Theoretical and Phenomenological Aspects of Projective-Invariant Metric-Affine Theories of Gravity

by

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To grandma and her legacy.

Abstract

Three models of parity violating modified gravity are considered in the metric-affine approach, their formal properties are discussed and new solutions are derived, together with predictions in cosmology, black holes and gravitational waves settings. An extension to generic metric-affine spacetimes of the Holst, Nieh-Yan and Chern-Simons topological terms is provided, accounting for the presence of nonmetricity and recovering the invariance under projective transformations acting on the affine connection. While the Holst term is projective invariant and vanishing on-half shell, even in presence of nonmetricity, the latter is responsible for the breaking of projective symmetry and the spoiling of the topological character both in the Nieh-Yan and Chern-Simons case. A generalization of these two topological invariants is provided, allowing to recover both properties independently. Specific models featuring the generalized terms are considered and the role of projective symmetry in relation to the absence of dynamical instabilities is thoroughly discussed. The absence of ghost degrees of freedom is proved for all three models. Ostrogradski instabilities in the Nieh-Yan model are ruled out by its dynamical equivalence with DHOST theories while new methods to avoid higher-order derivatives introduced by the Chern-Simons modification are devised. Then, several solutions are derived. New analytical hairy black hole solutions are presented for the Holst model and their thermodynamics is analysed, highlighting modifications to the black hole entropy and the role of the solution within a system of conjectures recently proposed in literature in the extended phase space approach. In models featuring the generalized Nieh-Yan term, semi-analytic, anisotropic cosmological solutions are obtained, exhibiting a big-bounce regularization of the initial singularity. The presence of future finite-time singularities is pointed out and their physical viability in terms of geodesic completeness and well-behaved scalar perturbations is proved. In the generalized Chern-Simons theory modifications to black hole perturbations and gravitational waves are derived. The quasinormal modes of Schwarzschild black holes are computed, showing their deviations from both General Relativity and the metric version of Chern-Simons gravity. New exact cosmological solutions with non-trivial affine structure are used as background configurations to study the propagation of gravitational waves, establishing the phenomenon of gravitational birefringence in metric-affine Chern-Simons gravity. The propagation in matter is also investigated proving the existence of gravitational Landau damping. Although observable in principle, the magnitude of the effects is too low for them to be tested given the sensitivity of present day experiments.

Keywords: Modified gravity, metric-affine theories, projective symmetry, torsion, nonmetricity, $f(R)$ theories, parity violation, Holst, Nieh-Yan, Chern-Simons, instabilities, ghosts, Ostrogradski, hairy black holes, black hole thermodynamics, big-bounce, quasinormal modes, gravitational waves birefringence, gravitational Landau damping.

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Contents

Author’s declaration

This thesis is a collection of the majority of the results I have obtained during the three years of the PhD program in Physics at the University of Trento. They are all presented in scientific papers written in collaboration with several professors and researchers. A detailed list of the papers follows. The papers containing results discussed in this thesis are:

- [1] S. Boudet, F. Bombacigno, G. Montani and M. Rinaldi, *Superentropic black hole with Immirzi hair*, Phys. Rev. D 103 (2021) 8, 084034;
- [2] F. Bombacigno, S. Boudet, G. J. Olmo and G. Montani, *Big bounce and future time singularity resolution in Bianchi I cosmologies: The projective invariant Nieh-Yan case*, Phys. Rev. D 103 (2021) 12, 124031;
- [3] S. Boudet, F. Bombacigno, G. J. Olmo and P. J. Porfirio, *Quasinormal modes of Schwarzschild black holes in projective invariant Chern-Simons modified gravity*, JCAP 05 (2022) 032;
- [4] S. Boudet, F. Bombacigno, F. Moretti and G.J. Olmo, *Torsional birefringence in metric-affine Chern-Simons gravity: gravitational waves in late-time cosmology*, JCAP 01 (2023) 026;
- [5] F. Bombacigno, F. Moretti, S. Boudet and G.J. Olmo, *Landau damping for gravitational waves in parity-violating theories*, JCAP 02 (2023) 009.

Papers not included in the thesis:

- [6] F. Bombacigno, S. Boudet and G. Montani, *Generalized Ashtekar variables for Palatini $f(R)$ models*, Nucl. Phys. B 963 (2021) 115281;
- [7] S. Boudet, M. Rinaldi and S. Silveravalle, *On the stability of scale-invariant black holes*, JHEP 01 (2023) 133.

Papers [6] and [7] are signed in alphabetical order.

Introduction

Science moves forward through the continuous improvement of the human understanding of natural phenomena, which is by its own nature only asymptotically tending to a complete description of the universe. Regarding the domain of physics, our current knowledge is based on the interaction between elementary particles mediated by four fundamental forces. Although the Newtonian concept of force has successfully been overcome in the last century, this process has not followed a single path, common to all interactions. While the electromagnetic, weak and strong forces are now described as the exchange of gauge bosons between fermions, the gravitational force results from the interaction of matter with spacetime curvature. Such duality is apparent in the mathematical description of the fundamental interactions, which relies on two complementary but incompatible frameworks. On one hand, the Standard Model of particle physics, with the theoretical background of quantum field theory, provides a coherent description of the strong and electroweak interactions as a gauge theory of the $SU(3) \times SU(2) \times U(1)$ group of symmetry, with a spontaneous symmetry breaking provided by the Higgs mechanism [B8, B9]. On the other hand, the best understanding of the gravitational phenomena is provided by Einstein's theory of General Relativity, in which the language of differential geometry is used to describe the interplay between matter fields and the geometry of spacetime, which is considered itself as a dynamical entity [B10, B11].

Gravity plays a very peculiar role within this big picture for several reasons: it is the only interaction in which the "charge" of particles, that is the gravitational mass, coincides with their inertial mass; it is extraordinarily weaker than the other interactions with a strength of order $\sim 10^{-38}$ compared to the strong force; by virtue of the mass-energy equivalence, it is the only interaction which affects every elementary particle. Finally, gravity is about the geometrodynamics of spacetime itself, which is not the mere stage upon which physical phenomena occurs, as it happens for all other interactions. All these features are embodied by the theory of General Relativity, in which the dynamics of tensor fields living on pseudo-Riemannian manifolds is described by differential equations covariant under general coordinate transformations.

General Relativity has provided some of the most accurate predictions in the history of physics [12]. It is worth mentioning the historical observation of the light bending during the solar eclipse of 1919 and the explanation of the precession of Mercury's perihelion. However, the most outstanding is undoubtedly the account of large part of the cosmological evolution given by the concordance model and in particular the prediction of the spectrum of the anisotropies in the Cosmic Microwave Background radiation [13]. Another merit of the theory is the remarkable prediction of new, previously unobserved, phenomena which were originally considered with skepticism but are now experimentally well-established: gravitational waves and black holes. The former, first detected in 2015 by the LIGO-Virgo collaboration [14], are well understood theoretically and are expected to yield new interesting observational inputs in the next years. The latter, whose image was captured for the first time four years ago by the Horizon telescope [15], are still evading a deep understanding but are well-described by General Relativity in the exterior region.
Despite all of the above, General Relativity is affected by some shortcomings both at the theoretical and observational level which call for a solution. On the theoretical side, General Relativity presents several issues when the quantization of the theory is addressed. Some of the obstacles towards its quantization are rooted in the aforementioned conflicts with the other fundamental interactions, e.g. the non renormalizability of the theory [16, 17], the impossibility of fully relying on a fixed background and the so-called problem of time [B18]. Another issue, which is inherent to General Relativity itself at the classical level, is the presence of singularities in some relevant solutions of the theory, namely the singularity at the center of black holes [19] and the big-bang singularity at the origin of the universe [20]. In both cases the theory is plagued by the presence of points in spacetime with infinite curvature and energy density. A common solution to these problems may be rooted in a yet unknown, well defined theory of quantum gravity which is widely believed to be required in high energy regimes, at scales of the order of the Planck length, i.e. $\sim 10^{-35}$ m.
Regarding discrepancies with experimental results, beside questions related to the dark components of the universe, namely the nature of dark matter [21] and the cause for the present acceleration of the universe expansion [22], which are still missing a satisfactory explanation, recent measurements added further challenges. In particular, we are referring to the H_0 and σ_8 tensions, resulting from the Planck 2018 measurements, which showed how probes at different redshift yield inconsistent estimates for these cosmological parameters [23].
Several solutions to such problems were proposed in literature, within a large variety of different frameworks and approaches. The quantum gravity problem has been addressed by string theory [24], loop quantum gravity [B25, B26], the asymptotic safety program [27] and causal dynamical triangulation theory [28,

29], to name a few. Loop quantum cosmology [30], classical bouncing cosmologies and the study of regular black hole spacetimes are all oriented towards a resolution of General Relativity singularities. The most promising solution to the dark matter problem is the hypothesis of the existence of Weakly Interactive Massive Particles [31], while the tensions in the cosmological parameters are currently being addressed following several paths [32].
Another paradigm in answering these open questions is modified gravity, a field that received increasing attention in the last years. In theories of modified gravity, or extended theories of gravity, some of the features of General Relativity are changed, with the aim of improving the theory. An unease aspect of the modified gravity approach is that it favours the proliferation of an excessive amount of gravitational models. To this day, countless theories have been proposed in the literature and though each one has its own theoretical and/or observational motivations, it is of paramount importance to select and falsify as many as possible of them. To this aim, two main aspects should be considered: internal theoretical consistency and the agreement with experimental observations. By the former, we basically mean the absence of unhealthy degrees of freedom such as ghosts, instabilities sourced by higher-order derivatives in the field equations or strong coupling problems [33] (see also [34, 35, 36, 37]). Regarding the latter instead, new models should deviate from General Relativity only in those regimes where it is lacking and should reduce to it where the theory offers experimentally viable predictions.
The most common way of modifying General Relativity consists in changing its action, either adding new curvature terms to the Einstein-Hilbert Lagrangian or considering new fields in the variational principle. Some relevant examples are quadratic gravity [38], Gauss-Bonnet gravity [39], Lovelock gravity [40], Brans-Dicke theory [41] and scalar-vector-tensor theories [42], Horndeski [43] and Degenerate-Higher-Order-Scalar-Tensor (DHOST) theories [44]. An important class of theories is represented by $f(R)$ gravity [45, 46, 47, 48], in which a general function of the Ricci scalar is chosen as Lagrangian, thus encompassing several models at the same time, constituting a very general and yet easy to deal with theory, especially when it is formulated in its scalar-tensor representation.
Another class of extended theories of gravity is given by metric-affine theories [49, 50, 51]. In metric-affine gravity the kinematics of General Relativity is modified allowing for manifolds with a richer geometric structure. Beside spacetime curvature, the gravitational field is also characterized by two additional geometric features: torsion and nonmetricity. Their presence is a consequence of a different perspective on the variational principle of the theory in which the two fundamental objects of General Relativity, i.e. the metric tensor and the connection, are considered as independent variables. This is known as first order or Palatini formulation, as opposed to the usual second order or metric formulation

of General Relativity, in which the connection is a priori fully determined by the metric.
Now, it turns out that considering the first order formulation of General Relativity is not enough to implement any modification in the theory, since the equivalence with the second order approach can be proved on-shell. However, several models of modified gravity based on the first order paradigm have been considered in literature. Among them we can recall Einstein-Cartan and Weyl theories, Poincaré gauge theory [49], teleparallel theories of gravity [52], Ricci based theories [53], Eddington inspire Born-Infeld gravity [54] and the Palatini version of $f(R)$ theories [48]. In all of them, the first order formulation comes with the inclusion of additional terms in the Lagrangian or additional fields, allowing for deviations from Einstein's gravity.

This thesis will focus on three curvature terms considered in a first order formulation: the Holst [55], Nieh-Yan [56, 57] and Chern-Simons terms [58]. A feature common to all of them is the parity breaking character, namely the fact that they change sign under a parity transformation. Among the symmetries of the universe, parity plays an important role and it is well-known to be violated in the Standard Model by electroweak interactions. Its role in gravity is less clear and the question of whether the gravitational interaction might break it is an open one in the literature. Moreover, the introduction of parity violating modifications usually entails peculiar observational signatures deviating from the General Relativity ones, in which the parity symmetry is preserved. This may offer interesting opportunities to test and falsify alternative models.
The origin of the first two terms mentioned above is rooted in the field of loop quantum gravity. This is a rather conservative approach to the quantum gravity problem, in which a nonperturbative, background independent, canonical quantization of General Relativity is addressed. The main idea consists in recasting the theory in terms of a new set of variables which allow to overcome some of the difficulties in the quantization process. These variables can be implemented in the theory adding the Holst [55] or the Nieh-Yan [59] term to the action. By virtue of the on-shell vanishing of the former and the topological character of the latter, their inclusion does not imply any modification at the level of the classical field equations, and they only play a significant role in the quantum framework.
Regarding the Chern-Simons term instead, its presence seems ubiquitous in different contexts, distant from one another. The gravitational anomaly in the Standard Model framework turns out to have the same structure as the Chern-Simons term, which must be considered to cancel the anomaly out. Counterterms of the same kind are also produced in string theories via the Green-Schwarz mechanism and they emerge in low energy effective string models [60]. Remarkably, connections with the loop quantum gravity approaches exist as well, since

Chern-Simons corrections arise when addressing the chiral anomaly of fermions together with the so-called Immirzi parameter ambiguity [61, 62].

Analogously to the Nieh-Yan one, also the Chern-Simons term is topological. Hence, in all three cases modifications at the classical level can only be introduced considering them in already extended frameworks, e.g. including them in $f(R)$-like models, or promoting the couplings of these terms to (pseudo-)scalar fields.

In literature, the Holst and Nieh-Yan terms have only been considered in presence of torsion, with vanishing nonmetricity, while the Chern-Simons term has mainly been studied in the purely metric formulation, and its implementation in the metric-affine framework has received little attention. In this sense there is a lack of generality in the existing works that should be remedied by extending these terms to the most general metric-affine setting, which will be the topic of the first part of this thesis.

During this process, close attention should be paid to two aspects. First, one may want to preserve the on-shell vanishing property of the Holst term and the topological character of the Nieh-Yan and Chern-Simons ones. Then, the resulting gravitational models should be endowed with invariance under projective transformations [63, 64]. These are transformations acting on the independent connection whose associated projective invariance is known to be related to the absence of dynamical instabilities in some metric-affine theories [34]. Neither of these features is a priori granted when nonmetricity is included or when a theory is promoted to the metric-affine framework and they will be checked case by case. Both aspects are sometimes trivially satisfied and in other cases we will need to provide a generalization of the terms.

After having proposed the generalizations of the parity violating terms and discussed their properties, they will be considered in some specific gravitational models. In particular, the Holst and Nieh-Yan terms will be analyzed within a Palatini $f(R)$ framework, while the Chern-Simons modification will be simply added to the Ricci scalar in the action. In both cases the coupling of the terms is promoted to a dynamical pseudo-scalar field, representing the so-called Immirzi field and the Chern-Simons scalar field, respectively.

Finally, we conclude with several concrete applications, obtaining results ranging from more theoretical considerations, obtained via semi-classical methods, to observationally oriented outcomes, derived both via analytical and numerical techniques first in cosmological settings and then, using perturbation theory, both in black holes and gravitational waves contexts.

In particular, the study of black hole spacetimes with scalar hair will be addressed in the Holst model, analyzing the thermodynamic properties of new non-asymptotically flat solutions. The presence of anisotropic bouncing cosmologies will be investigated in the presence of a generalized Nieh-Yan term and the possibility of resolving different kind of singularities will be discussed. Regarding the Chern-Simons term, its effect on the quasinormal modes of black holes and the propagation of gravitational waves will be computed, highlighting several observational signatures related to the parity breaking characterizing the model. The outline of the thesis is the following:

- The first chapter contains introductory material on metric-affine theories of gravity and a review of some gravitational models previously studied in literature. In the first section of the chapter (Sec. 1), after briefly recalling some aspects of General Relativity, the main features of metric-affine theories are introduced, including the concepts of non-Riemannian manifolds, the torsion and nonmetricity tensors and their geometrical meaning, the role of projective transformations and the differences between first and second order variational principles. Finally, motivations in support of such extended geometric settings are reported.
  In the second section (Sec. 2), $f(R)$ theories of gravity are reviewed, with special emphasis on their Palatini formulation. Then, we present the Holst, Nieh-Yan and Chern-Simons terms as they are known in the literature, discussing their main properties and their implementation in specific gravitational models.

- The original formal and theoretical results of this thesis are presented in the second chapter. We first address the projective-invariant extension to generic metric-affine spacetimes of the Holst and Nieh-Yan terms (Sec. 3), proposing a suitable generalization of the latter. The same is done for the Chern-Simons topological invariant (Sec. 4). Then, specific gravitational theories are constructed implementing the generalized Holst, Nieh-Yan and Chern-Simons terms in section 5, 6 and 7, respectively. In particular, we discuss to some extent the absence of instabilities and we derive the effective second order scalar-tensor theories, in a background independent and nonperturbative way for the Holst and Chern-Simons case and at the linearized level for the Chern-Simons theory.

- Original results regarding some concrete applications of the models are presented in the third chapter. In section 8 we report the hairy black hole solutions of the Holst model, together with their thermodynamic features. Section 9 is devoted to the Nieh-Yan theory and the big-bounce cosmological solutions found therein, with the analysis of their singularities. Finally, the results regarding black hole and gravitational wave phenomenology of the metric-affine Chern-Simons theory can be found in section 10, where we report the computations of the black hole's quasinormal modes and we derive the gravitational birefringence and Landau damping effects.

Finally, conclusions are drawn and future perspectives outlined in the last section.

