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$Sp(0)$ and Orientifolds

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Abstract

I discuss the geometric engineering perspective of the solution to the discrepancy between matrix model and field theory effective superpotential in the case of *None* gauge theories with matter in the antisymmetric representation.

I. INTRODUCTION

In a series of important papers Dijkgraaf and Vafa[1] formulated the conjecture that the exact effective Superpotential of a confining $N = 1$ supersymmetric gauge theory can be computed with the help of a simple matrix model. Much work has been devoted to apply this conjecture to theories with varying gauge groups and matter content. A rather puzzling result has been reported by Kraus and Shigemori in[2]. The authors investigated theories with orthogonal or symplectic gauge group and chiral matter multiplets in two index tensor representations. In the case of $Sp(N)$ gauge groups and matter in the antisymmetric representation they found that matrix-model computation did not reproduce the known results for low rank gauge groups and simple superpotentials.

Their calculation was a perturbative evaluation of the matrix-model partition function and this lead to the impression that maybe some important non-perturbative effects are missed. In two independent papers[3] and[4] the large N expansions of the matrix models were therefore compared with the Konishi-anomaly constraints of the field theory. Rather than showing a discrepancy a perfect match was found therefore even deepening the puzzle.

However, a solution could be found by Cachazo in[5]. The main ingredient of his solution was embedding the $Sp(N)$ theory in a theory with unitary gauge group. The embedding was such that $Sp(N)$ theory with tree-level superpotential of degree n was embedded into a theory with unitary gauge group $U(N + 2n)$ and adjoint matter. In particular this implied that the unbroken vacuum with $Sp(N)$ gauge group is embedded into a broken vacuum of the unitary theory with braking pattern $U(N + 2n) \rightarrow U(N + 2) \otimes U(2)^{n-1}$. This unexpected appearance of $U(2)^{n-1}$ has lead to the the notion of "$Sp(0)$" gauge group factors.

In this talk I will discuss how the specifics of the embedding, in particular the appearance of the $Sp(0)$ gauge group factors, can be understood as orientifold effects in the geometric engineering of the gauge theories[6].

II. REVIEW OF THE DISCREPANCY

Following Dijkgraaf and Vafa one expects that the exact effective superpotential of a confining $N = 1$ supersymmetric gauge theory can be computed using a matrix model whose action is given by the tree-level superpotential of the gauge theory. The partition function of the matrix model is given by

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$$Z = \frac{1}{|\mathrm{G}|} \int d \exp \left[\frac{1}{\kappa} (W_{\mathrm{tree}}()) \right] \tag{2.1}$$

where |G| denotes the volume of the gauge group and

$$W(z) = \sum_{k=1}^{n} \frac{t_k}{k+1} z^{k+1} , \tag{2.2}$$

is the tree level superpotential of the gauge theory. In a semiclassical large expansion we set

$$S_i = \kappa_i \tag{2.3}$$

which in field theory are interpreted as the gaugino bilinears of the gauge group factors of the breaking pattern $Sp(N) \to \prod_i Sp(N_i)$induced by a vev of an antisymmetric representation. $S_i = -\frac{1}{32\pi^2} W_i^\alpha W_{\alpha,i}$. From the matrix model free energy

$$F(S) = -\kappa^2 \log(Z) \tag{2.4}$$

we obtain the effective superpotential according to[1],[7]

$$W_{eff} = \sum_i N_i \frac{\partial F_2(S)}{\partial S_i} + F_2 + 4F_{\P^2} + \tau S. \tag{2.5}$$

To be specific let us look to the example of the $Sp(N)$ theory with cubic superpotential

$$W_{tree} = \frac{m}{2} \phi^2 + \frac{g}{3} \phi^3. \tag{2.6}$$

The field ϕ is transforming under the antisymmetric representation, more precisely we have $\phi.I = I.\phi^T$ such that $A = \phi.I$ is *antisymmetric*

$$I = \begin{pmatrix} 0 & \mathbf{1}_{N/2} \\ -\mathbf{1}_{N/2} & 0 \end{pmatrix} \tag{2.7}$$

