Schur Polynomials and Giant Graviton Dynamics

N. J. Park

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0.1 Declaration

I declare that this thesis is my own unaided work. It is being submitted for the Degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

Nicholas Joseph Park

June 9, 2010

0.2 Acknowledgments

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0.3 Abstract

We use the duality between Schur polynomials and half BPS operators in order to diagonalize the one loop dilatation operator studied in [1] and [2]. This problem has been studied for operators with $\mathcal{R}$-charge of $\mathcal{O}(1)$ and $\mathcal{O}(\sqrt{N})$, corresponding to Kaluza-Klein gravitons and strings respectively. Due to the complexity of the problem, there has been no prior attempt to study the problem with $\mathcal{R}$-charge of $\mathcal{O}(N)$ corresponding to D-branes solutions. In this work we study a large N limit where a particular sector decouples. This sector corresponds to two nearly maximal spherical giant gravitons.

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Chapter 1

Introduction

To date the best description of nature is given by the Standard Model. There are, however, a number of problems. One of the major problems is that the theory of the strong interaction (QCD) cannot be solved in the low energy limit using the existing perturbative methods of Quantum Field Theory. The coupling of this theory increases as we flow down to lower energy scales. Perturbation theory is not sensible for strong couplings. The origins of String Theory were an attempt to solve this problem. It was suggested that quarks were joined by a string, and as the distance between them grew, the force of attraction between them would grow, as was observed experimentally. This string corresponded to a gluon flux tube between these quarks. When a sufficient amount of energy was gained, the extra energy would result in a new string being created with a quark on each end. This model explained some aspects of quark confinement. Nambu and Goto realized that these results arose from a theory of relativistic strings. Although it was thought that this model does not explain reality, there is a renewed interest in this particular approach motivated by the AdS/CFT correspondence [3]. This conjecture

says that a Superstring Theory on an $AdS_5 \times S_5$ space is equivalent to an $\mathcal{N} = 4$ Super-Yang-Mills Theory (SYM) in 3+1 dimensions. There is a hope that we may, in time, be able to extend this correspondence so that we may be able to understand a general Yang-Mills Theory in 3+1 dimensions. If this can be achieved, then we will have found a way of describing QCD in the low energy limit.

A second major problem with the Standard Model is that it does not account for Gravity. We do not, as yet, have a consistent falsifiable theory that unites Quantum Mechanics and Gravity. A major hurdle is the fact that Newtons gravitational constant (G_N) has an inverse mass dimension ($\hbar = c = 1$). We can't expand the theory perturbatively in powers of G_N, since Quantum Field Theory tells us that such a coupling is not renormalizable. This means that we do not have a way of quantizing General Relativity. String Theory is a promising candidate as a theory consistent with both Quantum Mechanics and General Relativity.

We have seen two good reasons for studying String Theory, but we have not yet discussed how we would go about formulating such a theory. We already know how to build a theory of zero dimensional objects (particles). String Theory replaces these zero dimensional objects with one dimensional objects (strings), and later we will see that with certain boundary conditions we will be allowed to have various other higher dimensional objects (D-branes). String Theory starts with this basic premise, and attempts to solve the dynamics, while keeping the theory consistent. Consistency results in a number of difficult conceptual issues, such as 26 dimensions (10 for Superstrings). Along with these conceptual difficulties there have been

a number of promising results, such as predicting General Relativity and Quantum Field Theory. It is also consistent with the Standard Model at energies that we can achieve in the laboratory.

These strings are one dimensional, unlike strings that we see in our everyday world, which have a thickness and are made of a number of more 'fundamental' particles. The strings that we will study in String Theory are fundamental in that they are not composed of other more fundemental objects, in the way that a proton is reported to be made up of quarks.

When we consider a string that is made up of a number of particles, the tension of the string changes depending on how far the individual particles are pulled away from each other. As we stretch the string, the tension of the string will increase. When we consider this fundamental string, there are no basic particles that can move relative to each other. This means that this fundamental string must have a constant tension. Before beginning any further analysis, should String Theory prove to be correct, we will have a new fundamental constant of nature. The fundamental constant of Special Relativity relates lengths and times. The fundamental constant of Quantum Mechanics related distances and momentums. It is hoped that this new fundamental constant will relate Gravity (or curvature) with Quantum Field Theory.

All of this seems very promising, but there are a number of drawbacks to String Theory. The first is that performing many calculations is extremely difficult. Secondly there are a number of difficult conceptual problems that we do not fully understand. A final major problem is that we do not have many clues as to what is going on from experimental observations. For this

reason any possible source of insight about String Theory must be taken seriously.

In this work we study some aspects of the AdS/CFT correspondence.

$\mathcal{N} = 4$ SYM is unlike any physical theory of which we are aware. It is conformally invariant and super symmetric, but does shed some light on how to treat gauge theories. Although this duality does not match any physical theories, it does shed some insight required to understand String Theory as well as non-perturbative field theory. There is hope that we can extend this duality to breaking some of the super symmetry of the problem, and perhaps even the conformal invariance. There has been some progress in breaking some of the super symmetry by considering deformations of the S_5 background. If we can successfully break these symmetries, we may have a new way of considering QCD.

One of the major tests of this correspondence has been to consider the case where we hold $\lambda = g_{YM}^2 N$, fixed and take $N \to \infty$. Here we are considering an gauge theory with gauge group U(N). We consider the natural extension to this work. We keep N large, but finite. This leads to a number of new difficulties. The combinatorial factors associated with correlation functions grows with N, so that non-planar diagrams are no longer suppressed. This makes it very difficult to study operators with a large number of fields.

We can probe the correspondence, in this limit, by considering giant gravitons [4]. These giant gravitons non-perturbative objects in string theory (D_3-branes) that would be difficult to study without the duality proposed by Maldacena. In this report we consider these giant gravitons interacting. According to [3], [5] & [6], the Hamiltonian for this system is given by

the dilatation operator that is associated with the CFT. We will study the dilatation operator at one loop.

There are a number of technical difficulties in performing such a calculation. The half BPS states are given as multi trace operators. When considering a large number of fields in the large N limit, these combinatorial factors become difficult to work out. Also these multi trace operators are not always diagonal. We employ methods using Schur polynomials to diagonalize the two point correlation functions. When considering a particular limit of these states, we can make progress in solving this problem, since we have a dynamical decoupling of a subset of all possible operators.

There are three major sections to this dissertation:

- Chapters 2-4 gives a brief description of a number of background tools required to understand the questions that this work attempts to ask.
- The remainder of the report highlights a number of results that apply to solving our problem, as well as discussing the solution to the problem in chapter 7.
- The appendices contain the mathematical tools required to solve this particular problem.

The work from chapters 7 and 8 as well as appendices B,C,D,E & F is new, and the results will be reported in a paper that is in preparation.

Chapter 2

String Theory

In this section we will outline the basic idea behind Bosonic String Theory. We will start with the particle by choosing an action for it. We will then consider including a "metric" (einbein) into the action as a tool that can be used to remove the square root from the action. Then we will look at the Nambu-Goto action and its classical dynamics, followed by the introduction of the Polyakov action and describing the symmetries thereof. We then will outline a particularly convenient gauge for considering the conformal symmetries of the action.

Thereafter we will discuss some aspects of Conformal Field Theory that will be useful, both for considering the Polyakov action and later when considering SYM.

We will then consider the path integral quantization of the Polyakov action, which involves introducing ghost fields.

The section will be concluded by using some of the ideas from Conformal Field Theory to show that the stress energy tensor is anomalous, but that in 26 dimensions the ghost fields exactly cancel this anomaly [7], leaving the

quantum theory conformal.

2.1 Particle Theory

Consider a free particle moving in a d+1 dimensional flat spacetime with signature (-,+,+,...), and an action $S_0 = -m \int d\ell$ where $d\ell$ is the infinitesimal proper length of the worldline that the particle will follow. We can write $d\ell = \sqrt{-ds} = (-G_{\mu\nu} dX^\mu dX^\nu)^{1/2}$. We use G to describe the metric in spacetime. If we parameterize the particles worldline by $X^\mu(\lambda)$, we can write the action as:

$$S_o = -m \int d\lambda \left(-\frac{dX^\mu}{d\lambda} \frac{dX_\mu}{d\lambda} \right)^{1/2} \tag{2.1}$$

We have a first obvious symmetry. This action is invariant under the Poincaré group.

$$X^\mu \to \Lambda^\mu_\nu X^\nu + c^\mu$$

This is a global symmetry of the theory.

We also have a reparameterization invariance. $\lambda \to \lambda' = f(\lambda)$.

$$\begin{aligned} d\lambda &\to \frac{df}{d\lambda} d\lambda \\ \left(-\frac{dX^\mu}{d\lambda} \frac{dX_\mu}{d\lambda} \right)^{1/2} &\to \left(\frac{df}{d\lambda} \right)^{-1} \left(-\frac{dX^\mu}{d\lambda} \frac{dX_\mu}{d\lambda} \right)^{1/2} \\ S_0 &\to S_0 \end{aligned}$$

This is a local (gauge) symmetry of the theory. Since we have a gauge symmetry in which parameterization we use, we may as well choose a proper time parameterization. In what follows we will write $\frac{dq}{d\tau} = \dot{q}$

$$\begin{aligned}\delta S_0 =& m \int d\tau \left(-\dot{X}^\mu \dot{X}_\mu\right)^{-1/2} \dot{X}^\nu \delta \dot{X}_\nu \\ =& m \left(-\dot{X}^\mu \dot{X}_\mu\right)^{-1/2} \dot{X}^\nu \delta X_\nu \Big|_{\tau_i}^{\tau_f} \\ & - m \int d\tau \delta \dot{X}_\nu \frac{d}{d\tau} \left[\left(-\dot{X}^\mu \dot{X}_\mu\right)^{-1/2} \dot{X}^\nu\right]\end{aligned}$$

We usually set the variation at the boundary equal to zero $\delta X_\nu(\tau_f) = \delta X_{nu}(\tau_i) = 0$.

When we solve for $\delta S_0 = 0$, we get the equation of motion $\ddot{X}^\mu = 0$. This simply says that particles move along straight lines, or on a general background that strings move along geodesic lines.

If we consider a new field $g(\tau)$, the action eq.(2.1) can be written as:

$$S = \frac{1}{2} \int d\tau \left(g^{-1} \dot{X}^\mu \dot{X}_\mu - g m^2\right) \tag{2.2}$$

The equation of motion for g is given by:

$$g^2 m^2 = -\dot{X}^\mu \dot{X}_\mu$$

Putting this expression back into S, we recover S_0, showing that these actions are classically equivalent. Further, this new action preserves both the Poincaré invariance, as well as the reparameterization invariance.

Let's now consider a String action.

2.2 String Actions

The most important difference between a string and a particle is the fact that a string can be excited, whereas a particle cannot. When we consider the standard model, we are forced to define a large number of different particles (quarks, photons, etc.) It is hoped that with String Theory we will not have to define multiple different strings, but rather that the particles we are familiar with will emerge as various quantum modes of this one species of string. In this sense, String Theory hopes to explain why the standard model takes the form that it does.

2.2.1 Nambu-Goto Action

The dynamics of a particle was given by $X^\mu(\tau)$, which mapped out a worldline in spacetime. In the case of a string we now have an extra dimension, with an extra parameter σ. The dynamics of the string are now given by $X^\mu(\tau, \sigma)$. Just as we started considering the action of particles being the length of the world lines eq.(2.1), we now consider the action of a string to be the world area of the string. This is called the Nambu and Goto action. Just as we have $\frac{dq}{d\tau} \to \dot{q}$ we also define $\frac{dq}{d\sigma} \to q'$

$$S_0 = -T \int dA = -T \int d\tau d\sigma \left(-\det \partial_a X^\mu \partial_b X_\mu\right)^{1/2} \tag{2.3}$$

$$= -T \int d\tau d\sigma \left[\left(\dot{X}^\mu X'_\mu\right)^2 - \left(\dot{X}^\mu \dot{X}_\mu\right)\left(X'^\mu X'_\mu\right)\right]^{1/2} \tag{2.4}$$

We have introduced a new parameter T, which corresponds to the string tension. It is often written in terms of α', which is the Regge slope, or ℓ_s, which gives a fundamental length scale.

$$T = \frac{1}{2\pi\alpha'} = \frac{1}{2\pi\ell_s^2} \tag{2.5}$$

The first thing to notice here is that this action is invariant under reparameterizations. This means that if the length of the string is finite, we can scale the spacial parameter (σ) to run from $0 \to 2\pi$ (it can be shown that the energy of the string is proportional to its length, so any string of finite energy is a string of finite length). Define $\mathcal{L}(\dot{X}, X', \tau, \sigma)$ by $S = \int d\tau d\sigma \mathcal{L}$. Let's consider the classical dynamics of this theory.

$$\begin{aligned}\delta S_0 = &- \int d\tau d\sigma \left[\frac{\partial}{\partial\tau}\frac{\partial\mathcal{L}}{\partial\dot{X}^\mu} + \frac{\partial}{\partial\sigma}\frac{\partial\mathcal{L}}{\partial X'^\mu}\right]\delta X^\mu \\ &+ \int d\sigma \frac{\partial\mathcal{L}}{\partial\dot{X}^\mu}\delta X^\mu\Big|_{\tau_i}^{\tau_f} + \int d\tau \frac{\partial\mathcal{L}}{\partial X'^\mu}\delta X^\mu\Big|_{\sigma=0}^{\sigma=2\pi}\end{aligned}$$

The first boundary contribution is set equal to zero as was the case with the particle.

The second boundary term must also be equal to zero. There are a number of ways that this can be achieved. The first option is to identify $\sigma = 0$ with $\sigma = 2\pi$. This choice gives us a closed string. The second choice is to choose Dirichlet boundary conditions ($\delta X^\mu(\tau, 0) = \delta X^\mu(\tau, 2\pi) = 0$). this says that the end points of the string are attached to some fixed positions. Later we will see that this leads to the idea of a higher dimensional object, on which a string can start and end. This higher dimensional object has it own dynamics, and is called a D-brane. The final option is to use Neumann boundary conditions($\frac{\partial\mathcal{L}(\tau,0)}{\partial X'^\mu} = \frac{\partial\mathcal{L}(\tau,2\pi)}{\partial X'^\mu} = 0$)

Let's analyze the Neumann boundary conditions more carefully. We know that the canonical momentum (and in this case the actual momentum as well)

is defined by:

$$P^{\mu}_{\tau} = \frac{\partial \mathcal{L}}{\partial \dot{X}_{\mu}}$$
$$P^{\mu}_{\sigma} = \frac{\partial \mathcal{L}}{\partial X'_{\mu}}$$

Our equations of motion are given by:

$$\frac{\partial}{\partial \tau} P^{\mu}_{\tau} + \frac{\partial}{\partial \sigma} P^{\mu}_{\sigma} = 0 \text{ and } P^{\mu}_{\sigma}(\tau, 0) = P^{\mu}_{\sigma}(\tau, 2\pi) = 0$$

We have two different types of momentum in this problem. P^{μ}_{σ} is the momentum that is moving along the string. The boundary conditions say that no momentum is moving off the end point of the string, unless it is a closed string, in which case momentum can move around the string. The other type of momentum is the momentum that the string is carrying. The equation of motion that we see here is a statement about the conservation of energy and momentum of a free string.

When considering Dirichlet-type boundary conditions, momentum can move from the string. This does not makes sense unless the momentum from the string is flowing into another object. This is the first indication that D-Branes have their own dynamics. Energy (and momentum) can flow between D-branes and the string connected to them.

2.2.2 Polyakov Action

When considering the dynamics of a particle, we were able to get rid of the square root inside the action by introducing a new field (eq.(2.2)) in

such a way that the equations of motion remain unchanged . In a similar way we introduce the new action, understood by and named after Alexander Polyakov.

$$\begin{aligned} S &= -\frac{T}{2}\int d\sigma d\tau \sqrt{-g} g^{ab}\partial_a X^\mu \partial_b X_\mu \\ &= -\frac{T}{2}\int d^2\xi \sqrt{-g} g^{ab} h_{ab} \end{aligned} \tag{2.6}$$

Let's study this equation. First note that the indices here run over $a\&b$ corresponding to $\tau\&\sigma$. We have two different metrics. The metric $g = g(\tau, \sigma)$ refers to the co-ordinates or parameters on the world sheet. They give the metric on this 2 dimensional "target space" with d+1 scalar fields labeled by X^μ. This looks like a 2 dimensional theory of gravity, which we will see shortly can be quantized (without any anomalies) with 25+1 fields (X^μ). We then have a separate metric $G = G(X^\mu)$, which is the metric of the spacetime that the theory lives in.

Let's now consider some of the symmetries of this action.

We clearly have Poincaré invariance.

$$X^\mu \to \Lambda^\mu_\nu X^\nu + c^\mu$$

This is a global symmetry of the theory, which we had before.

We also have a reparameterization invariance.

$$\tau \to \xi^{(1)}(\tau, \sigma)$$
$$\sigma \to \xi^{(2)}(\tau, \sigma)$$

From this point on we will use the notation of $\xi^{(1)}$ and $\xi^{(2)}$ instead of σ and τ. This is a local (gauge) symmetry of the theory.

We also have a new type of symmetry in our theory.

$$g_{ab} \to e^{2\omega(\xi^{(1)},\xi^{(2)})} g_{ab}$$

This is called conformal (Weyl) symmetry, and is particularly useful in 2 dimensions. We have 2 free parameters ($\xi^{(i)}$), and 3 components in the most general metric on our target space. We can therefore use our freedom in ($\xi^{(i)}$) to set $g^{12} = 0$ and $g^{22} = g^{11}$, at least locally. With this choice we can write $g^{ab} \to g^{11}\eta^{\mu\nu}$. This is an important result. In 2 dimensions *all* metrics are conformally equivalent to the Minkowski metric, and therefore to any other metric.

This is not the case for higher dimensional spacetimes. This is the first indication that trying to build a theory of higher dimensional objects is harder than the theory of either particles or strings, since the metric on the worldsheet will not have as much freedom.

Let's now consider the equations of motion of the Polyakov action. First varying the action with respect to the metric, and using $\delta g = -g g_{ab}\delta g^{ab}$, where g_{ab} is defined by $g_{ab}g^{bc} = \delta^c_a$. We get:

$$\begin{aligned}
\delta S = &-\frac{T}{2}\int d^2\xi(-g)^{1/2}\delta g^{ab}\left[h_{ab} - \frac{1}{2}g_{ab}g^{cd}h_{cd}\right] \\
&\Rightarrow h_{ab} - \frac{1}{2}g_{ab}g^{cd}h_{cd} = 0 \\
&\Rightarrow -\frac{1}{4}\det(g_{ab})\left[g^{cd}h_{cd}\right]^2 = -det(h_{ab}) \\
&\sqrt{-h} = \frac{1}{2}\sqrt{-g}g^{ab}h_{ab}
\end{aligned}$$

If we put this expression back into the Polyakov action we recover the Nambu-Goto action, showing that these equations are classically equivalent.

At this point we recall a result from General relativity $T^{ab} \propto -\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g_{ab}}$. Here we define our stress energy tensor in the worldsheet as:

$$T^{ab} = -\frac{2\pi}{\sqrt{-g}}\frac{\delta S}{\delta g_{ab}} \tag{2.7}$$

Since the theory is conformally invarient, the equations of motion force $T^{ab} = 0$.

We are now in a position to solve the equations of motion for the Polyakov action in a particular gauge. We will choose the gauge where $g^{ab} = \eta^{ab}$.The equations of motion give:

$$\begin{aligned}
\ddot{X}^\mu - X''^\mu =& 0 \\
\dot{X}^\mu X'_\mu =& 0 \\
\dot{X}^\mu \dot{X}_\mu + X'^\mu X'_\mu =& 0 \\
\int d\tau X'^\mu \delta X_\mu \Big|_{\sigma=0}^{\sigma=2\pi} =& 0
\end{aligned}$$

An interesting consequence of the above equations of motion for the end-points of an open string with Neumann boundary condition is that:

$$\dot{X}(\tau, 2\pi)^\mu \dot{X}(\tau, 2\pi)_\mu = \dot{X}(\tau, 0)^\mu \dot{X}(\tau, 0)_\mu = 0$$

This says that the end points of a string with Neumann Boundary conditions are classically moving at the speed of light.

These equation are surprisingly equivalent to the equations of motion for the Nambu-Goto action. The simplicity of the equations of motion show the power of the Polykov action over the Nambu-Goto action. Although it is possible to quantize the Nambu-Goto action, the inconvenient square roots make this problem particularly cumbersome. We will consider the Polyakov action only from here on.

2.2.3 Fixing a Lightcone Gauge

In this section we go about fixing a lightcone gauge, that will be convenient for later use. We will consider the explicit case where the string is closed, so as to avoid awkward boundary conditions. Define the quantity $-i\sigma^{\pm} = \tau \pm \sigma$. This corresponds to doing two things. Going to a "light-cone" gauge, as well as Wick rotating the fields. We then fix the following by making this choice.

$$
\begin{aligned}
g_{+-} &= \frac{1}{2} \\
g^{+-} &= 2 \\
\partial_{\pm}\sigma^{\pm} &= 1 \\
\partial_{\pm}\sigma^{\mp} &= 0
\end{aligned}
$$

The action becomes:

$$S = -T \int d\sigma^{+} d\sigma^{-} \partial_{+} X^{\mu} \partial_{-} X_{\mu} \tag{2.8}$$

The stress energy tensor is given by:

$$
\begin{aligned}
T_{\pm\pm} &= -2\pi T \partial_\pm X^\mu \partial_\pm X_\mu \\
T_{\pm\mp} &= 0
\end{aligned}
\tag{2.9}
$$

And the equation of motion is given by:

$$
\partial_\pm \partial_\mp X^\mu = 0 \tag{2.10}
$$

When we come to studying 2 dimensional conformal theories, we will see why this particular gauge is so convenient.

2.3 Conformal field theories

We are interested in studying infinitesimal transformations of the form $x^\mu \to x'^\mu = x^\mu + \epsilon(x)^\mu$, which have the property that $g(x)^{\mu\nu} \to g'(x')^{\mu\nu} = e^{2\omega(x)} g(x)^{\mu\nu}$. If we consider an infinitesimal transformation, we have:

$$
\begin{aligned}
g'(x')_{\mu\nu} =& \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g(x)_{\alpha\beta} \\
=& \left(\delta^\alpha_\mu - \partial_\mu \epsilon^\alpha\right) \left(\delta^\beta_\nu - \partial_\nu \epsilon^\beta\right) g(x)_{\alpha\beta}
\end{aligned}
$$

Here we will consider the case where $g(x)^{\mu\nu}$ is the flat spacetime metric $\eta^{\mu\nu}$. If we require that the transformation be conformal, after contracting each side with $dx'^\mu dx'^\nu$, we get metric transforms according to:

$$
\begin{aligned}
\delta g_{\mu\nu} \propto& -\left(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu\right) \\
\frac{2}{d} \partial^\rho \epsilon_\rho \eta_{\mu\nu} =& \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu
\end{aligned}
\tag{2.11}
$$

We contract both sides of eq.(2.11) with $\eta^{\mu\nu}$ in order to determine the constant of proportionality, with d being the dimension of the space described by the metric g. We can now act on the above expression with $\partial^\mu, \partial^\nu$, or ∂^2. After collecting terms, we find:

$$\left[\partial_\mu \partial_\nu (d-2) + \eta_{\mu\nu}\partial^2\right] \partial^\rho \epsilon_\rho = 0 \tag{2.12}$$

Again we can see what is so special about a 1+1 dimensional space, such as the target space in String Theory. Conformal theories are much easier to build. We are now in a position to find the form of ϵ^μ for $d > 2$. From the above analysis, we can see that ϵ is at most quadratic in x. The transformation are given by:

$$\epsilon(x)^\mu = a^\mu + \omega^\mu_\nu x^\nu + \lambda x^\mu + 2(b \cdot x)x^\mu - x^2 b^\mu \tag{2.13}$$

We can easily interpret the terms in eq. (2.13). The first is a translation which has d generators (a^μ), the second is a Lorentz transformation with $\frac{d(d-1)}{2}$ generators(ω^μ_ν). These two sets of transformations form the Poincaré group. We then also have a scaling with 1 generator (λ), and finally we have the "special conformal transformation" ($2(b \cdot x)x^\mu - x^2 b^\mu$) which has d generators (b^μ). We now have a total of $\frac{(d+2)(d+1)}{2}$ generators. If we were considering a space time with 1 temporal component and d-1 spacial components, the Lorentz group would be isomorphic to SO(1,d-1). We could show that the "conformal group" in this space is isomorphic to S0(2,d). We have shown that at least there are the correct number of generators.

An important thing to note about Conformal field theories (where the

theory can be mapped into a hyperplane $\mathcal{R}^n$) is there is a one-to-one correspondence between states and local operators. We will use this relation in order to perform a number of calculations when we consider Giant Gravitons.

2.3.1 2D Conformal Field Theory

A large part of the above analysis does not apply to the case where we have a 2 dimensional metric. If we work in flat Euclidean space using eq.(2.13)

$$\begin{aligned}\partial^\rho \epsilon_\rho \delta_{\mu\nu} =& \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \\ \Rightarrow \partial_1 \epsilon_2 =& -\partial_2 \epsilon_1 \\ \partial_1 \epsilon_1 =& \partial_2 \epsilon_2 \end{aligned} \tag{2.14}$$

We can move to the complex plane by defining $z, \bar{z} = x_1 \pm i x_2$. This is the identical action to that used in fixing the gauge where $i\sigma^\pm = \tau \pm \sigma$, which was discussed in section 2.2.3.

Then $\epsilon, \bar{\epsilon} = \epsilon_1 \pm i\epsilon_2$. The above equations then take the form:

$$\partial_z \bar{\epsilon} = \partial_{\bar{z}} \epsilon = 0 \tag{2.15}$$

We see that the number of different functions that define a conformal transformation is infinite. If we consider the generators of conformal transformations, we define $l_n = -z^{n+1}\partial_z$ and $\bar{l}_m = -\bar{z}^{m+1}\partial_{\bar{z}}$ as the generators of the group, we have a Lie group with an infinite number of generators and the 'Virasoro' algebra:

$$[l_m, l_n] = (m-n)l_{m+n} \quad \left[\bar{l}_m, \bar{l}_n\right] = (m-n)\bar{l}_{m+n} \quad \left[l_m, \bar{l}_n\right] = 0 \tag{2.16}$$

We can now pick out a subgroup that can perform this mapping globally. It turns out that the subgroup that we want to consider is generated by $l_\pm, l_0$ & $\bar{l}_\pm, \bar{l}_0$. If we make this restriction we can identify the following: l_- generates a translation, l_0 generates a scaling, and l_+ generates a special conformal transformation, as we saw in the general case.

Using these generators the restricted conformal transformation is given by:

$$z \rightarrow \frac{az+b}{cz+d} \tag{2.17}$$

Although all conformal transformations are allowed in the 2 dimensional case, it is worth differentiating this sub-group from the rest.

We will return to using the conformal invariance, when we come to quantization, since these conformal transformations are a gauge symmetry of the theory.

