Relativistic Spectrum Generating Groups: Spectrum and Radiative Transitions in a Collective Model for Hadrons

A. Bohm, P. Kielanowski,* M. Kmiecik, M. Loewe
Center for Particle Theory, The University of Texas at Austin
Austin, Texas 78712

Abstract

It is shown how the spectrum generating group approach can be used in relativistic physics to describe hadron masses and transition rates without using approximation procedures. The electromagnetic potential and field operators of the model are chosen to depend on a relativistically covariant intrinsic collective position operator $\hat{\xi}_\mu$ having non-commuting components whose commutation relations follow directly from those of the spectrum generating group $SO(3,2)$.

Introduction

The spectrum generating group approach is based on that of the collective models. Collective models analyze the structures of physical systems in terms of their fundamental motions. Complementary to the collective models, atomistic models analyze the structures of physical systems in terms of their fundamental constituents. The standard models in various areas of physics have traditionally been chosen to be of the atomistic type even when collective models have provided simpler phenomenological descriptions of the data.

The standard model for molecules is the (N+M)-body Schrödinger equation for N electrons and M nuclei with Coulomb forces between these constituents. At low energies, especially when N+M is large, this equation is not of much practical value. If one looks at the work of practitioners in molecular physics[1] one sees that low energy spectra and structures of molecules are analyzed in terms of oscillators and rotators.

---

* Also at Institute of Theoretical Physics, Warsaw University and CINVESTAV del IPN, Mexico.

The standard model for nuclei, the microscopic theory of nuclear forces for many protons and neutrons, is more complicated than the standard model for molecules and the nuclear forces are not as universal as the Coulomb force. One resorts to collective models of oscillators and rotators in nuclear physics even more so than in molecular physics. The original Bohr-Mottelson model[2] (rotations and β- and γ-vibrations) and the interacting boson model[3] [U(6) subgroup chains for vibrational and rotational excitations] are the most famous examples of collective models.

The standard model for hadrons is quantum chromodynamics (QCD) of quarks and gluons;[4] in it hadrons are understood to be color singlet states of quarks with the forces between the quarks arising from the exchange of gluons. QCD has theoretical beauty and, like the standard models for molecules and nuclei, it is in principle solvable but predictions of experimental results have only come from its approximations which involve arbitrary assumptions: perturbative QCD has had impressive successes for hard processes (third jet in e^+e^- annihilation, cross section for jet production within a factor of three) but its results are uncertain due to renormalization scheme dependence and the presence of twists at today's energies; lattice QCD provides a computational scheme for, e.g., the hadron spectrum but the results are only qualitative or semi-quantitative (and dependent upon the quark masses). It therefore seems reasonable to also attempt to describe hadrons using collective models. In analogy to the collective models for molecules and nuclei we analyze low lying spectra and structures of (towers of) hadrons in terms of intrinsic collective oscillators and rotators except that for hadrons the oscillators and rotators must be relativistically covariant.

Non-relativistic collective models

We first illustrate the spectrum generating group approach of the collective models in a non-relativistic setting with the description of a molecule in mind. The quantum mechanical symmetry group of center of mass (c.m.) motion is then the extended Galilei group G^{ex} whose generators are the total angular momenta J_i, Galilean boosts K_i, total momenta P_i, total energy H, and total mass M. Its intrinsic energy and spin invariants are

$$U \equiv H - \frac{1}{2M}P^2 \quad \text{and} \quad S^2$$

where $\mathbf{S} \equiv \mathbf{J} - \mathbf{Q} \times \mathbf{P}$ is the spin angular momentum and $\mathbf{Q} \equiv \mathrm{K/M}$ is the c.m. position.

The molecule is considered to be an extended object with intrinsic collective motions that form a group—the spectrum generating group (SGG). For the SGG we consider the group SO(3,1) generated by the S_i and intrinsic position operators ξ_i with the commutation relations

$$[S_i,S_j] = i\varepsilon_{ijk}S_k\,, \quad [S_i,\xi_j] = i\varepsilon_{ijk}\xi_k\,, \quad [\xi_i,\xi_j] = -i\varepsilon_{ijk}S_k.$$

(Other possible SGGs are SO(4) for which $[\xi_i,\xi_j] = i\varepsilon_{ijk}S_k$ and E(3) for which $[\xi_i,\xi_j] = 0$, or even larger groups.)

The description of the mass, intrinsic energy, and spin spectrum of the molecule is obtained by choosing constraints that relate the mass, intrinsic energy, and spin operators to the generators of SO(3,1) and by choosing an irreducible representation of SO(3,1). For the mass we choose the constraint $M = m$ (i.e., we choose M to have a trivial spectrum). For the spin we have already chosen the S_i to be generators of the maximal compact subgroup K=SO(3) of SO(3,1). For the intrinsic energy we choose the rigid rotator constraint

$$U = \frac{1}{2I}S^2 \quad (\text{i.e., } H = \frac{1}{2M}\mathbf{P}^2 + \frac{1}{2I}S^2) \tag{1}$$

with the moment of inertia I and total mass M (= m) as system parameters.