Gauge transformations are $\phi \to U\phi U^\dagger$ and leave I invariant. Performing the matrix model calculation in the unbroken phase with $Sp(N)$ gauge group perturbatively around $\langle A \rangle = z_0.I$ leads to the result[2]:

$$W_{VY} = (N+2)\, S \left[1 - \log\left(\frac{\Lambda^3}{S} \right) \right] \tag{2.8}$$

$$W_{mm}^{pert} = (3-N)\alpha S^2 + \left(\frac{59}{3} - \frac{16}{3} N \right) \alpha^2 S^3 + \left(197 - \frac{140}{3} N \right) \alpha^3 S^4 \ldots$$

where $\alpha = \frac{g^2}{2m^3}$. Here W_{VY} is the Veneziano-Yankielowicz part of the superpotential and has to be added by hand. Alternatively it can be derived from the measure part of the Matrix model integral[1]. The exact superpotential is supposed to be

$$W_{exact} = W_{V.Y.} + W_{mm}^{pert} , \tag{2.9}$$

setting

$$\frac{\partial W_{exact}}{\partial S} = 0 \tag{2.10}$$

and eliminating S in terms of Λ and choosing the low rank gauge group $Sp(4)$.

$$W_{mm}(\Lambda, \alpha) = 3\Lambda^3 - \Lambda^6 \alpha - \Lambda^9 \alpha^2 - \frac{352}{27} \Lambda^{12} \alpha^3 + \ldots \tag{2.11}$$

For some simple superpotentials and low rank gauge groups pure field theory considerations based on holomorphicity of the superpotential have however lead to[8]

$$W_{gt}(\Lambda, \alpha) = 3\Lambda^3 - \Lambda^6 \alpha - 2\Lambda^9 \alpha^2 - \frac{187}{27} \Lambda^{12} \alpha^3 + \ldots \tag{2.12}$$

III. KONISHI ANOMALIES AND LOOP EQUATIONS

It is by now well known that generalized Konishi anomalies in field theory can be mapped in a one to one manner to the large expansion of the exact matrix model loop equations[9]. Since this is based on Ward-identities such a comparison between matrix model and field theory is non-perturbative in nature.

In the chiral ring the Konishi anomalies take the form

$$\delta \Phi_I \frac{\partial W}{\partial \Phi_I} = -\frac{1}{32\pi^2}{}_I{}^{\alpha}{}_{\alpha,J}{}^{J}{}^{K} \frac{\partial (\delta \Phi_K)}{\partial \Phi_I} \tag{3.1}$$

where $\delta\phi$ is a holomorphic field transformation and $^\alpha$ the superfield-strength. Capital indices denote a bases of the Representation of ϕ. Choosing

$$\delta\phi = \frac{1}{z-\phi} \quad , \quad \delta\phi = -\frac{1}{32\pi^2} \frac{{}^2}{z-\phi} \tag{3.2}$$

leads to Ward identities for the generating functions of the chiral correlators (ϕ^k) and $({}^2\phi^k)$:

$$\frac{1}{2} R^2(z) = W'(z) R(z) + \frac{1}{2} f(z)$$
$$T(z) R(z) = W'(z) T(z) + 2R'(z) + c(z)$$

where $f(z)$ and $c(z)$ are polynomials of order $n-1$ and $R(z) = -\frac{1}{32\pi^2}\left(\frac{2}{z-\phi}\right)$ and $T(z) = \left(\frac{1}{z-\phi}\right)$. Setting $u = R - W'$ they can be solved in terms of the hyperelliptic Riemann surface

$$u^2 = (W'(z))^2 + f(z) \tag{3.3}$$

The period integrals of R determine the gaugino condensates and the period integrals of $T(z)$ the ranks of the gauge group factors.

$$\oint_{A_i} R(z) =_i \quad , \quad \oint_{A_i} T(z) = N_i \tag{3.4}$$

The loop equations for the matrix model can be obtained from the Ward identity

$$\int d\left(\frac{\partial}{\partial}\frac{1}{z-}e^{-\frac{1}{\kappa}W()}\right) = 0 \tag{3.5}$$

Introducing the matrix model resolvent $\omega(z) = \kappa\left(\frac{1}{z-\phi}\right)$ they read

$$\left\langle \frac{1}{2}\omega(z)^2 - \frac{1}{2}\kappa\omega'(z) - W'(z)\omega(z) - \frac{1}{2}\tilde{f}(z)\right\rangle = 0 \tag{3.6}$$

In the large expansion $\langle\omega(z)\rangle = \omega_0(z) + \frac{\kappa}{}\omega_1(z) + \ldots$ we note that the loop equations reproduce the Konishi anomalies of the field theories if we identify $\omega_0(z) = R(z)$ and $\sum_i N_i \frac{\partial\omega_0(z)}{\partial S_i} + 4\omega_1(z) = T(z)$. Since $\langle\omega(z)\rangle$ determines the free energy of the matrix model and $T(z)$ the superpotential it also follows from these identifications that $W_{eff} = \sum_i N_i \frac{\partial F_0(z)}{\partial S_i} + 4F_1$ up to a piece that is independent of the tree-level couplings t_k.

