\mathcal{N} = 1$ heterotic background determined by ψ.

D.2 The Kähler Potential

The Kähler potential is

$$\mathcal{K} = \int_X \eta(\psi, \bar{\psi})^{1/2}, \tag{D.4}$$

where η is the symmetric pairing on sections of $\wedge^3 E$. We fix our conventions for this in terms of η on sections of E by examining how the usual inner product defined by g acts on tri-vectors. For $\alpha, \beta \in \Gamma(\wedge^3 T)$, the pairing is

$$\begin{aligned}g(\alpha, \beta) &= \tfrac{1}{3!}\tfrac{1}{3!}\alpha^{mnp}\beta^{qrs} g(\hat{e}_{mnp}, \hat{e}_{qrs}) \\ &\equiv \tfrac{1}{3!}\alpha^{mnp}\beta^{qrs} g(\hat{e}_m, \hat{e}_q) g(\hat{e}_n, \hat{e}_r) g(\hat{e}_p, \hat{e}_s) \\ &= \tfrac{1}{3!}\alpha^{mnp}\beta_{mnp} \\ &= \alpha \lrcorner \beta,\end{aligned} \tag{D.5}$$

where we have used $\hat{e}_{mnp} \lrcorner e^{qrs} = 3! \delta^q_{[m}\delta^r_n\delta^s_{p]}$. Similarly we define

$$\begin{aligned}\eta(\hat{E}^+_{mnp}, \hat{E}^+_{qrs}) &= 3!\eta(\hat{E}^+_m, \hat{E}^+_q)\eta(\hat{E}^+_n, \hat{E}^+_r)\eta(\hat{E}^+_p, \hat{E}^+_s) \\ &= 3!\delta_{mq}\delta_{nr}\delta_{ps},\end{aligned} \tag{D.6}$$

where an antisymmetrisation over mnp is assumed and for simplicity we take $\hat{e}_m$ to be an orthonormal frame, implying $\eta(\hat{E}^+_m, \hat{E}^+_n) = g_{mn} = \delta_{mn}$. With χ defined as in (3.83)

$$\chi = \frac{1}{3!} g^{1/4} \mathrm{e}^{-\varphi}\Omega^{mnp}\hat{E}^+_{mnp}, \tag{D.7}$$

the pairing $\eta(\chi, \bar{\chi})$ is given by

$$\begin{aligned}\eta(\chi, \bar{\chi}) &= \tfrac{1}{3!} g^{1/2} \mathrm{e}^{-2\varphi}\Omega^{mnp}\bar{\Omega}_{mnp} \\ &= g^{1/2}\mathrm{e}^{-2\varphi}\Omega^\sharp \lrcorner \bar{\Omega} \\ &= \mathrm{i}\,\mathrm{e}^{-2\varphi}\Omega \wedge \bar{\Omega},\end{aligned} \tag{D.8}$$

where we have used the standard SU(3) structure relations

$$\Omega^\sharp \lrcorner \bar{\Omega} = 8, \qquad g^{1/2} = \mathrm{vol} = \frac{\mathrm{i}}{8}\Omega \wedge \bar{\Omega}. \tag{D.9}$$

Integrated over X, this gives the expression for the Kähler potential given in the main text.

D.3 The Moment Map

The expression for the moment map given in the main text is

$$\mu(V) = -\frac{\mathrm{i}}{2}\int_X \eta(L_V\chi, \bar{\chi}). \tag{D.10}$$

To evaluate this, we need an expression for the Dorfman derivative of χ. For $V = \mathrm{e}^{-B}\mathrm{e}^{-A}(v + \lambda + \Lambda)$, where $v \in \Gamma(T)$, $\lambda \in \Gamma(T^*)$ and $\Lambda \in \Gamma(\operatorname{ad} P)$, we have

$$L_V\chi = \tfrac{1}{3!}\mathcal{L}_v(g^{1/4}\mathrm{e}^{-\varphi}\Omega^{mnp})\hat{E}^+_{mnp} + \tfrac{1}{2}g^{1/4}\mathrm{e}^{-\varphi}\Omega^{mnp}L_V\hat{E}^+_m \wedge \hat{E}^+_{np}, \tag{D.11}$$

$$L_V\hat{E}^+_m = \mathrm{e}^{-B}\mathrm{e}^{-A}\left(\mathcal{L}_v(\hat{e}_m + e_m) - \imath_{\hat{e}_m}\mathrm{d}\lambda - \imath_v\imath_{\hat{e}_m}H + 2\operatorname{Tr}(\Lambda\imath_{\hat{e}_m}F) - \imath_{\hat{e}_m}\mathrm{d}_A\Lambda - \imath_v\imath_{\hat{e}_m}F\right). \tag{D.12}$$

The expression for the moment map is then

$$\begin{aligned}\mu(V) &= -\tfrac{\mathrm{i}}{2}\int_X \eta\left(\tfrac{1}{3!}\mathcal{L}_v(g^{1/4}\mathrm{e}^{-\varphi}\Omega^{mnp})\hat{E}^+_{mnp} + \tfrac{1}{2}g^{1/4}\mathrm{e}^{-\varphi}\Omega^{mnp}L_V\hat{E}^+_m \wedge \hat{E}^+_{np}, \tfrac{1}{3!}g^{1/4}\mathrm{e}^{-\varphi}\bar{\Omega}^{qrs}\hat{E}^+_{qrs}\right)\\ &= -\tfrac{\mathrm{i}}{2}\int_X \tfrac{1}{3!}\mathcal{L}_v(g^{1/4}\mathrm{e}^{-\varphi}\Omega^{mnp})g^{1/4}\mathrm{e}^{-\varphi}\bar{\Omega}_{mnp}\\ &\quad - \tfrac{\mathrm{i}}{2}\int_X \tfrac{1}{2}g^{1/4}\mathrm{e}^{-\varphi}\Omega^{mnp}\tfrac{1}{3!}g^{1/4}\mathrm{e}^{-\varphi}\bar{\Omega}^{qrs}\eta(L_V\hat{E}^+_m \wedge \hat{E}^+_{np}, \hat{E}^+_{qrs}),\end{aligned} \tag{D.13}$$

where we have used

$$\eta(L_V\hat{E}^+_m \wedge \hat{E}^+_{np}, \hat{E}^+_{qrs}) = 3!\,\eta(L_V\hat{E}^+_m, \hat{E}^+_q)\delta_{nr}\delta_{ps}, \tag{D.14}$$

with an assumed antisymmetrisation over mnp and

$$\begin{aligned}\eta(L_V\hat{E}^+_m, \hat{E}^+_n) &= \eta\left(\mathcal{L}_v(\hat{e}_m + e_m) - \imath_{\hat{e}_m}\mathrm{d}\lambda - \imath_v\imath_{\hat{e}_m}H + 2\operatorname{Tr}(\Lambda\imath_{\hat{e}_m}F) - \imath_{\hat{e}_m}\mathrm{d}_A\Lambda - \imath_v\imath_{\hat{e}_m}F, \hat{e}_n + e_n\right)\\ &= \tfrac{1}{2}\imath_{\mathcal{L}_v\hat{e}_m}e_n + \tfrac{1}{2}\imath_{\hat{e}_n}\mathcal{L}_v e_m - \tfrac{1}{2}\imath_{\hat{e}_n}\imath_{\hat{e}_m}\mathrm{d}\lambda - \tfrac{1}{2}\imath_{\hat{e}_n}\imath_v\imath_{\hat{e}_m}H + \imath_{\hat{e}_n}\operatorname{Tr}(\Lambda\imath_{\hat{e}_m}F).\end{aligned} \tag{D.15}$$

Our task is now to find what conditions $\mu = 0$ imposes. To do this, we examine $\mu(V) = 0$ where V consists of an arbitrary vector, one-form or gauge parameter in turn. First, consider $V = \mathrm{e}^{-B}\mathrm{e}^{-A}\lambda$:

$$\begin{aligned}\int_X \eta(L_V\chi, \bar{\chi}) &= \int_X \tfrac{1}{2}g^{1/4}\mathrm{e}^{-\varphi}\Omega^{mnp}g^{1/4}\mathrm{e}^{-\varphi}\bar{\Omega}^{qrs}\left(-\tfrac{1}{2}\right)\imath_{\hat{e}_q}\imath_{\hat{e}_m}\mathrm{d}\lambda\,\delta_{nr}\delta_{ps}\\ &= -\tfrac{1}{4}\int_X \mathrm{e}^{-2\varphi}\Omega^{mnp}\bar{\Omega}^q{}_{np}\imath_{\hat{e}_q}\imath_{\hat{e}_m}\mathrm{d}\lambda\ \mathrm{vol}\\ &= 2\mathrm{i}\int_X \mathrm{e}^{-2\varphi}\mathrm{d}\lambda \wedge \omega \wedge \omega\\ &= 2\mathrm{i}\int_X \lambda \wedge \mathrm{d}(\mathrm{e}^{-2\varphi}\omega \wedge \omega),\end{aligned} \tag{D.16}$$

where we have used the SU(3) structure identity

$$\Omega^{mnp}\bar{\Omega}^{q}{}_{np}(\imath_{\hat{e}_q}\imath_{\hat{e}_m}\alpha_2)\ \mathrm{vol} = -8\mathrm{i}\alpha_2\wedge\omega\wedge\omega, \tag{D.17}$$

which holds for an arbitrary two-form α_2.

Next, consider $V = \mathrm{e}^{-B}\mathrm{e}^{-A}\Lambda$:

$$\begin{aligned}\int_X \eta(L_V\chi,\bar{\chi}) &= \int_X \tfrac{1}{2}g^{1/4}\mathrm{e}^{-\varphi}\Omega^{mnp}\tfrac{1}{3!}g^{1/4}\mathrm{e}^{-\varphi}\bar{\Omega}^{qrs}3!\left(\imath_{\hat{e}_q}\,\mathrm{Tr}(\Lambda\imath_{\hat{e}_m}F)\right)\delta_{nr}\delta_{ps}\\ &= \int_X \tfrac{1}{2}\,\mathrm{vol}\ \mathrm{e}^{-2\varphi}\Omega^{mnp}\bar{\Omega}^{q}{}_{np}\imath_{\hat{e}_q}\imath_{\hat{e}_m}\,\mathrm{Tr}(\Lambda F)\\ &= \int_X \tfrac{1}{2}\mathrm{e}^{-2\varphi}(-8\mathrm{i})\ \mathrm{Tr}(\Lambda F)\wedge\omega\wedge\omega\\ &= -4\mathrm{i}\int_X \mathrm{Tr}(\Lambda F)\wedge\mathrm{e}^{-2\varphi}\omega\wedge\omega,\end{aligned} \tag{D.18}$$

where again we have used (D.17).

Finally, consider $V = \mathrm{e}^{-B}\mathrm{e}^{-A}v$:

$$\begin{aligned}\int_X \eta(L_V\chi,\bar{\chi}) = &\int_X \tfrac{1}{3!}\mathcal{L}_v(g^{1/4})g^{1/4}\mathrm{e}^{-2\varphi}8\cdot 3! + \tfrac{1}{3!}\mathcal{L}_v(\mathrm{e}^{-\varphi}\Omega^{mnp})g^{1/2}\mathrm{e}^{-\varphi}\bar{\Omega}_{mnp}\\ &+\int_X \tfrac{1}{4}g^{1/2}\mathrm{e}^{-2\varphi}\Omega^{mnp}\bar{\Omega}^{q}{}_{np}\left(\imath_{\hat{e}_q}\mathcal{L}_v e_m - \imath_{\hat{e}_m}\mathcal{L}_v e_q - \imath_{\hat{e}_q}\imath_v\imath_{\hat{e}_m}H\right).\end{aligned} \tag{D.19}$$

Now note that the first term is real while $\int_X \eta(L_V\chi,\bar{\chi})$ is imaginary (after an integration by parts), so it cancels. The remaining terms can be rewritten as

$$\begin{aligned}\int_X \eta(L_V\chi,\bar{\chi}) = &\int_X \tfrac{1}{2}\tfrac{1}{3!}g^{1/2}\mathrm{e}^{-2\varphi}\mathcal{L}_v\Omega^{mnp}\bar{\Omega}_{mnp} + \tfrac{1}{2}\tfrac{1}{3!}g^{1/2}\mathrm{e}^{-2\varphi}\mathcal{L}_v\bar{\Omega}^{mnp}\Omega_{mnp}\\ &+\int_X \tfrac{1}{4}g^{1/2}\mathrm{e}^{-2\varphi}\left(2\bar{\Omega}\lrcorner\mathcal{L}_v\Omega - 2\Omega\lrcorner\mathcal{L}_v\bar{\Omega} - \Omega^{mnp}\bar{\Omega}^{q}{}_{np}\imath_{\hat{e}_q}\imath_v\imath_{\hat{e}_m}H\right),\end{aligned} \tag{D.20}$$

where we have the SU(3) structure identities $\Omega^\sharp\lrcorner\bar{\Omega} = 8$ and $8\,\mathrm{vol} = \mathrm{i}\Omega\wedge\bar{\Omega}$, and

$$\Omega^{mnp}\bar{\Omega}^{q}{}_{np} = 8g^{mq} + 8\mathrm{i}\,I^{mq} = 8g^{mq} - 8\mathrm{i}\,\omega^{mq}, \tag{D.21}$$

$$2\bar{\Omega}\lrcorner\mathcal{L}_v\Omega = \tfrac{1}{3}\mathcal{L}_v\Omega^{mnp}\bar{\Omega}_{mnp} + \Omega^{mnp}\bar{\Omega}^{q}{}_{np}\imath_{\hat{e}_q}\mathcal{L}_v e_m, \tag{D.22}$$

$$2\Omega\lrcorner\mathcal{L}_v\bar{\Omega} = \tfrac{1}{3}\mathcal{L}_v\bar{\Omega}^{mnp}\Omega_{mnp} + \Omega^{mnp}\bar{\Omega}^{q}{}_{np}\imath_{\hat{e}_m}\mathcal{L}_v e_q. \tag{D.23}$$

Again, note that the first two terms of (D.20) combine to give something real, and so they must cancel. We can then massage the remaining terms to give

$$\begin{aligned}\int_X \eta(L_V\chi,\bar{\chi}) &= \int_X \tfrac{1}{4}\mathrm{e}^{-2\varphi}\left(2\mathrm{i}\mathcal{L}_v\Omega\wedge\bar{\Omega} + 2\mathrm{i}\mathcal{L}_v\bar{\Omega}\wedge\Omega - 8\imath_v H\wedge\omega\wedge\omega\right)\\ &= \mathrm{i}\tfrac{1}{2}\int_X \mathrm{e}^{-2\varphi}(2\imath_v\bar{a} - 2\imath_v a + 2\imath_v\partial\varphi - 2\imath_v\bar{\partial}\varphi)\,\Omega\wedge\bar{\Omega}\\ &\quad + 2\mathrm{e}^{-2\varphi}\imath_v\partial\varphi\,\Omega\wedge\bar{\Omega} - 2\mathrm{e}^{-2\varphi}\imath_v\bar{\partial}\varphi\Omega\wedge\bar{\Omega}\\ &= \mathrm{i}\int_X \mathrm{e}^{-2\varphi}(\imath_v\bar{a} - \imath_v a + 2\imath_v\partial\varphi - 2\imath_v\bar{\partial}\varphi)\,\Omega\wedge\bar{\Omega}.\end{aligned} \tag{D.24}$$

To reach this result, we have integrated by parts and used $\mathrm{d}\Omega = \bar{a} \wedge \Omega$ for $\bar{a} \in \Omega^{0,1}(X)$, which is implied by integrability of the complex structure which in turn comes from involutivity of L_{-1}. We have also used $\bar{\Omega} \lrcorner \alpha_3 \,\mathrm{vol} = \mathrm{i}\, \alpha_3 \wedge \bar{\Omega}$ for an arbitrary three-form α_3. Summed up, the three contributions to $\mu(V)$ in (D.16), (D.18) and (D.24) give the expression for the moment map given in the main text.

Appendix E

Exceptional Complex Structures in $E_{6(6)} \times \mathbb{R}^+$ Geometry

Proposition 1. Any isotropic subbundle $L \subset E_{\mathbb{C}}$ has the form

$$e^{\alpha+\beta} \cdot (\Delta \oplus S_2 \oplus S_5) \tag{E.1}$$

where $\alpha \in \Omega^3(M)$ and$\beta \in \Omega^6(M)$ are arbitrary but fixed, and where $\Delta \subset T$, $S_2 \subset \wedge^2 T^*$, $S_5 \subset \wedge^5 T^*$ satisfy the following. For all $v \in \Delta$, $\omega, \omega' \in S_2$and $\sigma \in S_5$ we have

$$\begin{aligned} v \lrcorner \omega &= 0 & \qquad v \lrcorner \sigma &= 0 \\ \omega \wedge \omega' &= 0 & j\omega \wedge \sigma &= 0 \end{aligned} \tag{E.2}$$

To prove this, we follow a similar proof for isotropic bundles in $\mathrm{O}(d, d)$ geometry laid out in [165].

Proof. The condition for isotropy is $V_1 \times_N V_2 = 0$ for all $V_1, V_2 \in L$ which translates to

$$v_1 \lrcorner \omega_1 + v_2 \lrcorner \omega_1 = 0 \tag{E.3}$$

$$j\omega_1 \wedge \sigma_2 + j\omega_2 \wedge \sigma_1 = 0 \tag{E.4}$$

$$\omega_1 \wedge \omega_2 - v_1 \lrcorner \sigma_2 - v_2 \lrcorner \sigma_1 = 0 \tag{E.5}$$

It is a simple check to see that any L of the form (E.1) satisfies these conditions and hence defines an isotropic bundle. Hence it is left to show that any isotropic bundle takes that form.

