Group Contractions and Physical Applications

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Abstract

The paper is devoted to the description of the contraction (or limit transition) method in application to classical Lie groups and Lie algebras of orthogonal and unitary series. As opposed to the standard Wigner-Inönü contractions based on insertion of one or several zero tending parameters in group (algebra) structure the alternative approach, which is connected with consideration of algebraic structures over Pimenov algebra with commutative nilpotent generators is used. The definitions of orthogonal and unitary Cayley–Klein groups are given. It is shown that the basic algebraic constructions, characterizing Cayley–Klein groups can be found using simple transformations from the corresponding constructions for classical groups. The theorem on the classifications of transitions is proved, which shows that all Cayley–Klein groups can be obtained not only from simple classical groups. As a starting point one can choose any pseudogroup as well. As applications of the developed approach to physics the kinematics groups and contractions of the Electroweak Model at the level of classical gauge fields are regarded. The interpretations of kinematics as spaces of constant curvature are given. Two possible contractions of the Electroweak Model are discussed and are interpreted as zero and infinite energy limits of the modified Electroweak Model with the contracted gauge group.

1 Introduction

Group-Theoretical Methods are essential part of modern theoretical and mathematical physics. It is enough to remind that the most advanced theory of fundamental interactions, namely Standard Electroweak Model, is a gauge theory with gauge group $SU(2) \times U(1)$. All types of classical groups of infinite series: orthogonal, unitary and symplectic as well as inhomogeneous groups, which are semidirect products of their subgroups, are used in different areas of physics. Euclidean, Lobachevsky, Galilei, Lorentz, Poincaré, (anti) de Sitter groups are the bases for space and space-time symmetries. Supergroups and supersymmetric models in the field theory predict the existence of new supersymmetric partners of known elementary particles. Quantum deformations of Lie groups and Lie algebras lead to non-commutative space-time models (or kinematics).

Contractions of Lie groups is the method for receiving new Lie groups from the initial ones. In the standard E. Wigner and E. Inönü approach [34] continuous parameter ϵ is introduced in such a way that in the limit $\epsilon \to 0$ group operation is changed but Lie group structure and its dimension are conserved. It is well known that studying non-degenerate structures is easier then the degenerate ones. So one represent a general Lie group as semidirect product of semisimple and solvable groups and reduce the problem of Lie groups classification to the classifications of semisimple and solvable groups. But, while the classification of semisimple groups was established long ago there is no hope to find the classification of solvable groups [42]. In general, a contracted group is a semidirect product of its subgroups. In particular, a contraction of semisimple groups gives non-semisimple ones. Therefore, the contraction method is a tool for studying of non-semisimple groups starting from the well known semisimple (or simple) Lie groups.

The method of contractions (limit transitions) was extended later to other types of groups and algebras. Graded contractions [43, 44] additionally conserve grading of Lie algebra. Lie bialgebra contractions [3] conserve both Lie algebra structure and cocommutator. Contractions of Hopf algebras (or quantum groups) are introduced in such a way [8, 9] that in the limit $\epsilon \to 0$ new expressions for coproduct, counit and antipode appear which satisfies Hopf algebra axioms. All this gives rise to the following generalization of the notion of group contraction on contraction of algebraic structures [25].

Definition. Contraction of algebraic structure $(M, *)$ is the map ϕ_ϵ dependent on parameter ϵ

$$\psi_\epsilon : (M, *) \to (N, *'), \tag{1}$$

where $(N, *')$ is an algebraic structure of the same type, which is isomorphic $(M, *)$ when $\epsilon \neq 0$ and non-isomorphic when $\epsilon = 0$.

There is another approach [23] to the description of non-semisimple Lie groups (algebras) and corresponding quantum groups based on their consideration over Pimenov algebra $P_n(\iota)$ with nilpotent commutative generators. In this approach the motion groups of constant curvature spaces (or Cayley–Klein groups) are realized as matrix groups of special form over $P_n(\iota)$ and can be obtained from the simple classical orthogonal group by substitution of its matrix elements for Pimenov algebra elements. It turns out that such substitution coincides with the introduction of Wigner–Inönü contraction parameter ϵ [34]. So our approach demonstrates that the existence of the corresponding structures over algebra $P_n(\iota)$ is the mathematical base of the contraction method.

It should be noted that both approaches supplement each other and in the final analysis give the same results. Nilpotent generators are more suitable in the mathematical consideration of contractions whereas the contraction parameter continuously tending to zero more corresponds to physical intuition according to which a physical system continuously changes its state and smoothly goes into its limit state.

It is well known in geometry (see, for example, review [58]) that there are 3^n different geometries of dimension n, which admit the motion group of maximal order. R.I. Pimenov suggested [48, 51] a unified axiomatic description of all 3^n geometries of constant curvature (or Cayley–Klein geometries) and demonstrated that all these geometries can be locally simulated in some region of n-dimension spherical space with named coordinates, which can be real, imaginary and nilpotent ones. According to Erlanger programm by F. Klein the main content in geometry is its motion group whereas the properties of transforming objects are secondary. The motion group of n-dimensional spherical space is isomorphic to the orthogonal group $SO(n+1)$. In their turn the groups obtained from $SO(n+1)$ by contractions and analytical continuations are isomorphic to the motion groups of Cayley–Klein spaces. This correspondence provides the geometrical interpretation of Cayley–Klein contraction scheme. By analogy this interpretation is transferred to the contractions of other algebraic structures.

The method for achieving this goal is the method of transitions, which has clear geometrical meaning, and is based on the introduction of a set of contraction parameters $j = (j_1, \dots, j_n)$, each of them taking three values: a real unit, an imaginary unit and a nilpotent unit.

The method of transitions between groups apart from being of interest for group theory itself is of interest for theoretical physics too. If there is a group-theoretical description of a physical system then the contraction of its symmetry group corresponds to some limit case of the system under consideration. So the reformulation of the system description in terms of the transition method and the subsequent physical interpretations of contraction parameters j gives an opportunity to study different limit behaviours of the physical system. An example of such approach is given for the Electroweak Model of elementary particle interactions.

It is likely that developed formalism is an essential tool to construct "general theory of physical systems" according to which "it is necessary to turn from group-invariant study of a single physical theory in Klein understanding (i.e. characterized by symmetry group) to a simultaneous study of a set of limit theories. Then some physical and geometrical properties will be the invariant properties of all set of theories and they should be considered in the first place. Other properties will be relevant only for the particular representatives and will be changed under limit transition from one theory to another" [59].

2 Dual Numbers and Pimenov Algebra

2.1 Dual numbers

Dual numbers were introduced by W.K. Clifford [10] as far back as in the XIX century. They were used by A.P. Kotel'nikov [39] for constructing his theory of screws in three-dimensional spaces of Euclid, Lobachevsky and Riemann, by B.A. Rosenfeld [53, 54], for description of non-Euclidean spaces, by R.I. Pimenov [48, 49, 51] for axiomatic study of spaces with a constant curvature. Some applications of dual numbers in kinematics can be found in the work by I.M. Yaglom [57]. The applications of dual numbers in geometry and in theory of group representations were discussed by V.V. Kisil [36]. Fine distinctions between the quantum and classical mechanics were investigated with the help of dual numbers [37, 38]. The theory of dual numbers as number systems is exposed in monographs by D.N. Zeiliger [60] and A.Sh. Bloch [6]. Nevertheless, it is impossible to say that dual numbers are well-known, so we start with their description.

Definition. By the associative algebra of rank n over the real numbers field $\mathbf{R}$ we mean n-dimensional vector space over this field, on which the operation of multiplication is defined, associative $a(bc) = (ab)c$, distributive in respect to addition $(a+b)c = ac + bc$

and related with the multiplication of elements by real numbers as follows $(ka)b = k(ab) = a(kb)$, where a, b, c are the elements of algebra; k is a real number. If there is such element e of algebra that for any element a of algebra the relations $ae = ea = a$ are valid, then the element e is called a unit.

Definition. Dual numbers $a = a_0e_0 + a_1e_1$; $a_0, a_1 \in \mathbf{R}$ are the elements of associative algebra of rank 2 with the unit and the generators satisfying the following conditions: $e_0^2 = e_0,\ e_0e_1 = e_1e_0,\ e_1^2 = 0$.

This associative algebra is commutative and e_0 is its unit. Therefore, further we shall write 1 instead of e_0 and denote generator e_1 by ι_1 (the Greek letter "*iota*") and call it a (purely) dual unit.

For a sum, a product and a quotient of dual numbers a and b we have

$$a + b = (a_0 + \iota_1 a_1) + (b_0 + \iota_1 b_1) = a_0 + b_0 + \iota_1(a_1 + b_1),$$

$$ab = (a_0 + \iota_1 a_1)(b_0 + \iota_1 b_1) = a_0b_0 + \iota_1(a_1b_0 + a_0b_1),$$

$$\frac{a}{b} = \frac{a_0 + \iota_1 a_1}{b_0 + \iota_1 b_1} = \frac{a_0}{b_0} + \iota_1\left(\frac{a_1}{b_0} - a_0\frac{b_1}{b_0^2}\right). \tag{2}$$

Division can not always be carried out. Purely dual numbers $a_1\iota_1$ do not have an inverse element. Therefore dual numbers do not form a number field. As an algebraic structure they perform a ring. Dual numbers are equal $a = b$, if their real parts are equal $a_0 = b_0$ and their purely dual parts are equal $a_1 = b_1$. Thus, the equation $a_1\iota_1 = b_1\iota_1$ has the unique solution $a_1 = b_1$ for $a_1, b_1 \neq 0$. This fact can be written formally as $\iota_1/\iota_1 = 1$ and this is how the last relation has to be interpreted, because division $1/\iota_1$ is not defined.

Functions of dual variable $x = x_0 + \iota_1 x_1$ are defined by their Taylor expansion

$$f(x) = f(x_0) + \iota_1 x_1 \frac{\partial f(x_0)}{\partial x_0}, \tag{3}$$

where all terms with coefficients $\iota_1^2, \iota_1^3, \ldots$ are omitted. In particular, for dual x we have

$$\sin x = \sin x_0 + \iota_1 x_1 \cos x_0, \quad \sin(\iota_1 x_1) = \iota_1 x_1,$$

$$\cos x = \cos x_0 - \iota_1 x_1 \sin x_0, \quad \cos(\iota_1 x_1) = 1. \tag{4}$$

According to (3), the difference of two functions of dual variable can be presented as

$$f(x) - h(x) = f(x_0) - h(x_0) + \iota_1 x_1\left(\frac{\partial f(x_0)}{\partial x_0} - \frac{\partial h(x_0)}{\partial x_0}\right), \tag{5}$$

therefore, if real parts $f(x_0)$ and $h(x_0)$ of functions coincide, then functions $f(x)$ and $h(x)$ also coincide. Using this fact, D.N. Zeiliger shows [60] that in the domain of dual numbers all identities of algebra and trigonometry, all theorems of differential and integral calculus remain valid. In particular, the derivative of a function of a dual variable over a dual variable can be found as

$$\frac{df(x)}{dx} = \frac{\partial f(x_0)}{\partial x_0} + \iota_1 x_1\left(\frac{\partial^2 f(x_0)}{\partial x_0^2}\right). \tag{6}$$

2.2 Pimenov algebra

Let us consider now a more general situation, where several nilpotent units are taken as generators of associative algebra with a unit. (Further on we will use the name *nilpotent unit* instead of *dual unit*). R.I. Pimenov was the first who introduced [48, 49, 51] several nilpotent commutative units and used them for the unified axiomatic description of spaces with constant curvature. Therefore we name such algebra as a Pimenov algebra and denote it as $\mathbf{P}_n(\iota)$.

Definition. Pimenov algebra $\mathbf{P}_n(\iota)$ is an associative algebra with a unit and n nilpotent generators $\iota_1, \iota_2, \ldots, \iota_n$ with properties: $\iota_k\iota_p = \iota_p\iota_k \neq 0, \ k \neq p, \ \iota_k^2 = 0, \ p, k = 1, 2, \ldots, n$.

Any element of $\mathbf{P}_n(\iota)$ is a linear combination of monomials $\iota_{k_1}\iota_{k_2}\ldots\iota_{k_r}$, $k_1 < k_2 < \ldots < k_r$, which together with a unit element make a basis in algebra as in a linear space of dimension 2^n:

$$a = a_0 + \sum_{r=1}^{n} \sum_{k_1,\ldots,k_r=1}^{n} a_{k_1\ldots k_r}\iota_{k_1}\ldots\iota_{k_r}. \qquad (7)$$

This notation becomes unique, if we put an additional requirement $k_1 < k_2 < \ldots < k_r$ or condition of symmetry of coefficients $a_{k_1\ldots k_r}$ in respect to indices $k_1, \ldots k_r$. Two elements a, b of algebra $\mathbf{P}_n(\iota)$ coincide, if their coefficients in the expansion (7) are equal, i.e. $a_0 = b_0$, $a_{k_1\ldots k_r} = b_{k_1\ldots k_r}$. As in the case of dual numbers, this definition of equality of the elements of algebra $\mathbf{P}_n(\iota)$ is expressed in the possibility of cancellation of equal (with the same index) nilpotent units $\iota_k/\iota_k = 1, \ k = 1, 2, \ldots, n$ (but not ι_k/ι_m or $\iota_m/\iota_k, \ k \neq m$, as far as such expressions are not defined).

Here it is appropriate to compare Pimenov algebra $\mathbf{P}_n(\iota)$ with Grassmann algebra $\Gamma_{2n}(\epsilon)$, i.e. associative algebra with a unit, where a set of nilpotent generators $\epsilon_1, \epsilon_2, \ldots, \epsilon_{2n}, \ \epsilon$ 0 exhibits the properties of anticommutativity $\epsilon_k\epsilon_p = -\epsilon_p\epsilon_k \neq 0, \ p \neq k, \ p, k = 1, \ldots, 2n$. Any element f of Grassmann algebra $\Gamma_{2n}(\epsilon)$ can be expressed [5] as

$$f(\epsilon) = f(0) + \sum_{r=1}^{2n} \sum_{k_1,\ldots,k_r=1}^{2n} f_{k_1\ldots k_r}\epsilon_{k_1}\ldots\epsilon_{k_1}. \qquad (8)$$

The representation is unique, if one requires $k_1 < k_2 < \ldots < k_r$ or puts on condition of skew-symmetry $f_{k_1\ldots k_r}$ in respect to indices $k_1, \ldots, k_r$. If in the expansion (8) only the terms with an even r differ from zero, then the element f is called even in respect to the set of canonical generators ϵ_k, if in the expansion (8) only the terms with an odd r differ from zero, then f is called an odd element. As a linear space, Grassmann algebra splits into even Γ_{2n}^0 and odd Γ_{2n}' subspaces: $\Gamma_{2n}(\epsilon) = \Gamma_{2n}^0 + \Gamma_{2n}^1$, where Γ_{2n}^0 is not only subspace, but also a subalgebra.

Let us consider nonzero products $\epsilon_{2k-1}\epsilon_{2k}$, $k = 1, 2, \ldots, n$ of the generators of Grassmann algebra $\Gamma_{2n}(\epsilon)$. It is easy to see that these products possess the same properties as generators $\iota_k = \epsilon_{2k-1}\epsilon_{2k}$, $k = 1, 2, \ldots, n$. Thus Pimenov algebra $\mathbf{P}_n(\iota)$ is a subalgebra of the even part Γ_{2n}^0 of Grassmann algebra $\Gamma_{2n}(\epsilon)$. It is worth mentioning that even products of Grassmannian anticommuting generators are also called para-Grassmannian variables. The latter are employed for classical and quantum descriptions of massive and massless particlies with an integer spin [11, 14, 15] and in theory of strings [61].