Figure 1 shows the weight diagram and K-type of an irreducible representation[5] $(k_0{=}0,c)$ of SO(3,1) and the corresponding rigid rotator intrinsic energy diagram. Each dot in the weight diagram stands for an irreducible representation space of the SO(2) group generated by S_3 spanned by a single vector $|j\ j_3\rangle$ (j = eigenvalue of S^2, j_3 = eigenvalue of S_3). Each dot in the K-type stands for an irreducible representation space $R^{(j)}$ of the maximal compact subgroup K=SO(3) of SO(3,1) spanned by the vectors $|j\ j_3\rangle$ with $j_3 \in \{j, j-1,\ldots, -j\}$, whose vectors describe (for fixed c.m. momentum p_i) the physical states of an excited spin (rotational) level of the molecule. Each level of the rigid rotator intrinsic energy diagram stands for an eigenvalue $E_j^{int} = j(j+1)/(2I)$ of the rigid rotator intrinsic energy operator (1) which describes the intrinsic energy of the corresponding excited spin level of the molecule. The direct sum of the spaces $R^{(j)}$, with k_0 specifying the lowest j, gives the irreducible representation space

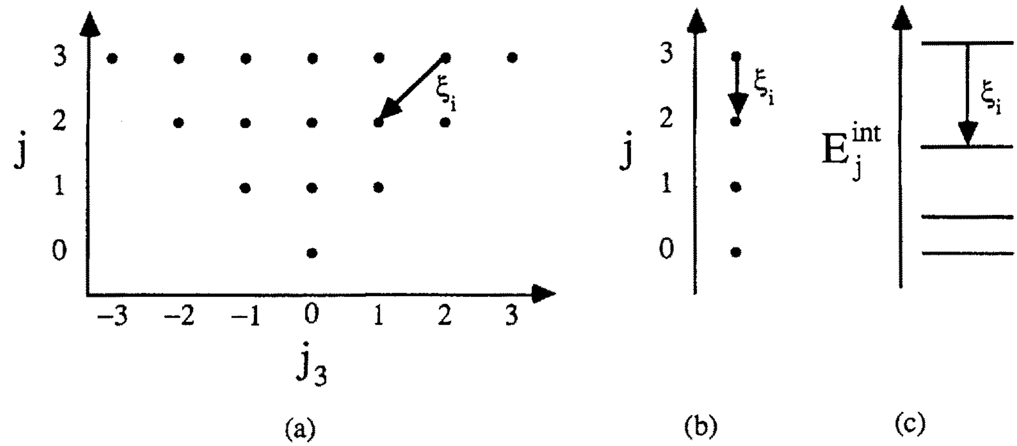

Figure 1. (a) Weight diagram of an irreducible representation (k_0=0,c) of SO(3,1); (b) K-type of the representation; (c) Corresponding rigid rotator intrinsic energy diagram.

$$\mathsf{H}(k_0=0,c) = \sum_{j=0,1,\ldots} \oplus \mathsf{R}^{(j)}$$

of SO(3,1) whose vectors describe all physical states of the molecule. (When c.m. momentum varies the space of physical states becomes the direct product $\mathsf{H}_{orb} \otimes \mathsf{H}(k_0=0,c)$ where H_{orb} is the orbital space spanned by the eigenvectors $|p_i\rangle$ of P_i.)

The compact generators S_i of SO(3,1) transform only within each subspace $\mathsf{R}^{(j)}$ while the non-compact generators ξ_i change the value of the spin quantum number j (by ± 1 or 0) to transform between these subspaces as shown by arrows in Figure 1. The ξ_i, and powers of them, can therefore be used to describe transitions between the intrinsic energy levels of the molecule as indicated in the energy diagram, e.g., to describe the physical process in which an excited state M* of the molecule decays into a lower energy state M with the emission of a photon: $M^* \rightarrow M + \gamma$. (The dipole moment operator is related to the intrinsic position operator by $\mathbf{d} = e\xi$.) The description of the motion of the molecule and of the rates of emitted radiation depends upon what operator is chosen as the interaction Hamiltonian. Later we will choose a Hamiltonian to describe radiative decays of nucleons in a relativistic setting.

Relativistic Collective Model for Hadrons

For hadrons we go to a relativistic setting. The symmetry group of c.m. motion is then the Poincaré group P with generators $J_{\mu\nu}$, P_μ. Its mass and spin invariants are

$$M \equiv (P_\mu P^\mu)^{1/2} \quad \text{and} \quad \hat{W} \equiv -\hat{w}_\mu \hat{w}^\mu = 1/2 \Sigma_{\mu\nu}\Sigma^{\mu\nu},$$

where $\Sigma_{\mu\nu} \equiv \varepsilon_{\mu\nu\rho\sigma}\hat{P}^\rho \hat{w}^\sigma$ is the spin tensor operator, $\hat{w}^\mu \equiv 1/2\varepsilon^{\mu\nu\rho\sigma}\hat{P}_\nu J_{\rho\sigma}$ is the Pauli-Lubanski vector operator, and $\hat{P}_\nu \equiv P_\nu M^{-1}$ is the c.m. momentum direction operator.