Clearly we have $\Delta = a(L)$. Suppose we have some

$$\omega_1, \omega_2 \in \pi_{\wedge^2 T^*}\left((\wedge^2 T^* \oplus \wedge^5 T^*) \cap L\right) \tag{E.6}$$

where $\pi_{\wedge^2 T^*} : E \to \wedge^2 T^*$, and similarly for π_T, $\pi_{\wedge^5 T^*}$. From (E.3) and (E.5) we see that for any $v \in \Delta$

$$\begin{aligned} v \lrcorner \omega_i = 0 \quad &\Rightarrow \quad \omega_i \in \mathcal{F}_1^2(\Delta) \\ \omega_1 \wedge \omega_2 = 0 \quad &\Rightarrow \quad \omega_i \in S_2 \end{aligned} \tag{E.7}$$

Now consider the element

$$\alpha(v) := \pi_{\wedge^2 T^*}\left(\pi_T^{-1}(v) \cap L\right) \in \frac{\wedge^2 T^*}{\pi_{\wedge^2 T^*}((\wedge^2 T^* \oplus \wedge^5 T^*) \cap L)} \tag{E.8}$$

From (E.3) for $V \times_N V = 0$, we see that we need

$$v \lrcorner \alpha(v) \quad \forall\, v \in \Delta \quad \Rightarrow \quad \alpha \in \wedge^3 T^* \text{ (WLOG)} \tag{E.9}$$

Then we can write any element $\lambda \in \pi_{\wedge^2 T^*}(L)$ as

$$\lambda = v \lrcorner \alpha + \omega \qquad v \in \Delta,\, \omega \in S_2 \tag{E.10}$$

No we consider any $\sigma \in \wedge^5 T^* \cap L$. From (E.4) and (E.5) we see that for all $v \in \Delta$, $\omega \in S_2$ we need

$$\begin{aligned} v \lrcorner \sigma = 0 \quad &\Rightarrow \quad \sigma \in \mathcal{F}_4^5(\Delta) \\ j\omega \wedge \sigma = 0 \quad &\Rightarrow \quad \sigma \in S_5 \end{aligned} \tag{E.11}$$

Note that we also need

$$j(v \lrcorner \alpha) \wedge \sigma = 0 \quad \Leftrightarrow \quad (\mathrm{vol}^\# \lrcorner \sigma) \lrcorner (v \lrcorner \alpha) = 0 \tag{E.12}$$

However, since $\mathcal{F}_4^5(\Delta) = 0$ if $\operatorname{rk} \Delta > 1$, one can check that

$$(\mathrm{vol}^\# \lrcorner \sigma) \lrcorner (v \lrcorner \alpha) \propto v \lrcorner (v \lrcorner \alpha) = 0 \tag{E.13}$$

Now consider the element

$$\theta(v, \omega) := \pi_{\wedge^5 T^*}\left((\pi_T^{-1}(v) + \pi_{\wedge^2 T^*}^{-1}(\omega)) \cap L\right) \in \frac{\wedge^5 T^*}{\wedge^5 T^* \cap L} \tag{E.14}$$

From (E.5) we need

$$(\omega_1 + v_1 \lrcorner \alpha) \wedge (\omega_2 + v_2 \lrcorner \alpha) - v_1 \lrcorner \theta(v_2, \omega_2) - v_2 \lrcorner \theta(v_1, \omega_1) \tag{E.15}$$

which has the general solution

$$\theta(v, \omega) = \frac{1}{2} v \lrcorner \alpha \wedge \alpha + v \lrcorner \beta + \lambda \wedge \alpha \tag{E.16}$$

where $\beta \in \wedge^6 T^*$ is arbitrary. It is a simple check to see that this also satisfies (E.4). Checking the action of $\mathrm{e}^{\alpha+\beta}$ we see that we have

$$L = \mathrm{e}^{\alpha+\beta} \cdot (\Delta \oplus S_2 \oplus S_5) \tag{E.17}$$

□

Proposition 4.

$$\dim_{\mathbb{C}} L = 6 \qquad \Leftrightarrow \qquad \operatorname{type} L = 0, 3, 6 \tag{E.18}$$

Proof. We will consider each type $L = k$ for $k = 0, 1, ..., 6$

k=0

All type 0 bundles are of the form $L = \mathrm{e}^{\alpha+\beta} \cdot T$ which is clearly 6 dimensional

k=1

If we have a type 1 bundle then $\mathrm{rk}\,\Delta = 5$ and $\mathrm{rk}\,\mathcal{F}_1^2(\Delta) = \mathrm{rk}\,\mathcal{F}_4^5(\Delta) = 0$. Hence the bundle looks like $\mathrm{e}^{\alpha+\beta} \cdot \Delta$ again. However, this is just 5 dimensional.

k=2

In this case we have $rk\Delta = 4$, $\mathrm{rk}\,\mathcal{F}_1^2(\Delta) = 1$, $\mathrm{rk}\,\mathcal{F}_4^5(\Delta) = 0$ and hence the isotropic bundle is of the form $\mathrm{e}^{\alpha+\beta} \cdot (\Delta \oplus \mathcal{F}_1^2(\Delta))$ which is 5 dimensional.

k=3

We have $\mathrm{rk}\,\Delta = 3$, $\mathrm{rk}\,\mathcal{F}_1^2 = 3$, $\mathrm{rk}\,\mathcal{F}_4^5 = 0$. Hence we can take $L = \mathrm{e}^{\alpha+\beta} \cot(\Delta \oplus \mathcal{F}_1^2(\Delta))$ which is 6 dimensional.

k = 4

We have $\mathrm{rk}\,\Delta = 2$, $\mathrm{rk}\,\mathcal{F}_1^2(\Delta) = 6$, $\mathrm{rk}\,\mathcal{F}_4^5(\Delta) = 0$. However, it is not that case that $\omega \wedge \omega' = 0$ for all $\omega, \omega' \in \mathcal{F}_1^2(\Delta)$. We take any subspace which satisfies this condition which has maximal rank 3. Hence the isotropic bundle of the form $\mathrm{e}^{\alpha+\beta} \cdot (\Delta \oplus S_2)$ has maximal dimension 5

k = 5

We have $\mathrm{rk}\,\Delta = 1$, $\mathrm{rk}\,\mathcal{F}_1^2 = 10$, $\mathrm{rk}\,\mathcal{F}_4^5(\Delta) = 1$. Again, we choose a maximal $S_2 \subset \mathcal{F}_1^2(\Delta)$ satisfying $\omega \wedge \omega' = 0$. This will have rank 3 and so the isotropic bundle $\mathrm{e}^{\alpha+\beta} \cdot (\Delta \oplus S_2 \oplus S_5)$ has dimension 5.

k = 6 In this case $\mathrm{rk}\,\Delta = 0$. It will be convenient to parameterise $\S_5 = \Gamma \lrcorner \mathrm{vol}$ where $\Gamma \subset T$. We will also choose a basis e^i of T^* with dual basis $\hat{e}_i$ of T. The only possible type 6 solutions are given in the table below.

Γ	S_2	L	$\dim L$
T	0	$\wedge^5 T^*$	6
$\langle \hat{e}_1, ..., \hat{e}_4 \rangle$	$\langle e^5 \wedge e^6 \rangle$	$\mathrm{S}_2 \oplus S_5$	5
$\langle \hat{e}_1, \hat{e}_2, \hat{e}_3 \rangle$	$\langle e^i \wedge e^j \mid i,j = 4,5,6 \rangle$	$\mathrm{s}_2 \oplus S_5$	6

□

Proposition 5. There are no exceptional complex structures of type 6

Proof. There are two different 6 dimensional isotropic spaces of type 6, which shown in the table above. We will show that these do not satisfy the remaining conditions in definition 20.

Firstly, let's consider $L_1 = \wedge^5 T^*$. Clearly, this does not satisfy condition (iii) as $\bar{L}_1 = L_1$. Therefore, this cannot be an ECS.

Secondly, let's consider $L_1 = \mathrm{e}^{\alpha} \cdot (S_2 \oplus S_5)$, where S_2, S_5 are as in the third row of the table above. We will show that $L_0 \cap (L_1 \oplus L_{-1}) \neq 0$ and hence this does not satisfy condition (iii).

To find L_0 we need to find the null space

$$A = \{Z \in E^* \mid \langle V, Z\rangle = 0 \,\forall\, V \in L_1 \oplus L_{-1}\} \tag{E.19}$$

Using the same notation as above, it is easy to see that

$$T^* \oplus \left\langle \hat{e}^i \wedge \hat{e}^j \mid i = 1, 2, 3,\ j = 4, 5, 6 \right\rangle \subseteq A \tag{E.20}$$

The left hand side of this is 15 dimensional. If L is to define an ECS then A must be 15 dimensional too and hence this must be the whole of A. In particular, this implies that

$$S_5 \oplus \bar{S}_5 = \wedge^5 T^* L_1 \oplus L_{-1} \tag{E.21}$$

Now taking any $\nu \in T^*$, and some $\mathrm{e}^\alpha \cdot \omega \in L_1$. Then we have

$$\mathrm{e}^\alpha \omega \times_{\mathrm{ad}} \nu = \mathrm{e}^\alpha \omega \wedge \nu \tag{E.22}$$

However, we have that

$$(\mathrm{e}^\alpha \omega \wedge \nu) \cdot L_{-1} \subseteq \wedge^5 T^* \subset L_1 \oplus L_{-1} \tag{E.23}$$

Hence we see that $L_0 \cap (L_1 \oplus L_1) \neq 0$ and so this cannot be an ECS $\square$

Appendix F

Proof of the Local Structure of Moduli of SU*(6) Structures

F.1 Type 3 Moduli

For the type 3 problem we have the subbundle

$$L_1 = \mathrm{e}^{\alpha+\beta} \cdot (\Delta \oplus \mathcal{F}_1^2(\Delta)) \tag{F.1}$$

where $\alpha \in \Omega^3(M)_{\mathbb{C}}$, $\beta \in \Omega^6(M)_{\mathbb{C}}$, and $\Delta \subset T$ all satisfy

$$[\Delta, \Delta] \subseteq \Delta \qquad v \lrcorner (w \lrcorner (x \lrcorner \mathrm{d}\alpha)) = 0 \tag{F.2}$$

for all $v, w, x \in \Gamma(\Delta)$. In what follows, it will be convenient to work with the $F_{\mathbb{C}}$-twisted Dorfman derivative, where locally $F_{\mathbb{C}} = \mathrm{d}\alpha$, and the untwisted bundle $\tilde{L}_1 = \Delta \oplus \mathcal{F}_1^2(\Delta)$. This is because, for type 3 solutions, the physical flux may be in a non-trivial cohomology class and hence the gauge potential A, which is implicit in the definition of α, may not be global. By working with $L_V^{F_{\mathbb{C}}}$, we can work only with globally defined objects.

We have the quotient spaces

$$E_{\mathbb{C}}/\tilde{L}_1 = \left(T/\mathfrak{F}_0^1\right) \oplus \left(\wedge^2 T^*/\mathcal{F}_1^2\right) \oplus \wedge^5 T^* \tag{F.3}$$

$$\mathfrak{Q}_{\mathbb{R}^+ \times \mathrm{U}^*(6)} = \left[\left(T/\mathfrak{F}_0^1\right) \otimes \left(T^*/\mathcal{F}_0^1\right)\right] \oplus \left(\wedge^3 T/\mathfrak{F}_2^3\right) \oplus \left(\wedge^3 T^*/\mathcal{F}_1^3\right) \oplus \wedge^6 T^* \tag{F.4}$$

We shall pick the following elements of the deformation space $\mathfrak{Q}$

$$\begin{aligned} &r \in \Gamma\left((T/\mathfrak{F}_0^1) \otimes (T^*/\mathcal{F}_0^1)\right) && \chi \in \wedge^3 T/\mathfrak{F}_2^3 \\ &\theta \in \wedge^3 T^*/\mathcal{F}_1^3 && \tau \in \wedge^6 T^* \end{aligned} \tag{F.5}$$

and write the deformation parameter as $R = X + r + \theta + \tau$. Hence the deformed bundle becomes

$$\tilde{L}_1' = (1 + R) \cdot \tilde{L}_1 = \mathrm{e}^{\theta+\tau}(1 + r + X) \cdot (\Delta \oplus \mathcal{F}_1^2(\Delta)) \tag{F.6}$$

where we are working to linear order in the deformation parameters only. We take sections

$V', W' \in \Gamma(\tilde{L}'_1)$ which are of the form

$$V' = \mathrm{e}^{\theta+\tau} \cdot (v + r \cdot v + X \lrcorner \lambda + \lambda + r \cdot \lambda) \qquad W' = \mathrm{e}^{\theta+\tau} \cdot (w + r \cdot w + x \lrcorner \mu + \mu + r \cdot \mu) \tag{F.7}$$

where $v, w \in \Gamma(\Delta)$, $\lambda, \mu \in \Gamma(\mathcal{F}^2_1)$. We will also denote by $\hat{V} = \hat{v} + \hat{\lambda} = (1 + r + X) \cdot (v + \lambda)$, and similarly for $\hat{W}$.

We want to determine when $\tilde{L}'_1$ is involutive under $L^{F_{\mathbb{C}}}_V$, to linear order in R. This is the statement that for all $v, w \in \Gamma(\Delta)$, $\lambda, \mu \in \Gamma(\mathcal{F}^2_1)$ we have

$$\begin{aligned} L^{F_{\mathbb{C}}}_{V'} W' &= \mathrm{e}^{\theta+\tau} \cdot \big(L_{\hat{V}} \hat{W} + \hat{w}(\lrcorner \hat{v} \lrcorner (F_{\mathbb{C}} + \mathrm{d}\theta)) + \hat{\mu} \wedge (\hat{v} \lrcorner (F_{\mathbb{C}} + \mathrm{d}\theta)) \big) && \text{(F.8)} \\ &= \mathrm{e}^{\theta+\tau} \cdot \Big([\hat{v}, \hat{w}] \\ &\qquad + \mathcal{L}_{\hat{v}} \hat{\mu} - \hat{w} \lrcorner \mathrm{d}\hat{\lambda} + \hat{w} \lrcorner (\hat{v} \lrcorner (F_{\mathbb{C}} + \mathrm{d}\theta)) && \text{(F.9)} \\ &\qquad - \hat{\mu} \wedge \mathrm{d}\hat{\lambda} + \hat{\mu} \wedge (\hat{v} \lrcorner (F_{\mathbb{C}} + \mathrm{d}\theta)) \Big) \\ &\in \Gamma(\tilde{L}'_1) && \text{(F.10)} \end{aligned}$$

Let us consider this term by term. We will use Greek letters $\alpha, \beta, \gamma, \ldots$ for Δ indices, and Latin letters $a, b, c, \ldots$ for the complement[1]. If we consider the vector piece only then we have, to linear order in R

$$\begin{aligned} \big(\mathrm{e}^{-\theta-\tau} \cdot L^{F_{\mathbb{C}}}_{V'} W'\big)|_T &= [v, w] + [r \cdot v + X \lrcorner \lambda, w] + [v, r \cdot w + X \lrcorner \mu] && \text{(F.11)} \\ &= [v, w] + \Big(r^b{}_\beta v^\beta \partial_b w^\alpha - r^b{}_\gamma w^\gamma \partial_b v^\alpha \Big) + \Big((X \lrcorner \nu)^b \partial_b w^\alpha - (X \lrcorner \lambda)^b \partial_b v^\alpha \Big) \\ &\quad + r \cdot [v, w] + X \lrcorner (v \lrcorner \mathrm{d}\mu - w \lrcorner \mathrm{d}\lambda) \\ &\quad w \lrcorner (v \lrcorner \mathrm{d}_\Delta r) + (v \lrcorner \mathrm{d}_\Delta X) \lrcorner \mu - (w \lrcorner \mathrm{d}_\Delta X) \lrcorner \lambda \\ & && \text{(F.12)} \\ &\stackrel{!}{=} z + r \cdot z + X \lrcorner \zeta && \text{(F.13)} \end{aligned}$$

where $z \in \Gamma(\Delta)$, $\zeta \in \Gamma(\mathcal{F}^2_1)$ are of the form[2]

$$z = [v, w] + O(R) \qquad \zeta = v \lrcorner \mathrm{d}\mu - w \lrcorner \mathrm{d}\lambda + w \lrcorner (v \lrcorner F_{\mathbb{C}}) + O(R) \tag{F.14}$$

For this to be true to linear order in R we need

$$\begin{aligned} w \lrcorner (v \lrcorner \mathrm{d}_\Delta r) + (v \lrcorner \mathrm{d}_\Delta X) \lrcorner \mu - (w \lrcorner \mathrm{d}_\Delta X) \lrcorner \lambda &= X \lrcorner (w \lrcorner (v \lrcorner (F_{\mathbb{C}}))) \\ &= w \lrcorner (v \lrcorner (j X \lrcorner j^2 F_{\mathbb{C}})) \end{aligned} \tag{F.15}$$

This must be true for all v, w, λ, μ and hence we have

$$\mathrm{d}_\Delta X = 0 \qquad \mathrm{d}_\Delta r - j X \lrcorner j^2 F_{\mathbb{C}} = 0 \tag{F.16}$$

[1] Here we are implicitly using the orthogonal complement under some metric. This is just for ease of the proof although it is not strictly needed to prove these results.

[2] The 0^{th} order piece of z, ζ should be given by the Dorfman derivative of the undeformed sections $V = v + \lambda$, $W = w + \mu$.

Now let's consider the 2-form piece. We have

$$\begin{aligned}\left(\mathrm{e}^{-\theta-\tau}\cdot L^{F_{\mathbb{C}}}_{V'}W'\right)|_{\wedge^2 T^*} = {} & v\lrcorner\mathrm{d}\mu - w\lrcorner\mathrm{d}\lambda + w\lrcorner(v\lrcorner F_{\mathbb{C}}) \\ & + (r\cdot v + X\lrcorner\lambda)\lrcorner\mathrm{d}\mu + v\lrcorner\mathrm{d}(r\cdot\mu) \\ & + \mathrm{d}((r\cdot v + X\lrcorner\lambda)\lrcorner\mu) + \mathrm{d}(v\lrcorner(r\cdot\mu)) \\ & - (r\cdot w + X\lrcorner\mu)\lrcorner\mathrm{d}\lambda - w\lrcorner\mathrm{d}(r\cdot\lambda) \\ & + (r\cdot w + X\lrcorner\mu)\lrcorner(v\lrcorner F_{\mathbb{C}}) + w\lrcorner((r\cdot v + X\lrcorner\lambda)\lrcorner F_{\mathbb{C}}) \\ & + w\lrcorner(v\lrcorner\mathrm{d}\theta)\end{aligned} \tag{F.17}$$

$$\stackrel{!}{=} \zeta + r\cdot\zeta \tag{F.18}$$

For now, let us set $\lambda, \mu = 0$. We are left with

$$w\lrcorner(v\lrcorner F_{\mathbb{C}})(r\cdot w)\lrcorner(v\lrcorner F_{\mathbb{C}}) + w\lrcorner((r\cdot v)\lrcorner F_{\mathbb{C}}) + w\lrcorner(v\lrcorner\mathrm{d}\theta) \tag{F.19}$$

$$= w\lrcorner(v\lrcorner F_{\mathbb{C}}) + r\cdot(w\lrcorner(v\lrcorner F_{\mathbb{C}})) + w\lrcorner(v\lrcorner(-r\cdot F_{\mathbb{C}} + \mathrm{d}\theta)) \tag{F.20}$$

$$\stackrel{!}{=} \zeta + r\cdot\zeta \tag{F.21}$$

For this to be the case, we need $\mathrm{d}\theta - r\cdot F_{\mathbb{C}} \in \Gamma(\mathcal{F}^4_1)$. This is equivalent to the statement that

$$\pi_1(\mathrm{d}\theta - r\cdot F_{\mathbb{C}}) = 0 \qquad \pi_k : \wedge^n T^* \longrightarrow \wedge^n T^*/\mathcal{F}^n_k \tag{F.22}$$

where we have introduced the projection operator π_k as defined above for definiteness.

Now let's set $v, \mu = 0$. We have

$$-w\lrcorner\mathrm{d}\lambda - (r\cdot w)\lrcorner\mathrm{d}\lambda - w\lrcorner\mathrm{d}(r\cdot\lambda) + w\lrcorner((X\lrcorner\lambda)\lrcorner F_{\mathbb{C}}) \tag{F.23}$$

$$\begin{aligned}= & -w\lrcorner\mathrm{d}\lambda - w^\alpha({r^c}_\alpha\partial_c\lambda_{ab} - \lambda_{bc}\partial_a {r^c}_\alpha - \lambda_{ca}\partial_b {r^c}_\alpha) \\ & - r\cdot(w\lrcorner\mathrm{d}\lambda) - (w\lrcorner\mathrm{d}_\Delta r)\cdot\lambda + w\lrcorner((X\lrcorner\lambda)\lrcorner F_{\mathbb{C}})\end{aligned} \tag{F.24}$$

$$\begin{aligned}= & -w\lrcorner\mathrm{d}\lambda - w^\alpha({r^c}_\alpha\partial_c\lambda_{ab} - \lambda_{bc}\partial_a {r^c}_\alpha - \lambda_{ca}\partial_b {r^c}_\alpha) \\ & - r\cdot(w\lrcorner\mathrm{d}\lambda) - (w\lrcorner(\mathrm{d}_\Delta r - jX\lrcorner j^2 F_{\mathbb{C}}))\cdot\lambda\end{aligned} \tag{F.25}$$

$$\stackrel{!}{=} \zeta + r\cdot\zeta \tag{F.26}$$

This is implied by the fact that $\mathrm{d}_\Delta r - jX\lrcorner j^2 F_{\mathbb{C}} = 0$.

