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A monstrous integral

Ivan Gonzalez[1] · John Lopez Santander[2] · Victor H. Moll[2] · Daniel Salinas-Arizmendi[3,4]

Dedicated to Krishna Alladi: thanks for creating and maintaining a place where evaluating Integrals is considered an interesting part of Mathematics.

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Abstract

The method of brackets originated from the so-called negative dimension approach for evaluating definite integrals arising in Feynman diagrams. Its fundamental concept, still heuristic, identifies a bracket with a divergent integral. The method converts the integral into a bracket series and prescribes a concise set of rules for its evaluation. In this work we present a ten-variable monstrous integral that combines Bessel functions, polylogarithms, and the exponential integral, and we evaluate it exactly by means of the method of brackets. This example illustrates the flexibility and power of the technique.

Keywords Method of brackets · Integrals · Bessel functions · Polylogarithms · Exponential integrals

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✉ Victor H. Moll
vhm@tulane.edu

Ivan Gonzalez
ivan.gonzalez@uv.cl

John Lopez Santander
jlopez12@tulane.edu

Daniel Salinas-Arizmendi
daniel.salinas@usm.cl

1 Instituto de Física y Astronomia, Universidad de Valparaiso, Valparaiso, Chile

2 Department of Mathematics, Tulane University, New Orleans, LA 70118, USA

3 Departamento de Física, Universidad Técnica Federico Santa Maria, Casilla 110-V, Valparaiso, Chile

4 Centro Cientifico-Tecnologico de Valparaiso, Valparaiso, Chile

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1 Introduction

Since the inception of integral calculus, the evaluation of definite integrals has remained an active field of research, an observation that may surprise the broader mathematical community, though certainly not the readers of this journal.

For some time now, the first and third author (and former students) have developed a new method to evaluate definite integrals (over products of half-lines) for a large class of integrands. It consists of a small number of rules (discussed in Sect. 4), some of which remain at a heuristic level. (Establishing these rules rigorously is work in progress. For instance, Rule E1 below is exactly the Ramanujan Master Theorem —see [1] for a rigorous discussion). The aforementioned method has its origin in the so-called `method of negative dimension` [2–9] used in the evaluation of integrals coming from Feynman diagrams. An introduction to these diagrams appears in [10] and [11]. The central object is the divergent integral

$$\langle a \rangle = \int_0^\infty x^{a-1}\, dx, \tag{1.1}$$

called the `bracket` of $a \in \mathbb{R}$. The method of brackets associates to the integral of a function over a product of half-lines $[0, \infty)$ a series of brackets. The operational properties of the method are described in Sects. 3, 4. In view of this, we have labeled this `the method of brackets`. The examples discussed in the literature lead to the following statement:

There is a large class of functions defined on $\mathbb{R}^+ = [0, \infty)$ *for which the evaluation of the integral over* $\mathbb{R}^+$ *reduces to the solution of a linear system of equations, usually of very small size.*

The goal of this note is to present one example using this method. The reader will find in [12–20] many more such evaluations. The integrand chosen here is rather complicated: it is a 10-dimensional integral involving some elementary functions (exponentials and trigonometric), a couple of Bessel functions (K_0, J_0), a polylogarithm function and the exponential integral. The arguments of these functions are rational functions of the variables of integration $\{x_j,\ 1 \leq j \leq 10\}$. The `monstrous integral` discussed here has simply been chosen to illustrate the power of the method brackets.

`Problem`: evaluate the integral

$$\begin{aligned} I := \int_0^\infty \cdots \int_0^\infty dx_1 \cdots dx_{10} \\ x_1^\beta K_0\left(\sqrt{\frac{x_2x_7x_8}{x_1x_3x_4x_{10}}}\right) J_0\left(\sqrt{\frac{x_2x_5x_7}{x_1x_3x_6x_8}}\right) \mathtt{Li}_4\left(-\frac{x_5x_7x_9}{x_3x_{10}}\right) \\ \times \arctan\left(\frac{x_2x_9}{x_3x_{10}}\right) \cos\left(\sqrt{\frac{x_1x_3x_5}{x_6x_{10}}}\right) \mathtt{Ei}\left(-\frac{x_2x_4x_7}{x_8x_9}\right) \end{aligned}$$

divided by

$$\exp\left(\frac{x_2x_5x_6}{x_1x_3x_8} + \frac{3x_6(x_7+x_8)}{x_1x_4x_5}\right)\left[1 + \frac{x_2x_3x_4x_5x_6x_{10}}{x_1x_7x_8x_9}\right]^\beta, \tag{1.2}$$

as a function of the single parameter β. The integrand does not appear from any physical context (say, the reduction of a single Feynman diagram).