2.4 Quantizing the String

We know from Quantum Field Theory that when we try to quantize a system, we usually need to introduce a regulator in order to understand the divergences. These regulators introduce a parameter that depends on the scale at which we are studying the theory. When we quantize the string we generally have to introduce some scale dependence. This scale dependence breaks the Conformal Symmetry. This is called the Weyl (trace) anomaly. If we break our conformal invariance, we have somehow introduced a scale on the parameters τ and σ, which describe the world sheet of the particle.

This is unphysical and will , in general, lead to inconsistencies. There are two ways of getting around this problem. We work with a carefully chosen number of fields (X^μ), or we introduce a new scalar field and modify the action. As we will see, if we work in 26 dimensions, the theory retains its scale invariance. This is called critical String Theory. The other option is working in any other dimension, introducing a new (Liouville) field, and modifying the action. We will only consider critical String Theory here.

We will consider the path integral quantization of the Polyakov action. Since the action is invariant under gauge transformation we will need to use the Fadeev-Popov method to quantize the system. This involves introducing ghost fields to the theory. These ghost fields contribute to the energy and momentum of the theory. The contribution from the ghosts and the contribution from the Weyl anomaly exactly cancel in 26 dimensions, as we will show shortly. We begin with

$$Z\left[X^\mu, g_{ab}\right] \propto \int \mathcal{D}X\mathcal{D}g e^{iS_{Pol}[X^\mu,\ g_{ab}]} \tag{2.18}$$

We need some sort of gauge fixing condition. We can choose any fiducial metric $\hat{g}$, and use the gauge fixing condition that $G = g - \hat{g} = 0$

The most general gauge transformation that we can write leaving the action invariant is given by:

$$\begin{aligned}\delta g_{ab} =& \partial_a\xi_b + \partial_b\xi_a + 2\Lambda\hat{g}_{ab}\\ =& (P\xi)_{ab} + 2\tilde{\Lambda}\hat{g}_{ab}\end{aligned}$$

$$(P\xi)_{ab} = \partial_a\xi_b + \partial_b\xi_a - (\partial^c\xi_c)\hat{g}_{ab} \quad \tilde{\Lambda} = \Lambda + \tfrac{1}{2}\partial^c\xi_c$$

$$
\begin{aligned}
Z &\propto \int \mathcal{D}X\mathcal{D}g\delta(G)e^{iS_{Pol}} \\
&\propto \int \mathcal{D}X\mathcal{D}\lambda det\left[\frac{\delta G}{\delta\lambda}\right] e^{iS_{Pol}}\Big|_{G=0} \\
&\propto \int \mathcal{D}X J e^{iS_{Pol}}\Big|_{G=0}
\end{aligned}
\tag{2.19}
$$

Here J is the Fadeev-Poppov determinant.

$$
\begin{aligned}
J^{-1} &= \int d\Lambda d\gamma_a \delta\left((P\gamma)_{ab} + 2\tilde{\Lambda}\hat{g}_{ab}\right) \\
&= \int d\Lambda d\gamma_a d\beta_{ab} exp\left[2i\pi \int d^2\xi\sqrt{-\hat{g}}\beta^{ab}\left((P\gamma)_{ab} - \partial_c\gamma^c\hat{g}_{ab} + 2\Lambda\hat{g}_{ab}\right)\right] \\
&= \int d\gamma_a d\beta_{ab}\delta(\beta^{ab}\hat{g}_{ab}) exp\left[2i\pi \int d^2\xi\sqrt{-\hat{g}}\beta^{ab}\left((P\gamma)_{ab} - \partial_c\gamma^c\hat{g}_{ab}\right)\right] \\
&= \int d\gamma_a d\hat{\beta}_{ab} exp\left[2i\pi \int d^2\xi\sqrt{-\hat{g}}\hat{\beta}^{ab}(P\gamma)_{ab}\right]
\end{aligned}
$$

We can invert J^{-1} to find J by using Grassmann valued $\gamma\&\hat{\beta}$. Introducing the ghost action with Grassmann valued fields $\hat{\beta}^{ab}$ and γ^a, we get:

$$
\begin{aligned}
S_{gh} =& \frac{1}{2}\int d^2\xi\sqrt{-\hat{g}}\hat{\beta}^{ab}(P\gamma)_{ab} \\
=& \int d^2\xi\sqrt{-g}\hat{\beta}^{ab}\partial_a\gamma_b
\end{aligned}
\tag{2.20}
$$

$$
Z \propto \int \mathcal{D}X\mathcal{D}\gamma\mathcal{D}\hat{\beta}e^{iS_{Pol}+iS_{gh}}\Big|_{G=0} \tag{2.21}
$$

At this point we are free to choose any fiducial metric. We will work in the gauge that we have already introduced in section 2.2.3. The properties associated with S_{gh} are:

$$
\begin{aligned}
S_{gh} &= \int d^2\xi \sqrt{\hat{g}} \hat{\beta}^{ab} \partial_a \gamma_b \\
&= \int dz\bar{z} \left(b\bar{\partial} c + \bar{b}\partial\bar{c} \right)
\end{aligned}
$$

Here we have used $\hat{\beta}^{zz} = b$, $\hat{\beta}^{\bar{z}\bar{z}} = \bar{b}$ and $\gamma^z = c$, $\gamma^{\bar{z}} = \bar{c}$

The equations of motion are given by:

$$
\begin{aligned}
\bar{\partial} c = \partial\bar{c} = 0 \\
\bar{\partial} b = \partial\bar{b} = 0
\end{aligned}
$$

And the stress energy tensor is given by:

$$
\begin{aligned}
T_{gh} =& c\partial b - 2ab\partial c \\
\bar{T}_{gh} =& a\bar{c}\bar{\partial}\bar{b} - 2a\bar{b}\bar{\partial}\bar{c}
\end{aligned}
$$

We can see that these are zero when the equations of motion are met.

We now return to arguments relating to conformal field theory in order to continue the analysis.

2.5 Conformally Invariant Field Theory

A conformally invariant quantum field theory is one, where the conformal invariance carries through to the quantum theory. This implies that the

quantum theory must have a vanishing expectation value for the trace over the stress energy tensor. It turns out that this is not the case for our theory. The Weyl anomaly spoils this invariance on a curved background. We will try to fix this anomaly in this section.

If we consider infinitesimal transformations among the parameters of the theory, we know that we can choose a basis for the local fields such that the field must transform in some way. We define a "quasi-primary" field as one which transforms under the restricted conformal transformations (eq.(2.17))

$$z \to z' = \frac{az+b}{cz+d}$$
$$\bar{z} \to \bar{z}' = \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}$$

as:

$$\Phi(z,\bar{z}) \to \left(\frac{\partial z'}{\partial x}\right)^{h} \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{\bar{h}} \Phi(z,\bar{z}) \tag{2.22}$$

Here we have introduced the numbers h and $\bar{h}$, which are called the conformal weights. If a field transforms as above for all conformal transformations, then it is said to be a "primary" field. As we will see shortly, the stress energy tensor is a quasi-primary field, and does not transform correctly under the full conformal group. There is an anomalous term that spoils the full conformal symmetry of the theory. If the stress energy tensor is only a quasi-primary field, then there are conformal transformations that will change value of the field. This anomaly spoils the theory.

Consider an infinitesimal transformation $z \to z' = z + \epsilon(z)$ and similarly for $\bar{z}$. If we consider how the field changes, we get:

$$\begin{aligned}\Phi(z,\bar{z}) &\to (1+\partial\epsilon)^h \left(1+\bar{\partial}\bar{\epsilon}\right)^{\bar{h}} \Phi\left(z+\epsilon,\bar{z}+\bar{\epsilon}\right)\\ &= (1+h\partial\epsilon)\left(1+\bar{h}\bar{\partial}\bar{\epsilon}\right)\left(1+\epsilon\partial+\bar{\epsilon}\bar{\partial}\right)\Phi(z,\bar{z})\\ \Rightarrow \delta\Phi(z,\bar{z}) &= \left(h\partial\epsilon+\epsilon\partial+\bar{h}\bar{\partial}\bar{\epsilon}+\bar{\epsilon}\bar{\partial}\right)\Phi(z,\bar{z}) \qquad (2.23)\end{aligned}$$

Let's now consider the stress energy tensor again. Conformal invariance requires $T^\mu_\mu = 0$. The conformal anomaly shows itself at this point. $T^\mu_\mu \propto R^{(2)}$, where $R^{(2)}$ is the Ricci scalar on the worldsheet. In our situation we find that $T_{z\bar{z}} = T_{\bar{z}z} = 0$, and similarly we find that $\partial^\mu T_{\mu\nu} = 0$ as we found classically. Since the stress energy tensor must be conserved at the quantum level, we will enforce these relations while quantizing the field. We promote the above expression to an operator equation.

We find that $\partial\bar{T} = \bar{\partial}T = 0$. Now, since $T(z)$ is conserved so must be $\epsilon(z)T(z)$. We therefore have an infinite number of conserved charges.

$$Q = \frac{1}{2i\pi}\oint dz\epsilon(z)T(z) + \frac{1}{2i\pi}\oint d\bar{z}\bar{\epsilon}(\bar{z})\bar{T}(\bar{z}$$

We have each of these Q being an operator, Since T is an operator.

The suggestive name ϵ is used because these ϵ are in fact the infinitesimal transformations of the parameters. Using eq.(2.23), we should find that:

$$\begin{aligned}\delta\Phi(z,\bar{z}) &= [Q,\Phi(z,\bar{z})]\\ &= \left(h\partial\epsilon+\epsilon\partial+\bar{h}\bar{\partial}\bar{\epsilon}+\bar{\epsilon}\bar{\partial}\right)\Phi(z,\bar{z})\end{aligned}$$

Here we will have the product of operators $Q, \Phi(z,\bar{z})$. When considering overlaps of operators, we always consider the time ordered product of the

operators. In these co-ordinates this corresponds to radial ordered operators. When writing any operator product from here we imply time (radial) ordering.

We now set about solving these expressions in terms of each other.

$$\begin{aligned}[Q, \Phi(z,\bar{z})] &= \frac{1}{2i\pi}\oint (dw\epsilon(w)T(w)\Phi(z,\bar{z}) + \text{“barred terms”}) \\ &= (h\partial\epsilon + \epsilon\partial + \text{“barred terms”})\,\Phi(z,\bar{z})\end{aligned}$$

Solving the above expression, we only gain information about terms with poles in them. The analytic terms give zero contributions when integrated over. We find that any primary field with conformal weights $h\&\bar{h}$, must obey.

$$\begin{aligned}T(z)\Phi(w,\bar{w}) =& \frac{h}{(z-w)^2}\Phi(w,\bar{w}) + \frac{1}{z-w}\partial\Phi(w,\bar{w}) + \text{analytic terms} \\ \bar{T}(\bar{z})\Phi(w,\bar{w}) =& \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\Phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\bar{\partial}\Phi(w,\bar{w}) + \text{analytic terms}\end{aligned} \tag{2.24}$$

We now have a way of testing whether or not a field is primary.

Using this test, we find that [8]:

$$T(z)T(y) = \frac{D}{2(z-y)^4} + \frac{2}{(z-w)^2}T(z) + \frac{1}{z-w}\partial T(z) + \text{analytic terms}$$

We get a similar expression for the barred terms. We see that there is an anomalous term in this theory, but at this point we have not included the contribution from the ghost fields [8].

$$T_{gh}(z)T_{gh}(y) = \frac{-13}{(z-y)^4} + \frac{2}{(z-w)^2}T_{gh}(z) + \frac{1}{z-w}\partial T_{gh}(z) + \text{analytic terms}$$

Therefore, the anomaly from the ghost fields exactly cancels the Weyl anomaly if we choose $D = 26$. This theory naturally predicts the dimension in which it should live. This is the first surprising yet elegant prediction of String Theory. It is a theory that makes a prediction about the spacetime in which it can live. All physics of the Standard Model is assumed to live in 3+1 dimensions. This is a potential stumbling block of String Theory, but as we will see shortly if some of these dimensions are compact, this problem need not be a factor that ruins String Theory, but potentially allows for experimental tests.

2.5.1 Critical Dimension of Superstrings

When we consider Superstrings, we are forced to introduce new ghost fields which are associated with fermionic fields (dimensions). We will therefore get a contribution that looks like $\frac{11}{2}$, which combines with the other ghost fields to give a central charge of $-26 + 11 + D^*$. Here I use D^*, because we no longer consider D bosonic fields, but rather D bosonic fields together with D fermionic fields. This gives $D^* = \frac{3}{2}D$. Therefore when considering Superstrings we find the critical dimension is $D = 10$.

2.5.2 Type IIB Superstrings

Later when we consider the AdS/CFT correspondence (5.1), we will look at the particular situation where $\mathcal{N} = 4$ SYM theory is dual to type IIB String Theory on an $\mathrm{AdS}_5 \times S^5$ background.

Type IIB Superstrings are closed strings (with chiral boundary condition [9]), together with open strings with Dirichlet boundary conditions. This naturally implies the existence of D_p-branes. These branes are stable for $p \in \{1, 3, 5, 7, 9\}$.

Chapter 3

Extensions and Implications

In this section we will briefly discuss some of the extensions and implications of String Theory. The Polykov action as we have written it is not particularly interesting. It is only when we consider the different ways in which these strings can interact, that we see the richness of String Theory. We will mention a few extensions to what we have already seen. Although we discuss these aspects with the bosonic string in mind, these ideas naturally extend to Superstrings.

We will start by considering the most general action that retains the symmetries of the Polyakov action and interpret the different fields that emerge, and briefly discuss the significance of each new field. We will then discuss compact dimensions, and the implications of them. This will include an outline of a Kaluza Klein type reduction. Then T-duality (specifically for a circle) will be brielfy discussed, leading to the introduction of an open string with Dirichlet boundary conditions as a dual theory to an open string with Neumann boundary conditions. We then introduce the D_p-Brane as a necessary consequence of Dirichlet boundary conditions.

3.1 Interacting Models

We have developed the basic tools needed to understand the Polyakov action. We will now try to discuss some of the extensions that can be made to the Polyakov action, which may reflect some interesting physics. It turns out that the most general second derivative co-ordinate invariant action that we can write down for bosonic strings on with a Euclidean signature rather than a Minkowski signature is given by [10]:

$$S = \frac{1}{8\pi} \int d^2\xi \left\{ \left(\sqrt{g} g^{ab} G(X)_{\mu\nu} + i\epsilon^{ab} B(X)_{\mu\nu} \right) \nabla_a X^\mu \nabla_b X^\nu + 2\sqrt{g} R^{(2)} \Phi(X) \right\} \tag{3.1}$$

We could have included a surface term, which is important when considering String Theory, since the boundary condition for strings can be fairly complicated.

These fields allow for a fairly natural interpretation, and all three survive in all String Theories, and thus are worth taking some time to understand.

$G_{\mu\nu}$ is a symmetric spin 2 field. Arguments by Feynman and Weinberg state that any theory with a massless interacting spin 2 field must be a gravitational theory. We interpret this as the metric of the spacetime, or a graviton, where the metric of spacetime is a superposition of a number of gravitons. This is analogous to the electric field being a superposition of photons in Quantum Field Theory. With this model we see that String Theory really is a quantum theory of Gravity.

$B_{\mu\nu}$ is the string analogue of the (non-)abelian gauge fields with which we are familiar from Quantum Field Theory. It is called the Kalb-Ramond field, and is determined by background on which a string propagates. In order to

understand this field let's consider a particle moving along a world line, in the presence of an electromagnetic potential.

$$S = -m \int d\ell \rightarrow -m \int d\ell + \int dx^\mu A_\mu(x)$$

Similarly introducing a Kalb-Ramond field into a String Theory corresponds to adding a term $(\int dx^\mu \wedge dx^\nu B_{\mu\nu})$to the action. This amounts to adding a 2-form to the action. When considering D_p-Branes, we need to introduce a p-form to play the role of the gauge fields in the theory. We will see this when we discuss Giant Gravitons.

The final scalar particle (Φ) is the dilaton. This particle is extremely interesting in terms of the hierarchy problem. In the Standard Model, we make a list of all fundamental particles, and consider all possible interactions (including mass). We then go into a laboratory and measure these parameters. There is no well established theoretical argument for why these parameters take in the values that they do. In String Theory this is no longer the case. The last term in the action contains no local dynamics, but is related to the topology of the string world sheet. If, for example, the endpoints of a string join, then the topology changes as we have a new handle. This changes the contribution from the last term, thereby setting the interaction strength of the process. In this way String Theory couplings are predicted, whereas particle couplings are fixed. This in some sense gives us an idea as to how to fix the Hierarchy problem.

$$g_s \sim e^{\langle \Phi \rangle} \tag{3.2}$$

3.2 Kaluza-Klein reductions

We have already seen a great number of good ideas as to why String Theory may be a physical description of nature, but a large conceptual problem is that consistency requires the theory to live in 26 dimensions (10 for the Superstring). Further, when quantizing String Theory, we quantize spacetime, since each dimension corresponded to a field. Space time co-ordinates will now experience some sort of spacetime uncertainty. We have blurred our spacetime. Worse than that, when we consider the Supersting some of the dimensions are fermionic. String Theory naturally predicts a non-commutative spacetime.

As we will see, changing the topology of these extra dimensions can result in them being very hard to detect. The idea is that if some of the dimensions are compact, then at the low energies at which we can perform experiments, these dimensions can't be detected. When we consider a particle in a box (of size R) from quantum mechanics, the eigen-energies $E_n \propto \frac{n^2}{R^2}$. The interesting thing to note is that the energy levels become very widely spaced as the size of the box decreases. So, if we were to consider an extremely small spherical dimension, the energy difference between the ground state and the first excited state of a quantum system will be very far away from each other, whereas the energy levels of the large dimensions that we perceive have energy levels fairly close together. Therefore for low energy phenomena, we can't detect these small extra dimensions. It is hoped that for certain high energy experiments, we may detect some energy leaking into some excited states in these compact dimensions as a test of String Theory.

We will briefly discuss the idea of Kaluza, later modified by Klein. It was

thought that Gravity and Electromagnetism could be unified into a single theory. Kaluza suggested that instead of considering a 3+1 dimensional space, we consider a 4+1 dimensional space. Since some of the components of the metric transform in a specific way, we recover Maxwell's equations as well as General Relativity. Later Klein interpreted these special transformations as being related to translations on a circle. We will briefly outline their ideas here. In what follows $\mu \in \{0, 1, 2, 3, 4\}$ and $\hat{\mu} \in \{0, 1, 2, 3\}$. with the co-ordinate transforms $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu$

$$G_{\mu\nu} \to G'_{\mu\nu} = G_{\mu\nu} - \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu$$

If we consider a special class of transformation $\epsilon_4(\hat{x})$ and $\epsilon_{\hat{\mu}} = 0$, which corresponds to rotating the compact dimension, then we find that $G_{\hat{\mu}\hat{\nu}}$ and G_{44} are invariant, while $G_{\hat{\mu}4}$ transforms as:

$$G_{\hat{\mu}4} \to G'_{\hat{\mu}4} = G_{\hat{\mu}4} - \partial_{\hat{\mu}} \epsilon_4$$

We should note that this theory need not depend on x^4. We should also recognize the transformation law. This looks like the transformation law for the photon. The metric can be decomposed as [8]:

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu = G_{\hat{\mu}\hat{\nu}} dx^{\hat{\mu}} dx^{\hat{\nu}} + G_{44} \left(dx^4 + A_{\hat{\mu}} dx^{\hat{\mu}} \right)^2$$

If we write without loss of generality $G_{44} = e^{2\Phi}$, the Einstein Hilbert action becomes [8]:

$$
\begin{aligned}
S &= \frac{1}{16\pi G_N^{(5)}} \int d^5x \sqrt{-G^{(5)}} R^{(5)} \\
&= \frac{1}{16\pi G_N^{(4)}} \int d^4x \sqrt{-G^{(4)}} \left(R^{(4)} - \frac{3}{2} \nabla_\mu \Phi \nabla^\mu \Phi - \frac{1}{4} e^{3\Phi} F_{\hat{\mu}\hat{\nu}} F^{\hat{\mu}\hat{\nu}} \right) \qquad (3.3) \\
G_N^{(4)} &= \frac{G_N^{(5)}}{2\pi R}
\end{aligned}
$$

Here we have defined the d dimensional Gravitational constant $G_N^{(d)}$.

We notice a few things here. Electrodynamics naturally arises here, as a result of compactifying the extra dimension. There is an extra field that also arises from the above compactifications, Φ, which looks like the dilaton, since it sets the strength of the electromagnetic interaction. The coupling between the electromagnetic field and gravity is set by a particle that we have seen emerging from reducing compact dimensions, as well as an extension of the Polyakov action.

3.3 T-duality

The ideas from Kaluza and Klein can naturally be applied to String Theory. There is a new feature that we need to take into account when considering strings though. The strings can "wrap" around these compact dimensions. Now since the tension in a string is constant, the energy of the string must be related to the length of the string. Therefore winding a string around a fairly large compact dimension will have a large amount of energy. Similarly, momentum excitations of a string in a small compact dimension will also have a large energy, as already discussed.

Consider a closed string wrapped around a circle. The first thing that we

must notice is the momentum moving around the circle can now only take on discrete value, so as to force the wave function to close. $p_{25} = \frac{n}{R}$. The other thing to notice is that the boundary conditions change. We now require $X_{25}(\sigma) = X_{25}(\sigma + 2\pi) + 2\pi mR$. Here m is just the number of times that the string has wrapped around the circle. The mass formula for the string is given by [8]:

$$
\begin{aligned}
M^2 &= \frac{n^2}{R^2} + \frac{m^2R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \bar{N} - 2) \\
nm &= -(N + \bar{N}) \\
\alpha' &= \frac{1}{2\pi T}
\end{aligned}
$$

N and $\bar{N}$ are the momentum excitations moving left and right along the string.

We can see that a simple interchange $m \leftrightarrow n$ and $R \leftrightarrow \frac{\alpha'}{R}$ leaves the theory invariant. This type of duality is called T-duality.

We know that in general a theory with closed strings must also contain open strings. When we perform a similar compactification for open strings with Neumann boundary conditions, we no longer require an integer number of total windings, since the end points need not be joined. If we consider the limit that $R \to 0$, it looks as if the theory lives in 1 less spacetime dimension. It looks as if closed strings live in D dimensions, whereas open strings live in D-1 dimensions. Now we cannot distinguish between the interior part of an open string or a closed string. The only things that we can tell the difference between are the endpoints of the open string, which only live on a D-1 dimensional space. This D-1 dimensional hyper surface is a D-Brane. Since we are considering a theory of gravity, it does not make sense to consider these

D-Branes as static. These branes should themselves be dynamic objects.

3.4 D-Branes

We have now introduced the higher dimensional dynamical objects called D-Branes. These D-Branes can also be open or closed. They can also live in compact dimensions, and so when we look at T-dual theories, we will again find hyper surfaces on which open Branes can start and end. This leads naturally to the idea of a D_p-Brane. p+1 is the number of co-ordinates needed to describe the world volume of the object. We naturally see that a D_0-Brane is a particle since its world line is parameterized by its proper time alone. A D_1-Brane is a string since it is parameterised by proper time and σ in our description.

Since these D-Branes are dynamical objects in their own right, they must have their own dynamics. These dynamics must be described by some action. Just as we took the action of a particle to be the length of the worldline and the action of a string the world sheet area, we use a worldvolume action for these higher dimensional object. This action is called the Born-Infeld action.

$$S_p = -T_p \int d^{p+1}\xi \sqrt{-\det(\partial_a X^\mu \partial_b X_\mu)} \tag{3.4}$$

This is a much more difficult problem to try to understand for a number of reasons.

We know that divergences in field theories come from large numbers of excitations. When considering string actions these divergences are renormalizable. On the other hand Born-Infeld actions are not renormalizable. This

makes any quantum theory of D-branes very difficult.

In this dissertation, we use a Hamiltonian to describe a particular type of D-brane system, which does not suffer from these problems.

Chapter 4

Super Symmetry(SUSY)

In this section we want to consider a natural way of extending the space time symmetry of a theory. Since we demand that a theory be invariant under the Lorentz (most case Poincaré) group, we want extensions that leave these symmetries alone. A natural extension is to include some sort of symmetry between the fields of the theory. SUSY is such an extension. It requires that the fermions and the bosons described in a theory are somehow related to each other. In this section we will discuss this symmetry, and then try to extend it further to include conformal symmetry as well.

We will begin with a brief discussion on Grassmann numbers and spinors. We will then describe the Poincaré Superalgebra, with a detailed description of an $\mathcal{N} = 1$ Lagrangian. The section will then conclude with a discussion of $\mathcal{N} = 4$ SYM.

4.1 Fermions and Spinors

We use Grassmann numbers to describe fermions. These have the following properties.

$$\psi_i \psi_j = -\psi_j \psi_i$$

We define derivatives and integrals that are also Grassmann valued (c is a scalar):

$$\tfrac{d}{d\psi_1}\psi_2\psi_1 = -\psi_2 \tfrac{d}{d\psi_1}\psi_1 = -\psi_2$$

$$\int d\psi_1 c = 0$$

$$\int d\psi_1 \psi_2 \psi_1 = -\psi_2 \int d\psi_1 \psi_1 = -\psi_2$$

We denote a left-handed Weyl spinor by ψ_a, and a right-handed spinor $[\psi_a]^\dagger = \psi^\dagger_{\dot{a}}$. We also make use of the Pauli matrices:

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

We build the invariant quantities $\sigma^\mu_{a\dot{a}} = (I, \vec{\sigma})_{a\dot{a}}$ and $.\bar{\sigma}^\mu_{\dot{a}a} = (I, -\vec{\sigma})_{\dot{a}a}$.

4.2 Poincaré Superalgebra

When considering the Poincaré group, we have generators associated with translations ($\mathcal{P}^\mu$) and Lorentz Transformations ($\Lambda^{\mu\nu}$). We want to extend the theory to include generators that transform between the fields. We introduce a $\mathcal{N}$ supercharges ($\mathcal{Q}_{aA}$), where the lower case Roman character is a spinor index, and the upper case Roman character labels the supercharge. The Superalgebra is defined as [14]:

$$[\mathcal{Q}_{aA}, \mathcal{P}^\mu] = 0 \tag{4.1}$$

$$\left[\mathcal{Q}^\dagger_{\dot{a}A}, \mathcal{P}^\mu\right] = 0 \tag{4.2}$$

$$[\mathcal{Q}_{aA}, \Lambda^{\mu\nu}] = (S_L^{\mu\nu})_{ac}\mathcal{Q}_{cA} \tag{4.3}$$

$$\left[\mathcal{Q}^\dagger_{\dot{a}A}, \Lambda^{\mu\nu}\right] = (S_R^{\mu\nu})_{\dot{a}\dot{c}}\mathcal{Q}^\dagger_{\dot{c}A} \tag{4.4}$$

$$\{\mathcal{Q}_{aA}, \mathcal{Q}_{bB}\} = Z_{AB}\epsilon_{ab} \tag{4.5}$$

$$\left\{\mathcal{Q}_{aA}, \mathcal{Q}^\dagger_{bB}\right\} = -2\delta_{AB}\sigma^\mu_{ab}\mathcal{P}_\mu \tag{4.6}$$

Here we have introduced the matrices:

$$(S_L^{ij})_{ac} = \frac{1}{2}\varepsilon_{ijk}\sigma^k_{ac}$$
$$(S_L^{0k})_{ac} = \frac{i}{2}\sigma^k_{ac}$$
$$(S_R^{\mu\nu})_{\dot{a}\dot{c}} = -\left[(S_L^{\mu\nu})_{ac}\right]^*$$

We want to build a theory that is invariant under each of these symmetries. To preserve translational invariance we must not have any explicit x^μ dependence. To preserve Lorentz invariance we can consider theories that have contracted Lorentz indices.