A tower of hadrons is considered to be a relativistic extended object with intrinsic collective motions and a corresponding SGG. For the SGG we consider the group SO(3,2) with generators $S_{\mu\nu}$, Γ_μ. The generators of the collective motions are the following relativistically covariant, "boosted" versions of the S_{ij}, S_{0i}, Γ_i, Γ_0, respectively:

$$\begin{array}{rlll}
\breve{g}_\mu{}^\rho \breve{g}_\nu{}^\sigma S_{\rho\sigma} & (\breve{g}_i{}^\rho \breve{g}_j{}^\sigma S_{\rho\sigma} \overset{\text{rest}}{=\!=} S_{ij}, & \breve{g}_0{}^\rho \breve{g}_i{}^\sigma S_{\rho\sigma} \overset{\text{rest}}{=\!=} 0), \\
\hat{\xi}_\mu \equiv -S_{\mu\rho}\hat{P}^\rho & (\quad \hat{\xi}_i \overset{\text{rest}}{=\!=} S_{0i}, & \hat{\xi}_0 \overset{\text{rest}}{=\!=} 0), \\
-\hat{\pi}_\mu \equiv \breve{g}_\mu{}^\rho \Gamma_\rho & (\quad -\hat{\pi}_i \overset{\text{rest}}{=\!=} \Gamma_i, & -\hat{\pi}_0 \overset{\text{rest}}{=\!=} 0), \\
\hat{P}_\rho \Gamma^\rho & (\quad \hat{P}_\rho \Gamma^\rho \overset{\text{rest}}{=\!=} \Gamma_0 &),
\end{array}$$

where $\breve{g}_\mu{}^\rho = \eta_\mu{}^\rho - \hat{P}_\mu \hat{P}^\rho$ projects onto the hyperplane perpendicular to $\hat{P}_\rho$. ($\hat{P}_\rho$ is assumed to commute with $S_{\mu\nu}$ and Γ_μ.) The commutation relations of the (dimensionless) intrinsic positions $\hat{\xi}_\mu$ and momenta $\hat{\pi}_\mu$ (and of the $\breve{g}_\mu{}^\rho \breve{g}_\nu{}^\sigma S_{\rho\sigma}$ and $\hat{P}_\rho \Gamma^\rho$) follow directly from the commutation relations of SO(3,2):

$$[\hat{\xi}_\mu, \hat{\xi}_\nu] = -i\, \breve{g}_\mu{}^\rho \breve{g}_\nu{}^\sigma S_{\rho\sigma}, \quad [\hat{\pi}_\mu, \hat{\pi}_\nu] = -i\, \breve{g}_\mu{}^\rho \breve{g}_\nu{}^\sigma S_{\rho\sigma}, \quad [\hat{\xi}_\mu, \hat{\pi}_\nu] = -i\, \breve{g}_{\mu\nu} \hat{P}_\rho \Gamma^\rho. \qquad (2)$$

That the $\breve{g}_\mu{}^\rho \breve{g}_\nu{}^\sigma S_{\rho\sigma}$, $\hat{\xi}_\mu$, and $\hat{\pi}_\mu$ satisfy

$$\hat{P}^\mu \breve{g}_\mu{}^\rho \breve{g}_\nu{}^\sigma S_{\rho\sigma} = \hat{P}^\mu \hat{\xi}_\mu = \hat{P}^\mu \hat{\pi}_\mu = 0$$

means that there are no collective motions along the direction of c.m. momentum (e.g., no ghosts). The properties of these intrinsic collective variables differ from the properties of intrinsic variables of more conventional models and, in particular, from those of the intrinsic positions and momenta of the canonical (3+1)-dimensional oscillator[6] which have the following commutation relations:

$$[\hat{\xi}_\mu, \hat{\xi}_\nu] = 0, \quad [\hat{\pi}_\mu, \hat{\pi}_\nu] = 0, \quad [\hat{\xi}_\mu, \hat{\pi}_\nu] = -i\,\eta_{\mu\nu}. \tag{3}$$

The description of the mass and spin spectrum of a hadron tower is obtained by choosing constraints that relate the mass and spin operators to the intrinsic collective variables and by choosing an irreducible representation of SO(3,2). For the spin we always choose the spin tensor constraint

$$\Sigma_{\mu\nu} = \check{g}_\mu{}^\rho \check{g}_\nu{}^\sigma S_{\rho\sigma}.$$

The spin tensor $\Sigma_{\mu\nu}$ is then a relativistically covariant version of the generators S_{ij} of the SO(3) subgroup of SO(3,2) and in the c.m. rest frame $\hat{p}_{\mu\,\mathrm{rest}} = (1,0,0,0)$ the spin $\hat{W}$, which equals $\mathbf{J}^2$ due to $\hat{w}_i \overset{\mathrm{rest}}{=\!=} 1/2\varepsilon_{ijk}J_{jk} \equiv J_i$ and $\hat{w}_0 \overset{\mathrm{rest}}{=\!=} 0$, also equals $\mathbf{S}^2 \equiv 1/2 S_{ij}S^{ij}$ due to $\Sigma_{ij} \overset{\mathrm{rest}}{=\!=} S_{ij}$ and $\Sigma_{0i} \overset{\mathrm{rest}}{=\!=} 0$:

$$\mathbf{J}^2 \overset{\mathrm{rest}}{=\!=} \hat{W} \overset{\mathrm{rest}}{=\!=} \mathbf{S}^2.$$