3 Cayley–Klein Orthogonal Groups and Algebras

3.1 Three fundamental geometries on a line

Let us introduce elliptic geometry on a line. Let us consider a circle $\mathbf{S}_1^* = \{x_0^{*2} + x_1^{*2} = 1\}$ on the Euclid plane $\mathbf{R}_2$. The rotations $x'^* = g(\varphi^*)x^*$, i.e.

$$x_0^{*'} = x_0^* \cos\varphi^* - x_1^* \sin\varphi^*,$$
$$x_1^{*'} = x_0^* \sin\varphi^* + x_1^* \cos\varphi^* \tag{9}$$

of group $SO(2)$ bring the circle into itself. Let us identify diametrically the opposite points of the circle and introduce an internal coordinate $w^* = x_1^*/x_0^*$. Then the following transformations correspond to the rotations (9) in $\mathbf{R}_2$ for $\varphi^* \in (-\pi/2, \pi/2)$:

$$w^{*'} = \frac{w^* - a^*}{1 + w^* a^*}, \quad a^* = \tan\varphi^*, \quad a^* \in \mathbf{R}. \tag{10}$$

These transformations make a group of translations (motions) G_1 of an elliptic line with the rule of composition

$$a^{*'} = \frac{a^* + a_1^*}{1 - a^* a_1^*}. \tag{11}$$

Let us consider the representation of the group $SO(2)$ in the space of differentiable functions on $\mathbf{R}_2$, defined by the left shifts

$$T(g(\varphi^*))f(x^*) = f(g^{-1}(\varphi^*)x^*). \tag{12}$$

The generator of the representation

$$X^* f(x^*) = \frac{d(T(g(\varphi^*))f(x^*))}{d\varphi^*}|_{\varphi^*=0}, \tag{13}$$

corresponding to the transformation (9), can be easily found:

$$X^*(x_0^*, x_1^*) = x_1^* \frac{\partial}{\partial x_0^*} - x_0^* \frac{\partial}{\partial x_1^*}. \tag{14}$$

For the representation of group G_1 by the left shifts in space of differentiable functions on elliptic line the generator Z^*, corresponding to the transformation (10), can be written as

$$Z^*(w^*) = (1 + w^{*2}) \frac{\partial}{\partial w^*}. \tag{15}$$

It is worth mentioning that matrix generator

$$Y^* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{16}$$

corresponds to rotations $g(\varphi^*) \in SO(2)$.

The transformation of Euclidean plane $\mathbf{R}_2$, consisting of multiplication of Cartesian coordinate x_1 by parameter j_1, namely

$$\phi : \mathbf{R}_2 \to \mathbf{R}_2(j_1)$$

$$\phi x_0^* = x_0, \quad \phi x_1^* = j_1 x_1, \tag{17}$$

where $j_1 = 1, \iota_1, i$, brings $\mathbf{R}_2$ into plane $\mathbf{R}_2(j_1)$; the geometry of the latter is defined by metrics $x^2(j_1) = x_0^2 + j_1^2 x_1^2$. It is easy to see that $R_2(j_1 = i)$ is Minkowski plane and $R_2(j_1 = \iota_1)$ is Galilean plane.

Our main idea is that the transformation of geometries (17) induces the transformation of the corresponding motion groups and their algebras. Let us show how to derive these transformations.

The definition of angle measure in Euclidean plane R_2 is determined by the ratio x_1^*/x_0^*, which under the transformation (17) turns into $j_1 x_1/x_0$, i.e. angles are transformed according to the rule $\phi\varphi^* = j_1\varphi$. The asterisk marks the initial quantities (coordinates, angles, generators and so on). The transformed quantities are denoted by the same symbols without asterisk. Changing the coordinates in (9) according to (17) and the angles according to the derived transformation rule and multiplying both sides of the second equation by j_1^{-1}, we get the rotations in the plane $\mathbf{R}_2(j_1)$:

$$\begin{aligned} x_0' &= x_0 \cos j_1\varphi - x_1 j_1 \sin j_1\varphi, \\ x_1' &= x_0 \tfrac{1}{j_1} \sin j_1\varphi + x_1 \cos j_1\varphi, \end{aligned} \tag{18}$$

which make group $SO(2; j_1)$. According to (4), $\cos \iota_1\varphi = 1$, $\sin \iota_1\varphi = \iota_1\varphi$, therefore the transformations of group $SO(2; \iota_1)$ are Galilean transformations and the elements of group $SO(2; i)$ are Lorentz transformations, if x_0 is interpreted as time, and x_1 as a spatial coordinate. The domain of definition $\Phi(j_1)$ of the group parameter φ is $\Phi(1) = (-\pi/2, \pi/2)$, $\Phi(\iota_1) = \Phi(i) = \mathbf{R}$.

The rotations (18) preserve the circle $\mathbf{S}_1(j_1) = \{x_0^2 + j_1^2 x_1^2 = 1\}$ (Fig. 1) in the plane $\mathbf{R}_2(j_1)$. the identification of diametrically opposite points gives the upper semicircle (for $j_1 = 1$) and the connected component of the sphere (circle), passing through the point $(x_0 = 1, x_1 = 0)$, for $j_1 = \iota_1, i$. The internal coordinate on the circle w^* is transformed according to the rule $\phi w^* = j_1 w$. Substituting in (10) and canceling j_1 out of both sides we get the formula for translations on a line:

$$w' = \frac{w - a}{1 + j_1^2 wa}, \quad a = \frac{1}{j_1} \tan j_1\varphi \in R, \tag{19}$$

which make group $G_1(j_1)$, i.e. the group of motions of the elliptic line $\mathbf{S}_1(1)$ for $j_1 = 1$, the parabolic line $\mathbf{S}_1(\iota_1)$ for $j_1 = \iota_1$, and the hyperbolic line $\mathbf{S}_1(i)$ for $j_1 = i$.

In the space of differentiable functions on $\mathbf{R}_2(j_1)$ the generator $X(x)$ of the representation of group $SO(2; j_1)$ is defined by the relation (13), where all quantities are taken without asterisks. Under the transformation (17) derivative $d/d\varphi^*$ turns $j_1^{-1}(d/d\varphi)$, therefore, to obtain derivative $d/d\varphi$ the generator X^* must be multiplied by j_1, i.e. the generators $X^*(\phi x)$ and $X(x)$ are interrelated by the transformation

$$X(x) = j_1 X^*(\phi x^*) = j_1^2 x_1 \frac{\partial}{\partial x_0} - x_0 \frac{\partial}{\partial x_1}. \tag{20}$$

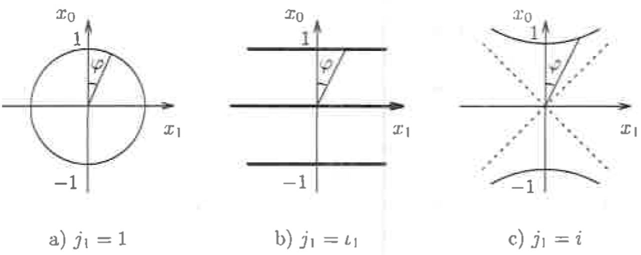

Figure 1: The circles of unit radius on the planes $\mathbf{R}_2(j_1)$

The generator Z is transformed according to the same rule:

$$Z(w) = j_1 Z^*(\phi w^*) = (1 + j_1^2 w^2)\frac{\partial}{\partial w}. \tag{21}$$

The transformation rule for the matrix generator of the rotation Y is as follows:

$$Y = j_1 Y^*(\to) = j_1 \begin{pmatrix} 0 & -j_1 \\ j_1^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -j_1^2 \\ 1 & 0 \end{pmatrix}. \tag{22}$$

Expressions (18)–(22) describe Cayley–Klein space and group in the traditional way with the help of real coordinates, generators and so on. Such approach was used in [23]. There is another way of describing Cayley–Klein spaces with the help of the named (i.e. having one of names: real, nilpotent, imaginary) coordinates of the form $j_1 x_1$, when under transformation (17) and the substitution $\phi\varphi^* = j_1\varphi$ in (9) both sides of the second equation are not multiplied by j_1^{-1}. Then the rotations on the plane $\mathbf{R}_2(j_1)$ with coordinates x_0, $j_1 x_1$ are written in the form

$$\begin{pmatrix} x_0' \\ j_1 x_1' \end{pmatrix} = \begin{pmatrix} \cos j_1\varphi & -\sin j_1\varphi \\ \sin j_1\varphi & \cos j_1\varphi \end{pmatrix} \begin{pmatrix} x_0 \\ j_1 x_1 \end{pmatrix}. \tag{23}$$

These rotations form group $SO(2; j_1)$, whose matrix generator is as follows

$$Y = j_1 Y^* = \begin{pmatrix} 0 & -j_1 \\ j_1 & 0 \end{pmatrix}. \tag{24}$$

The symbol Y^* instead of $Y^*(\to)$ in (22) means that the generator Y^* (16) is not transformed. It is the the second approach that we shall use in this book. One of its advantages is that for $j_1 = \iota_1$ the rotation matrix (23) from group $SO(2; \iota_1)$

$$\begin{pmatrix} 1 & -\iota_1\varphi \\ \iota_1\varphi & 1 \end{pmatrix}, \tag{25}$$

depend on group parameter φ, whereas for $j_1 \to 0$ it is equal to the unit matrix.

The group of motions $G_1(j_1)$ of one-dimensional Cayley–Klein space $\mathbf{S}_1(j_1)$ is closely connected with rotation group $SO(2; j_1)$ in space $\mathbf{R}_2(j_1)$. Therefore, under Cayley–Klein space we shall further mean both $\mathbf{S}_1(j_1)$ and $\mathbf{R}_2(j_1)$, and under their groups of motion — both $G_1(j_1)$ and $SO(2;j_1)$. We shall follow the same rule in the case of higher dimensions.

We have studied comprehensively the simplest case of groups $SO(2;j_1)$, $G_1(j_1)$ because here the main ideas of methods of transitions reveal themselves in the most clear way, not aggravated with mathematical calculations. These ideas are as follows: (a) to define the transformation (17) from Euclidean space to arbitrary Cayley–Klein space; (b) to find the rules of transformations of motion,generators etc. of the group; (c) using the approach exposed in (b), to derive motion,generators etc. of Cayley–Klein group from the corresponding quantities of classical orthogonal group. The method of transitions, in spite of its simplicity, enables us to describe all Cayley–Klein groups, being aware of only classical orthogonal ones.

3.2 Nine Cayley–Klein groups

Mapping

$$\phi : \mathbf{R}_3 \to \mathbf{R}_3(j)$$

$$\phi x_0^* = x_0, \quad \phi x_1^* = j_1 x_1, \quad \phi x_2^* = j_1 j_2 x_2, \tag{26}$$

where $j = (j_1, j_2)$, $j_1 = 1, \iota_1, i$, $j_2 = 1, \iota_2, i$, turns three-dimensional Euclidean space into spaces $\mathbf{R}_3(j)$, on the spheres (or connected components of spheres) of which

$$\mathbf{S}_2(j) = \{x_0^2 + j_1^2 x_1^2 + j_1^2 j_2^2 x_2^2 = 1\} \tag{27}$$

nine geometries of Cayley–Klein planes are realized. The interrelation of the geometries and values of parameters j is clear from Fig. 2.

Rotation angle φ_{rs} in the coordinate plane $\{x_r, x_s\}$, $r < s$, $r, s = 0, 1, 2$, is determined by the ratio x_s/x_r and under the mapping (26) is transformed as $\varphi_{rs}^* \to \varphi_{rs}(r,s)$, where

$$(i,k) = \prod_{l=\min(i,k)+1}^{\max i,k} j_l, \quad (k,k) = 1. \tag{28}$$

Therefore for one-parametric rotations in the plane $\{x_r, x_s\}$ of space $\mathbf{R}_3(j)$ the following relations are valid

$$(0,r)x_r' = x_r(0,r)\cos\left(\varphi_{rs}(r,s)\right) - x_s(0,s)\sin\left(\varphi_{rs}(r,s)\right),$$

$$(0,s)x_s' = x_r(0,r)\sin(\varphi_{rs}(r,s)) + x_s(0,s)\cos(\varphi_{rs}(r,s)). \tag{29}$$

The rest of the coordinates is not changed $x_p' = x_p$, $p \neq r, s$.

It is easy to find the matrix generators of the rotations (29)

$$Y_{01} = j_1 Y_{01}^* = \begin{pmatrix} 0 & -j_1 & 0 \\ j_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; Y_{12} = j_2 Y_{12}^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -j_2 \\ 0 & j_2 & 0 \end{pmatrix},$$

$$Y_{02} = j_1 j_2 Y_{02}^* = \begin{pmatrix} 0 & 0 & -j_1 j_2 \\ 0 & 0 & 0 \\ j_1 j_2 & 0 & 0 \end{pmatrix}. \tag{30}$$

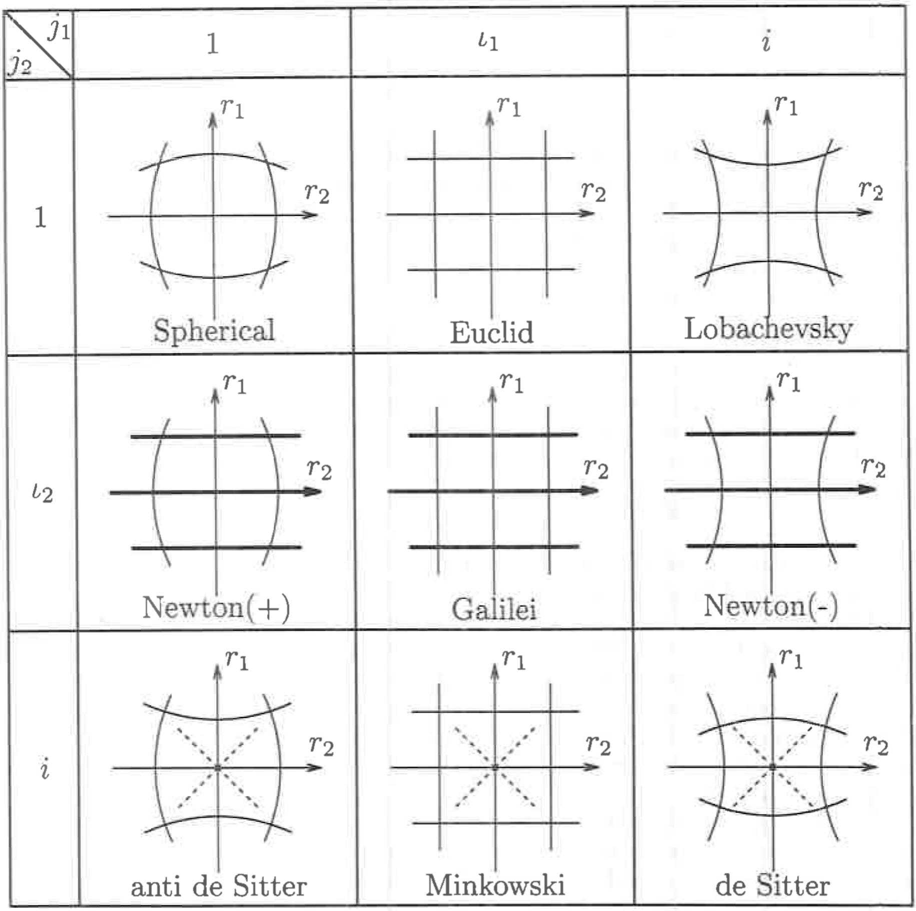

Figure 2: Cayley–Klein planes. The fibers are shown by thick lines and the light cone in $(1+1)$ kinematics are shown by dashed lines. Internal coordinates take values $r_1 = x_1/x_0$, $r_2 = x_2/x_0$