Notations and conventions

Spacetime signature:	$-+++$
4-dimensional spacetime indices:	$\mu, \nu, \rho, \alpha, \beta, \gamma, \ldots$
Spatial indices:	$i, j, k, \ldots$
Symmetrization:	$A_{(\mu\|\alpha\|\nu)} \equiv \frac{1}{2}\left(A_{\mu\alpha\nu} + A_{\nu\alpha\mu}\right)$
Antisymmetrization:	$A_{[\mu\|\alpha\|\nu]} \equiv \frac{1}{2}\left(A_{\mu\alpha\nu} - A_{\nu\alpha\mu}\right)$
Levi-Civita connection:	$\bar{\Gamma}^{\mu}{}_{\nu\rho}$
Levi-Civita covariant derivative:	$\bar{\nabla}_{\mu} V^{\nu} = \partial_{\mu} V^{\nu} + \bar{\Gamma}^{\nu}{}_{\sigma\mu} V^{\sigma}$
Metric Riemann tensor:	$R^{\mu}{}_{\nu\rho\sigma} = 2\left(\partial_{[\rho}\bar{\Gamma}^{\mu}{}_{\nu\|\sigma]} + \bar{\Gamma}^{\mu}{}_{\lambda[\rho}\bar{\Gamma}^{\lambda}{}_{\nu\|\sigma]}\right)$
d'Alembertian operator:	$\Box = \bar{\nabla}_{\mu}\bar{\nabla}^{\mu}$
Metric-affine connection:	$\Gamma^{\mu}{}_{\nu\rho}$
Metric-affine covariant derivative:	$\nabla_{\mu} V^{\nu} = \partial_{\mu} V^{\nu} + \Gamma^{\nu}{}_{\sigma\mu} V^{\sigma}$
Metric-affine Riemann tensor:	$\mathscr{R}^{\mu}{}_{\nu\rho\sigma} = 2\left(\partial_{[\rho}\Gamma^{\mu}{}_{\nu\|\sigma]} + \Gamma^{\mu}{}_{\lambda[\rho}\Gamma^{\lambda}{}_{\nu\|\sigma]}\right)$
Torsion tensor:	$T^{\mu}{}_{\nu\rho} \equiv 2\Gamma^{\mu}{}_{[\nu\rho]}$
Nonmetricity tensor:	$Q_{\mu\nu\rho} \equiv -\nabla_{\mu} g_{\nu\rho}$
Contorsion tensor:	$K^{\mu}{}_{\nu\rho} \equiv \frac{1}{2}\left(T^{\mu}{}_{\nu\rho} - T_{\nu}{}^{\mu}{}_{\rho} - T_{\rho}{}^{\mu}{}_{\nu}\right)$
Torsion trace:	$T_{\mu} \equiv T^{\nu}{}_{\mu\nu}$
Pseudo-axial trace:	$S_{\mu} \equiv \varepsilon_{\mu\nu\rho\sigma} T^{\nu\rho\sigma}$
Torsion rank-3 tensor component:	$q_{\mu\nu\rho}$
Weyl vector:	$Q_{\mu} \equiv Q_{\mu\nu}{}^{\nu}$
Second trace of nonmetricity:	$P_{\mu} \equiv Q^{\nu}{}_{\nu\mu}$
Nonmetricity rank-3 tensor component:	$\Omega_{\mu\nu\rho}$

Disformal tensor:	$D^{\mu}{}_{\nu\rho} \equiv \frac{1}{2}\left(Q^{\mu}{}_{\nu\rho} - Q_{\nu}{}^{\mu}{}_{\rho} - Q_{\rho}{}^{\mu}{}_{\nu}\right)$
Distorsion tensor:	$N^{\mu}{}_{\nu\rho} \equiv K^{\mu}{}_{\nu\rho} + D^{\mu}{}_{\nu\rho}$
Totally antisymmetric symbol:	$\epsilon_{\mu\nu\rho\sigma}$, with $\epsilon_{0123} = \epsilon^{0123} = 1$
Covariant Levi-Civita tensor:	$\varepsilon_{\mu\nu\rho\sigma} = \sqrt{-g}\epsilon_{\mu\nu\rho\sigma}$
Contravariant Levi-Civita tensor:	$\varepsilon^{\mu\nu\rho\sigma} = -\frac{1}{\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma}$
Immirzi parameter:	γ_0
Speed of light:	c
Newton's constant:	G
Einstein's constant:	$\kappa^2 \equiv 8\pi G$

Part I

Geometric theories of gravity

Chapter 1

Non-Riemannian geometries

The theory of General Relativity is based on a specific class of geometries, constructed on a pseudo-Riemannian manifold. However, the latter is not the only choice and it can be extended to a more general class of manifolds, leading to the so-called *non-Riemannian* geometries. These are the setting for the formulation of all metric-affine gravity models and their main features will be discussed in this chapter.
After a brief introduction on the basic features of General Relativity, the torsion and nonmetricity tensors will be presented and their properties and geometrical meaning will be discussed, together with a useful decomposition in terms of their irreducible components according to the Lorentz group. The role of projective transformations in the dynamical stability of metric-affine models and their physical meaning related to autoparallel trajectories will be introduced. We end the chapter explaining the differences between first and second order variational principles and providing several motivations in support of metric-affine theories of gravity.

1.1 The theory of General Relativity

In Einstein's theory of General Relativity [B10, B11], spacetime is considered as a dynamical entity, whose characteristics and evolution are determined by the distribution of matter and energy across the space. It is represented by a 4-dimensional differentiable manifold $\mathscr{M}$, on which the line element between two events, labelled by coordinates x^μ and $x^\mu + dx^\mu$, is computed via the metric tensor $g_{\mu\nu}$ as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{1.1}$$

The theory is covariant under arbitrary 4-dimensional diffeomorphisms which reparametrize spacetime coordinates as $x'^{\mu} = x'^{\mu}(y^{\rho})$.
Parallel transport of tensorial quantities throughout the manifold is defined via the connection $\bar{\Gamma}^{\nu}{}_{\rho\mu}$, which allows the introduction of covariant differentiation, i.e.

$$\bar{\nabla}_{\mu} V^{\nu} = \partial_{\mu} V^{\nu} + \bar{\Gamma}^{\nu}{}_{\rho\mu} V^{\rho}. \tag{1.2}$$

In General Relativity the expression for the connection is uniquely fixed, resulting in the Levi-Civita connection, which, once a local chart of coordinates is settled, is given by the Christoffel symbols of the metric, i.e.

$$\bar{\Gamma}^{\nu}{}_{\rho\mu} = \frac{1}{2} g^{\nu\sigma} \left(\partial_{\rho} g_{\sigma\mu} + \partial_{\mu} g_{\sigma\rho} - \partial_{\sigma} g_{\rho\mu} \right). \tag{1.3}$$

Thereby the kinematic content of the theory is established. The dynamics is instead determined by providing a specific Lagrangian. The theory of General Relativity is obtained via the simplest choice, represented by the Einstein-Hilbert action:

$$S_{EH}[g] = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} R, \tag{1.4}$$

where $\kappa^2 = 8\pi G$ is Einstein's constant and $R = g^{\mu\nu} R_{\mu\nu}$ is the Ricci scalar, obtained contracting the Ricci tensor $R_{\mu\nu}$ with the metric. The latter is related to the Riemann tensor $R^{\rho}{}_{\mu\nu\sigma}$ by $R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}$. The Riemann tensor, defined via the commutator of covariant derivatives acting on a vector as

$$\left[\bar{\nabla}_{\mu}, \bar{\nabla}_{\nu} \right] V^{\rho} = R^{\rho}{}_{\sigma\mu\nu} V^{\sigma}, \tag{1.5}$$

encodes the notion of spacetime curvature and is explicitly given by

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_{\rho} \bar{\Gamma}^{\mu}{}_{\nu\sigma} - \partial_{\sigma} \bar{\Gamma}^{\mu}{}_{\nu\rho} + \bar{\Gamma}^{\mu}{}_{\lambda\rho} \bar{\Gamma}^{\lambda}{}_{\nu\sigma} - \bar{\Gamma}^{\mu}{}_{\lambda\sigma} \bar{\Gamma}^{\lambda}{}_{\nu\rho}. \tag{1.6}$$

It satisfies the algebraic Bianchi identity

$$R_{\mu\nu\rho\sigma} + R_{\mu\sigma\nu\rho} + R_{\mu\rho\sigma\nu} = 0 \tag{1.7}$$

and the differential Bianchi identity

$$\bar{\nabla}_{\mu} R_{\nu\rho\sigma\lambda} + \bar{\nabla}_{\rho} R_{\mu\nu\sigma\lambda} + \bar{\nabla}_{\nu} R_{\rho\mu\sigma\lambda} = 0. \tag{1.8}$$

With these objects and the related properties one can build the Einstein tensor, a symmetric rank-2 tensor defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \tag{1.9}$$

which is covariantly conserved, i.e.

$$\bar{\nabla}_\mu G^\mu{}_\nu = 0. \tag{1.10}$$

Then, matter content can be included in the theory as

$$S[g,\psi] = S_{EH}[g] + S_m[\psi, g], \tag{1.11}$$

where the action of the matter fields, indicated collectively by ψ, reads

$$S_m[\psi, g] = \int d^4x \mathscr{L}_m(\psi, g), \tag{1.12}$$

where $\mathscr{L}_m$ represent the matter Lagrangian. Varying the above action with respect to $g^{\mu\nu}$ gives, modulo surface terms, Einstein's field equations, a set of second order partial differential equations in the metric tensor, which reads

$$G_{\mu\nu} = \kappa^2 T_{\mu\nu}, \tag{1.13}$$

where the energy-momentum tensor of matter is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta \mathscr{L}_m}{\delta g^{\mu\nu}}. \tag{1.14}$$

From (1.10) one can immediately see that the latter is conserved, i.e.

$$\bar{\nabla}_\mu T^\mu{}_\nu = 0. \tag{1.15}$$

On general grounds, the latter provides a dynamical equation for the matter sector, closing the system of differential equations for the metric tensor and the matter fields.
When the dynamics of test particles on a curved background are taken into account, their trajectories are described by the geodesic equation, which reads

$$u^\nu \bar{\nabla}_\nu u^\mu = 0, \tag{1.16}$$

being $u^\mu = dx^\mu/ds$ the vector tangent to the trajectory identified by $x^\mu = x^\mu(s)$, with affine parameter s. The last equation may also describe particles affected only by inertial forces, in absence of gravity. The effect of gravity is instead unequivocal in the geodesic deviation equation, describing the relative acceleration of two nearby geodesics separated by the vector η^μ:

$$u^\nu \bar{\nabla}_\nu \left(u^\rho \bar{\nabla}_\rho \eta^\mu \right) = R^\mu{}_{\nu\rho\sigma} u^\nu u^\rho \eta^\sigma, \tag{1.17}$$

which is non-trivial only in presence of spacetime curvature, i.e. for $R^\mu{}_{\nu\rho\sigma} \neq 0$. The possibility of summarizing most of the features of the theory in just two pages is part of the beauty of General Relativity which is relatively simple but complex enough to give an account of a huge variety of gravitational phenomena. All things considered, the next hundreds pages will simply contain several different variations on this common theme.

1.2 General affine connections

The manifold's metric and affine properties play crucial roles in the geometric description of the gravitational interaction and their physical interpretation ultimately allows to connect geometry with gravity. As we have seen in the previous section, two fundamental objects are the metric tensor and the connection. Although in General Relativity the latter is completely determined by the former and its derivatives, this possibility is not unique and one can actually consider the connection as an independent variable. We will denote the most general connection by ${\Gamma^{\mu}}_{\nu\rho}$.
Then, its identification with the Christoffel symbols of the metric tensor is just a (well-motivated) choice, which amounts to impose two requirements on the connection, namely the symmetry in the lower indices,

$$ {\Gamma^{\mu}}_{[\nu\rho]} = 0, \tag{1.18} $$

and the metric compatibility, i.e.

$$ \nabla_{\mu} g_{\nu\rho} = 0. \tag{1.19} $$

Indeed, adding cyclic permutations on the three indices of the latter and assuming (1.18), immediately leads to the identification ${\Gamma^{\mu}}_{\nu\rho} = {\bar{\Gamma}^{\mu}}_{\nu\rho}$, where

$$ {\bar{\Gamma}^{\mu}}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(\partial_{\nu} g_{\sigma\rho} + \partial_{\rho} g_{\sigma\nu} - \partial_{\sigma} g_{\nu\rho} \right) \tag{1.20} $$

is the Levi-Civita connection. Relaxing one or both of the two conditions above, allows for the presence of additional geometric objects called torsion and non-metricity, defined respectively by

$$ {T^{\mu}}_{\nu\rho} \equiv 2{\Gamma^{\mu}}_{[\nu\rho]} \tag{1.21} $$

and

$$ Q_{\mu\nu\rho} \equiv -\nabla_{\mu} g_{\nu\rho}. \tag{1.22} $$

We postpone the explanation of their geometrical and physical meaning to the next sections. For the time being we just note that their presence implies a more general expression for the connection which is now asymmetric and not metric compatible. Starting from (1.22) and performing cyclic permutations on the indices, it is easy to show that the most general form for the connection is now given by

$$ {\Gamma^{\mu}}_{\nu\rho} = {\bar{\Gamma}^{\mu}}_{\nu\rho} + {N^{\mu}}_{\nu\rho}\,, \tag{1.23} $$

where the distorsion tensor, defined by

$$N^{\mu}{}_{\nu\rho} \equiv K^{\mu}{}_{\nu\rho} + D^{\mu}{}_{\nu\rho}, \tag{1.24}$$

depends on the contorsion and disformal tensors, given by

$$K^{\mu}{}_{\nu\rho} \equiv \frac{1}{2}\left(T^{\mu}{}_{\nu\rho} - T_{\nu}{}^{\mu}{}_{\rho} - T_{\rho}{}^{\mu}{}_{\nu}\right), \tag{1.25}$$

$$D^{\mu}{}_{\nu\rho} \equiv -\frac{1}{2}\left(Q^{\mu}{}_{\nu\rho} - Q_{\nu}{}^{\mu}{}_{\rho} - Q_{\rho}{}^{\mu}{}_{\nu}\right), \tag{1.26}$$

respectively. Moreover, we have $K^{\mu\nu}{}_{\rho} = -K^{\nu\mu}{}_{\rho}$ and $D^{\mu}{}_{\nu\rho} = D^{\mu}{}_{\rho\nu}$. Finally, from (1.22) one can obtain the useful relations

$$\nabla_{\mu} g^{\nu\lambda} = g^{\alpha\nu} g^{\beta\lambda} Q_{\mu\alpha\beta} = Q_{\mu}{}^{\nu\lambda}, \tag{1.27}$$

$$\nabla_{\mu}\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\,Q_{\mu}, \tag{1.28}$$

$$\nabla_{\rho}\left(\sqrt{-g}\,\varepsilon_{\lambda}{}^{\sigma\mu\nu}\right) = -\sqrt{-g}\,\varepsilon^{\alpha\sigma\mu\nu} Q_{\rho\lambda\alpha}. \tag{1.29}$$

Torsion and nonmetricity are rank three tensors with specific symmetries that admit a useful decomposition into irreducible components according to the Lorentz group [65]. Regarding the torsion tensor, the independent components are the trace vector

$$T_{\mu} \equiv T^{\nu}{}_{\mu\nu}, \tag{1.30}$$

the pseudotrace axial vector

$$S_{\mu} \equiv \varepsilon_{\mu\nu\rho\sigma} T^{\nu\rho\sigma} \tag{1.31}$$

and the antisymmetric tensor $q_{\mu\nu\rho} = -q_{\mu\rho\nu}$, satisfying

$$\varepsilon^{\mu\nu\rho\sigma} q_{\nu\rho\sigma} = 0, \qquad q^{\mu}{}_{\nu\mu} = 0. \tag{1.32}$$

For what concerns the nonmetricity instead, we can define the Weyl vector

$$Q_{\rho} \equiv Q_{\rho}{}^{\mu}{}_{\mu}, \tag{1.33}$$

the second trace

$$P_{\rho} \equiv Q^{\mu}{}_{\mu\rho} = Q^{\mu}{}_{\rho\mu}, \tag{1.34}$$

and a completely traceless part $\Omega_{\rho\mu\nu}$, obeying

$$\Omega^{\mu}{}_{\mu\nu} = \Omega_{\nu\mu}{}^{\mu} = 0, \tag{1.35}$$

$$\Omega_{\rho\mu\nu} = \Omega_{\rho\nu\mu}. \tag{1.36}$$

In terms of these objects, torsion and nonmetricity are expressed by

$$T_{\mu\nu\rho} = \frac{1}{3}\left(T_\nu g_{\mu\rho} - T_\rho g_{\mu\nu}\right) + \frac{1}{6}\varepsilon_{\mu\nu\rho\sigma}S^\sigma + q_{\mu\nu\rho}, \tag{1.37}$$

$$Q_{\rho\mu\nu} = \frac{5Q_\rho - 2P_\rho}{18} g_{\mu\nu} - \frac{Q_{(\mu}g_{\nu)\rho} - 4P_{(\mu}g_{\nu)\rho}}{9} + \Omega_{\rho\mu\nu}, \tag{1.38}$$

as it is straightforward to check by applying the definitions of the individual components. This decomposition can be used to separate useful quantities into their Riemannian and non-Riemannian contributions. This procedure is sometimes called post-Riemannian expansion and allows to gain some insights on the role of the affine contributions contained in quantities defined in terms of the general affine connection. For instance, for the Riemann tensor we have

$$\mathscr{R}^\mu{}_{\rho\nu\sigma} = R^\mu{}_{\rho\nu\sigma} + \bar{\nabla}_\nu N^\mu{}_{\rho\sigma} - \bar{\nabla}_\sigma N^\mu{}_{\rho\nu} + N^\mu{}_{\lambda\nu} N^\lambda{}_{\rho\sigma} - N^\mu{}_{\lambda\sigma} N^\lambda{}_{\rho\nu}, \tag{1.39}$$

yielding the Ricci scalar

$$\begin{aligned}\mathscr{R} &= R + \bar{\nabla}_\mu\left(P^\mu - Q^\mu - 2T^\mu\right) + \frac{1}{24}S_\mu S^\mu - \frac{2}{3}T_\mu T^\mu + \frac{1}{18}P_\mu P^\mu - \frac{11}{72}Q_\mu Q^\mu \\ &+ \frac{2}{9}P_\mu Q^\mu + \frac{2}{3}P_\mu T^\mu - \frac{2}{3}Q_\mu T^\mu + \frac{1}{4}q_{\mu\nu\rho}q^{\mu\nu\rho} + \frac{1}{2}q^{\mu\nu\rho}q_{\nu\mu\rho} + \frac{1}{4}\Omega_{\mu\nu\rho}\Omega^{\mu\nu\rho} \\ &- \frac{1}{2}\Omega_{\mu\nu\rho}\Omega^{\nu\mu\rho} + q_{\mu\nu\rho}\Omega^{\nu\mu\rho},\end{aligned} \tag{1.40}$$

where the contribution from the purely metric Ricci scalar R is now apparent and separated from other affine contributions. Similar post-Riemannian decompositions will be used throughout the thesis.
The class of theories endowed with both torsion and nonmetricity as given by (1.37) and (1.38), with all vector and tensor components present, contains several sub-cases which are obtained setting to zero some of the quantities involved. From the most general geometry, which will be referred to as metric-affine geometry, one can reduce to the following ones:

- Einstein-Cartan ($Q_{\mu\nu\rho} = 0$);
- Weyl ($Q_\mu = 4P_\mu$, $\Omega_{\mu\nu\rho} = T_{\mu\nu\rho} = 0$);
- Teleparallel ($\mathscr{R}^\mu{}_{\nu\rho\sigma} = 0 = Q_{\mu\nu\rho}$);
- Symmetric Teleparallel ($\mathscr{R}^\mu{}_{\nu\rho\sigma} = 0 = T_{\mu\nu\rho}$);
- Riemannian ($Q_{\mu\nu\rho} = T_{\mu\nu\rho} = 0$).