For the mass we allow more flexibility in the choice of the constraint. A simple constraint which gives a reasonable fit to the masses of the mesons of the ρ/a-tower is the rotating-vibrator constraint

$$M^2 = m_0^2 + \frac{1}{\alpha'}\hat{P}_\rho \Gamma^\rho + \lambda^2 \frac{1}{2}\Sigma_{\mu\nu}\Sigma^{\mu\nu} \tag{4}$$

where m_0^2, $1/\alpha'$ (the Regge slope), and λ^2 are system parameters.

Figure 2 shows an assignment of mesons of the ρ/a-tower to the K-type of the irreducible representation $D(\mu_{min}=2, s=1)$ of SO(3,2).* Each dot stands for an irreducible

---

* All I=1, CP=+ mesons listed in the Meson Summary Table[7] are assigned except for ρ(1250) which is no longer established and $a_0(980)$ which cannot be accommodated by $D(\mu_{min}=2, s=1)$ but can be accommodated by any $D(\mu_{min}>2, s=1)$. The $D(\mu_{min}, s)$ denote the unitary irreducible representations of

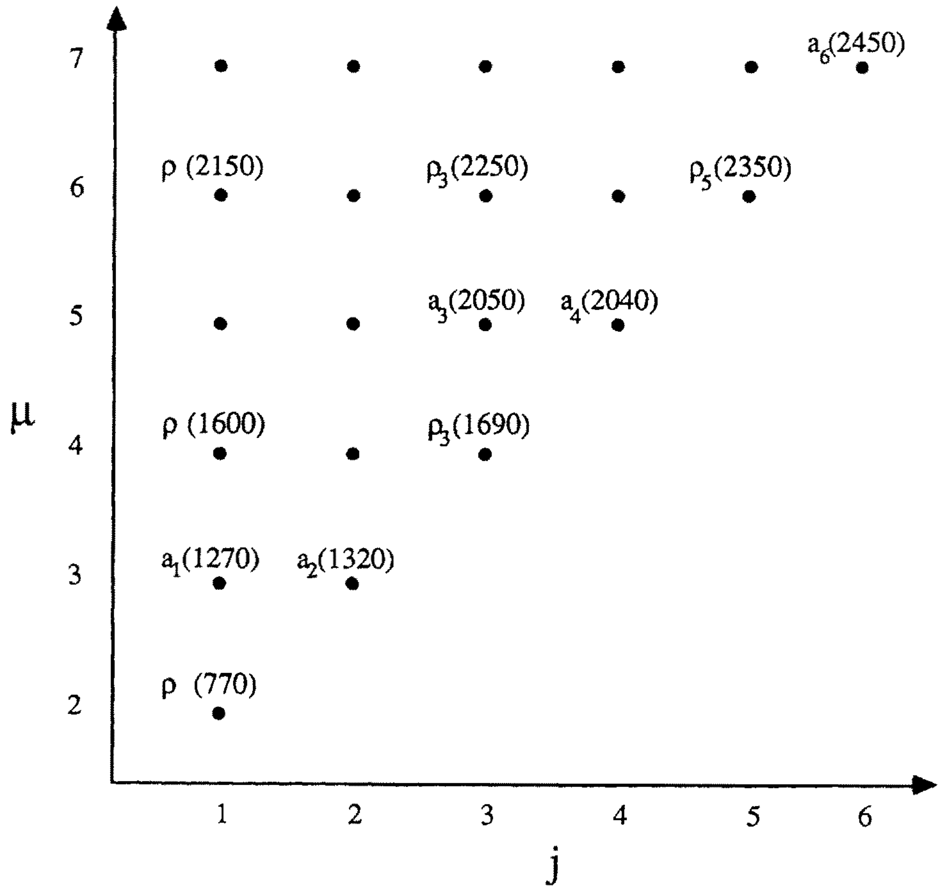

Figure 2. Mesons of the ρ/a-tower assigned to the K-type of the irreducible representation $D(\mu_{min}=2, s=1)$ of SO(3,2).

representation space $\mathsf{R}^{(\mu,j)}$ of the maximal compact subgroup K=SO(3)×SO(2) of SO(3,2) whose vectors describe (for fixed c.m. momentum direction $\hat{p}_\mu$, subject to $\hat{p}_\mu \hat{p}^\mu = 1$ for positive mass particles) the physical states of a ρ or an a meson with vibrational quantum number μ (μ = eigenvalue of $\hat{P}_\rho \Gamma^\rho$) and spin (rotational) quantum num-