They make a basis of Lie algebra $so(3;j)$. The rule of transformations for the generators of representation of group $SO(3;j)$ in the space of differentiable functions on $\mathbf{R}_3(j)$ by left shifts coincides with the rule of transformations for parameters φ_{rs} and can be written as follows [18, 20]:

$$X_{rs}(x) = (r,s)X^*_{rs}(\phi x^*), \tag{31}$$

and the generators themselves as

$$X_{rs}(x) = (r,s)^2 x_s \frac{\partial}{\partial x_r} - x_r \frac{\partial}{\partial x_s}. \tag{32}$$

Knowing the generators, one can evaluate their commutators. But we shall derive the commutators from the commutation relations of algebra $so(3)$. Let us introduce new notations for the generators $X^*_{01} = H^*$, $X^*_{02} = P^*$, $X^*_{12} = K^*$. As it is well-known, the commutators of Lie algebra $so(3)$ can be written as follows:

$$[H^*, P^*] = K^*, \quad [P^*, K^*] = H^*, \quad [H^*, K^*] = -P^*. \tag{33}$$

Generators of algebra $so(3)$ are transformed according to the rule $H = j_1 H^*, P = j_1 j_2 P^*$, $K = j_2 K^*$, i.e. $H^* = j_1^{-1} H$, $P^* = j_1^{-1} j_2^{-1} P$, $K^* = j_2^{-1} K$. Substituting these expressions in (33) and multiplying each commutator by a factor, equal to the denominator on the left side of each equation, i.e. the first — by $j_1^2 j_2$, the second — by $j_1 j_2^2$, and the third — by $j_1 j_2$, we get commutators of algebra Lie for group $SO(3;j)$:

$$[H, P] = j_1^2 K, \quad [P, K] = j_2^2 H, \quad [H, K] = -P. \tag{34}$$

Cayley–Klein spaces $\mathbf{S}_2(j)$ (or spaces of constant curvature) for $j_1 = 1, \iota_1, i$, $j_2 = \iota_2, i$ can serve as models of kinematics, i.e. space–time geometries. In this case internal coordinate $t = x_1/x_0$ can be interpreted as the temporal axis, and internal coordinate $r = x_2/x_0$ as the spatial one. Then H is the generator of the temporal shift, P is the generator of the spatial shift, and K is the generator of Galilean transformation for $j_2 = \iota_2$ or Lorentz transformation for $j_2 = i$. The semispherical group $SO(3; 1, \iota_2)$ (or Newton group) is isomorphic to the cylindrical group, which describes movement of a point on a cylindrical surface. This group is interpreted as the $E(2)$-like little group for massless particles [35].

The final relations should not involve division by a nilpotent number. This requirement suggests the way of finding the rule of transformations for algebraic constructions. Let an algebraic quantity $Q^* = Q^*(A_1^*, \ldots, A_k^*)$ be expressed in terms of quantities $A_1^*, \ldots, A_k^*$ with a known rule of transformation under mapping ϕ, for example, $A_1 = J_1 A_1^*, \ldots A_k = J_k A_k^*$, where coefficients $J_1, \ldots, J_k$ are some products of parameters j. Substituting $A_1^* = J_1^{-1} A_1, \ldots, A_k^* = J_k^{-1} A_k$ in the relation for Q^*, we get the formula $Q^*(J_1^{-1} A_1, \ldots, J_k^{-1} A_k)$, involving, in general, indeterminate expressions, when parameters j are equal to the nilpotent units. For this reason the last formula should be multiplied by such minimal coefficient J that the final formula would not involve indeterminate expressions:

$$Q = J Q^*(J_1^{-1} A_1, \ldots, J_k^{-1} A_k). \tag{35}$$

Then (35) is the rule of transformation for quantity Q under mapping ϕ.

Such method, stemmed out directly from the definition of coincidence of elements of Pimenov algebra $\mathbf{P}_n(\iota)$ turns out to be very useful and further will be widely employed. The rule of transformation (35) for algebraic quantity Q, derived from the requirement of absence of indeterminate expressions for nilpotent values of parameters j, is automatically satisfied for imaginary values of these parameters.

Let us exemplify this rule by Casimir operator. The only Casimir operator for algebra $so(3)$ is

$$C_2^*(H^*,\ldots) = H^{*2} + P^{*2} + K^{*2}. \tag{36}$$

Substituting $H^* = j_1^{-1}H$, $P^* = j_1^{-1}j_2^{-1}P$, $K^* = j_2^{-1}K$ in (36), we get

$$C_2^*(j_1^{-1}H,\ldots) = j_1^{-2}H^2 + j_1^{-2}j_2^{-2}P^2 + j_2^{-2}K^2. \tag{37}$$

The most singular factor for $j_1 = \iota_1$ and $j_2 = \iota_2$ is coefficient $(j_1j_2)^{-2}$ of the term P^2. Multiplying both sides of the equation (37) by $(j_1j_2)^2$, we get rid of the indeterminate expressions and derive the rule of transformation and Casimir operator for algebra $so(3;j)$:

$$C_2(j;H,\ldots) = j_1^2j_2^2C_2^*(j_1^{-1}H,\ldots) = j_2^2H^2 + P^2 + j_1^2K^2. \tag{38}$$

As it is known, Casimir operator for two dimensional Galilean algebra $so(3;\iota_1,\iota_2)$ is $C_2(\iota_1,\iota_2) = P^2$ (see, for example, [40]), for Poincaré algebra $so(3;\iota_1,i)$ is $C(\iota_1,i) = P^2 - H^2$, for algebra $so(3;i;1) = so(2,1)$ is $C_2(i,1) = H^2 + P^2 - K^2$ (see [45]). All these Casimir operators can be obtained from (38) for the corresponding values of parameters j.

The matrix generators (30) make the basis of fundamental representation of Lie algebra $so(3;j)$ of group $SO(3;j)$. Using exponential mapping one can put in correspondence to the general element

$$Y(\mathbf{r};j) = r_1Y_{01} + r_2Y_{02} + r_3Y_{12} = \begin{pmatrix} 0 & -j_1r_1 & -j_1j_2r_2 \\ j_1r_1 & 0 & -j_2r_3 \\ j_1j_2r_2 & j_2r_3 & 0 \end{pmatrix} \tag{39}$$

of algebra $so(3;j)$ the finite rotation $g(\mathbf{r};j) = \exp Y(\mathbf{r};j)$:

$$g(\mathbf{r};j) = E\cos(r) + Y(\mathbf{r};j)\frac{\sin r}{r} + Y'(\mathbf{r},j)\frac{1-\cos r}{r^2},$$

$$Y'(\mathbf{r};j) = \begin{pmatrix} j_2^2r_3^2 & -j_1j_2^2r_2r_3 & j_1j_2r_1r_3 \\ -j_1j_2^2r_2r_3 & j_1^2j_2^2r_2^2 & -j_1^2j_2r_1r_2 \\ j_1j_2r_1r_3 & -j_1^2j_2r_1r_2 & j_1^2r_1^2 \end{pmatrix},$$

$$r^2 = j_1^2r_1^2 + j_1^2j_2^2r_2^2 + j_2^2r_3^2, \tag{40}$$

acting on vector $(x_0, j_1x_1, j_1j_2x_2)^t \in \mathbf{R_3(j)}$ with the named components.

The disadvantage of the general parametrization (39), (40) is the complexity of the composition rule for parameters $\mathbf{r}$ under group multiplication. F.I. Fedorov [12] has proposed parametrization of rotation group $SO(3)$ for which the group composition law is especially simple. It turns out that it is possible to construct analogues of such parametrization for all groups $SO(3;j)$ [21]. The matrix of the finite rotations of group $SO(3;j)$ can be written as follows

$$g(\mathbf{c};j) = \frac{1+c^*(j)}{1-c^*(j)} = 1 + 2\frac{c^*(j)+c^{*2}(j)}{1+c^2(j)},$$

$$c^*(j) = \begin{pmatrix} 0 & -j_1^2 c_3 & j_1^2 j_2^2 c_2 \\ c_3 & 0 & -j_2^2 c_1 \\ -c_2 & c_1 & 0 \end{pmatrix},$$

$$c^2(j) = j_2^2 c_1^2 + j_1^2 j_2^2 c_2^2 + j_1^2 c_3^2, \tag{41}$$

and parameters $\mathbf{c}''$ correspond to matrix $g(\mathbf{c}''; j) = g(\mathbf{c}; j) g(\mathbf{c}'; j)$. These parameters can be expressed in terms of $\mathbf{c}$ and $\mathbf{c}'$ as follows

$$\mathbf{c}'' = \frac{\mathbf{c} + \mathbf{c}' + [\mathbf{c}, \mathbf{c}']_j}{1 - (\mathbf{c}, \mathbf{c}')_j}. \tag{42}$$

Here the scalar product of vectors $\mathbf{c}$ and $\mathbf{c}'$ is given by (41), and the vector product is given by

$$[\mathbf{c}, \mathbf{c}']_j = (j_1^2[\mathbf{c}, \mathbf{c}']_1,\ [\mathbf{c}, \mathbf{c}']_2,\ j_2^2[\mathbf{c}, \mathbf{c}']_3), \tag{43}$$

where $[\mathbf{c}, \mathbf{c}']_k$ are components of usual vector product in $\mathbf{R}_3$.

E.P. Wigner and E. Inönü [34] have introduced the operation of contraction (limit transition) of groups, algebras and their representations. Under this operation the generators of the initial group (algebra) undergo transformation, depending on a parameter ϵ, so that for $\epsilon \neq 0$ this transformation is non-singular and for $\epsilon \to 0$ it becomes singular. If the limits of the transformed generators exist for $\epsilon \to 0$, then they are the generators of a new (contracted) group (algebra), non isomorphic to the initial one. It is worth mentioning that the transformation (31) of the generators of algebra $so(3)$ for the nilpotent values of parameters j is Wigner–Inönü contraction. Really, $X^*_{rs}(\phi x^*)$ is the singularly transformed generator of initial algebra $so(3)$, the product (r, s) plays the role of parameter ϵ, tending to zero, and the resulted generators $X_{rs}(\mathbf{x})$ are the generators of the contracted algebra $so(3; j)$.

Comparing the rule of transformation for generators (31) and the expression (39) for a general element of algebra $so(3)$, we find that for the imaginary values of parameters j some of the real group parameters r_k become imaginary, i.e. they are analytically continued from the field of real numbers to the field of complex numbers. In this case orthogonal group $SO(3)$ is transformed into pseudoorthogonal group $SO(p, q)$, $p + q = 3$. When parameters j take nilpotent values, real group parameters r_k become elements of Pimenov algebra $\mathbf{P}(\iota)$ of the special form and we get the contraction of group $SO(3)$. Thus, from the point of view of the group transformation under mapping ϕ, both operations — analytical continuation of groups and contraction of groups different at first sight — have the same nature: the continuation of real group parameters to the complex numbers field or to Pimenov algebra $\mathbf{P}(\iota)$.

3.3 Extension to higher dimensions

Cayley–klein geometries of the dimension n are realized on spheres

$$\mathbf{S}_n(j) = \{(x, x) = x_0^2 + \sum_{k=1}^{n} (0, k)^2 x_k^2 = 1\} \tag{44}$$

in the spaces $\mathbf{R}_{n+1}(j)$ resulting from Euclidean space $\mathbf{R}_{n+1}$ under mapping

$$\phi : \mathbf{R}_{n+1} \to \mathbf{R}_{n+1}(j)$$

$$\phi x_0^* = x_0, \quad \phi x_k^* = (0,k)x_k, \quad k = 1,2,\ldots,n, \tag{45}$$

where $j = (j_1, \ldots, j_n)$, $j_k = 1, \iota_k, i$, $k = 1, 2, \ldots, n$. If all parameters are equal to one $j_k = 1$, then ϕ is identical mapping, if all or some parameters are imaginary $j_k = i$ and the other are equal to 1, then we obtain pseudoeuclidean spaces of different signature. The space $\mathbf{R}_{n+1}(j)$ is called non-fiber, if no of the parameters $j_1, \ldots, j_n$ take nilpotent value.

Definition. The space $\mathbf{R}_{n+1}(j)$ is called $(k_1, k_2, \ldots, k_p)$-fiber space, if $1 \le k_1 < k_2 < \ldots < k_p \le n$ and $j_{k_1} = \iota_{k_1}, \ldots, j_{k_p} = \iota_{k_p}$ and other $j_k = 1, i$.

These fiberings are trivial [7] and can be characterized by a set of consequently nested projections $pr_1, pr_2, \ldots, pr_p$, where for pr_1 the base is the subspace, spanned over the basis vector $\{x_0, x_1, \ldots, x_{k_1-1}\}$, and the fiber is the subspace, spanned over $\{x_{k_1}, x_{k_1+1}, \ldots x_n\}$; for pr_2 the base is the subspace $\{x_{k_1}, x_{k_1+1}, \ldots, x_{k_2-1}\}$, and the fiber is the subspace $\{x_{k_2}, x_{k_1+1}, \ldots, x_n\}$ and so on.

From the mathematical point of view the fibering in the space $\mathbf{R}_{n+1}(j)$ is trivial, i.e. its global and local structures are the same. From the physical point of view the fibering gives an opportunity to model quantities of different physical dimensions. For example, in Galilean space, which is realized on the sphere $\mathbf{S}_4(\iota_1, \iota_2, 1, 1)$, there are time $t = x_1$, $[t] = \text{sec}$ and space $\mathbf{R}_3 = \{x_2, x_3, x_4\}$, $[x_k] = \text{sm}$, $k = 2, 3, 4$ variables.

Definition. Group $SO(n+1; j)$ consists of all the transformations of the space $\mathbf{R}_{n+1}(j)$ with unit determinant, keeping invariant the quadratic form (44).

The totality of all possible values of parameters j gives 3^n different Cayley–Klein spaces $\mathbf{R}_{n+1}(j)$ and geometries $\mathbf{S}_n(j)$. It is customary to identify the spaces (and their group of motions), if their metrics have the same signature, i.e., for example, space $\mathbf{R}_3(1, i)$ with metric $x_0^2 + x_1^2 - x_2^2$ and space $\mathbf{R}_3(i, i)$ with metric $x_0^2 - x_1^2 + x_2^2$. But we have fixed Cartesian coordinate axes in $\mathbf{R}_{n+1}(j)$ ascribing to them fixed numbers, and for this reason in our case spaces $\mathbf{R}_3(1, i)$ and $\mathbf{R}_3(i, i)$ (and, correspondingly groups $SO(3; 1, i)$ and $SO(3; i, i)$) are different. Groups $SO(3; 1, i) \equiv SO(2, 1)$ and $SO(3; i, 1) \equiv SO(1, 2)$ are also considered to be different. This was made for convenience of applications of method being developed.

Really, the application of some mathematical formalism in a concrete science means first of all substantial interpretation of base mathematical constructions. For example, if we interpret in space $\mathbf{R}_4(i, 1, 1)$ with metric $x_0^2 - x_1^2 - x_2^2 - x_3^2$ the first Cartesian coordinate x_0 as the time axis and the other x_1, x_2, x_3 as the space axes, then we get relativistic kinematic (space-time model). In this example the substantial interpretation of coordinates is the numbers of Cartesian coordinate axes: axis number one, axis number two etc.