In order to preserve SUSY, we introduce superfields $\Phi(x^\mu, \theta_A, \theta_A^*)$, where θ_A and θ_A^* are Grassmann valued numbers.

In general, spacetime translations are generated by the momentum four-vector.

$$[\Phi, \mathcal{P}^\mu] = -i\partial^\mu \Phi$$

We expect similar relationships when considering the action of the $\mathcal{Q}$'s.

$$\begin{aligned} [\Phi, \mathcal{Q}_a] &= -i\mathcal{Q}_a \Phi \\ \left[\Phi, \mathcal{Q}_{\dot{a}}^\dagger\right] &= -i\mathcal{Q}_{\dot{a}}^\dagger \Phi \end{aligned} \tag{4.7}$$

4.2.1 $\mathcal{N} = 1$ Superfields

We now consider the case with one supercharge. The index on $\mathcal{Q}$ becomes redundant, and we look for a representation for $\mathcal{Q}$ the superalgebra with $Z = 0$. Defining $\partial_a \theta_c = \delta_{ac}$ and a similar expression for θ^*, we find that:

$$\begin{aligned} \mathcal{Q}_a &= \partial_a + i\sigma^\mu_{a\dot{c}}\theta^*_{\dot{c}}\partial_\mu \\ \mathcal{Q}_{\dot{a}}^\dagger &= -\partial_a^* - i\theta_c \sigma^\mu_{c\dot{a}}\partial_\mu \end{aligned}$$

Defining supercovarient derivatives:

$$\begin{aligned} \mathcal{D}_a &= \partial_a - i\sigma^\mu_{a\dot{c}}\theta^*_{\dot{c}}\partial_\mu \\ \mathcal{D}_{\dot{a}}^* &= -\partial_a^* + i\theta_c \sigma^\mu_{c\dot{a}}\partial_\mu \end{aligned}$$

To get the irreducible representations of SUSY, we can choose either $\mathcal{D}_a\Phi = 0$ (left-handed chiral superfield) or $\mathcal{D}^*_{\dot{a}}\Phi = 0$ (right-handed chiral superfield). Since θ and θ^* are Grassmann valued, we find that a left-handed chiral superfield obeys:

$$\mathcal{D}^*_{\dot{a}}\Phi(x,\theta,\theta^*) = 0$$

Consider the change in co-ordinates $y^\mu = x^\mu - i\theta_a\sigma^\mu_{a\dot{a}}\theta^*_{\dot{a}}$. We then have $\mathcal{D}^*_{\dot{a}}\theta_a = 0$ and $\mathcal{D}^*_{\dot{a}}y^\mu = 0$. We find that any superfield which is a function of y and θ alone must be a right-handed chiral superfield. Expanding Φ to all orders in θ:

$$\Phi(x,\theta,\theta^*) = \Phi(y,\theta) = \phi(y) + \sqrt{2}\theta_a\psi_a(y) + \theta\theta F(y)$$

Expanding y in terms of x gives:

$$\begin{aligned}\Phi(x,\theta,\theta^*) =& \phi(x) + \sqrt{2}\theta_a\psi_a(x) + \theta_a\theta_a F(x) - i\theta_a\sigma^\mu_{a\dot{a}}\theta^*_{\dot{a}}\partial_\mu\phi(x) \\ &- \frac{i}{\sqrt{2}}\theta_c\theta_c\theta^*_{\dot{a}}\sigma^\mu_{a\dot{a}}\partial_\mu\psi_a(x) + \frac{1}{4}\theta_c\theta_c\theta^*_{\dot{c}}\theta^*_{\dot{c}}\partial^\mu\partial_\mu\phi(x)\end{aligned}$$

We can calculate:

$$\begin{aligned}\mathcal{Q}_a\Phi(y,\theta) =& \sqrt{2}\psi_a(y) + 2\theta_a F(y) \\ \mathcal{Q}^*_{\dot{a}}\Phi(y,\theta) =& i\sqrt{2}\theta\theta\partial_\mu\psi_c(y)\sigma^\mu_{c\dot{a}} - 2i\theta_c\sigma^\mu_{c\dot{a}}\partial_\mu\phi(y)\end{aligned} \tag{4.8}$$

Now comparing eq.(4.7) to the above eq.(4.8) for each power of theta we find [14]:

$$\begin{aligned} [\phi, \mathcal{Q}_a] &= -i\sqrt{2}\psi_a & \left[\phi, \mathcal{Q}^\dagger_{\dot{a}}\right] &= 0 \\ [F, \mathcal{Q}_a] &= 0 & \left[F, \mathcal{Q}^\dagger_a\right] &= \sqrt{2}\partial_\mu \psi_a \sigma^\mu_{a\dot{a}} \\ \{\psi_c, \mathcal{Q}_a\} &= -i\sqrt{2}\varepsilon_{ac}F & \left\{\psi_c, \mathcal{Q}^\dagger_a\right\} &= -\sqrt{2}\sigma^\mu_{a\dot{a}}\partial_\mu \phi \end{aligned} \tag{4.9}$$

We can see that for both ϕ and ψ the transformation laws are complicated. When we consider F, however, we find that the the fields transform as a total derivative. Therefore in general the coefficient of $\theta\theta$ in the series expansion of a left-handed chiral superfield, when integrated over spacetime, will be invariant under SUSY.

Another useful result is the fact that any function (W) of left-handed chiral superfields (Φ_i) will remain a left-handed chiral superfield. We therefore want to choose the coefficient of $\theta\theta$ in the expansion as a term in our Lagrangian. This is denoted as:

$$W(\Phi)|_F = \frac{\partial W(\phi)}{\partial \phi_i} F_i - \frac{1}{2}\frac{\partial^2 W(\phi)}{\partial \phi_i \partial \phi_j}\psi_i \psi_j \tag{4.10}$$

By similar arguments we want the coefficient of $\theta\theta\theta^*\theta^*$ from some function of hermitian operators ($V(\Phi^\dagger\Phi)$). This is denoted by:

$$V(\Phi^\dagger\Phi)\big|_D$$

We are particularly interested in a kinetic term, which is quadratic in the fields. Although other forms exist, we will only consider $V(\Phi^\dagger\Phi) = \Phi^\dagger_i \Phi_i$.

$$\Phi^\dagger \Phi\Big|_D = -\partial^\mu \phi^\dagger \partial_\mu \phi + i\psi^\dagger_{\dot{a}} \bar{\sigma}^\mu_{\dot{a}a} \partial_\mu \psi_a + F^\dagger F \tag{4.11}$$

We can immediately integrate out the F terms since they are at most quadratic. This result is equivalent to inserting the classical equation of motion for F into the action.

$$\begin{aligned}
\mathcal{L} &= \Phi_i^\dagger \Phi_i\Big|_D + W(\Phi)|_F \\
&= -\partial^\mu \phi_i^\dagger \partial_\mu \phi_i + i\psi_i^\dagger \bar{\sigma}^\mu \partial_\mu \psi_i + F_i^\dagger F_i + \frac{\partial W(\phi)}{\partial \phi_i} F_i - \frac{1}{2} \frac{\partial^2 W(\phi)}{\partial \phi_i \partial \phi_j} \psi_i \psi_j \\
&= -\partial^\mu \phi_i^\dagger \partial_\mu \phi_i + i\psi_i^\dagger \bar{\sigma}^\mu \partial_\mu \psi_i - \left| \frac{\partial W(\phi)}{\partial \phi_i} \right|^2 - \frac{1}{2} \frac{\partial^2 W(\phi)}{\partial \phi_i \partial \phi_j} \psi_i \psi_j
\end{aligned}$$

This is a general form of the $\mathcal{N} = 1$ Lagrangian. The only assumption that we have made is the form of the kinetic term in the Lagrangian, but this can be generalized for other kinetic terms. If we consider a theory with $U(1)$ gauge invariance with $W = 0$, we recover the SUSY analogue of QED[14]:

$$\mathcal{L} = i\psi^\dagger \sigma^\mu \partial_\mu \psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \tag{4.12}$$

4.3 $\mathcal{N} = 4$ SYM

Even if nature is described by some sort of supersymmetric theory, we know that the SUSY must be broken in some way. None-the-less it is particularly useful to consider SUSY as a toy model that may lead to some deeper understanding of certain theories. It may well turn out that a broken SUSY

describes nature and solves the hierarchy problem. At very least, it does give us an insight into the nature of String Theory via the AdS/CFT correspondence (sec:5.1).

There are a great many parallels between this theory and QCD, with a major difference. $\mathcal{N} = 4$ SYM is conformally invariant. This means that the coupling constant does not change. This allows us to study some of the aspects that exist in QCD without the problem of having a large coupling constant at low energy. This makes $\mathcal{N} = 4$ SYM an interesting toy model.

$\mathcal{N} = 4$ SYM, is the 4 dimensional theory that contains the maximum allowed symmetry without including gravitational interaction. It can be derived by considering as similar analysis to the above (sec.4.2.1) with 4 different supercharges. It also has 16 generators. The Lagrangian is given as[1]:

$$\mathcal{L} = \frac{N}{\lambda} Tr \Big[F^{\mu\nu} F_{\mu\nu} + \theta F^{\mu\nu} \tilde{F}_{\mu\nu} + D_\mu \phi D^\mu \phi + \psi \sigma^\mu D_\mu \psi - \psi \phi \psi - [\phi_i, \phi_j][\phi_i, \phi_j] \Big] \tag{4.13}$$

Without considering all of the details here, the action has 1 gauge field, 4 spinor fields and 6 Higgs fields. The action allows for all renormalizable interactions. In this dissertation we will be concerned primarily with the 2 of the Higgs fields. We see that this interaction has couplings $\theta, \lambda \& N$. λ is called the t'Hooft coupling. If we consider a large N limit, we can associate some surface to each Feynman diagram. The genus of the surface gives us the N dependence of the diagram. This leads naturally to an expansion in $\frac{1}{N}$ [15]. The t'Hooft coupling is related to the Yang-Mills coupling by:

[1]Indices on the fields have been omitted. Such interaction exist for each ϕ_i and ψ_j.

$$g_{YM}^2 N = \lambda \tag{4.14}$$

Although the theory has a conformal invariance, there is still an anomalous dimension associated with the quantum theory. This anomalous dimension is measured by the dilatation operator. It is the goal of this report to diagonalize this dilatation operator on the string side of the AdS/CFT correspondence.

It is interesting to note that the first derivation of the $\mathcal{N} = 4$ SYM Lagrangian came from toroidal compactifications of a 10 dimensional $\mathcal{N} = 1$ SYM [16] [17].

4.3.1 R Charge and BPS states

For a general theory that is invariant under the poincaré group, we have the spacetime symmetries, $\Lambda_{\mu\nu}$ and $\mathcal{P}_\mu$. Such a theory will have a number of internal symmetries. The charges U associated with such symmetries are conserved.

$$[\Lambda_{\mu\nu}, U] = [\mathcal{P}_\mu, U] = 0$$

We have similar expressions for the charges U in the SUSY extension.

$$[\Lambda_{\mu\nu}, U] = [\mathcal{P}_\mu, U] = [\mathcal{Q}_a, U] = 0$$

We can also consider another type of charge when considering SUSY recall equations (4.5) & (4.6)

$$\{\mathcal{Q}_{aA}, \mathcal{Q}_{bB}\} = Z_{AB}\epsilon_{ab}$$
$$\left\{\mathcal{Q}_{aA}, \mathcal{Q}^{\dagger}_{bB}\right\} = -2\delta_{AB}\sigma^{\mu}_{ab}\mathcal{P}_{\mu}$$

These equations are invariant under $U(\mathcal{N})$ transformations.

$$\mathcal{Q}_{aA} \to U^{B}_{A}\mathcal{Q}_{aB}$$

This symmetry is called an $\mathcal{R}$-symmetry, and will have corresponding charges. These $\mathcal{R}$-charges obey:

$$[\Lambda_{\mu\nu}, \mathcal{R}] = [\mathcal{P}_{\mu}, \mathcal{R}] = 0$$
$$[\mathcal{Q}_{a}, \mathcal{R}] \neq 0$$

$\mathcal{R}$-charges are not internal symmetries of the theory, since even though they commute with the bosonic spacetime symmetry generators ($\Lambda\&\mathcal{P}$), they do not commute with the fermionic spacetime symmetry generators ($Q\&Q^{\dagger}$).

We define BPS (Bogomol'nyi Prasad Sommerfeld) states as states that preserve all SUSY generators. A 1/2 BPS state preserves only half the SUSY generators. This idea will not be explored further here. More details are available in [18].

Chapter 5

AdS/CFT and D-Branes

We have briefly described a number of features of bosonic String Theory. Although our main problem does not apply to the bosonic string, all of the important features highlighted in section 3 apply to Superstrings as well. On of the major differences is the dimension in which these live (10 dimensions). It turns out that there is more than one consistent Superstring theory, although we won't discuss these here. Our interest is in type IIB strings. These strings are limited to closed strings and open strings with Dirichlet boundary conditions.

In this section we will look briefly at the Maldacena conjecture. This will involve looking at the symmetry aspects of the theory, and then highlighting the strong/weak duality and its difficulties, as well as potential beauty.

After that we will briefly describe the Giant Gravitons, which have shown to be an important probe into the AdS/CFT correspondence.

5.1 AdS/CFT Correspondence

In order to understand this correspondence, we first need to understand what an AdS space is. In order to do this we first consider S_n. This is most easily considered by embedding the geometry in a higher dimensional flat space. For example, we can consider the surface of a 3 ball as S_2. In general we can write this as:

$$X_1^2 + X_2^2 + \cdots + X_{n+1}^2 = R^2 \tag{5.1}$$

The simplest way to think of an AdS_n space is in a similar way to how we considered S_n above.

$$-X_0^2 + X_1^2 + X_2^2 + \cdots + X_{n-1}^2 - X_n^2 = -R^2 \tag{5.2}$$

The AdS/CFT correspondence is a general statement that for any conformally invariant field theory there is a higher dimensional theory on an AdS space, which is dual to the conformal field theory [21]. If we consider a conformal field theory in a 3+1 dimensional Minkowski space, the field theory is invariant under SO(2,4) (see section 2.3). When considering an AdS_5 theory, this subspace is invariant under SO(2,4). This indicates that these two theories may be related.

It is generally difficult to find conformally invariant field theories in physics. The most studied version of this correspondence is to consider an $\mathcal{N} = 4$ SYM theory. This theory is also invariant under $\mathcal{R}$-symmetry $SU(4) \sim SO(6)$. We need to adjust the AdS side of the duality to contain

the same symmetry. The simplest way is to extend the space to include an S^5. The proposed duality is between $\mathcal{N} = 4$ SYM and type IIB string theory on an $\mathrm{AdS}_5 \times S^5$ background.

For a more technical review see [3],[21].

Since the AdS/CFT correspondence is predicted to hold at all energy scales, we must identify the coupling constants (see eq.(2.5), eq.(3.2)). We get:

$$\begin{aligned} g_s &= g_{YM}^2 \\ \frac{R^4}{\ell_s^4} &= g_{YM}^2 N = \lambda \end{aligned} \tag{5.3}$$

Here we define R as the radius of curvature of the space. For distances much smaller than R, the space looks flat, and gravitational effects can be ignored. So if we have $R >> \ell_s$, then we can perform perturbative calculations on the String Theory side of the equations. This particular choice corresponds to a large t'Hooft coupling. This means that we cannot perform sensible calculation on the field theory side. At the same time when $\lambda << 1$, we can perform perturbative calculation on the field theory side of the correspondence. This corresponds to a highly curved geometry for the string side, making string calculations difficult. This is called a strong/weak duality.

This strong/weak duality is an extremely useful tool. A major problem with physics for the Standard Model is that we do not know how to perform QCD calculations. If this duality is shown to be correct, and we can find a way of breaking some of the symmetries, we will have found a way to perform accurate calculations in QCD.

The strength of this conjecture is also its weakness. The correspondence

has not been proven, and until it has been rigorously proven, testing this duality remains difficult. Only protected quantities can be used to test this correspondence.

5.2 Giant Gravitons

The popular model that is often used from the AdS/CFT correspondence, is the duality between String Theory on an $AdS_5 \times S^5$ background and $\mathcal{N} = 4$ SYM. In this section we will discuss a particular type of D-brane solution that arises on this background. Further details can be found in [4].

We consider the motion of massless particles on the S^5 of $AdS_5 \times S^5$. These particles correspond to particles such as gravitons. These particles have an angular momentum L. We usually consider particles as point particles regardless of their angular momentum. When we consider extended objects that arise from String Theory, it seems natural to allow the size to change as the angular momentum changes, which is a polarization effect. We will see that the size of the object increases with the angular momentum. These particles extend until they fill the S^5. We naturally build a bound for the size of these objects. When considering matrix models we see that the Kaluza-Klein states terminate when these particles reach the size of the S^5. This results in a stringy exclusion principle [22], [23] & [24].

We will briefly outline some of the aspects of these giant gravitons below.

5.2.1 Dipole in a magnetic field

We will briefly consider a toy model to describe the basic idea behind giant gravitons.

Consider two oppositely charged particles that are joined together by a perfect spring. We will ignore the electric effect, as well as kinetic contributions. When these particles are not moving, the spring will naturally shrink to zero size. When moving in a magnetic field, these particles will be pulled away from each. As the particles move faster, they will move away from each other.

Now consider constraining this problem to the surface of a sphere, with a magnetic monopole placed at the center of the sphere, as shown in fig 5.1. Again when the particles are not moving in the magnetic field, they will shrink to a point, due to the spring. When we begin to rotate the system the charges will separate. As the angular momentum of the system is increased, the size of the spring will stretch further, until the charges are at the opposite ends of the sphere. The spring cannot expand further than this.

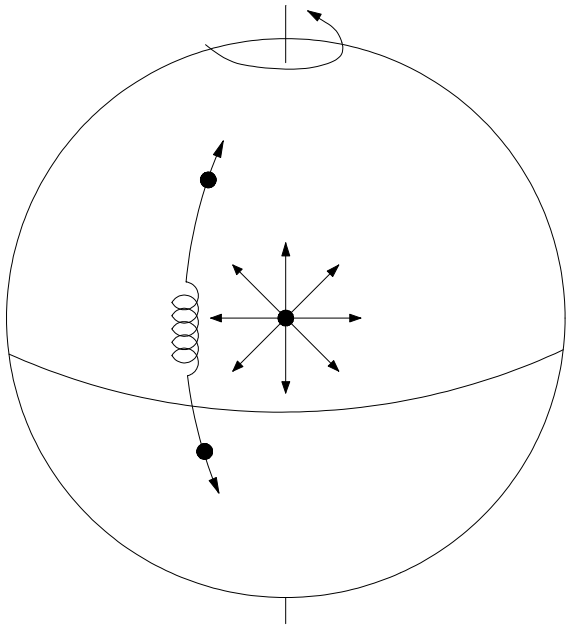


Figure 5.1: Dipole constrained to move on the surface of a Sphere, in the presence of a magnetic monopole

5.2.2 Sphere Giants

Consider the action of a spherical D_3-brane (S_3) moving in the S_5 part of the background. We recall that the Lagrangian describing such a field is given by the Born-Infeld action (eq.(3.4)). If we couple in a Chern-Simons term (four-form potential, analogous to a gauge potential when considering particles), we get the action:

$$\begin{aligned}\mathcal{L} &= L_{BI} + L_{CS} \\ &= -T\Omega_3 r^3\sqrt{1-\dot{\phi}^2(R^2-r^2)} + \dot{\phi}N\frac{r^4}{R^4}\end{aligned} \tag{5.4}$$

In this relation T is the tension of the brane, g_s is the string coupling, Ω_3 is the volume of a unit 3-sphere, $\dot{\phi}$ is the angular velocity with which the brane is moving around the S_5, R is the radius of the background 5-sphere and r is the radius of the brane.

The angular momentum is found to be:

$$L = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{Nr^3(R^2-r^2)}{R^4\sqrt{1-(R^2-r^2)\dot{\phi}^2}}\dot{\phi} + \frac{Nr^4}{R^4} \tag{5.5}$$

The energy is given by:

$$H = L\dot{\phi} - \mathcal{L} = \frac{Nr^3}{R^4\sqrt{1-\dot{\phi}^2(R^2-r^2)}} \tag{5.6}$$

There is a stable minimum for the energy at fixed angular momentum:

$$r^2 = \frac{L}{N} R^2$$
$$E = \frac{L}{R} \tag{5.7}$$

The expression for the energy (eq.(5.7)) is the same as the energy of a Kaluza-Klein (kk) graviton, with angular momentum L around the S^5. When we consider large N, quantum corrections are suppressed. The kk graviton is a BPS state, and the energy does not change if it blows up into a giant state. Hence the size of the state is determined by the angular momentum. Since the maximum size that the state can reach is R, there is a maximum angular momentum of the state.

Another minimum for the energy occurs when $r = 0$. This corresponds to a massless point particle moving around the equator. This solution is gravitationally singular. This singularity is avoided by blowing up the solution into a large membrane.

Chapter 6

Schur Polynomial D-brane Correspondence

We know that type IIB String Theory on an $AdS_5 \times S_5$ background is dual to $\mathcal{N} = 4$ SYM. Therefore we can use techniques on one side of the duality to solve problems on the other side. This is the main idea of this dissertation. We can use conformal techniques from the CFT side to perform calculations on the string side.

The Kaluza-Klein states in the theory are built up by considering all possible multi-trace operators that we can build from the field of $\mathcal{N} = 4$ SYM. Here we will only consider the Scalar fields. In order the keep the $U(N)$ gauge freedom these are considered to be $N \times N$ matrix fields described by the action:

$$S = \frac{N}{(2\pi)^3 \lambda} \int d^4x Tr \left(\frac{1}{2} D_\mu \phi^i D^\mu \phi_i + \frac{1}{4} ([\phi^i, \phi^j])^2 \right)$$

We can group these 6 real valued scalar fields into 3 complex valued

scalars:

$$Z = \phi^1 + i\phi^2 \quad Y = \phi^3 + i\phi^4 \quad X = \phi^5 + i\phi^6$$

In the free field limit ($N \to \infty; \lambda =$ constant) the propagators are given by:

$$\left\langle Z^{\dagger}{}^i_j Z^l_m \right\rangle = \left\langle Y^{\dagger}{}^i_j Y^l_m \right\rangle = \left\langle X^{\dagger}{}^i_j X^l_m \right\rangle = \frac{4\pi\lambda}{N}\delta^i_m \delta^l_j$$

The coefficient $\frac{4\pi\lambda}{N}$ will not be carried through the calculation, as it is easily inserted at the end.

Half BPS operators are built from only one of the complex fields, and will remain supersymmetric, even when the coupling is switched on. When considering a total of n Zs, it can be shown that there is a distinct operator for each partition of these Zs. For instance when considering 3 Zs we can portion them according to $Tr(Z^3); Tr(Z)Tr(Z^2); Tr(Z)^3$. These operators are not diagonal even in the large N limit. We must also note that there is a cutoff when considering such operators. $Tr(Z^m), m \geq N$ can be written in terms of other operators with the power inside the trace smaller than N.

When we consider $\mathcal{O}(1)$ fields, we are studying something dual to a point graviton. When considering something that is built from $\mathcal{O}(\sqrt{N})$ fields we get something dual to a string. When considering $\mathcal{O}(N)$ fields we are studying operators dual to giant gravitons. We find a new geometry for operators with $\mathcal{O}(N^2)$ fields.

Previous calculation have studied operators with $\mathcal{O}(1)$ as well as $\mathcal{O}(\sqrt{N})$ fields by considering the large N limit. In that limit, a large number of "non-planar" interactions can be ignored since they are suppressed by a factor of

$\frac{1}{N}$. When considering $\mathcal{O}(N)$ or more, we are no longer allowed to ignore the non-planar contributions, due to the large combinatoric factors. This makes the problem extremely difficult.

When we write a multi-trace of n fields Z we can write it as $Tr(\sigma Z^{\otimes n})$ where $\sigma \in S_n$ is an element of the symmetric group. Using the cycle notation, (1) is the identity element, which will not mix the Zs. $Tr((1)Z^{\otimes n}) = Tr(Z)^n$. When considering elements that cycle between elements, the corresponding Z's are joined. $Tr((12)Z^{\otimes n}) = Tr(Z^2)Tr(Z)^{n-2}$.

The Schur Polynomial is defined as:

$$\chi_R(Z^{\otimes n}) = \frac{1}{n!} \sum_{\sigma \in S_n} Tr(\Gamma_R(\sigma)) Tr(\sigma Z^{\otimes n}) \tag{6.1}$$

Here $\Gamma_R(\sigma)$ is a representation of σ, corresponding to the Young diagram R, with n boxes. The two point function is given by:

$$\left\langle \chi_R(Z^{\otimes n}) \chi_S(Z^{\dagger^{\otimes n}}) \right\rangle = \delta_{RS} f_R \tag{6.2}$$

Here f_R is the product of the weights of the Young diagram R. The weight of the box in the ith row and j^{th} column is given by $N - i + j$.

It turns out that there is a one-to-one correspondence between the Schur Polynomials and the BPS operators. Further than this we have the added convenience that the Schur Polynomials are diagonal to all orders in N when the coupling is switched off. There is also a simple rule for evaluating 2 point functions in the zero coupling limit. For this reason it seems natural to consider this problem in the Schur polynomial basis.

When we have a long column (with $\mathcal{O}(N)$ boxes) in the Young diagram, we associate this state with a giant graviton, growing in the S_5. The size of the giant is determined by the number of boxes in the young diagram. We saw in section 5.2, that the size of a giant graviton is determined by its angular momentum. A Young diagram with a large number of boxes in a column represents a giant graviton with a large amount of angular momentum, and therefore a larger size. The Young diagram can only contain N boxes in a column. This corresponds to the cutoff associated with the stringy exclusion principle we saw in section 5.2. It is natural to associate the length of the column with the angular momentum, and therefore the size of the giant.

If we consider 2 columns we can interpret them as a state with 2 giant gravitons. If the columns are the same length, we could also interpret this as the giant winding around the S_5.

When considering a large number of boxes in a row, we can interpret the state as a giant graviton in the AdS_5. Notice that the size of these giants is not bounded, since the corresponding space is not compact.

In the situation where $\mathcal{O}(N)$ fields are considered, we can no longer consider only the planar diagrams, since the large number of non-planar contributions makes them significant. This has mean't that previously no calculations have been done previously in this sector.

In our problem we want to consider the case where we have two nearly maximal giants expanding into the S_5. These membranes are nearly half BPS. This means that there are a large number of Zs, with a small number of impurities. These impurities complicate the problem again. In order to study the problem with these impurities, we need to study an extension of

the Schur polynomial, the restricted Schur polynomial.