---

SO(3,2) for which the spectrum of Γ_0 is bounded from below, where μ_{min} is the minimum value of Γ_0 and s(s+1) is the eigenvalue of S^2 for vectors for which Γ_0 has eigenvalue μ_{min}; such representations exist for $\mu_{min} \geq s+1/2$ when $s \in \{0,1/2\}$ and for $\mu_{min} \geq s+1$ when $s \in \{1,3/2,2,5/2,...\}$.[8]

ber j [j(j+1) = eigenvalue of $1/2\Sigma_{\mu\nu}\Sigma^{\mu\nu}$]. The direct sum of the spaces $\mathsf{R}^{(\mu,j)}$ gives the irreducible representation space

$$\mathsf{H}(\mu_{min}=2,s=1) = \sum_{(\mu,j)\text{ of Fig.2}} \oplus\, \mathsf{R}^{(\mu,j)}$$

of SO(3,2) whose vectors describe all physical states of the ρ/a-tower. (When the c.m. momentum direction varies the space of physical states becomes the direct product $\mathsf{H}_{orb}\otimes\mathsf{H}(\mu_{min}=2,s=1)$ where H_{orb} is the orbital space spanned by the eigenvectors $|\hat{p}_\mu\rangle$ of $\hat{P}_\mu$.) For $D(\mu_{min}=2,s=1)$ the vibrating-rotator constraint (4) yields the mass formula

$$m^2(\mu,j) = m_0^2 + \frac{1}{\alpha'}\,\mu + \lambda^2 j(j+1)$$

and a least squares fit to the masses of the mesons of the ρ/a-tower, assigned as shown in Figure 2, yields the following values for the parameters:

$$(m_0^2 + \frac{1}{\alpha'}) = -\,0.51\ (\text{GeV})^2, \qquad \frac{1}{\alpha'} = 1.06\ (\text{GeV})^2, \qquad \lambda^2 = 0.02\ (\text{GeV})^2.$$

The small value of λ^2 means that the masses are described almost as well without the spin term $\lambda^2 j(j+1)$. With these values of the parameters and the spectrum of (μ,j) shown in Figure 2, M^2 has a positive definite spectrum; there are no negative mass states (no tachyons).

The operators $\Sigma_{\mu\nu}$ and $\hat{P}_\rho\Gamma^\rho$ transform only within each subspace $\mathsf{R}^{(\mu,j)}$ since they commute with $\hat{P}_\rho\Gamma^\rho$ and $\hat{W}$. The operators $\hat{\xi}_\mu$ and $\hat{\pi}_\mu$, however, do not commute with $\hat{P}_\rho\Gamma^\rho$ or $\hat{W}$; they change the vibrational quantum number μ (by ± 1) and spin quantum number j (by ± 1 or 0) and transform between the subspaces $\mathsf{R}^{(\mu,j)}$. The $\hat{\xi}_\mu$ and $\hat{\pi}_\mu$, and powers of them, can therefore be used to describe transitions between the vibrational and spin levels of the hadron tower in complete analogy to the description of transitions between spin levels of the molecule. In the relativistic setting, like in the non-relativistic setting, the description of decay rates will depend upon the choice of an interaction Hamiltonian.

Radiative Decays

We now consider the radiative decays of nucleon resonances N* into the proton or neutron. The decay rate for the process $N^* \rightarrow N + \gamma$ is usually expressed in terms of two independent (after using symmetry) photoelectric amplitudes $A_{j3}^{N^*}$:[9]

$$\Gamma = \frac{2k^2}{\pi(2j+1)} \frac{m_N}{m_{N^*}} \left(|A_{1/2}^{N^*}|^2 + |A_{3/2}^{N^*}|^2 \right)$$

where $j_3 \in \{1/2, 3/2\}$ is the helicity of the decaying state N*, j is the spin quantum number of N*, m_{N^*} and m_N are the masses of N* and N, and $k=[(m^2_{N^*} - m^2_N)/(2m_{N^*})]$ is the photon energy. Experimental values of the photoelectric amplitudes are obtained from partial wave analysis of single pion photoproduction, $N\gamma \rightarrow N^* \rightarrow N\pi$.