Although the perturbative calculation in the matrix model shows a clear discrepancy from field theory the loop equations reproduce the Konishi anomalies perfectly.

IV. GEOMETRIC ENGINEERING

The puzzle of the discrepancy between the the field theory results and the matrix model has been solved by Cachazo in[5]. His main tool was an embedding of the $Sp(N)$ theory into a theory with unitary gauge group. We want to give a geometric engineering explanation of his results. But we can even do a bit more. We will consider the complete set of theories with orthogonal or symplectic gauge groups with two-index matter representations, i.e. we will consider the two cases

(A) $SO(N)$ + symmetric or
$SP(N)$ + antisymmetric
(B) $SO(N)$ + adjoint or $SP(N)$ + adjoint

In both cases we can start with the singular A_1 fibration in IIB string theory

$$X_0: \quad xy = (u - W'(z))(u + W'(z)) \quad , \tag{4.1}$$

The blow up $\hat{X}$ can be described as

$$\beta(u - W'(z)) = \alpha x$$
$$\alpha(u + W'(z)) = \beta y$$
$$(u - W'(z))(u + W'(z)) = xy$$

The exceptional divisors are parametrized by the projective coordinates $[\alpha, \beta]$ and lie above $z = z_j$ with $W'(z_j) = 0$ and will be denoted by D_j.

We will also consider the two holomorphic $_2$-orientifold actions:

$$\hat{k}_A: \quad ([\alpha,\beta], z, u, x, y) \longrightarrow ([-\beta,\alpha], z, -u, y, x)$$
$$\hat{k}_B: \quad ([\alpha,\beta], z, u, x, y) \longrightarrow ([-\beta,\alpha], -z, u, -y, -x) \quad ,$$

In the case B we have to demand that $W(z)$ is an even polynomial. Correspondingly there are then $n = 2m+1$ zeros of $W'(z)$ that come in pairs $z_{-j} = -z_j$ except for the fixed point $z_0 = 0$. We also note that $\hat{k}_A$ acts on each exceptional divisor, whereas $\hat{k}_B$ acts on D_0 but exchanges D_j with D_{-j}!

The fixed point loci are 5-Orientifolds:

$$\hat{O}_A: \quad x - y = u = x^2 + W'(z)^2 = 0 \quad , \quad \alpha/\beta = \pm i$$
$$\hat{O}_B = \quad x + y = z = x^2 + u^2 = 0 \quad , \quad \alpha/\beta = \pm i$$

where $O_A = O_5^{-\epsilon}$ and $O_B = O_5^{+\epsilon}$ and ϵ keeps track of the sign of the orientifold charge.
Wrapping D5-branes over the exceptional divisors we have a realization of the gauge groups:

	$\epsilon = +1$	$\epsilon = -1$
$\hat{k}_A$:	$\prod_{i=1}^n SO(N_i)$	$\prod_{i=1}^n SP(N_i)$
$\hat{k}_B$:	$SP(N_0) \otimes \prod_{i=1}^{n/2} U(N_i)$	$SO(N_0) \otimes \prod_{i=1}^{n/2} U(N_i)$

In case (A) we find the breaking patter of a symmetric representation for orthogonal gauge groups and the breaking pattern of an antisymmetric representation for symplectic gauge groups whereas in case (B) we recognize the breaking pattern that is induced by a vacuum expectation value of a field in the adjoint representation.

The geometry with the wrapped branes is supposed to undergo a geometric transition[10]. The gauge dynamics lets the $\P^1$'s shrink to zero size and instead three-cycles (S^3) grow. The D5-branes disappear but their original presence is now given by RR-flux on the S^3's. The geometry is then given by the deformation of X_0

$$xy = u^2 - W'(z)^2 - f(z) \tag{4.2}$$

where $f(z)$ polynomial of order $n-1$. Notice that the right hand side describes a hyperelliptic Riemann surface of precisely the same form as it is found in the solution of the Konishi anomalies (3.3). We chose a basis of three-cycles (A_i, B_i) with symplectic intersection number

$$\langle A_i, B_j\rangle = \delta_{ij} \tag{4.3}$$

All the B-cycles are non-compact. Indeed one can reduce the period integrals on the Calabi-Yau to period integrals on the Riemann surface (3.3)[11]