1.3 Geometric effects of torsion and nonmetricity

As it is well-known, curvature affects the parallel propagation of vectors along closed loops [B18]. In a similar way, torsion and nonmetricity have their own geometric effects on the parallel transport of vectors.

Indeed, in presence of torsion the parallelogram formed by parallely transporting two infinitesimal vectors along each other does not close. To see this we can consider two vectors, A^μ and B^μ and write the equations for the parallel transport of the first one along the second and vice versa, i.e.

$$B^\nu \nabla_\nu A^\mu = B^\nu \partial_\nu A^\mu + \Gamma^\mu{}_{\rho\nu} A^\rho B^\nu = 0, \tag{1.41}$$

$$A^\nu \nabla_\nu B^\mu = A^\nu \partial_\nu B^\mu + \Gamma^\mu{}_{\rho\nu} B^\rho A^\nu = 0. \tag{1.42}$$

For infinitesimal vectors we can approximate the partial derivatives as

$$\partial_\nu A^\mu \approx \frac{A^\mu(x^\rho + \delta x^\rho) - A^\mu(x^\rho)}{\delta x^\nu}, \tag{1.43}$$

$$\partial_\nu B^\mu \approx \frac{B^\mu(x^\rho + \delta x^\rho) - B^\mu(x^\rho)}{\delta x^\nu}, \tag{1.44}$$

where the displacement can be identified with $\delta x^\rho = B^\rho$ and $\delta x^\rho = A^\rho$, respectively. The quantity we are interested in is

$$\Delta_{AB} \equiv A^\mu(x^\rho) + B^\mu(x^\rho + \delta x^\rho) - (B^\mu(x^\rho) + A^\mu(x^\rho + \delta x^\rho)), \tag{1.45}$$

which quantifies the failure in closing the infinitesimal parallelogram. Using the expressions above it can be written as

$$\Delta_{AB} = 2\Gamma^\mu{}_{[\rho\nu]} A^\rho B^\nu = T^\mu{}_{\rho\nu} A^\rho B^\nu. \tag{1.46}$$

In absence of torsion we would have $\Delta_{AB} = 0$ and the two operations of transporting A^μ along B^μ and B^μ along A^μ would commute, leading to the same point in spacetime. When torsion is present instead, the infinitesimal parallelogram fails to close, the discrepancy being proportional to the torsion tensor.

The geometric effect of nonmetricity is instead related to the non conservation of the angle between two vectors upon parallel transport. Consider again the vectors A^μ and B^μ and their parallel transport along a curve with tangent vector u^μ:

$$u^\nu \nabla_\nu A^\mu = 0, \tag{1.47}$$

$$u^\nu \nabla_\nu B^\mu = 0. \tag{1.48}$$

Using these conditions and the definition of nonmetricity, the variation of the angle between the two vectors along the curve can be written as

$$u^\rho \nabla_\rho \left(g_{\mu\nu} A^\mu B^\nu \right) = B_\mu u^\rho \nabla_\rho A^\mu + A_\mu u^\rho \nabla_\rho B^\mu + A^\mu B^\nu u^\rho \nabla_\rho g_{\mu\nu}$$
$$= -Q_{\rho\mu\nu} u^\rho A^\mu B^\nu, \tag{1.49}$$

so that the nonmetricity tensor quantifies the rate of change of the angle and for vanishing nonmetricity we recover the simple case in which the angle is constant along the curve. A particular case is obtained setting $B^\mu = A^\mu$, resulting in the non conservation of the norm of vectors upon parallel transport:

$$u^\rho \nabla_\rho \left(A^\mu A_\mu \right) = -Q_{\rho\mu\nu} u^\rho A^\mu A^\nu. \tag{1.50}$$

1.4 Projective transformations

In metric-affine theories of gravity the presence of an additional field, namely the independent connection, allows for the existence of a new symmetry, beside the usual invariance under four-dimensional diffeomorphisms. The associated transformation is called *projective transformation* and it consists of a shift of the independent connection by an arbitrary one-form degree ξ_μ:

$$\Gamma^\mu{}_{\nu\rho} \to \tilde{\Gamma}^\mu{}_{\nu\rho} = \Gamma^\mu{}_{\nu\rho} + \delta^\mu_\nu \xi_\rho. \tag{1.51}$$

From the latter, one can easily obtain the transformation properties of other quantities such as the Riemann tensor,

$$\mathscr{R}^\rho{}_{\mu\sigma\nu} \to \tilde{\mathscr{R}}^\rho{}_{\mu\sigma\nu} = \mathscr{R}^\rho{}_{\mu\sigma\nu} - 2\delta^\rho{}_\mu \partial_{[\sigma} \xi_{\nu]}, \tag{1.52}$$

and the vector components of torsion and nonmetricity,

$$T^\rho \to \tilde{T}^\rho = T^\rho - 3\xi^\rho, \tag{1.53}$$
$$S^\rho \to \tilde{S}^\rho = S^\rho, \tag{1.54}$$
$$Q^\rho \to \tilde{Q}^\rho = Q^\rho + 8\xi^\rho, \tag{1.55}$$
$$P^\rho \to \tilde{P}^\rho = P^\rho + 2\xi^\rho, \tag{1.56}$$

while both rank-3 tensors are invariant:

$$q_{\mu\nu\rho} \to \tilde{q}_{\mu\nu\rho} = q_{\mu\nu\rho}, \tag{1.57}$$
$$\Omega_{\mu\nu\rho} \to \tilde{\Omega}_{\mu\nu\rho} = \Omega_{\mu\nu\rho}. \tag{1.58}$$

A particular case of projective transformations is given by *special* projective transformations, in which the one-form degree can be expressed as the gradient of a scalar field:

$$\xi_\mu = \partial_\mu \lambda. \tag{1.59}$$

From the Riemann tensor transformation we immediately see that the Ricci tensor is only invariant under the latter, since

$$\mathscr{R}_{\mu\nu} \to \tilde{\mathscr{R}}_{\mu\nu} = \mathscr{R}_{\mu\nu} - 2\partial_{[\mu}\xi_{\nu]}, \tag{1.60}$$

while the Ricci scalar is invariant under general projective transformations:

$$\mathscr{R} \to \tilde{\mathscr{R}} = \mathscr{R}. \tag{1.61}$$

In particular, the last condition implies that the first order formulation of General Relativity (see section 1.5) is invariant under projective transformations. In general, the projective symmetry in metric-affine theories entails the presence of a redundant vector degree which can always be gauged away performing a projective transformation. A common choice consists in choosing ξ^μ such that one among the vectors $\tilde{T}^\mu$, $\tilde{Q}^\mu$ and $\tilde{P}^\mu$ is vanishing. By doing this, one is just working with different representations of the same theory, completely equivalent both at the level of the action and field equations, that sometimes allow for a great simplification in the computations. This mechanism plays also a role in proving the equivalence of first and second order formulations of General Relativity, as we will see in the next section.

Another important aspect related to projective transformations is that the breaking of projective symmetry may be related to the arising of dynamical instabilities. In particular, this is always the case for the specific subclass of metric-affine theories identified by $f(\mathscr{R}_{\mu\nu})$ Lagrangians, built only with the Ricci tensor. As recently shown in [35], ghost-like degrees of freedom propagate in the non-projectively invariant sector of such models. These pathologies can be cured either by restoring the projective symmetry or by imposing constraints on the affine structure of the theory, e.g. the torsionless condition, as in [35]. These results show the importance of this symmetry for the well-behaviour of degrees of freedom in metric-affine theories. Although only a specific subclass of these is addressed in [35], one should expect that adding more complicated terms to the Lagrangian the outcome can only get worse. In general, projective invariance deserves special attention when metric-affine models are considered and especially when promoting a purely metric theory of gravity to its first order formulation.

Beside this theoretical argument, projective transformations also have a physical meaning that can be appreciated by considering their effect on autoparallel trajectories [66]. These are curves along which the tangent vector is parallel transported along itself, i.e.

$$u^\nu \nabla_\nu u^\mu = 0, \tag{1.62}$$

where

$$u^\mu = \frac{dy^\mu}{d\lambda}. \tag{1.63}$$

While in General Relativity they coincide with geodesics ($u^\nu \bar{\nabla}_\nu u^\mu = 0$), in presence of torsion and/or nonmetricity they represent different trajectories. Equation (1.62) can be written as

$$\frac{d^2 y^\mu}{d\lambda^2} + \Gamma^\mu{}_{\nu\rho} \frac{dy^\nu}{d\lambda} \frac{dy^\rho}{d\lambda} = 0. \tag{1.64}$$

Performing a projective transformation from $\Gamma^\mu{}_{\nu\rho}$ to a new connection defined by $\tilde{\Gamma}^\mu{}_{\nu\rho} = \Gamma^\mu{}_{\nu\rho} + \delta^\mu_\nu \xi_\rho$ the autoparallel equation becomes

$$\frac{d^2 y^\mu}{d\lambda^2} + \tilde{\Gamma}^\mu{}_{\nu\rho} \frac{dy^\nu}{d\lambda} \frac{dy^\rho}{d\lambda} = f(\lambda) \frac{dy^\mu}{d\lambda}, \tag{1.65}$$

where

$$f(\lambda) \equiv \xi_\rho \frac{dy^\rho}{d\lambda}. \tag{1.66}$$

That is, λ is no more an affine parameter for the new connection $\tilde{\Gamma}^\mu{}_{\nu\rho}$. Now, it is always possible to find a reparametrization $s = s(\lambda)$ such that the new parameter is an affine parameter for $\tilde{\Gamma}^\mu{}_{\nu\rho}$, namely such that

$$\frac{d^2 y^\mu}{ds^2} + \tilde{\Gamma}^\mu{}_{\nu\rho} \frac{dy^\nu}{ds} \frac{dy^\rho}{ds} = 0. \tag{1.67}$$

To see this one can make use of the following relations

$$\frac{dy^\mu}{d\lambda} = \frac{dy^\mu}{ds} \dot{s}, \tag{1.68a}$$

$$\frac{d^2 y^\mu}{d\lambda^2} = \frac{d^2 y^\mu}{ds^2} \dot{s}^2 + \frac{dy^\mu}{ds} \ddot{s}, \tag{1.68b}$$

where a dot represents a derivative with respect to λ. Substituting into (1.65) yields

$$\frac{d^2 y^\mu}{ds^2} + \tilde{\Gamma}^\mu{}_{\nu\rho} \frac{dy^\nu}{ds} \frac{dy^\rho}{ds} = \frac{1}{\dot{s}^2} \left(f(\lambda) \dot{s} - \ddot{s} \right) \frac{dy^\mu}{ds}. \tag{1.69}$$

Then, (1.67) is obtained imposing

$$f(\lambda) \dot{s} - \ddot{s} = 0, \tag{1.70}$$

namely fixing the reparametrization to be

$$s(\lambda) = c_1 \int^{\lambda} e^{\int^u f(v)dv} du + c_2, \tag{1.71}$$

with c_1 and c_2 arbitrary constants. Therefore, projective transformations act on autoparallel trajectories changing their affine parameter to a non-affine one. However, the latter can always be transformed to a new affine parameter via (1.71). In other words, autoparallel trajectories are invariant under projective transformations upon reparametrization of the parameter labeling points along the curve. Hence, projective transformations can be thought of as rules defining a *class* of connections that have the property of leaving invariant the parallel transport on a metric-affine manifold.

1.5 First and second order formulations

Another, more meaningful perspective on the possibility of having a connection more general than the Levi-Civita one, is offered by considering two alternative paradigms in formulating variational principles for geometric theories of gravity. In the standard formulation of General Relativity one considers the action as a functional of the metric tensor alone,

$$S[g] = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} R, \tag{1.72}$$

obtaining the field equations via variations with respect to the inverse metric $\delta g^{\mu\nu}$. This landscape can be referred to as *second order* formulation, since the action and the equations of motion contain second order derivatives of the metric tensor. An alternative is instead represented by the following choice

$$S[g, \Gamma] = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \mathscr{R}, \tag{1.73}$$

where the connection has been promoted to an independent variable. The Riemann tensor is now considered as a function of the independent connection and field equations are derived via independent variations of the metric $\delta g^{\mu\nu}$ and the connection $\delta \Gamma^{\mu}{}_{\nu\rho}$. This formulation is known as *first order* formulation, since now only first derivatives of the metric and of the connection appear.
Considering the connection as an independent variable, not only introduces modifications in the gravitational sector but it also may affect the way matter couples to gravity. Indeed, alongside the stress-energy tensor introduced in section 1.1, which is obtained varying the matter Lagrangian $\mathscr{L}_m$ with respect to the metric

tensor, matter is described by an additional object in a first order framework, the hyper-momentum tensor [67, 68]. It is defined as

$$\Delta_\lambda{}^{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta\mathscr{L}_m}{\delta\Gamma^\lambda{}_{\mu\nu}} \tag{1.74}$$

and it is a priori allowed by the fact that the matter Lagrangian may be a function of the independent connection as well.

Several commonly used matter Lagrangian, describing physically relevant types of matter, actually yield a vanishing hyper-momentum. For instance, this is the case of a canonical scalar field, the electromagnetic field and the perfect fluid, all examples of matter not coupling to the affine sector. An important case in which the matter Lagrangian depends on the connection is given by spinor fields, describing fermionic particles. They can be introduced in the tetradic formalism and turn out to feature the spin-connection in their Lagrangian [69]. Beside fermions, one can also consider perfect hyper-fluids [70] as a direct generalization of standard perfect fluids, in which microscopic characteristic of matter are considered, such as spin, shear and dilation.

Throughout this thesis we will assume that the matter sector does not couple to the independent connection, and hence $\Delta_\lambda{}^{\mu\nu} = 0$. Such hypothesis, quite common in literature, allows to simplify the computations and it is adequate since it does not exclude the relevant matter contents listed above from the picture, thus still allowing the investigations of interesting enough physical scenarios.

Now, as long as matter contributions do not depend on the connection, the above first and second order formulations both yield the same result. To see this, let us first evaluate the field equations for the connection. The computation can be performed using the generalized Palatini identity for the variation of the Riemann tensor, i.e.

$$\delta\mathscr{R}^\rho{}_{\mu\sigma\nu} = \nabla_\sigma\delta\Gamma^\rho{}_{\mu\nu} - \nabla_\nu\delta\Gamma^\rho{}_{\mu\sigma} - T^\lambda{}_{\sigma\nu}\,\delta\Gamma^\rho{}_{\mu\lambda}, \tag{1.75}$$

and the following relation

$$\begin{aligned}\int d^4x\nabla_\mu\left(\sqrt{-g}V^\mu\right) &= \int d^4x\partial_\mu\left(\sqrt{-g}V^\mu\right) + \int d^4x\sqrt{-g}T^\rho{}_{\mu\rho}V^\mu\\ &= \int d^4x\sqrt{-g}T^\rho{}_{\mu\rho}V^\mu,\end{aligned} \tag{1.76}$$

holding for any vector density $\sqrt{-g}V^\mu$. Then, varying (1.73) with respect to $\Gamma^\mu{}_{\nu\rho}$ one obtains

$$-\nabla_\lambda\left(\sqrt{-g}g^{\mu\nu}\right) + \delta^\nu_\lambda\nabla_\rho\left(\sqrt{-g}g^{\mu\rho}\right) + \sqrt{-g}\left(g^{\mu\nu}T^\rho{}_{\lambda\rho} - \delta^\nu_\lambda T^{\rho\mu}{}_\rho + T^{\nu\mu}{}_\lambda\right) = 0, \tag{1.77}$$

or, equivalently,

$$\frac{1}{2}g^{\mu\nu}Q_\lambda + \left(P^\mu - \frac{1}{2}Q^\mu\right)\delta^\nu_\lambda - Q_\lambda{}^{\mu\nu} + g^{\mu\nu}T_\lambda - \delta^\nu_\lambda T^\mu + T^{\nu\mu}{}_\lambda = 0. \tag{1.78}$$