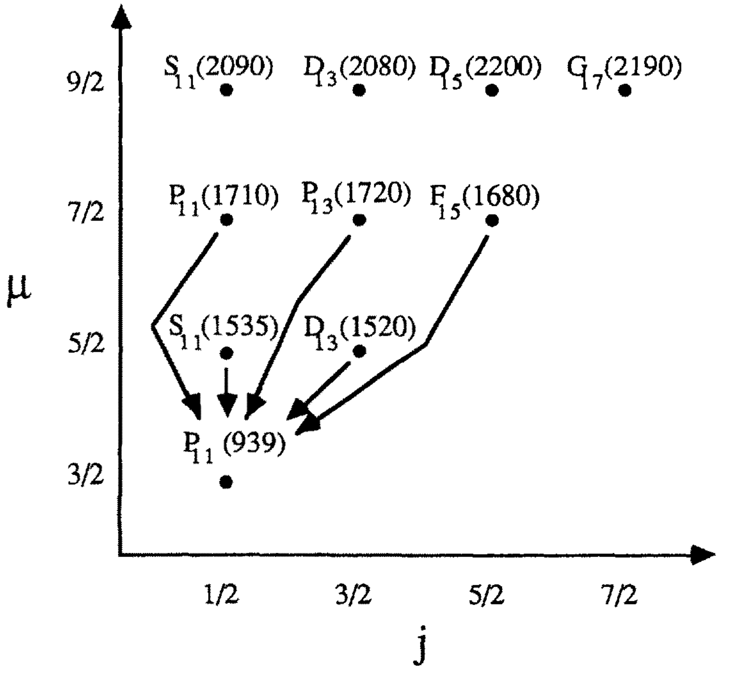

Figure 3. Some nucleon resonances assigned to the K-type of the irreducible representation $D(\mu_{min}=3/2, s=1/2)$ of SO(3,2). Arrows indicate for which of these resonances experimental values of the photoelectric amplitudes are known.

Figure 3 shows an assignment of some of the nucleon resonances to the K-type of the irreducible representation $D(\mu_{min}=3/2, s=1/2)$ of SO(3,2). Arrows indicate for which of these resonances experimental values for the photoelectric amplitudes are known;[7] these values are listed in Column 4 of Table I.

In order to describe these radiative decays we must choose an interaction Hamiltonian that couples the photon observables to the intrinsic collective observables $\hat{\xi}_\mu$ and $\hat{\pi}_\mu$ of the nucleon-tower. We obtain one in the following way: Starting with the free Hamiltonian

$$H_{free} = \varphi(P_\mu P^\mu - m_0^2 - \frac{1}{\alpha'}\hat{P}_\rho \Gamma^\rho - \lambda^2 \frac{1}{2}\Sigma_{\mu\nu}\Sigma^{\mu\nu}),$$

Table I

Resonance	Target	j_3	Experimental photoelectric amplitudes	SO(3,2) collective model	QCD inspired quark model	Relativistic oscillating quark model
$S_{11}(1535)$	p	1/2	73 ± 14	106	147	
	n	1/2	− 76 ± 32	− 59	− 119	
$D_{13}(1520)$	p	1/2	− 22 ± 10	− 26	− 23	34
	p	3/2	167 ± 10	167	128	109
	n	1/2	− 65 ± 13	− 66	− 45	− 31
	n	3/2	− 144 ± 14	− 138	− 122	− 109
$P_{11}(1710)$	p	1/2	5 ± 16	17	− 47	
	n	1/2	− 5 ± 23	− 1	− 21	
$P_{13}(1720)$	p	1/2	52 ± 39	59	− 133	
	p	3/2	− 35 ± 24	− 28	46	
	n	1/2	− 2 ± 26	− 10	57	
	n	3/2	− 43 ± 94	5	− 10	
$F_{15}(1680)$	p	1/2	− 17 ± 10	2	0	− 10
	p	3/2	127 ± 12	99	91	59
	n	1/2	31 ± 16	− 16	26	35
	n	3/2	− 30 ± 14	− 26	− 25	0

which is related to the constraint (4) with φ a Lagrange multiplier of constrained Hamiltonian mechanics,[10,11] we make the minimal coupling substitutions

$$P_\mu \rightarrow P_\mu - eA_\mu,$$
$$\hat{P}_\mu \rightarrow \hat{P}_\mu - e\, g_\mu{}^\rho \frac{1}{M} A_\rho$$

and the Pauli coupling[11] substitution

$$\Sigma_{\mu\nu} \rightarrow \Sigma_{\mu\nu} - \gamma F_{\mu\nu}$$

where A_μ and $F_{\mu\nu}$ are the electromagnetic potential and electromagnetic field operators and e and γ are coupling constants. We also postulate that A_μ and $F_{\mu\nu}$ act in both the photon and hadron spaces such that, for the emission of one photon with momentum k_μ and helicity λ, taking their photon space matrix elements leaves the following operators that act only in the hadron space:

$$\langle k_\sigma,\lambda|\, A_\mu\, |0\rangle = \varepsilon^*_\mu(k_\sigma,\lambda)\, \exp(i\beta k_\rho \hat{\xi}^\rho),$$

$$\langle k_\sigma,\lambda|\, F_{\mu\nu}\, |0\rangle = ik_\nu \langle k_\sigma,\lambda|\, A_\mu\, |0\rangle - ik_\mu \langle k_\sigma,\lambda|\, A_\nu\, |0\rangle$$

$$= i[k_\nu \varepsilon^*_\mu(k_\sigma,\lambda) - k_\mu \varepsilon^*_\nu(k_\sigma,\lambda)]\, \exp(i\beta k_\rho \hat{\xi}^\rho),$$

where $\varepsilon^*_\mu(k_\sigma,\lambda)$ is the photon polarization and β is a system parameter; unlike the conventional case, A_μ and $F_{\mu\nu}$ then depend on non-commuting intrinsic position operators $\hat{\xi}^\rho$ that act in the hadron space. These substitutions give, to first order in A_μ and $F_{\mu\nu}$ (sufficient when considering one photon processes), the interaction Hamiltonian

$$H_{int} = -\,\varphi(e\{P_\mu,A^\mu\} + e\frac{1}{\alpha'}\frac{1}{M}A_\mu\hat{\pi}^\mu - \gamma\lambda^2\frac{1}{2}\{\Sigma_{\mu\nu},F^{\mu\nu}\}).$$