The rotations in the two-dimensional plane $\{x_r, x_s\}$, the rule of transformation for representation generators and the generators themselves are given, correspondingly, by (29), (31), (32), where $r, s = 0, 1, \ldots, n$, $r < s$. For the non-zero elements of the matrix generators of rotations the following relations are valid: $(Y_{rs})_{sr} = -(Y_{rs})_{rs} = (r, s)$. The commutation relations for Lie algebra $so(n+1; j)$ can be most simply derived from the commutators of algebra $so(n+1)$, as it has been done in section 3.2. The non-zero commutators are

$$[X_{r_1s_1}, X_{r_2s_2}] = \begin{cases} (r_1, s_1)^2 X_{s_1s_2}, & r_1 = r_2,\ s_1 < s_2, \\ (r_2, s_2)^2 X_{r_1r_2}, & r_1 < r_2,\ s_1 = s_2, \\ -X_{r_1s_2}, & r_1 < r_2 = s_1 < s_2. \end{cases} \tag{46}$$

Algebra $so(n+1)$ has $[(n+1)/2]$ independent Casimir operators, where $[x]$ is the integer part of a number x. As it is known [4], for even $n=2k$ Casimir operators are given by

$$\hat{C}^*_{2p}(X^*_{rs}) = \sum_{a_1,\ldots,a_p=0}^{n} X^*_{a_1a_2}X^*_{a_2a_3}\ldots X^*_{a_{2p}a_1}, \tag{47}$$

where $p=1,2,\ldots,k$. For odd $n=2k+1$ the operator

$$C'^*_n(X^*_{rs}) = \sum_{a_1,\ldots,a_n=0}^{n} \epsilon_{a_1a_2\ldots a_n} X^*_{a_1a_2}X^*_{a_3a_4}\ldots X^*_{a_na_{n+1}}, \tag{48}$$

where $\epsilon_{a_1\ldots a_n}$ is a completely antisymmetric unit tensor, must be added to the operators (47).

Casimir operators C^*_{2p} can be defined in another way [13] as a sum of principal minors of order $2p$ for antisymmetric matrix A, composed of generators X^*_{rs}, i.e. $(A)_{rs} = X^*_{rs}, (A)_{sr} = -X^*_{rs}$. To obtain Casimir operators of algebra $so(n+1;j)$ we use the method of section 3.2. We find $X^*_{rs} = (r,s)^{-1}X_{rs}$ from (31) and substitute in (47). The most singular coefficient $(0,n)^{-2p}$ is that of the term $X_{0n}X_{n0}\ldots X_{n0}$ in (47). To eliminate it in the minimal manner we multiply $\hat{C}^*_{2p}$ by $(0,n)^{2p}$. Thus, the rule of transformation for Casimir operators $\hat{C}_{2p}$ is

$$\hat{C}_{2p}(j;X_{rs}) = (0,n)^{2p}\hat{C}^*_{2p}((r,s)^{-1}X_{rs}), \tag{49}$$

and Casimir operators themselves turn out to be

$$\hat{C}_{2p}(j) = \sum_{a_1,\ldots,a_{2p}=0}^{n} (0,n)^{2p}\prod_{v=1}^{2p}(r_v,s_v)^{-1}X_{a_1a_2}\ldots X_{a_{2p}a_1}, \tag{50}$$

where $r_v=\min(a_v,a_{v+1})$, $s_v=\max(a_v,a_{v+1})$, $v=1,2,\ldots,2p-1$, $r_{2p}=\min(a_1,a_{2p})$, $s_{2p}=\max(a_1,a_{2p})$.

For operators C_{2p} and C'_n the expression without singular terms can be obtained, multiplying them by factor q, equal to the least common denominator of coefficients of terms, arising after the substitution of generators X for X^*. This least common denominator can be found by induction [19]. We restrict ourselves with the final expressions for the rule of transformations for these Casimir operators:

$$C_{2p}(j;X_{rs}) = \left(\prod_{m=1}^{p-1} j_m^{2m} j_{n-m+1}^{2m} \prod_{l=p}^{n-p+1} j_l^{2p}\right) C^*_{2p}(X_{rs}(r,s)^{-1}),$$

$$p=1,2,\ldots,k,$$

$$C'_n(j;X_{rs}) = \left(j_{(n+1)/2}^{(n+1)/2}\prod_{m=1}^{(n-1)/2} j_m^m j_{n-m+1}^m\right) C'^*_n(X_{rs}(r,s)^{-1}). \tag{51}$$

Operator $C_{2p}(j)$ (or $C'(j)$) commutes with all generators X_{rs} of algebra $so(n+1;j)$. Really, evaluating zero commutator $[C^*_{2p}, X^*_{rs}]$, we get the same terms with the opposite signs. Under the transformations (31), (49) both terms are multiplied by the same combination of parameters, which is a product of even powers of parameters. Therefore, both terms either change their sign, or vanish, or do not change their sign, but in all cases their sum is equal to zero. Moreover, operators $C_{2p}(j)$ for $p = 1, 2, \ldots, k$ are linearly independent because they consist of the different powers of generators X_{rs}.

The next question to be cleared up is as follows: do $[(n+1)/2]$ Casimir operators (51) exhaust all the invariant operators of algebra $so(n+1; j)$? The answer is given by the following theorem.

Theorem. For any set of values of parameters j the number of invariant operators of algebra $so(n+1; j)$ is $[(n+1)/2]$.

The proof is given in [23]. Thus, all invariant operators of algebra $so(n+1; j)$ are polynomial and are given by (51).

4 Cayley–Klein Unitary Groups and Algebras

4.1 Definitions, generators, commutators

Special unitary groups $SU(n+1; j)$ are connected with complex Cayley–Klein spaces $\mathbf{C}_{n+1}(j)$ which come out from $(n+1)$-dimensional complex space $\mathbf{C}_{n+1}$ under the mapping

$$\phi : \mathbf{C}_{n+1} \to \mathbf{C}_{n+1}(j)$$

$$\phi z_0^* = z_0^*, \quad \phi z_k^* = (0, k) z_k, \quad k = 1, 2, \ldots, n, \tag{52}$$

where $z_0^*, z_k^* \in \mathbf{C}_{n+1}$, $z_0, z_k \in \mathbf{C}_{n+1}(j)$ are complex Cartesian coordinates; $j = (j_1, \ldots, j_n)$, each of parameters j_k takes three values: $j_k = 1, \iota_k, i$. Quadratic form $(z^*, z^*) = \sum_{m=0}^{n} |z_m^*|^2$ of the space $\mathbf{C}_{n+1}$ turns into quadratic form

$$(z, z) = |z_0|^2 + \sum_{k=1}^{n} (0, k)^2 |z_k|^2 \tag{53}$$

of the space $\mathbf{C}_{n+1}(j)$ under the mapping (52). Here $|z_k| = (x_k^2 + y_k^2)^{1/2}$ is absolute value (modulus) of complex number $z_k = x_k + j y_k$, and z is complex vector: $z = (z_0, z_1, \ldots, z_n)$.

Definition of complex fiber space is similar to the real fiber space in section 3.3.

Definition. Group $SU(n+1; j)$ consists of all transformations of space $\mathbf{C}_{n+1}(j)$ with unit determinant, keeping invariant the quadratic form (53).

In the $(k_1, k_2, \ldots, k_p)$-fiber space $\mathbf{C}_{n+1}(j)$ we have $p+1$ quadratic forms, which remains invariant under transformations of group $SU(n+1; j)$. Under transformations of group $SU(n+1; j)$, which do not affect coordinates $z_0, z_1, \ldots, z_{k_s - 1}$, the form

$$(z, z)_{s+1} = \sum_{a=k_s}^{k_{s+1}-1} (k_s, a)^2 |z_a|^2, \tag{54}$$

where $s = 0, 1, \ldots, p$, $k_0 = 0$, remains invariant. For $s = p$ the summation over a goes up to n.

The mapping (52) induces the transition of classical group $SU(n+1)$ into group $SU(n+1; j)$, correspondingly, of algebra $su(n+1)$ into algebra $su(n+1; j)$. All $(n+1)^2 - 1$

generators of algebra $su(n+1)$ are Hermitian matrices. However, because the commutators for Hermitian generators are not symmetric, usually one prefers matrix generators A^*_{km}, $k, m = 0, 1, 2, \ldots, n$ of general linear algebra $gl_{n+1}(R)$, such that $(A^*_{km})_{km} = 1$ and all other matrix elements vanish. (The asterisk means that A^* is a generator of a classical algebra.) The commutators of generators A^* satisfy the following relations

$$[A^*_{km}, A^*_{pq}] = \delta_{mp} A^*_{kq} - \delta_{kq} A^*_{pm}, \tag{55}$$

where δ_{mp} is Kronecker symbol. Independent Hermitian generators of algebra $su(n+1)$ are given by the equations

$$Q^*_{rs} = \frac{i}{2}(A^*_{rs} + A^*_{sr}), \quad L^*_{rs} = \frac{1}{2}(A^*_{sr} - A^*_{rs}),$$

$$P^*_k = \frac{i}{2}(A^*_{k-1,k-1} - A^*_{kk}), \tag{56}$$

where $r = 0, 1, \ldots, n-1,\ s = r+1, r+2, \ldots, n,\ k = 1, 2, \ldots, n$.

Matrix generators A^* are transformed under the mapping (52) as follows:

$$A_{rs}(j) = (r, s) A^*_{rs}, \quad A_{kk}(j) = A^*_{kk}. \tag{57}$$

The commutators of generators $A(j)$ can be easily found [31]:

$$[A_{km}, A_{pq}] = (k, m)(p, q) \left(\delta_{mp} A_{kq} (k, q)^{-1} - \delta_{kp} A_{pm} (m, p)^{-1} \right). \tag{58}$$

Hermitian generators (56) are transformed in the same way under transition from algebra $su(n+1)$ to algebra $su(n+1; j)$. This enables to find matrix generators of algebra $su(n+1; j)$ for the case, when group $SU(n+1; j)$ acts in the space $C_{n+1}(j)$ with named coordinates

$$Q_{rs}(j) = (r, s) Q^*_{rs}, \quad L_{rs}(j) = (r, s) L^*_{rs}, \quad P_k(j) = P^*_k. \tag{59}$$

We do not cite the commutation relations for these generators because they are cumbersome. They can be found, using (58).

Let us cite one more realization of generators for unitary group. If group GL_{n+1} acts via left translations in the space of analytic functions on $\mathbf{C}_{n+1}$, then the generators of its algebra are $X^*_{ab} = z^{*b} \partial^*_a$, where $\partial^*_a = \frac{\partial}{\partial z^{*a}}$. Hermitian generators of algebra $su(n+1)$ can be expressed in terms of X^*_{ab} using (56), in which A^* must be changed for X^*. Under the mapping ψ they are transformed according to the rule

$$Z_{ab} = (a, b) Z^*_{ab}(\psi z^*), \tag{60}$$

where $Z_{ab} = Q_{rs}, L_{rs}, P_k = P_{kk}$. Generators X^*_{ab} are transformed in a similar way, and this gives us

$$X_{kk} = z_k \partial_k,\ X_{sr} = z_r \partial_s,\ X_{rs} = (r, s)^2 z_s \partial_r, \tag{61}$$

where $k = 1, 2, \ldots, n,\ r, s = 0, 1, \ldots, n,\ r < s$.

The matrix generators (59) make a basis of Lie algebra $su(n+1; j)$. To the general element of the algebra

$$Z(\mathbf{u}, \mathbf{v}, \mathbf{w}; j) = \sum_{t=1}^{n(n+1)/2} (u_t Q_t(j) + v_t L_t(j)) + \sum_{k=1}^{n} w_k P_k, \tag{62}$$

where index t is connected with the indices r, s, $r < s$, by relation

$$t = s + r(n-1) - \frac{r(r-1)}{2}, \qquad (63)$$

and the group parameters u_t, v_t, w_k are real, corresponds a finite group transformation of group $SU(n+1;j)$

$$W(\mathbf{u}, \mathbf{v}, \mathbf{w}; j) = \exp\{Z(\mathbf{u}, \mathbf{v}, \mathbf{w}; j)\}. \qquad (64)$$

According to Cayley–Hamilton theorem, matrix W can be algebraically expressed in terms of matrices Z^m, $m = 0, 1, 2, \ldots, n$, but one can to derive it explicitly only for groups $SU(2; j_1)$ and $SU(3; j_1, j_2)$, which will be discussed in the next sections.

4.2 Unitary group $SU(2; j_1)$

The group $SU(2; j_1)$ is the simplest one from unitary Cayley–Klein groups. **Definition.** The set of all transformations of the space $\mathbf{C}_2(j_1)$, leaving invariant the quadratic form $|z_0|^2 + j_1^2|z_1|^2$, make up the special unitary Cayley–Klein group $SU(2; j_1)$.

The group $SU(2; j_1)$ acts on the space $\mathbf{C}_2(j)$

$$z'(j_1) = \begin{pmatrix} z_0' \\ j_1 z_1' \end{pmatrix} = \begin{pmatrix} \alpha & j_1\beta \\ -j_1\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} z_0 \\ j_1 z_1 \end{pmatrix} = u(j_1)z(j_1),$$

$$\det u(j_1) = |\alpha|^2 + j_1^2|\beta|^2 = 1, \quad u(j_1)u^{\dagger}(j_1) = 1. \qquad (65)$$

Here the bar notes complex conjugation. Constructing generators of algebra $su(2; j_1)$ according to (59), we get

$$P_1 = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q_{01} = \frac{i}{2}\begin{pmatrix} 0 & j_1 \\ j_1 & 0 \end{pmatrix}, \quad L_{01} = \frac{1}{2}\begin{pmatrix} 0 & -j_1 \\ j_1 & 0 \end{pmatrix}, \qquad (66)$$

and find commutation relations

$$[P_1, Q_{01}] = L_{01}, \quad [L_{01}, P_1] = Q_{01}, \quad [Q_{01}, L_{01}] = j_1^2 P_1. \qquad (67)$$

The generators (66) for $j_1 = 1$ up to factors coincide with Pauli matrices. It is also worth mentioning that if under contraction $j_1 = \iota_1$ the dimension (number of linearly independent generators) of general linear group $GL(2; j_1)$ (or its algebra) diminishes, because the generator $A_{01}(\iota_1)$ vanishes, then for special unitary groups (algebras) in complex Cayley–Klein spaces the dimension of the groups (algebras) for any (including nilpotent) values of parameters remains unchanged.

One-dimensional subgroup, corresponding to the generators (66), are as follows:

$$g_1(r; j_1) = \exp rQ_{01}(j_1) = \begin{pmatrix} \cos\frac{1}{2}j_1 r & i\sin\frac{1}{2}j_1 r \\ i\sin\frac{1}{2}j_1 r & \cos\frac{1}{2}j_1 r \end{pmatrix},$$

$$g_2(s; j_1) = \exp sL_{01}(j_1) = \begin{pmatrix} \cos\frac{1}{2}j_1 s & -\sin\frac{1}{2}j_1 s \\ \sin\frac{1}{2}j_1 s & \cos\frac{1}{2}j_1 s \end{pmatrix},$$

$$g_3(w) = \exp wP_1 = \begin{pmatrix} e^{iw/2} & 0 \\ 0 & e^{-iw/2} \end{pmatrix}, \qquad (68)$$

and to general element $Z = rQ_{01} + sL_{01} + wP_1$ of algebra $su(2; j_1)$ we, using exponential mapping, put in correspondence the matrix of finite transformation of group $SU(2; j_1)$, which can be easily found

$$g(\zeta, w; j_1) = \exp Z = \begin{pmatrix} \cos\frac{v}{2} + i\frac{w}{v}\sin\frac{v}{2} & -j_1\frac{\zeta}{v}\sin\frac{v}{2} \\ j_1\frac{\zeta}{v}\sin\frac{v}{2} & \cos\frac{v}{2} - i\frac{w}{v}\sin\frac{v}{2} \end{pmatrix},$$

$$v^2(j_1) = w^2 + j_1^2|\zeta|^2, \quad \zeta = s + ir. \tag{69}$$

In Euler parametrization [56] transformations from group $SU(2; j_1)$ can be written as

$$g(\varphi, \theta, \omega; j_1) = g_3(\varphi; j_1)g_1(\theta; j_1)g_3(\omega; j_1) =$$

$$= \begin{pmatrix} e^{i\frac{\omega+\varphi}{2}}\cos j_1\frac{\theta}{2} & e^{-i\frac{\omega-\varphi}{2}}\, i\sin j_1\frac{\theta}{2} \\ e^{i\frac{\omega-\varphi}{2}}\, i\sin j_1\frac{\theta}{2} & e^{-i\frac{\omega+\varphi}{2}}\cos j_1\frac{\theta}{2} \end{pmatrix}, \tag{70}$$

where group parameters (Euler angles) are in the bounds

$$0 \le \varphi < 2\pi, \quad -2\pi \le \omega \le 2\pi, \quad \theta \in \Theta(j) = \begin{cases} (0, \pi), & j_1 = 1 \\ (0, \infty), & j_1 = \iota \\ (-\infty, 0), & j_1 = i. \end{cases} \tag{71}$$

Let us note, that for $j_1 = 1$ matrices $g(\varphi, \theta, \omega; j_1)$ coincide with matrices (1.1.3–4), ch. III in [56], for $j_1 = i$ they coincide with the matrices (1.3.4–5), ch. VI in [56], and for $j_1 = \iota_1$ they describe Euclidean group $SU(2; \iota_1)$ in Euler parametrization.