The literature contains a large variety of methods to evaluate definite integrals. For the class of integrals arising from Feynman diagrams, the reader will find complete information in [21] and [22]. A comparison with the existing methods to evaluate definite integrals coming from Feynman diagrams and the method of brackets appears in [7]. The advantage of the method of brackets is that it is a general purpose integration method; it applies to a large class of integrands.

The structure of the paper is as follows: Sect. 2 presents series expansions of each of the parts of the integrand in (1.2). These expansions will be central to the application of the method of brackets. Section 3 introduces the concept of a bracket and it shows how the evaluation of a definite integral is transformed into the evaluation of a bracket series. Section 4 describes all the operational rules of the method of brackets. A couple of simple examples are presented here to illustrate the procedure. Section 5 presents the bracket series for the monstrous integral. Finally, we state our conclusions in Sect. 6.

2 The expansion of the components of the integrand

The method of brackets applies to functions that can be expanded in a formal power series

$$f(x) = \sum_{n=0}^{\infty} a(n) x^{\alpha n + \beta - 1}, \tag{2.1}$$

where $\alpha, \beta \in \mathbb{R}$ and the coefficients $a(n) \in \mathbb{R}$. (The extra -1 in the exponent is for a convenient formulation of the operational rules). The adjective *formal* refers to the fact that the expansion is used to integrate over $[0, \infty)$, even though it might be valid only on a proper subset of the half-line.

The integrand in (1.2) contains several special functions. The goal of this section is to present some series expansions for them. Some of them are convergent series. Some other are *non-classical series representations* for functions f, which do not have expansions like (2.1). These representations are formally of the type (2.1) but some of the coefficients $a(n)$ might be null or divergent. These non-classical series are classified according to the following types:

1) `Totally (partially) divergent series`. Each term (some of the terms) in the series is a divergent value. For example,

$$\sum_{n=0}^{\infty} \Gamma(-n) x^n \text{ and } \sum_{n=0}^{\infty} \frac{\Gamma(n-3)}{n!} x^n. \tag{2.2}$$

2) `Totally (partially) null series`. Each term (some of the terms) in the series vanishes. For example,

$$\sum_{n=0}^{\infty} \frac{1}{\Gamma(-n)} x^n \text{ and } \sum_{n=0}^{\infty} \frac{1}{\Gamma(3-n)} x^n. \tag{2.3}$$

This type includes series where all but finitely many terms vanish. These are polynomials in the corresponding variable.

3) `Formally divergent series`. This is a classical divergent series: the terms are finite but the sum of the series diverges. For example,

$$\sum_{n=0}^{\infty} \frac{n!^2}{(n+1)\,(2n)!} 5^n. \tag{2.4}$$

The method of brackets makes use of these expansions to provide an evaluation of the monstrous integral. The reader will find in [23] some more elementary examples using these ideas.

In the application of the method, it is convenient to introduce the symbol

$$\phi_k = \frac{(-1)^k}{k!} \tag{2.5}$$

called the `indicator` of the index k.

- `Cosine`: this is an elementary function, with power series

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}. \tag{2.6}$$

The series converges for all $x \in \mathbb{R}$. In terms of the indicator, this series can be rewritten as

$$\cos x = \sum_{k=0}^{\infty} \phi_k \frac{k!}{(2k)!} x^{2k}. \tag{2.7}$$

- `The arctangent`: this is also an elementary function, with power series

$$\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}, \tag{2.8}$$

and in terms of an indicator, this is written as

$$\arctan x = \sum_{k=0}^{\infty} \phi_k \frac{k!}{2k+1} x^{2k+1}. \tag{2.9}$$

This series converges for $|x| < 1$.

- The `exponential` and the `power function` appearing in the denominator are also elementary and we use their classical expansions

$$e^{-x} = \sum_{k=0}^{\infty} \phi_k x^k, \tag{2.10}$$

which converges for all $x \in \mathbb{R}$ and

$$(1+x)^{-\beta} = \sum_{k=0}^{\infty} \phi_k \frac{\Gamma(k+\beta)}{\Gamma(\beta)} x^k, \tag{2.11}$$

converging for $|x| < 1$.

- `The Bessel function` $J_\nu(x)$ is one of the standard classical functions, defined as the solution of the differential equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \nu^2) = 0, \tag{2.12}$$

which has asymptotic behavior

$$\left(\frac{x}{2}\right)^\nu \left[\frac{1}{\Gamma(\nu+1)} + \mathcal{O}(x^2)\right].$$

The example considered here is the special case $\nu = 0$. The series expansion

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k} \tag{2.13}$$

comes directly from (2.12). The indicator appears directly in this expansion, so we can write

$$J_\nu(x) = \sum_{k=0}^{\infty} \phi_k \frac{1}{\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k+\nu}. \tag{2.14}$$

In particular,

$$J_0(x) = \sum_{k=0}^{\infty} \phi_k \frac{1}{\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k}. \tag{2.15}$$

For $\nu \in \mathbb{Z}$ the function $J_\nu(x)$ is an entire function, otherwise it is a multi-valued function with singularities at the origin.