On c.m. rest frame states H_{int} may be written as

$$H_{int} = -\,\varphi[e\{P_0,A_0\} + e\frac{1}{\alpha'}\frac{1}{M}\mathbf{A}\cdot\Gamma - \gamma\lambda^2(\mathbf{B}\cdot\mathbf{S} + \mathbf{S}\cdot\mathbf{B})]$$

where $S_i \equiv 1/2\varepsilon_{ijk}S_{jk}$ and $B_i \equiv 1/2\varepsilon_{ijk}F_{jk}$; the first term is an electrostatic coupling which becomes zero in the gauge $A_0 = 0$, the second term is an electric dipole coupling (with zero electric dipole moment due to Zitterbewegung of the expectation values of Γ, but non-zero dipole transition moments), and the third term is a magnetic moment coupling.

The theoretical expressions for the photoelectric amplitudes can be derived from H_{int} in much the same way as in atomic or nuclear physics; the result of a lengthy calculation is

$$A_{j3}^{N*(\mu,j)} \underset{\text{phase}}{=} \frac{\pi}{\sqrt{2k\, m_N^3\, m_{N*}^3}} \langle \mu'=3/2\ j'=1/2\ j_3'=j_3-1 |\ J_{-1}\ | \mu\ j\ j_3 \rangle$$

where (μ,j) are the vibrational and spin quantum numbers of the decaying state N* and $J_{-1} = 1/\sqrt{2}\,(J_1 - J_2)$ is a spherical component of the 3-vector operator

$$J_m = a\ \exp(i\beta k S^{03})\,\Gamma_m + bki\ \{\exp(i\beta k S^{03}), S_{m3}\}$$

with $a = \varphi e/(\alpha' m_N)$ and $b = -\varphi\gamma\lambda^2$. The matrix elements $\langle \mu'=3/2\ j'=1/2\ j_3'=j_3-1 |\ J_{-1}\ | \mu\ j\ j_3 \rangle$ are calculated from group theory; they depend on the parameters a, b, β, and also on phase factors [since the physical state vectors are related to the basis vectors $|\mu\ j\ j_3\rangle$ of the space $\mathcal{H}(\mu_{min}=3/2, s=1/2)$ only up to arbitrary phases] which are chosen to give the best fit to the data. A least squares fit using experimentally determined values for the masses and with the further restriction $b_{proton}/b_{neutron} = g_{proton}/g_{neutron} \approx 1.5$ yields the following values for the parameters:

$$\begin{aligned} a_{proton} &= -1.98, & b_{proton} &= 0.77\ (\text{GeV})^{-1}, & \beta_{proton} &= 1.52\ (\text{GeV})^{-1} = 0.3\ \ \text{fm}, \\ a_{neutron} &= 0.77, & b_{neutron} &= -0.50\ (\text{GeV})^{-1}, & \beta_{neutron} &= 0.32\ (\text{GeV})^{-1} = 0.06\ \text{fm}. \end{aligned}$$

The predictions calculated using these values are listed in Column 5 of Table I. For comparison, the predictions of a QCD inspired non-relativistic potential model[12] and of a relativistic oscillating quark model based on the commutation relations of Eq.(3)[6] are listed in columns 6 and 7, respectively.

Supergroups

Physical systems with more complicated spectra can be understood as more complicated extended objects having more complicated intrinsic collective motions and larger SGGs. Supergroups can also be used as SGGs with the difference that they would describe spectra containing both integer and half-odd integer spins. The simplest such supergroup is Osp(1,2) generated by the S_i of SO(3) and by a two-component spinor operator χ_α satisfying

$$[S_i,\chi_\alpha] = -1/2(\sigma_i)_{\alpha\beta}\chi_\beta, \quad \{\chi_\alpha,\chi_\beta\} = -(\sigma_i\sigma_2)_{\alpha\beta}S_i,$$

where σ_i, $i \in \{1,2,3\}$, are the Pauli matrices. Unlike the ξ_i of SO(3,1), the χ_α of Osp(1,2) transform between irreducible representation spaces $\mathcal{R}^{(j)}$ of SO(3) with j values differing by 1/2.