Next, let's consider when $w = \lambda = 0$. We find

$$v\lrcorner\mathrm{d}\mu + (r\cdot v)\lrcorner\mathrm{d}\mu + v\lrcorner\mathrm{d}(r\cdot\mu) + \mathrm{d}((r\cdot v)\lrcorner\mu) + \mathrm{d}(v\lrcorner(r\cdot\mu)) + (X\lrcorner\mu)\lrcorner(v\lrcorner F_{\mathbb{C}}) \tag{F.27}$$

$$\begin{aligned}= & v\lrcorner\mathrm{d}\mu + v^\alpha({r^c}_\alpha\partial_c\mu_{ab} - \mu_{bc}\partial_a {r^c}_\alpha - \mu_{ca}\partial_b {r^c}_\alpha) \\ & + r\cdot(v\lrcorner\mathrm{d}\mu) + (v\lrcorner\mathrm{d}_\Delta r)\cdot\mu + (X\lrcorner\mu)\lrcorner(v\lrcorner F_{\mathbb{C}})\end{aligned} \tag{F.28}$$

$$\begin{aligned}= & v\lrcorner\mathrm{d}\mu + v^\alpha({r^c}_\alpha\partial_c\mu_{ab} - \mu_{bc}\partial_a {r^c}_\alpha - \mu_{ca}\partial_b {r^c}_\alpha) \\ & + r\cdot(v\lrcorner\mathrm{d}\mu) + (v\lrcorner(\mathrm{d}_\Delta r - jX\lrcorner j^2 F_{\mathbb{C}}))\cdot\mu\end{aligned} \tag{F.29}$$

$$\stackrel{!}{=} \zeta + r\cdot\zeta \tag{F.30}$$

This is again implied by the fact that $\mathrm{d}_\Delta r - jX\lrcorner j^2 F_{\mathbb{C}} = 0$.

Finally for the 2-forms, we consider the case when $v, w = 0$. We find that

$$\begin{aligned}
&(X\lrcorner\lambda)\lrcorner\mathrm{d}\mu - (X\lrcorner\mu)\lrcorner\mathrm{d}\lambda + \mathrm{d}((X\lrcorner\lambda)\lrcorner\mu) && \text{(F.31)}\\
&= 3(X\lrcorner\lambda)^a\partial_{[a}\mu_{bc]} - 3(X\lrcorner\mu)^a\partial_{[a}\lambda_{bc]} + 2\partial_{[b|}((X\lrcorner\lambda)^a\mu_{a|c]} \\
&\quad - ((\mathrm{d}_\Delta X)\lrcorner\lambda)\cdot\mu && \text{(F.32)}\\
&\overset{!}{=} \zeta + r\cdot\zeta = \zeta && \text{(F.33)}
\end{aligned}$$

Note that we can consider $(\mathrm{d}_\Delta X)\lrcorner\lambda$ as an adjoint element and hence it has a natural action on μ. Also, the final equality holds to linear order in R since $\zeta \sim O(R)$ in this case. This case holds because $\mathrm{d}_\Delta X = 0$.

Now we just need to consider the 5-form pieces and show that they vanish. That is, we need

$$\begin{aligned}
\left(\mathrm{e}^{-\theta-\tau}\cdot L^{F_{\mathbb{C}}}_{V'}W'\right)|_{\wedge^5 T^*} &= -\mu\wedge\mathrm{d}\lambda - (r\cdot\mu)\wedge\mathrm{d}\lambda - \mu\wedge\mathrm{d}(r\cdot\lambda) \\
&\quad + \mu\wedge(v\lrcorner(F_{\mathbb{C}}+\mathrm{d}\theta)) + (r\cdot\mu)\wedge(v\lrcorner F_{\mathbb{C}}) \\
&\quad + \mu\wedge((r\cdot v + X\lrcorner\lambda)\lrcorner F_{\mathbb{C}}) && \text{(F.34)}\\
&\overset{!}{=} 0 && \text{(F.35)}
\end{aligned}$$

Let us first set $v = 0$. Then we have

$$\begin{aligned}
&-\mu\wedge\mathrm{d}\lambda - (r\cdot\mu)\wedge\mathrm{d}\lambda - \mu\wedge\mathrm{d}(r\cdot\lambda) + \mu\wedge((X\lrcorner\lambda)\lrcorner F_{\mathbb{C}}) && \text{(F.36)}\\
&= -(r\cdot\mu)\wedge\mathrm{d}_\Delta\lambda - \mu\wedge\mathrm{d}_\Delta(r\cdot\lambda) + \mu\wedge((jX\lrcorner j^2F_{\mathbb{C}})\cdot\lambda) && \text{(F.37)}\\
&= -(r\cdot\mu)\wedge\mathrm{d}_\Delta\lambda - \mu\wedge((\mathrm{d}_\Delta r)\cdot\lambda) - \mu\wedge r\cdot\mathrm{d}_\Delta\lambda + \mu\wedge((jX\lrcorner j^2F_{\mathbb{C}})\cdot\lambda) && \text{(F.38)}\\
&= -r\cdot(\mu\wedge\mathrm{d}_\Delta\lambda) - \mu\wedge((\mathrm{d}_\Delta r - jX\lrcorner j^2F_{\mathbb{C}})\cdot\lambda) && \text{(F.39)}\\
&= 0 && \text{(F.40)}
\end{aligned}$$

This holds because any term $\sim \mu\wedge\mathrm{d}\lambda = 0$ by virtue of the integrability of Δ, and by the fact that $\mathrm{d}_\Delta r - jX\lrcorner j^2F_{\mathbb{C}} = 0$. Now let's consider instead $\lambda = 0$. We have

$$\begin{aligned}
&\mu\wedge(v\lrcorner(F_{\mathbb{C}}+\mathrm{d}\theta)) + (r\cdot\mu)\wedge(v\lrcorner F_{\mathbb{C}}) + \mu\wedge((r\cdot v)\lrcorner F_{\mathbb{C}}) && \text{(F.41)}\\
&= \mu\wedge v\lrcorner\pi_1(\mathrm{d}\theta) + (r\cdot\mu)\wedge(v\lrcorner F_{\mathbb{C}}) + \mu\wedge((r\cdot v)\lrcorner F_{\mathbb{C}}) && \text{(F.42)}\\
&= \mu\wedge(v\lrcorner(r\cdot F_{\mathbb{C}})) + (r\cdot\mu)\wedge(v\lrcorner F_{\mathbb{C}}) + \mu\wedge((r\cdot v)\lrcorner F_{\mathbb{C}}) && \text{(F.43)}\\
&= (r\cdot\mu)\wedge(v\lrcorner F_{\mathbb{C}}) + \mu\wedge r\cdot(v\lrcorner F_{\mathbb{C}}) && \text{(F.44)}\\
&= r\cdot(\mu\wedge(v\lrcorner F_{\mathbb{C}})) && \text{(F.45)}\\
&= 0 && \text{(F.46)}
\end{aligned}$$

This vanishes because $\mu\wedge(v\lrcorner F_{\mathbb{C}}) = 0$ by restrictions on $F_{\mathbb{C}}$ imposed by integrability of L_1.

Hence, we have found the integrability conditions for the deformations, and they are given by

$$0 = \mathrm{d}_\Delta X \qquad \text{(F.47)}$$

$$0 = \mathrm{d}_\Delta r - jX \lrcorner j^2 F_{\mathbb{C}} \tag{F.48}$$

$$0 = \pi_1(\mathrm{d}\theta - r \cdot F_{\mathbb{C}}) \tag{F.49}$$

Now we need to consider the exactness conditions. These are given by

$$\tilde{L}'_1 = (1 + L_V^{F_{\mathbb{C}}})\tilde{L}_1 \tag{F.50}$$

where $V = -v - \omega - \sigma \in \Gamma((T/\mathfrak{F}_0^1) \oplus (\wedge^2 T^*/\mathcal{F}_1^2) \oplus \wedge^5 T^*)$. The minus signs are for convenience. Given $W = w + \mu \in \Gamma(\tilde{L}_1)$, we have

$$\begin{aligned}(1 + L_V^{F_{\mathbb{C}}})W = w &- [v, w] \\ &+ \mu - \mathcal{L}_v \mu + w \lrcorner \mathrm{d}\omega - w \lrcorner (v \lrcorner F_{\mathbb{C}}) \\ &+ w \lrcorner \mathrm{d}\sigma + \mu \wedge \mathrm{d}\omega + w \lrcorner (\omega \wedge F_{\mathbb{C}}) - \mu \wedge (v \lrcorner F_{\mathbb{C}})\end{aligned} \tag{F.51}$$

$$\begin{aligned}&= (w - v^a \partial_a w^\alpha) + (\mathrm{d}_\Delta v) \cdot w \\ &\quad + (\mu - 3v^a \partial_{[a}\mu_{bc]} - 2\partial_{[b|}(v^a \mu_{a|c]}) + (\mathrm{d}_\Delta v) \cdot \mu + w \lrcorner (\mathrm{d}\omega - v \lrcorner F_{\mathbb{C}}) \\ &\quad + w \lrcorner (\mathrm{d}\sigma + \omega \wedge F_{\mathbb{C}}) + \mu \wedge (\mathrm{d}\omega - v \lrcorner F_{\mathbb{C}})\end{aligned} \tag{F.52}$$

$$= \mathrm{e}^{\pi_1(\mathrm{d}\omega - v \lrcorner F_{\mathbb{C}}) + (\mathrm{d}\sigma + \omega \wedge F_{\mathbb{C}})}(1 + \mathrm{d}_\Delta v) \cdot (\tilde{w} + \tilde{\mu}) \tag{F.53}$$

where $\tilde{W} = \tilde{w} + \tilde{\mu} \in \Gamma(\tilde{L}_1)$, and we have introduced the projection $\pi_1 : \wedge^n T^* \to \wedge^n T^*/\mathcal{F}_1^n$ for definiteness. Hence, the exactness conditions are given by

$$r = \mathrm{d}_\Delta v \tag{F.54}$$

$$\theta = \pi_1(\mathrm{d}\omega - v \lrcorner F_{\mathbb{C}}) \tag{F.55}$$

$$\tau = \mathrm{d}\sigma + \omega \wedge F_{\mathbb{C}} \tag{F.56}$$

This reproduces the results at the end of section 4.3.3.

If we assume the flux is trivial, which is not the case for any AdS solution, then we can take the complex twist $\mathrm{e}^{\alpha+\beta}$ to be globally well-defined. In this case, then it is possible to show using the following deformation parameter

$$R = \mathrm{e}^{\alpha+\beta}(r + jX \lrcorner j\alpha + X + \theta + r \cdot \hat{\alpha} - \tfrac{1}{2} jX \lrcorner j\alpha \cdot \hat{\alpha} + \mu)\mathrm{e}^{-\alpha-\beta} \tag{F.57}$$

that the cohomology defined by the flux-twisted derivatives above is isomorphic to

$$H^0_\Delta(M, \wedge^3 T/\mathfrak{F}_2^3) \oplus H^1_\Delta(M, T/\mathfrak{F}_0^1) \oplus H^3_{\mathcal{F}_1}(M) \oplus H^6_{\mathrm{d}}(M) \tag{F.58}$$

We have had to assume something slightly stronger about the integrability conditions to show this. Namely that there exists some $\hat{\alpha} \in \Gamma(\mathcal{F}_1^3)$ such that

$$\mathrm{d}\alpha = \mathrm{d}\hat{\alpha} \tag{F.59}$$

This is not directly implied by the involutivity conditions (which just states that $\mathrm{d}\alpha \in \Gamma(\mathcal{F}_1^4)$)

but it may be implied by the vanishing of the moment map.

F.2 Generalised $\partial\bar{\partial}$ Moduli

Following the notation from section 4.3.4, the moduli of generic $\mathrm{SU}^*(6)$ structures is counted by the cohomology of the following complex.

$$\begin{array}{ccccc} \Gamma(\wedge^2\mathfrak{X}^*) & \xrightarrow{D_{-1}} & \Gamma(\wedge^3\mathfrak{X}^*_{-1}) & \xrightarrow{D_{-1}} & \Gamma(\wedge^4\mathfrak{X}^*_{-2}) \\ & \nearrow^{D_0} & & \nearrow^{D_0} & \\ \Gamma(\wedge^5\mathfrak{X}^*_{-1}) & \xrightarrow{D_{-1}} & \Gamma(\wedge^6\mathfrak{X}^*_{-2}) & & \end{array} \tag{F.60}$$

Here D_0 and D_{-1} are operators coming from any torsion free $\mathrm{USp}(6)$ connection[3] decomposed into $\mathrm{SU}^*(6)$ representations.

$$D = D_1 + D_0 + D_{-1} \tag{F.61}$$

In this section, we give give the cohomology of the complex above in terms of the cohomology of D_{-1} provided the background satisfies the generalised $\partial\bar{\partial}$-lemma.

Definition 30. D_0, D_{-1} are said to satisfy the *generalised $\partial\bar{\partial}$-lemma* if the satisfy the following

$$\operatorname{im} D_0 \cap \ker D_{-1} \subseteq \operatorname{im} D_{-1}D_0 \tag{F.62}$$

With this we can prove the following result.

Proposition 6. If a background satisfies the generalised $\partial\bar{\partial}$-lemma, and D_0 defines a chain homomorphism $D_0 : \Gamma(\wedge^\bullet\mathfrak{X}^*_\bullet) \to \Gamma(\wedge^{\bullet-2}\mathfrak{X}^*_\bullet)$, then the cohomology of the complex (F.60) is given by

$$H^3_{D_{-1}} \oplus H^6_{D_{-1}} \tag{F.63}$$

where $H^p_{D_{-1}}$ is the p^{th} cohomology of the differential D_{-1}

Proof. The cohomology of the complex (F.60) is given by[4]

$$\mathcal{H} = \frac{\{A + B \in \Gamma(\wedge^3\mathfrak{X}^*_{-1} \oplus \wedge^6\mathfrak{X}^*_{-2}) \,|\, D_{-1}A + D_0B = 0\}}{\{A = D_{-1}C + D_0E,\, B = \frac{1}{2}D_{-1}E \,|\, C \in \Gamma(\wedge^2\mathfrak{X}^*_0),\, E \in \Gamma(\wedge^5\mathfrak{X}^*_{-1})\}} \tag{F.64}$$

Let us define a new quotient group by

$$\mathcal{K} = \frac{\{B \in \Gamma(\wedge^6\mathfrak{X}^*_{-2}) \,|\, D_0B = 0\}}{\{B = D_{-1}E \,|\, E \in \Gamma(\wedge^5\mathfrak{X}^*_{-1}),\, D_0E = 0\}} \tag{F.65}$$

and two maps

$$\begin{array}{rclcrcl} \theta : H^3_{D_-} \oplus \mathcal{K} & \longrightarrow & \mathcal{H} & \qquad & \psi : \mathcal{H} & \longrightarrow & H^3_{D_-} \oplus \mathcal{K} \\ {[A]_3 + [B]_{\mathcal{K}}} & \longmapsto & [A+B]_{\mathcal{H}} & & [A+B]_{\mathcal{H}} & \longmapsto & [\tilde{A}]_3 + [\tilde{B}]_{\mathcal{K}} \end{array} \tag{F.66}$$

[3]We will always assume that we are deforming around a full supergravity background

[4]The factor of $\frac{1}{2}$ in the quotient is due to the precise form of the projection $D \times_{\mathrm{ad}} V$ in $\mathrm{SU}^*(6)$ indices.

where $A, \tilde{A} \in \Gamma(\wedge^3 \mathfrak{X}^*_{-1})$, $B, \tilde{B} \in \Gamma(\wedge^6 \mathfrak{X}^6_{-2})$ and where the subscript denotes the cohomology group that class is a member of. $\tilde{A}, \tilde{B}$ are defined from $[A+B]_{\mathcal{H}}$ in the following way. We have

$$0 = D_- A + D_0 B \tag{F.67}$$

$$\Rightarrow \quad 0 = D_- D_0 B \tag{F.68}$$

So, using the generalised $\partial\bar{\partial}$-lemma we can write $D_0 B = D_- D_0 E$ for some $E \in \Gamma(\wedge^5 \mathfrak{X}^*_{-1})$. We then define

$$\tilde{A} = A + D_0 E \qquad \tilde{B} = B + \tfrac{1}{2} D_- E \tag{F.69}$$

We need to check that these do define elements of $H^3_{D_-}$ and $\mathcal{K}$ respectively, and if the map ψ is well defined. Firstly, we note that

$$\begin{aligned} D_- \tilde{A} &= D_- A + D_- D_0 E \\ &= D_- A + D_0 B \\ &= 0 \end{aligned} \qquad \begin{aligned} D_0 \tilde{B} &= D_0 B + \tfrac{1}{2} D_0 D_- E \\ &= D_0 B - D_- D_0 E \\ &= D_0 B - D_0 B \\ &= 0 \end{aligned} \tag{F.70}$$

This shows that $[\tilde{A}]_3 \in H^3_{D_-}$ and $[\tilde{B}]_{\mathcal{K}} \in \mathcal{K}$. Note here we have used the fact that, when evaluated on $\Gamma(\wedge^5 \mathfrak{X}^*_-)$

$$D_- D_0 + \tfrac{1}{2} D_0 D_- = 0 \tag{F.71}$$

which follows from the complex (F.60). The factor of $\frac{1}{2}$ comes from the way the Dorfman derivative acts. Now suppose that $[A = B]_{\mathcal{H}} = [A' + B']_{\mathcal{H}}$. Then, there exists $c \in \Gamma(\wedge^2 \mathfrak{X}^*_0)$, $e \in \Gamma(\wedge^5 \mathfrak{X}^*_-)$ such that

$$A' = D_- c + D_0 e \qquad B' = \tfrac{1}{2} D_- e \tag{F.72}$$

From these, we define E' such that $D_0 B' = D_- D_0 E'$. It is a simple check to see that we can choose $E' = E - e$. Then we have

$$\begin{aligned} \tilde{A}' &= A' + D_0 E' \\ &= A + D_- c + D_0 e + D_0 (E - e) \\ &= A + D_0 E + D_- c \\ &= \tilde{A} + D_- c \end{aligned} \qquad \begin{aligned} \tilde{B}' &= B' + \tfrac{1}{2} D_- E' \\ &= B + \tfrac{1}{2} D_- e + \tfrac{1}{2} D_- (E - e) \\ &= B + \tfrac{1}{2} D_- E \\ &= \tilde{B} \end{aligned} \tag{F.73}$$

Hence we see that

$$[A+B]_{\mathcal{H}} = [A' + B']_{\mathcal{H}} \qquad \Rightarrow \qquad [\tilde{A}]_3 = [\tilde{A}']_3 \quad [\tilde{B}]_{\mathcal{K}} = [\tilde{B}']_{\mathcal{K}} \tag{F.74}$$

Finally, since E as defined above is not unique, we need to check that the map does not depend on the choice. Indeed, suppose

$$D_0 B = D_- D_0 E = D_- D_0 E' \qquad \Rightarrow \qquad D_- D_0 (E - E') = 0 \tag{F.75}$$

Using the generalised $\partial\bar{\partial}$-lemma again, we can write $D_0(E' - E) = D_- D_0 F$ for some $F \in$

$\Gamma(\wedge^4 \mathfrak{X}_0^*)$. Then we have

$$\begin{aligned} \tilde{A}' &= A + D_0 E' \\ &= A + D_0 E + D_0(E' - E) \\ &= A + D_0 E + D_- D_0 F \\ &= \tilde{A} + D_- c \end{aligned} \qquad \begin{aligned} \tilde{B}' &= B + \tfrac{1}{2} D_- E' \\ &= B + \tfrac{1}{2} D_- E + \tfrac{1}{2} D_-(E - E') \\ &= \tilde{B} + D_- e \end{aligned} \tag{F.76}$$

where $c = D_0 F \in \Gamma(\wedge^2 \mathcal{X}_0^*)$, and $e = \frac{1}{2}(E - E' + D_- F) \in \Gamma(\wedge^5 \mathfrak{X}_{-1}^*)$ is such that $D_0 e = 0$. Hence we have

$$D_- D_0 E = D_- D_0 E' \qquad \Rightarrow \qquad [\tilde{A}']_3 = [\tilde{A}]_3 \quad [\tilde{B}']_{\mathcal{K}} = [\tilde{B}]_{\mathcal{K}} \tag{F.77}$$

Hence, the map ψ is well defined. It is a simple check to see that θ is also well defined.