5 Classification of Transitions between Cayley–Klein Spaces and Groups

In the previous sections we have found orthogonal and unitary groups in Cayley–Klein spaces and shown that their generators, Casimir operators and other algebraic constructions can be obtained by transformation of the corresponding constructions for classical groups. Such approach is natural and is justified by the fact that classical groups and their characteristic algebraical constructions are well studied. But is such approach the only one? Is it possible to take one of the groups in Cayley–Klein space as the initial one? The positive answer to this question is given by the following theorem on the structure of transitions between groups.

Let us define (formally) the transition from the space $\mathbf{C}_{n+1}(j)$ and the generators $Z_{ab}(\mathbf{z}; j)$ of unitary group $SU(n+1; j)$ to the space $\mathbf{C}_{n+1}(j')$ and the generators $Z_{ab}(\mathbf{z}'; j')$ via transformations, which can be obtained from the transformations (52) and (60), substituting in the latter the parameters j_k for $j'_k j_k^{-1}$:

$$\phi' : \mathbf{C}_{n+1}(j) \to \mathbf{C}_{n+1}(j')$$

$$\phi' z_0 = z'_0,\ \phi' z_k = z'_k \prod_{m=1}^{k} j'_m j_m^{-1}, \quad k = 1, 2, \ldots, n,$$

$$Z_{ab}(\mathbf{z}';j') = (\prod_{l=1+\min(a,b)}^{\max(a,b)} j'_l j_l^{-1}) Z_{ab}(\phi'\mathbf{z};j). \qquad (72)$$

The inverse transitions can be obtained from (72) by the change of the dashed parameters j' for the undashed parameters j and vice versa. Applying (72) to the quadratic form (53) and the generators (61), we obtain

$$(z',z') = |z'_0|^2 + \sum_{k=1}^{n} |z'_k|^2 \prod_{m=1}^{k} j'^2_m,$$

$$X_{kk} = z'_k\partial'_k,\ X_{sr} = z'_r\partial'_s,\ X_{rs} = (\prod_{l=1+r}^{s} j'^2_l) z'_s\partial'_r, \qquad (73)$$

i.e. quadratic form in space $C_{n+1}(j')$ and generators of group $SU(n+1;j')$.

However, the constructed transitions do not make sense for all groups and spaces, because for the nilpotent values of parameters j the expressions ι_k^{-1}, $\iota_m \cdot \iota_k^{-1}$ for $k \neq m$ are not defined. We have defined in section 2 only the expressions $\iota_k \cdot \iota_k^{-1} = 1$, $k = 1, 2, \ldots, n$. So if for some k we put $j_k = \iota_k$, then the transformations (72) will be defined and give us (73) only in the case when the dashed parameter with the same number is equal to the same nilpotent number, i.e. $j'_k = \iota_k$.

The transitions from space $\mathbf{R}_{n+1}(j)$ to space $\mathbf{R}_{n+1}(j')$, and from groups $SO(n+1;j)$, $Sp(n;j)$ to groups $SO(n+1;j')$, $Sp(n;j')$ as well, can be, correspondingly, obtained from the transition (45), (31) by the same substitution of parameters j_k for $j'_k j_k^{-1}$. Similarly can be justified the permissibility of these relations. Let us introduce the notations: $G(j) = SO(n+1;j)$, $SU(n+1;j)$, $Sp(n;j)$, $\mathbf{R}(j) = \mathbf{R}_{n+1}(j)$, $\mathbf{C}_{n+1}(j)$, $\mathbf{R}_n(j) \times \mathbf{R}_n(j)$ and denote the transformation of group generators by the symbol $\Phi G(j) = G(j')$. Easy analysis of the transformations (72) and their inverse transformations from the point of view of admissibility of the transitions [24] implies the following theorem.

Transitions classification theorem. I. Let $G(j)$ be a group in non-fiber space $\mathbf{R}(j)$ and $G(j')$ be a group in arbitrary space $\mathbf{R}(j')$, then $G(j') = \Psi G(j)$. If $\mathbf{R}(j')$ is a non-fiber space, then Ψ is one-to-one mapping, and $G(j) = \Psi^{-1}G(j')$.

II. Let $G(j)$ be a group in $(k_1, k_2, \ldots, k_p)$-fiber space $\mathbf{R}(j)$ and $G(j')$ be a group in $(m_1, m_2, \ldots, m_q)$-fiber space $\mathbf{R}(j')$, then $G(j') = \Psi G(j)$, if the set of integers $(k_1, \ldots, k_p)$ is involved in the set of numbers $(m_1, \ldots, m_q)$. The inverse transition $G(j) = \Phi^{-1}G(j')$ is valid if and only if $p = q$, $k_1 = m_1, \ldots, k_p = m_p$.

It follows from the theorem that the group $G(j)$ for any set of values of the parameters j can be obtained not only from a classical group, but from a group in an arbitrary non-fiber Cayley–Klein space, i.e. from pseudoorthogonal, pseudounitary or pseudosymplectic groups. It is naturally that the transitions between other algebraic constructions, in particular between Casimir operators, are described by this theorem as well.

6 Kinematics as Spaces of Constant Curvature

Possible kinematic groups, i.e. groups of motion for four-dimensional models of space-time (kinematics), satisfying natural physical postulates: 1) space is isotropic; 2) spatial property of being even inversion of time are automorphisms of kinematic groups; 3) boosts (rotations in spatial-temporal plane) make a non-compact subgroups are described by H. Bacry and J.-M. Levy-Leblond [1]. In [2] H. Bacry and J. Nuyts rejected postulates 2) and 3) and obtained a more wider set of groups with spatial isotropy. Now we shall bring the geometric interpretation of kinematics [22, 33].

All kinematic groups are 10-dimensional; for this reason kinematics from the geometrical point of view, should be among four-dimensional maximally homogeneous spaces — spaces of constant curvature, which groups of motions are of dimension 10. These spaces are realized on the connected components of the sphere

$$S_4(j) = \{x_0^2 + \sum_{k=1}^{4}(0,k)x_k^2 = 1\}. \tag{74}$$

Let us introduce internal (Beltramian) coordinates $\xi_k = x_k/x_0,\ k = 1,2,3,4$ on $S_4(j)$. The generators (32) of group $SO(4;j)$ can be expressed in terms of the internal coordinates ξ via formulas

$$X_{0s}(\xi) = -\partial_1 - (0,s)^2\xi_s \sum_{k=1}^{4}\xi_k\partial_k, \quad \partial_k = \partial/\partial\xi_k,$$

$$X_{rs}(u) = -\xi_r\partial_s + (r,s)^2\xi_s\partial_r, \quad r < s,\ r,s = 1,2,3,4 \tag{75}$$

and satisfy the commutation relations (46). The generator $X_{0s}(u)$ has a meaning of generator for translation along the s-th Beltrami axis, and $X_{rs}(u)$ is generator of rotation in two-dimensional plane $\{\xi_r, \xi_s\}$.

Physical postulates 1)–3) can be expressed in terms of parameters j. Postulate 1) means that under the transformations (45) three Beltrami coordinates should be multiplied by the same quantity and interpreted as a temporal axis of kinematics. It is possible in two cases:

A) for $j_3 = j_4 = 1$, when coordinates ξ_2, ξ_3, ξ_4 are multiplied by the product $j_1 j_2$ and called spatial and ξ_1 is multiplied by j_1 and called temporal;

B) for $j_2 = j_3 = 1$, when the spatial coordinates $\xi_k = r_k,\ k = 1,2,3$ are multiplied by j_1, and temporal coordinate $\xi_4 = t$ is multiplied by the product $j_1 j_4$.

Postulate 3) imposes restrictions on the character of rotations in two-dimensional planes, spanned over temporal and spatial axes of kinematics, requiring these rotations to be Lorentzian and Galilean. In terms of parameters j this gives $j_2 = \iota_2, i$ in the case A) and $j_4 = \iota_4, i$ in the case B). The requirements of postulate 2) can be taken into account by the definition of space with the constant curvature as a connected component of the sphere (74).

In the case A) the kinematic generators H, $\mathbf{P} = (P_1, P_2, P_3)$ (spatial-temporal translations), $\mathbf{J} = (J_1, J_2, J_3)$ (rotations), $\mathbf{K} = (K_1, K_2, K_3)$ (boosts) are expressed in terms of generators (75) in accordance with above mentioned interpretation by the relations $H = -X_{01},\ P_k = -X_{0,k+1},\ K_k = -X_{1,k+1},\ J_1 = X_{34},\ J_2 = -X_{24},\ J_3 = X_{23},\ k = 1,2,3,$ and satisfy the commutation relations

$$[H, \mathbf{J}] = 0, \quad [H, \mathbf{K}] = \mathbf{P}, \quad [H, \mathbf{P}] = -j_1^2\mathbf{K}$$

$$[\mathbf{P}, \mathbf{P}] = j_1^2 j_2^2 \mathbf{J}, \quad [\mathbf{K}, \mathbf{K}] = j_2^2\mathbf{J}, \quad [P_k, K_l] = -j_2^2\delta_{kl}H. \tag{76}$$

Here $[\mathbf{X}, \mathbf{Y}] = \mathbf{Z}$ means $[X_k, Y_l] = e_{klm} Z_m$, where e_{klm} is the antisymmetric unit tensor. The spaces of constant curvature $S_4(j_1, j_2, 1, 1) \equiv S_4(j_1, j_2)$, $j_1 = 1, \iota_1, i$, $j_2 = \iota_2, i$ are shown at Fig. 2 (see section 3.2), where the spatial axis r should be imagined as a three-dimensional space. Semispherical group $SO(5; 1, \iota_2)$ and semihyperbolic group $SO(5; i, \iota_2)$ correspond to Newton groups $N_\pm$ (sometimes the latter are called Hooke groups). The interpretation of other groups is well-known.

In the case of B) the temporal and spatial axes of kinematics are expressed in another way in terms of Beltramian coordinates of space with the constant curvature; correspondingly, the geometrical generators $X(\xi)$ obtain another kinematic interpretation: $H = X_{04}$, $P_k = -X_{0k}$, $K_k = X_{k4}$, $J_1 = X_{23}$, $J_2 = -X_{13}$, $J_3 = X_{12}$ and satisfy the commutation relations

$$[\mathbf{J}, \mathbf{J}] = \mathbf{J}, \quad [\mathbf{J}, \mathbf{P}] = \mathbf{P}, \quad [\mathbf{J}, \mathbf{K}] = \mathbf{K},$$
$$[H, \mathbf{J}] = 0, \quad [H, \mathbf{K}] = -j_4^2 \mathbf{P}, \quad [H, \mathbf{P}] = j_1^2 \mathbf{K},$$
$$[\mathbf{P}, \mathbf{P}] = j_1^2 \mathbf{J}, \quad [\mathbf{K}, \mathbf{K}] = j_4^2 \mathbf{J}, \quad [P_k, K_l] = \delta_{kl} H. \qquad (77)$$

The value of parameter $j_4 = i$, as it can be readily understood, does not lead to new kinematics, because $SO(5; j_1, 1, 1, i)$ for $j_1 = 1, \iota_1, i$ is, correspondingly, de Sitter group, Poincaré group and anti-de Sitter group.

Kinematic Carroll group [19] of motions of the flat Carroll space, first described in physical terms by J.-M. Levy-Leblond [40] corresponds to the values of parameters $j_1 = \iota_1$, $j_4 = \iota_4$. Comparing the commutators (77) with the commutators in the paper [1] by H. Bacry and J.-M. Levy-Leblond, we find that group $SO(5; 1, 1, 1, \iota_4)$ coincides with kinematic group $ISO(4)$, and group $SO(5; i, 1, 1, \iota_4)$ is "para-Poincaré" group P'. As parameter j_1 determines the sign of the space curvature (curvature is positive for $j_1 = 1$, zero for $j_1 = \iota_1$ and negative for $j_1 = i$) we conclude that group $SO(5; 1, 1, 1, \iota_4)$ (or $ISO(4)$) is the group of motions of Carroll kinematics with a positive curvature, group $SO(5; 1, 1, 1, \iota_4)$ (or P') is the group of motions of Carroll kinematics with a negative curvature. Such interpretation of kinematic groups $ISO(4)$ and P', as far as it can be seen, was not recognized by the authors of [1], and this fact, by the way, is reflected in the names and notations of these groups. Further Carroll kinematics will be denoted as $C_4(j_1)$, and their kinematic groups as $G(j_1) = SO(5; j_1, 1, 1, \iota_4)$.

H. Bacry and J.-M. Levy-Leblond [1] have described 11 kinematical groups. Nine of them have obtained geometrical interpretation as spaces of constant curvature. The rest two kinematics — "para-Galilean" and static — can not be identified with any of the spaces of constant curvature. For example, kinematic "para-Galilean" group is obtained from Galilean group $SO(5; \iota_1, \iota_2)$ by substitution $\mathbf{P} \to \mathbf{K}$, $\mathbf{K} \to \mathbf{P}$, i.e. under the new interpretation of generators , in which the generators of spatial translations of Galilean kinematics are claimed to be the generators of boosts of "para-Galilean" kinematics, and the generators of Galilean boosts — to be the generators of spatial "para-Galilean" translations.

7 Standard Electroweak Model

The standard Electroweak Model (Weinberg-Glashow-Salam theory) is a gauge theory based on the group $SU(2) \times U(1)$ and gives a good description of electroweak processes

[46, 47, 55]. Mathematically this theory is very complicated with nonlinear dynamics of the involved fields.