Restricted Schur polynomials form a complete basis of multiple matrices. When considering restricted Schur polynomials in q different matrices, we label the problem by $q+1$ Young diagrams. The restricted Schur polynomials with two impurities which are relevant to this study are defined by:

$$\chi_{R,\{r,s\}}(Z^{\otimes n}, X^{\otimes 2}) = \frac{1}{n!2!} \sum_{\sigma \in S_{n+2}} Tr_{r,s}(\Gamma_R(\sigma)) Tr(\sigma Z^{\otimes n} X^{\otimes 2}) \tag{6.3}$$

Here most of the notation is kept from the regular Schur polynomial. The new feature is $Tr_{r,s}(\Gamma_R(\sigma))$, which is an instruction to trace over only a particular subspace of the representation $\Gamma_R(\sigma)$. This is achieved by considering projection operators which are discussed more in appendix A

Chapter 7

Dilatation Operator

In this section we describe the dilatation operator, and highlight the results obtained from the diagonalization.

According to [2][1], when considering a conformal field theory, we must have:

$$\langle \mathcal{O}_a(x)\mathcal{O}_b(y)\rangle = \frac{\delta_{ab}}{(x-y)^{2\Delta_a}} \tag{7.1}$$

Here Δ_a is the conformal dimension of the operator $\mathcal{O}_a$. A standard way to calculate this would be to consider a perturbative expansion. There are a number of difficulties associated with such a computation. Some of these difficulties are the problem of renormalization, and operator mixing. When considering perturbative techniques, this amounts to solving this complex problem for each loop. When we consider operators built from a large number of fields, the Feynman diagrams are complicated and tedious.

The anomalous dimensions are the eigenvalues of the dilatation operator

[1] see also references therein

$\hat{D}$.

$$\hat{D}\mathcal{O}_a = \Delta_a \mathcal{O}_a \tag{7.2}$$

The dilatation operator $\hat{D}$ is computed up to two loops in [1] & [2].

We will restrict our discussion to dealing with operators built from the 6 Higgs fields alone [25] & [26]. We will use complex representations of these fields.

$$\begin{aligned} X &= \Phi_1 + i\Phi_2 \\ Y &= \Phi_3 + i\Phi_4 \\ Z &= \Phi_5 + i\Phi_6 \end{aligned} \tag{7.3}$$

In the free theory we have that:

$$\left\langle X^{\dagger}{}^{i}_{j}(x) X^{l}_{m}(0) \right\rangle = \frac{\delta^i_m \delta^l_j}{x^2}$$

We have similar expressions for the other fields. We therefore must have that the zero'th order dilatation operator counts the number of fields in the theory.

$$D_0 = Tr(XD_X) + Tr(YD_Y) + Tr(ZD_Z)$$

Here $D_X{}^{i}_{j} X^{l}_{m} = \delta^i_m \delta^l_j$.

The zeroth order dilatation operator is diagonalized by Schur polynomials. We will therefore perform our calculation in the Schur polynomial basis.

The calculation for the one loop correction is more tricky. The result is quoted below.

$$D_2 = Tr([X,Y][D_Y, D_X] + [X,Z][D_Z, D_X] + [Y,Z][D_Z, D_Y]) \tag{7.4}$$

The dilatation operator is given to one loop by:

$$\hat{D} = D_0 + \frac{g_{YM}^2}{16\pi^2} D_2 \tag{7.5}$$

We now wish to find the eigenvalues and eigenoperators for eq.(7.2). This gives the spectrum of the CFT. Since these form a basis, they must be a linear combination of the restricted Schur polynomials (see appendix B).

We have considered the interpretation of this problem on the field theory side. What is the corresponding question on the String Theory side? It turns out that D_2 is the Hamiltonian operator on the String Theory side.

Therefore solving eq.(7.2) corresponds to solving a time independent Schroedinger equation. Conceptually this seems a simple enough problem.

In solving this problem we know how to calculate the action of H on $|\Psi' >$.

$$H|\Psi'_n > = \sum_m \beta_{nm} |\Psi'_m > \tag{7.6}$$

We then look for a basis that diagonalizes the operator:

$$H \sum_n \alpha_{nl} |\Psi'_n > = \sum_{mn} \alpha_{nl} \beta_{nm} |\Psi'_m > \tag{7.7}$$

$$H|\Psi_l > = E_l |\Psi_l > \tag{7.8}$$

where

$$|\psi_l >= \sum_n \alpha_{nl} |\psi'_n > \tag{7.9}$$

The calculation of $\alpha_{mn} \& \beta_{mn}$ is difficult, and the subject of the appendices of this report.

It is natural to interpret these states as "wave functions". We know from Quantum Mechanics that the wave functions that diagonalize the Hamiltonian do not localize the particles.

Chapter 8

Results and Conclusions

Here we outline the results that have been obtained in this research. For details see the appendices.

8.1 Action of the Dilatation Operator on Operators with Two Impurities

We consider the one loop dilatation operator

$$\hat{D} = g_{YM}^2 Tr[X, Z][D_Z, D_X]$$

acting on the restricted Schur polynomial with 2 impurities.

$$\chi_{R;\{r,s\}}(Z^{\otimes n}, X^{\otimes 2}) = \frac{1}{n!2!} \sum_{\sigma \in S_{n+2}} Tr_{(r,s)}(\Gamma_R(\sigma)) Z^{i_1}_{i\phi_1} ... Z^{i_n}_{i\phi_n} X^{i_{n+1}}_{i\phi_{n+1}} X^{i_{n+2}}_{i\phi_{n+2}}$$

Here R is a Young diagram with n+2 boxes, r is a young diagram with n boxes and s is a young diagram with 2 boxes. The action of the dilatation

operator is given by (see appendix D):

$$\frac{g_{YM}^2}{(n-1)!}\sum_{\sigma\in S_{n+2}} Tr_{(r,s)}(\Gamma_R([(n,n+2),\sigma]))Z^{i_1}_{i\phi_1}...Z^{i_{n-1}}_{i\phi_{n-1}}[X,Z]^{i_n}_{i\phi_n}X^{i_{n+1}}_{i\phi_{n+1}}\delta^{i_{n+2}}_{i\phi_{n+2}}$$

We now wish to write this expression in terms of our original Schur polynomial basis. This is done by performing the following steps.

- Change basis from the $S_n \times S_2$ basis to the $S_n \times S_1 \times S_1$ basis. (See appendix B.4)
- Evaluate the action of (n,n+2). (See appendix B.4)
- Perform the reduction, removing the delta to get a Schur polynomial labeled by $S_{n-1} \times S_1 \times S_1$. (See appendix B.4, and also [27])
- Separate the XZ and ZX in the expression. (See appendix D.2, and also [28])
- Change basis from the $S_n \times S_1 \times S_1$ basis to the $S_n \times S_2$ basis. (See appendix B.4)

We chose to perform the calculation in the $S_n \times S_1 \times S_1$ basis despite this basis being overdetermined. It is overdetermined since the two impurities are indistinguishable, we therefore move back to the $S_n \times S_2$ basis as soon as the calculation is complete.

8.2 Single Membrane States

Single membranes states are found by considering R labeled by a Young diagram with either a single row (AdS_5 giant) or single column (S_5 giant).

These both correspond to one dimensional representations of R, and are thus abelian. This implies that $\Gamma_R([(n, n+2), \sigma]) = 0$. We therefore find that single membrane states remain supersymmetric. This agrees with the results of [29] & [30].

8.3 Two membranes with two impurities

In this section we consider the case where R corresponds to a Young diagram with 2 columns. This situation corresponds to 2 interacting S_5 giant gravitons.

8.3.1 Labeling Conventions

We consider Schur polynomials with n Z's and 2 impurities ($X's$). We restrict our analysis to considering only Schur polynomials labeled by young diagrams with boxes in the first two columns. We can write our polynomials as:

$$\chi_{R;\{r,s\}}(Z^{\otimes n}, X^{\otimes 2}) = \frac{1}{n!2!} \sum_{\phi \in S_{n+2}} Tr_{\{r,s\}}(\Gamma_R(\phi)) Z^{i_1}_{i\phi_1} Z^{i_2}_{i\phi_2} ... Z^{i_n}_{i\phi_n} X^{i_{n+1}}_{i\phi_{n+1}} X^{i_{n+2}}_{i\phi_{n+2}} \tag{8.1}$$

Here R is a young diagram with p boxes in the first column and q boxes in the second column, r is any young diagram with two boxes removed from the original diagram R. We also use the description that r is a young diagram with $b_1 + b_0$ boxes in the first column, and b_0 boxes in the second column. Finally s is a diagram such that R is one of the diagrams obtained from considering $s \otimes r$.

$$\chi_{\begin{array}{l}\square\square\\ \square\square\\ \square\square\\ \square\\ \#\\ \#\end{array}}\,;_{\begin{array}{l}\square\\ \square\end{array}}(Z^{\otimes n}, X^{\otimes 2}) = O_a(b_1, b_0)$$

$$\chi_{\begin{array}{l}\square\square\\ \square\#\\ \square\#\\ \square\\ \square\\ \square\end{array}}\,;_{\begin{array}{l}\square\\ \square\end{array}}(Z^{\otimes n}, X^{\otimes 2}) = O_b(b_1, b_0)$$

$$\chi_{\begin{array}{l}\square\square\\ \square\square\\ \square\#\\ \square\\ \square\\ \#\end{array}}\,;_{\square\square}(Z^{\otimes n}, X^{\otimes 2}) = O_d(b_1, b_0)$$

$$\chi_{\begin{array}{l}\square\square\\ \square\square\\ \square\#\\ \square\\ \square\\ \#\end{array}}\,;_{\begin{array}{l}\square\\ \square\end{array}}(Z^{\otimes n}, X^{\otimes 2}) = O_e(b_1, b_0)$$

We can interpret $O_a(b_1, b_0)$ as an operator dual to the state with two giant gravitons, and two strings attached to the larger of the two giants, and similarly $O_b(b_1, b_0)$ can be considered has dual to two gravitons with 2 strings attached to the smaller giant. We can see that $O_d(b_1, b_0)\&O_e(b_1, b_0)$ correspond to states where the strings are attached to both gravitons. There are two states that we will see emerging from this situation. The first is where there is a string attached to the larger membrane, and a sting attached to the smaller membrane ($\sim O_d(b_1, b_0) + O_e(b_1, b_0)$). The seconds state corresponds the two strings stretching between the giants ($\sim O_d(b_1, b_0) - O_e(b_1, b_0)$).

8.3.2 Dilatation Operator

The exact expression for the action of the dilatation operator on normalized states is given in appendix E. We see that the dilatation operator acting on a two membrane state will generate a two membrane state with a box in the

third column. This is potentially a fatal feature, since the action of a two column state generates a three column state, similarly a three column state will generate a four column state, and so on until we are forced to consider all representations of S_{n+2}. This makes the problem extremely difficult.

It turns out that there is a particular limit in which the 2 membrane states decouple from the rest of the dynamics. We take a large N limit, where the number of boxes in each column is very large, then the three column contributions generated by the dilatation operator is suppressed (when considering normalized states) by a factor of $\mathcal{O}(1/\sqrt{b_0})$ where b_0 is roughly the length of the second column.

If we take the particular limit where both b_1 and b_0 are large, the dilatation operator takes the form:

$$
\begin{aligned}
&\hat{D}\,|b_1, b_0, a\rangle \sim \lambda\mathcal{O}(\frac{1}{b_1}) \\
&\hat{D}\,|b_1, b_0, b\rangle \sim \lambda\mathcal{O}(\frac{1}{b_1}) \\
&\hat{D}\,|b_1, b_0, d\rangle \approx \\
&\quad \lambda(1-\frac{b_0}{N})\,(2\,|b_1, b_0, d\rangle - |b_1-2, b_0+1, d\rangle - |b_1+2, b_0-1, d\rangle) \\
&-\lambda(1-\frac{b_0}{N})\,(2\,|b_1, b_0, e\rangle - |b_1-2, b_0+1, e\rangle - |b_1+2, b_0-1, e\rangle) + \lambda\mathcal{O}(\frac{1}{b_1}) \\
&\hat{D}\,|b_1, b_0, e\rangle \approx -\hat{D}\,|b_1, b_0, d\rangle
\end{aligned}
\tag{8.2}
$$

We see that for the two giants well separated, $\hat{D}\,|b_1, b_0, a\rangle\,, \hat{D}\,|b_1, b_0, b\rangle$ both remain nearly supersymmetric. These states have impurities only in the smaller (or larger) membrane. Recalling that single membrane states remain supersymmetric (section 8.2), it seems natural that when the membranes are well separated, these states will again remain supersymmetric. We can

also see that $\hat{D}\,|b_1, b_0, d\rangle + \hat{D}\,|b_1, b_0, e\rangle$ will also remain nearly supersymmetric, suggesting that there is a supersymmetric way to deform a pair of membranes.

Notice that $\hat{D}\,|b_1, b_0, d\rangle + \hat{D}\,|b_1, b_0, e\rangle$ looks like an operator with one string attached to the larger membrane, and a string attached to the smaller membrane (appendix B.4). We would guess that when the membranes are well separated, and there are no strings between the membranes, that these would nearly decouple.

We would therefore expect about 3/4 of our states to remain supersymmetric. Finally consider the combination $\hat{D}\,|b_1, b_0, d\rangle - \hat{D}\,|b_1, b_0, e\rangle = \hat{O}_{d-e}(b_1, b_0)$. Again comparing this to appendix B.4, this looks like the states where the strings are stretched between the membranes. We would expect that this would be the most significant contribution, and we find that:

$$\hat{D}\hat{O}_{d-e}(b_1, b_0) \approx -2\lambda g_s^2 \left(\hat{O}_{d-e}(b_1+2, b_0-1) - 2\hat{O}_{d-e}(b_1, b_0) + \hat{O}_{d-e}(b_1-2, b_0+1)\right) \tag{8.3}$$

Here $g_s = \sqrt{1 - \frac{b_0}{N}}$ is the string coupling. This takes on the form of a discretized second derivative operator. Diagonalizing this we would expect to find evenly space energy levels, as we are looking to solve a discretized form of:

$$\hat{D}\hat{O}_{d-e}(x) = -2\lambda g_s^2 \frac{d^2}{dx^2}\hat{O}_{d-e}(x) = \Lambda \hat{O}_{d-e}(x)$$

We see that the young diagram forms the lattice, and represents the geometry of the space.

8.3.3 Numerical Results

When considering Young diagrams , we can not have columns that have more that N boxes in them. Therefore if we consider the situation where there are nearly 2N boxes in the young diagram, then b_0 remains a large number, and the three column contributions decouple from the rest of the problem. In this cases we have a dynamic decoupling of states. Note that there is no limit to the length of a row, so there is no analogue to this treatment when considering AdS giants.

We want to consider the situation where the giants are separated by a distance that is about the length of a string. This means that $p_{max} - q_{min} \sim \mathcal{O}(\sqrt{N})$. We also know that the size of the larger membrane is bound by the size of $S_5 \Rightarrow p_{max} = N$

This problem can be separated into one with an even number of boxes, and one with an odd number of boxes. We will consider the case where the number of boxes (M) is even.

We have $\sigma = p - q \in \{0, 2, 4, ...2N - M\}$. When $\sigma = 0$, we cannot define $|0, b_0, a>$ or $|0, b_0, e>$. All other states are well defined. For each value of σ, there are four linearly independent operators, except for the above mentioned exceptions. The total number of states is given by $N_{total} = 4(\frac{2N-M}{2}+1)-2 = 2(2N - M + 1)$.

The action of the dilatation operator on these states is degenerate, and we only get one non-zero eigenvalue for each σ. We therefore have a total of $N_{massive} = N - \frac{M}{2} + 1$ massive states.

We find that $N_{massless} = \frac{3}{4}(N_{total} - 2)$.This seems to agree with the results of the previous section.

Using the methods of appendix F, we solve the problem for even values of σ. The graph 8.1 shows the eigenvalues for $N = 62500 \& M = 124500$

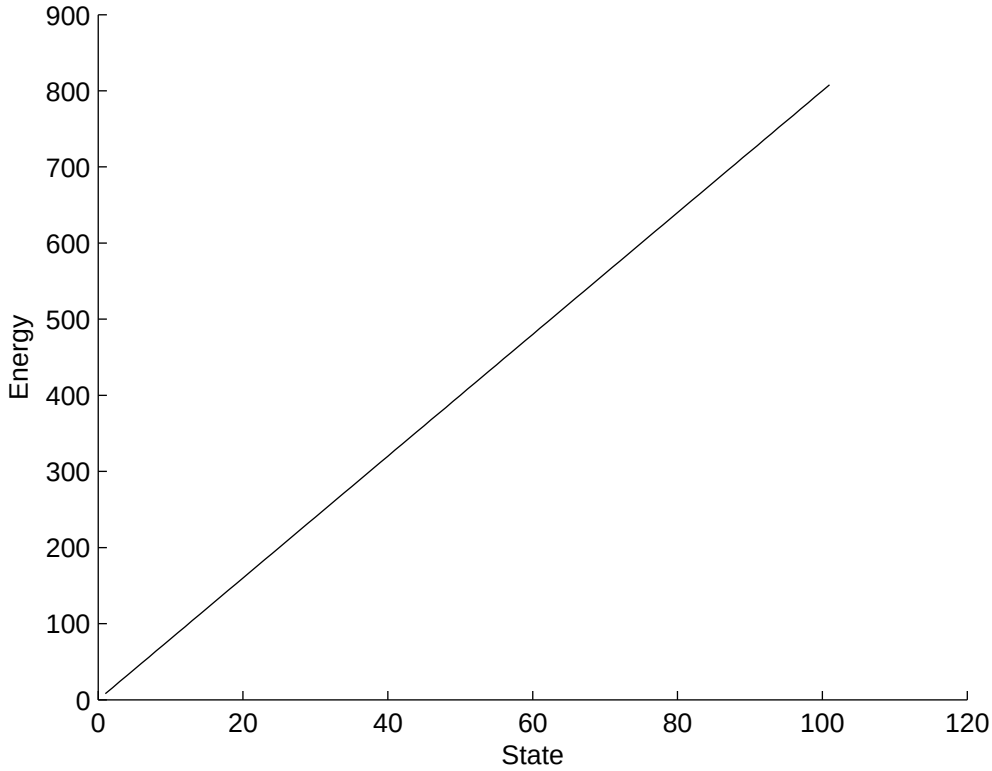


Figure 8.1: Eigenvalue spectrum

We describe some features that we see in fig.8.1. The first is that the eigenvalues are nearly evenly spaced, showing that the system is an harmonic oscillator. It is almost surprising that these eigenvalues are so evenly spaced. We also notice that there is no degeneracy among the non-zero states. This suggests that this is a one-dimensional harmonic oscillator, as we expected. How do we find the emergence of the radial direction. Recalling that we can interpret b_1 as the difference in size between the two membranes, and b_0 as

the size of the larger membrane, we plot the expectation value of b_1 with the corresponding eigenvalue in fig.8.2.

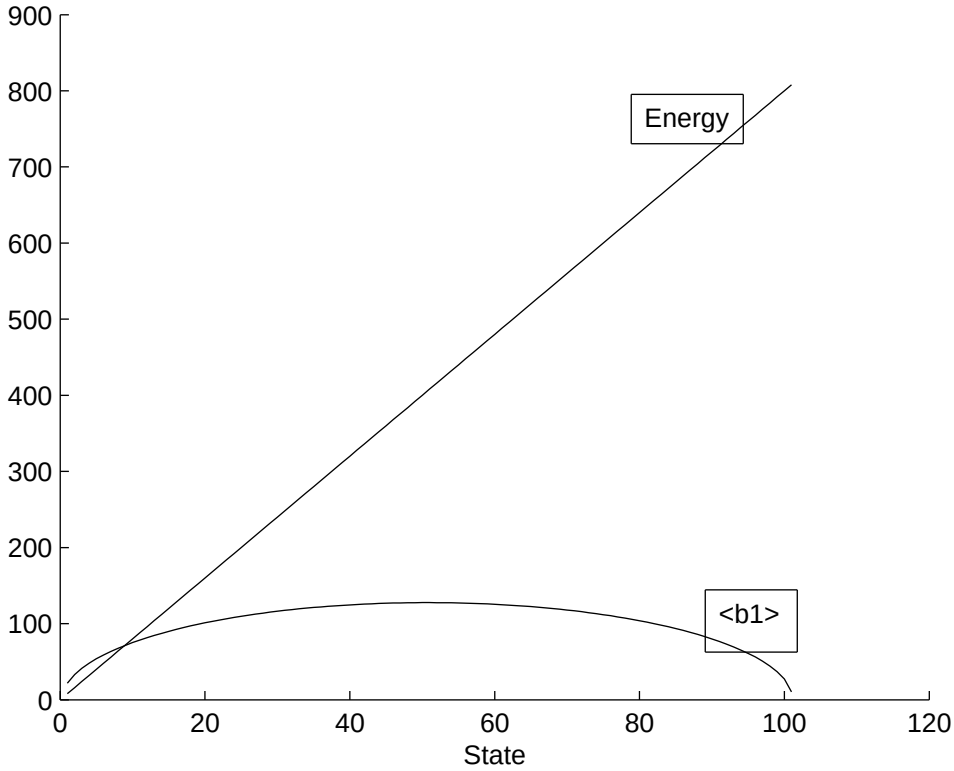


Figure 8.2: Eigenvalue spectrum, and $\langle b_1 \rangle$

We consider each membrane as an harmonic oscillator, with a string stretched between them. The stretched string couples these oscillators. These oscillators have two modes, corresponding to the membranes oscillating in phase or out of phase. When the membranes are in phase, the string between them will hardly be excited. When the membranes are out of phase, the string between them will have the largest energies. This is shown diagrammatically in fig. 8.3 & 8.4

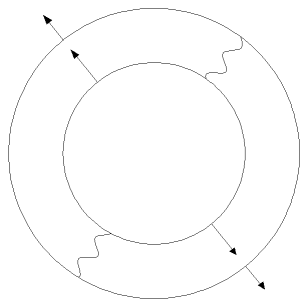


Figure 8.3: Low Energy Modes

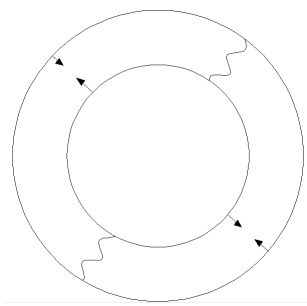


Figure 8.4: High Energy Modes

8.4 Conclusions

In this thesis we have determined the anomalous dimension of operators built from $\mathcal{O}(N)$ fields. These operators can be interpreted via the AdS/CFT correspondence as a membrane which appears in String theory, with the energy corresponding to the anomalous dimension of the corresponding operator.

In particular we studied two nearly maximal sphere giants, which are labeled by a Young diagram with 2 columns and $\mathcal{O}(N)$ boxes in each column. When considering the case where these giants are nearly maximal, we see that there is a dynamical decoupling. This corresponds to the membrane number being conserved for large N (or alternatively at weak coupling). The contribution which would build a third column is suppressed. We found that these giants behave like two couple harmonic oscillators, with low energy modes corresponding to the membranes oscillating in phase, and the high energy modes corresponding to the membranes oscillating out of phase. We

also saw the emergence of a discretized radial direction, with the lattice labeled by the young diagram associated with the state.

We have also explicitly shown that single membrane states remain supersymmetric.

Appendix A

Properties of Schur Polynomials

A great deal of work had been dedicated to developing Schur polynomials. In this appendix, we state some of the results together with some examples to illustrate them. We do not, however, give detailed proofs or derivations. The reader is referred to [31],[27],[32],[28],[26] & [33] for further details.

A.1 Schur polynomial

Schur polynomials are meant to form a basis for matrix polynomials. We begin with a definition of Schur polynomials.

$$\chi_R(Z^{\otimes n}) = \frac{1}{n!} \sum_{\phi \in S_n} Tr\left(\Gamma_R(\phi)\right) Tr(\phi Z^{\otimes n}) \tag{A.1}$$

Here $\Gamma_R(\phi)$ is representation of the element ϕ in the irrep R. The factor $Tr(\phi Z^{\otimes n})$ is just a multi trace operator. ϕ tells us which multitrace to

consider. If we have the identity, then we have $Tr((1)Z^{\otimes n}) = Tr(Z)^n$. If we have the elements that swaps between the first and second element, we have $Tr((12)Z^{\otimes n}) = Tr(Z^2)Tr(Z)^{n-2}$. Since we sum over all elements of the symmetric group, we build all possible partitions of the n Z's, and therefore all possible multi-trace operators.

Example 1

$$\begin{aligned}
\chi_{\ydiagram{2}} &= \frac{1}{2}Tr(\Gamma_{\ydiagram{2}}(1))Z^{i_1}_{i_1}Z^{i_2}_{i_2} + \frac{1}{2}Tr(\Gamma_{\ydiagram{2}}(12))Z^{i_1}_{i_2}Z^{i_2}_{i_1} \\
&= \frac{1}{2}Tr(Z)^2 + \frac{1}{2}Tr(Z^2) \\
\chi_{\ydiagram{1,1}} &= \frac{1}{2}Tr(\Gamma_{\ydiagram{1,1}}(1))Z^{i_1}_{i_1}Z^{i_2}_{i_2} + \frac{1}{2}Tr(\Gamma_{\ydiagram{1,1}}(12))Z^{i_1}_{i_2}Z^{i_2}_{i_1} \\
\chi_{\ydiagram{1,1}} &= \frac{1}{2}Tr(Z)^2 - \frac{1}{2}Tr(Z^2)
\end{aligned}$$

A.1.1 Reduction Formula

If we define the operator $D_{Z^i_j}$ by its action $D_{X^i_j}X^l_m = \delta^i_m\delta^l_j$. One can show that the action of the operator on a polynomial is:

$$\begin{aligned}
D_{X^i_i}\chi_R(Z^{\otimes n}) &= \frac{1}{n-1!}\sum_{\phi\in S_n} Tr\left(\Gamma_R(\phi)\right) Z^{i_1}_{i\phi_1}Z^{i_2}_{i\phi_2}...\delta^{i_n}_{i\phi_n} \\
&= \sum_{\alpha} c_\alpha \chi_{R'_\alpha}(Z^{\otimes n-1})
\end{aligned} \tag{A.2}$$

Where c_α is the weight of the box labeled by α. If a box is in row i and column j, then the weight of that box is N+j-i.

R'_α is the diagram obtained from R by removing box α.