Now we show that ψ, θ are inverses of each other. Firstly,

$$\theta \circ \psi([A + B]_{\mathcal{H}}) = \theta([\tilde{A}]_3 + [\tilde{B}]_{\mathcal{K}}) \tag{F.78}$$
$$= [\tilde{A} + \tilde{B}]_{\mathcal{H}} \tag{F.79}$$
$$= [A + D_0 E + B + \tfrac{1}{2} D_- E]_{\mathcal{H}} \tag{F.80}$$
$$= [A + B] \tag{F.81}$$

Therefore, $\theta \circ \psi = \mathbb{I}_{\mathcal{H}}$. Next consider,

$$\psi \circ \theta([A]_3 + [B]_{\mathcal{K}}) = \psi([A + B]_{\mathcal{H}}) \tag{F.82}$$
$$= [\tilde{A}]_3 + [\tilde{B}]_{\mathcal{K}} \tag{F.83}$$
$$= [A + D_0 E]_3 + [B + \tfrac{1}{2} D_- E] \tag{F.84}$$

But since $D_0 B = 0$ by assumption, we can choose $E = 0$. Hence,

$$\psi \circ \theta([A]_3 + [B]_{\mathcal{K}}) = [A]_3 + [B]_{\mathcal{K}} \tag{F.85}$$

So $\psi \circ \theta = \mathbb{I}_{H^3 \oplus \mathcal{K}}$ and hence $\psi = \theta^{-1}$. Clearly θ and ψ are homomorphisms. Hence,

$$\mathcal{H} \cong H^3_{D_-} \oplus \mathcal{K} \tag{F.86}$$

Now we want to show that $\mathcal{K} \cong H^6_{D_-}$. Again, let's define some maps

$$\begin{aligned} \eta : \mathcal{K} &\longrightarrow H^6_{D_-} \\ [B]_{\mathcal{K}} &\longmapsto [B]_6 \end{aligned} \qquad \begin{aligned} \zeta : H^6_{D_-} &\longrightarrow \mathcal{K} \\ [B]_6 &\longmapsto [\tilde{B}]_{\mathcal{K}} \end{aligned} \tag{F.87}$$

where $\tilde{B}$ is defined by the following. For any $B \in \Gamma(\wedge^6 \mathfrak{X}_{-2}^*)$ we have $D_- B = 0$. But we also assume that D_0 is a chain homomorphism, meaning that $D_-(D_0 B) = 0$. Hence, using the generalised $\partial\bar{\partial}$-lemma, we can define an E such that

$$D_0 B = D_- D_0 E \tag{F.88}$$

We therefore define $\tilde{B}$ as

$$\tilde{B} = B + \tfrac{1}{2}D_{-}E \qquad \Rightarrow \qquad D_0\tilde{B} = D_0 B + \tfrac{1}{2}D_0 D_{-}E = 0 \tag{F.89}$$

A similar proof as above shows that these maps are well defined and are inverses of each other. Hence we have

$$\mathcal{H} \cong H^3_{D_-} \oplus \mathcal{K} \cong H^3_{D_-} \oplus H^6_{D_-} \tag{F.90}$$

□

F.3 Calabi-Yau Moduli

Here we will show that the Calabi-Yau satisfies the generalised $\partial\bar{\partial}$-lemma and hence we can calculate its moduli using the formula above. The proof involves using a compact Calabi-Yau but the result holds more generally as one can calculate the moduli using a type 0 presentation of the exceptional complex structure instead.

The ECS for a Calabi-Yau is

$$\tilde{J} = \frac{1}{2}\Big(I - \mathrm{vol} - \mathrm{vol}^{\#}\Big) \qquad L_1 = \mathrm{e}^{\mathrm{i}\,\mathrm{vol}} \cdot (T^{1,0} \oplus \wedge^{0,2}T^*) \tag{F.91}$$

Using the adjoint action of $\tilde{J}$, we can decompose $E_{\mathbb{C}}$ and $\mathrm{ad}\,\tilde{F}_{\mathbb{C}}$ into eigenbundles

$$E\_\mathbb{C} = L_1 \oplus L_0 \oplus L_{-1} \qquad \mathrm{ad}\,\tilde{F}_{\mathbb{C}} = \mathrm{ad}\,P_{\mathbb{R}^+\times \mathrm{U}^*(6)} \oplus S_1 \oplus S_{-1} \oplus S_2 \oplus S_{-2} \tag{F.92}$$

The eigenbundles needed for the deformation problem laid out in the previous section are given explicitly by

$$\wedge^5\mathfrak{X}^*_{-1} = L_{-} = \left\{ \begin{array}{c|c} \bar{w} - \mathrm{i}\bar{w}\lrcorner\,\mathrm{vol} & \bar{w} \in T^{0,1} \\ \omega & \omega \in \wedge^{2,0}T^* \end{array} \right\} \tag{F.93}$$

$$\wedge^2\mathfrak{X}^*_{0} = L_0 = \left\{ \begin{array}{c|c} v - \mathrm{i}v\lrcorner\,\mathrm{vol} & v \in T^{1,0} \\ \bar{v} + \mathrm{i}\bar{v}\lrcorner\,\mathrm{vol} & \bar{v} \in T^{0,1} \\ \theta & \theta \in \wedge^{1,1}T^* \end{array} \right\} \tag{F.94}$$

$$\wedge^3\mathfrak{X}^*_{-1} = S_{-1} = \left\{ \begin{array}{c|c} \alpha(\frac{2}{3} + \frac{1}{3}\mathbb{I} + \mathrm{i}\,\mathrm{vol} - \mathrm{i}\,\mathrm{vol}^{\#}) & \alpha \in \mathbb{C} \\ r & r \in T^{0,1}\otimes T^{*1,0} \\ \beta + \mathrm{i}\,\mathrm{vol}^{\#}\lrcorner\beta & \beta \in \wedge^{2,1}T^* \\ \gamma - \mathrm{i}\,\mathrm{vol}^{\#}\lrcorner\gamma & \gamma \in \wedge^{3,0}T^* \end{array} \right\} \tag{F.95}$$

$$\wedge^6\mathfrak{X}^*_{-2} = S_{-2} = \left\{ \begin{array}{c|c} \lambda + \mathrm{i}\,\mathrm{vol}^{\#}\lrcorner\lambda & \lambda \in \wedge^{3,0}T^* \end{array} \right\} \tag{F.96}$$

Using the holomorphic 3-form Ω of the Calabi-Yau, we can define a chain isomorphism $\mathfrak{X}^* \simeq T^*$.

Indeed, we have

$$\wedge^5\mathfrak{X}^*_{-1} \to \left\{ \begin{array}{c} \bar{w}\lrcorner\,\mathrm{vol} \in \wedge^{3,2}T^* \\ \omega\wedge\bar{\Omega} \in \wedge^{2,3}T^* \end{array} \right\} \sim \wedge^5 T^* \tag{F.97}$$

$$\wedge^2\mathfrak{X}^*_{0} \to \left\{ \begin{array}{c} v\lrcorner\Omega \in \wedge^{2,0}T^* \\ \bar{v}\lrcorner\Omega \in \wedge^{0,2}T^* \\ \theta \in \wedge^{1,1}T^* \end{array} \right\} \sim \wedge^2 T^* \tag{F.98}$$

$$\wedge^3\mathfrak{X}^*_{-1} \to \left\{ \begin{array}{c} \alpha\Omega \in \wedge^{3,0}T^* \\ r\cdot\bar{\Omega} \in \wedge^{1,2}T^* \\ \beta \in \wedge^{2,1}T^* \\ (\bar{\Omega}^{\#}\lrcorner\gamma)\bar{\Omega} \in \wedge^{0,3}T^* \end{array} \right\} \sim \wedge^3 T^* \tag{F.99}$$

$$\wedge^6\mathfrak{X}^*_{-2} \to \left\{ \lambda\wedge\bar{\Omega} \in \wedge^{3,3}T^* \right\} \sim \wedge^6 T^* \tag{F.100}$$

We can also take the torsion free compatible connection ∇, and lift it to a generalised connection D as in [181]. With this lift, and with the isomorphism above we find

$$D_- \to \partial \qquad D_0 \to \Omega^{\#}\lrcorner\bar{\partial} + \bar{\Omega}^{\#}\lrcorner\partial \tag{F.101}$$

where here $\partial, \bar{\partial}$ denote the projection of ∇ onto the $T^{*1,0}, T^{*0,1}$ piece respectively.

We need to show that these operators satisfy the generalised $\partial\bar{\partial}$-lemma. We just need to show this for elements in $\wedge^5 T*$ and $\wedge^6 T^*$ for the proof to hold.

Proof. First take $\alpha \in \wedge^{2,3}T^*$. Then we have

$$D_0\alpha = (\Omega^{\#}\lrcorner\bar{\partial})\lrcorner\alpha + (\bar{\Omega}^{\#}\lrcorner\partial)\alpha \tag{F.102}$$

We can consider only the second term which is just $\bar{\Omega}^{\#}\lrcorner(\partial\alpha)$. Suppose further that $D_0\alpha \in \ker D_- \sim \ker\partial$. Then each term individually has to be in $\ker\partial$. Since $h^{0,3} = 1$, we must have that, up to ∂ exact terms

$$\bar{\Omega}^{\#}\lrcorner(\partial\alpha) = c\bar{\Omega} \qquad \Rightarrow \qquad \partial\alpha = \tilde{c}\,\mathrm{vol} \tag{F.103}$$

for some constants $c, \tilde{c}$. However, vol is not ∂-exact and hence we must have $c = \tilde{c} = 0$. Therefore, $\partial\alpha = 0$ and so $[\alpha] \in H^{2,3}_{\partial} = 0$. Therefore, α is ∂-exact and so $D_0\alpha = D_0 D_- a \sim D_- D_0 a$ for some $a \in \wedge^4 T^*$.

Now take $\beta \in \wedge^{3,2}T^*$. Here we automatically have $\partial\beta = 0$ and so $[\beta] \in H^{3,2}_{\partial} = 0$. Therefore β is ∂-exact and so $D_0\beta = D_0 D_- b \sim D_- D_0 b$ for some $b \in \wedge^4 T^*$.

Finally, we take $\gamma \in \wedge^{3,3}T^*$ and write this as $\gamma = c\,\mathrm{vol} + \partial\psi$ for some constant c and some $\psi \in \wedge^{2,3}T^*$. For any constant, we have $D_0(c\,\mathrm{vol}) = 0$ since D_0 is built from the compatible connection ∇. Therefore, we have

$$D_0\gamma = D_0\partial\psi = D_0 D_-\psi \sim D_- D_0\psi \tag{F.104}$$

This gives the result. □

Using the results of the previous section on the moduli of a background satisfying the generalised $\partial\bar{\partial}$-lemma, we see that the moduli of the Calabi-Yau are given by

$$\mathcal{H} = H^3_{D_-} \oplus H^6_{D_-} \cong H^3_{\partial} \oplus H^6_{\partial} \tag{F.105}$$

Note that, since $H^p_{\partial} \cong H^p_{\mathrm{d}}$ for a Calabi-Yau manifold, this agrees with the result obtained for the moduli of the Calabi-Yau calculated through a type 0 ECS, as discussed in section 4.3.5.

Appendix G

Detailed calculation of GMPT moduli

We have the parameterisation

$$E_{\mathbb{C}}/L_3 = \mathrm{e}^{\Sigma}\big([L_{-1}^{\mathcal{J}_-} \oplus \bar{\mathcal{U}}_{\mathcal{J}_-}] \oplus S_1 \oplus S_{-1} \oplus (\wedge^6 T^* \otimes [L_1^{\mathcal{J}_-} \oplus L_{-1}^{\mathcal{J}_-}])\big), \tag{G.1}$$

$$\mathfrak{Q}_{\mathbb{R}^+\times \mathrm{U}(7)} = \mathrm{e}^{\Sigma}\big(\wedge^2(L_1^{\mathcal{J}_-})^* \oplus (S_0 \oplus S_{-2}) \oplus r^i \wedge^6 T^*\big)\mathrm{e}^{-\Sigma}, \tag{G.2}$$

where we have used $\wedge^5 T^* \simeq \wedge^6 T^* \otimes T$. We take $\chi = \chi_0 + \chi_{-2} \in \Gamma(S_0 \oplus S_{-2})$, $\varepsilon \in \Gamma(\wedge^2(L_1^{\mathcal{J}_-})^*)$ and $\Theta \in \Gamma(\wedge^6 T^*)$ and consider the following generic deformation

$$L_3 = \mathrm{e}^{\Sigma}[L_1^{\mathcal{J}_-} \oplus \mathcal{U}_{\mathcal{J}_-}] \quad \to \quad L_3' = \mathrm{e}^{\Sigma + \not{\varepsilon}\mu + \chi + r^i(\Theta + \frac{1}{2}(\Sigma, \not{\varepsilon}\mu - \chi))}[L_1^{\mathcal{J}_-^{\varepsilon}} \oplus \mathcal{U}_{\mathcal{J}_-^{\varepsilon}}], \tag{G.3}$$

where

$$\Sigma = C + 8\,\mathrm{i}\,\mathrm{e}^{-3A}\,\mathrm{im}\,\Phi_{\mp}, \qquad L_1^{\mathcal{J}_-^{\varepsilon}} = (1+\varepsilon)L_1^{\mathcal{J}_-}, \qquad \mathcal{U}_{\mathcal{J}_-^{\varepsilon}} = (1+\not{\varepsilon})\mathcal{U}_{\mathcal{J}_-} \tag{G.4}$$

The latter two define a deformed generalised complex structure $\mathcal{J}_-^{\varepsilon}$. Note that $\Phi_-^{\varepsilon} = (1+\not{\varepsilon})\Phi_-$ is indeed the pure spinor associated to $L_1^{\mathcal{J}_-^{\varepsilon}}$. We define $\mu \in \Gamma(S_2)$ in the following manner. Firstly, in what follows we will make the same simplification as in [203] and assumed that the the generalised complex structure $\mathcal{J}_-$ satisfies the $\mathrm{d}\mathrm{d}^{\mathcal{J}_-}$-lemma [280]. For us this will mean that

$$\mathrm{im}\,\partial \cap \ker\bar{\partial} = \mathrm{im}\,\bar{\partial} \cap \ker\partial = \mathrm{im}\,\bar{\partial}\partial, \tag{G.5}$$

where ∂ and $\bar{\partial}$ are the generalised Dolbeault operators of $\mathcal{J}_-$. With this we have

$$\begin{aligned} (\mathrm{d}\Sigma)_{-1} = 0 \quad &\Rightarrow & \bar{\partial}\Sigma_0 &= -\partial\Sigma_{-2} \\ &\Rightarrow & \partial\bar{\partial}\Sigma_0 &= 0 \\ &\Rightarrow & \partial\Sigma_0 &= \bar{\partial}\partial\alpha_1 \end{aligned} \tag{G.6}$$

for some $\alpha_1 \in \Gamma(S_1)$. We then define

$$\mu = \Sigma_2 + \partial\alpha_1. \tag{G.7}$$

Note that $\partial\alpha_1$ is not uniquely defined – the ambiguity is some element of $\Gamma(S_2)$ that is closed under $\bar{\partial}$. As we will see, this ambiguity can be absorbed in the definition of χ. For definiteness, one can see that the deformation (G.3) to linear order is given by

$$\mathrm{e}^{\Sigma}[\varepsilon + (\not\mu + \chi) + r^i(\Theta - (\Sigma, \chi))]\mathrm{e}^{-\Sigma} \in \Gamma(\mathfrak{Q}_{\mathbb{R}^+ \times \mathrm{U}(7)}). \tag{G.8}$$

It is important to note that this is a globally well defined section of $\mathfrak{Q}_{\mathbb{R}^+\times 7}$ because F is in a trivial cohomology class. This is guaranteed by the generalised $\partial\bar{\partial}$-lemma and means that the gauge potential C is a global polyform.

We now calculate the conditions for integrability of L_3'. Following the results of section 5.2.2, we find that we have integrability only if

$$[\![L_1^{\mathcal{J}_-^\varepsilon}, L_1^{\mathcal{J}_-^\varepsilon}]\!]_{\mathrm{O}(6,6)} \subseteq L_1^{\mathcal{J}_-^\varepsilon} \tag{G.9}$$

From [165] this implies

$$\mathrm{d}_L \varepsilon = 0, \tag{G.10}$$

where $\mathrm{d}_L\colon \Gamma(\wedge^p(L_1^{\mathcal{J}_-})^*) \to \Gamma(\wedge^{p+1}(L_1^{\mathcal{J}_-})^*)$ is the differential associated to the Lie algebroid structure $L_1^{\mathcal{J}_-}$. This means that $\not\varepsilon$ and $\bar{\partial}$ commute as operators on S:

$$\bar{\partial}\not\varepsilon = \not\varepsilon\bar{\partial}. \tag{G.11}$$

Letting S_n, S_n^ε be the eigenspaces of S with respect to $\mathcal{J}_-, \mathcal{J}_-^\varepsilon$ respectively, we further require

$$[\mathrm{d}(\Sigma + \not\varepsilon\mu + \chi + r^i\Theta)]_{S_{-1}^\varepsilon} = [\mathrm{d}(\Sigma + \not\varepsilon\mu + \chi + r^i\Theta)]_{S_{-3}^\varepsilon} = 0, \tag{G.12}$$

where the notation above means the projection of the polyform onto S_{-1}^ε and S_{-3}^ε respectively. We will still use subscript indices to denote projection onto S_n. Working to linear order in the deformation parameters and using the integrability of L_3, we find

$$\begin{aligned} 0 &= (1 + \not\varepsilon + \not\bar{\varepsilon})[\mathrm{d}(\Sigma + \not\varepsilon\mu + \chi + r^i\Theta)]_{-1} - \not\varepsilon[\mathrm{d}\Sigma]_1 - \not\bar{\varepsilon}[\mathrm{d}\Sigma]_{-3} \\ &= [\mathrm{d}\not\varepsilon\mu]_{-1} + [\mathrm{d}\chi]_{-1} - \not\varepsilon[\mathrm{d}\Sigma]_1 \\ &= \bar{\partial}\not\varepsilon\mu_2 + \bar{\partial}\chi_0 + \partial\chi_{-2} - \not\varepsilon\bar{\partial}\Sigma_2 - \not\varepsilon\partial\Sigma_0 \\ &= \not\varepsilon\bar{\partial}\Sigma_2 + \not\varepsilon\bar{\partial}\partial\alpha_1 - \not\varepsilon\bar{\partial}\Sigma_2 - \not\varepsilon\partial\Sigma_0 + \bar{\partial}\chi_0 + \partial\chi_{-2} \\ &= \not\varepsilon\partial\Sigma_0 - \not\varepsilon\partial\Sigma_0 + \bar{\partial}\chi_0 + \partial\chi_{-2} \\ &= \bar{\partial}\chi_0 + \partial\chi_{-2}. \end{aligned} \tag{G.13}$$

We also have

$$\begin{aligned} 0 &= (1 + \not\varepsilon + \not\bar{\varepsilon})[\mathrm{d}(\Sigma + \varepsilon\mu + \chi + r^i\Theta)]_{-3} - \not\bar{\varepsilon}[\mathrm{d}\Sigma]_{-1} \\ &= [\mathrm{d}\chi]_{-3} \\ &= \bar{\partial}\chi_{-2}. \end{aligned} \tag{G.14}$$

Taken together, we see that the integrability conditions are

$$\mathrm{d}_L\varepsilon = 0, \qquad \bar{\partial}\chi_0 + \partial\chi_{-2} = 0, \qquad \bar{\partial}\chi_{-2} = 0. \tag{G.15}$$

We can simplify this further. Using the $\mathrm{dd}^{\mathcal{J}_-}$-lemma we see that we can write $\partial\chi_{-2} = \bar{\partial}\partial\eta_{-1}$ for some $\eta_{-1} \in \Gamma(S_{-1})$. Then, defining $\tilde{\chi}_0 = \chi_0 + \partial\eta_{-1}$, we see that the integrability conditions become

$$\mathrm{d}_L\varepsilon = 0, \qquad \bar{\partial}\tilde{\chi}_0 = 0, \qquad \bar{\partial}\chi_{-2} = 0. \tag{G.16}$$

Note again that $\partial\eta_{-1}$ is only defined up to a term that is $\bar{\partial}$-exact. We will see shortly that these terms correspond to trivial deformations.