The Electroweak Model involve particles with integer spins: photon, responsible for electromagnetic interactions, neutral Z^0 and charged $W^{\pm}$ bosons, which are week interaction carriers. For each subgroup $SU(2)$ and $U(1)$ of the gauge group its own coupling constants g and g' are introduced. Complex space $\mathbf{C}_2$ of the fundamental representation of the group $SU(2)$ is interpreted as the space of matter fields $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in \mathbf{C}_2$. Gauge fields $A_\mu(x)$ for the group $SU(2)$ take their values in Lie algebra $su(2)$

$$A_\mu(x) = -ig \sum_{k=1}^{3} T_k A_\mu^k(x), \tag{78}$$

where matrices T_k, connected with Pauli matrices τ^k by the following relations

$$T_1 = \frac{1}{2}\tau^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T_2 = \frac{1}{2}\tau^2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$T_3 = \frac{1}{2}\tau^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{79}$$

submit commutation relations $[T_k, T_p] = i\epsilon_{kpr}T_r$ and represent the algebra $su(2)$ with structure constants $C_{kpr} = \epsilon_{kpr}$. The gauge fields (78) are as follows in the matrix form:

$$A_\mu(x) = -i\frac{g}{2}\begin{pmatrix} A_\mu^3 & A_\mu^1 - iA_\mu^2 \\ A_\mu^1 + iA_\mu^2 & -A_\mu^3 \end{pmatrix}. \tag{80}$$

For the group $U(1)$ with generator $Y = \frac{1}{2}\mathbf{1}$ the gauge field takes the form

$$B_\mu(x) = -i\frac{g'}{2}\begin{pmatrix} B_\mu & 0 \\ 0 & B_\mu \end{pmatrix} \tag{81}$$

and has stress tensor $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. For the field $A_\mu(x)$ its stress tensor is given by

$$F_{\mu\nu}(x) = \mathcal{F}_{\mu\nu}(x) + [A_\mu(x), A_\nu(x)] \tag{82}$$

and has the components

$$F_{\mu\nu}^1 = \mathcal{F}_{\mu\nu}^1 + g(A_\mu^2 A_\nu^3 - A_\mu^3 A_\nu^2) = \mathcal{F}_{\mu\nu}^1 + g\sum_{k,m=1}^{3} \epsilon_{1km} A_\mu^k A_\nu^m,$$

$$F_{\mu\nu}^2 = \mathcal{F}_{\mu\nu}^2 + g(A_\mu^3 A_\nu^1 - A_\mu^1 A_\nu^3) = \mathcal{F}_{\mu\nu}^2 + g\sum_{k,m=1}^{3} \epsilon_{2km} A_\mu^k A_\nu^m,$$

$$F_{\mu\nu}^3 = \mathcal{F}_{\mu\nu}^3 + g(A_\mu^1 A_\nu^2 - A_\mu^2 A_\nu^1) = \mathcal{F}_{\mu\nu}^3 + g\sum_{k,m=1}^{3} \epsilon_{3km} A_\mu^k A_\nu^m, \tag{83}$$

where $\mathcal{F}^k_{\mu\nu} = \partial_\mu A^k_\nu - \partial_\nu A^k_\mu$ is the stress tensor for Abel group. Boson sector of Electroweak Model is characterized by Lagrangian

$$L_B = L_A + L_\phi, \tag{84}$$

which comprises two parts: the gauge fields Lagrangian

$$L_A = \frac{1}{8g^2}\mathrm{Tr}(F_{\mu\nu})^2 - \frac{1}{4}(B_{\mu\nu})^2 =$$

$$= -\frac{1}{4}[(F^1_{\mu\nu})^2 + (F^2_{\mu\nu})^2 + (F^3_{\mu\nu})^2] - \frac{1}{4}(B_{\mu\nu})^2, \tag{85}$$

and the matter fields Lagrangian

$$L_\phi = \frac{1}{2}(D_\mu\phi)^\dagger D_\mu\phi - V(\phi). \tag{86}$$

The potential is taken in the special form

$$V(\phi) = \frac{\lambda}{4}\left(\phi^\dagger\phi - v^2\right)^2, \tag{87}$$

where λ, v are constants. Covariant derivative

$$D_\mu\phi = \partial_\mu\phi - ig\left(\sum_{k=1}^{3} T_k A^k_\mu\right)\phi - ig'YB_\mu\phi \tag{88}$$

for the matter fields ϕ_1, ϕ_2 is given by

$$D_\mu\phi_1 = \partial_\mu\phi_1 - \frac{i}{2}(gA^3_\mu + g'B_\mu)\phi_1 - \frac{ig}{2}(A^1_\mu - iA^2_\mu)\phi_2,$$

$$D_\mu\phi_2 = \partial_\mu\phi_2 + \frac{i}{2}(gA^3_\mu - g'B_\mu)\phi_2 - \frac{ig}{2}(A^1_\mu + iA^2_\mu)\phi_1. \tag{89}$$

Space-time variables are numbered by Greek indexes $\mu, \nu, \ldots = 0, 1, 2, 3$.

To obtain vector boson masses the special mechanism of spontaneous symmetry breaking (or Higgs mechanism) is used. One of Lagrangian L_B ground states

$$\phi^{vac} = \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad A^k_\mu = B_\mu = 0 \tag{90}$$

is taken as vacuum of the model, and then small field excitations

$$\phi_1(x), \quad \phi_2(x) = v + \chi(x), \quad A^a_\mu(x), \quad B_\mu(x) \tag{91}$$

with respect to this vacuum are regarded. Matrix $Q = Y + T_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, which annihilates the ground state $Q\phi^{vac} = 0$, is the generator of electromagnetic subgroup $U(1)_{em}$. New gauge fields are introduced

$$W^\pm_\mu = \frac{1}{\sqrt{2}}\left(A^1_\mu \mp iA^2_\mu\right), \quad Z_\mu = \frac{1}{\sqrt{g^2 + g'^2}}\left(gA^3_\mu - g'B_\mu\right),$$

$$A_\mu = \frac{1}{\sqrt{g^2+g'^2}}\left(g'A_\mu^3 + gB_\mu\right), \tag{92}$$

where $W_\mu^\pm$ are complex fields $\bar{W}_\mu^- = W_\mu^+$, and Z_μ, A_μ are real fields.

Boson Lagrangian (84) can be rewritten in the form

$$L_B = L_B^{(2)} + L_B^{int}. \tag{93}$$

As usual, the second order terms

$$L_B^{(2)} = \frac{1}{2}(\partial_\mu\chi)^2 - \frac{1}{2}m_\chi^2\chi^2 - \frac{1}{4}\mathcal{Z}_{\mu\nu}\mathcal{Z}_{\mu\nu} + \frac{1}{2}m_Z^2 Z_\mu Z_\mu -$$
$$-\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu} - \frac{1}{2}\mathcal{W}_{\mu\nu}^+\mathcal{W}_{\mu\nu}^- + m_W^2 W_\mu^+ W_\mu^-, \tag{94}$$

where $\mathcal{Z}_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu$, $\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\mathcal{W}^\pm{}_{\mu\nu} = \partial_\mu W_\nu^\pm - \partial_\nu W_\mu^\pm$, describe the spectrum of boson particles, and higher-order terms L_B^{int} are interpreted as their interactions. So, Lagrangian (94) describes charged W-bosons with identical mass $m_W = \frac{1}{2}gv$, massless photon A_μ, $m_A = 0$, neutral Z-boson with mass $m_Z = \frac{v}{2}\sqrt{g^2+g'^2}$ and Higgs boson χ, $m_\chi = \sqrt{2\lambda}v$. W- and Z-bosons were observed experimentally and have masses $m_W = 80\,\mathrm{GeV}$, $m_Z = 91\,\mathrm{GeV}$. Higgs boson with the mass of 125 GeV was detected at LHC in 2012.

Besides gauge bosons, there are fermions in the Electroweak Model. The fermion sector is represented by leptons and quarks. Leptons are fermions, which do not interact strongly. There are three types of charged leptons in Nature: electron e^-, muon μ^-, τ-muon τ^- and three types of neutrinos ν_e, ν_μ, ν_τ, as well as the corresponding antiparticles. Neutrino masses, if they exist, are extremely small, therefore in the Electroweak Model neutrinos are considered as massless particles. Neutrinos are fermions of left chirality, i.e. their spin projection is opposite to the direction of movement. The name "left fermion" , is used in this case. Pairs (or generations) of leptons (ν_e, e^-), (ν_μ, μ^-), (ν_τ, τ^-) have identical properties with respect to all interactions. Therefore it is sufficient to discuss only one generation, for example, (ν_e, e^-).

The lepton Lagrangian is taken in the form

$$L_L = L_l^\dagger i\tilde{\tau}_\mu D_\mu L_l + e_r^\dagger i\tau_\mu D_\mu e_r - h_e[e_r^\dagger(\phi^\dagger L_l) + (L_l^\dagger\phi)e_r], \tag{95}$$

where $L_l = \begin{pmatrix} \nu_l \\ e_l \end{pmatrix}$ is $SU(2)$-doublet (vector in the space $\mathbf{C}_2$), e_r is $SU(2)$-singlet (scalar with respect of $SU(2)$), h_e is a constant. All fields e_r, e_l, ν_l are in their turn two-component Lorentz spinors. Here τ_μ are Pauli matrixes, $\tau_0 = \tilde{\tau}_0 = \mathbf{1}$, $\tilde{\tau}_k = -\tau_k$. The above mentioned division of the fields on doublets and singlets is based on the experimental fact that only the *left* components of an electron and a neutrino interact with $W^\pm$-boson fields, but the right components do not interacted with $W^\pm$-boson.

The covariant derivatives of the lepton fields $D_\mu L_l$ in (95) are given by the formula (88) for $Y = -\frac{1}{2}$ with L_l instead of ϕ, and $D_\mu e_r = (\partial_\mu + ig'B_\mu)e_r$. For the new fields (92) these derivatives are as follows

$$D_\mu e_r = \partial_\mu e_r + ig'A_\mu e_r\cos\theta_w - ig'Z_\mu e_r\sin\theta_w,$$

$$D_\mu = \partial_\mu - i\frac{g}{\sqrt{2}}\left(W_\mu^+ T_+ + W_\mu^- T_-\right) -$$

$$-i\frac{g}{\cos\theta_w} Z_\mu \left(T_3 - Q\sin^2\theta_w\right) - ieA_\mu Q, \tag{96}$$

where $T_\pm = T_1 \pm iT_2$, and e is electron charge

$$O = Y + T_3|_{Y=-\frac{1}{2}} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \frac{gg'}{\sqrt{g^2+g'^2}},$$

$$g = \frac{e}{\sin\theta_w}, \quad \cos\theta_w = \frac{g}{\sqrt{g^2+g'^2}}, \quad \sin\theta_w = \frac{g'}{\sqrt{g^2+g'^2}}. \tag{97}$$

Then lepton Lagrangian (95) can be rewritten in the form

$$L_L = e_l^\dagger i\tilde{\tau}_\mu \partial_\mu e_l + \nu_l^\dagger i\tilde{\tau}_\mu \partial_\mu \nu_l + e_r^\dagger i\tau_\mu \partial_\mu e_r + \frac{g}{2\cos\theta_w}\nu_l^\dagger \tilde{\tau}_\mu Z_\mu \nu_l +$$

$$+\frac{g}{\sqrt{2}} e_l^\dagger \tilde{\tau}_\mu W_\mu^- \nu_l - e e_l^\dagger \tilde{\tau}_\mu A_\mu e_l + \frac{g\cos 2\theta_w}{2\cos\theta_w} e_l^\dagger \tilde{\tau}_\mu Z_\mu e_l +$$

$$+\frac{g}{\sqrt{2}}\nu_l^\dagger \tilde{\tau}_\mu W_\mu^+ e_l - g'\cos\theta_w e_r^\dagger \tau_\mu A_\mu e_r + g'\sin\theta_w e_r^\dagger \tau_\mu Z_\mu e_r -$$

$$-h_e[e_r^\dagger \phi_2^\dagger e_l + e_l^\dagger \phi_2 e_r + e_r^\dagger \phi_1^\dagger \nu_l + \nu_l^\dagger \phi_1 e_r]. \tag{98}$$

The first three terms are kinetic terms of the left electron, the left neutrino and the right electron. The last four terms with the multiplier h_e are mass terms of the electron. The rest of the terms describe the electron and neutrino interactions with the gauge bosons $A_\mu, Z_\mu, W_\mu^\pm$.

The next two lepton generations are introduced in the same way. They are left $SU(2)$-doublets

$$\begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_l, \quad \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_l; \quad Y = -\frac{1}{2} \tag{99}$$

and right $SU(2)$-singlets

$$\mu_r, \quad \tau_r, \quad Y = -1. \tag{100}$$

The complete lepton Lagrangian is given by the sum

$$L_L = L_{L,e} + L_{L,\mu} + L_{L,\tau}, \tag{101}$$

where each summand has the structure (98) with its own constants h_e, h_μ, h_τ.

Quarks are strong interacting fermions. Six types of quarks are known. From the viewpoint of electroweak interactions all known quarks are divided into three generations: (u,d), (c,s) and (t,b). Electroweak interactions of all quark generations are identical, therefore we discuss quarks of the first generation in the beginning. The quark Lagrangian is given by

$$L_Q = Q_l^\dagger i\tilde{\tau}_\mu D_\mu Q_l + u_r^\dagger i\tau_\mu D_\mu d_r -$$

$$-h_d[d_r^\dagger(\phi^\dagger Q_l) + (Q_l^\dagger \phi)d_r] - h_u[u_r^\dagger(\tilde{\phi}^\dagger Q_l) + (Q_l^\dagger \tilde{\phi})u_r], \tag{102}$$

where the left fields u- and d-quark of the first generation form doublet $Q_l = \begin{pmatrix} u_l \\ d_l \end{pmatrix}$ relative to the electroweak group $SU(2)$, and the right fields u_r, d_r are $SU(2)$-singlets, $\tilde{\phi}_i = \epsilon_{ik}\bar{\phi}_k, \epsilon_{00} = 1, \epsilon_{ii} = -1$ are conjugate representation of the group $SU(2)$, at last, h_u, h_d are constant multipliers for mass terms. All fields u_l, d_l, u_r, d_r are two-component Lorentz spinors.

The left fields of the next quark generations

$$\begin{pmatrix} c_l \\ s_l \end{pmatrix}, \quad \begin{pmatrix} t_l \\ b_l \end{pmatrix}, \quad Y = \frac{1}{6}, \tag{103}$$

are described by $SU(2)$-doublets, and the right fields are $SU(2)$-singlets

$$c_r,\ t_r, \quad Y = \frac{2}{3}; \quad s_r,\ b_r, \quad Y = -\frac{1}{3}. \tag{104}$$

The covariant derivatives are given by the formulae

$$D_\mu Q_l = \left(\partial_\mu - ig\sum_{k=1}^{3}\frac{\tau_k}{2}A^k_\mu - ig'\frac{1}{6}B_\mu\right)Q_l,$$

$$D_\mu a_r = \left(\partial_\mu - ig'\frac{2}{3}B_\mu\right)a_r, \quad D_\mu f_r = \left(\partial_\mu + ig'\frac{1}{3}B_\mu\right)f_r, \tag{105}$$

where $a = u, c, t$ and $f = d, s, b$, but Q_l now denote the left fields of all three quark generations. The complete quark Lagrangian is the sum

$$L_Q = L_{Q,(u,d)} + L_{Q,(c,s)} + L_{Q,(t,b)}, \tag{106}$$

where each term has the structure (102) with its own constants $h_u, h_d, h_c, h_s, h_t, h_b$.

Lagrangian of the Standard Electroweak Model is the sum

$$L = L_B + L_L + L_Q, \tag{107}$$

of boson L_B (84), (93), lepton L_L (98), (101) and quark L_Q (102), (106) Lagrangians.

8 The Electroweak Model with Contracted Gauge Group

As far as all three lepton and quark generations behave in the same way, we shall further consider only the first generations. Contracted gauge group $SU(2;j) \times U(1)$ acts in the boson, lepton and quark sectors. The contracted group $SU(2;j)$ is obtained by the consistent rescaling of the fundamental representation of the group $SU(2)$ and the space $\mathbf{C}_2$ [28, 29]:

$$z'(j) = \begin{pmatrix} jz'_1 \\ z'_2 \end{pmatrix} = \begin{pmatrix} \alpha & j\beta \\ -j\bar{\beta} & \bar{\alpha} \end{pmatrix}\begin{pmatrix} jz_1 \\ z_2 \end{pmatrix} = u(j)z(j),$$

$$\det u(j) = |\alpha|^2 + j^2|\beta|^2 = 1, \quad u(j)u^\dagger(j) = 1, \tag{108}$$

where contraction parameter $j \to 0$ or is equal to the nilpotent unit $j = \iota$. The hermitian form $z^\dagger z(j) = j^2|z_1|^2 + |z_2|^2$. remains invariant under this rescaling. The actions of the unitary group $U(1)$ and the electromagnetic subgroup $U(1)_{em}$ in the fiber space $\mathbf{C}_2(\iota)$ with the base $\{z_2\}$ and the fiber $\{z_1\}$ are given by the same matrices as in the space $\mathbf{C}_2$.