- `The Bessel function` $K_0(x)$. The modified Bessel equation

$$z^2 \frac{d^2 y}{dz^2} + z \frac{dy}{dz} - (z^2 + \nu^2) = 0, \tag{2.16}$$

comes from (2.12) by replacing x by $\pm iz$. A standard first solution is given by the power series

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}z^2\right)^k}{k!\Gamma(\nu+k+1)}. \tag{2.17}$$

A second solution, denoted by $K_\nu(z)$, is obtained by imposing the asymptotic behavior

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z}, \tag{2.18}$$

as $z \to \infty$ (with the phase bounded by $\frac{3}{2}\pi - \delta < \frac{3}{2}\pi$) or by the expression

$$K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \pi \nu} \tag{2.19}$$

when ν is not integer. For ν integer, a limiting version of (2.19) is used.

The function K_ν, for $\nu = n \in \mathbb{N}$, admits the expansion

$$\begin{aligned} K_n(x) = {} & \frac{1}{2} \left(\frac{x}{2} \right)^{-n} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(-\frac{x}{2} \right)^{2k} + (-1)^{n+1} I_n(z) \ln \left(\frac{x}{2} \right) \\ & + \frac{(-1)^n}{2} \left(\frac{x}{2} \right)^n \sum_{k=0}^{\infty} \left[\psi(k+1) + \psi(k+n+1) \right] \frac{\left(\frac{x}{2} \right)^{2k}}{k!\,(n+k)!}, \end{aligned} \tag{2.20}$$

showing that $K_n(x)$ has a logarithmic singularity at $x = 0$.

A process to produce series representations of a function f, to which the method of brackets is applicable, has been expanded in [24]. The starting point is the Mellin transform of f. Example 4.3 shows how to produce the expansion of $J_\nu(x)$ from the Mellin transform

$$\int_0^\infty x^{s-1} J_\nu(x)\, dx = 2^{s-1} \frac{\Gamma\left(\frac{\nu+s}{2} \right)}{\Gamma\left(\frac{2+\nu+s}{2} \right)} \tag{2.21}$$

appearing as entry 6.541.14 in [25]. The Mellin transform of $K_\nu(x)$ is entry 6.561.16:

$$\int_0^\infty x^{s-1} K_\nu(x)\, dx = 2^{s-2} \Gamma\left(\frac{s+\nu}{2} \right) \Gamma\left(\frac{s-\nu}{2} \right) \tag{2.22}$$

gives the representation

$$K_\nu(x) = \sum_{k=0}^{\infty} \phi_k \frac{\Gamma(-k-\nu)}{2^{2k+\nu+1}} x^{2k+\nu}. \tag{2.23}$$

In particular, for $\nu = 0$, it follows that

$$K_0(x) = \sum_{k=0}^{\infty} \phi_k \frac{\Gamma(-k)}{2} \left(\frac{x}{2} \right)^{2k}. \tag{2.24}$$

Euler's reflection formula for the gamma function $\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x$ shows that the gamma function has a singularity at the non-positive integers. Therefore (2.24) is a totally divergent series, consistent with (2.20). Despite the divergence of each summand, the authors have used this representation combined with the method of brackets to evaluate a series of examples of definite integrals involving K_ν. See Section 7 of [23] for details.

• The polylogarithm function $\text{Li}_s(z)$ is defined by the power series

$$\text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} = \sum_{k=1}^{\infty} \phi_k \frac{k!}{k^s} (-z)^k. \tag{2.25}$$

This is a generalization of $\text{Li}_1(z) = -\ln(1-z)$. The series is valid for $|z| < 1$ and it can be extended for $|z| \geq 1$ by analytic continuation. The name of the function comes from the fact that it may also be defined by an iterated integral

$$\text{Li}_{s+1}(z) = \int_0^z \frac{\text{Li}_s(t)}{t} \, dt. \tag{2.26}$$

In the case $s = -n$ a negative integer, this is a rational function with an expansion involving the Stirling numbers of the second kind. The special case $s = 4$ appears in the integral (1.2), with the expansion

$$\text{Li}_4(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^4}. \tag{2.27}$$

• The final function appearing in (1.2) is the exponential integral function defined by the integral representation

$$\text{Ei}(x) = -\int_{-x}^{\infty} \frac{\exp(-t)}{t} \, dt, \quad \text{for } x > 0. \tag{2.28}$$

and for $x < 0$ the integral has to be interpreted as a Cauchy principal value. The Risch algorithm shows that Ei is not an elementary function.