Contracting the latter with δ^λ_ν, $g_{\mu\nu}$ and $\varepsilon_{\alpha\nu\mu}{}^\lambda$ yields three equations for the four vector components T_μ, S_μ, Q_μ and P_μ:

$$3P_\mu - \frac{3}{2}Q_\mu - 2T_\mu = 0, \tag{1.79}$$

$$P_\mu + \frac{1}{2}Q_\mu + 2T_\mu = 0, \tag{1.80}$$

$$S_\mu = 0, \tag{1.81}$$

while contraction with $\delta^\lambda{}_\mu$ results in the trivial identity $0 = 0$. As a result, the above system is not closed and one needs to provide an additional condition to completely determine the solution. This is a consequence of the projective invariance of the theory. Indeed, one of the vector degrees is redundant and one can always get rid of it performing a suitable projective transformation (see section 1.4). In particular, one can choose $\xi_\mu = T_\mu/3$, such that $\tilde{T}_\mu = 0$. In terms of the new variables, the above system reduces to

$$3\tilde{P}_\mu - \frac{3}{2}\tilde{Q}_\mu = 0, \tag{1.82}$$

$$\tilde{P}_\mu + \frac{1}{2}\tilde{Q}_\mu = 0, \tag{1.83}$$

$$\tilde{S}_\mu = 0, \tag{1.84}$$

implying the vanishing of all affine vector components. Decomposing torsion and nonmetricity in (1.78), the equation becomes

$$q_{\nu\mu\lambda} - \Omega_{\lambda\mu\nu} = \frac{2}{3}\left(T_\mu g_{\nu\lambda} - T_\nu g_{\mu\lambda}\right) - \frac{1}{6}\varepsilon_{\nu\mu\lambda\sigma}S^\sigma + \\ - \frac{1}{9}\left(g_{\mu\nu}\left(2Q_\lambda + P_\lambda\right) - g_{\nu\lambda}\left(4Q_\mu - 7P_\mu\right) - g_{\mu\lambda}\left(-\frac{1}{2}Q_\nu + 2P_\nu\right)\right), \tag{1.85}$$

where we dropped the tilde notation for the sake of clarity. Setting the vectors to zero, the last equation implies

$$q_{[\nu\mu]\lambda} = 0, \tag{1.86}$$

$$\Omega_{(\lambda\mu)\nu} = 0. \tag{1.87}$$

In turn, these conditions allow to write

$$q_{\nu\mu\lambda} = -q_{\nu\lambda\mu} = -q_{\lambda\nu\mu} = q_{\lambda\mu\nu} = q_{\mu\lambda\nu} = -q_{\mu\nu\lambda} = -q_{\nu\mu\lambda}, \tag{1.88}$$

$$\Omega_{\nu\mu\lambda} = \Omega_{\nu\lambda\mu} = -\Omega_{\lambda\nu\mu} = -\Omega_{\lambda\mu\nu} = \Omega_{\mu\lambda\nu} = \Omega_{\mu\nu\lambda} = -\Omega_{\nu\mu\lambda}, \tag{1.89}$$

resulting in the vanishing of the tensor parts as well. The absence of torsion and nonmetricity implies the validity of condition (1.19), which is solved by the Christoffel symbols in (1.20). In this way, we identify the independent connection with the Levi-Civita one dynamically, solving the field equations of the theory.
Let us now consider the metric equations, obtained varying the action with respect to $\delta g^{\mu\nu}$. The outcome is[1]

$$\mathscr{R}_{(\mu\nu)} - \frac{1}{2}\mathscr{R}\, g_{\mu\nu} = 0, \tag{1.90}$$

Note that the affine Ricci tensor is a priori asymmetric, resulting in the Ricci tensor symmetrization in the above equation, which stems from the fact that it is contracted with the symmetric variation of the metric $\delta g^{\mu\nu}$ during the variational procedure. When dealing with metric-affine theories, it is common to work *on half-shell*. By this we mean that the solution for the connection is taken into account and substituted into the remaining equations, leaving only metric, and possibly scalar (as in the models presented in chapter II), degrees of freedom to be dealt with. In the present case, given the vanishing of torsion and nonmetricity we have that $\mathscr{R}_{\mu\nu} = R_{\mu\nu}$ and the on half-shell metric equations simply reduce to the vacuum Einstein's equations, thus proving the dynamical equivalence of the first and second order frameworks.
This result is quite peculiar and it only applies to General Relativity, i.e. choosing the Lagrangian to be simply given by the Ricci scalar, and with connection independent matter Lagrangians. If the gravitational action is modified the equivalence usually breaks down and first and second order formulations give rise to completely distinct theories, as it happens for $f(R)$ theories (see section 2.1). Therefore, the two paradigms are not a mere conventional choice but rather they entail different kinematical as well as dynamical scenarios. There are several arguments in support of both choices, here we will review some of the reasons why it is interesting to consider the metric-affine framework, both from a theoretical and phenomenological point of view.

- First, from an historical perspective, it is relevant to mention that a first order formulation of gravity was already considered soon after the breakthrough of General Relativity, first by Palatini [71], although in a slightly different way than the one considered here, and later by Einstein himself [72].

- One of the merits of first order formulations can already be appreciated considering the previous discussion. From a first order perspective indeed,

[1] We are here considering the vacuum case, with $T_{\mu\nu} = 0$. The extension to the presence of a non-vanishing stress-energy tensor is trivial and does not change the results discussed here.

conditions (1.18) and (1.19) are consequences of the dynamics of the theory rather then being imposed by hand a priori on the geometric structure of the manifold. In the same way, instead of assuming that the connection be given by the Christoffel symbols of the metric, one obtains the same result dynamically, as a solution of the field equations.

- A strong theoretical motivation comes from the fact that the metric-affine paradigm admits a formulation of gravity as a gauge theory of the general affine group, allowing to put the equations in a suggestive Yang-Mills form [73]. This seems particularly interesting in relation to the difficulties in the quantization of the gravitational interaction, given that other gauge theories can be consistently quantized.

- It is to the last statement that the quantization scheme carried out in the loop quantum gravity program is inspired [B25, B26]. In that approach, the implementation of a new set of variables within General Relativity allows to recast the theory as a $SU(2)$ gauge theory, giving the opportunity to apply already existing quantization methods which are successful in other frameworks. The first order paradigm necessarily comes into play since a Lagrangian formulation of the theory can only be devised introducing additional terms in the Einstein-Hilbert action which are non-trivial only if the connection is considered as an independent variable.

Further arguments can be put forward in support of the inclusion of torsion and nonmetricity, necessarily present in metric-affine gravity models. Some of them are summarized here:

- A relevant aspect of torsion and nonmetricity is that these objects allow for equivalent formulations of General Relativity. We are not referring to the aforementioned equivalence between first and second order formulations, but rather to the so-called geometric trinity of gravity [52]. According to this idea, one can formulate the same theory, i.e. General Relativity, in three different ways, by ascribing gravity to curvature, as usual, or also to torsion or nonmetricity. The first case amounts to consider the affine Riemann tensor but setting $T^{\mu}{}_{\nu\rho} = Q^{\mu}{}_{\nu\rho} = 0$, reducing to the usual metric Riemann tensor $R^{\mu}{}_{\nu\rho\sigma}$. However, there are two alternative options consisting in setting the latter to zero, i.e. $R^{\mu}{}_{\nu\rho\sigma} = 0$, together with only one between nonmetricity and torsion, resulting in the so-called Teleparallel and Symmetric Teleparallel theories of gravity, respectively. Beside this fact, relevant in itself, teleparallel formulations offer interesting alternatives to modify General Relativity beyond the ones summarized in the next section.

- Another less known fact regarding torsion is that it provides a successful renormalization scheme in Quantum Field Theory [74], alongside the better known dimensional regularization. In this approach the presence of torsion implies the non-commutation of momentum components, allowing to replace diverging integrals in Feynman diagrams with summations over the momentum eigenvalues, ultimately yielding finite results.

- A more heuristic justification for torsion and nonmetricity resides in a parallel that can be traced with ordinary matter physics systems. In particular, torsion and nonmetricity can be shown to be related to defects in regular structures like the ones present in liquid crystals or dislocated metals, corresponding to densities of point defects and line defects, respectively [75]. Then, an analogy can be drawn with gravity if one advocates for a non-trivial discrete micro-structure of spacetime at very small scales, with torsion and nonmetricity playing analogous roles.
  Remarkably, analogies between condensed matter physics systems and Einstein-Cartan spacetimes were also discussed in [76, 77, 78, 79, 80, 81], where a crucial role is played by the Nieh-Yan term that will be thoroughly discussed in the following.

- A cornerstone idea of General Relativity is that gravitational phenomena arise from the intertwined interaction of the stress-energy tensor of matter and spacetime curvature, the latter being determined by the former and the dynamics of matter following from the spacetime configuration. When matter with intrinsic spin is considered, e.g. spinor fields, it is natural to put forward the idea that spin be associated to its own independent geometric object, which turns out to be torsion itself, just as the energy density is associated to curvature. Similarly, one can also consider other microscopic characteristics of matter such as dilation and shear which can be in turn associated to the nonmetricity tensor [49]. The spin, dilation and shear currents are all encoded in the so-called hypermomentum tensor (see section 1.5), defined in terms of the variation of the matter Lagrangian with respect to the independent connection, by close analogy with the standard energy momentum tensor.

Chapter 2

Modified Gravity models

The introduction of geometric structures such as torsion and nonmetricity amounts to modify the kinematic content of General Relativity, which is basically relying on tensor fields living on a pseudo-Riemannian manifold. Beside the kinematics of the theory, one can also extend General Relativity by modifying its dynamics. The latter is only determined after a particular Lagrangian is specified, and the Einstein-Hilbert action is not a priori the only possible choice compatible with general covariance. Rather, it just consists of the simplest choice. Indeed, soon after the formulation of General Relativity, Lagrangians containing higher-order curvature invariants were proposed in order to address the non-renormalizability of the theory [82, 83]. Furthermore, over the last decades also dark matter and dark energy problems have been addressed via modified gravity approaches, in order to explain astrophysical and cosmological observations, as for instance the accelerated expansion of the universe [84, 85, 86]. Clearly, alternative descriptions of gravitational phenomena cannot spoil well-established General Relativity predictions [87] and some tests must be used for singling out viable models. In this chapter we will review some extended theories of gravity that are relevant for the original results presented in the next chapter. First, the main properties of $f(R)$ theories of gravity will be recalled, both from a metric and Palatini perspective. Then, we will introduce the Holst and Nieh-Yan terms and their motivations coming from the loop quantum gravity framework. Their implementation within Palatini $f(R)$ theories will be discussed, presenting models already investigated in literature. We will also explain the issue related to the Immirzi parameter ambiguity, present in loop quantum gravity, and introduce models endowed with a dynamical Immirzi field. Finally, we consider the Chern-Simons term and the main properties of the associated modified theory of gravity in its purely metric version, which is the most studied in literature.

2.1 $f(R)$ theories of gravity

Among the possible extensions of General Relativity, $f(R)$ theories of Gravity [45, 46, 47] stem out for their generality and handiness. In these theories, the Einstein-Hilbert action is modified by replacing the Ricci scalar by a general function of it. The cosmological and astrophysical implications of $f(R)$ gravity have attracted a flurry activity in the past decades, see e.g. [88, 89] for reviews. Among specific $f(R)$ models considered in literature, it is worth mentioning quadratic gravity, whose classical and quantum properties were thoroughly investigated already in the late 70's [90, 91], initially in the context of compact and spherically symmetric objects (see also [92, 93, 94, 95] for further developments). In particular, the simplest quadratic Lagrangian, often called Starobinski model, has the form $L \propto \sqrt{-g}(R + R^2/m^2)$ [96] and, when used to describe the inflationary expansion of the Universe, turns out to be among the most accurate models when compared to observations [97]. Moreover, the linear and quadratic terms can also be interpreted as the truncation of a Taylor expansion of any analytic function $f(R)$ around flat space.

As for General Relativity, also $f(R)$ theories can either be formulated at first or second order, resulting in two inequivalent theories denoted Palatini $f(R)$ and metric $f(R)$ theory of gravity, respectively. It has to be stressed that, contrary to what happens in General Relativity, where the first and second order formulations are dynamically equivalent under suitable assumptions on the matter action (see section 1.5), in the case of $f(R)$ theories the two formulations are inequivalent, even in vacuum. We will now give a brief introduction to both formulations, recalling their basic features.

Metric $f(R)$ gravity

In metric $f(R)$ gravity the Einstein-Hilbert action is replaced by the second order variational principle

$$S_f[g] = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(R), \tag{2.1}$$

where the metric is the only independent variable and the connection is given by the usual Levi-Civita connection. The equations of motion for the metric tensor constitute a set of fourth order partial differential equations which reduces to Einstein's equations in the case $f(R) = R$. Including also the matter action (1.12), they are given by

$$f'(R)R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} + \left(g_{\mu\nu} \Box - \bar{\nabla}_\mu \bar{\nabla}_\nu \right) f'(R) = \kappa^2 T_{\mu\nu}, \tag{2.2}$$

where a prime denotes differentiation with respect to the argument of the function and $\Box = g^{\mu\nu}\bar{\nabla}_\mu\bar{\nabla}_\nu$. Tracing the previous equation yields

$$3\Box f'(R) + f'(R)R - 2f(R) = \kappa^2 T, \tag{2.3}$$

where $T = g^{\mu\nu}T_{\mu\nu}$ is the trace of the stress-energy tensor. The last equation shows that, in metric $f(R)$ theory, there is a differential relation between the Ricci scalar and the trace of matter, as opposed to the case of General Relativity, in which they are merely proportional to each other. Although higher-order derivatives are present in the field equations, this does not imply the arising of dynamical instabilities since the Ricci scalar turns out to be an independent, well-behaved scalar degree of freedom. This is apparent in the Jordan frame formulation presented below for the Palatini case and discussed for the metric case at the end of this section, where Brans-Dicke theories are briefly recalled.

2.1.1 Palatini $f(R)$ gravity

Since some of the models considered in this thesis are based on the first order formulation of $f(R)$ theories, we will now indulge more on the main features of Palatini $f(R)$ gravity [48]. In this case, the action reads

$$S_f[g,\Gamma] = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}f(\mathcal{R}) \tag{2.4}$$

and, according to a first order formalism, the connection is considered as an independent variable. The equations of motion for the metric and the connection stemming from (2.4) (completed with (1.12)) can be derived as in [48], resulting in

$$f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = \kappa^2 T_{\mu\nu}, \tag{2.5}$$

$$\nabla_\mu\left(\sqrt{-g}f'(\mathcal{R})g^{\rho\sigma}\right) = 0. \tag{2.6}$$

Tracing the first equation yields

$$f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = \kappa^2 T. \tag{2.7}$$

This equation shows how, in the Palatini $f(R)$ theory, the relation between the Ricci scalar and the trace of matter is merely an algebraic one and allows to consider $\mathcal{R}$, as well as $f(\mathcal{R})$, as algebraic functions of T. Equation (2.6) can be solved for the independent connection giving

$$\Gamma^\mu{}_{\nu\rho} = \frac{1}{2f'(\mathcal{R})}g^{\mu\sigma}\left[\partial_\nu\left(f'(\mathcal{R})g_{\sigma\rho}\right) + \partial_\rho\left(f'(\mathcal{R})g_{\sigma\nu}\right) - \partial_\sigma\left(f'(\mathcal{R})g_{\nu\rho}\right)\right], \tag{2.8}$$

where the dependence on the connection contained in the factors $f'(\mathscr{R})$ can be eliminated substituting the expression of $\mathscr{R}$ as a function of T obtained from the solution of equation (2.7). Thus, the right hand side of equation (2.8) depends only on T and the metric tensor, showing how the connection behaves as a sort of auxiliary variable on half shell, i.e. along the solution of the equations of motion. Moreover, equation (2.8) can be interpreted saying that the independent connection is actually the Christoffel symbol of a conformally rescaled metric defined as

$$\tilde{g}_{\mu\nu} = f'(\mathscr{R})g_{\mu\nu}. \tag{2.9}$$

Thus, $\mathscr{R}_{\mu\nu}$ can be considered as the Ricci tensor of the conformally rescaled metric. In order to relate $\mathscr{R}$ with R in the Palatini $f(R)$ action, one can use the relation holding under conformal transformations [B10], i.e.

$$\mathscr{R} = R + \frac{3}{2(f')^2} g^{\mu\nu}\bar{\nabla}_\mu f' \bar{\nabla}_\nu f' - \frac{3}{f'} g^{\mu\nu}\bar{\nabla}_\mu \bar{\nabla}_\nu f'. \tag{2.10}$$

As we shall now see, substituting this last result into the action does *not* allow to recover the second order formulation, namely metric $f(R)$ theory, proving the dynamical inequivalence between the two formalisms, contrary to what happens in General Relativity.
The handiness of $f(R)$ theories relies on the fact that both metric and Palatini $f(R)$ theories can equivalently be formulated in the so-called Jordan frame, where the degree of freedom added to the theory through the function f is embodied by a scalar field ϕ, often called scalaron. In this way the theory can be recast in the form of a scalar-tensor theory which is easier to deal with.
The transition to the Jordan frame is done via the introduction of an auxiliary scalar field χ in the variational principle. Let us first discuss the case of metric $f(R)$ gravity and then extend the procedure to the Palatini $f(R)$ theory we are interested in. In the metric case the action takes the form

$$S_f[g,\chi] = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\,[f(\chi) + f'(\chi)(R-\chi)]\,. \tag{2.11}$$

The vanishing of the variation of the action with respect to χ gives

$$f''(\chi)(R-\chi) = 0, \tag{2.12}$$

which, for $f''(\chi) \neq 0$, implies $\chi = R$. Substituting back this result into (2.11), the equivalence with (2.1) is proved. Eventually, the action in the Jordan frame reads

$$S_f[g,\phi] = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\,[\phi R - V(\phi)]\,, \tag{2.13}$$

where the scalar field ϕ is defined as

$$\phi \equiv f'(\chi) \tag{2.14}$$

and the potential reads $V(\phi) = \phi\chi(\phi) - f(\chi(\phi))$, where χ has to be considered as a function of ϕ through the inversion of (2.14), which is possible as long as the condition $f''(\chi) \neq 0$ holds.
This seems to exclude the special case $f(\chi) = \chi$, namely General Relativity. However, the condition for the second derivative to be non vanishing is not strictly necessary. As shown in [98], $f(R)$ theories can be recast in the Jordan frame just assuming the weaker condition that $f'(\chi)$ has to be continuous and one-to-one.
The equations of motion for the metric tensor and the scalar field are derived by varying the action (2.13) and read, respectively

$$\phi G_{\mu\nu} + g_{\mu\nu}\Box\phi - \bar{\nabla}_\mu\bar{\nabla}_\nu\phi + \frac{1}{2}g_{\mu\nu}V(\phi) = \kappa^2 T_{\mu\nu}, \tag{2.15}$$

$$R = V'(\phi), \tag{2.16}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ and we included the stress-energy tensor for matter fields. The first equation coincides with (2.2) once the reparametrization (2.14) is taken into account. Substituting its trace in the second equation allows to eliminate the Ricci scalar from it, yielding the analogue of (2.3):

$$3\Box\phi + 2V(\phi) - \phi V'(\phi) = \kappa^2 T, \tag{2.17}$$

where the trace of the stress-energy tensor of matter acts as a source term, dictating the dynamics of the scalar field.
Moreover, the terms in equation (2.15) can be rearranged in order to express it in the form of Einstein's equations with a modified source and an effective coupling constant as

$$G_{\mu\nu} = \kappa^2_{eff}\left(T_{\mu\nu} + T^{(eff)}_{\mu\nu}\right), \tag{2.18}$$

where $\kappa^2_{eff} = \kappa^2/\phi$ and the effective stress-energy tensor is defined as

$$T^{(eff)}_{\mu\nu} = \frac{1}{\kappa^2}\left(-\frac{1}{2}g_{\mu\nu}V(\phi) + \bar{\nabla}_\mu\bar{\nabla}_\nu\phi - g_{\mu\nu}\Box\phi\right). \tag{2.19}$$