To find the form of trivial deformations we take

$$V = \mathrm{e}^{\Sigma}(W + c\,\Phi_- + U + \nu + r^i\sigma + \tau), \tag{G.17}$$

where $W \in \Gamma(L_1^{\mathcal{J}_-})$, $U \in \Gamma(L_{-1}^{\mathcal{J}_-})$, $c \in C^\infty(M)$, $\nu = \nu_1 + \nu_{-1} + \nu_{-3} \in \Gamma(S_1 \oplus S_{-1} \oplus S_{-3})$, $\sigma \in \Gamma(\wedge^5 T^*)$ and $\tau \in \Gamma(T^* \otimes \wedge^7 T^*)$. Then we consider

$$L_3' = (1 + L_V)L_3. \tag{G.18}$$

After a lengthy calculation we find that to linear order in V this deformation is given by

$$\mathrm{e}^{\Sigma}[\mathrm{d}_L U + (\mathrm{d}_L U)\mu + (\mathrm{d}\nu)_0 + (\mathrm{d}\nu)_{-2} + r^i(\mathrm{d}\tilde{\sigma} - (\Sigma, (\mathrm{d}\nu)_0 + (\mathrm{d}\nu)_{-2}))]\mathrm{e}^{-\Sigma}. \tag{G.19}$$

which is a section of $\Gamma(\mathfrak{Q}_{\mathbb{R}^+\times\mathrm{U}(7)})$. Here $\tilde{\sigma}$ is a 5-form that depends on σ, ν and U and is of the form $\tilde{\sigma} = \sigma + f(\nu, U)$ where f is some function whose form we do not need. A deformation is trivial if and only if

$$\begin{aligned} \varepsilon &= \mathrm{d}_L U, \\ \chi_0 &= \bar{\partial}\nu_1 + \partial\nu_{-1}, \\ \chi_{-2} &= \bar{\partial}\nu_{-1} + \partial\nu_{-3}, \\ \Theta &= \mathrm{d}\tilde{\sigma}. \end{aligned} \tag{G.20}$$

We can simplify this further using the $\mathrm{dd}^{\mathcal{J}_-}$-lemma. Notice that we can write $\partial\nu_{-3} = \bar{\partial}\partial\eta_{-2}$ for some $\eta_{-2} \in \Gamma(S_{-2})$ and hence χ_{-2} is trivial if $\chi_{-2} = \bar{\partial}(\nu_{-1} + \partial\eta_{-2}) = \bar{\partial}\tilde{\nu}_{-1}$. Moreover, if we calculate $\tilde{\chi}_0$ from these we find that $\tilde{\chi}_0 = \bar{\partial}\tilde{\nu}_1$ for some $\tilde{\nu}_1 \in \Gamma(S_1)$. Hence trivial deformations are given by $\bar{\partial}$-exact $\tilde{\chi}_0$ and χ_{-2}.

All of this shows that the inequivalent deformations are controlled by the following disjoint complex

$$\begin{aligned}
&(L_1^{\mathcal{J}_-})^* \xrightarrow{\quad \mathrm{d}_L \quad} \wedge^2(L_1^{\mathcal{J}_-})^* \xrightarrow{\quad \mathrm{d}_L \quad} \wedge^3(L_1^{\mathcal{J}_-})^* \\
&S_1 \xrightarrow{\qquad \bar{\partial} \qquad} S_0 \xrightarrow{\qquad \bar{\partial} \qquad} S_{-1} \\
&S_{-1} \xrightarrow{\qquad \bar{\partial} \qquad} S_{-2} \xrightarrow{\qquad \bar{\partial} \qquad} S_{-3} \\
&\wedge^5 T^* \xrightarrow{\qquad \mathrm{d} \qquad} \wedge^6 T^*
\end{aligned} \tag{G.21}$$

and so the deformations are counted by the cohomology

$$H^2_{\mathrm{d}_L}(M) \oplus H^0_{\bar{\partial}}(M) \oplus H^{-2}_{\bar{\partial}}(M) \oplus H^6_{\mathrm{d}}(M, \mathbb{C}). \tag{G.22}$$

Appendix H

Determinants of Laplacians

In this appendix, we will review some results about Laplace operators detailed in [3], and [277, 292] for G_2 manifolds.

Throughout chapter 6 we use the ζ-regularisation method. concretely, take a positive operator $\mathcal{D}$ on a real bosonic space V with inner product $\langle\cdot,\cdot\rangle$. Say the spectrum of the operator $\mathcal{D}$ is

$$A = \{a_n \,|\, n = 1, 2, 3, ...\} \qquad 0 < a_1 < a_2 < a_3 < ... \tag{H.1}$$

we define the (ζ regularised) determinant of $\mathcal{D}$ to be the *zeta* regularised product of the spectrum. That is

$$\int D\phi \, \mathrm{e}^{\langle\phi,\mathcal{D}\phi\rangle} := (\det{}'\,\mathcal{D})^{-1/2} \qquad \det{}'\,\mathcal{D} := \mathrm{e}^{\zeta_A'(0)} \tag{H.2}$$

where, for large $\mathrm{re}(s)$, the ζ_A function is defined to be

$$\sum_{n=1}^{\infty} \frac{1}{a_n^s} \tag{H.3}$$

and is defined for arbitrary $s \in \mathbb{C}$ through analytic continuation. ζ_A' denotes the derivative of ζ_A with respect to its complex argument, whereas $\det{}'$ denotes the ζ regularised determinant. If the space V is fermionic then we instead get a factor of $(\det{}'\,\mathcal{D})^{1/2}$. Moreover, if the space is complex then we get an exponent of 1 rather than $1/2$.

We can make a slight generalisation of this following [3]. If $w : V \to W$ is a linear map between two real bosonic vector spaces V, W, each with an inner product denoted by $\langle\cdot,\cdot\rangle$, then we can define the determinant of w via

$$\int D\phi D\psi \, \mathrm{e}^{\langle\phi,w\psi\rangle} = |\det{}'\,w|^{-1/2} \qquad |\det{}'\,w| = \sqrt{\det{}'\,(w^\dagger w)} \tag{H.4}$$

where $w^\dagger$ denotes the adjoint operator under the inner product $\langle\cdot,\cdot\rangle$. We can use these properties to determine identities of Laplacian operators on different manifolds.

H.1 Riemannian Manifolds

The Laplacian of the de Rham operator is defined to be

$$\Delta = \mathrm{d}^\dagger \mathrm{d} + \mathrm{d}\mathrm{d}^\dagger \tag{H.5}$$

We will add a subscript to denote the degree of the form that this operator is acting on. So we write

$$\Delta_p = \mathrm{d}^\dagger_{p+1}\mathrm{d}_p + \mathrm{d}_{p-1}\mathrm{d}^\dagger_p \tag{H.6}$$

Note that, in general, there are 0 modes of d_p, $\mathrm{d}^\dagger_p$, and Δ_p and so we assume they are acting on the quotient spaces $\Omega^p(M)/H^p(M)$[1] so that we can properly define the ζ-regularised determinant. The adjoint is defined with respect to the usual inner product on differential forms

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta \tag{H.7}$$

where $*$ is the Hodge operator defined with respect to some predefined metric g.

Using the Hodge decomposition theorem, one can write the quotient space $\Omega^p(M)/H^p = \mathrm{d}_{p-1}\Omega^{p-1}(M) \oplus \mathrm{d}^\dagger_{p+1}\Omega^{p+1}(M)$. Then the Laplacian decomposes into

$$\Delta_p = {}^*\Delta_p + \Delta^*_p \qquad \text{where} \qquad {}^*\Delta_p = \mathrm{d}_{p-1}\mathrm{d}^\dagger_p \quad \Delta^*_p = \mathrm{d}^\dagger_{p+1}\mathrm{d}_p \tag{H.8}$$

where ${}^*\Delta_p$, Δ^*_p is non-zero only on $\mathrm{d}_{p-1}\Omega^{p-1}(M)$ and $\mathrm{d}^\dagger_{p+1}\Omega^{p+1}(M)$ respectively. From these definitions we see that

$$|\det{}'\,\mathrm{d}_p| = \left(\det{}'({}^*\Delta_p)\right)^{1/2} \tag{H.9}$$

$$\det{}'\,{}^*\Delta_p = \det{}'\,\Delta^*_{p-1} \tag{H.10}$$

$$\det{}'\,\Delta_p = (\det{}'\,{}^*\Delta_p)(\det{}'\,\Delta^*_p) \tag{H.11}$$

$$\det{}'\,\Delta_p = \det{}'\,\Delta_{n-p} \tag{H.12}$$

where the last identity follows from Hodge symmetry induced by $*$, and where n is the dimension of the manifold.

H.2 Calabi-Yau Manifolds

We will focus on Calabi-Yau 3-folds as those will be the ones of interest to us in this thesis. We can define a Laplacian with respect to the two Dolbeault operators $\partial, \bar\partial$ by

$$\Delta_{\bar\partial} = \bar\partial\bar\partial^\dagger + \bar\partial^\dagger\bar\partial \qquad \Delta_\partial = \partial\partial^\dagger + \partial^\dagger\partial \tag{H.13}$$

It turns out that, for a Kähler manifold we have

$$\Delta_{\bar\partial} = \Delta_\partial = \frac{1}{2}\Delta_{\mathrm{d}} \tag{H.14}$$

[1] We may need a slightly different quotient given by $\Omega^p(M)/\Omega^p_{\text{closed}}$ or $\Omega^p(M)/\Omega^p_{\text{coclosed}}$ if we are talking about just d or $\mathrm{d}^\dagger$.

Hence, we will freely refer to each of these as Δ without confusion. Moreover, these map $\Omega^{p,q}(M) \to \Omega^{p,q}(M)$ so we will again denote the restriction of Δ to $\Omega^{p,q}(M)$ by $\Delta_{p,q}$. Again, one actually needs to consider the quotient $\Omega^{p,q}(M)/H^{p,q}$ to properly define det′.

Many of the same identities hold as above. However, since we have two operators $\partial, \bar{\partial}$, each obeying a type of Hodge decomposition theorem, we can refine the Laplacians further. Let us first define some notation.

$$\begin{aligned} \bar{\partial}\Omega^{p,q-1} &= {}^{\bullet}\Omega^{p,q} & \qquad \partial\Omega^{p-1,q} &= \Omega^{p,q\bullet} \\ \bar{\partial}^{\dagger}\Omega^{p,q+1} &= \Omega^{p,q}_{\bullet} & \qquad \partial^{\dagger}\Omega^{p+1,q} &= {}_{\bullet}\Omega^{p,q} \end{aligned} \tag{H.15}$$

We can then further define

$$\begin{aligned} \overset{\bullet}{\Omega} &= {}^{\bullet}\Omega \cap \Omega^{\bullet} & \qquad \underset{\bullet}{\Omega} &= {}_{\bullet}\Omega \cap \Omega_{\bullet} \\ \bullet\Omega &= {}^{\bullet}\Omega \cap {}_{\bullet}\Omega & \qquad \Omega\bullet &= \Omega^{\bullet} \cap \Omega_{\bullet} \end{aligned} \tag{H.16}$$

We then have $\Omega^{p,q}/H^{p,q} = \overset{\bullet}{\Omega} \oplus \bullet\Omega \oplus \underset{\bullet}{\Omega} \oplus \Omega\bullet$ and the Laplacian decomposes as

$$\Delta_{p,q} = \overset{\bullet}{\Delta}_{p,q} + \bullet\Delta_{p,q} + \Delta_{p,q}\bullet + \underset{\bullet}{\Delta}_{p,q} \tag{H.17}$$

$$\Rightarrow \quad \det{}'\Delta_{p,q} = (\det{}'\overset{\bullet}{\Delta}_{p,q})(\det{}' \bullet \Delta_{p,q})(\det{}'\underset{\bullet}{\Delta}_{p,q})(\det{}'\Delta_{p,q}\bullet) \tag{H.18}$$

Following [3], there are many identities between the different values of the determinants of the operators defined above. In fact, there are only 3 independent values that these can take. These are given by

$$A = \det{}'\Delta_{0,0} = \det{}'\Delta_{0,3} = \det{}'\Delta_{3,0} = \det{}'\Delta_{3,3} \tag{H.19}$$

$$\begin{aligned} AB &= \det{}'\Delta_{1,0} = \det{}'\Delta_{2,0} = \det{}'\Delta_{0,1} = \det{}'\Delta_{0,2} \\ &= \det{}'\Delta_{3,1} = \det{}'\Delta_{3,2} = \det{}'\Delta_{1,3} = \det{}'\Delta_{2,3} \end{aligned} \tag{H.20}$$

$$AB^2C = \det{}'\Delta_{1,1} = \det{}'\Delta_{1,2} = \det{}'\Delta_{2,1} = \det{}'\Delta_{2,2} \tag{H.21}$$

One can even express det′ of each of the suboperators defined in (H.17) as one of A, B, C. However, we won't do so here. It follows from using (H.18) and the identities above.

H.3 $\mathbf{G_2}$ Manifolds

On a G_2 manifold, one can decompose the differential forms into irreducible G_2 representations as follows.

$$\wedge^0 T^* = \wedge^0_{\mathbf{1}} T^* \tag{H.22}$$

$$\wedge^1 T^* = \wedge^1_{\mathbf{7}} T^* \tag{H.23}$$

$$\wedge^2 T^* = \wedge^2_{\mathbf{7}} T^* \oplus \wedge^2_{\mathbf{14}} T^* \tag{H.24}$$

$$\wedge^3 T^* = \wedge^3_{\mathbf{1}} T^* \oplus \wedge^3_{\mathbf{7}} T^* \oplus \wedge^3_{\mathbf{27}} T^* \tag{H.25}$$

Here, the bold subscript denotes the G_2 representation, and they are given more explicitly in appendix J. We also have the same decompositions for forms of degree 4,5,6,7 by Hodge symmetry. As is noted in [85], the de Rham Laplacian preserves the G_2 representation meaning we can decompose

$$\Delta_0 = \Delta_{0,\mathbf{1}} \tag{H.26}$$

$$\Delta_1 = \Delta_{1,\mathbf{7}} \tag{H.27}$$

$$\Delta_2 = \Delta_{2,\mathbf{7}} + \Delta_{2,\mathbf{14}} \tag{H.28}$$

$$\Delta_3 = \Delta_{3,\mathbf{1}} + \Delta_{3,\mathbf{7}} + \Delta_{3,\mathbf{27}} \tag{H.29}$$

The precise decomposition is given in a lot of detail in [277, Tables 1,2,3] Moreover, as is highlighted in [292], the determinant of the Laplacians is constant on representations meaning

$$\det{}' \Delta_{0,\mathbf{1}} = \det{}' \Delta_{3,\mathbf{1}} = \det{}' \Delta_{\mathbf{1}} \tag{H.30}$$

$$\det{}' \Delta_{1,\mathbf{7}} = \det{}' \Delta_{2,\mathbf{7}} = \det{}' \Delta_{3,\mathbf{7}} = \det{}' \Delta_{\mathbf{7}} \tag{H.31}$$

$$\det{}' \Delta_{2,\mathbf{14}} = \det{}' \Delta_{\mathbf{14}} \tag{H.32}$$

$$\det{}' \Delta_{3,\mathbf{27}} = \det{}' \Delta_{\mathbf{27}} \tag{H.33}$$

and similarly for the Laplacians on forms of degree 4,5,6,7.

Appendix I

The Metric on the Space of SU*(6) Structures

I.1 Metric on $T_{\tilde{J}}Q_{\mathrm{U}^*(6)\times\mathbb{R}^+}$

We have a Calabi-Yau manifold (M,ρ,ω), where $\rho\in\Omega^3(M)_{\mathbb{R}}$ defines the $\mathrm{SL}(3,\mathbb{C})$ structure and is the real part of the holomorphic 3-form Ω. The imaginary part of Ω is $\hat{\rho}$ and these satisfy the following useful identities

$$\begin{aligned}
\rho^{\#}\lrcorner\rho &= 4 & \hat{\rho}^{\#}\lrcorner\hat{\rho} &= 4\\
j\rho^{\#}\lrcorner j\rho &= 2\mathbb{I} & j\hat{\rho}^{\#}\lrcorner j\hat{\rho} &= 2\mathbb{I}\\
\rho\wedge\hat{\rho} &= 4\,\mathrm{vol} & \rho^{\#}\lrcorner\hat{\rho} &= 0\\
j\rho^{\#}\lrcorner j\hat{\rho} &= -2I & j\hat{\rho}^{\#}\lrcorner j\rho &= 2I\\
\rho^{\#}\lrcorner\,\mathrm{vol} &= \hat{\rho} & \hat{\rho}^{\#}\lrcorner\,\mathrm{vol} &= -\rho\\
\mathrm{vol}^{\#}\lrcorner\rho &= -\hat{\rho}^{\#} & \mathrm{vol}^{\#}\lrcorner\hat{\rho} &= \rho^{\#}
\end{aligned}\tag{I.1}$$

where $\mathbb{I}$ is the identity endomorphism on T, and I is the complex structure on T induced by ρ. The $\mathrm{SL}(3,\mathbb{C})$ structure that ρ induces defines a reduction of the exterior algebra into irreducible representations. In particular we find

$$\wedge^3T^*\simeq\wedge^{3,0}T^*\oplus\wedge^{2,1}T^*\oplus\wedge^{1,2}T^*\oplus\wedge^{0,3}T^*\tag{I.2}$$

$$\wedge^6T^*\simeq\wedge^{3,3}T^*\tag{I.3}$$

We can translate this into the language of exceptional complex structures in $\mathrm{E}_{6(6)}\times\mathbb{R}^+$ geometry via the following assignments

$$\chi=\mathrm{e}^{\mathrm{i}\rho}\cdot 1\qquad\tilde{J}=-\frac{1}{2}\rho+\frac{1}{2}\rho^{\#}\tag{I.4}$$

Using this structure, we can decompose the adjoint bundle into eigenbundles of $\tilde{J}$. We find

$$\mathrm{ad}\,\tilde{F}_{\mathbb{C}}=S_{-2}\oplus S_{-1}\oplus S_1\oplus S_2\oplus\mathrm{ad}\,P_{\mathbb{R}^+\times\mathrm{U}^*(6)}\tag{I.5}$$

where S_n is the ni-eigenbundle. We have

$$S_{-2} \simeq \left\{ \alpha(\tfrac{\mathrm{i}}{2}I - \hat{\rho} + \hat{\rho}^{\#} - \tfrac{\mathrm{i}}{2}\,\mathrm{vol} - \tfrac{\mathrm{i}}{2}\,\mathrm{vol}^{\#}) \,|\, \alpha \in C^{\infty}(M) \right\} \tag{I.6}$$

$$\begin{aligned} S_{-1} \simeq & \left\{ \beta(\tfrac{4\mathrm{i}}{3} + \tfrac{2\mathrm{i}}{3}\mathbb{I} + \rho + \rho^{\#}) \,|\, \beta \in C^{\infty}(M) \right\} \\ & \oplus \left\{ \gamma(\hat{\rho} + \hat{\rho}^{\#} + 2\mathrm{i}\,\mathrm{vol} - 2\mathrm{i}\,\mathrm{vol}^{\#}) \,|\, \gamma \in C^{\infty}(M) \right\} \\ & \oplus \left\{ \tfrac{\mathrm{i}}{2} j\bar{\Omega}^{\#} \lrcorner j\delta + \delta - \mathrm{i}\,\mathrm{vol}^{\#} \lrcorner \delta \,|\, \delta \in \Omega^{2,1}(M) \right\} \\ & \oplus \left\{ \tfrac{\mathrm{i}}{2} j\Omega^{\#} \lrcorner j\epsilon + \epsilon + \mathrm{i}\,\mathrm{vol}^{\#} \lrcorner \epsilon \,|\, \epsilon \in \Omega^{1,2}(M) \right\} \end{aligned} \tag{I.7}$$

We can consider the adjoint orbit of $\tilde{J}$, at one particular point on M. The tangent space at $\tilde{J}$ will be spanned by the elements

$$T_{\tilde{J}} Q_{\mathrm{U}^*(6)\times\mathbb{R}^+} = [S_{+2} \oplus S_{+1} \oplus S_{-1} \oplus S_{-2}]_{\mathbb{R}} \tag{I.8}$$

where the subscript $\mathbb{R}$ means the real part of this vector space. The grading provides a natural complex structure $\mathcal{J}$ on this vector space given by

$$T_{\tilde{J}} Q_{\mathrm{U}^*(6)\times\mathbb{R}^+} = \underbrace{(S_{+2} \oplus S_{+1})}_{-\mathrm{i}} \oplus \underbrace{(S_{-1} \oplus S_{-2})}_{+\mathrm{i}} \tag{I.9}$$

Note that $\mathcal{J}$ has the opposite sign conventions to $\tilde{J}$. This is to ensure that χ, which is charged $+2\mathrm{i}$ under $\tilde{J}$, is left invariant by antiholomorphic transformations (i.e. those charged $-\mathrm{i}$ under $\mathcal{J}$). There is also a natural symplectic structure on the space of adjoint orbits given by the Kirillov-Kostant-Souriau symplectic form

$$\omega_{\tilde{J}}(X,Y) = -\operatorname{Tr}(\tilde{J},[X,Y]) \qquad X,Y \in T_{\tilde{J}} Q_{\mathrm{U}^*(6)\times\mathbb{R}^+} \tag{I.10}$$

We can use this and the complex structure to find a hermitian metric on $T_{\tilde{J}} Q_{\mathrm{U}^*(6)\times\mathbb{R}^+}$ via

$$\tilde{g}(X,Y) = -\operatorname{Tr}(\tilde{J},[X,\mathcal{J}Y]) \tag{I.11}$$

We will now calculate this explicitly.