The method of brackets was used in [23] to produce the divergent series representation[1]

$$\text{Ei}(-x) = \sum_{n=0}^{\infty} \phi_n \frac{x^n}{n} \tag{2.29}$$

and evaluate with it formulas such as

$$\int_0^{\infty} \text{Ei}(-x) J_0(2\sqrt{zx}) \, dx = \frac{e^{-z} - 1}{z}, \tag{2.30}$$

appearing as 6.782.1 in [25].

This provides series expansions for all the functions appearing in the integrand of (1.2).

[1] This is a partially divergent series representation, where the zeroth summand is a divergent value for $x \neq 0$.

3 The method of brackets

The method of brackets is a procedure to evaluate definite integrals of the form

$$I(f) = \int_0^\infty f(x)\,dx \tag{3.1}$$

for a class of functions satisfying some natural restrictions. The method consists of a small number of `heuristic rules`.

An initial class of functions for which the method has been applied are those admitting an expansion of the form

$$f(x) = \sum_{n=0}^{\infty} C_n x^{\alpha n+\beta-1} \tag{3.2}$$

with some $\alpha,\ \beta \in \mathbb{R}_n, C_n \in \mathbb{C}$. (The additional -1 in the exponent is included for later notational convenience). The class $\mathcal{F}$ of functions for which the method applies is not determined yet.

As given in the introduction, the basic concept is the definition of the `bracket` by the divergent integral

$$\langle b \rangle = \int_0^\infty x^{b-1}\,dx, \quad b \in \mathbb{R}, \tag{3.3}$$

and using linearity (and ignoring issues of convergence) it follows that

$$I = \int_0^\infty f(x)\,dx = \int_0^\infty \sum_{n=0}^{\infty} C_n x^{\alpha n+\beta-1} = \sum_{n=0}^{\infty} C_n \langle \alpha n + \beta \rangle, \tag{3.4}$$

where the expression on the right is called a `bracket series`.

The method consists of a sequence of rules to produce and evaluate such series. These are presented in the next section.

4 Rules for the method of brackets

The method of brackets is based on the method of negative dimension. For background, the reader will find in [26] a detailed calculation of an integral corresponding to the two-loop massless sunset diagram illustrating the method of negative dimension and in [27] an introduction to the method of brackets and its applications.

The expression (3.4) shows, at a formal level, that the evaluation of the definite integral I is given by a bracket series. The goal of this section is to enumerate a small list of rules to evaluate this series. For reasons that will become immediately clear, the symbol C_n is written as $C(n)$.

`Rule` E_1. The bracket series in (3.4) is assigned a number via

$$\sum_n \phi_n C(n) \langle \alpha n + \beta \rangle \mapsto \frac{1}{|\alpha|} C(n^*) \Gamma(-n^*), \tag{4.1}$$

where n^* is the unique solution of the equation $\alpha n + \beta = 0$; that is, $n^* = -\beta/\alpha$.

Remark 4.1 In order to illustrate the power of the method of brackets, we present the evaluation of an elementary integral. (A simple change of variables reduces this example to the definition of the gamma function). Let

$$I = \int_0^\infty x^2 e^{-2x^{5/3}}\, dx. \tag{4.2}$$

Use the power series for the exponential to obtain the expansion

$$x^2 e^{-2x^{5/3}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} 2^n x^{5n/3+2} = \sum_{n=0}^{\infty} \phi_n 2^n x^{5n/3+2}. \tag{4.3}$$

Then $C(n) = 2^n$, $\alpha = \frac{5}{3}$ and $\beta = 2$. The integral is associated the bracket series

$$\int_0^\infty x^2 e^{-2x^{5/3}}\, dx \mapsto \sum_n \phi_n 2^n \langle \tfrac{5}{3} n + 3 \rangle. \tag{4.4}$$

Solving the linear equation $\frac{5}{3} n + 3 = 0$ yields $n^* = -\frac{9}{5}$ and Rule $E1$ now gives

$$\int_0^\infty x^2 e^{-2x^{5/3}}\, dx \mapsto \frac{3}{5} 2^{-9/5} \Gamma\left(\tfrac{9}{5}\right), \tag{4.5}$$

the correct value.

Remark 4.2 An important point is that in Rule $E1$, the value of the bracket series is given by $C(n^*)$. Initially the function C is defined only at the positive integers and since, in general $n^* \notin \mathbb{N}$, an extension of C is needed for the application of this rule.