The supergroup Osp(1,4) contains SO(3,2) as its even subgroup and has, for $s \in \{1/2,1,3/2,2,\ldots\}$, irreducible representations that reduce into the representations $D(\mu_{min}{=}s+1,s)\oplus D(\mu_{min}{=}s+3/2,s+1/2)$ of SO(3,2).[13]* There exists some evidence that $D(\mu_{min}{=}3/2,s{=}1/2)\oplus D(\mu_{min}{=}2,s{=}1)$, along with the constraint

$$M^2 = m_0^2 + \frac{1}{\alpha'}\,\frac{1}{4}\sum_{\beta=1}^{4}\{\hat{Q}_\beta,\hat{Q}_\beta^\dagger\}$$

where the $\hat{Q}_\beta$ are "boosted" versions of the fermionic generators (a Majorana spinor operator) of Osp(1,4), describes nucleons and ρ/a mesons with a single slope $1/\alpha'$ for their Regge trajectories.[14] Evidence for supermultiplets also appears in atomic and in nuclear physics.[15]

Conclusion

The specific models considered above have served their purpose to illustrate the conceptual simplicity of the spectrum generating group approach and to show that it can be used in relativistic physics to describe hadron masses and transition rates without using

---

* These representations are also irreducible representations of the supergroup SU(2,2/1).

approximation procedures. Theoretical assumptions underlying the choices of these models have been kept to a minimum. Different choices which rely, perhaps, on further or different theoretical assumptions may give better descriptions of the data.

One feature that connects our SO(3,2) relativistic collective models with atomistic models is that the invariant s that characterizes the representations $D(\mu_{min},s)$ can be interpreted to be the total intrinsic spin of the hadron tower's fundamental constituents—e.g., of its quarks. With this interpretation our choices s=1 for the ρ/a mesons and s=1/2 for the nucleons are consistent with those of the standard non-relativistic quark model.[7] In the non-relativistic limit $c\rightarrow\infty$ and $\mu_{min}(c)\rightarrow\infty$ one has the group contractions $\mathcal{P}\times U(1)\rightarrow G^{ex}$ and $SO(3,2)\rightarrow SO(3)\circledS HO(3)$ and one may obtain from the operators $\hat{\xi}_\mu$ and $\hat{\pi}_\mu$ with the commutation relations of Eq.(2) operators $\hat{\xi}_i^{(\infty)}$ and $\hat{\pi}_i^{(\infty)}$ with the commutation relations of the 3-dimensional oscillator group HO(3):[16]

$$[\hat{\xi}_i^{(\infty)},\hat{\xi}_j^{(\infty)}] = 0, \quad [\hat{\pi}_i^{(\infty)},\hat{\pi}_j^{(\infty)}] = 0, \quad [\hat{\xi}_i^{(\infty)},\hat{\pi}_j^{(\infty)}] = i\,\delta_{ij}I.$$

In the limit, s(s+1) is the eigenvalue of the quarks' total intrinsic spin operator $(\mathbf{S} - \hat{\xi}^{(\infty)} \times \hat{\pi}^{(\infty)})^2$ where **S** is the spin angular momentum operator of the hadron tower and $\hat{\xi}^{(\infty)} \times \hat{\pi}^{(\infty)}$ is the operator for the orbital angular momentum of the quarks around each other.

References

[1] G. Herzberg, **Molecular Spectra and Molecular Structure,** D. van Nostrand Publishers (1966).

[2] A. Bohr, B. Mottelson, **Nuclear Structure,** Vol. II, Benjamin (1969).

[3] A. Arima, F. Iachello, Ann. Rev. Nucl. Part. Sci. **31,** 75 (1981).

[4] H. D. Politzer, Proceedings 21st International Conference on High Energy Physics, Journal de Physique **43,** p. C3-659 (1982).

[5] M. A. Naimark, **Linear Representations of the Lorentz Group,** Pergamon Press, (1964); A. Bohm, **Quantum Mechanics,** 2nd ed., Springer-Verlag, (1986).

[6] R. P. Feynman, M. Kislinger, F. Ravndal, Phys. Rev. D**3,** 2706 (1971). See also T. Takabayashi, Prog. Theor. Phys. Suppl. **67,** 1 (1979) and Y. S. Kim, E. Noz, **Theory and Applications of the Poincaré Group,** Reidel Publishing (1986) for additional references.

[7] Particle Data Group: Review of Particle Properties, Phys. Lett. **170B,** April 1986

[8] N. T. Evans, J. Math. Phys. **8**, 170 (1967); C. Fronsdal, Phys. Rev. D**12**, 3819 (1975).

[9] Particle Data Group: Review of Particle Properties, Rev. Mod. Phys. **48**, No. 2, Part II, April 1976.

[10] P. A. M. Dirac, Can. J. Math. **2**, 129 (1950).

[11] A. J. Hanson, T. Regge, Annals Phys. **87**, 498 (1974), section 4B.

[12] K. R. Koniuk, N. Isgur, Phys. Rev. D**21**, 1868 (1980).

[13] W. Heidenreich, Phys. Lett. **110B**, 461 (1982).

[14] A. Bohm, Phys. Rev.Lett. **57**, 1203 (1986).

[15] A. B. Balantekin, K. Bars, and F. Iachello, Phys. Rev. Lett. **47**, 19 (1981); H. -Z. Sun, M. Vallieres, D. H. Feng, R. Gilmore, and R. F. Casten, Phys. Rev. C **29**, 352 (1984); V. A. Kostelecky and M. M. Nieto, Phys. Rev. Lett. **53**, 2285 (1984).

[16] A. Bohm, M. Loewe, P. Magnollay, M. Tarlini, R. R. Aldinger, L. C. Biedenharn, H. van Dam Phys. Rev. D**32**, 2828 (1985).