Thus, the theory is always characterized by the presence of an effective stress-energy tensor, even in vacuum, where $T_{\mu\nu} = 0$.
The procedure for the Palatini case is formally the same but the connection is considered as an independent variable. The outcome of the transition to the Jordan frame is

$$S_f[g, \Gamma, \phi] = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\,[\phi\mathcal{R} - V(\phi)]\,. \tag{2.20}$$

The auxiliary character of the connection implies that the dynamics is completely determined by the equations of motion for the metric tensor and the scalar field, which, once equations (2.8) and (2.14) are taken into account, read, respectively

$$\phi G_{\mu\nu} + \left(g_{\mu\nu}\Box - \bar{\nabla}_\mu \bar{\nabla}_\nu\right)\phi + \frac{1}{2} g_{\mu\nu} V(\phi) + \frac{3}{2\phi}\left(\bar{\nabla}_\mu \phi \bar{\nabla}_\nu \phi - \frac{1}{2} g_{\mu\nu} \bar{\nabla}^\rho \phi \bar{\nabla}_\rho \phi\right) = \kappa^2 T_{\mu\nu}, \tag{2.21}$$

$$\Box\phi = \frac{\phi}{3}\left(R - V'(\phi)\right) + \frac{1}{2\phi} \bar{\nabla}^\mu \phi \bar{\nabla}_\mu \phi. \tag{2.22}$$

Tracing the first equation and substituting the result in the second one, yields

$$2V(\phi) - \phi V'(\phi) = \kappa^2 T. \tag{2.23}$$

This is called *structural equation* and has important consequences in the Palatini $f(R)$ theory as well as in more general models that will be considered in the following. Given a specific function $f(R)$, which determines the form of the potential $V(\phi)$, the relation $\phi = \phi(T)$ is obtained solving (2.23) algebraically for ϕ. This implies that ϕ is not a real degree of freedom of the theory. Indeed, it affects the way in which matter appears in the equations and generates the spacetime curvature, rather than representing part of the dynamics of the gravitational field itself.

This is a peculiarity of Palatini $f(R)$ theory since in metric $f(R)$, equation (2.23) is replaced by (2.17), which has the additional term proportional to $\Box\phi$. In that equation, T acts as a source term for the dynamics of ϕ, as it happens for the metric tensor in Einstein's equations, giving a proper dynamics to the scalar field and making it an independent field $\phi = \phi(x^\mu)$ instead of an algebraic relation in T. In particular, in vacuum, where $T = 0$, the structural equation yields the trivial outcome $\phi =$ *constant*, while it holds identically if cases $f(R) = R$ ($\phi = 1$, $V(\phi) = 0$) or $f(R) = R^2$ ($V(\phi) = \phi^2/4$), are considered.

Also in the Palatini case, the ϕ-dependent terms featuring equation (2.21) can be moved to the right hand side and an effective stress-energy tensor can be defined as

$$T^{(eff)}_{\mu\nu} = \frac{1}{\kappa^2}\left[-\frac{1}{2} g_{\mu\nu} V(\phi) + \left(\bar{\nabla}_\mu \bar{\nabla}_\nu - g_{\mu\nu}\Box\right)\phi - \frac{3}{2\phi}\left(\bar{\nabla}_\mu \phi \bar{\nabla}_\nu \phi - \frac{1}{2} g_{\mu\nu} \bar{\nabla}^\rho \phi \bar{\nabla}_\rho \phi\right)\right], \tag{2.24}$$

such that the equations of motion take the form (2.18). In vacuum, where the structural equation implies that the scalar field is a constant ϕ_0, the theory reduces to General Relativity with a cosmological constant given by

$$\Lambda = \frac{V_0}{2\phi_0}, \tag{2.25}$$

where $V_0 = V(\phi_0)$, and a modified coupling constant $\kappa_0^2 = \kappa^2/\phi_0$.
If instead $T_{\mu\nu} \neq 0$ and $T \neq 0$, the presence of $T_{\mu\nu}^{(eff)}$ in the equation of motion implies a modification in the coupling between the metric tensor and matter. Indeed, derivatives of the stress-energy tensor arise via the terms featuring equation (2.24). This feature was shown to give rise to curvature singularities at the surface of stars modelled in the Palatini $f(R)$ theory [99, 100, 101]. However, it was later proved that, using the appropriate matching conditions at the stellar surface, pathologies are only present for unphysical equations of state [102].
Also equation (2.10) can be rewritten in the Jordan frame as

$$\mathcal{R} = R + \frac{3}{2\phi^2} g^{\mu\nu}\bar{\nabla}_\mu\phi\bar{\nabla}_\nu\phi - \frac{3}{\phi} g^{\mu\nu}\bar{\nabla}_\mu\bar{\nabla}_\nu\phi. \tag{2.26}$$

As already mentioned, this gives the possibility to eliminate the independent connection, allowing, modulo surface terms, to express the action (2.20) in the form

$$S_f[g,\phi] = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[\phi R + \frac{3}{2\phi}\bar{\nabla}^\mu\phi\bar{\nabla}_\mu\phi - V(\phi)\right]. \tag{2.27}$$

Comparison with (2.13) shows the inequivalence between metric and Palatini $f(R)$ theories on a dynamical level. Moreover, this last expression for the action allows to introduce the following feature of $f(R)$ theories.
Both metric and Palatini $f(R)$ theories, when expressed in the Jordan frame, belong to a wider class of scalar-tensor theories [B103], named Brans-Dicke theories [104]. The action of such theories reads

$$S_{BD}[g,\phi] = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[\phi R - \frac{\omega_{BD}}{\phi}\bar{\nabla}^\mu\phi\bar{\nabla}_\mu\phi - V(\phi)\right], \tag{2.28}$$

where ω_{BD} is an arbitrary parameter. Comparison with (2.13) and (2.27) immediately shows that metric and Palatini $f(R)$ theories are equivalent to Brans-Dicke theories with parameter $\omega_{BD} = 0$ and $\omega_{BD} = -3/2$, respectively.
It is possible to show that in Brans-Dicke theories the field ϕ is governed by the equation

$$(3 + 2\omega_{BD})\Box\phi + 2V(\phi) - \phi V'(\phi) = \kappa^2 T. \tag{2.29}$$

This equation correctly reproduces the structural equation of Palatini and metric cases when $\omega_{BD} = -3/2$ and $\omega_{BD} = 0$, respectively but also shows how $\omega_{BD} = -3/2$ is the only case in which the field ϕ is non dynamical in the sense explained above. For every other choice of ω_{BD} the term proportional to $\Box\phi$ survives and ϕ behaves as a proper dynamical degree of freedom. A further extension of Brans-Dicke theory consists of promoting the Brans-Dicke parameter to an arbitrary function of the scalaron in the action, i.e. $\omega_{BD} \rightarrow \Omega(\phi)$. This class of theories is often referred to as scalar-tensor theories and as we will see some of the models we will consider in the following are dynamically equivalent to theories obtained from specific choices of the function $\Omega(\phi)$.

2.2 Parity violating topological terms

Another perspective on the extension of General Relativity is offered by topological terms. By topological terms we mean quantities $\mathscr{T}$ that can be written as the (Levi-Civita) divergence of some current J^μ, namely $\mathscr{T} = \bar{\nabla}_\mu J^\mu$. Hence, by virtue of the Gauss theorem, the corresponding additional contribution included in the action reduces to a boundary term $\mathscr{B}$:

$$\int_{\mathscr{M}} d^4x\sqrt{-g}\mathscr{T} = \int_{\mathscr{M}} d^4x\sqrt{-g}\bar{\nabla}_\mu J^\mu = \int_{\partial\mathscr{M}} d^3x\sqrt{h}\mathscr{B}, \tag{2.30}$$

where h is the determinant of the 3-metric induced on the boundary of the manifold $\partial\mathscr{M}$. In the variational principle, one usually imposes the vanishing of the variation of the fields on the boundary. Hence topological terms by themselves do not affect the equations of motion of the theory.

However, they can offer an interesting mechanism to modify General Relativity via the introduction of additional scalar fields. By now, the existence of scalar fields is well-established from different theoretical and phenomenological settings. Among them, the observation of the Higgs boson certainly is the most important [105, 106]. Scalar fields also play a fundamental role in the inflation paradigm in cosmology [B107]. Moreover, they are predicted by string theory models and have been proposed in loop quantum gravity contexts as well [108, 109]. Finally, going back to the content of the last section, they are useful tools in $f(R)$ modified theories of gravity, which include a plethora of gravitational models allowing to address numerous open problems, all of them featuring a scalar degree of freedom in their dynamical content, as it is apparent in their formulation in the Jordan frame.

Now, a boundary term is usually included in the action multiplied by a suitable coupling constant. Whenever the latter is promoted to a scalar field, the topological nature of the boundary term is spoiled. This allows for a simple and direct way of modifying General Relativity, including additional contributions in the field equations that are proportional to the scalar field derivatives. As a consequence, the resulting extended theory of gravity will reduce to General Relativity in the limit of constant scalar field. This offers the possibility of considering physical settings where a non-trivial behavior for the scalar field is possible in some high energy regime, where deviations from General Relativity are expected, while a relaxation to the trivial constant configuration can be attained in low energy asymptotic limits, recovering General Relativity where it gives the correct predictions.

Another important role boundary terms fulfill has to do with the well-posedness of variational principles. These are always formulated assigning specific conditions on the boundary. In particular, if one imposes that the variation of the metric tensor (but not its derivatives) be vanishing on the boundary, then the Einstein-Hilbert action does not represent a well-posed variational principle, in the sense that a non-vanishing boundary term survives beside the bulk term yielding Einstein's field equations. This additional unwanted contribution can be compensated by including the so-called Gibbons-Hawking-York boundary term in the variational principle [B110]. Beside that, the role this boundary term turns out to play in the thermodynamic description of Schwarzschild black holes is remarkable [111]. Indeed, in the Euclidean path integral formulation, the vanishing of the bulk term of the Euclidean on-shell action implies that the Gibbons-Hawking-York boundary term is the only one contributing to the non-trivial thermodynamic properties of the black hole [112].

The terms we are about to present in the next sections are all characterized by the fact that they are parity violating terms. We can define parity transformations as purely spatial reflections of the tetrad defining the coordinate system [113]. Then a quantity has even (odd) parity or it is said to be parity preserving (violating) if it has eigenvalue $+1$ (-1) with respect to the parity operator.

Theories of gravity exhibiting parity violation are recently receiving increasing attention. Among them, beside the models considered here we can recall degenerate higher-order scalar-tensor theories (DHOST) [114, 115], bumblebee models [116, 117, 118, 119, 120, 121, 122] and Hořava-Lifshitz gravity [123, 124]. The prominent role of parity violation in modern physics relies in its possible effects in CMB polarization [125, 126, 127, 128], primordial gravitational waves [129, 130, 131, 132, 133, 134, 135], the baryon asymmetry problem [136, 137, 138, 139], black hole perturbations [140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150] and gravitational wave birefringence [151, 115, 152, 153, 154, 155, 156].

In this section we will set nonmetricity to zero as we are reviewing previous results derived with this assumption. In sections 2.2.1 we will work in the Einstein-Cartan framework, where only curvature and torsion are present, while in section 2.2.2 also the torsionless condition will be imposed, restricting the analysis to purely metric spacetimes. The extension to the case of generic metric-affine spacetimes will be addressed in chapter II.

2.2.1 Holst and Nieh-Yan terms

Since the Holst and Nieh-Yan terms were discovered in Einstein-Cartan spacetimes, in this section we will restrict to manifolds where nonmetricity is vanishing and only torsion is a priori allowed. Let us start the discussion introducing the Holst term. Strictly speaking, it is not a topological term, in the sense that it cannot be written as a total derivative. However, as we will now show, it does not affect the field equations by virtue of its main property: it is vanishing on half-shell. The Holst term is defined by the following contraction of the Levi-Civita tensor with the Riemann tensor of an asymmetric, metric-compatible connection:

$$\mathscr{H} \equiv -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\mathscr{R}_{\mu\nu\rho\sigma}. \tag{2.31}$$

The simplest gravitational model featuring the Holst term is obtained introducing it in the Hilbert-Palatini action of General Relativity, yielding the Holst action

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left(\mathscr{R} + \gamma_0\mathscr{H}\right), \tag{2.32}$$

where γ_0 is a dimensionless parameter, referred to as Immirzi parameter in the context of loop quantum gravity [157]. The metric field equations in vacuum are given by

$$\mathscr{R}_{(\mu\nu)} - \frac{1}{2}\mathscr{R}\,g_{\mu\nu} = -\frac{\gamma_0}{2}\varepsilon_{(\mu}{}^{\lambda\rho\sigma}\mathscr{R}_{\nu)\lambda\rho\sigma}, \tag{2.33}$$

while varying with respect to the connection one obtains (1.78) with vanishing nonmetricity and a non vanishing right hand side:

$$g^{\mu\nu}T_\lambda - \delta^\nu_\lambda T^\mu + T^{\nu\mu}{}_\lambda = -\gamma_0\left[\varepsilon_\lambda{}^{\rho\mu\nu}T_\rho + \frac{1}{2}\varepsilon_\lambda{}^{\mu\rho\sigma}T^\nu{}_{\rho\sigma}\right]. \tag{2.34}$$

Contractions of the latter with δ^λ_ν, $g_{\mu\nu}$ and $\varepsilon_{\alpha\nu\mu}{}^\lambda$ yield two independent equations:

$$2T_\mu + \frac{\gamma_0}{2}S_\mu = 0, \tag{2.35}$$

$$S_\mu + 2\gamma_0 T_\mu = 0, \tag{2.36}$$

resulting in

$$T_\mu = -\frac{\gamma_0}{4}S_\mu, \tag{2.37}$$

$$\left(1 - \frac{\gamma_0^2}{2}\right)S_\mu = 0. \tag{2.38}$$

Provided[1] that $\gamma_0 \neq \pm\sqrt{2}$, this implies $T_\mu = S_\mu = 0$. Then, the equation for the tensorial components becomes

$$q_{\nu\mu\lambda} - \frac{\gamma_0}{2}\varepsilon_{\mu\lambda\sigma\rho}q_\nu{}^{\sigma\rho} = 0, \tag{2.39}$$

which may be written as

$$\Delta_{\mu\lambda}{}^{\alpha\beta}q_{\nu\alpha\beta} = 0, \tag{2.40}$$

where we defined

$$\Delta_{\mu\lambda}{}^{\alpha\beta} \equiv \delta^\alpha_{[\mu}\delta^\beta_{\lambda]} - \frac{\gamma_0}{2}\varepsilon_{\mu\lambda}{}^{\alpha\beta}. \tag{2.41}$$

We conclude that $q_{\mu\nu\rho} = 0$. This can be shown either inverting the operator $\Delta_{\mu\lambda}{}^{\alpha\beta}$, i.e.

$$\left(\Delta^{-1}\right)_{\mu\lambda}{}^{\alpha\beta} \equiv \frac{1}{1+\gamma_0^2}\left(\delta^\alpha_{[\mu}\delta^\beta_{\lambda]} + \frac{\gamma_0}{2}\varepsilon_{\mu\lambda}{}^{\alpha\beta}\right), \tag{2.42}$$

or expanding all the components of equation (2.39) and showing that they are all trivially vanishing. In both cases the outcome is valid for every real value of γ_0 and only the $\gamma_0 = \pm i$ cases are excluded[2].
Therefore, the inclusion of the Holst term does not allow for non-trivial torsion. Moreover, the Holst term is vanishing on half-shell since, if $\Gamma^\mu{}_{\nu\rho} = \bar{\Gamma}^\mu{}_{\nu\rho}$, then

$$\mathscr{H} = -\frac{\gamma_0}{2}\varepsilon^{\mu\nu\rho\sigma}\mathscr{R}_{\mu\nu\rho\sigma} = -\frac{\gamma_0}{2}\varepsilon^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = 0, \tag{2.43}$$

which is vanishing by virtue of the algebraic Bianchi identities. Hence, the metric field equations are not modified either and one recovers General Relativity with a Levi-Civita connection.
The failure of the Holst term of being a topological term can be compensated extending its definition and leading to the Nieh-Yan topological invariant. In order to introduce it, it is instructive to look at the Holst term post-Riemannian expansion when only torsion is present:

$$\mathscr{H} = -\frac{1}{2}\bar{\nabla}_\mu S^\mu - \frac{1}{3}S^\mu T_\mu - \frac{1}{4}\varepsilon_{\mu\nu\rho\sigma}q_\lambda{}^{\mu\nu}q^{\lambda\rho\sigma}. \tag{2.44}$$

We see that the last two terms prevent it from being a boundary term. This observation leads us to the topological term known as Nieh-Yan term. It is defined by

$$\mathscr{N}\mathscr{Y} \equiv \frac{\gamma_0}{2}\varepsilon^{\mu\nu\rho\sigma}\left(\frac{1}{2}T^\lambda{}_{\mu\nu}T_{\lambda\rho\sigma} - \mathscr{R}_{\mu\nu\rho\sigma}\right). \tag{2.45}$$

[1]The condition $\gamma_0 \neq \pm\sqrt{2}$ is not strictly necessary. Indeed, one can obtain the same result without restrictions on γ_0 via a projective transformation, since the Holst term is already projective invariant in the Einstein-Cartan framework, as we will show in section 3.1.