In all the bracket series arising from integration one obtains multi-dimensional sums where the number of sums is at least the number of brackets. The difference is called the `index of the representation`, or simply the `index of the sum`; that is,

$$\text{index of a sum} = \text{number of sums} - \text{number of brackets}.$$

For example, the bracket series in Rule E_1 has index 0.

The second rule is about the evaluation of a multi-dimensional bracket series of index 0.

`Rule` E_2. The multi-dimensional bracket series of index 0, with the notation $\phi_{n_1\cdots n_k} = \phi_{n_1}\cdots\phi_{n_k}$,

$$\sum_{n_1,\cdots,n_k} \phi_{n_1\cdots n_k} C(n_1,\cdots,n_k)\langle\alpha_{11}n_1+\cdots+\alpha_{1k}n_k+\beta_1\rangle\cdots\langle\alpha_{k1}n_1+\cdots+\alpha_{kk}n_k+\beta_k\rangle,$$

is assigned the value

$$\frac{1}{|\det(A)|}C(n_1^*,\cdots,n_k^*)\prod_{j=1}^{k}\Gamma(-n_j^*). \tag{4.6}$$

Here A is the $k\times k$ matrix with entries α_{ij} and $(n_1^*,\ldots,n_k^*)$ is the unique solution of the linear system $A\vec{n}+\vec{\beta}=0$, indicating the vanishing of the brackets. The method does not apply if the matrix A is not invertible.

The next rule deals with the situation of a brackets series of positive index.

`Rule` E_3. Assume a bracket series has k series and b brackets, with positive index $i=k-b$. Among the k indices n_j, every choice of i of them produces a brackets series with $k-i=b$ sums and b brackets. This gives a collection of bracket series of index zero. These will be called `basis series`.

The value of the integral is obtained by summing the basis series which converge in a common region.

Example 4.3 The rule above is illustrated with the evaluation of the elementary integral

$$I=\int_0^\infty e^{-ax}\sin(bx)\,dx=\frac{b}{a^2+b^2},\quad\text{with } a>0. \tag{4.7}$$

Start with the expansions

$$e^{-ax}=\sum_{n_1=0}^{\infty}\frac{(-1)^{n_1}}{n_1!}a^{n_1}x^{n_1}=\sum_{n_1=0}^{\infty}\phi_{n_1}a^{n_1}x^{n_1} \tag{4.8}$$

and

$$\sin(bx)=\sum_{n_2=0}^{\infty}\frac{(-1)^{n_2}}{(2n_2+1)!}b^{2n_2+1}x^{2n_2+1}=\sum_{n_1=0}^{\infty}\phi_{n_2}\frac{\Gamma(n_2+1)}{\Gamma(2n_2+2)}b^{2n_2+1}x^{2n_2+1}. \tag{4.9}$$

Then

$$\begin{aligned} I &= \int_0^\infty\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\phi_{n_1n_2}\frac{\Gamma(n_2+1)}{\Gamma(2n_2+2)}a^{n_1}b^{2n_2+1}x^{n_1+2n_2+1}\,dx \\ &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\phi_{n_1n_2}\frac{\Gamma(n_2+1)}{\Gamma(2n_2+2)}a^{n_1}b^{2n_2+1}\langle n_1+2n_2+2\rangle, \end{aligned} \tag{4.10}$$

is a bracket series representation for the integral. The index is 1. The integral is now obtained by choosing a selection of free indices.
`Case 1`. Choose n_1 as the free parameter. Then $n_2 = -\frac{n_1}{2} - 1$ and (4.10) evaluates to

$$S_1 = \frac{1}{2}\sum_{n_1=0}^{\infty} \phi_{n_1} \frac{\Gamma(-n_1/2)}{\Gamma(-n_1)} a^{n_1} b^{-n_1-1} \Gamma(\tfrac{1}{2}n_1 + 1) \tag{4.11}$$

$$= \frac{1}{2b}\sum_{n_1=0}^{\infty} \frac{(-1)^{n_1}\Gamma(-n_1/2)\Gamma(n_1/2+1)}{\Gamma(-n_1)\Gamma(n_1+1)} \left(\frac{a}{b}\right)^{n_1}. \tag{4.12}$$

The quotient $\Gamma(-n_1/2)/\Gamma(-n_1)$ has been left in the sum, even though it might be singular. It is clear that the terms with n_1 odd vanish. Now write $n_1 = 2k$ to obtain

$$S_1 = \frac{1}{2b}\sum_{k=0}^{\infty} \frac{\Gamma(-k)\Gamma(k+1)}{\Gamma(-2k)\Gamma(2k+1)} \left(\frac{a^2}{b^2}\right)^{k}. \tag{4.13}$$