Example 2

$$D\chi_{\ydiagram{3,2,1,1}} = N\chi_{\ydiagram{2,1,1,1}} + (N-3)\chi_{\ydiagram{3,2,1}}$$

A.1.2 Correlation Functions

We normalize the free theory so that:

$$\begin{aligned}\left\langle Z^i_j Z^{*l}{}_m \right\rangle =& \delta^i_m \delta^l_j \\ \left\langle Z^i_j Z^l_m \right\rangle =& 0 \\ \left\langle Z^{*i}{}_j Z^{*l}{}_m \right\rangle =& 0\end{aligned}$$

We can show that the correlation function between two Schur polynomials is [31]:

$$\begin{aligned}\left\langle \chi_R(Z)\chi_S(Z^\dagger)\right\rangle =& \frac{1}{n_R! n_S!}\sum_{\phi\in S_n}\sum_{\phi'\in S_n}\Gamma_R(\phi)\Gamma_S(\phi) \\ & \times \left\langle Z^{i_1}_{i\phi_1} Z^{i_2}_{i\phi_2}...Z^{i_n}_{i\phi_n} Z^{\dagger j_1}{}_{j\phi'_1} Z^{\dagger j_2}{}_{j\phi'_2}...Z^{\dagger j_n}{}_{j\phi'_n}\right\rangle \\ =& \delta_{RS} f_R \end{aligned} \tag{A.3}$$

Here f_R is the product of the weights associated with diagram R, and the delta function ensures that the correlation function is zero, unless R and S are the same rep of S_n

Example 3

$$\left\langle \chi_{\ydiagram{2,1,1}}(Z)\chi_{\ydiagram{2,1,1}}(Z^\dagger)\right\rangle = N^2(N+1)(N-1)(N-2)$$

$$\left\langle \chi_{\ydiagram{2,1,1,1}}(Z)\chi_{\ydiagram{2,1,1}}(Z^\dagger)\right\rangle = 0$$

A.2 Restricted Schurs

In this subsection we briefly describe how a basis of matrix polynomials in 2 different matrices ($Z\&W$) can be computed

Consider the action of the operator $Tr(ZD_Z)$.

$$Tr(ZD_Z)\chi_R(Z^{\otimes n}) = n\chi_R(Z^{\otimes n})$$

We can see that the action of this operator is to remove each possible box, and replace it with a box that is associated with Z. Now consider the action of the operator ${W^{(1)}}^i_j D_{Z}{}^j_i$.This will be the same as above. Remove a box associated with Z, and replace it with a box associated with W. This must be done with each box.

$$\sum_\alpha \chi_{R,R_\alpha}(Z^{\otimes n-1}, W^{(1)}) = Tr(W^{(1)}D_Z)\ \chi_R(Z^{\otimes n})$$

This expression tells us how to deal with the sum over α. How do we treat each of the diagrams R_α? It turns out that this is done by tracing over each of the subspaces of R.

We need to develop a consistent way of tracing over each of the subspaces.

We know that a rep $\Gamma_R(\phi)$ is only well defined up to a change of basis. We can therefore manipulate a rep by shifting $\Gamma_R(\phi) \Rightarrow M^{-1}\Gamma_R(\phi)M$ for any matrix M. The choice of the matrix M fixes the basis. We need to construct an operator that will trace over the correct subspace of $\Gamma_R(\phi)$ for a fixed choice of M. We will consider this problem first in a convenient basis, and then try to develop a general method for constructing "Projection operators".

A.2.1 Young-Yamanouchi Basis

If R is a rep of S_n, and R_α is the diagram with the box labeled α removed from R. The Young-Yamanouchi basis is partially defined by the relation.[34]

$$\Gamma_R(\phi) = \sum_{\oplus\,\alpha} \Gamma_{R_\alpha}(\phi) \quad \forall \phi \in S_{n-1}$$

Here it is important to define the order of the 'addition'. The first term in the sum corresponds to first removing the *lowest* box that can be removed. Continue in this way until all the boxes have been removed.

Example 4

$$\Gamma_{\ydiagram{2,1,1}}(\phi) = \Gamma_{\ydiagram{2,1}}(\phi) \oplus \Gamma_{\ydiagram{2,1,1}}(\phi) \forall \phi \in S_4 \subset S_5$$

$$= \begin{bmatrix} \Gamma_{\ydiagram{2,1}}(\phi) & 0 \\ 0 & \Gamma_{\ydiagram{2,1,1}}(\phi) \end{bmatrix}$$

We can easily see how to trace over the relevant subspaces in this particular example. We now need to understand how to build the remaining elements of the group. Since we can build all the elements of the symmetric group from various compositions of the 2-cycle elements, we only need to find $\Gamma_R(n-1, n)$ and we can generate all the elements of the group.

The first step is to label all the different orders of composing the Young diagram. We then associate the Yamanouchi symbol with each diagram. This is found by listing the row in which each number appears in the labeling, starting with the largest number.

Example 5

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array}$$

[32211] [32121] [23211] [23121] [21321]

Choose another label m which will label rows and columns of the rep ($m = 1$ corresponding to the largest Yamanouchi symbol), we define the state $\left|Y_R^{(m)}\right\rangle$, by the labeling convention associated with m. In this labeling convention, the label i is found in the row r_i, and column c_i, and similarly for the label $i-1$. Then we have

$$(i-1,i)\left|Y_R^m\right\rangle = \frac{1}{c_i - c_{i-1} + r_{i-1} - r_i}\left|Y_R^m\right\rangle + \sqrt{1 - \frac{1}{c_i - c_{i-1} + r_{i-1} - r_i}}\left|Y_R^{m'}\right\rangle$$

Here m' is the diagram with the labels $i-1\&i$ swapped. If we contract each of these with each of $\langle Y_n|$ we have

$$\Gamma_R(\phi)_{(n,m)} = \langle Y_R^n|\,\phi\,|Y_R^m\rangle$$

Example 6

$$\Gamma_{\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline \ \\ \cline{1-1}\end{array}}(4,5) = \begin{bmatrix} \frac{-1}{2} & 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & \frac{-1}{2} & 0 & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

In this particular basis, the idea of tracing over sub-spaces is very easy and natural. We simply define the projection operator

$$P_{\substack{\square\square\\\square\square\\\square}\,,\,\substack{\square\square\\\square\square}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.4}$$

Then

$$Tr_{\substack{\square\square\\\square\square}}\left(\Gamma_{\substack{\square\square\\\square\square\\\square}}(\phi)\right) = Tr\left(P_{\substack{\square\square\\\square\square\\\square}\,;\,\substack{\square\square\\\square\square}}\,\Gamma_{\substack{\square\square\\\square\square\\\square}}(\phi)\right) \tag{A.5}$$

$$= \left\langle \substack{\square\square\\\square\square\\\boxed{1}} {}_m \middle| (\phi) \middle| \substack{\square\square\\\square\square\\\boxed{1}} {}_m \right\rangle \tag{A.6}$$

There is a particularly convenient graphical notation that is introduced from eq.(A.6). We define restricted Schur polynomials by the order that boxes are to be removed from the rep R. This notation continues when we consider polynomials obtained from tracing over off diagonal blocks in appendix A.3.1

A.2.2 Projection Operators

We have seen how to take the trace over a particular subspace in a particular basis. We now need to consider how to take the trace over a subspace for a general basis.

We know that the sum over all elements in a particular class is proportional to the identity. What if we consider all the elements in a particular class of a subgroup.

If we have $\hat{O}$ given by the sum of all elements in a class that leaves the index n inert, then acting on an irrep of S_{n-1} will be proportional to acting with the identity in S_{n-1}.

$$\begin{aligned}
\hat{O}\left|R_i'\right\rangle &= \lambda_i \left|R_i'\right\rangle \\
\Rightarrow \hat{O} &= \sum_i \lambda_i \left|R_i'\right\rangle \left\langle R_i'\right| \\
\& \ \Gamma_R(1) &= \sum_i \left|R_i'\right\rangle \left\langle R_i'\right|
\end{aligned}$$

The eigenvalue is dependent on which subspace we are considering. Notice that in the situation that one of the eigenvalues is repeated, we can distinguish between the different subspaces, by considering the next class. We can use these relations to build a projection operator $P_i = \left|R_i'\right\rangle \left\langle R_i'\right|$.

The eigenvalue for a 2-cycle class of S_n with Young diagram with r_i boxes in the ith row and c_j boxes in the jth column is given by:

$$\lambda = \frac{1}{2}\sum_i r_i(r_i - 1) - \frac{1}{2}\sum_i c_i(c_i - 1)$$

Example 7

$$\hat{O} = \Gamma_{\begin{smallmatrix}\square\square\\\square\square\\\square\end{smallmatrix}}(12) + \Gamma_{\begin{smallmatrix}\square\square\\\square\square\\\square\end{smallmatrix}}(13) + \Gamma_{\begin{smallmatrix}\square\square\\\square\square\\\square\end{smallmatrix}}(14) + \Gamma_{\begin{smallmatrix}\square\square\\\square\square\\\square\end{smallmatrix}}(23) + \Gamma_{\begin{smallmatrix}\square\square\\\square\square\\\square\end{smallmatrix}}(24) + \Gamma_{\begin{smallmatrix}\square\square\\\square\square\\\square\end{smallmatrix}}(34)$$

$$= 0P_{\begin{smallmatrix}\square\square\\\square\square\\\square\end{smallmatrix},\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}} - 2P_{\begin{smallmatrix}\square\square\\\square\square\\\square\end{smallmatrix},\begin{smallmatrix}\square\square\\\square\\\square\end{smallmatrix}}$$

$$\Gamma_{\begin{smallmatrix}\square\square\\\square\square\\\square\end{smallmatrix}}(1) = P_{\begin{smallmatrix}\square\square\\\square\square\\\square\end{smallmatrix},\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}} + P_{\begin{smallmatrix}\square\square\\\square\square\\\square\end{smallmatrix},\begin{smallmatrix}\square\square\\\square\\\square\end{smallmatrix}}$$

$$\Rightarrow P_{\begin{smallmatrix}\square\square\\\square\square\\\square\end{smallmatrix},\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}} = \Gamma_{\begin{smallmatrix}\square\square\\\square\square\\\square\end{smallmatrix}}(1) + \frac{1}{2}\hat{O}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The last result was evaluated in the Young-Yamanouchi basis, and is in agreement with what we expected from eq. A.4. The result in any other basis is related by conjugation by M.

A.2.3 Reduction Formula

We now know how to trace over the different subspaces. It is now a simple task to see that

$$Tr(D_{W^{(1)}})\chi_{R,R'}(Z^{\otimes n-1}, W^{(1)}) = c\ \chi_{R'}(Z^{\otimes n-1})$$

Where c is just the weight of the box removed from R to give R'

Example 8

$$Tr(D_{W^{(1)}})\chi_{\begin{array}{c}\ydiagram{2,2}\\ \ydiagram{2,2}\\ \boxed{1}\end{array}}(Z^{\otimes 4}, W^{(1)}) = (N-2)\chi_{\ydiagram{2,2}}(Z^{\otimes 4})$$

A.3 Many Distinguishable Matrices

Having multiple distinguishable matrices is treated similarly to the way that we treated one. The important thing to realize is that in general $\chi_{R,R',R''}(Z^{\otimes n}, W^{(1)}, W^{(2)}) \neq \chi_{R,R',R''}(Z^{\otimes n}, W^{(2)}, W^{(1)})$ and that different ways of going from R to R'' lead to different polynomials. In general, the last argument of the Schur polynomial will be the first one to be removed.

Example 9

$$P_{\begin{array}{c}\ydiagram{2,2}\\ \ydiagram{2,2}\\ \ydiagram{1}\end{array},\ydiagram{2,2}}P_{\ydiagram{2,2},\ydiagram{2,1}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P_{\begin{array}{c}\ydiagram{2,2}\\ \ydiagram{2,1}\\ \ydiagram{1}\end{array},\begin{array}{c}\ydiagram{2,1}\\ \ydiagram{1}\end{array}}P_{\begin{array}{c}\ydiagram{2,1}\\ \ydiagram{1}\end{array},\ydiagram{2,1}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A.3.1 Twisted States

When we consider tracing over two subspaces that are of the same shape, but were obtained by different paths, we get non-zero polynomials. We interpret these tracing over an off-diagonal subspace. Using the notation adopted earlier, this corresponds to states that look like $\left\langle Y^m_{R,R',R''} \middle| \phi \middle| Y^m_{R,S',R''} \right\rangle \neq \left\langle Y^m_{R,S',R''} \middle| \phi \middle| Y^m_{R,R',R''} \right\rangle$

This is done by using introducing the operator [28]:

$$\begin{aligned} P_{AB} =& \aleph\ P_A \Gamma_R(n-1,n) P_B \\ \aleph =& \sqrt{\frac{dim(R'')}{Tr(P_A \Gamma_R(n-1,n) P_B \Gamma_R(n-1,n)}} \end{aligned} \tag{A.7}$$

Example 10

$$\chi_{\substack{\square\square \\ \square\,\overset{2}{\square} \\ \overset{1}{\square}}}(Z^{\otimes 3}, W^{(2)}, W^{(1)})$$

$$P_{AB} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that when performing reductions, we must take care considering the order in which the W's are being removed. If we are reducing with respect to the first W that is to be removed, the result for twisted states is zero.

Example 11

$$Tr(D_{W^{(1)}})\chi_{\substack{\square\square\\ \square\,\overset{2}{\square}\\ \overset{1}{\square}}}(Z^{\otimes 3}, W^{(2)}, W^{(1)}) = 0$$

$$Tr(D_{W^{(2)}})\chi_{\substack{\square\square\\ \square\,\overset{2}{\square}\\ \overset{1}{\square}}}(Z^{\otimes 3}, W^{(2)}, W^{(1)}) \neq 0$$

The second line follows since we need to perform a subgroup swap so that $W^{(2)}$ is removed first. This will be discussed in section A.3.2.

A.3.2 Sub group swap rule

Consider $(1,2)\phi[1,2,3,...,n](1,2)$, where ϕ is some element of the symmetric group. The two elements on either side of ϕ simply swap the arguments 1&2. The result is $\phi[2,1,3,...,n]$. This simple result is the key to the sub group-swap rule. Lets consider a Schur polynomial in n Z's and 2 distinguishable X's ($W^{(1)}$ & $W^{(2)}$). Here we will consider a subspace that leaves the Z's invariant, and has $W^{(2)}$ removed before $W^{(1)}$. This polynomial can be written as:

$$\begin{aligned}
\chi_{R,R',R''}(Z^{\otimes n-2}, W^{(1)}, W^{(2)}) =& \frac{1}{n-2!}\sum_{\sigma\in S_n} Tr_{R',R''}\left(\Gamma_R(\sigma)\right)\times \\
&\times Z^{i_1}_{i\sigma_1} Z^{i_2}_{i\sigma_2}...Z^{i_{n-2}}_{i\sigma_{n-2}}(W^{(1)})^{i_{n-1}}_{i\sigma_{n-1}}(W^{(2)})^{i_n}_{i\sigma_n} \\
=& \frac{1}{n-2!}\sum_{\sigma\in S_{n+2}} Tr_{R',R''}\left(\Gamma_R((n+1,n+2)\sigma(n+1,n+2))\right)\times \\
&\times Z^{i_1}_{i\sigma_1} Z^{i_2}_{i\sigma_2}...Z^{i_{n-2}}_{i\sigma_{n-2}}(W^{(2)})^{i_{n-1}}_{i\sigma_{n-1}}(W^{(1)})^{i_n}_{i\sigma_n} \quad \text{(A.8)}
\end{aligned}$$

Example 12

$$\chi_{\begin{array}{ll}\square & \square\square\\ \square & \square\,{\scriptstyle 1}\\ {\scriptstyle 2} & \end{array}}(Z^{\otimes 3}, W^{(1)}, W^{(2)}) = \frac{1}{4}\chi_{\begin{array}{ll}\square & \square\square\\ \square & \square\,{\scriptstyle 1}\\ {\scriptstyle 2} & \end{array}}(Z^{\otimes 3}, W^{(2)}, W^{(1)}) + \frac{3}{4}\chi_{\begin{array}{ll}\square & \square\square\\ \square & \square\,{\scriptstyle 2}\\ {\scriptstyle 1} & \end{array}}(Z^{\otimes 3}, W^{(2)}, W^{(1)}) +$$

$$- \frac{\sqrt{3}}{4}\chi_{\begin{array}{ll}\square & \square\square\\ \square & \square\,{\scriptstyle 1}\\ {\scriptstyle 2} & {\scriptstyle 2}\\ {\scriptstyle 1} & \end{array}}(Z^{\otimes 3}, W^{(2)}, W^{(1)}) - \frac{\sqrt{3}}{4}\chi_{\begin{array}{ll}\square & \square\square\\ \square & \square\,{\scriptstyle 2}\\ {\scriptstyle 1} & {\scriptstyle 1}\\ {\scriptstyle 2} & \end{array}}(Z^{\otimes 3}, W^{(2)}, W^{(1)})$$

A.4 Coset Expansions

We can write all the elements of a group in terms of products of two cycles and a subgroup as

$$\begin{aligned}\sum_{\phi\in S_{n+2}} F(\phi) &= \sum_{\phi\in S_{n+1}} \sum_{i=1}^{n+2} F((i,n+2)\phi) \\ &= \sum_{\phi\in S_{n+1}} \sum_{i=1}^{n+2} F(\phi(i,n+2))\end{aligned}$$

for any function F.

Applying this result to the restricted schur polynomial, we find:

$$\begin{aligned}\chi_{R,r,s}(Z^{\otimes n}, W^{(1)}, W^{(2)}) =& \frac{1}{n!}\sum_{\phi\in S_{n+2}} Tr_{R,r,s}(\phi) Z^{i_1}_{i\phi_1}...Z^{i_n}_{i\phi_n}(W^{(1)})^{i_{n+1}}_{i\phi_{n+1}}(W^{(2)})^{i_{n+2}}_{i\phi_{n+2}} \\ =& \frac{1}{n!}\sum_{\phi\in S_{n+1}} Tr_{R,r,s}((1,n+2)\phi)(W^{(2)}Z)^{i_1}_{i\phi_1}...Z^{i_n}_{i\phi_n}(W^{(1)})^{i_{n+1}}_{i\phi_{n+1}} \\ &+ ... + \frac{1}{n!}\sum_{\phi\in S_{n+1}} Tr_{R,r,s}((n,n+2)\phi) Z^{i_1}_{i\phi_1}...(W^{(2)}Z)^{i_n}_{i\phi_n}(W^{(1)})^{i_{n+1}}_{i\phi_{n+1}} \\ &+ \frac{1}{n!}\sum_{\phi\in S_{n+1}} Tr_{R,r,s}((n+1,n+2)\phi) Z^{i_1}_{i\phi_1}...Z^{i_n}_{i\phi_n}(W^{(2)}W^{(1)})^{i_{n+1}}_{i\phi_{n+1}} \\ &+ \frac{1}{n!}\sum_{\phi\in S_{n+1}} Tr_{R,r,s}(\phi) Z^{i_1}_{i\phi_1}...Z^{i_n}_{i\phi_n}(W^{(1)})^{i_{n+1}}_{i\phi_{n+1}} Tr(W^{(2)}) \\ =& \frac{1}{n!}\sum_{\phi\in S_{n+1}} Tr_{R,r,s}(\phi(1,n+2))(ZW^{(2)})^{i_1}_{i\phi_1}...Z^{i_n}_{i\phi_n}(W^{(1)})^{i_{n+1}}_{i\phi_{n+1}} \\ &+ ... + \frac{1}{n!}\sum_{\phi\in S_{n+1}} Tr_{R,r,s}(\phi(n,n+2)) Z^{i_1}_{i\phi_1}...(ZW^{(2)})^{i_n}_{i\phi_n}(W^{(1)})^{i_{n+1}}_{i\phi_{n+1}} \\ &+ \frac{1}{n!}\sum_{\phi\in S_{n+1}} Tr_{R,r,s}(\phi(n+1,n+2)) Z^{i_1}_{i\phi_1}...Z^{i_n}_{i\phi_n}(W^{(1)}W^{(2)})^{i_{n+1}}_{i\phi_{n+1}} \\ &+ \frac{1}{n!}\sum_{\phi\in S_{n+1}} Tr_{R,r,s}(\phi) Z^{i_1}_{i\phi_1}...Z^{i_n}_{i\phi_n}(W^{(1)})^{i_{n+1}}_{i\phi_{n+1}} Tr(W^{(2)})\end{aligned}$$

The last thing left to realize is that all of the Z's are indistinguishable, and live in the same sub-space. Therefore the first n terms in each of the expressions group together. We are left with:

$$\begin{aligned}
\chi_{R,r,s}(Z^{\otimes n}, W^{(1)}, W^{(2)}) =& \frac{1}{n!} \sum_{\phi \in S_{n+2}} Tr_{R,r,s}(\phi) Z^{i_1}_{i\phi_1} ... Z^{i_n}_{i\phi_n} (W^{(1)})^{i_{n+1}}_{i\phi_{n+1}} (W^{(2)})^{i_{n+2}}_{i\phi_{n+2}} \\
=& \frac{1}{(n-1)!} \sum_{\phi \in S_{n+1}} Tr_{R,r,s}((n, n+2)\phi) Z^{i_1}_{i\phi_1} ... (W^{(2)} Z)^{i_n}_{i\phi_n} (W^{(1)})^{i_{n+1}}_{i\phi_{n+1}} \\
&+ \frac{1}{n!} \sum_{\phi \in S_{n+1}} Tr_{R,r,s}((n+1, n+2)\phi) Z^{i_1}_{i\phi_1} ... Z^{i_n}_{i\phi_n} (W^{(2)} W^{(1)})^{i_{n+1}}_{i\phi_{n+1}} \\
&+ \frac{1}{n!} \sum_{\phi \in S_{n+1}} Tr_{R,r,s}(\phi) Z^{i_1}_{i\phi_1} ... Z^{i_n}_{i\phi_n} (W^{(1)})^{i_{n+1}}_{i\phi_{n+1}} Tr(W^{(2)})
\end{aligned} \tag{A.9}$$

$$\begin{aligned}
=& \frac{1}{(n-1)!} \sum_{\phi \in S_{n+1}} Tr_{R,r,s}(\phi(n, n+2)) Z^{i_1}_{i\phi_1} ... (Z W^{(2)})^{i_n}_{i\phi_n} (W^{(1)})^{i_{n+1}}_{i\phi_{n+1}} \\
&+ \frac{1}{n!} \sum_{\phi \in S_{n+1}} Tr_{R,r,s}(\phi(n+1, n+2)) Z^{i_1}_{i\phi_1} ... Z^{i_n}_{i\phi_n} (W^{(1)} W^{(2)})^{i_{n+1}}_{i\phi_{n+1}} \\
&+ \frac{1}{n!} \sum_{\phi \in S_{n+1}} Tr_{R,r,s}(\phi) Z^{i_1}_{i\phi_1} ... Z^{i_n}_{i\phi_n} (W^{(1)})^{i_{n+1}}_{i\phi_{n+1}} Tr(W^{(2)})
\end{aligned} \tag{A.10}$$

Example 13

$$
\begin{aligned}
\chi_{\begin{array}{|c|c|}\hline & \\ \hline & 1 \\ \hline 2 \\ \cline{1-1}\end{array}}(Z^{\otimes 3}, W^{(1)}, W^{(2)}) =& Tr(W^{(2)})\chi_{\begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}}(Z^{\otimes 3}, W^{(1)}) \\
&+ \frac{1}{2}\chi_{\begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}}(Z^{\otimes 3}, (W^{(2)}W^{(1)})) \\
&- \frac{5}{6}\chi_{\begin{array}{|c|c|}\hline & 2 \\ \hline \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}}(Z^{\otimes 2}, (W^{(2)}Z), W^{(1)}) \\
&+ \frac{1}{2}\chi_{\begin{array}{|c|c|}\hline & \\ \hline 2 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}}(Z^{\otimes 2}, (W^{(2)}Z), W^{(1)}) \\
&- \frac{\sqrt{2}}{3}\chi_{\begin{array}{|c|c|}\hline & 1 \\ \hline & 2 \\ \hline \\ \cline{1-1} 2 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}}(Z^{\otimes 2}, (W^{(2)}Z), W^{(1)})
\end{aligned}
$$

A.5 Correlation Functions

We now want to consider the two point function of restricted Schur polynomials. We will consider polynomials with one W, as principles remain the same when considering many different W's. The most general form that the problem can take is:

$$
\begin{aligned}
\left\langle {W^{(1)}}^i_j {W^{(1)\dagger}}^l_m \right\rangle =& F_0^{(1)}\delta^i_m\delta^l_j + F_1^{(1)}\delta^i_j\delta^l_m \\
\left\langle Z^i_j {Z^\dagger}^l_m \right\rangle =& \delta^i_m\delta^l_j
\end{aligned}
$$

All other correlation functions are zero. The coefficients $F_0 \& F_1$ can only be calculated for specific forms of W. We will not discuss this in this section, since for our problem, we have a particularly simple expression. We can write this as [27]:

$$
\begin{aligned}
&\left\langle \chi_{R,R'}(Z^{\otimes n}, W^{(1)})\chi^{\dagger}_{S,S'}(Z^{\otimes n}, W^{(1)})\right\rangle \\
=&F_1^{(1)}\left\langle Tr(D_{W^{(1)}})\chi_{R,R'}(Z^{\otimes n}, W^{(1)})Tr(D_{W^{(1)\dagger}})\chi^{\dagger}_{S,S'}(Z^{\otimes n}, W^{(1)})\right\rangle \\
&+F_0^{(1)}\left(\frac{(n+1)!}{n!}\right)^2\left\langle \chi_{R,R'}(Z^{\otimes n+1})\chi^{\dagger}_{S,S'}(Z^{\otimes n+1})\right\rangle |_{n+1} \\
=&F_1^{(1)}\left\langle Tr(D_{W^{(1)}})\chi_{R,R'}(Z^{\otimes n}, W^{(1)})Tr(D_{W^{(1)\dagger}})\chi^{\dagger}_{S,S'}(Z^{\otimes n}, W^{(1)})\right\rangle \\
&+F_0^{(1)}\left\langle \chi_{R,\vec{R'}}(Z^{\otimes n+1})\chi^{\dagger}_{S,\vec{S'}}(Z^{\otimes n+1})\right\rangle |_{n+1}
\end{aligned}
\tag{A.11}
$$

The notation $|_{n+1}$ means that we perform the reduction by forcing the Z associated with $n+1$, to be contracted with the $Z^{\dagger}$ associated with $n+1$. This is referred to as glueing.

Clearly in this situation we will find the answer proportional to $\delta_{R,S}\delta_{R',S'}$. We already know how to perform the reduction, so all we need to do is understand how to solve for the second term. The result from [27] is given by

$$
\begin{aligned}
\left\langle \chi_{R,\vec{R'}}(Z^{\otimes n+1})\chi^{\dagger}_{S,\vec{S'}}(Z^{\otimes n+1})\right\rangle |_{n+1} =&\frac{(n+1)!}{n!}\frac{d_{R'}}{d_R}f_R \\
&\frac{Hooks_R}{Hooks_{R'}}f_R\delta_{R,S}\delta_{R',S'}
\end{aligned}
\tag{A.12}
$$

Example 14

$$
\begin{aligned}
\left\langle \chi_{\substack{\square\square\\ \square\,\square\\ \boxed{1}}}\chi^{\dagger}_{\substack{\square\square\\ \square\,\square\\ \boxed{1}}}\right\rangle =&F_1\left\langle Tr(D_W)\chi_{\substack{\square\square\\ \square\,\square\\ \boxed{1}}}Tr(D_{W^{\dagger}})\chi^{\dagger}_{\substack{\square\square\\ \square\,\square\\ \boxed{1}}}\right\rangle + F_0\left\langle \chi_{\substack{\square\square\\ \square\,\square\\ \vec{\boxed{1}}}}\chi^{\dagger}_{\substack{\square\square\\ \square\,\square\\ \vec{\boxed{1}}}}\right\rangle \\
=&F_1N^2(N^2-1)(N-2)^2+10F_0N^2(N^2-1)(N-2)
\end{aligned}
$$

This is easily generalized for multiple $W's$, and is illustrated in the following example.