[2]The $\gamma_0 = \pm i$ case corresponds to the original complex-valued Ashtekar variables which were later discarded [158].

It was first discovered in [56, 57] in the context of Einstein-Cartan theory and then shown to have the same role as the Holst term in the implementation of Ashtekar variables in loop quantum gravity [159].
We see that its expression is obtained completing the Holst term with a torsion squared contribution. The main consequence of the latter is to modify the decomposition in (2.44), which now simply reads

$$\mathcal{N}\mathcal{Y} = -\frac{1}{2}\bar{\nabla}_\mu S^\mu. \tag{2.46}$$

Hence, the Nieh-Yan term is a proper topological term which can be written as a total divergence and reduces to a boundary term in the action principle. As a consequence, it does not affect the field equations at all.

Immirzi parameter ambiguity and the Immirzi field

As already anticipated, the two terms presented in this section are mainly motivated by the fact that they find application in one of the existing attempt to quantize gravity, that is loop quantum gravity [B25, B26]. This is a non perturbative and background independent canonical quantization of General Relativity. It relies on the possibility of recasting Einstein's theory in terms of a new set of canonical conjugate variables called Ashtekar-Barbero-Immirzi variables [158, 160, 161, 162] (often just called Ashtekar variables). Their implementation in the theory can be carried out at the Lagrangian level by introducing either the Holst or the Nieh-Yan term in the action. A direct consequence of the on half-shell vanishing of the former and the topological character of the latter is that their presence does not modify the theory at the classical level. The field equations are still given by Einstein's equations and the classical theory is left untouched. However, the reformulation in terms of Ashtekar variables, induces a new structure in the canonical phase space of the theory allowing to consider it as a gauge theory of the $SU(2)$ group, a crucial ingredient for the implementation of nonperturbative quantization techniques proper of the loop quantum gravity approach. This recipe led to some progress in the canonical quantization of General Relativity and to some intriguing results regarding the nature of spacetime at the Planck scale [163]. However, the loop quantum gravity program is still plagued by some unsolved issues. Among them, we will briefly discuss here the so-called Immirzi parameter ambiguity. The parameter γ_0 enters the theory as an arbitrary real parameter, either coupling the Holst and Nieh-Yan terms in the action at the Lagrangian level, or labeling the canonical transformation leading to Ashtekar variables in the Hamiltonian formalism [B26]. In any case, it does not appear in the classical field equations. Despite that, it turns out to enter the definition of the quantum observables of the theory. In loop quantum gravity these are

the quanta of area and volume, constituting the spectra of geometrical operators at the quantum level [163]. In this way, the Immirzi parameter labels different quantum sectors of the theory that have been shown not to be related by unitary transformations. Hence, a one parameter quantum ambiguity is present in the theory [157].
This issue has been addressed in several works with different proposals. A possible explanation is that the Immirzi parameter is a new fundamental constant which should set the size of the quanta of space with respect to the Planck scale. It has been proposed to fix its value in order to match the value of black holes entropy predicted by Loop Quantum Gravity with the one given by the Bekenstein-Hawking formula [164], but later works seemed to disregard this hypothesis [165, 166]. Its meaning has also been investigated in the presence of spinor fields [167, 168].
Here instead, we want to focus on the idea put forward in [108], where it was proposed to promote the Immirzi parameter to a dynamical scalar field:

$$\gamma_0 \to \gamma(x^\mu). \tag{2.47}$$

In this landscape, the Immirzi field is a new fundamental field that may have had a non-trivial behavior in the primordial universe or in other high energy scenarios, and later have relaxed to a constant value during the cosmological evolution, providing a natural interpretation of the Immirzi parameter γ_0 as the vacuum expectation value of the field $\gamma(x)$. The extensions of the Holst or Nieh-Yan action to this case read as

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g}\, (\mathscr{R} + \gamma L)\,, \tag{2.48}$$

where L is either the Holst or the Nieh-Yan term. Now, the results presented in the previous section cease to hold, and the irrelevance of the Holst and Nieh-Yan terms in the field equations is only recovered in the special case $\gamma(x) \equiv \gamma_0$. In general instead, deviations from General Relativity are expected, offering a way to obtain new theoretical insights as well as to derive observational signature for the existence of the Immirzi field. In this regard, several investigations have been carried out. Models like (2.48) have been studied revealing interesting phenomenology, such as bouncing solutions in isotropic cosmological models [108, 169, 170] and the presence of additional gravitational waves polarizations [171, 172], together with implications at a more fundamental level regarding the strong CP problem [173, 174], the chiral anomaly [61], and the implementation of Ashtekar variables [169, 6].

Palatini $f(R)$ models with Holst and Nieh-Yan terms

The main intent of the loop quantum gravity program is the attempt to quantize the gravitational interaction, as it is classically described by the theory of General Relativity. It is for this purpose that the Holst and Nieh-Yan terms were studied within the loop quantum gravity community.
However, as we have seen there are several arguments for studying theories beyond General Relativity that should be considered as the correct theory of gravity, reducing to General Relativity only in certain limits. On this basis, it is natural to investigate the outcomes of quantization programs originally applied only to General Relativity when they are implemented in modified gravity theories. This is not just a necessary step, if one considers General Relativity only as a low energy limit of some other theory of gravity, but it actually may offer new insights into difficulties and shortcomings that are impeding a consistent quantization of Einstein's gravity.
As a first attempt in this direction, it is convenient to consider $f(R)$ theories of gravity, that encompass a wide set of models and at the same time are still easy to deal with. The extension of the loop quantum gravity approach to metric $f(R)$ theories was originally considered in [175] (see also [176, 177]). However, when it comes to the loop quantum gravity quantization scheme, in view of its first order formulation, the most natural implementation should take place in the Palatini version of the $f(R)$ theory, modifying its action via the inclusion of either the Holst or the Nieh-Yan term [178, 6]. Their mere addition to the $f(R)$ action already entails an ambiguity, since summing them directly to the function f or to its argument yields different inequivalent outcomes, namely

$$S_o = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\,[f(\mathscr{R}) + \gamma_0 L]\,, \tag{2.49}$$

$$S_i = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\,f(\mathscr{R} + \gamma_0 L), \tag{2.50}$$

where L is either $\mathscr{H}$ or $\mathscr{N}\mathscr{Y}$. In the following we will proceed performing computations starting from both theories in parallel and without specifying the form of L. The above actions can be transformed to the Jordan frame by applying the same method introduced in section 2.1.1. Let us then introduce an auxiliary scalar field χ as

$$S_o = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[f(\chi) + f_\chi(\chi)(\mathscr{R} - \chi) + L\right], \tag{2.51}$$

$$S_i = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[f(\chi) + f_\chi(\chi)(\mathscr{R} + L - \chi)\right], \tag{2.52}$$

where a subscript denotes a derivative with respect to the argument. Provided[3] $f_{\chi\chi} \neq 0$, variation with respect to χ yields the conditions

$$\chi = \mathscr{R}, \tag{2.53}$$

$$\chi = \mathscr{R} + L, \tag{2.54}$$

for S_o and S_i, respectively. Reinserting them into the actions proves the equivalence with (2.49) and (2.50). Then, introducing the scalaron field defined as $\phi \equiv f_\chi(\chi)$, we can write the actions in the equivalent scalar-tensor representation

$$S_o = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\,[\phi\mathscr{R} + L - V(\phi)]\,, \tag{2.55}$$

$$S_i = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\,[\phi(\mathscr{R} + L) - V(\phi)]\,, \tag{2.56}$$

where the potential is still given by $V(\phi) = \phi\chi(\phi) - f(\chi(\phi))$ and $\chi(\phi)$ is obtained inverting the definition of the scalaron. In the Jordan frame, the way in which the terms are included (added to the function f or to its argument) simply amounts to multiply them by the scalaron ϕ or not. Taking into account the decomposition of the Ricci scalar and the Holst and Nieh-Yan contributions in terms of the irreducible components of torsion, it is easy to compute the field equations for the vector and tensor components (see [6]), which can be algebraically solved, yielding $q_{\mu\nu\rho} = 0$ and

$$T_\mu = \frac{3}{2\phi}\left[\frac{1 + b_1 b_2 \Phi\gamma^2/\phi}{1 + b_2\Phi^2\gamma^2/\phi^2}\right]\partial_\mu\phi, \tag{2.57}$$

$$S_\mu = \frac{6\gamma}{\phi}\left[\frac{b_1 - b_2\Phi/\phi}{1 + b_2\Phi^2\gamma^2/\phi^2}\right]\partial_\mu\phi., \tag{2.58}$$

where we introduced two parameters b_1 and b_2 which can take values 0 or 1 according to the specific model considered (see Table 2.1). If the Holst term is taken into account, then $b_2 = 1$, while $b_1 = 0$ and $b_1 = 1$ for $f(\mathscr{R}) + \mathscr{H}$ and $f(\mathscr{R} + \mathscr{H})$, respectively. When the action features the Nieh-Yan contribution, $b_2 = 0$, while $b_1 = 0$ and $b_1 = 1$ for $f(\mathscr{R}) + \mathscr{N}\mathscr{Y}$ and $f(\mathscr{R} + \mathscr{N}\mathscr{Y})$, respectively. Finally, Φ is coincident with ϕ in the $f(\mathscr{R} + \mathscr{H})$ case and identically equal to 1 in the $f(\mathscr{R}) + \mathscr{H}$ one.

[3] As already mentioned in section 2.1.1, the condition for the second derivative to be non vanishing is not strictly necessary [98].

Model	b_1	b_2	Φ
$f(\mathscr{R})+\mathscr{H}$	0	1	1
$f(\mathscr{R}+\mathscr{H})$	1	1	ϕ
$f(\mathscr{R})+\mathscr{N}\mathscr{Y}$	0	0	–
$f(\mathscr{R}+\mathscr{N}\mathscr{Y})$	1	0	–

Table 2.1: Values of the parameters identifying possible Holst and Nieh-Yan modifications to Palatini $f(R)$ gravity.

Substituting these expressions back into the actions, the four theories can be described by the following effective second order scalar tensor action:

$$S = \frac{1}{2\kappa^2}\int \sqrt{-g}\left(\phi R - \frac{\Omega(\phi)}{\phi}\bar{\nabla}_\mu\phi\bar{\nabla}^\mu\phi - V(\phi)\right), \tag{2.59}$$

where the expression for the function $\Omega(\phi)$ depends on the specific choice considered. In particular, we have

$$f(\mathscr{R}+\mathscr{H}) \quad \text{and} \quad f(\mathscr{R})+\mathscr{N}\mathscr{Y}\ : \qquad \Omega = -\frac{3}{2}, \tag{2.60}$$

$$f(\mathscr{R}+\mathscr{N}\mathscr{Y})\ : \qquad \Omega = -\frac{3}{2}\left(1-\gamma_0^2\right), \tag{2.61}$$

$$f(\mathscr{R})+\mathscr{H}\ : \qquad \Omega = -\frac{3}{2}\frac{\phi^2}{\phi^2+\gamma_0^2}, \tag{2.62}$$

We see that only including the Holst term inside the argument of the function or adding the Nieh-Yan term to the function itself, one retains Palatini $f(R)$ gravity, while the other two options lead to different scalar tensor theories. Note that setting $\gamma_0 = 0$ all choices consistently reduce to $\Omega = -3/2$. Thus, as one might expect, the topological character of the Nieh-Yan term is preserved if it is added directly to the Lagrangian and otherwise spoiled by the multiplication by the scalaron. Less trivial, instead, is the outcome for the Holst models. In this case, indeed, the vanishing of the Holst term on half-shell is to some extent recovered only if it is included in the argument of the function f.

2.2.2 Metric Chern-Simons gravity

The last topological, parity odd term we need to introduce is the so-called Chern-Simons term. Here we will discuss its purely metric version, thus setting both torsion and nonmetricity to zero. The role of the Chern-Simons term in the arena of modified gravity theories is strongly motivated by several arguments arising

from different physical backgrounds, where the presence of a Chern-Simons term seems to be ubiquitous (see [58] for a review). In particle physics, for instance, the gravitational anomaly turns out to be proportional to the Pontryagin density, and a Chern-Simons-like counterterm must be included in the action to cancel the anomaly out. Counterterms of this kind can be actually produced also in string theory via the Green-Schwarz mechanism and emerge in low energy effective string models [179, 60]. Remarkably, some analogies can be outlined with loop quantum gravity approaches [180] as well, where Chern-Simons corrections arise in addressing the chiral anomaly of fermions and the Immirzi field ambiguity [167, 181, 59, 182, 62]. Moreover, this theory may help in designing new strategies to probe the (local) Lorentz/CPT symmetry breaking in gravitation, which is expected to receive new observational inputs in the next few years. Indeed, Chern-Simons parity violation effects are already well-established in contexts such as amplitude birefringence for gravitational wave propagation [183, 184, 185, 186, 187], CMB polarization [125, 126, 127, 128] and the baryon asymmetry problem [136, 137, 138].
Another motivation supporting Chern-Simons gravity concerns Kerr-like black hole solutions. Deviations from the usual Kerr spacetime can be implemented by introducing additional parameters by hand in the metric, to be compared with observational constraints from strong field tests [188]. This modifications are introduced irrespective of the actual theory generating them. In this regard, Chern-Simons gravity can spontaneously generate Kerr deformations, offering a theoretical justification to approaches like the one carried on in [188].
The theory was originally proposed by Jackiw and Pi [183] inspired by the Chern-Simons modification of electrodynamics [189]. As for the $U(1)$ gauge theory, the Maxwell Lagrangian is modified by introducing a (pseudo)-scalar field, $\theta(x)$, coupled to the $U(1)$ gauge topological Pontryagin density, i.e. ${}^*F^{\mu\nu}F_{\mu\nu}$, where ${}^*F^{\mu\nu}$ denotes the Hodge dual. Even maintaining the gauge invariance, such a modification allows for Lorentz/CPT symmetry violation [189]. It can be readily seen by casting the modified term into the Carroll-Field-Jackiw form, i.e., $v_\mu F^{\mu\nu}A_\nu$, where $v_\mu \equiv \partial_\mu\theta$ is the axial vector responsible for Lorentz/CPT symmetry breaking. In a wider picture, v_μ corresponds to one of the coefficients for Lorentz/CPT violation in the Standard Model Extension (SME) [190, 191, 116]. In much the same way as in the Chern-Simons modification of Maxwell electrodynamics, one can add to the Lagrangian of General Relativity a non-minimal coupling between $\theta(x)$ and the gravitational Pontryagin density which, in turn, is defined by

$$P = \varepsilon^{\mu\nu\rho\sigma} R^{\alpha}{}_{\beta\mu\nu} R^{\beta}{}_{\alpha\rho\sigma}. \tag{2.63}$$

The resulting action may be written as

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left(R + \frac{\alpha}{8}\theta(x)\varepsilon^{\mu\nu\rho\sigma}R^{\alpha}{}_{\beta\mu\nu}R^{\beta}{}_{\alpha\rho\sigma} - \frac{\beta}{2}\bar{\nabla}_{\mu}\theta\bar{\nabla}^{\mu}\theta - \beta V(\theta)\right), \quad (2.64)$$

where α and β are coupling constants and we included a potential term as well. The Pontryagin density is a topological density, namely it can be written as a total divergence as

$$P = \bar{\nabla}_{\mu}K^{\mu}, \quad (2.65)$$

where

$$K^{\mu} \equiv \varepsilon^{\alpha\beta\gamma\delta}\bar{\Gamma}^{\nu}{}_{\beta\mu}\left(\partial_{\gamma}\bar{\Gamma}^{\mu}{}_{\delta\nu} + \frac{2}{3}\bar{\Gamma}^{\mu}{}_{\gamma\lambda}\bar{\Gamma}^{\lambda}{}_{\delta\nu}\right). \quad (2.66)$$

This assures a topological character to the Chern-Simons term which reduces to a boundary term in the case of constant θ. For a non-trivial scalar field instead, the vacuum field equations for the metric tensor are modified as

$$G_{\mu\nu} + C_{\mu\nu} = \beta T^{sf}_{\mu\nu}(\theta) - \frac{\beta}{2}g_{\mu\nu}V(\theta), \quad (2.67)$$

where the C-tensor, often referred to as Cotton tensor, is defined by

$$C_{\alpha\beta} \equiv \varepsilon^{\gamma\mu\nu}{}_{(\alpha}\left(\bar{\nabla}_{\nu}R_{\beta)\mu}\bar{\nabla}_{\gamma}\theta - \frac{1}{2}R^{\delta}{}_{|\beta)\mu\nu}\bar{\nabla}_{\gamma}\bar{\nabla}_{\delta}\theta\right), \quad (2.68)$$

while $T^{sf}_{\mu\nu}$ represents the scalar field stress-energy tensor, i.e.

$$T^{sf}_{\mu\nu} = \frac{1}{2}\left(\bar{\nabla}_{\mu}\theta\bar{\nabla}_{\nu}\theta - \frac{1}{2}g_{\mu\nu}\bar{\nabla}_{\rho}\theta\bar{\nabla}^{\rho}\theta\right). \quad (2.69)$$

The scalar field is instead governed by

$$\beta\Box\theta + \frac{\alpha}{8}\varepsilon^{\mu\nu\rho\sigma}R^{\alpha}{}_{\beta\mu\nu}R^{\beta}{}_{\alpha\rho\sigma} = \beta\frac{dV}{d\theta}. \quad (2.70)$$

Two versions of the theory have been considered in literature. The original formulation of the Chern-Simons theory is characterized by the absence of the kinetic and potential terms for the scalar field in the action and is obtained from (2.64) setting $\beta = 0$. This model was later dubbed *non-dynamical* Chern-Simons theory, since in this case the scalar field is a purely external quantity and it has not a proper dynamical character. Indeed, for $\beta = 0$ its equation of motion reduces to the Pontryagin constraint

$$\varepsilon^{\mu\nu\rho\sigma}R^{\alpha}{}_{\beta\mu\nu}R^{\beta}{}_{\alpha\rho\sigma} = 0, \quad (2.71)$$

which restricts solutions of the theory to those metrics satisfying (2.71). Common solutions such as Schwarzschild or Friedmann-Lemaître-Robertson-Walker

(FLRW) spacetimes annihilate the constraint which is instead violated by axially symmetric spacetimes such as the Kerr black hole, which is not a vacuum solution in Chern-Simons gravity.
However, the presence of the Pontryagin constraint causes issues in the dynamics of black hole perturbations, even if the simple Schwarzschild case is considered [192]. Indeed, (2.71) imposes an additional condition on the metric perturbations yielding an over-constrained system of equations which is eventually inconsistent. This shortcoming led to the alternative version of the theory, which is known as *dynamical* Chern-Simons gravity (first introduced in [179]). In this case $\beta \neq 0$ and one allows for the presence of a standard kinetic term for $\theta(x)$ in the action. Within this setting, the scalar field obeys the differential equation (2.70) and has a proper dynamical character [140].
Another aspect of Chern-Simons gravity, present in both the dynamical and non-dynamical versions, is that the C-tensor contains third order derivatives of the metric. This implies that the theory is fully consistent only in an effective field theory approach, namely in the limit of small coupling $|\alpha^2/\beta| << 1$. Actually, in a perturbative framework at linear order higher derivatives play no role because they are multiplied by derivatives of θ, thus forbidding terms linear in the metric perturbation and its derivatives (if the scalar field is expanded around a constant value). Moreover, they disappear when restricting to specific solutions, such as Schwarzschild or FLRW spacetimes. However, third order derivatives may cause issues in more general settings, e.g. when discussing the initial value formulation of the theory, where one is forced to resort to the effective field theory framework [193].
Until now we considered the purely metric formulation of Chern-Simons gravity, which is the most studied one in literature, while its alternative metric-affine version only received little attention [194]. In [195, 196, 197, 198, 199], the first-order Chern-Simons theory has been discussed within the Cartan formalism, with a focus mainly on theoretical aspects, while the derivation of observable effects has received little attention. The most straightforward generalization of Chern-Simons gravity to the metric-affine framework would consist in the following theory

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left(\mathscr{R} + \frac{\alpha}{8}\theta(x)\varepsilon^{\mu\nu\rho\sigma}\mathscr{R}^{\alpha}{}_{\beta\mu\nu}\mathscr{R}^{\beta}{}_{\alpha\rho\sigma} - \frac{\beta}{2}\nabla_\mu\theta\nabla^\mu\theta - \beta V(\theta)\right), \tag{2.72}$$

where the independent connection is asymmetric and not metric compatible. However, as we will see in chapter II, this is not the only possibility nor the most correct one.