The effective superpotential of the gauge theory can then be computed as the flux-superpotential

$$W_{eff} = \int_{A_i}\Omega\int_{B_i}H - \int_{A_i}H\int_{B_i}\Omega \tag{4.4}$$

The period Integrals are

$$\begin{aligned} &\int_{A_i}\Omega = S_i \ , \quad \int_{A_i}H = N_i \ , \\ &\int_{B_i}\Omega = \frac{\partial F_0}{\partial S_i} \ , \quad \int_{B_i}H = \tau_i \end{aligned} \tag{4.5}$$

Such that the flux-superpotential can be written as

$$W_{eff} = \sum_i\left(N_i\frac{\partial F_0}{\partial S_i} - \tau_i S_i\right) \tag{4.6}$$

Let us investigate the relation to the Konishi-anomalies now and concentrate first on the case (A). The orientifold plane is given by

$$O_A: \quad x = y \ , \quad x^2 + W'(z)^2 + f(z) = 0\ . \tag{4.7}$$

The Riemann surface (3.3) can be extracted by just setting $x=y=0$. Notice that the orientifold pierces the Riemann surface precisely in the branch points $u=0$!

In[3],[4] it was shown that the Konishi-Anomalies for these gauge theories are

$$\begin{aligned} &\frac{1}{2}R^2 = W'(z)R(z) + \frac{f}{2} \ , \\ &W'T = TR - 2\epsilon R' + c. \end{aligned} \tag{4.8}$$

The equation for R can be solved in terms of the hyperelliptic curve

$$u^2 = W'(z)^2 + f(z) \tag{4.9}$$

where $R = W' - u$. T can be obtained then as

$$T = \frac{c}{u} - 2\epsilon\frac{W'' - u'}{u} = \tilde{T} + \Psi \tag{4.10}$$

where $\tilde{T} = \frac{\tilde{c}}{u}$ with $\tilde{c} = c - 2\epsilon W''$ and $\Psi = 2\epsilon\frac{u'}{u}$

$\Psi(z)$ is precisely the contribution of the orientifold!

The pair $(R, \tilde{T})$ satisfies the Konishi anomaly relations of unitary gauge groups with adjoint matter:

$$W'R = \frac{1}{2}R^2 - \frac{f}{2} \quad , \quad W'\tilde{T} = \tilde{T}R + \tilde{c}$$

Since we have subtracted the orientifold contribution we find that in this unitary theory the flux numbers are shifted according to

$$\tilde{N}_j = \oint_{A_j}\tilde{T} = \oint_{A_j}T - \oint_{A_j}\Psi = N_j - 2\epsilon \tag{4.11}$$

The breaking pattern of the Sp/SO theory is therefore mapped as

$$\begin{matrix} \prod_{j=1}^n SP(N_j) \\ \prod_{j=1}^n SO(N_j) \end{matrix} \rightarrow \begin{matrix} \prod_{j=1}^n U(N_j+2) \\ \prod_{j=1}^n U(N_j-2) \end{matrix} \tag{4.12}$$

onto the unitary theory. For a degree $n+1$ tree-level superpotential the unbroken phase of $SP(N)$ is mapped to a broken phase of $U(N+2n)$!

In particular for the cubic superpotential we find $SP(N)\otimes SP(0) \rightarrow U(N+2)\otimes U(2)$ There are two gaugino-condensates S_1 and S_2! Correspondingly the superpotential should be computed as

$$\begin{aligned} W_{eff}^{SP} &= N_1\frac{\partial F_0}{\partial S_1} + N_2\frac{\partial F_0}{\partial S_2} + 4F_1 \\ W_{eff}^{U} &= (N_1+2)\frac{\partial F_0}{\partial S_1} + (N_2+2)\frac{\partial F_0}{\partial S_2} \end{aligned}$$

$$\Rightarrow F_1 = \frac{1}{2}\left(\frac{\partial F_0}{\partial S_1} + \frac{\partial F_0}{\partial S_2}\right) \tag{4.13}$$

Even in the perturbative calculation one has to keep $S_2 = \kappa_2$ different from zero!

$$\langle A\rangle = \begin{pmatrix} z_0 I_1 & 0 \\ 0 & z_1 I_2 \end{pmatrix} \tag{4.14}$$

Indeed the calculation done in this manner for $SP(4)$ leads to

$$W_{gt}(\Lambda,\alpha) = 3\Lambda^3 - \Lambda^6\alpha - 2\Lambda^9\alpha^2 - \frac{187}{27}\Lambda^{12}\alpha^3 + \ldots \tag{4.15}$$

which now coincides with the field theory result!