Example 15

$$\left\langle \chi_{\begin{array}{|c|c|}\hline \square & \square \\ \hline \square & \square \\ \hline \end{array}\,\begin{array}{c}2\\1\end{array}} \chi^{\dagger}_{\begin{array}{|c|c|}\hline \square & \square \\ \hline \square & \square \\ \hline \end{array}\,\begin{array}{c}2\\1\end{array}} \right\rangle$$

$$= F_1^{(1)} F_1^{(2)} \left\langle Tr(D_W^{(2)}) Tr(D_W^{(1)}) \chi_{\begin{array}{|c|c|}\hline \square & \square \\ \hline \square & \square \\ \hline \end{array}\,\begin{array}{c}2\\1\end{array}} Tr(D_{W^{(2)\dagger}}) Tr(D_{W^{(1)\dagger}}) \chi^{\dagger}_{\begin{array}{|c|c|}\hline \square & \square \\ \hline \square & \square \\ \hline \end{array}\,\begin{array}{c}2\\1\end{array}} \right\rangle + F_0^{(1)} F_0^{(2)} \left\langle \chi_{\begin{array}{|c|c|}\hline \square & \square \\ \hline \square & \square \\ \hline \end{array}\,\begin{array}{c}\vec{2}\\\vec{1}\end{array}} \chi^{\dagger}_{\begin{array}{|c|c|}\hline \square & \square \\ \hline \square & \square \\ \hline \end{array}\,\begin{array}{c}\vec{2}\\\vec{1}\end{array}} \right\rangle$$

$$+ F_0^{(1)} F_1^{(2)} \left\langle Tr(D_W^{(2)}) \chi_{\begin{array}{|c|c|}\hline \square & \square \\ \hline \square & \square \\ \hline \end{array}\,\begin{array}{c}\vec{2}\\\vec{1}\end{array}} Tr(D_{W^{(2)\dagger}}) \chi^{\dagger}_{\begin{array}{|c|c|}\hline \square & \square \\ \hline \square & \square \\ \hline \end{array}\,\begin{array}{c}\vec{2}\\\vec{1}\end{array}} \right\rangle + F_1^{(1)} F_0^{(2)} \left\langle Tr(D_W^{(1)}) \chi_{\begin{array}{|c|c|}\hline \square & \square \\ \hline \square & \square \\ \hline \end{array}\,\begin{array}{c}\vec{2}\\\vec{1}\end{array}} Tr(D_{W^{(1)\dagger}}) \chi^{\dagger}_{\begin{array}{|c|c|}\hline \square & \square \\ \hline \square & \square \\ \hline \end{array}\,\begin{array}{c}\vec{2}\\\vec{1}\end{array}} \right\rangle$$

$$= F_1^{(1)} F_1^{(2)} N^3 (N^2-1)(N-2)^2 + 8 F_0^{(1)} F_0^{(2)} N^2 (N^2-1)(N-2)$$

$$+ \frac{8}{3} F_0^{(1)} F_1^{(2)} N^3 (N^2-1)(N-2) + 4 F_1^{(1)} F_0^{(2)} N^2 (N^2-1)(N-2)^2$$

Appendix B

Relationships

B.1 Labeling Conventions

For our particular problem we must consider distinguishable matrices $Z\&X$. These matrices are distinct and independent, and of size $(N \times N)$. We consider Schur polynomials with n Z's and 2 $X's$. We restrict our analysis to considering only Schur polynomials labeled by young diagrams with boxes in the first two columns only. This will be motivated further later, when we show that contributions from a third column are small contributions. We can write our polynomials as:

$$\chi_{R;\{r,s\}}(Z^{\otimes n}, X^{\otimes 2}) = \frac{1}{n!2!} \sum_{\phi \in S_{n+2}} Tr_{R;\{r,s\}}(\phi) Z^{i_1}_{i\phi_1} Z^{i_2}_{i\phi_2} ... Z^{i_n}_{i\phi_n} X^{i_{n+1}}_{i\phi_{n+1}} X^{i_{n+2}}_{i\phi_{n+2}} \tag{B.1}$$

Here R is a young diagram with p boxes in the first column and q boxes in the second column, r is any young diagram with any two boxes removed from the original diagram R. Finally s is a diagram such that R is one of the

diagrams obtained from considering $s \otimes r$. We want to find a way of tracing over both the subspace r and s. This can be done by considering projection operators (one onto r and another onto s). We already know how to build restricted polynomials with 2 distinguishable matrices $W^{(1)}\&W^{(2)}$, we will need to develop a way of considering indistinguishable matrices for the reps of S_2. We therefore need to consider a new basis. We define the following notation for later use.

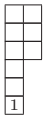
$$|a;\{b_1(1),b_0(3)\}\rangle = |\begin{array}{c}\square\square\\ \square\square\\ \square\square\\ \square\\ \#\\ \#\end{array};\begin{array}{c}\square\\ \square\end{array}\rangle = |\begin{array}{c}\square\square\\ \square\square\\ \square\square\\ \square\\ 2\\ 1\end{array}\rangle = |1;\{b_1,b_0\}\rangle \tag{B.2}$$

$$|b;\{b_1(5),b_0(1)\}\rangle = |\begin{array}{c}\square\square\\ \square\#\\ \square\#\\ \square\\ \square\\ \square\end{array};\begin{array}{c}\square\\ \square\end{array}\rangle = |\begin{array}{c}\square\square\\ \square 2\\ \square 1\\ \square\\ \square\\ \square\end{array}\rangle = |2;\{b_1,b_0\}\rangle \tag{B.3}$$

$$|d;\{b_1(3),b_0(2)\}\rangle = |\begin{array}{c}\square\square\\ \square\square\\ \square\#\\ \square\\ \square\\ \#\end{array};\square\square\rangle = \alpha\,|\begin{array}{c}\square\square\\ \square\square\\ \square 2\\ \square\\ \square\\ 1\end{array}\rangle + \beta\,|\begin{array}{c}\square\square\\ \square\square\\ \square 1\\ \square\\ \square\\ 2\end{array}\rangle = \alpha\,|5;\{b_1,b_0\}\rangle + \beta\,|4;\{b_1,b_0\}\rangle \tag{B.4}$$

$$|e;\{b_1(3),b_0(2)\}\rangle = |\begin{array}{c}\square\square\\ \square\square\\ \square\#\\ \square\\ \square\\ \#\end{array};\begin{array}{c}\square\\ \square\end{array}\rangle = \widetilde{\alpha}\,|\begin{array}{c}\square\square\\ \square\square\\ \square 2\\ \square\\ \square\\ 1\end{array}\rangle + \widetilde{\beta}\,|\begin{array}{c}\square\square\\ \square\square\\ \square 1\\ \square\\ \square\\ 2\end{array}\rangle = \widetilde{\alpha}\,|5;\{b_1,b_0\}\rangle + \widetilde{\beta}\,|4;\{b_1,b_0\}\rangle \tag{B.5}$$

We will also need to consider polynomials containing nZ's and one other matrix. For later use, we define:

$$|R_1\{b_1, b_0\}\rangle = |$$  $$\rangle \tag{B.6}$$

$$|R_2\{b_1, b_0\}\rangle = |$$ $$\rangle \tag{B.7}$$

$\alpha, \beta, \widetilde{\alpha}\&\widetilde{\beta}$ are all functions that need to be determined to ensure that we have the correct symmetry for the rep of S_2. b_0 is the number of empty boxes in the second column, and b_1 is the number of empty boxes in the first column less the number of empty boxes in the second column. The number in brackets next to the definition is an example corresponding to the particular Young diagrams. Each of the empty boxes correspond to one of the Z's, and each of the filled boxes corresponding to one of the Y's. We therefore have $n = 2b_0 + b_1$. We define the following Schur polynomials:

$$\chi_{\begin{array}{l}\square\square\\\square\square\\\square\square\\\square\\\#\\\#\end{array}}\,;_{\begin{array}{l}\square\\\square\end{array}}(Z^{\otimes n}, X^{\otimes 2}) = O_a(b_1, b_0) = O_1(b_1, b_0)$$

$$\chi_{\begin{array}{l}\square\square\\\square\#\\\square\#\\\square\\\square\\\square\end{array}}\,;_{\begin{array}{l}\square\\\square\end{array}}(Z^{\otimes n}, X^{\otimes 2}) = O_b(b_1, b_0) = O_2(b_1, b_0)$$

$$\chi_{\begin{array}{l}\square\square\\\square\square\\\square\#\\\square\\\square\\\#\end{array}}\,;_{\square\square}(Z^{\otimes n}, X^{\otimes 2}) = O_d(b_1, b_0) = \alpha^2 O_5(b_1, b_0) + \beta^2 O_4(b_1, b_0)$$

$$+ \alpha\beta\left[O_{4,5}(b_1, b_0) + O_{5,4}(b_1, b_0)\right]$$

$$\chi_{\begin{array}{l}\square\square\\\square\square\\\square\#\\\square\\\square\\\#\end{array}}\,;_{\begin{array}{l}\square\\\square\end{array}}(Z^{\otimes n}, X^{\otimes 2}) = O_e(b_1, b_0) = \widetilde{\alpha}^2 O_5(b_1, b_0) + \widetilde{\beta}^2 O_4(b_1, b_0)$$

$$+ \widetilde{\alpha}\widetilde{\beta}\left[O_{4,5}(b_1, b_0) + O_{5,4}(b_1, b_0)\right]$$

$O_{4,5}$ is the Schur polynomial, where we trace over off diagonal elements. The rows are labeled by the shape of O_4, and the columns by the shape of O_5.

B.2 Building (anti-)Symmetric States

We have that $|d; \{b_1, b_0\}\rangle = \alpha\,|5; \{b_1, b_0\}\rangle + \beta\,|4; \{b_1, b_0\}\rangle$. The conditions that we require this state to satisfy are:

$$\langle d; \{b_1, b_0\} \, | d; \{b_1, b_0\} \rangle = 1$$

$$(n+1, n+2) \, | d; \{b_1, b_0\} \rangle = | d; \{b_1, b_0\} \rangle$$

The first condition gives $\alpha^2 + \beta^2 = 1$. The second gives

$$\begin{aligned}
&(n+1, n+2) \left[\alpha \, | 5; \{b_1, b_0\} \rangle + \beta \, | 4; \{b_1, b_0\} \rangle \right] \\
&= \frac{-\alpha}{b_1 + 1} \, | 5; \{b_1, b_0\} \rangle + \beta \sqrt{1 - \frac{1}{(b_1+1)^2}} \, | 5; \{b_1, b_0\} \rangle \\
&+ \frac{\beta}{b_1 + 1} \, | 4; \{b_1, b_0\} \rangle + \alpha \sqrt{1 - \frac{1}{(b_1+1)^2}} \, | 4; \{b_1, b_0\} \rangle
\end{aligned}$$

We find that:

$$\begin{aligned}
\alpha =& \sqrt{\frac{b_1}{2(b_1+1)}} \\
\beta =& \sqrt{\frac{b_1+2}{2(b_1+1)}}
\end{aligned} \tag{B.8}$$

Similarly by requiring that $(n+1, n+2) \, | e; \{b_1, b_0\} \rangle = - \, | e; \{b_1, b_0\} \rangle$, we find that $\widetilde{\alpha} = \beta$ and $\widetilde{\beta} = -\alpha$

We therefore have:

$$\begin{aligned}
O_d(b_1, b_0) =& \frac{b_1}{2(b_1+1)} O_5(b_1, b_0) + \frac{b_1+2}{2(b_1+1)} O_4(b_1, b_0) \\
&+ \frac{\sqrt{b_1(b_1+2)}}{2(b_1+1)} \left[O_{4,5}(b_1, b_0) + O_{5,4}(b_1, b_0) \right]
\end{aligned} \tag{B.9}$$

$$\begin{aligned}
O_e(b_1, b_0) =& \frac{b_1+2}{2(b_1+1)} O_5(b_1, b_0) + \frac{b_1}{2(b_1+1)} O_4(b_1, b_0) \\
&- \frac{\sqrt{b_1(b_1+2)}}{2(b_1+1)} \left[O_{4,5}(b_1, b_0) + O_{5,4}(b_1, b_0) \right]
\end{aligned} \tag{B.10}$$

Example 16

$$\chi_{\begin{smallmatrix}\square\square\\ \square\#\\ \#\end{smallmatrix};\square\square} = \frac{1}{4}\chi_{\begin{smallmatrix}\square\square\\ \square\,2\\ 1\end{smallmatrix}} + \frac{3}{4}\chi_{\begin{smallmatrix}\square\square\\ \square\,1\\ 2\end{smallmatrix}} + \frac{\sqrt{3}}{4}\left[\chi_{\begin{smallmatrix}\square\square\\ \square\,{}^{1}_{2}\\ {}^{2}_{1}\end{smallmatrix}} + \chi_{\begin{smallmatrix}\square\square\\ \square\,{}^{2}_{1}\\ {}^{1}_{2}\end{smallmatrix}}\right]$$

$$\chi_{\begin{smallmatrix}\square\#\\ \square\\ \square\\ \#\end{smallmatrix};\begin{smallmatrix}\square\\ \square\end{smallmatrix}} = \frac{5}{8}\chi_{\begin{smallmatrix}\square\,2\\ \square\\ \square\\ 1\end{smallmatrix}} + \frac{3}{8}\chi_{\begin{smallmatrix}\square\,1\\ \square\\ \square\\ 2\end{smallmatrix}} + \frac{\sqrt{15}}{8}\left[\chi_{\begin{smallmatrix}\square\,{}^{1}_{2}\\ \square\\ \square\\ {}^{2}_{1}\end{smallmatrix}} + \chi_{\begin{smallmatrix}\square\,{}^{2}_{1}\\ \square\\ \square\\ {}^{1}_{2}\end{smallmatrix}}\right]$$

B.3 Twisted States

For our problem the X's are indistinguishable. Therefore if we apply the subgroup swap rule to swap between the two X's we will get the same polynomial back again.

$$\begin{aligned}\chi_{R;\{r,s\}} =& \frac{1}{n!2!}\sum_{\phi\in S_{n+2}} Tr_{R;\{r,s\}}(\phi) Z^{i_1}_{i\phi_1} Z^{i_2}_{i\phi_2} ... Z^{i_n}_{i\phi_n} X^{i_{n+1}}_{i\phi_{n+1}} X^{i_{n+2}}_{i\phi_{n+2}} \\ =& \frac{1}{n!2!}\sum_{\phi\in S_{n+2}} Tr_{R;\{r,s\}}((n+1,n+2)\phi(n+1,n+2)) Z^{i_1}_{i\phi_1} Z^{i_2}_{i\phi_2} ... Z^{i_n}_{i\phi_n} X^{i_n+1}_{i\phi_{n+1}} X^{i_n+2}_{i\phi_{n+2}}\end{aligned}$$

After performing all the relevant contractions, we can write:

$$O_5(b_1,b_0) = \frac{1}{(b_1+1)^2}O_5(b_1,b_0) + \left(1-\frac{1}{(b_1+1)^2}\right)O_4(b_1,b_0)$$
$$+\frac{-1}{b_1+1}\sqrt{1-\frac{1}{(b_1+1)^2}}\,[O_{5,4}(b_1,b_0)+O_{4,5}(b_1,b_0)]$$

In a similar way we can show that:

$$O_{5,4}(b_1,b_0) = \frac{-1}{(b_1+1)^2}O_{5,4}(b_1,b_0) + \left(1-\frac{1}{(b_1+1)^2}\right)O_{4,5}(b_1,b_0)$$
$$+\frac{1}{b_1+1}\sqrt{1-\frac{1}{(b_1+1)^2}}\,[O_4(b_1,b_0)-O_5(b_1,b_0)]$$

These can be re-written to give the relations:

$$\Rightarrow O_{5,4}(b_1,b_0) = O_{4,5}(b_1,b_0) = \frac{\sqrt{b_1(b_1+2)}}{2}\,[O_4(b_1,b_0)-O_5(b_1,b_0)] \quad \text{(B.11)}$$

Example 17

$$\chi_{\substack{\ydiagram{2,1,1,1}\\ \ytableausetup{mathtoc}\ytableau{\ ,1,2}\\ \ytableau{2,1}}} = \chi_{\substack{\ydiagram{2,1,1,1}\\ \ytableausetup{mathtoc}\ytableau{\ ,2,1}\\ \ytableau{1,2}}} = \frac{\sqrt{15}}{2}\left[\chi_{\substack{\ydiagram{2,1,1}\\ \ytableausetup{mathtoc}\ytableau{\ ,1}\\ \ytableau{2}}} - \chi_{\substack{\ydiagram{2,1,1}\\ \ytableausetup{mathtoc}\ytableau{\ ,2}\\ \ytableau{1}}}\right]$$

$$\chi_{\substack{\ydiagram{2,2}\\ \ytableausetup{mathtoc}\ytableau{\ ,1,2}\\ \ytableau{2,1}}} = \chi_{\substack{\ydiagram{2,2}\\ \ytableausetup{mathtoc}\ytableau{\ ,2,1}\\ \ytableau{1,2}}} = \frac{\sqrt{3}}{2}\left[\chi_{\substack{\ydiagram{2,2}\\ \ytableausetup{mathtoc}\ytableau{\ ,1}\\ \ytableau{2}}} - \chi_{\substack{\ydiagram{2,2}\\ \ytableausetup{mathtoc}\ytableau{\ ,2}\\ \ytableau{1}}}\right]$$

B.4 Swapping Basis

We have from eq.'s(B.9),(B.10), together with eq.(B.11):

$$
\begin{aligned}
O_d(b_1,b_0) =& \alpha^2 O_5(b_1,b_0) + \alpha\beta\left(O_{4,5}(b_1,b_0) + O_{5,4}(b_1,b_0)\right) + \beta^2 O_4(b_1,b_0) \\
=& \frac{b_1}{2(b_1+1)} O_5(b_1,b_0) + \frac{b_1(b_1+2)}{2(b_1+1)}\left[O_4(b_1,b_0) - O_5(b_1,b_0)\right] + \frac{b_1+2}{2(b_1+1)} O_4(b_1,b_0) \\
=& \frac{b_1+2}{2} O_4(b_1,b_0) - \frac{b_1}{2} O_5(b_1,b_0) \qquad \text{(B.12)} \\
O_e(b_1,b_0) =& \frac{b_1+2}{2} O_5(b_1,b_0) - \frac{b_1}{2} O_4(b_1,b_0) \qquad \text{(B.13)}
\end{aligned}
$$

We can invert this to see that:

$$
\begin{aligned}
O_5(b_1,b_0) =& \frac{b_1+2}{2(b_1+1)} O_e(b_1,b_0) + \frac{b_1}{2(b_1+1)} O_d(b_1,b_0) \qquad \text{(B.14)} \\
O_4(b_1,b_0) =& \frac{b_1+2}{2(b_1+1)} O_d(b_1,b_0) + \frac{b_1}{2(b_1+1)} O_e(b_1,b_0) \qquad \text{(B.15)} \\
O_{4,5}(b_1,b_0) = O_{5,4}(b_1,b_0) =& \frac{1}{2}\sqrt{1 - \frac{1}{(b_1+1)^2}}\left[O_d(b_1,b_0) - O_e(b_1,b_0)\right] \qquad \text{(B.16)}
\end{aligned}
$$

Now using equations (B.12),(B.13),(B.14),(B.15) & (B.16), we have a simple way of translating between the various bases.

Example 18

$$
\begin{aligned}
\chi_{\substack{\square\square\square \\ \square\square\,{}^{2} \\ {}^{1}\square}} &= \frac{3}{4}\chi_{\substack{\square\square\square \\ \square\square\square \\ \square}};\substack{\square \\ \square} + \frac{1}{4}\chi_{\substack{\square\square\square \\ \square\square\square \\ \square}};\square\square \\
\chi_{\substack{\square\square\square \\ \square\square\,{}^{1} \\ {}^{2}\square}} &= \frac{3}{4}\chi_{\substack{\square\square\square \\ \square\square\square \\ \square}};\square\square + \frac{1}{4}\chi_{\substack{\square\square\square \\ \square\square\square \\ \square}};\substack{\square \\ \square} \\
\chi_{\substack{\square\,{}^{2} \\ \square \\ \square \\ {}^{1}\square}} &= \frac{5}{8}\chi_{\substack{\square\square\square \\ \square \\ \square \\ \square}};\substack{\square \\ \square} + \frac{3}{8}\chi_{\substack{\square\square\square \\ \square \\ \square \\ \square}};\square\square \\
\chi_{\substack{\square\,{}^{1} \\ \square \\ \square \\ {}^{2}\square}} &= \frac{5}{8}\chi_{\substack{\square\square\square \\ \square \\ \square \\ \square}};\square\square + \frac{3}{8}\chi_{\substack{\square\square\square \\ \square \\ \square \\ \square}};\substack{\square \\ \square}
\end{aligned}
$$

Appendix C

Correlation Functions

In this section we want to build up the technology to compute exact correlation functions between our polynomials in 2 indistinguishable X's, with:

$$\left\langle X^i_j X^{\dagger}{}^l_m \right\rangle = \delta^i_m \delta^l_j$$
$$\left\langle Z^i_j Z^{\dagger}{}^l_m \right\rangle = \delta^i_m \delta^l_j$$

We know how to solve correlation functions for the X's being distinguishable, but not yet for the indistinguishable ones. Also, we know how to solve correlation functions for the basis $O_1, O_2, O_4 \& O_5$. We will try to develop a means of considering correlation functions in our basis with indistinguishable X's in terms of correlation function in the basis $O_1, O_2, O_4 \& O_5$ between distinguishable X's.

C.1 Regular Shapes

Let's start by considering the correlation function between indistinguishable X's in the old basis.

$$
\begin{aligned}
&\left\langle \chi_{R,R',R''}(Z^{\otimes n}, X^{\otimes 2})\ \chi_{S,S',S''}(Z^{\dagger\otimes n}, X^{\dagger\otimes 2})\right\rangle \\
=&\frac{1}{n!2!}\frac{1}{n!2!}\sum_{\phi\in S_{n+2}}\sum_{\phi'\in S_{n+2}} Tr_{R,R',R''}(\phi)Tr_{S,S',S''}(\phi')\left\langle Z^{i_1}_{i\phi_1}Z^{i_2}_{i\phi_2}...Z^{i_n}_{i\phi_n}Z^{\dagger j_1}_{j\phi'_1}Z^{\dagger j_2}_{j\phi'_2}...Z^{\dagger j_n}_{j\phi'_n}\right\rangle \\
&\times\left\langle X^{i_{n+1}}_{i\phi_{n+1}}X^{i_{n+2}}_{i\phi_{n+2}}X^{\dagger j_{n+1}}_{j\phi'_{n+1}}X^{\dagger j_{n+2}}_{j\phi'_{n+2}}\right\rangle \\
=&\frac{1}{n!2!}\frac{1}{n!2!}\sum_{\phi\in S_{n+2}}\sum_{\phi'\in S_{n+2}} Tr_{R,R',R''}(\phi)Tr_{S,S',S''}(\phi')\left\langle Z^{i_1}_{i\phi_1}Z^{i_2}_{i\phi_2}...Z^{i_n}_{i\phi_n}Z^{\dagger j_1}_{j\phi'_1}Z^{\dagger j_2}_{j\phi'_2}...Z^{\dagger j_n}_{j\phi'_n}\right\rangle \\
&\times\left[\left\langle X^{i_{n+1}}_{i\phi_{n+1}}X^{\dagger j_{n+1}}_{j\phi'_{n+1}}\right\rangle\left\langle X^{i_{n+2}}_{i\phi_{n+2}}X^{\dagger j_{n+2}}_{j\phi'_{n+2}}\right\rangle+\left\langle X^{i_{n+1}}_{i\phi_{n+1}}X^{\dagger j_{n+2}}_{j\phi'_{n+2}}\right\rangle\left\langle X^{i_{n+2}}_{i\phi_{n+2}}X^{\dagger j_{n+1}}_{j\phi'_{n+1}}\right\rangle\right] \\
=&\frac{1}{n!2!}\frac{1}{n!2!}\sum_{\phi\in S_{n+2}}\sum_{\phi'\in S_{n+2}} Tr_{R,R',R''}(\phi)Tr_{S,S',S''}(\phi')\left\langle Z^{i_1}_{i\phi_1}Z^{i_2}_{i\phi_2}...Z^{i_n}_{i\phi_n}Z^{\dagger j_1}_{j\phi'_1}Z^{\dagger j_2}_{j\phi'_2}...Z^{\dagger j_n}_{j\phi'_n}\right\rangle \\
&\times\left[\left\langle X^{(2)}{}^{i_{n+1}}_{i\phi_{n+1}}X^{(2)}{}^{\dagger j_{n+1}}_{j\phi'_{n+1}}\right\rangle\left\langle X^{(1)}{}^{i_{n+2}}_{i\phi_{n+2}}X^{(1)}{}^{\dagger j_{n+2}}_{j\phi'_{n+2}}\right\rangle+\left\langle X^{(2)}{}^{i_{n+1}}_{i\phi_{n+1}}X^{(2)}{}^{\dagger j_{n+2}}_{j\phi'_{n+2}}\right\rangle\left\langle X^{(1)}{}^{i_{n+2}}_{i\phi_{n+2}}X^{(1)}{}^{\dagger j_{n+1}}_{j\phi'_{n+1}}\right\rangle\right] \\
=&\frac{1}{4}\left\langle \chi_{R,R',R''}(Z^{\otimes n}, X, Y)\ \chi_{S,S',S''}(Z^{\dagger\otimes n}, X, Y)\right\rangle \\
&+\frac{1}{4}\left\langle \chi_{R,R',R''}(Z^{\otimes n}, X, Y)\ \chi_{S,S',S''}(Z^{\dagger\otimes n}, Y, X)\right\rangle
\end{aligned}
$$

We already know how to perform these calculations from section A eq.(A.12). In the second factor we simply apply the sub-group swap rule before we can use the standard methods for determining the correlation function. Note that we are now allowed to have different shapes having a non zero correlation function.This means that the basis $O_1, O_2, O_4 \& O_5$ are no longer orthogonal. This is another good motivation for building the (anti-)symmetric basis since it is diagonal.

The results relevant to our problem where the boxes are limited to the first column only, are:

$$
\begin{aligned}
\left\langle O_1 O_1^\dagger \right\rangle^{\text{indis}} &= \frac{1}{2}\left\langle O_1 O_1^\dagger \right\rangle^{\text{dis}} \\
\left\langle O_2 O_2^\dagger \right\rangle^{\text{indis}} &= \frac{1}{2}\left\langle O_2 O_2^\dagger \right\rangle^{\text{dis}} \\
\left\langle O_4 O_4^\dagger \right\rangle^{\text{indis}} &= \frac{1}{4}\left[1+\frac{1}{(b_1+1)^2}\right]\left\langle O_4 O_4^\dagger \right\rangle^{\text{dis}} \\
\left\langle O_5 O_5^\dagger \right\rangle^{\text{indis}} &= \frac{1}{4}\left[1+\frac{1}{(b_1+1)^2}\right]\left\langle O_5 O_5^\dagger \right\rangle^{\text{dis}} \\
\left\langle O_4 O_5^\dagger \right\rangle^{\text{indis}} &= \frac{1}{4}\left[1-\frac{1}{(b_1+1)^2}\right]\left\langle O_4 O_4^\dagger \right\rangle^{\text{dis}} \\
\left\langle O_5 O_4^\dagger \right\rangle^{\text{indis}} &= \frac{1}{4}\left[1-\frac{1}{(b_1+1)^2}\right]\left\langle O_5 O_5^\dagger \right\rangle^{\text{dis}}
\end{aligned}
$$

Here "dis" implies that the X's are distinguishable, and "indis" implies the X's are indistinguishable. Now using $\left\langle O_4 O_4^\dagger \right\rangle^{\text{dis}} = \left\langle O_5 O_5^\dagger \right\rangle^{\text{dis}}$

$$
\begin{aligned}
\left\langle O_a O_a^\dagger \right\rangle &= \frac{(b_1+1)(b_1+b_0+3)(b_1+b_0+2)}{2(b_1+3)}\frac{N!}{(N-b_0-b_1-2)!}\frac{(N+1)!}{(N-b_0+1)!} \\
\left\langle O_b O_b^\dagger \right\rangle &= \frac{(b_1+1)(b_0+2)(b_0+1)}{2(b_1-1)}\frac{N!}{(N-b_0-b_1)!}\frac{(N+1)!}{(N-b_0-1)!} \\
\left\langle O_d O_d^\dagger \right\rangle &= \frac{(b_1+b_0+2)(b_0+1)}{2}\frac{N!}{(N-b_0-b_1-1)!}\frac{(N+1)!}{(N-b_0)!} \\
\left\langle O_e O_e^\dagger \right\rangle &= \frac{(b_1+b_0+2)(b_0+1)}{2}\frac{N!}{(N-b_0-b_1-1)!}\frac{(N+1)!}{(N-b_0)!}
\end{aligned}
\tag{C.1}
$$

All other correlation functions are zero, showing that this basis is orthogonal under correlation functions.