The terms $\Gamma(-2k)$ and $\Gamma(2k+1)$ are transformed using the duplication formula for the gamma function to obtain

$$S_1 = \frac{1}{2b}\sum_{k=0}^{\infty} \frac{2^{1+2k}\sqrt{\pi}\,\Gamma(k+1)2^{-2k}\sqrt{\pi}}{\Gamma(-k+1/2)\Gamma(k+1/2)\Gamma(k+1)} \left(\frac{a^2}{b^2}\right)^{k}. \tag{4.14}$$

The reflection formula for the gamma function

$$\Gamma(-k+1/2)\Gamma(k+1/2) = \frac{\pi}{\sin \pi(k+1/2)} = (-1)^k \pi, \tag{4.15}$$

then yields

$$S_1 = \frac{1}{b}\sum_{k=0}^{\infty} \left(-\frac{a^2}{b^2}\right)^{k} = \frac{b}{a^2+b^2} \tag{4.16}$$

provided $|a| < |b|$.
n_2 `as a free index`. Then $n_1 = -2n_2 - 2$ and the bracket series evaluates to

$$S_2 = \sum_{n_2=0}^{\infty} \phi_{n_2} \frac{\Gamma(n_2+1)}{\Gamma(2n_2+2)} a^{-2n_2-2} b^{2n_2+1} \Gamma(2n_2+2) = \frac{b}{a^2}\sum_{n_2=0}^{\infty} (-1)^{n_2} \frac{b^{2n_2}}{a^{2n_2}} = \frac{b}{a^2+b^2}, \tag{4.17}$$

provided $|b| < |a|$. The sums S_1 and S_2 give the same value, namely $b/(a^2+b^2)$, but the regions of convergence are distinct. The value of the integral for missing values $a = \pm b$ is obtained by continuity.

5 The bracket series for the monstrous integral

The expansions of the functions appearing in the integrand of (1.2) are now used to produce a bracket series for the integral.

(1) The factor K_0 has the expansion

$$K_0(M_1) = \sum_{n_1=0}^{\infty} \phi_{n_1} \frac{\Gamma(-n_1)}{2} \left(\frac{M_1}{2} \right)^{2n_1}, \text{ with } M_1 = \sqrt{\frac{x_2 x_7 x_8}{x_1 x_3 x_4 x_{10}}}. \tag{5.1}$$

(2) The factor involving J_0 has the expansion

$$J_0(M_2) = \sum_{n_2=0}^{\infty} \frac{\phi_{n_2}}{n_2!} \left(\frac{M_2}{2} \right)^{2n_2}, \text{ with } M_2 = \sqrt{\frac{x_2 x_5 x_7}{x_1 x_3 x_6 x_8}}. \tag{5.2}$$

(3) The polylogarithm $\mathtt{Li}_4$ is given by the expansion

$$\mathtt{Li}_4(-M_3) = -\sum_{n_3=0}^{\infty} \phi_{n_3} \frac{n_3!}{(n_3+1)^4} M_3^{n_3+1}, \text{ with } M_3 = \frac{x_5 x_7 x_9}{x_3 x_{10}}. \tag{5.3}$$

(4) The arctangent function is given by the series

$$\arctan(M_4) = \sum_{n_4=0}^{\infty} \phi_{n_4} \frac{n_4!}{2n_4+1} M_4^{2n_4+1}, \text{ with } M_4 = \frac{x_2 x_9}{x_3 x_{10}}. \tag{5.4}$$

(5) The cosine term is given by its power series

$$\cos(M_5) = \sum_{n_5=0}^{\infty} \phi_{n_5} \frac{n_5!}{(2n_5)!} M_5^{2n_5}, \quad \text{with } M_5 = \sqrt{\frac{x_1 x_3 x_5}{x_6 x_{10}}}. \tag{5.5}$$

(6) The exponential integral is given by its (divergent) series representation

$$\mathtt{Ei}(-M_6) = \sum_{n_6=0}^{\infty} \phi_{n_6} \frac{M_6^{n_6}}{n_6}, \quad \text{with } M_6 = \frac{x_2 x_4 x_7}{x_8 x_9}. \tag{5.6}$$

(7) The exponential term is given by the product of three series

$$\exp(-M_7 - 3M_8 - 3M_9) = \sum_{n_7=0}^{\infty} \sum_{n_8=0}^{\infty} \sum_{n_9=0}^{\infty} \phi_{n_7} \phi_{n_8} \phi_{n_9} 3^{n_8+n_9} M_7^{n_7} M_8^{n_8} M_9^{n_9} \tag{5.7}$$

with

$$M_7 = \frac{x_2 x_5 x_6}{x_1 x_3 x_8}, \quad M_8 = \frac{x_6 x_7}{x_1 x_4 x_5}, \quad M_9 = \frac{x_6 x_8}{x_1 x_4 x_5}. \tag{5.8}$$