The space $\mathbf{C}_2(j)$ of the fundamental representation of $SU(2;j)$ group can be obtained from $\mathbf{C}_2$ substituting z_1 by jz_1. The substitution $z_1 \to jz_1$ induces the substitution of the Lie algebra generators

$$T_1 \to jT_1, \quad T_2 \to jT_2, \quad T_3 \to T_3. \tag{109}$$

As far as the gauge fields take their values in Lie algebra, we can substitute the gauge fields instead of transforming the generators (109), namely:

$$A_\mu^1 \to jA_\mu^1, \quad A_\mu^2 \to jA_\mu^2, \quad A_\mu^3 \to A_\mu^3, \quad B_\mu \to B_\mu. \tag{110}$$

Indeed, due to commutativity and associativity of multiplication by j we have

$$su(2;j) \ni \left\{A_\mu^1(jT_1) + A_\mu^2(jT_2) + A_\mu^3 T_3\right\}$$
$$= \left\{(jA_\mu^1)T_1 + (jA_\mu^2)T_2 + A_\mu^3 T_3\right\}. \tag{111}$$

For the gauge fields (92) the substitutions (110) are as follows:

$$W_\mu^\pm \to jW_\mu^\pm, \quad Z_\mu \to Z_\mu, \quad A_\mu \to A_\mu. \tag{112}$$

The left lepton $L_l = \begin{pmatrix} \nu_l \\ e_l \end{pmatrix}$ and quark $Q_l = \begin{pmatrix} u_l \\ d_l \end{pmatrix}$ fields are $SU(2)$-doublets, so their components are transformed in the similar way as the components of the vector z, namely:

$$\nu_l \to j\nu_l, \quad e_l \to e_l, \quad u_l \to ju_l, \quad d_l \to d_l. \tag{113}$$

The right lepton and quark fields are $SU(2)$-singlets and therefore are not changed.

After the transformations (112), (113) and spontaneous symmetry breaking (90) the boson Lagrangian (84)–(86) can be represented in the form [27, 29]:

$$L_B(j) = L_B^{(2)}(j) + L_B^{int}(j) =$$
$$= \frac{1}{2}\left(\partial_\mu \chi\right)^2 - \frac{1}{2}m_\chi^2 \chi^2 - \frac{1}{4}\mathcal{Z}_{\mu\nu}\mathcal{Z}_{\mu\nu} + \frac{1}{2}m_Z^2 Z_\mu Z_\mu - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu} +$$
$$+ j^2\left\{-\frac{1}{2}\mathcal{W}_{\mu\nu}^+\mathcal{W}_{\mu\nu}^- + m_W^2 W_\mu^+ W_\mu^-\right\} + L_B^{int}(j), \tag{114}$$

where as usual the second order terms describe the boson particles content of the model. Higher order terms

$$L_B^{int}(j) = \frac{gm_z}{2\cos\theta_W}\left(Z_\mu\right)^2\chi - \lambda v \chi^3 + \frac{g^2}{8\cos^2\theta_W}\left(Z_\mu\right)^2\chi^2 - \frac{\lambda}{4}\chi^4 +$$

$$+ j^2\left\{-2ig\left(W_\mu^+ W_\nu^- - W_\mu^- W_\nu^+\right)\left(\mathcal{F}_{\mu\nu}\sin\theta_W + \mathcal{Z}_{\mu\nu}\cos\theta_W\right) + gW_\mu^+ W_\mu^- \chi - \right.$$

$$-\frac{i}{2}e\left[A_\mu\left(\mathcal{W}^+_{\mu\nu}W^-_\nu-\mathcal{W}^-_{\mu\nu}W^+_\nu\right)-A_\nu\left(\mathcal{W}^+_{\mu\nu}W^-_\mu-\mathcal{W}^-_{\mu\nu}W^+_\mu\right)\right]-$$

$$-\frac{i}{2}g\cos\theta_W\left[Z_\mu\left(\mathcal{W}^+_{\mu\nu}W^-_\nu-\mathcal{W}^-_{\mu\nu}W^+_\nu\right)-Z_\nu\left(\mathcal{W}^+_{\mu\nu}W^-_\mu-\mathcal{W}^-_{\mu\nu}W^+_\mu\right)\right]+$$

$$+\frac{g^2}{4}\left[\left(W^+_\mu W^-_\nu-W^-_\mu W^+_\nu\right)^2+W^+_\mu W^-_\nu\chi^2\right]-\frac{e^2}{4}\left\{\left[\left(W^+_\mu\right)^2+\left(W^-_\mu\right)^2\right](A_\nu)^2-\right.$$

$$\left.-2\left(W^+_\mu W^+_\nu+W^-_\mu W^-_\nu\right)A_\mu A_\nu+\left[\left(W^+_\nu\right)^2+\left(W^-_\nu\right)^2\right](A_\mu)^2\right\}-$$

$$-\frac{g^2}{4}\cos\theta_W\left\{\left[\left(W^+_\mu\right)^2+\left(W^-_\mu\right)^2\right](Z_\nu)^2-2\left(W^+_\mu W^+_\nu+W^-_\mu W^-_\nu\right)Z_\mu Z_\nu+\right.$$

$$+\left[\left(W^+_\nu\right)^2+\left(W^-_\nu\right)^2\right](Z_\mu)^2\Big\}-eg\cos\theta_W\Big\{W^+_\mu W^-_\mu A_\nu Z_\nu+W^+_\nu W^-_\nu A_\mu Z_\mu-$$

$$\left.\left.-\frac{1}{2}\left(W^+_\mu W^-_\nu+W^+_\nu W^-_\mu\right)(A_\mu Z_\nu+A_\nu Z_\mu)\right\}\right\} \qquad (115)$$

are regarded as their interactions. The lepton Lagrangian (98) in terms of electron and neutrino fields takes the form [30]

$$L_L(j)=e_l^\dagger i\tilde{\tau}_\mu\partial_\mu e_l+e_r^\dagger i\tau_\mu\partial_\mu e_r-m_e(e_r^\dagger e_l+e_l^\dagger e_r)+$$

$$+\frac{g\cos 2\theta_w}{2\cos\theta_w}e_l^\dagger\tilde{\tau}_\mu Z_\mu e_l-ee_l^\dagger\tilde{\tau}_\mu A_\mu e_l-g'\cos\theta_w e_r^\dagger\tau_\mu A_\mu e_r+$$

$$+g'\sin\theta_w e_r^\dagger\tau_\mu Z_\mu e_r+j^2\Big\{\nu_l^\dagger i\tilde{\tau}_\mu\partial_\mu\nu_l+\frac{g}{2\cos\theta_w}\nu_l^\dagger\tilde{\tau}_\mu Z_\mu\nu_l+$$

$$+\frac{g}{\sqrt{2}}\left[\nu_l^\dagger\tilde{\tau}_\mu W^+_\mu e_l+e_l^\dagger\tilde{\tau}_\mu W^-_\mu\nu_l\right]\Big\}=L_{L,b}+j^2L_{L,f}. \qquad (116)$$

The quark Lagrangian (102) in terms of u- and d-quarks fields can be written as

$$L_Q(j)=d^\dagger i\tilde{\tau}_\mu\partial_\mu d+d_r^\dagger i\tau_\mu\partial_\mu d_r-m_d(d_r^\dagger d+d^\dagger d_r)-\frac{e}{3}d^\dagger\tilde{\tau}_\mu A_\mu d-$$

$$-\frac{g}{\cos\theta_w}\left(\frac{1}{2}-\frac{2}{3}\sin^2\theta_w\right)d^\dagger\tilde{\tau}_\mu Z_\mu d-\frac{1}{3}g'\cos\theta_w d_r^\dagger\tau_\mu A_\mu d_r+$$

$$+\frac{1}{3}g'\sin\theta_w d_r^\dagger\tau_\mu Z_\mu d_r+j^2\Big\{u^\dagger i\tilde{\tau}_\mu\partial_\mu u+u_r^\dagger i\tau_\mu\partial_\mu u_r-$$

$$-m_u(u_r^\dagger u+u^\dagger u_r)+\frac{g}{\cos\theta_w}\left(\frac{1}{2}-\frac{2}{3}\sin^2\theta_w\right)u^\dagger\tilde{\tau}_\mu Z_\mu u+$$

$$+\frac{2e}{3}u^\dagger\tilde{\tau}_\mu A_\mu u+\frac{g}{\sqrt{2}}\left[u^\dagger\tilde{\tau}_\mu W^+_\mu d+d^\dagger\tilde{\tau}_\mu W^-_\mu u\right]+$$

$$+\frac{2}{3}g'\cos\theta_w u_r^\dagger\tau_\mu A_\mu u_r-\frac{2}{3}g'\sin\theta_w u_r^\dagger\tau_\mu Z_\mu u_r\Big\}=$$

$$=L_{Q,b}+j^2L_{Q,f}, \qquad (117)$$

where $m_e=h_e v/\sqrt{2}$ and $m_u=h_u v/\sqrt{2},\ m_d=h_d v/\sqrt{2}$ represent electron and quark masses.

The complete Lagrangian of the modified model is the sum

$$L(j) = L_B(j) + L_Q(j) + L_L(j) = L_b + j^2 L_f. \tag{118}$$

The boson Lagrangian $L_B(j)$ was discussed in [27, 29], where it was shown that masses of all particles involved in the Electroweak Model remain the same under the contraction $j^2 \to 0$ in both formulations: the standard one [27] and without Higgs boson [29]. In this limit the contribution $j^2 L_f$ of neutrino, W-boson and u-quark fields as well as their interactions with the other fields to the Lagrangian (118) become vanishingly small in comparison with the contribution L_b of electron, d-quark and the remaining boson fields. So Lagrangian (118) describes a very rare interaction of neutrino fields with the matter, which consists of quarks and leptons in the Standard Electroweak Model. On the other hand, the contribution of the neutrino part $j^2 L_f$ to the complete Lagrangian is risen when the parameter j^2 is increased, which corresponds to the experimental facts. It follows from this that the contraction parameter is connected with neutrino energy and this dependence can be obtained from the experimental data.

9 Description of Physical Systems and Group Contractions

The standard way of describing a physical system in the field theory is its decomposition on independent more or less simple subsystems, which can be exactly described, and then introducing interactions between them. In Lagrangian formalism this implies that some terms describe independent subsystems (free fields) and the rest of the terms correspond to interactions between the fields. When the subsystems do not interact with each other the composed system is a formal unification of the subsystems and symmetry group of the whole system is the direct product $G = G_1 \times G_2$, where G_1 and G_2 are symmetry groups of the subsystems. The Electroweak Model gives a nice example of such approach. Indeed, there are free boson, lepton and quark fields in Lagrangian and the terms which describe interactions between these fields.

The operation of group contraction transforms a simple or semisimple group G to a non-semisimple one with the structure of a semidirect product $G = A \otimes G_1$, where A is Abel and $G_1 \subset G$ is an untouched subgroup. At the same time the fundamental representation space of the group G is fibered under the contraction in such a way that the subgroup G_1 acts in the fiber. The gauge theory with a contracted gauge group describes a physical system, which is divided on two subsystems S_b and S_f. One subsystem S_b includes all fields from the base and the other subsystem S_f is built from fiber fields. S_b forms a closed system since according to semi-Riemannian geometry [50, 26] the properties of the base do not depend on the points of the fiber, which physically means that the fields from the fiber do not interact with the fields from the base. On the contrary the properties of the fiber depend on the points of the base, therefore the subsystem S_b exerts influence upon S_f. More precisely, the fields from the base are outer (or background) fields for the subsystem S_f and specify outer conditions in every fiber.

In particular, the simple group $SU(2)$ is contracted to the non-semisimple group $SU(2;\iota)$, which is isomorphic to the Euclid group $E(2) = A_2 \otimes SO(1)$, where Abel subgroup A_2 is generated by the translations [27, 28, 29]. The fields space of the Standard Electroweak Model is fibered after the contraction in such a way that neutrino, W-boson and u-quark fields are in the fiber, whereas all the other fields are in the base.

The simple and the best known example of fiber space is the non-relativistic space-time with one-dimensional base, which is interpreted as time, and three-dimensional fiber, which is interpreted as proper space. It is well known, that in non-relativistic physics the time is absolute and does not depend on the space coordinates, while the space properties can be changed in time. The simplest demonstration of this fact is Galilei transformation $t' = t,\ x' = x + vt$. The space-time of the special relativity is transformed to the non-relativistic space-time when a dimensional parameter — the velocity of light c — tends to the infinity and a dimensionless parameter tends to zero $\frac{v}{c} \to 0$.

10 Rarely Neutrino-Matter Interactions

To discover the connection of gauge group contraction with the limiting case of the Electroweak Model and to establish the physical meaning of the contraction parameter we consider neutrino elastic scattering on electrons and quarks. The corresponding diagrams for the neutral and charged currents interactions are represented in Fig. 3 and Fig. 4.

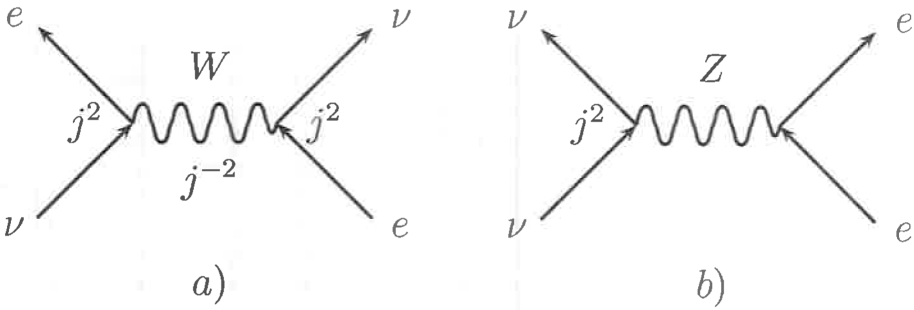

Figure 3: Neutrino elastic scattering on electron

Under substitutions (112), (113) both vertices of diagram in Fig. 3, a) are multiplied by j^2, as it follows from lepton Lagrangian (116). The propagator of virtual fields W according to boson Lagrangian (114) is multiplied by j^{-2}. Indeed, a propagator is an inverse operator toan operator of a free field, but the later for W-fields is multiplied by j^2.

So on the whole the probability amplitude for charged weak current interactions is transformed as $\mathcal{M}_W \to j^2\mathcal{M}_W$. For the diagram in Fig. 3, b) only one vertex is multiplied by j^2, whereas the second vertex and the propagator of Z virtual field do not change, so the corresponding amplitude for neutral weak current interactions is transformed in a similar way $\mathcal{M}_Z \to j^2\mathcal{M}_Z$. Cross-section is proportionate to squared amplitude, so neutrino-electron scattering cross-section is proportionate to j^4. For low energies $s \ll m_W^2$ this cross-section makes a principal contribution to the electron-neutrino interaction and is as follows [46]

$$\sigma_{\nu e} = G_F^2 s f(\xi) = \frac{g^4}{m_w^4}\bar{f}(\xi), \tag{119}$$

where $G_F = 10^{-5}\frac{1}{m_p^2} = 1,17 \cdot 10^{-5}$ GeV^{-2} is Fermi constant, s is squared energy in center-of-mass system, $\xi = \sin\theta_w$, $\bar{f}(\xi) = f(\xi)/32$ is the function of Weinberg angle. The cross-section in the laboratory system for neutrino energy $m_e \ll E_\nu \ll m_W$ is given by [52]

$$\sigma_{\nu e} = G_F^2 m_e E_\nu \tilde{g}(\xi). \tag{120}$$

On the other hand, taking into account that the contraction parameter j is dimensionless, we can write down

$$\sigma_{\nu e} = j^4\sigma_0 = (G_F s)(G_F f(\xi)) \tag{121}$$

and obtain

$$j^2(s) = \sqrt{G_F s} \approx \frac{g\sqrt{s}}{m_W}. \tag{122}$$

So the contraction parameter is expressed in terms of Fermi constant and the fundamental parameters of the Electroweak Model.