Let us also briefly look to case (B) Here the orientifold is given by

$$O_B: \quad x = -y \quad , \quad z = 0 \quad , \quad x^2 + u^2 - f(0) = 0 \; . \tag{4.16}$$

The Konishi anomalies for these gauge theories are

$$W'R = \frac{1}{2}R^2 - \frac{f}{2} \quad , \quad W'T = TR + \frac{2\epsilon}{z}R + c \tag{4.17}$$

again they can be solved by

$$R = W' - u \quad , \quad T = \frac{c}{u} + \frac{2\epsilon}{z}\left[\frac{W'}{u} - 1\right] = \tilde{T} + \Psi \tag{4.18}$$

Which gives $\Psi(z) = 2\epsilon\frac{1}{z}$ now!
There is an orientifold contribution only at the cut around $z = 0$! R and $\tilde{T}$ fulfill again the Konishi anomaly relations of a unitary theory and the map of the breaking patterns is now

$$\begin{aligned} &SO(N_0)\times \prod_{j=1}^{n} U(N_j) \\ &\quad\to \quad U(N_0 - 2) \times \prod_{j=1}^{n} (U(N_j) \times U(N_j)) \\ &SP(N_0)\times \prod_{j=1}^{n} U(N_j) \\ &\quad\to \quad U(N_0 + 2) \times \prod_{j=1}^{n} (U(N_j) \times U(N_j)) \end{aligned} \tag{4.19}$$

V. CONCLUSION

We have seen that the solution to the matrix model puzzle in the case of Sp gauge groups with antisymmetric matter can be understood easily in terms of the geometric engineering. The somewhat mysterious "$Sp(0)$" gauge group factors can be traced back to flux contributions of the orientifold. Note that in the case of SO theories $SO(2)$ groups factors are mapped to $U(0)$ factors which confirms the absence of non-trivial gauge dynamics in this case.

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[1] R. Dijkgraaf and C. Vafa, "Matrix models, topological strings, and supersymmetric gauge theories," Nucl. Phys. B **644** (2002) 3 [arXiv:hep-th/0206255];
R. Dijkgraaf and C. Vafa, "On geometry and matrix models," Nucl. Phys. B **644** (2002) 21 [arXiv:hep-th/0207106];
R. Dijkgraaf and C. Vafa, "A perturbative window into non-perturbative physics," arXiv:hep-th/0208048.

[2] P. Kraus and M. Shigemori, JHEP **0304**, 052 (2003) [arXiv:hep-th/0303104].

[3] L. F. Alday and M. Cirafici, "Effective superpotentials via Konishi anomaly," JHEP **0305**, 041 (2003) [arXiv:hep-th/0304119].

[4] P. Kraus, A. V. Ryzhov and M. Shigemori, "Loop equations, matrix models, and N = 1 supersymmetric gauge theories," JHEP **0305**, 059 (2003) [arXiv:hep-th/0304138].

[5] F. Cachazo, "Notes on supersymmetric Sp(N) theories with an antisymmetric tensor," arXiv:hep-th/0307063.

[6] K. Landsteiner and C. I. Lazaroiu, "On Sp(0) factors and orientifolds," arXiv:hep-th/0310111.

[7] S. K. Ashok, R. Corrado, N. Halmagyi, K. D. Kennaway and C. Romelsberger, "Unoriented strings, loop equations, and N = 1 superpotentials from matrix models," Phys. Rev. D **67** (2003) 086004 [arXiv:hep-th/0211291];
H. Ita, H. Nieder and Y. Oz, "Perturbative computation of glueball superpotentials for SO(N) and USp(N)," JHEP **0301** (2003) 018 [arXiv:hep-th/0211261].

[8] P. L. Cho and P. Kraus, "Symplectic SUSY gauge theories with antisymmetric matter," Phys. Rev. D **54** (1996) 7640 [arXiv:hep-th/9607200].

[9] F. Cachazo, M. R. Douglas, N. Seiberg and E. Witten, "Chiral rings and anomalies in supersymmetric gauge theory," JHEP **0212**, 071 (2002) [arXiv:hep-th/0211170].

[10] C. Vafa, "Superstrings and topological strings at large N," J. Math. Phys. **42**, 2798 (2001) [arXiv:hep-th/0008142].

[11] A. Klemm, W. Lerche, P. Mayr, C. Vafa and N. P. Warner, "Self-Dual Strings and N=2 Supersymmetric Field Theory," Nucl. Phys. B **477**, 746 (1996) [arXiv:hep-th/9604034].