(8) Finally, the binomial term is given by

$$(1+M_{10})^{-\beta} = \sum_{n_{10}=0}^{\infty} \phi_{n_{10}} \frac{\Gamma(n_{10}+\beta)}{\Gamma(\beta)} M_{10}^{n_{10}}, \text{ with } \beta \in \mathbb{R} \text{ and } M_{10} = \frac{x_2x_3x_4x_5x_6x_{10}}{x_1x_7x_8x_9}. \tag{5.9}$$

Replacing these expansions in (1.2) it follows that the monstrous integral is given by

$$I = \sum_{n_1,\ldots,n_{10}=0}^{\infty} \phi_{n_1\cdots n_{10}} C_{n_1,\cdots,n_{10}} \int_0^\infty \cdots \int_0^\infty P(x_1,\ldots,x_{10}) dx_1 \cdots dx_{10} \tag{5.10}$$

where

$$C_{n_1,\cdots,n_{10}} = \frac{-3^{n_8+n_9} n_3! n_4! n_5! \Gamma(-n_1)\Gamma(n_{10}+\beta)}{2^{2n_1+2n_2+1} n_2!(2n_5)!(n_3+1)^4(2n_4+1)n_6\Gamma(\beta)} \tag{5.11}$$

and

$$P(x_1,\ldots,x_{10}) = x_1^{\beta} M_1^{2n_1} M_2^{2n_2} M_3^{n_3+1} M_4^{2n_4+1} M_5^{2n_5} M_6^{n_6} M_7^{n_7} M_8^{n_8} M_9^{n_9} M_{10}^{n_{10}}$$

(also recall the multi-index notation $\phi_{n_1\cdots n_{10}} = \phi_{n_1}\cdots\phi_{n_{10}}$).

Writing the expressions M_j, $1 \le j \le 10$, in detail it gives

$$\begin{aligned} P(x_1,\ldots,x_{10}) = {} & x_1^{-n_1-n_2+n_5-n_7-n_8-n_9-n_{10}+\beta} x_2^{n_1+n_2+2n_4+n_6+n_7+n_{10}+1} \\ & \times x_3^{-n_1-n_2-n_3-2n_4+n_5-n_7+n_{10}-2} x_4^{-n_1+n_6-n_8-n_9+n_{10}} \\ & \times x_5^{n_2+n_3+n_5+n_7-n_8-n_9+n_{10}+1} x_6^{-n_2-n_5+n_7+n_8+n_9+n_{10}} \\ & \times x_7^{n_1+n_2+n_3+n_6+n_8-n_{10}+1} x_8^{n_1-n_2-n_6-n_7+n_9-n_{10}} \\ & \times x_9^{n_3+2n_4-n_6-n_{10}+2} x_{10}^{-n_1-n_3-2n_4-n_5+n_{10}-2}. \end{aligned} \tag{5.12}$$

It follows that

$$\begin{aligned} I = {} & \sum_{n_1,\ldots,n_{10}=0}^{\infty} \phi_{n_1,\ldots,n_{10}} C(n_1,\ldots,n_{10}) \\ & \times \int x_1^{-n_1-n_2+n_5-n_7-n_8-n_9-n_{10}+\beta} \, dx_1 \\ & \cdots \int x_{10}^{-n_1-n_3-2n_4-n_5+n_{10}-2} \, dx_{10}. \end{aligned} \tag{5.13}$$

Each of the integrals appearing above is now converted into a bracket to obtain an expression for the monstrous integral as a bracket series

$$I = \sum_{n_1,\ldots,n_{10}=0}^{\infty} \phi_{n_1,\ldots,n_{10}} C_{n_1,\ldots,n_{10}} B(n_1,\ldots,n_{10}) \tag{5.14}$$

where B is obtained as the product of all the brackets, one per integral:

$$\begin{aligned} B = & \langle -n_1 - n_2 + n_5 - n_7 - n_8 - n_9 - n_{10} + \beta + 1\rangle \, \langle n_1 + n_2 + 2n_4 + n_6 + n_7 + n_{10} + 2\rangle \\ & \times \langle -n_1 - n_2 - n_3 - 2n_4 + n_5 - n_7 + n_{10} - 1\rangle \, \langle -n_1 + n_6 - n_8 - n_9 + n_{10} + 1\rangle \\ & \times \langle n_2 + n_3 + n_5 + n_7 - n_8 - n_9 + n_{10} + 2\rangle \, \langle -n_2 - n_5 + n_7 + n_8 + n_9 + n_{10} + 1\rangle \\ & \times \langle n_1 + n_2 + n_3 + n_6 + n_8 - n_{10} + 2\rangle \, \langle n_1 - n_2 - n_6 - n_7 + n_9 - n_{10} + 1\rangle \\ & \times \langle n_3 + 2n_4 - n_6 - n_{10} + 3\rangle \, \langle -n_1 - n_3 - 2n_4 - n_5 + n_{10} - 1\rangle. \end{aligned} \tag{5.15}$$