Let $a \in S_{-2}$, $b,c,d,e \in S_{-1}$ represent elements in each of the sets decomposing S_{-1}, S_{-2} in (I.6), (I.7). Then we have

$$\begin{aligned} \tilde{g}(a,\bar{a}) &= -\operatorname{Tr}(\tilde{J},[a,-\mathrm{i}\bar{a}]) && \text{(I.12)} \\ &= \mathrm{i}\alpha\bar{\alpha}\operatorname{Tr}(\tilde{J},[\tfrac{\mathrm{i}}{2}I - \hat{\rho} + \hat{\rho}^{\#} - \tfrac{\mathrm{i}}{2}\,\mathrm{vol} - \tfrac{\mathrm{i}}{2}\,\mathrm{vol}^{\#}, -\tfrac{\mathrm{i}}{2}I - \hat{\rho} + \hat{\rho}^{\#} + \tfrac{\mathrm{i}}{2}\,\mathrm{vol} + \tfrac{\mathrm{i}}{2}\,\mathrm{vol}^{\#}]) && \text{(I.13)} \\ &= \tfrac{1}{2}\mathrm{i}\alpha\bar{\alpha}\operatorname{Tr}(-\rho + \rho^{\#}, ... + 4\mathrm{i}\rho - 4\mathrm{i}\rho^{\#} + ...) && \text{(I.14)} \\ &= -\alpha\bar{\alpha}(\rho^{\#}\lrcorner\rho + \rho^{\#}\lrcorner\rho) && \text{(I.15)} \\ &= -8\alpha\bar{\alpha} && \text{(I.16)} \end{aligned}$$

$$\tilde{g}(b,\bar{b}) = -\operatorname{Tr}(\tilde{J},[b,-\mathrm{i}\bar{b}]) \tag{I.17}$$

$$= \mathrm{i}\beta\bar{\beta}\,\mathrm{Tr}(\tilde{J}, [\tfrac{4\mathrm{i}}{3} + \tfrac{2\mathrm{i}}{3}\mathbb{I} + \rho + \rho^{\#}, -\tfrac{4\mathrm{i}}{3} - \tfrac{2\mathrm{i}}{3}\mathbb{I} + \rho + \rho^{\#}]) \tag{I.18}$$

$$= \tfrac{1}{2}\mathrm{i}\beta\bar{\beta}\,\mathrm{Tr}(-\rho + \rho^{\#}, ... - 4\mathrm{i}\rho + 4\mathrm{i}\rho^{\#} + ...) \tag{I.19}$$

$$= 2\beta\bar{\beta}\tfrac{1}{2}(\rho^{\#}\lrcorner\rho + \rho^{\#}\lrcorner\rho) \tag{I.20}$$

$$= 8\beta\bar{\beta} \tag{I.21}$$

$$\tilde{g}(c,\bar{c}) = -\,\mathrm{Tr}(\tilde{J}, [c, -\mathrm{i}\bar{c}]) \tag{I.22}$$

$$= \mathrm{i}\gamma\bar{\gamma}\,\mathrm{Tr}(\tilde{J}, [\hat{\rho} + \hat{\rho}^{\#} + 2\mathrm{i}\,\mathrm{vol} - 2\mathrm{i}\,\mathrm{vol}^{\#}, \hat{\rho} + \hat{\rho}^{\#} - 2\mathrm{i}\,\mathrm{vol} + 2\mathrm{i}\,\mathrm{vol}^{\#}]) \tag{I.23}$$

$$= \tfrac{1}{2}\mathrm{i}\gamma\bar{\gamma}\,\mathrm{Tr}(-\rho + \rho^{\#}, ... - 4\mathrm{i}\rho + 4\mathrm{i}\rho^{\#} + ...) \tag{I.24}$$

$$= \gamma\bar{\gamma}(\rho^{\#}\lrcorner\rho + \rho^{\#}\lrcorner\rho) \tag{I.25}$$

$$= 8\gamma\bar{\gamma} \tag{I.26}$$

$$\tilde{g}(d,\bar{d}) = -\,\mathrm{Tr}(\tilde{J}, [d, -\mathrm{i}\bar{d}]) \tag{I.27}$$

$$= \mathrm{i}\,\mathrm{Tr}(\tilde{J}, [\tfrac{\mathrm{i}}{2}j\bar{\Omega}^{\#}\lrcorner j\delta + \delta - \mathrm{i}\,\mathrm{vol}^{\#}\lrcorner\delta, -\tfrac{\mathrm{i}}{2}j\Omega^{\#}\lrcorner j\bar{\delta} + \bar{\delta} + \mathrm{i}\,\mathrm{vol}^{\#}\lrcorner\bar{\delta}]) \tag{I.28}$$

$$\begin{aligned} &= \tfrac{1}{2}\mathrm{i}\,\mathrm{Tr}(-\rho + \rho^{\#}, ... + \tfrac{\mathrm{i}}{2}(j\bar{\Omega}^{\#}\lrcorner j\delta)\cdot\bar{\delta} + \tfrac{\mathrm{i}}{2}(j\Omega\lrcorner j\bar{\delta})\cdot\delta \\ &\qquad + \tfrac{1}{2}(j\bar{\Omega}^{\#}\lrcorner j\delta)\cdot(\mathrm{vol}^{\#}\lrcorner\bar{\delta}) + \tfrac{1}{2}(j\Omega^{\#}\lrcorner j\bar{\delta})\cdot(\mathrm{vol}^{\#}\lrcorner\delta) + ...) \end{aligned} \tag{I.29}$$

$$= \mathrm{i}\,\mathrm{vol}^{\#}\lrcorner(\delta\wedge\bar{\delta}) \tag{I.30}$$

$$\tilde{g}(e,\bar{e}) = -\,\mathrm{Tr}(\tilde{J}, [e, -\mathrm{i}\bar{e}]) \tag{I.31}$$

$$= \mathrm{i}\,\mathrm{Tr}(\tilde{J}, [\tfrac{\mathrm{i}}{2}j\Omega^{\#}\lrcorner j\epsilon + \epsilon + \mathrm{i}\,\mathrm{vol}^{\#}\lrcorner\epsilon, -\tfrac{\mathrm{i}}{2}j\bar{\Omega}^{\#}\lrcorner j\bar{\epsilon} + \bar{\epsilon} - \mathrm{i}\,\mathrm{vol}^{\#}\lrcorner\bar{\epsilon}]) \tag{I.32}$$

$$\begin{aligned} &= \tfrac{1}{2}\mathrm{i}\,\mathrm{Tr}(-\rho + \rho^{\#}, ... + \tfrac{\mathrm{i}}{2}(j\Omega^{\#}\lrcorner j\epsilon)\cdot\bar{\epsilon} + \tfrac{\mathrm{i}}{2}(j\bar{\Omega}^{\#}\lrcorner j\bar{\epsilon})\cdot\epsilon \\ &\qquad - \tfrac{1}{2}(j\Omega\lrcorner j\epsilon)\cdot(\mathrm{vol}^{\#}\lrcorner\bar{\epsilon}) - \tfrac{1}{2}(j\bar{\Omega}\lrcorner j\bar{\epsilon})\cdot(\mathrm{vol}^{\#}\lrcorner\epsilon) + ...) \end{aligned} \tag{I.33}$$

$$= -\mathrm{i}\,\mathrm{vol}^{\#}\lrcorner(\epsilon\wedge\bar{\epsilon}) \tag{I.34}$$

It is a simple check to see that this metric has complex signature $(14, 7)$

I.2 Metric on $T_\chi\mathcal{Z}_{\mathrm{SU}^*(6)}$

What we would really like to calculate is the metric on $T_\chi\mathcal{Z}_{\mathrm{SU}^*(6)}$[1]. We can find this by first finding the metric on $T_\chi Q_{\mathrm{SU}^*(6)}$, and then extending. The space $Q_{\mathrm{SU}^*(6)}$ is a complex cone over $Q_{\mathrm{U}^*(6)\times\mathbb{R}^+}$ so we need to find the metric on the remaining $\mathbb{C}^*$ part of the adjoint action generated by $J, \mathbb{R}^+$. Schematically, the metric on $T_\chi W_{\mathrm{SU}^*(6)}$ is given by

$$\mathrm{d}s^2 = \mathrm{d}z\mathrm{d}\bar{z} + \lambda\mathrm{d}s^2_{\mathcal{X}} \tag{I.35}$$

where $\mathrm{d}z$ is some holomorphic cone direction given by a combination of J and $\mathbb{R}^+$, $\mathrm{d}s^2_{\mathcal{X}}$ is the metric $\tilde{g}$ on the base, and λ is some relative constant.

We can write the (complexified) tangent space at a point in terms of an (anti)holomorphic

[1] This is the metric on the space of global $\mathrm{SU}^*(6)$ structures at the point $\chi \in \mathcal{Z}_{\mathrm{SU}^*(6)}$ corresponding to the Calabi-Yau.

basis

$$T_\chi Q_{\mathrm{SU}^*(6)} = \underbrace{\left(\left(-\frac{2\mathrm{i}}{3}\mathbf{1} + \tilde{J}\right) \oplus S_{+1} \oplus S_{+2}\right)}_{-\mathrm{i}} \oplus \underbrace{\left(\left(+\frac{2\mathrm{i}}{3}\mathbf{1} + \tilde{J}\right) \oplus S_{-1} \oplus S_{-2}\right)}_{+\mathrm{i}} \tag{I.36}$$

where $\mathbf{1}$ is the generator of the $\mathbb{R}^+$ action (so $\mathbf{1} \cdot \chi = 3\chi$). The underbraces provide the complex structure $\mathcal{I}$ on the tangent space. Note that we have an explicit metric on $T_\chi Q_{\mathrm{SU}^*(6)}$ evaluated in $\det T^*$ found in chapter 4. It is given by

$$\begin{aligned} G(\alpha, \beta) = \frac{\mathrm{i}}{\mathrm{Tr}(\chi\bar{\chi})^{1/2}} \Bigg(& \frac{1}{2} \mathrm{Tr}\,(\alpha \cdot \chi, (\mathcal{I}\beta) \cdot \bar{\chi}) - \frac{1}{2} \mathrm{Tr}\,((\mathcal{I}\beta) \cdot \chi, \alpha \cdot \bar{\chi}) \\ & + \frac{1}{4} \frac{\mathrm{Tr}\,((\mathcal{I}\beta) \cdot \chi, \bar{\chi})\, \mathrm{Tr}\,(\chi, \alpha \cdot \bar{\chi})}{\mathrm{Tr}(\chi\bar{\chi})} - \frac{1}{4} \frac{\mathrm{Tr}\,(\alpha \cdot \chi, \bar{\chi})\, \mathrm{Tr}\,(\chi, (\mathcal{I}\beta) \cdot \bar{\chi})}{\mathrm{Tr}(\chi\bar{\chi})} \Bigg) \end{aligned} \tag{I.37}$$

where α, β are tangent vectors viewed as adjoint elements.

To calculate this metric, we need the following results. If we take $z = \frac{2\mathrm{i}}{3}\mathbf{1} + \tilde{J}$ to be the holomorphic cone direction then we have

$$z \cdot \chi = 4\mathrm{i}\chi \qquad\qquad z \cdot \bar{\chi} = 0 \tag{I.38}$$

$$\bar{z} \cdot \chi = 0 \qquad\qquad \bar{z} \cdot \bar{\chi} = -4\mathrm{i} \tag{I.39}$$

We also have

$$\mathbb{I} \cdot \chi = \mathbb{I} \cdot \mathrm{e}^{\mathrm{i}\rho} \cdot 1 \tag{I.40}$$

$$= \mathbb{I} \cdot \left(\sum_{n=0}^{4} \frac{1}{n!} (\mathrm{i}\rho)^n \right) \tag{I.41}$$

$$= \sum_{n=0}^{4} \frac{1}{n!} \mathbb{I} \cdot (\mathrm{i}\rho)^n \tag{I.42}$$

$$= \sum_{n=0}^{4} \frac{-3n}{n!} (\mathrm{i}\rho)^n \tag{I.43}$$

$$= -3\mathrm{i}\rho \cdot \left(\sum_{n=0}^{3} \frac{1}{n!} (\mathrm{i}\rho)^n \right) \tag{I.44}$$

$$= -3\mathrm{i}\rho \cdot \left(\sum_{n=0}^{4} \frac{1}{n!} (\mathrm{i}\rho)^n \right) \tag{I.45}$$

$$= -3\mathrm{i}\rho \cdot \chi \tag{I.46}$$

$$\mathbb{I} \cdot \bar{\chi} = 3\mathrm{i}\rho \cdot \bar{\chi} \tag{I.47}$$

Note also that since Tr is an $\mathrm{E}_{6(6)}$ invariant, we have the following.

$$\mathrm{Tr}(\chi, \bar{\chi}) = \mathrm{Tr}(\mathrm{e}^{\mathrm{i}\rho}, \mathrm{e}^{-\mathrm{i}\rho}) \tag{I.48}$$

$$= \mathrm{Tr}(1, \mathrm{e}^{-2\mathrm{i}\rho}) \tag{I.49}$$

$$= \operatorname{Tr}\left(1, \frac{(-2\mathrm{i})^4\rho^4}{4!}\right) \tag{I.50}$$

$$= \frac{2}{3}\operatorname{Tr}(1, \rho^4) \tag{I.51}$$

$$\operatorname{Tr}(\rho\cdot\chi, \bar{\chi}) = -\operatorname{Tr}\left(\chi, \rho\cdot\bar{\chi}\right) \tag{I.52}$$

$$= -\operatorname{Tr}\left(1, \mathrm{e}^{-\mathrm{i}\rho}\rho\mathrm{e}^{-\mathrm{i}\rho}\right) \tag{I.53}$$

$$= -\operatorname{Tr}\left(1, (1 - \mathrm{i}\rho - \tfrac{1}{2}\rho^2 + \tfrac{\mathrm{i}}{6}\rho^3)\rho(1 - \mathrm{i}\rho - \tfrac{1}{2}\rho^2 + \tfrac{\mathrm{i}}{6}\rho^3)\right) \tag{I.54}$$

$$= -\operatorname{Tr}\left(1, \frac{4\mathrm{i}}{3}\rho^4\right) \tag{I.55}$$

$$= -2\mathrm{i}\operatorname{Tr}(\chi, \bar{\chi}) \tag{I.56}$$

$$\operatorname{Tr}(\chi, \rho\cdot\bar{\chi}) = 2\mathrm{i}\operatorname{Tr}(\chi, \bar{\chi}) \tag{I.57}$$

$$\operatorname{Tr}(\rho\cdot\chi, \rho\cdot\bar{\chi}) = -\operatorname{Tr}(\chi, \rho^2\cdot\bar{\chi}) \tag{I.58}$$

$$= -\operatorname{Tr}(1, \mathrm{e}^{-\mathrm{i}\rho}\rho^2\mathrm{e}^{-\mathrm{i}\rho}) \tag{I.59}$$

$$= -\operatorname{Tr}\left(1, (1 - \mathrm{i}\rho - \tfrac{1}{2}\rho^2)\rho^2(1 - \mathrm{i}\rho - \tfrac{1}{2}\rho^2)\right) \tag{I.60}$$

$$= \operatorname{Tr}(1, 2\rho^4) \tag{I.61}$$

$$= 3\operatorname{Tr}(\chi, \bar{\chi}) \tag{I.62}$$

We want to calculate $G(\rho, \rho)$. To do this we need to find what $\mathcal{I}\rho$ is. We write

$$\begin{aligned} \rho &= \frac{1}{2}(\rho + \rho^{\#}) - \frac{1}{2}(-\rho + \rho^{\#}) \\ &= \frac{1}{4}\left(\rho + \rho^{\#} + \frac{4\mathrm{i}}{3} + \frac{2\mathrm{i}}{3}\mathbb{I}\right) + \frac{1}{4}\left(\rho + \rho^{\#} - \frac{4\mathrm{i}}{3} - \frac{2\mathrm{i}}{3}\mathbb{I}\right) \\ &\quad - \frac{1}{4}\left(-\rho + \rho^{\#} + \frac{2\mathrm{i}}{3}\right) - \frac{1}{4}\left(-\rho + \rho^{\#} - \frac{2\mathrm{i}}{3}\right) \\ &= b(\tfrac{1}{4}) + \bar{b}(\tfrac{1}{4}) - z(\tfrac{1}{4}) - \bar{z}(\tfrac{1}{4}) \end{aligned} \tag{I.63}$$

$$\begin{aligned} \Rightarrow \mathcal{I}\rho &= \mathrm{i}b(\tfrac{1}{4}) - \mathrm{i}\bar{b}(\tfrac{1}{4}) - \mathrm{i}z(\tfrac{1}{4}) + \mathrm{i}\bar{z}(\tfrac{1}{4}) \\ &= \frac{\mathrm{i}}{2}\left(\frac{4\mathrm{i}}{3} + \frac{2\mathrm{i}}{3}\mathbb{I}\right) - \frac{\mathrm{i}}{2}\left(\frac{2\mathrm{i}}{3}\right) \\ &= -\frac{1}{3}(\mathbf{1} + \mathbb{I}) \end{aligned} \tag{I.64}$$

$$\begin{aligned} \Rightarrow (\mathcal{I}\rho)\cdot\chi &= -\frac{1}{3}(\mathbf{1} + \mathbb{I})\cdot\chi \\ &= -\frac{1}{3}(3 - 3\mathrm{i}\rho)\chi \\ &= \mathrm{i}\rho\cdot\chi - \chi \end{aligned} \tag{I.65}$$

$$\Rightarrow (\mathcal{I}\rho)\cdot\bar{\chi} = -\mathrm{i}\rho\cdot\bar{\chi} - \bar{\chi} \tag{I.66}$$

where the numbers in the brackets correspond to the value of the coefficient in from of the corresponding term in $T_\chi Q_{\mathrm{SU}^*(6)}$. Using this, we can now calculate

$$\begin{aligned} G(\rho,\rho) = \frac{\mathrm{i}}{\mathrm{Tr}(\chi\bar\chi)^{1/2}}\bigg(&\frac{1}{2}\,\mathrm{Tr}\,(\rho\cdot\chi, -\mathrm{i}\rho\cdot\bar\chi - \bar\chi) - \frac{1}{2}\,\mathrm{Tr}\,(\mathrm{i}\rho\cdot\chi - \chi, \rho\cdot\bar\chi) \\ &+ \frac{1}{4}\frac{\mathrm{Tr}\,(\mathrm{i}\rho\cdot\chi-\chi,\bar\chi)\,\mathrm{Tr}\,(\chi,\rho\cdot\bar\chi)}{\mathrm{Tr}(\chi\bar\chi)} - \frac{1}{4}\frac{\mathrm{Tr}\,(\rho\cdot\chi,\bar\chi)\,\mathrm{Tr}\,(\chi,-\mathrm{i}\rho\cdot\bar\chi-\bar\chi)}{\mathrm{Tr}(\chi\bar\chi)}\bigg) \end{aligned} \tag{I.67}$$

$$\begin{aligned} = \frac{1}{\mathrm{Tr}(\chi\bar\chi)^{1/2}}\bigg(&\mathrm{Tr}(\rho\cdot\chi,\rho\cdot\bar\chi) - \frac{\mathrm{i}}{2}\,\mathrm{Tr}(\rho\cdot\chi,\bar\chi) + \frac{\mathrm{i}}{2}\,\mathrm{Tr}(\chi,\rho\cdot\bar\chi) \\ &- \frac{1}{2}\frac{\mathrm{Tr}(\rho\cdot\chi,\bar\chi)\,\mathrm{Tr}(\chi,\rho\cdot\bar\chi)}{\mathrm{Tr}(\chi\bar\chi)} - \frac{\mathrm{i}}{4}\,\mathrm{Tr}(\chi,\rho\cdot\bar\chi) + \frac{\mathrm{i}}{4}\,\mathrm{Tr}(\rho\cdot\chi,\bar\chi)\bigg) \end{aligned} \tag{I.68}$$

$$= \frac{1}{\mathrm{Tr}(\chi\bar\chi)^{1/2}}\bigg(3\,\mathrm{Tr}(\chi\bar\chi) - \frac{1}{2}\,\mathrm{Tr}(\chi\bar\chi) - \frac{1}{2}\,\mathrm{Tr}(\chi\bar\chi) - 2\,\mathrm{Tr}(\chi\bar\chi)\bigg) \tag{I.69}$$

$$= 0 \tag{I.70}$$

However, we can also express $G(\rho,\rho)$ in another way.