Example 19

$$\left\langle \chi_{\begin{array}{ll}\square & \square\\ \square & \#\\ \# & \end{array};\square\square}\, \chi^{\dagger}_{\begin{array}{ll}\square & \square\\ \square & \#\\ \# & \end{array};\square\square} \right\rangle = 4N^2(N^2-1)(N-2)$$

$$\left\langle \chi_{\begin{array}{ll}\square & \#\\ \square & \\ \square & \\ \# & \end{array};\begin{array}{l}\square\\ \square\end{array}}\, \chi^{\dagger}_{\begin{array}{ll}\square & \#\\ \square & \\ \square & \\ \# & \end{array};\begin{array}{l}\square\\ \square\end{array}} \right\rangle = \frac{5}{2}N(N^2-1)(N-2)(N-3)$$

$$\left\langle \chi_{\begin{array}{ll}\square & \#\\ \square & \#\\ \square & \end{array};\begin{array}{l}\square\\ \square\end{array}}\, \chi^{\dagger}_{\begin{array}{ll}\square & \#\\ \square & \#\\ \square & \end{array};\begin{array}{l}\square\\ \square\end{array}} \right\rangle = 2N^2(N^2-1)(N-2)$$

Shortly we will need to consider polynomials with a box associated with X in the third column. We will also need the correlation functions associated with these. The symmetric state with a box in the first and third column to be removed will be called f, and its antisymmetric counterpart called g. Then the symmetric state with boxes in the second and third column to be removed called h, and its symmetric counterpart called i.

Example 20

$$\chi_f(2,3) = \chi_{\begin{array}{lll}\square & \square & \#\\ \square & \square & \\ \square & \square & \\ \square & & \\ \square & & \\ \# & & \end{array};\square\square}$$

$$\chi_g(2,3) = \chi_{\begin{array}{lll}\square & \square & \#\\ \square & \square & \\ \square & \square & \\ \square & & \\ \square & & \\ \# & & \end{array};\begin{array}{l}\square\\ \square\end{array}}$$

$$\chi_h(2,3) = \chi_{\begin{array}{lll}\square & \square & \#\\ \square & \square & \\ \square & \square & \\ \square & \# & \\ \square & & \end{array};\square\square}$$

$$\chi_i(2,3) = \chi_{\begin{array}{lll}\square & \square & \#\\ \square & \square & \\ \square & \square & \\ \square & \# & \\ \square & & \end{array};\begin{array}{l}\square\\ \square\end{array}}$$

We get the relationships:

$$\begin{aligned}
\left\langle O_f O_f^\dagger \right\rangle &= \frac{(b_0+1)(b_1+1)(b_1+b_0+3)}{2b_0(b_1+2)} \frac{N!}{(N-b_0-b_1-1)!} \frac{(N+1)!}{(N-b_0+1)!}(N+2) \\
\left\langle O_g O_g^\dagger \right\rangle &= \frac{(b_0+1)(b_1+1)(b_1+b_0+3)}{2b_0(b_1+2)} \frac{N!}{(N-b_0-b_1-1)!} \frac{(N+1)!}{(N-b_0+1)!}(N+2) \\
\left\langle O_h O_h^\dagger \right\rangle &= \frac{(b_1+1)(b_1+b_0+2)(b_0+2))}{2b_1(b_1+b_0+1)} \frac{N!}{(N-b_0-b_1)!} \frac{(N+1)!}{(N-b_0)!}(N+2) \\
\left\langle O_i O_i^\dagger \right\rangle &= \frac{(b_1+1)(b_1+b_0+2)(b_0+2))}{2b_1(b_1+b_0+1)} \frac{N!}{(N-b_0-b_1)!} \frac{(N+1)!}{(N-b_0)!}(N+2)
\end{aligned} \tag{C.2}$$

C.2 Correlation Functions for Irregular Polynomials

Later, when we solve for the dilatation operator in the original basis, we will be left with polynomials in X^2, as well as polynomials proportional to $Tr(X)$. The treatment of these terms is very similar for both cases.

We expect that all these polynomials in 2 X's, should be expressed in terms of the symmetrized basis. We therefore search for a way of writing these irregular polynomials in terms of the original basis.

$$\begin{aligned}
O_{R'}(Z^{\otimes n}, X^2) &= \sum_{R;\{r,s\}} \varsigma_{R;\{r,s\}} O_{R;\{r,s\}}(Z^{\otimes n}, X^{\otimes 2}) \\
Tr(X) O_{R'}(Z^{\otimes n}, X) &= \sum_{R;\{r,s\}} \varepsilon_{R;\{r,s\}} O_{R;\{r,s\}}(Z^{\otimes n}, X^{\otimes 2})
\end{aligned}$$

We need to solve for the expressions $\varsigma_{R;\{r,s\}} \& \varepsilon_{R;\{r,s\}}$. This is done by using the fact that the $O_{(R;\{r,s\})}$ are orthogonal under correlation functions.

$$\left\langle O_{R;\{r,s\}} O^{\dagger}_{R,R'}(Z^{\otimes n}, X^2) \right\rangle = \varsigma_{R;\{r,s\}} \left\langle O_{R;\{r,s\}} O^{\dagger}_{R;\{r,s\}} \right\rangle$$

We get a similar expression for $\varepsilon_{R;\{r,s\}}$. Let's consider

$$\begin{aligned}
&\left\langle \chi_{R,R'}(Z^{\otimes n}, X^2)\ \chi_{S,S'}(Z^{\dagger\otimes n}, X^{\dagger 2}) \right\rangle \\
=&\frac{1}{n!}\frac{1}{n!}\sum_{\phi\in S_{n+1}}\sum_{\phi'\in S_{n+1}} Tr_{R,R'}(\phi) Tr_{S,S'}(\phi') \left\langle Z^{i_1}_{i\phi_1} Z^{i_2}_{i\phi_2} ... Z^{i_n}_{i\phi_n} Z^{\dagger j_1}_{j\phi'_1} Z^{\dagger j_2}_{j\phi'_2} ... Z^{\dagger j_n}_{j\phi'_n} \right\rangle \\
&\times \left\langle X^{i_{n+1}}_{l} X^{l}_{i\phi_{n+1}} X^{\dagger j_{n+1}}_{m} X^{\dagger m}_{j\phi'_{n+1}} \right\rangle \\
=&\frac{1}{n!}\frac{1}{n!}\sum_{\phi\in S_{n+1}}\sum_{\phi'\in S_{n+1}} Tr_{R,R'}(\phi) Tr_{S,S'}(\phi') \left\langle Z^{i_1}_{i\phi_1} Z^{i_2}_{i\phi_2} ... Z^{i_n}_{i\phi_n} Z^{\dagger j_1}_{j\phi'_1} Z^{\dagger j_2}_{j\phi'_2} ... Z^{\dagger j_n}_{j\phi'_n} \right\rangle \\
&\times \left[\left\langle X^{i_{n+1}}_{l} X^{\dagger j_{n+1}}_{m} \right\rangle \left\langle X^{l}_{i\phi_{n+1}} X^{\dagger m}_{j\phi'_{n+1}} \right\rangle + \left\langle X^{i_{n+1}}_{l} X^{\dagger m}_{j\phi'_{n+1}} \right\rangle \left\langle X^{l}_{i\phi_{n+1}} X^{\dagger j_{n+1}}_{m} \right\rangle \right] \\
=&\frac{1}{n!}\frac{1}{n!}\sum_{\phi\in S_{n+1}}\sum_{\phi'\in S_{n+1}} Tr_{R,R'}(\phi) Tr_{S,S'}(\phi') \left\langle Z^{i_1}_{i\phi_1} Z^{i_2}_{i\phi_2} ... Z^{i_n}_{i\phi_n} Z^{\dagger j_1}_{j\phi'_1} Z^{\dagger j_2}_{j\phi'_2} ... Z^{\dagger j_n}_{j\phi'_n} \right\rangle \\
&\times \left[\delta^{i_{n+1}}_{i\phi_{n+1}} \delta^{j_{n+1}}_{j\phi'_{n+1}} + \delta^{i_{n+1}}_{j\phi'_{n+1}} \delta^{j_{n+1}}_{i\phi_{n+1}} N \right] \\
=&\left\langle \chi_{R,R'}(Z^{\otimes n}, W)\ \chi_{S,S'}(Z^{\dagger\otimes n}, W^{\dagger}) \right\rangle \\
&\text{Using} F_0 = N \& F_1 = 1
\end{aligned}$$

Similarly we find that:

$$\begin{aligned}
&\left\langle Tr(X)\chi_{R,R'}(Z^{\otimes n}, X)\ Tr(X^{\dagger})\chi_{S,S'}(Z^{\dagger\otimes n}, X^{\dagger}) \right\rangle \\
&= \left\langle \chi_{R,R'}(Z^{\otimes n}, X^2)\ \chi_{S,S'}(Z^{\dagger\otimes n}, X^{\dagger 2}) \right\rangle
\end{aligned}$$

If we consider each of the 2 cases, we have

$$
\begin{aligned}
&\left\langle \chi_{R_1'}(b_1,b_0)(Z^{\otimes n},X^2)\ \chi_{R_1'}(b_1,b_0)(Z^{\dagger\otimes n},X^{\dagger 2})\right\rangle \\
&= \left[(N-b_0-b_1)+\frac{N(b_1+1)(b_0+b_1+2)}{b_1+2}\right]\frac{N!}{(N-b_0-b_1-1)!}\frac{(N+1)!}{(N-b_0+1)!} \\
&\left\langle \chi_{R_2'}(b_1,b_0)(Z^{\otimes n},X^2)\ \chi_{R_2'}(b_1,b_0)(Z^{\dagger\otimes n},X^{\dagger 2})\right\rangle \\
&= \left[(N-b_0+1)+\frac{N(b_1+1)(b_0+1)}{b_1}\right]\frac{N!}{(N-b_0-b_1)!}\frac{(N+1)!}{(N-b_0)!}
\end{aligned}
\tag{C.3}
$$

We now want to solve for the $\varsigma_{R;\{r,s\}}$. This involves solving problems of the form:

$$
\begin{aligned}
&\left\langle \chi_{S,S'}(Z^{\otimes n},X^2)\ \chi_{R;\{r,s\}}(Z^{\dagger\otimes n},X^{\dagger\otimes 2})\right\rangle \\
=&\frac{1}{n!}\frac{1}{n!2!}\sum_{\phi\in S_{n+1}}\sum_{\phi'\in S_{n+2}} Tr_{S,S'}(\phi)Tr_{R;\{r,s\}}(\phi')\left\langle Z^{i_1}_{i\phi_1}Z^{i_2}_{i\phi_2}...Z^{i_n}_{i\phi_n}Z^{\dagger j_1}_{j\phi'_1}Z^{\dagger j_2}_{j\phi'_2}...Z^{\dagger j_n}_{j\phi'_n}\right\rangle \\
&\times\left\langle X^{i_{n+1}}_{l}X^{l}_{i\phi_{n+1}}X^{\dagger j_{n+1}}_{j\phi'_{n+1}}X^{\dagger j_{n+2}}_{j\phi'_{n+2}}\right\rangle \\
=&\frac{1}{n!}\frac{1}{n!}\sum_{\phi\in S_{n+1}}\sum_{\phi'\in S_{n+2}} Tr_{S,S'}(\phi)Tr_{R;\{r,s\}}(\phi')\left\langle Z^{i_1}_{i\phi_1}Z^{i_2}_{i\phi_2}...Z^{i_n}_{i\phi_n}Z^{\dagger j_1}_{j\phi'_1}Z^{\dagger j_2}_{j\phi'_2}...Z^{\dagger j_n}_{j\phi'_n}\right\rangle \\
&\times\left\langle X^{i_{n+1}}_{l}X^{\dagger j_{n+1}}_{j\phi'_{n+1}}\right\rangle\left\langle X^{l}_{i\phi_{n+1}}X^{\dagger j_{n+2}}_{j\phi'_{n+2}}\right\rangle \\
=&\frac{1}{n!}\frac{1}{n!}\sum_{\phi\in S_{n+1}}\sum_{\phi'\in S_{n+1}} Tr_{S,S'}(\phi)Tr_{R;\{r,s\}}(\phi')\left\langle Z^{i_1}_{i\phi_1}Z^{i_2}_{i\phi_2}...Z^{i_n}_{i\phi_n}Z^{\dagger j_1}_{j\phi'_1}Z^{\dagger j_2}_{j\phi'_2}...Z^{\dagger j_n}_{j\phi'_n}\right\rangle \\
&\times\delta^{i_{n+1}}_{j\phi'_{n+1}}\delta^{j_{n+1}}_{j\phi'_{n+2}}\delta^{j_{n+2}}_{i\phi_{n+1}} \\
=&\frac{1}{n!}\frac{1}{n!}\sum_{\phi\in S_{n+1}}\sum_{\phi'\in S_{n+1}} Tr_{S,S'}(\phi)Tr_{R;\{r,s\}}((n+1,n+2)\phi')\left\langle Z^{i_1}_{i\phi_1}Z^{i_2}_{i\phi_2}...Z^{i_n}_{i\phi_n}Z^{\dagger j_1}_{j\phi'_1}Z^{\dagger j_2}_{j\phi'_2}...Z^{\dagger j_n}_{j\phi'_n}\right\rangle \\
&\times\delta^{i_{n+1}}_{j\phi'_{n+2}}\delta^{j_{n+1}}_{j\phi'_{n+1}}\delta^{j_{n+2}}_{i\phi_{n+1}} \\
=&\Gamma_s((1,2))\left\langle \chi_{S,S'}(Z^{\otimes n},X)\ Tr(D_{X^\dagger})\chi_{R;\{r,s\}}(Z^{\dagger\otimes n},X^{\dagger\otimes 2})\right\rangle
\end{aligned}
$$

Similarly we get:

$$\begin{aligned}&\left\langle \chi_{S,S'}(Z^{\otimes n}, X^2)\ \chi_{R;\{r,s\}}(Z^{\dagger\otimes n}, X^{\dagger\otimes 2})\right\rangle\\ =&\left\langle \chi_{S,S'}(Z^{\otimes n}, X)\ Tr(D_{X^\dagger})\chi_{R;\{r,s\}}(Z^{\dagger\otimes n}, X^{\dagger\otimes 2})\right\rangle\end{aligned}$$

We immediately see that $\left\|\varsigma_{R;\{r,s\}}\right\| = \varepsilon_{R;\{r,s\}}$.

We can use these expressions to find the ς's. The final solution is quoted below.

$$\varsigma_{R;\{r,s\}} = \frac{\left\langle O_{R;\{r,s\}} O^\dagger_{R,R'}(Z^{\otimes n}, X^2)\right\rangle}{\left\langle O_{R;\{r,s\}} O^\dagger_{R;\{r,s\}}\right\rangle}$$

$$\begin{aligned}
\varsigma_{R1,a}(b_1, b_0) &= \frac{-2(b_1+3)}{(b_0+b_1+3)(b_1+2)}\\
\varsigma_{R1,d}(b_1, b_0) &= \frac{1}{b_0+1}\\
\varsigma_{R1,e}(b_1, b_0) &= \frac{-b_1}{(b_1+2)(b_0+1)}\\
\varsigma_{R1,f}(b_1, b_0) &= \frac{b_0}{b_0+1}\\
\varsigma_{R1,g}(b_1, b_0) &= \frac{-b_0(b_0+b_1+1)}{(b_0+3)(b_0+b_1+3)}\\
\varsigma_{R2,b}(b_1, b_0) &= \frac{-2(b_1-1)}{b_1(b_0+2)}\\
\varsigma_{R2,d}(b_1, b_0) &= \frac{1}{b_0+b_1+2}\\
\varsigma_{R2,e}(b_1, b_0) &= \frac{-(b_1+2)}{b_1(b_0+b_1+2)}\\
\varsigma_{R2,h}(b_1, b_0) &= \frac{b_0+b_1+1}{b_0+b_1+2}\\
\varsigma_{R2,i}(b_1, b_0) &= \frac{-b_0(b_0+b_1+1)}{(b_0+2)(b_0+b_1+2)}
\end{aligned} \tag{C.4}$$

Appendix D

Dilatation Operator

We use the following operator as a candidate for our Hamiltonian operator.

$$\hat{D} = [X, Z]^i_j \, [D_Z, D_X]^j_i \tag{D.1}$$

We see that the action of the operator is to remove and then immediately add an X and a Z when acting on our basis. We should be able to write the action of this operator in terms of our original basis. In this appendix we find the exact result for the action of the above operator on each of our states.

D.1 Action of the $\hat{D}$

$$
\begin{aligned}
\hat{D}\chi_{R,\{r,s\}} =& (XZ - ZX)^i_j \left(D_{Z}{}^j_l D_{X}{}^l_i - D_{X}{}^j_l D_{Z}{}^l_i\right) \frac{1}{n!2!} \sum_{\phi \in S_{n+2}} \\
& \times Tr_{R,\{r,s\}}(\phi) Z^{i_1}_{i\phi_1} Z^{i_2}_{i\phi_2} ... Z^{i_n}_{i\phi_n} X^{i_{n+1}}_{i\phi_{n+1}} X^{i_{n+2}}_{i\phi_{n+2}} \\
=& (XZ - ZX)^i_j \frac{1}{(n-1)!} \sum_{\phi \in S_{n+2}} Tr_{R,\{r,s\}}(\phi) Z^{i_1}_{i\phi_1} ... Z^{i_n - 1}_{i\phi_{n-1}} X^{i_{n+2}}_{i\phi_{n+2}} \times \\
+ \times & \left[\delta^j_{i\phi_n} \delta^{i_n}_l \delta^l_{i\phi_{n+1}} \delta^{i_{n+1}}_i - \delta^j_{i\phi_{n+1}} \delta^{i_{n+1}}_l \delta^l_{i\phi_n} \delta^{i_n}_i\right] \\
=& \frac{1}{(n-1)!} \sum_{\phi \in S_{n+2}} Tr_{R,\{r,s\}}(\phi) Z^{i_1}_{i\phi_1} ... Z^{i_n - 1}_{i\phi_{n-1}} X^{i_{n+2}}_{i\phi_{n+2}} \times \\
& \times \left[(XZ - ZX)^{i_n+1}_{i\phi_n} \delta^{i_n}_{i\phi_{n+1}} - (XZ - ZX)^{i_n}_{i\phi_{n+1}} \delta^{i_{n+1}}_{i\phi_n}\right]
\end{aligned}
$$

We can now shift $\phi \to \phi(n, n+1)$ to get

$$
\begin{aligned}
\hat{D}\chi_{R,\{r,s\}} =& \frac{1}{(n-1)!} \sum_{\phi \in S_{n+2}} Tr_{R,\{r,s\}}(\phi(n, n+1)) Z^{i_1}_{i\phi_1} ... Z^{i_n - 1}_{i\phi_{n-1}} X^{i_{n+2}}_{i\phi_{n+2}} \times \\
& \times \left[(XZ - ZX)^{i_{n+1}}_{i\phi_{n+1}} \delta^{i_n}_{i\phi_n} - (XZ - ZX)^{i_n}_{i\phi_n} \delta^{i_{n+1}}_{i\phi_{n+1}}\right]
\end{aligned}
$$

Now we can use the sub-group swap rule to swap the δ from the n position to the $n+1$ position in the first term. The second term will be left alone.

$$
\begin{aligned}
\hat{D}\chi_{R,\{r,s\}} =& \frac{1}{(n-1)!} \sum_{\phi \in S_{n+2}} Tr_{R,\{r,s\}}([(n, n+1), \phi]) Z^{i_1}_{i\phi_1} ... Z^{i_n - 1}_{i\phi_{n-1}} X^{i_{n+2}}_{i\phi_{n+2}} \times \\
& \times \left[(XZ - ZX)^{i_n}_{i\phi_n} \delta^{i_{n+1}}_{i\phi_{n+1}}\right]
\end{aligned}
$$

The final step is to use the subgroup swap rule between to swap the δ and the X. We also could have used the subgroup swap rule before we started the calculation, since swapping the X's has no effect.

$$\begin{aligned}\hat{D}\chi_{R,\{r,s\}} =& \frac{1}{(n-1)!}\sum_{\phi\in S_{n+2}} Tr_{R,\{r,s\}}((n,n+2)\phi-\phi(n,n+2)))\times \\ &\times Z^{i_1}_{i\phi_1}...Z^{i_n-1}_{i\phi_{n-1}}\,(XZ-ZX)^{i_n}_{i\phi_n}\,X^{i_{n+1}}_{i\phi_{n+1}}\delta^{i_{n+2}}_{i\phi_{n+2}} \\ =& D_V{}^i_i\frac{1}{(n-1)!}\sum_{\phi\in S_{n+2}} Tr_{R,\{r,s\}}((n,n+2)\phi-\phi(n,n+2)))\times \\ &\times Z^{i_1}_{i\phi_1}...Z^{i_n-1}_{i\phi_{n-1}}\,(XZ-ZX)^{i_n}_{i\phi_n}\,X^{i_{n+1}}_{i\phi_{n+1}}V^{i_{n+2}}_{i\phi_{n+2}}\end{aligned}$$

We cannot perform this reduction until we have looked at the action of $(n, n+2)$. At this point we need to split up our basis, and then perform the reduction. Some of the states can be ignored, because they mix V, and will be removed after the reduction. The diagonal states can be ignored since they give the same contribution for both terms. We are left with:

$$\begin{aligned}\hat{D}O_a(b_1,b_0)(Z^{\otimes n},X^{\otimes 2}) =& \frac{(N-b_0-b_1-1)\sqrt{(b_1+1)(b_1+3)}}{(b_1+2)(b_1+3)}\times \\ &\times\big[O_{4,5}(b_1+1,b_0-1)(Z^{\otimes n-1},XZ-ZX,X) \\ &\quad - O_{5,4}(b_1+1,b_0-1)(Z^{\otimes n-1},XZ-ZX,X)\big]\end{aligned} \tag{D.2}$$

$$\begin{aligned}\hat{D}O_b(b_1,b_0)(Z^{\otimes n},X^{\otimes 2}) =& \frac{(N-b_0)\sqrt{(b_1+1)(b_1-1)}}{b_1(b_1-1)}\times \\ &\times\big[O_{4,5}(b_1-1,b_0)(Z^{\otimes n-1},XZ-ZX,X) \\ &\quad - O_{5,4}(b_1-1,b_0)(Z^{\otimes n-1},XZ-ZX,X)\big]\end{aligned} \tag{D.3}$$

$$= \frac{N-b_0}{N-b_0-b_1}\frac{b_1+1}{b_1-1}\hat{D}O_a(b_1-2,b_0+1) \tag{D.4}$$

$$\begin{aligned}\hat{D}O_d(b_1,b_0) =& \frac{b_1}{2}\hat{D}O_a(b_1-2,b_0+1)\\ &-\frac{b_1+2}{2}\hat{D}O_b(b_1+2,b_0-1) \qquad \text{(D.5)}\\ \hat{D}O_e(b_1,b_0) =& \frac{b_1}{2}\hat{D}O_b(b_1+2,b_0-1)\\ &-\frac{b_1+2}{2}\hat{D}O_a(b_1-2,b_0+1) \qquad \text{(D.6)}\end{aligned}$$

Example 21

$$\hat{D}\chi_{\begin{array}{l}\square\,\boxed{2}\\ \square\,\boxed{1}\\ \square\end{array}} = \frac{N\sqrt{2}}{3}\left[\chi_{\begin{array}{l}\square\,\boxed{1}\\ \square\\ \boxed{2}\end{array}} - \chi_{\begin{array}{l}\square\,\boxed{2}\\ \square\\ \boxed{1}\end{array}}\right]$$

$$\hat{D}\chi_{\begin{array}{l}\square\,\square\\ \square\\ \boxed{2}\\ \boxed{1}\end{array}} = \frac{N-3}{3\sqrt{2}}\left[\chi_{\begin{array}{l}\square\,\boxed{1}\\ \square\\ \boxed{2}\end{array}} - \chi_{\begin{array}{l}\square\,\boxed{2}\\ \square\\ \boxed{1}\end{array}}\right]$$

D.2 Splitting XZ & ZX

We now have an expression for the action of $\hat{D}$ in terms of polynomials in $XZ\&ZX$. These polynomials need to be split and returned to the original basis. Using the techniques of [28], and eq.'s (B.14),(B.15) & (B.16) to write:

$$
\begin{aligned}
DO_a(b_1,b_0) = 2(N-b_0-b_1-1)\Bigg[& \frac{2}{(b_1+2)^2}O_a(b_1,b_0) \\
& \frac{-1}{b_1+2}O_d(b_1,b_0) \\
& \frac{b_1}{(b_1+2)^2}O_e(b_1,b_0) \\
& \frac{b_1+1}{(b_1+2)(b_1+3)}O_d(b_1+2,b_0-1) \\
& \frac{-(b_1+4)(b_1+1)}{(b_1+3)(b_1+2)^2}O_e(b_1+2,b_0-1) \\
& \frac{2(b_1+1)}{(b_1+3)(b_1+2)^2}O_b(b_1+2,b_0-1) \\
& \frac{1}{b_1+2}O_{R_1}(b_1,b_0)(X^2) \\
& \frac{-(b_1+1)}{(b_1+2)(b_1+3)}O_{R_2}(b_1+2,b_0-1)(X^2)\Bigg]
\end{aligned}
\tag{D.7}
$$

Now using the relations (C.4), we can write the irregular polynomials in terms of our original basis:

$$
\begin{aligned}
DO_a(b_1,b_0) = 2(N-b_0-b_1-1)\Bigg[& \frac{2}{(b_1+2)^2}\left(1-\frac{b_1+3}{b_0+b_1+3}\right)O_a(b_1,b_0) \\
& \frac{-1}{b_1+2}\left(1-\frac{1}{b_0+1}\right)O_d(b_1,b_0) \\
& \frac{b_1}{(b_1+2)^2}\left(1-\frac{1}{b_0+1}\right)O_e(b_1,b_0) \\
& \frac{b_1+1}{(b_1+2)(b_1+3)}\left(1-\frac{1}{b_0+b_1+3}\right)O_d(b_1+2,b_0-1) \\
& \frac{-(b_1+4)(b_1+1)}{(b_1+3)(b_1+2)^2}\left(1-\frac{1}{b_0+b_1+3}\right)O_e(b_1+2,b_0-1) \\
& \frac{2(b_1+1)}{(b_1+3)(b_1+2)^2}\left(1+\frac{b_1+1}{b_0+1}\right)O_b(b_1+2,b_0-1) \\
& \frac{1}{b_1+2}\frac{b_0}{b_0+1}O_f(b_1,b_0) \\
& \frac{-1}{b_1+2}\frac{b_0(b_0+b_1+1)}{(b_0+1)(b_0+b_1+3)}O_g(b_1,b_0) \\
& \frac{-(b_1+1)}{(b_1+2)(b_1+3)}\frac{b_0+b_1+2}{b_0+b_1+3}O_h(b_1+2,b_0-1) \\
& \frac{b_1+1}{(b_1+2)(b_1+3)}\frac{(b_0-1)(b_0+b_1+2)}{(b_0+1)(b_0+b_1+3)}O_i(b_1+2,b_0-1)\Bigg]
\end{aligned}
$$

Here we have defined $O_f, O_g, O_h, \& O_i$ in section C.1

We still obviously have the relations (D.4), (D.5) & (D.6).