Part II

Holst, Nieh-Yan and Chern-Simons terms: projective-invariant generalization

In chapter 1, we outlined the features of the most generic pseudo-Riemannian spacetimes, showing how they allow to consider the connection as a truly independent field, without imposing any restriction on it. The importance of projective invariance in metric-affine theories has also been discussed and motivated. In this regard, all the topological terms and related gravitational models introduced in the previous chapters lack in generality, in one way or another.
Indeed, they are considered in literature only by imposing restrictions on the connection (notably metric compatibility) and their projective invariance has never been discussed before. This is not surprising since these two aspects are somehow related. Indeed, it is actually inaccurate to analyse projective invariance in presence of torsion (or nonmetricity) alone, since even starting with vanishing nonmetricity, one could generate it via a projective transformation.
Specifically, the Holst and Nieh-Yan terms were originally introduced in the limited context of Einstein-Cartan spacetimes, that is with vanishing nonmetricity. Neglecting the presence of nonmetricity amounts to select a specific subclass of manifolds belonging to the general metric-affine ones. This choice may seem arbitrary, especially considering the role of the full metric-affine group in the gauge theory formulation of gravity. By extending the Holst and Nieh-Yan terms to the presence of nonmetricity as well, the correspondent modifications of General Relativity can be framed in the context of the most general metric-affine framework. Moreover, nonmetricity is finding interesting applications in teleparallel formulations of gravity (Symmetric Teleparallel Gravity) [200], especially in cosmological [201] and black hole settings [202].
The inclusion of nonmetricity is not necessarily straightforward and some aspects deserve special care. In particular, the on half-shell vanishing and topological character are two fundamental properties of the terms we are considering and must be preserved when allowing for a non-vanishing nonmetricity tensor. This may either be automatically assured or one might need to modify the definition of the term in order to restore the spoilt property.
Regarding the Chern-Simons term instead, although a metric-affine formulation was already discussed to some extent [194, 195, 196, 197, 198, 199], the theory was mainly studied in purely metric case, where the connection is simply given by the Levi-Civita one and there is no non-trivial affine structure. However, the metric-affine version of the Chern-Simons term should be considered closer to a gauge theory than the purely metric one [49, 203] since the Chern-Simons term is constructed using the connection of the corresponding gauge field and it is rather surprising that the literature on Chern-Simons modified gravity has mainly focused on the study of its metric version, while the metric-affine formulation has received only timid attention.
Moreover, a discussion on projective invariance is missing, both for the Chern-Simons and the Holst and Nieh-Yan terms. Given its relation to the dynamical

stability of metric-affine theories [34] it seems of paramount importance to address this topic. We will be mainly interested in two kind of instabilities (see [33] for a comprehensive overview of instabilities in modified gravity models).

The first one is known as ghost instability. Its simplest manifestation affects scalar degrees of freedom and consists of an exponential divergence of the scalar field perturbation due to complex frequency modes in the solution of the linearized field equation on some given background. Moreover, it can be shown that the Hamiltonian has a linear dependence on momentum. This in turn implies that the Hamiltonian is unbounded from below and the system has not a stable ground state. Another possible source of pathologies, known as Ostrogradski instabilities, is the presence of derivatives of order higher than second in the field equations. Also in this case, the Hamiltonian is unbounded from below and the theory is unstable.

Ghosts can be easily spotted by inspecting the action, where a wrong sign[4] of the scalar field's kinetic term allows for negative values of the frequency's radicand. Although higher-order derivatives in the field equations may seem equally easy to detect, they may be hidden by the presence of auxiliary fields. In simple terms, one can always introduce a variable defined as the derivative of a field thus reducing the order of the differential equations, giving the deceiving impression that no higher-order derivatives are present. However, the physically relevant scenario is always obtained on-shell, where the solution for such auxiliary fields must be taken into account.

This aspect is crucial when metric-affine theories of gravity are considered. Indeed, the affine sector usually contains auxiliary variables. In particular, on-half shell torsion and nonmetricity are often expressed in terms of the other fields and their derivatives (metric tensor and scalar fields, in the models relevant to the present discussion). This feature has often led to the misleading conclusion that first order formulations of gravitational models are automatically deprived of Ostrogradski instabilities, but as we will see this is not necessarily the case.

This chapter contains part of the original work of this thesis. After briefly discussing the simplest case of the Holst term, we move to the less trivial Nieh-Yan and Chern-Simons ones, providing a generalized version of the terms in which topologicity and projective symmetry are ensured. Then, the Holst and Nieh-Yan terms are considered in Palatini $f(R)$ models featuring an Immirzi field. The resulting gravitational models are a generalization of some models presented in section 2.2.1 which allows to unify some of them up to a rescaling of the Immirzi field. We provide a detailed discussion on the stability of the models, their reduction to previously known theories and the dynamical equivalence with scalar-tensor

[4]With the conventions adopted in this thesis the kinetic term must have a minus sign to avoid ghosts.

theories and with the experimental compatible subclass of DHOST theories.
Next, we focus on the metric-affine version of the Chern-Simons term. After proposing its topological and projective-invariant generalization, we consider the simplest non-trivial model with and without a kinetic term for the scalar field in the action. We derive the field equations at the nonperturbative level and identify which affine components may be responsible for the arising of third order derivatives, providing also recipes to guarantee their absence. Finally, we compute a linearized solution for the connection in terms of the scalar field and metric perturbation which allows to recast the theory at the perturbative level as an effective scalar tensor theory. The outcome allows to determine the dynamical character of the scalar field in relation to the presence or absence of its kinetic term in the action.

Chapter 3

Holst and Nieh-Yan terms in presence of nonmetricity

3.1 The Holst case

Let us start with the simplest case, namely the Holst term. In presence of nonmetricity its definition is formally equivalent to the one provided in section 2.2.1, i.e.

$$\mathscr{H} \equiv -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\mathscr{R}_{\mu\nu\rho\sigma}. \tag{3.1}$$

However, now the Riemann tensor is built with the most general affine connection which is neither symmetric nor metric-compatible. As a result, the metric field equations obtained from the Holst action are unchanged while the connection field equation now reads

$$\begin{aligned} &\frac{1}{2}g^{\mu\nu}Q_\lambda + \left(P^\mu - \frac{1}{2}Q^\mu\right)\delta^\nu_\lambda - Q_\lambda{}^{\mu\nu} + g^{\mu\nu}T_\lambda - \delta^\nu_\lambda T^\mu + T^{\nu\mu}{}_\lambda = \\ &= -\gamma_0\left[\varepsilon^{\alpha\rho\mu\nu}Q_{\rho\lambda\alpha} + \varepsilon_\lambda{}^{\rho\mu\nu}T_\rho + \frac{1}{2}\varepsilon_\lambda{}^{\mu\rho\sigma}T^\nu{}_{\rho\sigma}\right]. \end{aligned} \tag{3.2}$$

It is easy to show that the Holst term as defined above and the field equations are invariant under projective transformations. Indeed, using (1.52) one has

$$\mathscr{H} \to \tilde{\mathscr{H}} = -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\left(\mathscr{R}_{\mu\nu\rho\sigma} - 2g_{\mu\nu}\partial_{[\rho}\xi_{\sigma]}\right) = -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\mathscr{R}_{\mu\nu\rho\sigma} = \mathscr{H}. \tag{3.3}$$

So that projective symmetry is already included in the model and no further generalizations are needed. We would also like to check that the on half-shell vanishing of the Holst term is not spoilt by nonmetricity. To this aim we can contract the connection equation with δ^λ_ν, $g_{\mu\nu}$ and $\varepsilon_{\alpha\nu\mu}{}^\lambda$, yielding

$$3P_\mu - \frac{3}{2}Q_\mu - 2T_\mu = \frac{\gamma_0}{2}S_\mu, \tag{3.4}$$

$$P_\mu + \frac{1}{2}Q_\mu + 2T_\mu = -\frac{\gamma_0}{2}S_\mu, \tag{3.5}$$

$$S_\mu = \gamma_0\left(6P_\mu - 2T_\mu\right). \tag{3.6}$$

Now, using a projective transformation, which leaves the above system of equations invariant, we can impose $T_\mu = 0$. Then, the above equations imply

$$S_\mu = 6\gamma_0 P_\mu, \tag{3.7}$$

$$Q_\mu = 4P_\mu, \tag{3.8}$$

$$(\gamma_0^2 + 1)P_\mu = 0. \tag{3.9}$$

For $\gamma_0 \neq \pm i$, one has $S_\mu = Q_\mu = P_\mu = 0$ and the equation for the tensorial components becomes

$$q_{\nu\mu\lambda} - \Omega_{\lambda\mu\nu} = \gamma_0\left(\varepsilon_{\mu\nu\alpha\rho}\Omega^{\alpha}{}_{\lambda}{}^{\rho} - \frac{1}{2}\varepsilon_{\lambda\mu\alpha\rho}q_\nu{}^{\alpha\rho}\right), \tag{3.10}$$

which only has the trivial solution $q_{\mu\nu\rho} = \Omega_{\mu\nu\rho} = 0$, as in the Einstein-Cartan case. Therefore, the inclusion of the Holst term does not allow for non-trivial affine structures.
Moreover, the Holst term is still vanishing on half-shell since $\Gamma^\mu{}_{\nu\rho} = \bar{\Gamma}^\mu{}_{\nu\rho}$ still implies

$$\mathscr{H} = -\frac{\gamma_0}{2}\varepsilon^{\mu\nu\rho\sigma}\mathscr{R}_{\mu\nu\rho\sigma} = -\frac{\gamma_0}{2}\varepsilon^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = 0, \tag{3.11}$$

which is again vanishing by virtue of the algebraic Bianchi identities. We therefore reach the same result as in the Einstein-Cartan case, i.e. one recovers the theory of General Relativity with the usual Levi-Civita connection.

3.2 The generalized Nieh-Yan term

Things are more complicated when the Nieh-Yan term is considered. Its straightforward extension to generic metric-affine spacetimes would be formally equivalent to (2.45), i.e.

$$\mathscr{N}\mathscr{Y} \equiv \frac{\gamma_0}{2}\varepsilon^{\mu\nu\rho\sigma}\left(\frac{1}{2}T^\lambda{}_{\mu\nu}T_{\lambda\rho\sigma} - \mathscr{R}_{\mu\nu\rho\sigma}\right), \tag{3.12}$$

but with the Riemann tensor containing both torsion and nonmetricity contributions. Its decomposition in terms of irreducible components would now read

$$\mathcal{N}\mathcal{Y} = -\frac{1}{2}\bar{\nabla}_\mu S^\mu + \frac{1}{6}\left(P^\mu - Q^\mu\right) S_\mu - \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} q^{\lambda\mu\nu} \Omega^{\rho}{}_{\lambda}{}^{\sigma}, \tag{3.13}$$

from which we immediately see that the presence of nonmetricity spoils the topological character of the Nieh-Yan term. In literature this feature could always be neglected by simply disregarding nonmetricity from the very beginning (see [204, 168, 205, 206, 61, 169, 172, 171]).

The other feature we want to ensure is the projective symmetry. We already observed that the Holst term is invariant, so that any violation should come from the torsion squared term. Indeed, one has that

$$\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma} T^{\lambda}{}_{\mu\nu} T_{\lambda\rho\sigma} \to \frac{1}{4}\varepsilon^{\mu\nu\rho\sigma} \tilde{T}^{\lambda}{}_{\mu\nu} \tilde{T}_{\lambda\rho\sigma} = \frac{1}{4}\varepsilon^{\mu\nu\rho\sigma} T^{\lambda}{}_{\mu\nu} T_{\lambda\rho\sigma} - S_\mu \xi^\mu, \tag{3.14}$$

implying the breaking of projective symmetry. Note that the symmetry is violated even if only special projective transformations are considered, i.e. even if $\xi_\mu = \partial_\mu \lambda$ for some scalar field λ. Contrary to the spoiling of the topological character, this result is not related to the presence of nonmetricity, so that the original Nieh-Yan term violates projective symmetry already in the context of Einstein-Cartan theory.

We now want to propose a generalization of the Nieh-Yan term, in which additional contributions are included in its definition acting as counterterms to restore both topologicity and projective invariance. The unwanted terms in (3.13) suggest that a mixed term featuring both torsion and nonmetricity is needed to recover topologicty. Indeed, the last two terms in (3.13) can be written in the following way

$$\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} T^{\lambda}{}_{\mu\nu} Q_{\rho\sigma\lambda} = -\frac{1}{6}\left(P^\mu - Q^\mu\right) S_\mu + \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} q^{\lambda\mu\nu} \Omega^{\rho}{}_{\lambda}{}^{\sigma}, \tag{3.15}$$

so that we may consider the inclusion of the latter to take care of topologicity. Remarkably, under projective transformations one also have

$$\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} T^{\lambda}{}_{\mu\nu} Q_{\rho\sigma\lambda} \to \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} \tilde{T}^{\lambda}{}_{\mu\nu} \tilde{Q}_{\rho\sigma\lambda} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} T^{\lambda}{}_{\mu\nu} Q_{\rho\sigma\lambda} + S_\mu \xi^\mu, \tag{3.16}$$

which precisely cancel the contribution coming from (3.14). Therefore, a generalized Nieh-Yan term defined as

$$\mathcal{N}\mathcal{Y}^* = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\left(\frac{1}{2} T^{\lambda}{}_{\mu\nu} T_{\lambda\rho\sigma} + T^{\lambda}{}_{\mu\nu} Q_{\rho\sigma\lambda} - \mathcal{R}_{\mu\nu\rho\sigma}\right), \tag{3.17}$$

would be a topological term and it would preserve projective symmetry as well. However, it turns out that projective symmetry and topologicity are not necessarily related to each other and they can be implemented independently. To see this, let us instead define the generalized Nieh-Yan term as

$$\boxed{\mathcal{N}\mathcal{Y}^* \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\left(\frac{\lambda_1}{2}T^{\lambda}{}_{\mu\nu}T_{\lambda\rho\sigma} + \lambda_2 T^{\lambda}{}_{\mu\nu}Q_{\rho\sigma\lambda} - \mathcal{R}_{\mu\nu\rho\sigma}\right),} \tag{3.18}$$

where we introduced the real parameters λ_1 and λ_2. The above expression can be written as

$$\mathcal{N}\mathcal{Y}^* = -\frac{1}{2}\bar{\nabla}_\mu S^\mu + \frac{(\lambda_1 - 1)}{4}\varepsilon^{\mu\nu\rho\sigma}T^{\lambda}{}_{\mu\nu}T_{\lambda\rho\sigma} + \frac{(\lambda_2 - 1)}{2}\varepsilon^{\mu\nu\rho\sigma}T^{\lambda}{}_{\mu\nu}Q_{\rho\sigma\lambda}, \tag{3.19}$$

and we have that a projective transformation yields

$$\mathcal{N}\mathcal{Y}^* \to \mathcal{N}\mathcal{Y}^* - (\lambda_1 - \lambda_2)\xi^\mu S_\mu. \tag{3.20}$$

Hence, for $\lambda_1 = \lambda_2 = \lambda$, the newly defined quantity $\mathcal{N}\mathcal{Y}^*$ is projective invariant, despite topologicity being still violated. If in addition we set $\lambda = 1$, then also the topological character is recovered. Bottom line, one can have projective symmetry without topologicity but the latter always implies the former. Finally, let us note that this generalized Nieh-Yan term allows to recover all the already known terms, as the Holst ($\lambda_1 = \lambda_2 = 0$) or the standard Nieh-Yan ($\lambda_1 = 1, \lambda_2 = 0$) terms.