$$G(\rho,\rho) = G(b(\tfrac14)+\bar b(\tfrac14)-z(\tfrac14)-\bar z(\tfrac14), b(\tfrac14)+\bar b(\tfrac14)-z(\tfrac14)-\bar z(\tfrac14)) \tag{I.71}$$

$$= 2G(b(\tfrac14),\bar b(\tfrac14)) + 2G(z(\tfrac14),\bar z(\tfrac14)) \tag{I.72}$$

$$= 2\lambda\tilde g(b(\tfrac14),\bar b(\tfrac14)) + 2G(z(\tfrac14),\bar z(\tfrac14)) \tag{I.73}$$

$$= \lambda + 2G(z(\tfrac14),\bar z(\tfrac14)) \tag{I.74}$$

$$= 0 \tag{I.75}$$

$$\Rightarrow \lambda = -2G(z(\tfrac14),\bar z(\tfrac14)) \tag{I.76}$$

However, we can calculate $G(z,\bar z)$ directly. We have

$$\begin{aligned} G(z,\bar z) = \frac{\mathrm{i}}{\mathrm{Tr}(\chi\bar\chi)^{1/2}}\bigg(&\frac{1}{2}\,\mathrm{Tr}\,(z\cdot\chi, -\mathrm{i}\bar z\cdot\bar\chi) - \frac{1}{2}\,\mathrm{Tr}\,(-\mathrm{i}\bar z\cdot\chi, z\cdot\bar\chi) \\ &+ \frac{1}{4}\frac{\mathrm{Tr}\,(-\mathrm{i}\bar z\cdot\chi,\bar\chi)\,\mathrm{Tr}\,(\chi,z\cdot\bar\chi)}{\mathrm{Tr}(\chi\bar\chi)} - \frac{1}{4}\frac{\mathrm{Tr}\,(z\cdot\chi,\bar\chi)\,\mathrm{Tr}\,(\chi,-\mathrm{i}\bar z\cdot\bar\chi)}{\mathrm{Tr}(\chi\bar\chi)}\bigg) \end{aligned} \tag{I.77}$$

$$= \frac{\mathrm{i}}{\mathrm{Tr}(\chi\bar\chi)^{1/2}}\bigg(\frac{1}{2}\,\mathrm{Tr}\,(4\mathrm{i}\chi,-4\bar\chi) - \frac{1}{4}\frac{\mathrm{Tr}\,(4\mathrm{i}\chi,\bar\chi)\,\mathrm{Tr}\,(\chi,-4\bar\chi)}{\mathrm{Tr}(\chi\bar\chi)}\bigg) \tag{I.78}$$

$$= \frac{1}{\mathrm{Tr}(\chi\bar\chi)^{1/2}}\bigg(8\,\mathrm{Tr}(\chi\bar\chi) - 4\,\mathrm{Tr}(\chi\bar\chi)\bigg) \tag{I.79}$$

$$= 4\,\mathrm{Tr}(\chi\bar\chi)^{1/2} \tag{I.80}$$

Hence we have

$$\lambda = -\frac{1}{2}\,\mathrm{Tr}(\chi\bar\chi)^{1/2} \tag{I.81}$$

This gives us the metric on $T_\chi Q_{\mathrm{SU}^*(6)}$. Of course, what we need is the metric on $T_\chi \mathcal{Z}_{\mathrm{SU}^*(6)}$ but this is obtained by integrating G over the manifold.

$$\mathcal{G}(\alpha,\beta) = \int_M G(\alpha,\beta) \tag{I.82}$$

I.2.1 More useful Form of the Metric

We can rewrite this metric in a form more suitable to performing path integrals to calculate 1-loop corrections. We have the freedom to rescale χ by some constant. We can therefore choose χ such that

$$\mathrm{Tr}(\chi\bar{\chi})^{1/2} = \mathrm{vol} \tag{I.83}$$

where vol is the volume form defined by ρ. With this choice, we find that we can write the metric as

$$\mathcal{G}(z, \bar{z}') = \int_M \zeta\bar{\zeta}'\rho \wedge \hat{\rho} \tag{I.84}$$

$$\mathcal{G}(a, \bar{a}') = \int_M \alpha\bar{\alpha}'\rho \wedge \hat{\rho} \tag{I.85}$$

$$\mathcal{G}(b, \bar{b}') = -\int_M \beta\bar{\beta}'\rho \wedge \hat{\rho} \tag{I.86}$$

$$\mathcal{G}(c, \bar{c}') = -\int_M \gamma\bar{\gamma}'\rho \wedge \hat{\rho} \tag{I.87}$$

$$\mathcal{G}(d, \bar{d}') = -\frac{\mathrm{i}}{2}\int_M \delta \wedge \bar{\delta}' \tag{I.88}$$

$$\mathcal{G}(e, \bar{e}') = \frac{\mathrm{i}}{2}\int_M \epsilon \wedge \bar{\epsilon}' \tag{I.89}$$

where we have added an arbitrary function in the cone direction $z = \zeta(\frac{2\mathrm{i}}{3}\mathbf{1} + \tilde{J})$.

Appendix J

The Metric on the Space of SU(7) Structures

J.1 Metric on $T_J Q_{U(7)\times\mathbb{R}^+}$

We have an almost G_2-manifold (M, ϕ) where $\phi \in \wedge^3 T^*$ and M is a 7-dimensional manifold. ϕ is a stable form in the sense of Hitchin [294]. It also satisfies the following

$$(v \lrcorner \phi) \wedge (w \lrcorner \phi) \wedge \phi = 6 g_\phi(v, w) \mathrm{Vol}_\phi \qquad \forall\, v, w \in T \tag{J.1}$$

and hence it defines a canonical metric g_ϕ. There also exists an orthonormal frame $e^1, ..., e^7$ such that ϕ takes the canonical form

$$\phi = e^{123} - e^{145} - e^{167} - e^{246} + e^{257} - e^{347} - e^{356} \tag{J.2}$$

With this, one can see that we have the identities

$$\phi^{\#} \lrcorner \phi = 7 \qquad j\phi^{\#} \lrcorner j\phi = 3\mathbb{I} \tag{J.3}$$

The group that preserves ϕ is G_2 and hence we can decompose the spaces of differential forms in G_2 irreps. We have

$$\wedge^1 T^* \simeq \wedge^1_7 T^* \tag{J.4}$$
$$\wedge^2 T^* \simeq \wedge^2_7 T^* \oplus \wedge^2_{14} T^* \tag{J.5}$$
$$\wedge^3 T^* \simeq \wedge^3_1 T^* \oplus \wedge^3_7 T^* \oplus \wedge^3_{27} T^* \tag{J.6}$$

where

$$\wedge^2_7 T^* \simeq \{v \lrcorner \phi \mid v \in T\} \tag{J.7}$$
$$\wedge^2_{14} T^* \simeq \{\alpha \mid \alpha \wedge *\phi = 0\} \tag{J.8}$$
$$\wedge^3_1 T^* \simeq \langle \phi \rangle \tag{J.9}$$
$$\wedge^3_7 T^* \simeq \{v \lrcorner (*\phi) \mid v \in T\} \tag{J.10}$$

$$\wedge^3_{27}T^* \simeq \{\beta \mid \beta \wedge \phi = 0 \quad \beta \wedge *\phi = 0\} \tag{J.11}$$

and similarly for the others via $\wedge^p T^* \simeq \wedge^{7-p}T^*$.

We can translate this structure into the language of exceptional complex structures with the following definitions

$$\psi = \mathrm{e}^{\mathrm{i}\phi} \cdot 1 \qquad\qquad J = -\phi + \phi^{\#} \tag{J.12}$$

We can decompose both the complexified generalised tangent bundle and adjoint bundle $E_{\mathbb{C}}$, $\mathrm{ad}\,\tilde{F}_{\mathbb{C}}$ into eigenbundles of J. The decomposition works as follows.

$$E_{\mathbb{C}} \simeq E_{+3} \oplus E_{+1} \oplus E_{-1} \oplus E_{-3} \tag{J.13}$$

We find that

$$E_{+3} \simeq \mathrm{e}^{\mathrm{i}\phi} \cdot T \tag{J.14}$$

$$E_{+1} \simeq E_{+1,7} \oplus E_{+1,14} \tag{J.15}$$

$$= \left\{ v + \frac{1}{3}\mathrm{i}(v \lrcorner \phi) + \frac{1}{6}(v \lrcorner \phi) \wedge \phi + \frac{1}{6}\mathrm{i}j\phi \wedge \phi \wedge (v \lrcorner \phi) \,|\, v \in T \right\} \tag{J.16}$$

$$\oplus \left\{ \omega + i\omega \wedge \phi \,|\, \omega \in \wedge^2_{14}T^* \right\} \tag{J.17}$$

For the adjoint bundle we have

$$\mathrm{ad}\,\tilde{F}_{\mathbb{C}} \simeq S_{+4} \oplus S_{+2} \oplus S_{-2} \oplus S_{-4} \oplus \mathrm{ad}(P_{U(7)\times\mathbb{R}^+}) \tag{J.18}$$

where

$$S_{-4} \simeq \left\{ \frac{\mathrm{i}}{4}(j\phi^{\#} \lrcorner ja - ja^{\#} \lrcorner j\phi) + a + \frac{\mathrm{i}}{4}\phi \wedge a - a^{\#} + \frac{\mathrm{i}}{4}\phi^{\#} \wedge a^{\#} \,|\, a \in \wedge^3_7 T^* \right\} \tag{J.19}$$

$$S_{-2} \simeq S_{-2,1} \oplus S_{-2,7} \oplus S_{-2,27} \tag{J.20}$$

$$= \left\{ k \left(\frac{7\mathrm{i}}{3} + \frac{2\mathrm{i}}{3}\mathbb{I} + \phi + \phi^{\#} \right) \,|\, k \in \mathbb{R} \right\} \tag{J.21}$$

$$\oplus \left\{ b + \frac{\mathrm{i}}{2}\phi \wedge b + b^{\#} - \frac{\mathrm{i}}{2}\phi^{\#} \wedge b^{\#} \,|\, b \in \wedge^3_7 T^* \right\} \tag{J.22}$$

$$\oplus \left\{ \frac{\mathrm{i}}{2}(j\phi^{\#} \lrcorner jc + jc^{\#} \lrcorner j\phi) + c + c^{\#} \,|\, c \in \wedge^3_{27} T^* \right\} \tag{J.23}$$

where $\mathbb{I} = \mathrm{id}_T$.

We can then consider the adjoint orbit of J at a point on the manifold. The tangent space at the point will be spanned by the elements

$$T_J Q_{U(7)\times\mathbb{R}^+} = [S_{+4} \oplus S_{+2} \oplus S_{-2} \oplus S_{-4}]_{\mathbb{R}} \tag{J.24}$$

where the subscript $\mathbb{R}$ means the real part of this space. Note that the grading provides a natural complex structure $\mathcal{J}$ on this tangent space given by

$$T_J Q_{U(7)\times\mathbb{R}^+} = \underbrace{(S_{+4}\oplus S_{+2})}_{-\mathrm{i}} \oplus \underbrace{(S_{-2}\oplus S_{-2})}_{+\mathrm{i}} \tag{J.25}$$

Note that $\mathcal{J}$ has the opposite sign convention to J. This will ensure later that ψ (which is charged $+7\mathrm{i}$ under J) will not transform under antiholomorphic transformations (those charged $-\mathrm{i}$ under $\mathcal{J}$).

We also have a natural symplectic structure on the tangent space given by the Kirillov-Kostant-Souriau symplectic form

$$\omega_J(X,Y) = -\mathrm{Tr}(J,[X,Y]) \qquad X,Y\in T_J Q_{U(7)\times\mathbb{R}^+} \tag{J.26}$$

From this we can find the metric on $T_J Q_{U(7)\times\mathbb{R}^+}$ via

$$\tilde{g}(X,Y) = -\mathrm{Tr}(J,[X,\mathcal{J}Y]) \tag{J.27}$$

We will now calculate what this is explicitly.

First we will introduce some notation for the elements of $T_J Q_{U(7)\times\mathbb{R}^+}$. We will write

$$\begin{aligned} T_J Q_{U(7)\times\mathbb{R}^+} = \Big\langle & u_1 = k(\phi+\phi^{\#}),\ u_2 = k\left(\frac{7}{3}+\frac{2}{3}\mathbb{I}\right) \\ & v_1 = b+b^{\#},\ v_2 = \frac{1}{2}\phi\wedge b - \frac{1}{2}\phi^{\#}\wedge b^{\#} \\ & w_1 = c+c^{\#},\ w_2 = \frac{1}{2}(j\phi^{\#}\lrcorner jc + jc^{\#}\lrcorner j\phi) \\ & x_1 = a-a^{\#},\ x_2 = \frac{1}{4}(j\phi^{\#}\lrcorner ja - ja^{\#}\lrcorner j\phi) + \frac{1}{4}\phi\wedge a + \frac{1}{4}\phi^{\#}\wedge a^{\#}\Big\rangle \end{aligned} \tag{J.28}$$

where we have used the same notation as in (J.19)-(J.23). Note that with this notation we have

$$\begin{aligned} \mathcal{J}u_1 &= -u_2 & \mathcal{J}u_2 &= u_1 \\ \mathcal{J}v_1 &= -v_2 & \mathcal{J}v_2 &= v_1 \\ \mathcal{J}w_1 &= -w_2 & \mathcal{J}w_2 &= w_1 \\ \mathcal{J}x_1 &= -x_2 & \mathcal{J}x_2 &= x_1 \end{aligned} \tag{J.29}$$

Using the formula for the bracket and trace from [187] we have

$$\tilde{g}(u_1,u_1') = -\mathrm{Tr}(J,[u_1',-u_2']) \tag{J.30}$$
$$= \mathrm{Tr}(J, 2kk'\phi - 2kk'\phi^{\#}) \tag{J.31}$$
$$= -2\frac{1}{2}kk'(-2\phi^{\#}\lrcorner\phi) \tag{J.32}$$
$$= 2kk'\langle\phi,\phi\rangle \tag{J.33}$$

$$\tilde{g}(u_2,u_2') = -\mathrm{Tr}(J,[u_2,u_1']) \tag{J.34}$$
$$= \mathrm{Tr}(J,[u_1',u_2]) \tag{J.35}$$

$$= 2kk' \langle \phi, \phi \rangle \tag{J.36}$$

$$\begin{aligned}
\tilde{g}(v_1, v_1') &= -\mathrm{Tr}(J, [v_1, -v_2']) && \text{(J.37)}\\
&= \mathrm{Tr}(J, -\frac{1}{2} b^{\#} \lrcorner (\phi \wedge b') - \frac{1}{2}(\phi^{\#} \wedge b'^{\#}) \lrcorner b) && \text{(J.38)}\\
&= -\frac{1}{2}\frac{1}{2}\left(\phi^{\#} \lrcorner (b'^{\#} \lrcorner (\phi \wedge b)) - ((\phi^{\#} \wedge b'^{\#}) \lrcorner b) \lrcorner \phi\right) && \text{(J.39)}\\
&= 2\left\langle b, b' \right\rangle && \text{(J.40)}
\end{aligned}$$

$$\begin{aligned}
\tilde{g}(v_2, v_2') &= -\mathrm{Tr}(J, [v_2, v_1']) && \text{(J.41)}\\
&= \mathrm{Tr}(J, [v_1', v_2]) && \text{(J.42)}\\
&= 2\left\langle b, b' \right\rangle && \text{(J.43)}
\end{aligned}$$

$$\begin{aligned}
\tilde{g}(w_1, w_1') &= -\mathrm{Tr}(J, [w_1, -w_2']) && \text{(J.44)}\\
&= \mathrm{Tr}(J, -\frac{1}{2}(j\phi^{\#} \lrcorner jc' + jc'^{\#} \lrcorner j\phi) \cdot (c + c^{\#})) && \text{(J.45)}\\
&= -\frac{1}{2}\frac{1}{2}(\phi^{\#} \lrcorner ((j\phi^{\#} \lrcorner jc' + jc'^{\#} \lrcorner j\phi) \cdot c) - ((j\phi^{\#} \lrcorner jc' + jc'^{\#} \lrcorner j\phi) \cdot c^{\#}) \lrcorner \phi) && \text{(J.46)}\\
&= 2\left\langle c, c' \right\rangle && \text{(J.47)}
\end{aligned}$$

$$\begin{aligned}
\tilde{g}(w_2, w_2') &= -\mathrm{Tr}(J, [w_2, w_1']) && \text{(J.48)}\\
&= \mathrm{Tr}(J, [w_1', w_2]) && \text{(J.49)}\\
&= 2\left\langle c, c' \right\rangle && \text{(J.50)}
\end{aligned}$$

$$\begin{aligned}
\tilde{g}(x_1, x_1') &= -\mathrm{Tr}(J, [x_1, -x_2']) && \text{(J.51)}\\
&= \mathrm{Tr}(J, \frac{1}{4}(-(j\phi^{\#} \lrcorner ja' - ja'^{\#} \lrcorner j\phi) \cdot a + a^{\#} \lrcorner (\phi \wedge a') && \text{(J.52)}\\
&\qquad + (j\phi^{\#} \lrcorner ja' - ja'^{\#} \lrcorner j\phi) \cdot a^{\#} + (\phi^{\#} \wedge a'^{\#}) \lrcorner a)) && \text{(J.53)}\\
&= \frac{1}{2}\frac{1}{4}(\phi^{\#} \lrcorner (-(j\phi^{\#} \lrcorner ja' - ja'^{\#} \lrcorner j\phi) \cdot a) + \phi^{\#} \lrcorner (a^{\#} \lrcorner (\phi \wedge a')) && \text{(J.54)}\\
&\qquad - ((j\phi^{\#} \lrcorner ja' - ja'^{\#} \lrcorner j\phi) \cdot a^{\#}) \lrcorner \phi - ((\phi^{\#} \wedge a'^{\#}) \lrcorner a) \lrcorner \phi) && \text{(J.55)}\\
&= -4\left\langle a, a' \right\rangle && \text{(J.56)}
\end{aligned}$$

$$\begin{aligned}
\tilde{g}(x_2, x_2') &= -\mathrm{Tr}(J, [x_2, x_1']) && \text{(J.57)}\\
&= \mathrm{Tr}(J, [x_1', x_2]) && \text{(J.58)}\\
&= -4\left\langle a, a' \right\rangle && \text{(J.59)}
\end{aligned}$$

Here the $\langle , \rangle$ is an inner product on $\wedge^3 T^*$ and is given by

$$\left\langle \eta, \eta' \right\rangle := \frac{1}{3!}\eta^{abc}\eta'_{abc} \qquad \eta, \eta' \in \wedge^3 T^* \tag{J.60}$$

The indices are raised and lowered using the metric g_ϕ defined by ϕ. One can show that these elements are orthogonal in the sense that

$$\tilde{g}(\mu_i, \nu_j) \propto \delta_{\mu\nu}\delta_{ij} \tag{J.61}$$

where $\mu, \nu = u, v, w, x$ and $i, j = 1, 2$.