There are 10 series and the same number of brackets, so this is a problem of index 0. A linear system is now formed by the vanishing of the brackets. This is $A \cdot \vec{n} = \vec{b}$ and in detail

$$\begin{aligned} A \cdot \vec{n} = & \begin{pmatrix} -1 & -1 & 0 & 0 & 1 & 0 & -1 & -1 & -1 & -1 \\ 1 & 1 & 0 & 2 & 0 & 1 & 1 & 0 & 0 & 1 \\ -1 & -1 & -1 & -2 & 1 & 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & -1 & -1 & 1 \\ 0 & -1 & 0 & 0 & -1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 2 & 0 & -1 & 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & -2 & -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \\ n_7 \\ n_8 \\ n_9 \\ n_{10} \end{pmatrix} \\ = & \begin{pmatrix} -\beta - 1 \\ -2 \\ 1 \\ -1 \\ -2 \\ -1 \\ -2 \\ -1 \\ -3 \\ 1 \end{pmatrix}. \end{aligned} \tag{5.16}$$

The determinant of the matrix A is $\det A = -16$, so system has a unique solution. It is given by

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \\ n_7 \\ n_8 \\ n_9 \\ n_{10} \end{pmatrix} = \begin{pmatrix} -18-7\beta/2 \\ 10+9\beta/4 \\ 1+\beta \\ -3/2-\beta/8 \\ 9+3\beta/2 \\ 11+2\beta \\ 8+3\beta/4 \\ -16-3\beta \\ 36+29\beta/4 \\ -10-5\beta/4 \end{pmatrix}. \tag{5.17}$$

Using (4.6) we obtain the value of the integral as

$$I = \frac{2^{11+5\beta/2}3^{20+17\beta/4}\pi^3}{\sin(\pi\beta)\sin\left(\frac{3\pi\beta}{2}\right)\cos\left(\frac{\pi\beta}{8}\right)}\frac{\Gamma\mathrm{Num}(\beta)}{\Gamma\mathrm{Den}(\beta)} \tag{5.18}$$

where

$$\begin{aligned}\Gamma\mathrm{Num}(\beta) = {} & \Gamma\left(-36-\tfrac{29}{4}\beta\right)\Gamma\left(-10-\tfrac{9}{4}\beta\right)\Gamma\left(-11-2\beta\right)\Gamma\left(-8-\tfrac{3}{4}\beta\right) \\ & \times\Gamma\left(-10-\tfrac{1}{4}\beta\right)\Gamma\left(10+\tfrac{5}{4}\beta\right)\Gamma\left(16+3\beta\right)\Gamma\left(18+\tfrac{7}{2}\beta\right)^2\end{aligned}$$

and

$$\Gamma\mathrm{Den}(\beta) = (\beta+2)^4(2\beta+11)\Gamma(19+3\beta)\Gamma\left(-1-\frac{\beta}{4}\right)\Gamma(\beta)\Gamma\left(11+\frac{9\beta}{4}\right).$$

Finally, we have obtained the value of the integral (1.2). The process is reviewed next: Starting from the individual expansions, the multidimensional integral was converted into a bracket series of index 0, and the associated linear system was subsequently solved. The final expression (5.18) is written in terms of gamma functions, trigonometric functions, and the parameter β. The result is meromorphic in β; the only simple poles arise at the zeros of the trigonometric prefactor and at the negative integers indicated by the gamma functions in the denominator, so the formula is valid for all $\beta \in \mathbb{R}$ outside this discrete set.

6 Conclusions

An exact evaluation has been presented for a highly intricate ten-variable integral that combines Bessel functions, polylogarithms, the exponential integral, and elementary functions. By expanding each factor into a bracket series and applying Rule E_1, the multidimensional integral was reduced to a linear system of ten equations, yielding a closed form expression in terms of powers and gamma functions that depends solely on the parameter β. This result clearly demonstrates the efficiency and power of the method of brackets in converting high dimensional integrals into tractable algebraic problems, thereby fulfilling the primary objective of the study.

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Declarations

Competing interests The authors declare no competing interests.

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