Appendix E

Cuntz Chain States

In any conformal field theory there is a map between states and operators. In this section we want to trade our operators for a set of normalized Cuntz chain states. These normalization factors are easily inserted, since the dilatation operator is linear. $\hat{D}(cO) = c\hat{D}(O)$. Consider

$$\aleph_a(b_1, b_0) = \frac{1}{\sqrt{\left\langle O_a^\dagger(b_1, b_0) O_a(b_1, b_0) \right\rangle}}$$

then

$$|b_1, b_0, a\rangle = \aleph_a(b_1, b_0) O_a(b_1, b_0)$$

$$|b_1, b_0, b\rangle = \aleph_b(b_1, b_0) O_b(b_1, b_0)$$

$$|b_1, b_0, d\rangle = \aleph_d(b_1, b_0) O_d(b_1, b_0)$$

$$|b_1, b_0, e\rangle = \aleph_e(b_1, b_0) O_e(b_1, b_0)$$

From here on, we will absorb the weight into a factors that we call $W_a(b_1, b_0) = N - b_0 - b_1 - 1$ and $W_b(b_1, b_0) = N - b_0$.

$$
\begin{aligned}
D\,|b_1,b_0,a\rangle = 2W_a(b_1,b_0)\Bigg[&\frac{2}{(b_1+2)^2}\frac{b_0}{b_0+b_1+3}\,|b_1,b_0,a\rangle\\
&\frac{-1}{b_1+2}\frac{b_0}{b_0+1}\frac{\aleph_a(b_1,b_0)}{\aleph_d(b_1,b_0)}\,|b_1,b_0,d\rangle\\
&\frac{b_1}{(b_1+2)^2}\frac{b_0}{b_0+1}\frac{\aleph_a(b_1,b_0)}{\aleph_e(b_1,b_0)}\,|b_1,b_0,e\rangle\\
&\frac{b_1+1}{(b_1+2)(b_1+3)}\frac{b_0+b_1+2}{b_0+b_1+3}\\
&\qquad\frac{\aleph_a(b_1,b_0)}{\aleph_d(b_1+2,b_0-1)}\,|b_1+2,b_0-1,d\rangle\\
&\frac{-(b_1+4)(b_1+1)}{(b_1+3)(b_1+2)^2}\frac{b_0+b_1+2}{b_0+b_1+3}\\
&\qquad\frac{\aleph_a(b_1,b_0)}{\aleph_e(b_1+2,b_0-1)}\,|b_1+2,b_0-1,e\rangle\\
&\frac{2(b_1+1)}{(b_1+3)(b_1+2)^2}\frac{b_0+b_1+2}{b_0+1}\\
&\qquad\frac{\aleph_a(b_1,b_0)}{\aleph_b(b_1+2,b_0-1)}\,|b_1+2,b_0-1,b\rangle\\
&\frac{1}{b_1+2}\frac{b_0}{b_0+1}\frac{\aleph_a(b_1,b_0)}{\aleph_f(b_1,b_0)}\,|b_1,b_0,f\rangle\\
&\frac{-1}{b_1+2}\frac{b_0(b_0+b_1+1)}{(b_0+1)(b_0+b_1+3)}\frac{\aleph_a(b_1,b_0)}{\aleph_g(b_1,b_0)}\,|b_1,b_0,g\rangle\\
&\frac{-(b_1+1)}{(b_1+2)(b_1+3)}\frac{b_0+b_1+2}{b_0+b_1+3}\\
&\qquad\frac{\aleph_a(b_1,b_0)}{\aleph_h(b_1+2,b_0-1)}\,|b_1+2,b_0-1,h\rangle\\
&\frac{b_1+1}{(b_1+2)(b_1+3)}\frac{(b_0-1)(b_0+b_1+2)}{(b_0+1)(b_0+b_1+3)}\\
&\qquad\frac{\aleph_a(b_1,b_0)}{\aleph_i(b_1+2,b_0-1)}\,|b_1+2,b_0-1,i\rangle\Bigg]
\end{aligned}
$$

$$
\hat{D}\,|b_1,b_0,b\rangle = \frac{N-b_0}{N-b_0-b_1}\frac{b_1+1}{b_1-1}\frac{\aleph_b(b_1,b_0)}{\aleph_a(b_1-2,b_0+1)}\hat{D}\,|b_1-2,b_0+1,a\rangle
$$

$$\begin{aligned}\hat{D}\left|b_1,b_0,d\right\rangle =&\frac{b_1}{2}\frac{\aleph_d(b_1,b_0)}{\aleph_a(b_1-2,b_0+1)}\hat{D}\left|b_1-2,b_0+1,a\right\rangle\\ &-\frac{N-b_0+1}{N-b_0-b_1-1}\frac{(b_1+2)(b_1+3)}{2(b_1+1)}\frac{\aleph_d(b_1,b_0)}{\aleph_a(b_1,b_0)}\hat{D}\left|b_1,b_0,a\right\rangle\end{aligned}$$

$$\begin{aligned}\hat{D}O_e(b_1,b_0) =&\frac{N-b_0+1}{N-b_0-b_1-1}\frac{b_1(b_1+3)}{2(b_1+1)}\frac{\aleph_e(b_1,b_0)}{\aleph_a(b_1,b_0)}\hat{D}\left|b_1,b_0,a\right\rangle\\ &-\frac{b_1+2}{2}\frac{\aleph_e(b_1,b_0)}{\aleph_a(b_1-2,b_0+1)}\hat{D}\left|b_1-2,b_0+1,a\right\rangle\end{aligned}$$

We are now in a position to start making approximations. We are interested in the large N limit for nearly maximal giants. This implies that b_0 is also large. We can make the approximations that $\frac{1}{\sqrt{N}}\sim\frac{1}{\sqrt{b_0}}\sim 0$. Using the normalization calculated in eq.'s (C.1) & (C.2)

$$\begin{aligned}\frac{\aleph_a(b_1,b_0)}{\aleph_f(b_1,b_0)} &= \sqrt{\frac{(b_0+1)(b_1+3)}{b_0(b_0+b_1+2)(b_1+2)}\frac{N+2}{N-b_0-1}}\\ &= \sqrt{\frac{(b_0+1)(b_1+3)}{b_0(b_0+b_1+2)(b_1+2)}}+\mathcal{O}(\frac{1}{\sqrt{N}})\\ &\sim \frac{1}{\sqrt{b_0}}\sim 0\end{aligned}$$

We can drop the contributions from all the states that are labeled by a Young diagram with a box in the third column. We can also make the approximations that:

$$\frac{b_0+\mathcal{O}(1)}{b_0}\approx\frac{b_0+b_1+\mathcal{O}(1)}{b_0+b_1}\approx\frac{b_0+b_1+\mathcal{O}(1)}{b_0}\approx 1 \tag{E.1}$$

We can use the approximations (eq.(E.1)) to get the following identities:

$$\aleph_d(b_1, b_0) = \aleph_e(b_1, b_0) \tag{E.2}$$

$$\frac{\aleph_a(b_1, b_0)}{\aleph_d(b_1, b_0)} = \sqrt{\frac{(b_1+3)}{(b_1+1)}\frac{(N-b_0+1)}{(N-b_1-b_0-1)}} \tag{E.3}$$

$$\frac{\aleph_a(b_1, b_0)}{\aleph_d(b_1+2, b_0-1)} = \sqrt{\frac{(b_1+3)}{(b_1+1)}} \tag{E.4}$$

$$\frac{\aleph_a(b_1, b_0)}{\aleph_b(b_1+2, b_0-1)} = \frac{(b_1+3)}{(b_1+1)}\sqrt{\frac{(N-b_0+1)}{(N-b_1-b_0-1)}} \tag{E.5}$$

Using these normalization factors we find that:

$$\begin{aligned} D\,|b_1, b_0, a\rangle =& \frac{4}{(b_1+2)^2} W_a(b_1, b_0)\,|b_1, b_0, a\rangle \\ & \frac{-2}{b_1+2}\sqrt{\frac{b_1+3}{b_1+1}}\sqrt{W_a(b_1, b_0)W_b(b_1+2, b_0-1)}\,|b_1, b_0, d\rangle \\ & \frac{2b_1}{(b_1+2)^2}\sqrt{\frac{b_1+3}{b_1+1}}\sqrt{W_a(b_1, b_0)W_b(b_1+2, b_0-1)}\,|b_1, b_0, e\rangle \\ & \frac{2}{b_1+2}\sqrt{\frac{b_1+1}{b_1+3}} W_a(b_1, b_0)\,|b_1+2, b_0-1, d\rangle \\ & \frac{-2(b_1+4)}{(b_1+2)^2}\sqrt{\frac{b_1+1}{b_1+3}} W_a(b_1, b_0)\,|b_1+2, b_0-1, e\rangle \\ & \frac{4}{(b_1+2)^2}\sqrt{W_a(b_1, b_0)W_b(b1+2, b_0-1)}\,|b_1+2, b_0-1, b\rangle \end{aligned} \tag{E.6}$$

$$\hat{D}\,|b_1, b_0, b\rangle = \sqrt{\frac{W_b(b_1, b_0)}{W_a(b_1-2, b_0+1)}}\hat{D}\,|b_1-2, b_0+1, a\rangle \tag{E.7}$$

$$\begin{aligned} \hat{D}\,|b_1, b_0, d\rangle =& \frac{b_1}{2}\sqrt{\frac{b_1-1}{b_1+1}}\hat{D}\,|b_1-2, b_0+1, a\rangle \\ & - \frac{b_1+2}{2}\sqrt{\frac{W_b(b_1+2, b_0-1)}{W_a(b_1, b_0)}}\sqrt{\frac{b_1+3}{b_1+1}}\hat{D}\,|b_1, b_0, a\rangle \end{aligned} \tag{E.8}$$

$$\begin{aligned}\hat{D}\left|b_1, b_0, e\right\rangle =& \frac{b_1}{2}\sqrt{\frac{W_b(b_1+2, b_0-1)}{W_a(b_1, b_0)}}\sqrt{\frac{b_1+3}{b_1+1}}\hat{D}\left|b_1, b_0, a\right\rangle \\ &- \frac{b_1+2}{2}\sqrt{\frac{b_1-1}{b_1+1}}\hat{D}\left|b_1-2, b_0+1, a\right\rangle \end{aligned} \tag{E.9}$$

Appendix F

Diagonalizing $\hat{D}$

In this section we want to diagonalize $\hat{D}$. Since we wish to consider $\hat{D}$ as a Hamiltonian, we should have real valued eigenvalues, which are at least bounded below.

We know that the Schur polynomial basis is a complete basis for the problem. This does not necessarily mean that the Schur polynomial basis will diagonalize the dilatation operator. Since we wish to consider the dilatation operator as the Hamiltonian of the system, we need to build states that diagonalize it. These states will also form a basis for the problem. Our goal in this appendix is to write these states in terms of the original Schur polynomial, which we think we know how to interpret.

Since the total number of boxes in the Young diagram remains the same after the action of $\hat{D}$, we only need to label our states by one changing parameter and shape. Since b_1 appears most regularly it seems natural to retain this when labeling the different states. The structure of the problem means that b_1 only changes by multiples of 2. We can therefore split the problem into solving for an even number of boxes and an odd number of

boxes. Let's consider the number of boxes even, and for simplicity N even. The only other freedom we have is in our choice of N. In order for us to justify dropping all the polynomials, let N be a large number, and the total number of boxes be slightly less than $2N$. This means that the weight of the last box in the first column will decrease to zero, as b_1 increases. When the weight of the last box in the first column goes to zero, this state has a zero norm. This effectively truncates our sum, forcing b_0 to always be a large number.

At this point we will change notation slightly. Instead of labeling the polynomials by $b_0 \& b_1$, we will now label them by the length of each column $p \& q$, which includes the boxes corresponding to X.

$$
\begin{aligned}
|b_1, b_0, a\rangle &\rightarrow |p = b_0 + b_1 + 2, q = b_0, a\rangle \\
|b_1, b_0, b\rangle &\rightarrow |p = b_0 + b_1, q = b_0 + 2, b\rangle \\
|b_1, b_0, d\rangle &\rightarrow |p = b_0 + b_1 + 1, q = b_0 + 1, d\rangle \\
|b_1, b_0, e\rangle &\rightarrow |p = b_0 + b_1 + 1, q = b_0 + 1, e\rangle \\
W_a(b_1, b_0) &\rightarrow W_a(p, q) = N + 1 - p \\
W_b(b_1, b_0) &\rightarrow W_b(p, q) = N + 2 - q
\end{aligned}
$$

We are interested in the large N limit where we are considering 2 nearly maximal Giant gravitons, with strings attached to them. We know that strings have a length of $\mathcal{O}(N)$ therefore we must consider the situation where the difference in size between the giants is $\mathcal{O}(N)$. We can naturally interpret the size of the larger giant as p and the the size of the smaller one as q. The

number $p-q$ gives the difference in size of the two giants.

Due to the fact that we can't define a state $|p,p,a\rangle$, we shift from writing expressions in terms of $D\,|p,q,a\rangle$ to $D\,|p,q,b\rangle$

The relations (eq.'s (E.6), (E.7), (E.8) & (E.9)) from the previous section become($\sigma = p-q$):

$$
\begin{aligned}
D\,|p,q,b\rangle =& \frac{4}{(\sigma+2)^2}\sqrt{W_a(p+1,q-1)W_b(p,q)}\,|p+1,q-1,a\rangle \\
& \frac{-2}{\sigma+2}\sqrt{\frac{\sigma+3}{\sigma+1}}W_b(p,q)\,|p,q,d\rangle \\
& \frac{2\sigma}{(\sigma+2)^2}\sqrt{\frac{\sigma+3}{\sigma+1}}W_b(p,q)\,|p,q,e\rangle \\
& \frac{2}{\sigma+2}\sqrt{\frac{\sigma+1}{\sigma+3}}\sqrt{W_a(p+1,q-1)W_b(p,q)}\,|p+1,q-1,d\rangle \\
& \frac{-2(\sigma+4)}{(\sigma+2)^2}\sqrt{\frac{\sigma+1}{\sigma+3}}\sqrt{W_a(p+1,q-1)W_b(p,q)}\,|p+1,q-1,e\rangle \\
& \frac{4}{(\sigma+2)^2}W_b(p,q)\,|p,q,b\rangle
\end{aligned}
$$

$$
\hat{D}\,|p,q,a\rangle = \sqrt{\frac{W_a(p,q)}{W_b(p-1,q+1)}}\hat{D}\,|p-1,q+1,b\rangle
$$

$$
\begin{aligned}
\hat{D}\,|p,q,d\rangle =& \frac{\sigma}{2}\sqrt{\frac{\sigma-1}{\sigma+1}}\sqrt{\frac{W_a(p,q)}{W_b(p-1,q+1)}}\hat{D}\,|p-1,q+1,b\rangle \\
& -\frac{\sigma+2}{2}\sqrt{\frac{\sigma+3}{\sigma+1}}\hat{D}\,|p,q,b\rangle
\end{aligned}
$$

$$
\begin{aligned}
\hat{D}\,|b_1,b_0,e\rangle =& \frac{\sigma}{2}\sqrt{\frac{\sigma+3}{\sigma+1}}\hat{D}\,|p,q,b\rangle \\
& -\frac{\sigma+2}{2}\sqrt{\frac{\sigma-1}{\sigma+1}}\sqrt{\frac{W_a(p,q)}{W_b(p-1,q+1)}}\hat{D}\,|p-1,q+1,b\rangle
\end{aligned}
$$

We now define the number of boxes to be $M = p + q$. Truncating the sum is now easy. To avoid having a box with zero weight, we have $p_{\text{max}} = N$. The minimum value of σ changes depending on the shape. The following table gives the ranges of p and σ.

Shape	p_{max}	σ_{max}	p_{min}	σ_{min}
a	N	$2N - M$	$\frac{M+2}{2}$	2
b	N	$2N - M$	$\frac{M}{2}$	0
d	N	$2N - M$	$\frac{M}{2}$	0
e	N	$2N - M$	$\frac{M+2}{2}$	2

We can see that all the information about the polynomial is kept in N, M, σ and the shape. Since N and M are the same for all of the polynomials, we need only label states by σ ,the number of boxes and the shape. Since the number of boxes does not change by the action of $\hat{D}$, we only need to keep track of σ and the shape.

$$
\begin{aligned}
\hat{D} \sum_{j=0}^{\frac{2N-M}{2}} b_{2j} \left|2j, b\right\rangle = & \sum_{j=0}^{\frac{M-2N}{2}} \frac{4}{(2j+2)^2} \sqrt{W_a(2j+2)W_b(2j)} b_{2j} \left|2j+2, a\right\rangle \\
& - \sum_{j=0}^{\frac{2N-M}{2}} \frac{2}{2j+2} \sqrt{\frac{2j+3}{2j+1}} W_b(2j) b_{2j} \left|2j, d\right\rangle \\
& + \sum_{j=0}^{\frac{2N-M}{2}} \frac{2(2j)}{(2j+2)^2} \sqrt{\frac{2j+3}{2j+1}} W_b(2j) b_{2j} \left|2j, e\right\rangle \\
& + \sum_{j=0}^{\frac{2N-M}{2}} \frac{2}{2j+2} \sqrt{\frac{2j+1}{2j+3}} \sqrt{W_a(2j+2)W_b(2j)} b_{2j} \left|2j+2, d\right\rangle \\
& - \sum_{j=0}^{\frac{2N-M}{2}} \frac{2(2j+4)}{(2j+2)^2} \sqrt{\frac{2j+1}{2j+3}} \sqrt{W_a(2j+2)W_b(2j)} b_{2j} \left|2j+2, e\right\rangle \\
& + \sum_{j=0}^{\frac{2N-M}{2}} \frac{4}{(2j+2)^2} W_b(2j) b_{2j} \left|2j, b\right\rangle \\
= & \sum_{j=1}^{\frac{M-2N}{2}} \frac{4}{(2j)^2} \sqrt{W_a(2j)W_b(2j-2)} b_{2j-2} \left|2j, a\right\rangle \\
& - \sum_{j=0}^{\frac{2N-M}{2}} \frac{2}{2j+2} \sqrt{\frac{2j+3}{2j+1}} W_b(2j) b_{2j} \left|2j, d\right\rangle \\
& + \sum_{j=1}^{\frac{2N-M}{2}} \frac{2(2j)}{(2j+2)^2} \sqrt{\frac{2j+3}{2j+1}} W_b(2j) b_{2j} \left|2j, e\right\rangle \\
& + \sum_{j=0}^{\frac{2N-M}{2}} \frac{2}{2j} \sqrt{\frac{2j-1}{2j+1}} \sqrt{W_a(2j)W_b(2j-2)} b_{2j-2} \left|2j, d\right\rangle \\
& - \sum_{j=1}^{\frac{2N-M}{2}} \frac{2(2j+2)}{(2j)^2} \sqrt{\frac{2j-1}{2j+1}} \sqrt{W_a(2j)W_b(2j-2)} b_{2j-2} \left|2j, e\right\rangle \\
& + \sum_{j=0}^{\frac{2N-M}{2}} \frac{4}{(2j+2)^2} W_b(2j) b_{2j} \left|2j, b\right\rangle
\end{aligned}
$$

Here we have used the fact that $W_a(\frac{2N-M}{2}+1)$=0, and forced $\frac{b_{-2}}{0}=0$. The contributions for the other shapes are:

$$
\begin{aligned}
\hat{D}\sum_{j=1}^{\frac{2N-M}{2}} a_{2j}\left|2j,a\right\rangle =&\hat{D}\sum_{j=1}^{\frac{2N-M}{2}}\sqrt{\frac{W_a(2j)}{W_b(2j-2)}}a_{2j}\left|2j-2,b\right\rangle\\
=&\hat{D}\sum_{j=0}^{\frac{2N-M}{2}}\sqrt{\frac{W_a(2j+2)}{W_b(2j)}}a_{2j+2}\left|2j,b\right\rangle\\
=&\sum_{j=0}^{\frac{2N-M}{2}} b_{2j}^{(2)}\hat{D}\left|2j,a\right\rangle\\
\hat{D}\sum_{j=0}^{\frac{2N-M}{2}} d_{2j}\left|2j,d\right\rangle =&\hat{D}\sum_{j=0}^{\frac{2N-M}{2}}\frac{2j}{2}\sqrt{\frac{2j-1}{2j+1}}\sqrt{\frac{W_a(2j)}{W_b(2j-2)}}d_{2j}\left|2j-2,b\right\rangle\\
&-\hat{D}\sum_{j=0}^{\frac{2N-M}{2}}\frac{2j+2}{2}\sqrt{\frac{2j+3}{2j+1}}d_{2j}\left|2j,b\right\rangle\\
=&\hat{D}\sum_{j=0}^{\frac{2N-M}{2}}\frac{2j+2}{2}\sqrt{\frac{2j+1}{2j+3}}\sqrt{\frac{W_a(2j+2)}{W_b(2j)}}d_{2j+2}\left|2j,b\right\rangle\\
&-\hat{D}\sum_{j=0}^{\frac{2N-M}{2}}\frac{2j+2}{2}\sqrt{\frac{2j+3}{2j+1}}d_{2j}\left|2j,b\right\rangle\\
=&\sum_{j=0}^{\frac{2N-M}{2}} b_{2j}^{(3)}\hat{D}\left|2j,b\right\rangle\\
\hat{D}\sum_{j=1}^{\frac{2N-M}{2}} e_{2j}\left|2j,e\right\rangle =&\hat{D}\sum_{j=1}^{\frac{2N-M}{2}}\frac{2j}{2}\sqrt{\frac{2j+3}{2j+1}}e_{2j}\left|2j,b\right\rangle\\
&-\hat{D}\sum_{j=1}^{\frac{2N-M}{2}}\frac{2j+2}{2}\sqrt{\frac{2j-1}{2j+1}}\sqrt{\frac{W_a(2j)}{W_b(2j-2)}}e_{2j}\left|2j-2,b\right\rangle\\
=&\hat{D}\sum_{j=1}^{\frac{2N-M}{2}}\frac{2j}{2}\sqrt{\frac{2j+3}{2j+1}}e_{2j}\left|2j,b\right\rangle\\
&-\hat{D}\sum_{j=0}^{\frac{2N-M}{2}}\frac{2j+4}{2}\sqrt{\frac{2j+1}{2j+3}}\sqrt{\frac{W_a(2j+2)}{W_b(2j)}}e_{2j+2}\left|2j,b\right\rangle\\
=&\sum_{j=0}^{\frac{2N-M+2}{2}} b_{2j}^{(4)}\hat{D}\left|2j,b\right\rangle
\end{aligned}
$$

We are now in a position to diagonalize $\hat{D}$. We define $\Gamma_{2j} = b_{2j} + b_{2j}^{(2)} + b_{2j}^{(3)} + b_{2j}^{(4)}$. We can then write our eigenvalue problem as:

$$a_{2j}\Lambda = \frac{4}{(2j)^2}\sqrt{W_a(2j)W_b(2j-2)}\Gamma_{2j-2} \tag{F.1}$$

$$b_{2j}\Lambda = \frac{4}{(2j+2)^2}W_b(2j)\Gamma_{2j} \tag{F.2}$$

$$\begin{aligned} d_{2j}\Lambda =& \frac{2}{2j}\sqrt{\frac{2j-1}{2j+1}}\sqrt{W_a(2j)W_b(2j-2)}\Gamma_{2j-2} \\ &- \frac{2}{2j+2}\sqrt{\frac{2j+3}{2j+1}}W_b(2j)\Gamma_{2j} \end{aligned} \tag{F.3}$$

$$\begin{aligned} e_{2j}\Lambda =& \frac{2(2j)}{(2j+2)^2}\sqrt{\frac{2j+3}{2j+1}}W_b(2j)\Gamma_{2j} \\ &- \frac{2(2j+2)}{(2j)^2}\sqrt{\frac{2j-1}{2j+1}}\sqrt{W_a(2j)W_b(2j-2)}\Gamma_{2j-2} \end{aligned} \tag{F.4}$$

We already know that there is a great deal of degeneracy in the problem, and degeneracy corresponds to zero eigenvalues. The non-zero eigenvalues and eigenvectors are of interest to us here.

$$a_{2j} = \sqrt{\frac{W_a(2j)}{W_b(2j-2)}}b_{2j-2} \tag{F.5}$$

$$d_{2j} = \frac{2j}{2}\sqrt{\frac{2j-1}{2j+1}}\sqrt{\frac{W_a(2j)}{W_b(2j-2)}}b_{2j-2} - \frac{2j+2}{2}\sqrt{\frac{2j+3}{2j+1}}b_{2j} \tag{F.6}$$

$$e_{2j} = \frac{2j}{2}\sqrt{\frac{2j+3}{2j+1}}b_{2j} - \frac{2j+2}{2}\sqrt{\frac{2j-1}{2j+1}}\sqrt{\frac{W_a(2j)}{W_b(2j-2)}}b_{2j-2} \tag{F.7}$$

We can use eq.'s (F.5),(F.6),(F.7) to solve for eq. (F.1). We get the following:

$$b_{2j}\Lambda = f_-(2j)b_{2j-2} + f_0(2j)b_{2j} + f_+(2j)b_{2j+2} \tag{F.8}$$

$$\begin{aligned}
f_-(2j) &= -\frac{2(2j)\sqrt{(2j+3)(2j-1)}}{(2j+2)(2j+1)}\sqrt{\frac{W_a(2j)}{W_b(2j-2)}}W_b(2j) \\
f_0(2j) &= \frac{2((2j)^2+5(2j)+8)}{(2j+3)(2j+2)}W_a(2j+2) + \frac{2((2j)^2+3(2j)+4)}{(2j+2)(2j+1)}W_b(2j) \\
f_+(2j) &= -\frac{2(2j+4)\sqrt{(2j+5)(2j+1)}}{(2j+3)(2j+2)}\sqrt{W_a(2j+2)W_b(2j)}
\end{aligned} \tag{F.9}$$

These expressions are very easily put into a matrix which can be diagonalized to find the eigenvalues and eigenvectors.

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