J.2 Metric on $T_\psi \mathcal{Z}$

Of course what we really need for the calculation is the metric on the space $T_\psi \mathcal{Z}$. However, using the fact that $Q_{SU(7)}$ is a complex cone over $Q_{U(7)\times\mathbb{R}^+}$, we only need to find the metric on the remaining $\mathbb{C}^*$ part of the adjoint action generated by $J, \mathbb{R}^+$. The metric on $T_\psi Q_{SU(7)}$ is schematically given by

$$ds^2 = \mathrm{d}z\mathrm{d}\bar{z} + \lambda\, \mathrm{d}s^2_{\mathcal{X}} \tag{J.62}$$

where $\mathrm{d}z$ is some holomorphic cone direction given by a combination of the J and $\mathbb{R}^+$ action. $ds^2_{\mathcal{X}}$ is the metric $\tilde{g}$ generated above and, λ is some relative constant.

First note that the (complexified) tangent space at a point is spanned by the following adjoint elements.

$$T_\psi Q_{SU(7)} = \underbrace{\left(\left(-\frac{7\mathrm{i}}{3}\mathbf{1} + J\right) \oplus S_{+2} \oplus S_{+4}\right)}_{-\mathrm{i}} \oplus \underbrace{\left(\left(+\frac{7\mathrm{i}}{3}\mathbf{1} + J\right) \oplus S_{-2} \oplus S_{-4}\right)}_{+\mathrm{i}} \tag{J.63}$$

where $\mathbf{1}$ is the generator of the $\mathbb{R}^+$ action. The underbraces provide the complex structure $\mathcal{I}$ on the tangent space. We have an explicit metric on $T_\psi Q_{SU(7)}$ evaluated in $\det T^*$ [1]. For vectors α, β we have

$$\begin{aligned} G(\alpha,\beta) = \frac{\mathrm{i}}{3}\frac{1}{\left(\mathrm{i}s(\psi,\bar{\psi})\right)^{2/3}}\Bigg(&\mathrm{i}s(\alpha\cdot\psi,(\mathcal{I}\beta)\cdot\bar{\psi}) - \mathrm{i}s((\mathcal{I}\beta)\cdot\psi,\alpha\cdot\bar{\psi}) \\ &- \frac{2}{3}\frac{\mathrm{i}s(\alpha\cdot\psi,\bar{\psi})\,\mathrm{i}s(\psi,(\mathcal{I}\beta)\cdot\bar{\psi})}{\mathrm{i}s(\psi,\bar{\psi})} + \frac{2}{3}\frac{\mathrm{i}s((\mathcal{I}\beta)\cdot\psi,\bar{\psi})\,\mathrm{i}s(\psi,\alpha\cdot\bar{\psi})}{\mathrm{i}s(\psi,\bar{\psi})}\Bigg) \end{aligned} \tag{J.64}$$

where $\alpha\cdot\psi$ means the action of α on ψ, viewing it as a section of the adjoint bundle. (In the generalising G_2 paper we wrote this as $\imath_\alpha\delta\psi$.) As before we will introduce notation for some of the tangent vectors. The tangent space will be spanned by $u_1, u_2, v_1, v_2, w_1, w_2, x_1, x_2, y_1, y_2$, where everything is as in the previous section and

$$y_1 = J \quad y_2 = \frac{7}{3}\mathbf{1} \qquad \Rightarrow \qquad \mathcal{I}y_1 = -y_2 \tag{J.65}$$

To calculate G we will need the following results. By definition of ψ and the $\mathbb{R}^+$ action we have

$$y_1\cdot\psi = J\cdot\psi = 7\mathrm{i}\psi \qquad\qquad y_2\cdot\psi = \frac{7}{3}\mathbf{1}\cdot\psi = 7\psi \tag{J.66}$$

$$y_1\cdot\bar{\psi} = J\cdot\bar{\psi} = -7\mathrm{i}\bar{\psi} \qquad\qquad y_2\cdot\bar{\psi} = \frac{7}{3}\mathbf{1}\cdot\bar{\psi} = 7\bar{\psi} \tag{J.67}$$

We also have

$$\mathbb{I} \cdot \psi = \mathbb{I} \cdot \left(\sum_{n=0}^{7} \frac{1}{n!} (\mathrm{i}\phi)^n \right) \tag{J.68}$$

$$= \sum_{n=0}^{7} \frac{1}{n!} \mathbb{I} \cdot (\mathrm{i}\phi)^n \tag{J.69}$$

$$= \sum_{n=0}^{7} \frac{-3n}{n!} (\mathrm{i}\phi)^n \tag{J.70}$$

$$= -3\mathrm{i}\phi \cdot \left(\sum_{n=0}^{6} \frac{1}{n!} (\mathrm{i}\phi)^n \right) \tag{J.71}$$

$$= -3\mathrm{i}\phi \cdot \left(\sum_{n=0}^{7} \frac{1}{n!} (\mathrm{i}\phi)^n \right) \tag{J.72}$$

$$= -3\mathrm{i}\phi \cdot \psi \tag{J.73}$$

$$\mathbb{I} \cdot \bar{\psi} = 3\mathrm{i}\phi \cdot \bar{\psi} \tag{J.74}$$

Note also that s is $E_{7(7)}$ invariant, and ϕ commutes with itself. Hence we can see that

$$s(\psi, \bar{\psi}) = s(\mathrm{e}^{\mathrm{i}\phi}, \mathrm{e}^{-\mathrm{i}\phi}) \tag{J.75}$$

$$= s(1, \mathrm{e}^{-2\mathrm{i}\phi}) \tag{J.76}$$

$$= s\left(1, \frac{2^7(-\mathrm{i})^7}{7!}\phi^7\right) \tag{J.77}$$

$$= \frac{2^7\mathrm{i}}{7!} s(1, \phi^7) \tag{J.78}$$

$$s(\phi, \cdot\psi, \bar{\psi}) = -s(\psi, \phi \cdot \bar{\psi}) \tag{J.79}$$

$$= -s(\mathrm{e}^{\mathrm{i}\phi}, \phi \cdot \mathrm{e}^{-\mathrm{i}\phi}) \tag{J.80}$$

$$= -s(1, \phi \cdot \mathrm{e}^{-2\mathrm{i}\phi}) \tag{J.81}$$

$$= -s\left(1, \frac{2^6(-\mathrm{i})^6}{6!}\phi^7\right) \tag{J.82}$$

$$= \frac{2^6}{6!} s(1, \phi^7) \tag{J.83}$$

$$= -\mathrm{i}\frac{7}{2}\left(\frac{2^7\mathrm{i}}{7!} s(1, \phi^7)\right) \tag{J.84}$$

$$= -\mathrm{i}\frac{7}{2} s(\psi, \bar{\psi}) \tag{J.85}$$

$$s(\phi \cdot \psi, \phi \cdot \bar{\psi}) = -s(\psi, \phi^2 \cdot \mathrm{e}^{-\mathrm{i}\phi}) \tag{J.86}$$

$$= -s(1, \phi^2\mathrm{e}^{-2\mathrm{i}\phi}) \tag{J.87}$$

$$= -s\left(1, \frac{2^5(-\mathrm{i})^5}{5!}\phi^7\right) \tag{J.88}$$

$$= \frac{2^5\mathrm{i}}{5!} s(1, \phi^7) \tag{J.89}$$

$$= \frac{7}{2}\frac{6}{2}\left(\frac{2^7\mathrm{i}}{7!}s(1,\phi^7)\right) \tag{J.90}$$

$$= \frac{7}{2}\frac{6}{2}s(\psi,\bar{\psi}) \tag{J.91}$$

We can now use these to calculate what the metric is. Note that we can calculate the metric directly in some cases. We have

$$\begin{aligned} G(y_1,y_1) = \frac{\mathrm{i}}{3}\frac{1}{\left(\mathrm{i}s(\psi,\bar{\psi})\right)^{2/3}}\Bigg(&\mathrm{i}s(y_1\cdot\psi,-y_2\cdot\bar{\psi}) - \mathrm{i}s(-y_2\cdot\psi,y_1\cdot\bar{\psi}) \\ &- \frac{2}{3}\frac{\mathrm{i}s(y_1\cdot\psi,\bar{\psi})\,\mathrm{i}s(\psi,-y_2\cdot\bar{\psi})}{\mathrm{i}s(\psi,\bar{\psi})} + \frac{2}{3}\frac{\mathrm{i}s(-y_2\cdot\psi,\bar{\psi})\,\mathrm{i}s(\psi,y_1\cdot\bar{\psi})}{\mathrm{i}s(\psi,\bar{\psi})}\Bigg) \end{aligned} \tag{J.92}$$

$$\begin{aligned} = \frac{\mathrm{i}}{3}\frac{1}{\left(\mathrm{i}s(\psi,\bar{\psi})\right)^{2/3}}\Bigg(&\mathrm{i}s(7\mathrm{i}\psi,-7\bar{\psi}) - \mathrm{i}s(-7\psi,-7\mathrm{i}\bar{\psi}) \\ &- \frac{2}{3}\frac{\mathrm{i}s(7\mathrm{i}\psi,\bar{\psi})\,\mathrm{i}s(\psi,-7\bar{\psi})}{\mathrm{i}s(\psi,\bar{\psi})} + \frac{2}{3}\frac{\mathrm{i}s(-7\psi,\bar{\psi})\,\mathrm{i}s(\psi,-7\mathrm{i}\bar{\psi})}{\mathrm{i}s(\psi,\bar{\psi})}\Bigg) \end{aligned} \tag{J.93}$$

$$= \frac{\mathrm{i}}{3}\frac{1}{\left(\mathrm{i}s(\psi,\bar{\psi})\right)^{2/3}}\left(98s(\psi,\bar{\psi}) - \frac{2}{3}\frac{98\,s(\psi,\bar{\psi})\,s(\psi,\bar{\psi})}{s(\psi,\bar{\psi})}\right) \tag{J.94}$$

$$= \frac{98}{9}(\mathrm{i}s(\psi,\bar{\psi}))^{1/3} \tag{J.95}$$

$$G(y_2,y_2) = G(-\mathcal{I}y_1,-\mathcal{I}y_1) \tag{J.96}$$

$$= G(y_1,y_1) \tag{J.97}$$

$$= \frac{98}{9}(\mathrm{i}s(\psi,\bar{\psi}))^{1/3} \tag{J.98}$$

Now we need to find the constant λ. To do so we will calculate the following

$$G(\phi,2\phi) = G(\phi,(\phi+\phi^{\#}) - (-\phi+\phi^{\#})) \tag{J.99}$$

$$= G(\phi,u_1 - y_1) \tag{J.100}$$

$$\begin{aligned} = \frac{\mathrm{i}}{3}\frac{1}{\left(\mathrm{i}s(\psi,\bar{\psi})\right)^{2/3}}\Bigg(&\mathrm{i}s(\phi\cdot\psi,(-u_2+y_2)\cdot\bar{\psi}) - \mathrm{i}s((-u_2+y_2)\cdot\psi,\phi\cdot\bar{\psi}) \\ &- \frac{2}{3}\frac{\mathrm{i}s(\phi\cdot\psi,\bar{\psi})\,\mathrm{i}s(\psi,(-u_2+y_2)\cdot\bar{\psi})}{\left(\mathrm{i}s(\psi,\bar{\psi})\right)} + \frac{2}{3}\frac{\mathrm{i}s((-u_2+y_2)\cdot\psi,\bar{\psi})\,\mathrm{i}s(\psi,\phi\cdot\bar{\psi})}{\left(\mathrm{i}s(\psi,\bar{\psi})\right)}\Bigg) \end{aligned} \tag{J.101}$$

But $-u_2 + y_2 = -\frac{2}{3}\mathbb{I}$ and so

$$\begin{aligned} G(\phi,2\phi) = \frac{\mathrm{i}}{3}\frac{1}{\left(\mathrm{i}s(\psi,\bar{\psi})\right)^{2/3}}\Bigg(&\mathrm{i}s(\phi\cdot\psi,-\frac{2}{3}\mathbb{I}\cdot\bar{\psi}) - \mathrm{i}s(-\frac{2}{3}\mathbb{I}\cdot\psi,\phi\cdot\bar{\psi}) \\ &- \frac{2}{3}\frac{\mathrm{i}s(\phi\cdot\psi,\bar{\psi})\,\mathrm{i}s(\psi,-\frac{2}{3}\mathbb{I}\cdot\bar{\psi})}{\left(\mathrm{i}s(\psi,\bar{\psi})\right)} + \frac{2}{3}\frac{\mathrm{i}s(-\frac{2}{3}\mathbb{I}\cdot\psi,\bar{\psi})\,\mathrm{i}s(\psi,\phi\cdot\bar{\psi})}{\left(\mathrm{i}s(\psi,\bar{\psi})\right)}\Bigg) \end{aligned} \tag{J.102}$$

$$
\begin{aligned}
&= \frac{\mathrm{i}}{3}\frac{1}{\big(\mathrm{i}s(\psi,\bar\psi)\big)^{2/3}}\bigg(\mathrm{i}s(\phi\cdot\psi,-2\mathrm{i}\phi\cdot\bar\psi)-\mathrm{i}s(2\mathrm{i}\phi\cdot\psi,\phi\cdot\bar\psi) \\
&\qquad -\frac{2}{3}\frac{\mathrm{i}s(\phi\cdot\psi,\bar\psi)\,\mathrm{i}s(\psi,-2\mathrm{i}\phi\cdot\bar\psi)}{\big(\mathrm{i}s(\psi,\bar\psi)\big)}+\frac{2}{3}\frac{\mathrm{i}s(2\mathrm{i}\phi\cdot\psi,\bar\psi)\,\mathrm{i}s(\psi,\phi\cdot\bar\psi)}{\big(\mathrm{i}s(\psi,\bar\psi)\big)}\bigg) && \text{(J.103)}\\
&= \frac{\mathrm{i}}{3}\frac{1}{\big(\mathrm{i}s(\psi,\bar\psi)\big)^{2/3}}\left(4s(\phi\cdot\psi,\phi\cdot\bar\psi)-\frac{8\mathrm{i}}{3}\frac{s(\phi\cdot\psi,\bar\psi)\,s(\psi,\phi\cdot\bar\psi)}{\big(\mathrm{i}s(\psi,\bar\psi)\big)}\right) && \text{(J.104)}\\
&= \frac{\mathrm{i}}{3}\frac{1}{\big(\mathrm{i}s(\psi,\bar\psi)\big)^{2/3}}\left(4\frac{7}{2}\frac{6}{2}s(\psi,\bar\psi)-\frac{8\mathrm{i}}{3}\frac{1}{\mathrm{i}s(\psi,\bar\psi)}\left(-\frac{7\mathrm{i}}{2}s(\psi,\bar\psi)\right)\left(\frac{7\mathrm{i}}{2}s(\psi,\bar\psi)\right)\right) && \text{(J.105)}\\
&= \frac{\mathrm{i}}{3}\frac{1}{\big(\mathrm{i}s(\psi,\bar\psi)\big)^{2/3}}\left(42s(\psi,\bar\psi)-\frac{98}{3}s(\psi,\bar\psi)\right) && \text{(J.106)}\\
&= \frac{1}{3}\frac{126-98}{3}\frac{\mathrm{i}s(\psi,\bar\psi)}{\big(\mathrm{i}s(\psi,\bar\psi)\big)^{2/3}} && \text{(J.107)}\\
&= \frac{28}{9}\big(\mathrm{i}s(\psi,\bar\psi)\big)^{1/3} && \text{(J.108)}
\end{aligned}
$$

But we can also evaluate this via

$$
\begin{aligned}
G(\phi,2\phi) &= \frac{1}{2}G((\phi+\phi^{\#})-(-\phi+\phi^{\#}),(\phi+\phi^{\#})-(-\phi+\phi^{\#})) && \text{(J.109)}\\
&= \frac{1}{2}G(u_1-y_1,u_1-y_1) && \text{(J.110)}\\
&= \frac{1}{2}G(u_1,u_1)+\frac{1}{2}G(y_1,y_1) && \text{(J.111)}\\
&= \frac{1}{2}\lambda\tilde g+\frac{49}{9}\big(\mathrm{i}s(\psi,\bar\psi)\big)^{1/3} && \text{(J.112)}\\
&= 7\lambda+\frac{49}{9}\big(\mathrm{i}s(\psi,\bar\psi)\big)^{1/3} && \text{(J.113)}
\end{aligned}
$$

So we have

$$
\begin{aligned}
7\lambda &= \frac{28}{9}\big(\mathrm{i}s(\psi,\bar\psi)\big)^{1/3}-\frac{49}{9}\big(\mathrm{i}s(\psi,\bar\psi)\big)^{1/3} && \text{(J.114)}\\
&= -\frac{21}{9}\big(\mathrm{i}s(\psi,\bar\psi)\big)^{1/3} && \text{(J.115)}\\
\Rightarrow\quad \lambda &= -\frac{1}{3}\big(\mathrm{i}s(\psi,\bar\psi)\big)^{1/3} && \text{(J.116)}
\end{aligned}
$$

This is the metric on the cone at apoint $T_\psi Q_{SU(7)}$. Of course we need the metric on $T_\psi\mathcal{Z}$ but this is obtained by just integrating over the manifold

$$
\mathcal{G}(\alpha,\beta)=\int_M G(\alpha,\beta) \tag{J.117}
$$

J.2.1 More Useful Form of Metric

We can write this in a form which will be more useful for the quantization later. By definition, for any 3-forms $\eta,\eta'\in\wedge^3T^*$ we have

$$
\langle\eta,\eta'\rangle\,\mathrm{Vol}_\phi=\eta\wedge*\eta' \tag{J.118}
$$

So we have

$$\tilde{g}(\mu_i,\nu_i)\mathrm{Vol}_\phi = \begin{cases} 2\eta_\mu \wedge *\eta_\nu & \mu=\nu=u,v,w \qquad i=j \\ -4\eta_\mu \wedge *\eta_\nu & \mu=\nu=x \qquad i=j \\ 0 & \text{otherwise} \end{cases} \tag{J.119}$$

where η_μ, η_ν are the 3-forms labelling μ_i, ν_j respectively. Note also that

$$\left(\mathrm{i}s(\psi,\bar{\psi})\right)^{1/3} \propto \mathrm{Vol}_\phi \tag{J.120}$$

We have the freedom to rescale ψ by some constant and so we can choose it such that

$$\mathcal{G}(u_i,u'_j) = -\delta_{ij}\int_M kk'\phi\wedge *\phi \tag{J.121}$$

$$\mathcal{G}(v_i,v'_j) = -\delta_{ij}\int_M b\wedge *b' \tag{J.122}$$

$$\mathcal{G}(w_i,w'_j) = -\delta_{ij}\int_M c\wedge *c' \tag{J.123}$$

$$\mathcal{G}(x_i,x'_j) = \delta_{ij}2\int_M a\wedge *a' \tag{J.124}$$

$$\mathcal{G}(ly_i,l'y_j) = \delta_{ij}21\int_M ll'\phi\wedge *\phi \tag{J.125}$$

I get 7/3 for the ll' coefficient - which I think is what you get too. Typo? where $k,k',l,l' \in C^\infty(M)$ and $a,b,c \in \Gamma(\wedge^3 T^*)$ as defined in (J.19)-(J.23). Any other combination of u,v,w,x,y will be 0. To get this we choose the scaling

$$\left(\mathrm{i}s(\psi,\bar{\psi})\right)^{1/3} = \frac{3}{2}\mathrm{Vol}_\phi \tag{J.126}$$