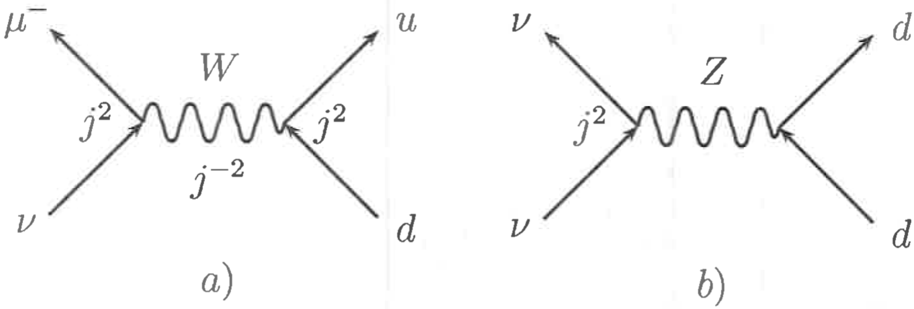

Figure 4: Neutrino elastic scattering on quarks

Neutrino elastic scattering on quarks by means of neutral and charged currents is pictured in Fig. 4. Cross-sections for neutrino-quarks scattering are obtained in a way similar to the lepton case and are as follows [46]

$$\sigma_\nu^W = G_F^2 s\hat{f}(\xi)), \quad \sigma_\nu^Z = G_F^2 s\, h(\xi). \tag{123}$$

Nucleons are some composite constructions of quarks, therefore some form-factors appear in the expressions for neutrino-nucleons scattering cross-sections. The final expression

$$\sigma_{\nu n} = G_F^2 s\hat{F}(\xi) \tag{124}$$

coincides with (119), i.e. this cross-section is transformed as (121) with the contraction parameter (122). At low energies scattering interactions make the leading contribution to the total neutrino-matter cross-section, therefore it has the same properties (121), (122) with respect to contraction of the gauge group.

We have shown that contraction of the gauge group of the Standard Electroweak Model corresponds to its low-energy limit. The zero tending contraction parameter depends on neutrino energy and determines the energy dependence of the neutrino-matter interaction cross-section.

The limit transition $c \to \infty$ in special relativity resulted in the notion of group contraction [34]. In the Electroweak Model the notion of group contraction is used on the contrary to explain the experimentally verified fundamental limit process of nature: a decrease of the neutrinos-matter cross-section when neutrino energy tends to zero.

11 Electroweak Model at Infinite Energy

In the previous section we have shown that contraction of the gauge group of the Standard Electroweak Model corresponds to its low-energy limit. In this limit the first components of the lepton and quark doublets become infinitely small in comparison with their second components. On the contrary, when energy increases the first components of the doublets become greater then their second ones. In the infinite energy limit the second components of the lepton and quark doublets will be infinitely small as compared with their second components. To describe this limit we introduce instead of (108) new contraction parameter ϵ and *new consistent rescaling* of the group $SU(2)$ and the space C_2 as follows

$$z'(\epsilon) = \begin{pmatrix} z'_1 \\ \epsilon z'_2 \end{pmatrix} = \begin{pmatrix} \alpha & \epsilon\beta \\ -\epsilon\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} z_1 \\ \epsilon z_2 \end{pmatrix} = u(\epsilon)z(\epsilon),$$

$$\det u(\epsilon) = |\alpha|^2 + \epsilon^2|\beta|^2 = 1, \quad u(\epsilon)u^{\dagger}(\epsilon) = 1, \tag{125}$$

where $\epsilon \to 0$. Both contracted groups $SU(2;j)$ (108) and $SU(2;\epsilon)$ (125) are the same and are isomorphic to Euclid group $E(2)$, but the space $C_2(\epsilon)$ is splited in the limit $\epsilon \to 0$ on the one-dimension base $\{z_1\}$ and the one-dimension fiber $\{z_2\}$. From the mathematical point of view it is not important if the first or the second Cartesian axis forms the base of fibering and in this sence constructions (108) and (125) are equivalent. But the doublet components are interpreted as certain physical fields, therefore the fundamental representations (108) and (125) of the same contracted unitary group lead to different limit cases of the Electroweak Model, namely, its zero energy and infinite energy limits.

In the second contraction scheme (125) all gauge bosons are transformed according to the rules (112) with the natural substitution of j by ϵ. Instead of (113) the lepton and quark fields are transformed now as follows

$$e_l \to \epsilon e_l, \quad d_l \to \epsilon d_l, \quad \nu_l \to \nu_l, \quad u_l \to u_l. \tag{126}$$

The next reason for inequality of the first and second doublet components is the special mechanism of spontaneous symmetry breaking, which is used to generate mass of vector bosons and other elementary particles of the model. In this mechanism one of Lagrangian ground states $\phi^{vac} = \begin{pmatrix} 0 \\ v \end{pmatrix}$ is taken as vacuum of the model and then small field excitations $v + \chi(x)$ with respect to this vacuum are regarded. So Higgs boson field χ and the constant v are multiplied by ϵ. As far as masses of all particlies are proportionate to v we obtain the following transformation rule for contraction (125)

$$\chi \to \epsilon\chi, \quad v \to \epsilon v, \quad m_p \to \epsilon m_p, \tag{127}$$

where $p = \chi, W, Z, e, u, d$.

After transformations (112), (126)–(127) the boson Lagrangian of the Electroweak Model can be represented in the form

$$L_B(\epsilon) = -\frac{1}{4}\mathcal{Z}_{\mu\nu}^2 - \frac{1}{4}\mathcal{F}_{\mu\nu}^2 + \epsilon^2 L_{B,2} + \epsilon^3 g W_\mu^+ W_\mu^- \chi + \epsilon^4 L_{B,4},$$

$$L_{B,4} = m_W^2 W_\mu^+ W_\mu^- - \frac{1}{2}m_\chi^2\chi^2 - \lambda v \chi^3 - \frac{\lambda}{4}\chi^4 + \frac{g^2}{4}W_\mu^+ W_\nu^- \chi^2 +$$

$$+\frac{g^2}{4}\left(W_\mu^+ W_\nu^- - W_\mu^- W_\nu^+\right)^2,$$

$$L_{B,2} = \frac{1}{2}\left(\partial_\mu \chi\right)^2 + \frac{1}{2}m_Z^2\left(Z_\mu\right)^2 -$$

$$-\frac{1}{2}\mathcal{W}_{\mu\nu}^+\mathcal{W}_{\mu\nu}^- + \frac{g m_z}{2\cos\theta_W}\left(Z_\mu\right)^2\chi + \frac{g^2}{8\cos^2\theta_W}\left(Z_\mu\right)^2\chi^2 -$$

$$-2ig\left(W_\mu^+ W_\nu^- - W_\mu^- W_\nu^+\right)\left(\mathcal{F}_{\mu\nu}\sin\theta_W + \mathcal{Z}_{\mu\nu}\cos\theta_W\right) -$$

$$-\frac{i}{2}e\left[A_\mu\left(\mathcal{W}_{\mu\nu}^+ W_\nu^- - \mathcal{W}_{\mu\nu}^- W_\nu^+\right) + \frac{i}{2}eA_\nu\left(\mathcal{W}_{\mu\nu}^+ W_\mu^- - \mathcal{W}_{\mu\nu}^- W_\mu^+\right)\right] -$$

$$-\frac{i}{2}g\cos\theta_W\left[Z_\mu\left(\mathcal{W}_{\mu\nu}^+ W_\nu^- - \mathcal{W}_{\mu\nu}^- W_\nu^+\right) - Z_\nu\left(\mathcal{W}_{\mu\nu}^+ W_\mu^- - \mathcal{W}_{\mu\nu}^- W_\mu^+\right)\right] -$$

$$-\frac{e^2}{4}\left\{\left[\left(W_\mu^+\right)^2 + \left(W_\mu^-\right)^2\right](A_\nu)^2 - 2\left(W_\mu^+ W_\nu^+ + W_\mu^- W_\nu^-\right)A_\mu A_\nu +\right.$$

$$\left.\left[\left(W_\nu^+\right)^2 + \left(W_\nu^-\right)^2\right](A_\mu)^2\right\} - \frac{g^2}{4}\cos\theta_W\left\{\left[\left(W_\mu^+\right)^2 + \left(W_\mu^-\right)^2\right](Z_\nu)^2 -\right.$$

$$\left.-2\left(W_\mu^+ W_\nu^+ + W_\mu^- W_\nu^-\right)Z_\mu Z_\nu + \left[\left(W_\nu^+\right)^2 + \left(W_\nu^-\right)^2\right](Z_\mu)^2\right\} -$$

$$-eg\cos\theta_W\left[W_\mu^+ W_\mu^- A_\nu Z_\nu + W_\nu^+ W_\nu^- A_\mu Z_\mu -\right.$$

$$\left.-\frac{1}{2}\left(W_\mu^+ W_\nu^- + W_\nu^+ W_\mu^-\right)\left(A_\mu Z_\nu + A_\nu Z_\mu\right)\right]. \tag{128}$$

In terms of electron and neutrino fields the lepton Lagrangian takes the form

$$L_L(\epsilon) = L_{L,0} + \epsilon^2 L_{L,2} = \nu_l^\dagger i\tilde{\tau}_\mu\partial_\mu\nu_l + e_r^\dagger i\tau_\mu\partial_\mu e_r + g'\sin\theta_w e_r^\dagger\tau_\mu Z_\mu e_r -$$

$$-g'\cos\theta_w e_r^\dagger\tau_\mu A_\mu e_r + \frac{g}{2\cos\theta_w}\nu_l^\dagger\tilde{\tau}_\mu Z_\mu\nu_l + \epsilon^2\left\{e_l^\dagger i\tilde{\tau}_\mu\partial_\mu e_l - m_e(e_r^\dagger e_l + e_l^\dagger e_r) +\right.$$

$$\left.+\frac{g\cos 2\theta_w}{2\cos\theta_w}e_l^\dagger\tilde{\tau}_\mu Z_\mu e_l - e e_l^\dagger\tilde{\tau}_\mu A_\mu e_l + \frac{g}{\sqrt{2}}\left(\nu_l^\dagger\tilde{\tau}_\mu W_\mu^+ e_l + e_l^\dagger\tilde{\tau}_\mu W_\mu^-\nu_l\right)\right\}. \tag{129}$$

In terms of u- and d-quarks fields the quark Lagrangian can be written as

$$L_Q(\epsilon) = L_{Q,0} - \epsilon\, m_u(u_r^\dagger u_l + u_l^\dagger u_r) + \epsilon^2 L_{Q,2},$$

$$L_{Q,0} = d_r^\dagger i\tau_\mu \partial_\mu d_r + u_l^\dagger i\tilde{\tau}_\mu \partial_\mu u_l + u_r^\dagger i\tau_\mu \partial_\mu u_r - \frac{1}{3} g' \cos\theta_w d_r^\dagger \tau_\mu A_\mu d_r +$$

$$+\frac{1}{3} g' \sin\theta_w d_r^\dagger \tau_\mu Z_\mu d_r + \frac{2e}{3} u_l^\dagger \tilde{\tau}_\mu A_\mu u_l + \frac{g}{\cos\theta_w}\left(\frac{1}{2} - \frac{2}{3}\sin^2\theta_w\right) u_l^\dagger \tilde{\tau}_\mu Z_\mu u_l +$$

$$+\frac{2}{3} g' \cos\theta_w u_r^\dagger \tau_\mu A_\mu u_r - \frac{2}{3} g' \sin\theta_w u_r^\dagger \tau_\mu Z_\mu u_r,$$

$$L_{Q,2} = d_l^\dagger i\tilde{\tau}_\mu \partial_\mu d_l - m_d(d_r^\dagger d_l + d_l^\dagger d_r) - \frac{e}{3} d_l^\dagger \tilde{\tau}_\mu A_\mu d_l -$$

$$-\frac{g}{\cos\theta_w}\left(\frac{1}{2} - \frac{2}{3}\sin^2\theta_w\right) d_l^\dagger \tilde{\tau}_\mu Z_\mu d_l + \frac{g}{\sqrt{2}}\left[u_l^\dagger \tilde{\tau}_\mu W_\mu^+ d_l + d_l^\dagger \tilde{\tau}_\mu W_\mu^- u_l\right]. \qquad (130)$$

The complete Lagrangian of the modified model is given by the sum $L(\epsilon) = L_B(\epsilon) + L_L(\epsilon) + L_Q(\epsilon)$ and for the infinite energy (for $\epsilon = 0$) is equal to

$$L_\infty = -\frac{1}{4}\mathcal{Z}_{\mu\nu}^2 - \frac{1}{4}\mathcal{F}_{\mu\nu}^2 + \nu_l^\dagger i\tilde{\tau}_\mu \partial_\mu \nu_l + u_l^\dagger i\tilde{\tau}_\mu \partial_\mu u_l + e_r^\dagger i\tau_\mu \partial_\mu e_r +$$

$$+d_r^\dagger i\tau_\mu \partial_\mu d_r + u_r^\dagger i\tau_\mu \partial_\mu u_r + L_\infty^{int}(A_\mu, Z_\mu),$$

$$L_\infty^{int}(A_\mu, Z_\mu) = \frac{g}{2\cos\theta_w}\nu_l^\dagger \tilde{\tau}_\mu Z_\mu \nu_l + \frac{g}{\cos\theta_w}\left(\frac{1}{2} - \frac{2}{3}\sin^2\theta_w\right) u_l^\dagger \tilde{\tau}_\mu Z_\mu u_l +$$

$$+\frac{2e}{3} u_l^\dagger \tilde{\tau}_\mu A_\mu u_l + g' \sin\theta_w e_r^\dagger \tau_\mu Z_\mu e_r - g' \cos\theta_w e_r^\dagger \tau_\mu A_\mu e_r - \frac{1}{3} g' \cos\theta_w d_r^\dagger \tau_\mu A_\mu d_r +$$

$$+\frac{1}{3} g' \sin\theta_w d_r^\dagger \tau_\mu Z_\mu d_r + \frac{2}{3} g' \cos\theta_w u_r^\dagger \tau_\mu A_\mu u_r - \frac{2}{3} g' \sin\theta_w u_r^\dagger \tau_\mu Z_\mu u_r. \qquad (131)$$

The limit model includes only massless particles: neutral massless Z-bosons Z_μ and photons A_μ, massless right electrons e_r and neutrinos ν_l, and massless left and right quarks u_l, u_r, d_r. The electroweak interactions become long-range because they are mediated by the massless neutral Z-bosons and photons. There are no interactions between particlies of different kind, for example neutrinos interact only with each other by neutral currents. Similar higher energies can exist in the early Universe after inflation and reheating on the first stages of the Hot Big Bang [17, 41]. The electroweak phase transition and neutrino decoupling which take place during the first second after the Big Bang [16] are apparently in correspondence with the infinity energy limit of the Electroweak Model (131). The mass term of u-quark in the complete Lagrangian is proportional to ϵ whereas the mass terms of electron and d-quark are multiplied by ϵ^2, so u-quark first restores its mass in the evolution of the Universe.

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