$\mathcal{N}\mathcal{Y}^*$	λ_1	λ_2	λ_3
Projective symmetry	λ_2	$\mathbb{R}\backslash\{0\}$	$\mathbb{R}$
Projective symmetry and topologicity	1	1	0

Table 3.1: Values of the parameters featuring the generalized Nieh-Yan term (including the quadratic nonmetricity term in (3.21)) allowing just projective symmetry or projective symmetry and topologicity simultaneously.

Let us end this section discussing a further possible extension which consists in including a term quadratic in nonmetricity and considering the quantity

$$\mathcal{N}\mathcal{Y}^* + \lambda_3\varepsilon^{\mu\nu\rho\sigma}Q_{\mu\nu}{}^{\lambda}Q_{\rho\sigma\lambda}. \tag{3.21}$$

In this case, since the Levi-Civita tensor is actually selecting only the tensorial component of nonmetricity, i.e.

$$\varepsilon^{\mu\nu\rho\sigma}Q_{\mu\nu}{}^{\lambda}Q_{\rho\sigma\lambda} = \lambda_3\varepsilon^{\mu\nu\rho\sigma}\Omega_{\mu\nu}{}^{\lambda}\Omega_{\rho\sigma\lambda}, \tag{3.22}$$

and being the latter invariant, the projective symmetry would be also assured. However, the topological character would be ineluctably lost. Hence, we will only consider expression (3.18) in the following. All possible cases are summarized in Table 3.1.

Chapter 4

Generalized metric-affine Chern-Simons term

Let us consider the generalization of the Chern-Simons term to metric-affine spacetimes already presented in section 2.2.2. In particular, we are interested in the metric-affine version of the Pontryagin density,

$$\mathscr{P} = \varepsilon^{\mu\nu\rho\sigma} \mathscr{R}^{\alpha}{}_{\beta\mu\nu} \mathscr{R}^{\beta}{}_{\alpha\rho\sigma}, \tag{4.1}$$

differing from the standard expression (2.63) by the fact that the connection is now the most general metric-affine connection, endowed with both torsion and nonmetricity. It is not difficult to show that also in this case one can write

$$\mathscr{P} = \bar{\nabla}_{\mu} \mathscr{K}^{\mu}, \tag{4.2}$$

where

$$\mathscr{K}^{\mu} = \varepsilon^{\alpha\beta\gamma\delta} \Gamma^{\nu}{}_{\beta\mu} \left(\partial_{\gamma} \Gamma^{\mu}{}_{\delta\nu} + \frac{2}{3} \Gamma^{\mu}{}_{\gamma\lambda} \Gamma^{\lambda}{}_{\delta\nu} \right), \tag{4.3}$$

so that the topological character of the Chern-Simons term is preserved when passing to the metric-affine framework. On the other hand, applying a projective transformation yields

$$\mathscr{P} \to \widetilde{\mathscr{P}} = \mathscr{P} - \varepsilon^{\mu\nu\rho\sigma} \left(4\hat{\mathscr{R}}_{\mu\nu} \bar{\nabla}_{\rho} \xi_{\sigma} - 16 \bar{\nabla}_{\mu} \xi_{\nu} \bar{\nabla}_{\rho} \xi_{\sigma} \right). \tag{4.4}$$

The resulting breaking of projective symmetry is somehow weakened with respect to the Nieh-Yan case. First, if one restricts to special projective transformations then the Pontryagin density is invariant. Indeed, setting $\xi_{\mu} = \partial_{\mu} \lambda$, all additional terms on the right hand side of (4.4) vanish because of the contractions with the Levi-Civita tensor. Moreover, even considering the most general scenario, we can rewrite the parity breaking terms as a boundary term. This can be seen considering that the homotetic curvature can be written in terms of the Weyl vector as

$$\hat{\mathscr{R}}_{\mu\nu} = \partial_{[\mu}Q_{\nu]}. \tag{4.5}$$

The transformation property (4.4) can then be rewritten as

$$\mathscr{P} \to \widetilde{\mathscr{P}} = \mathscr{P} + \bar{\nabla}_{\mu}\mathscr{B}^{\mu}, \tag{4.6}$$

where

$$\mathscr{B}^{\mu} \equiv \varepsilon^{\mu\nu\rho\sigma}\left(16\xi_{\nu} - 4Q_{\nu}\right)\bar{\nabla}_{\rho}\xi_{\sigma}. \tag{4.7}$$

Therefore, the Chern-Simons term is invariant up to a boundary term. This is however not negligible when the coupling θ is promoted to a pseudo-scalar field, as in (2.72).

From (4.4) is clear that terms involving the homotetic curvature must be included to recover projective invariance. Let us consider then the following general expression

$$\mathscr{C}\mathscr{S}^{*} \equiv \epsilon^{\mu\nu\rho\sigma}\left(\mathscr{R}^{\alpha}{}_{\beta\mu\nu}\mathscr{R}^{\beta}{}_{\alpha\rho\sigma} + \lambda_1\,\mathscr{R}_{\mu\nu}\mathscr{R}_{\rho\sigma} + \lambda_2\,\hat{\mathscr{R}}_{\mu\nu}\hat{\mathscr{R}}_{\rho\sigma} + \lambda_3\,\mathscr{R}_{\mu\nu}\hat{\mathscr{R}}_{\rho\sigma}\right), \tag{4.8}$$

where λ_1, λ_2 and λ_3 are arbitrary real parameters and we included all possible quadratic combinations of Ricci and homotetic curvature terms. Now, acting with a projective transformation gives

$$\begin{aligned}\mathscr{C}\mathscr{S}^{*} \to \widetilde{\mathscr{C}\mathscr{S}}^{*} = \mathscr{C}\mathscr{S}^{*} - 2\varepsilon^{\mu\nu\rho\sigma}\Big[&(2 + 8\lambda_2 + \lambda_3)\hat{\mathscr{R}}_{\rho\sigma}\bar{\nabla}_{\mu}\xi_{\nu} + 2(\lambda_1 + 2\lambda_3)\mathscr{R}_{\mu\nu}\bar{\nabla}_{\rho}\xi_{\sigma}\\ &-2(4 + \lambda_1 + 16\lambda_2 + 4\lambda_3)\bar{\nabla}_{\mu}\xi_{\nu}\bar{\nabla}_{\rho}\xi_{\sigma}\Big],\end{aligned} \tag{4.9}$$

so that we need to impose

$$2 + 8\lambda_2 + \lambda_3 = 0, \tag{4.10}$$

$$\lambda_1 + 2\lambda_3 = 0, \tag{4.11}$$

$$4 + \lambda_1 + 16\lambda_2 + 4\lambda_3 = 0. \tag{4.12}$$

One of the above condition is redundant, while the others give two parameters in terms of the third one:

$$\lambda_2 = \frac{\lambda_1 - 4}{16}, \tag{4.13}$$

$$\lambda_3 = -\frac{\lambda_1}{2}. \tag{4.14}$$

Therefore, there is a one parameter family of projectively invariant Chern-Simons terms.

Regarding topologicity instead, we can gain some insight by recalling (4.5), which implies that the term quadratic in the homotetic curvature can be written as a total divergence as

$$\varepsilon^{\mu\nu\rho\sigma}\hat{\mathscr{R}}_{\mu\nu}\hat{\mathscr{R}}_{\rho\sigma} = \bar{\nabla}_{\mu}\left(\varepsilon^{\mu\nu\rho\sigma}Q_{\nu}\bar{\nabla}_{\rho}Q_{\sigma}\right). \tag{4.15}$$

The same is not true for terms involving the Ricci tensor, which cannot be recast in the same form. Therefore, in order to impose topologicity, the parameters λ_1 and λ_3 must vanish, while λ_2 remains arbitrary as long as only the topologicity is imposed.
Requiring both topologicity and projective symmetry yields

$$\lambda_1 = \lambda_3 = 0, \tag{4.16}$$

$$\lambda_2 = -\frac{1}{4}, \tag{4.17}$$

and the respective generalized Chern-Simons term:

$$\boxed{\mathscr{C}\mathscr{S}^{*} \equiv \epsilon^{\mu\nu\rho\sigma}\left(\mathscr{R}^{\alpha}{}_{\beta\mu\nu}\mathscr{R}^{\beta}{}_{\alpha\rho\sigma} - \frac{1}{4}\hat{\mathscr{R}}_{\mu\nu}\hat{\mathscr{R}}_{\rho\sigma}\right).} \tag{4.18}$$

Summing up, the situation looks more involved than the Nieh-Yan one. Now topologicity and projective invariance are truly independent and one can have the former ($\lambda_1 = \lambda_3 = 0$) still violating the latter ($\lambda_2 \neq -1/4$), or vice versa, imposing (4.13) and (4.14), but requiring $\lambda_1 \neq 0$ and/or $\lambda_3 \neq 0$ (see Table 4.1).
This concludes the projective-invariant extension to metric-affine spacetimes of the parity violating topological terms considered. We will now implement these terms in specific gravitational models and discuss their properties.

$\mathscr{C}\mathscr{S}^{*}$	λ_1	λ_2	λ_3
Topologicity	0	$\mathbb{R}\backslash\left\{-\frac{1}{4}\right\}$	0
Projective symmetry	$\mathbb{R}\backslash\{0\}$	$\frac{\lambda_1-4}{16}$	$-\frac{\lambda_1}{2}$
Both	0	$-\frac{1}{4}$	0

Table 4.1: Values of the parameters featuring the generalized Chern-Simons term allowing topologicity, projective symmetry or both properties simultaneously.

Chapter 5

Holst term in $f(R)$ models

We will now consider concrete applications of the gravitational terms presented above. Regarding the Holst and Nieh-Yan terms, their linearity in the curvature tensor makes them relatively easy to deal with. It is therefore possible to consider them in quite general models, that can then be reduced to simpler ones via adequate limits. Hence, we will consider a generalization of the models presented in sections 2.2.1, obtained taking both ideas into account simultaneously, namely introducing the Holst or Nieh-Yan term in Palatini $f(R)$ theory *and* promoting the Immirzi parameter to a dynamical field.
For the time being we will add the Holst term to the argument of the function f, in parallel with (2.50). This choice seems more in line with the original purpose of the Holst term, namely avoiding modifications at the classical level and only recasting the same theory in a way more suitable for canonical quantization. In this case, even including the Holst term in the argument of the function f, the presence of the Immirzi field prevents the theory to be equivalent to Palatini $f(R)$ gravity. However, if the latter has to be recovered in the limit $\gamma(x) \to \gamma_0$, then the right action to be considered is:

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[f(\mathscr{R} + \gamma \mathscr{H}) - W(\gamma) \right], \tag{5.1}$$

where a potential term for the Immirzi field has been included as well. It is again convenient to formulate the theory in the Jordan frame. The method is analogous to the one illustrated in section 2.1.1 and 2.2.1 and yields the Jordan frame action

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[\phi(\mathscr{R} + \gamma \mathscr{H}) - V(\phi) - W(\gamma) \right], \tag{5.2}$$

where

$$V(\phi) = \phi \chi(\phi) - f(\chi(\phi)). \tag{5.3}$$

We can now reduce the theory to its effective scalar tensor counterpart, by working on half-shell. To proceed, we compute the field equations for the connection,

$$
\begin{aligned}
&-\nabla_\lambda\left(\sqrt{-g}\phi g^{\mu\nu}\right)+\delta^\nu_\lambda\nabla_\rho\left(\sqrt{-g}\phi g^{\mu\rho}\right)+\sqrt{-g}\phi\left(g^{\mu\nu}T_\lambda-\delta^\nu{}_\lambda T^\mu+T^{\nu\mu}{}_\lambda\right)\\
&-\nabla_\rho\left(\sqrt{-g}\gamma\phi\varepsilon_\lambda{}^{\rho\mu\nu}\right)+\sqrt{-g}\gamma\phi\left[\varepsilon_\lambda{}^{\rho\mu\nu}T_\rho+\frac{1}{2}\varepsilon_\lambda{}^{\mu\rho\sigma}T^\nu{}_{\rho\sigma}\right]=0,
\end{aligned}
\tag{5.4}
$$

and we extract the equations for the vector components:

$$
\begin{cases}
3\left(\frac{1}{2}Q_\mu-P_\mu\right)+2T_\mu+\frac{1}{2}\gamma S_\mu=3\bar\nabla_\mu\ln\phi\\
\left(\frac{1}{2}Q_\mu+P_\mu\right)+2T_\mu+\frac{1}{2}\gamma S_\mu=3\bar\nabla_\mu\ln\phi\\
2\gamma\left(P_\mu-Q_\mu\right)-4\gamma T_\mu+S_\mu=6\bar\nabla_\mu\gamma+6\gamma\bar\nabla_\mu\ln\phi.
\end{cases}
\tag{5.5}
$$

Subtracting the first two we see that the geometry is always characterized by a Weyl configuration, i.e.

$$
Q_\mu=4P_\mu,
\tag{5.6}
$$

so that, both nonmetricity vectors can be set to zero with a projective transformation. Then, the system is solved by

$$
T_\mu=\frac{3}{2\phi}\bar\nabla_\mu\phi+\frac{3\gamma}{2(\gamma^2+1)}\bar\nabla_\mu\gamma,
\tag{5.7}
$$

$$
S_\mu=-\frac{6}{(\gamma^2+1)}\bar\nabla_\mu\gamma.
\tag{5.8}
$$

Substituting back into the connection equation yields the equation for the rank-3 tensor components

$$
\phi\left(\Omega_{\lambda\mu\nu}-q_{\nu\mu\lambda}\right)=\gamma\left(\frac{1}{2}\varepsilon^{\rho\sigma}{}_{\lambda\mu}q_{\nu\rho\sigma}-\varepsilon^{\rho\sigma}{}_{\mu\nu}\Omega_{\rho\sigma\lambda}\right),
\tag{5.9}
$$

which still implies the vanishing of the tensor parts. Substituting back into the action, the effective second order theory is

$$
S=\frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[\phi R+\frac{3}{2\phi}\bar\nabla_\mu\phi\bar\nabla^\mu\phi-\frac{\phi}{2}\bar\nabla_\mu\psi\bar\nabla^\mu\psi-V(\phi)-W(\psi)\right],
\tag{5.10}
$$

where we defined the scalar field ψ as

$$
\psi(x)\equiv\sqrt{3}\sinh^{-1}\gamma(x).
\tag{5.11}
$$

The metric and Immirzi field equations in vacuum reduce to

$$
G_{\mu\nu}=\frac{1}{\phi}\left(\bar\nabla_\mu\bar\nabla_\nu\phi-g_{\mu\nu}\Box\phi\right)-\frac{3}{2\phi^2}K_{\mu\nu}(\phi)+\frac{1}{2}K_{\mu\nu}(\psi)-\frac{1}{2\phi}\left(V(\phi)+W(\psi)\right),
\tag{5.12}
$$

$$
\Box\psi=-\bar\nabla^\mu\psi\bar\nabla_\mu\ln\phi+\frac{1}{\phi}\frac{dW}{d\psi},
\tag{5.13}
$$

where $\Box = \bar{\nabla}_\mu \bar{\nabla}^\mu$ is the d'Alembert operator built from the Levi-Civita connection and

$$K_{\mu\nu}(\cdot) \equiv \bar{\nabla}_\mu(\cdot)\bar{\nabla}_\nu(\cdot) - \frac{1}{2} g_{\mu\nu}\bar{\nabla}^\rho(\cdot)\bar{\nabla}_\rho(\cdot). \tag{5.14}$$

Regarding the scalaron, the structural equation is modified with respect to the standard Palatini $f(R)$ case, and it now reads

$$2V(\phi) - \phi\frac{dV}{d\phi} = -2W(\gamma). \tag{5.15}$$

The main consequence of this is that the scalaron is not forced anymore to the constant configuration (in vacuum) but rather it can acquire a dynamics sourced by a non-trivial behavior of the Immirzi field.
The latter is a well-behaved dynamical degree of freedom and the theory is safe from ghosts. This can be appreciated in the Einstein frame defined by the conformal rescaling $\tilde{g}_{\mu\nu} = \phi\, g_{\mu\nu}$ (see [B103, 207] for details). In the Einstein frame the nonminimal coupling of ϕ with the Ricci scalar is removed along with its kinetic term, and we can just look at the kinetic term for the Immirzi field which takes the standard form

$$-\frac{1}{2}\tilde{g}^{\mu\nu}\bar{\nabla}_\mu\psi\bar{\nabla}_\nu\psi, \tag{5.16}$$

with the correct sign.
Let us end this section comparing these results with the outcome obtained adding the Holst term outside the function f as in

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[f(\mathscr{R}) + \gamma\mathscr{H} - W(\gamma)\right], \tag{5.17}$$

In this case, with similar steps we would obtain the following effective theory

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\Big[\phi R + \frac{3}{2}\frac{\phi}{\phi^2+\gamma^2}\left(\bar{\nabla}_\mu\phi\bar{\nabla}^\mu\phi - \bar{\nabla}_\mu\gamma\bar{\nabla}^\mu\gamma\right) + \frac{3\gamma}{\phi^2+\gamma^2}\bar{\nabla}_\mu\phi\bar{\nabla}^\mu\gamma$$
$$-V(\phi) - W(\gamma)\Big]. \tag{5.18}$$

As for the models with $\gamma = \gamma_0$ discussed in section 2.2.1, the two theories differ. However, the presence of the Immirzi field now allows for a rescaling of the latter that can be used to transform one model into the other. Indeed, if we define a new Immirzi field by

$$\tilde{\gamma} = \phi\gamma, \tag{5.19}$$

it is trivial to show that (5.10) computed in $\gamma = \tilde{\gamma}/\phi$ is equivalent to (5.18) evaluated at $\gamma = \tilde{\gamma}$. Therefore, the framework in which the Immirzi parameter is promoted to a dynamical field offers a possibility to reconcile the two models which is otherwise unavailable for a constant Immirzi parameter. The price to pay is a more complicated form of the potentials which are no more separable.

Chapter 6

Generalized Nieh-Yan term in $f(R)$ models

Let us consider now an extension of Palatini $f(R)$ gravity as a specific implementation of the generalized Nieh-Yan term introduced in section 3.2. Following the same argument that led us to (5.1), we may now consider adding the Nieh-Yan term outside the function f, as in $f(\mathscr{R}) + \mathscr{N}\mathscr{Y}^*$, and promoting the Immirzi parameter to a scalar field, adding its own potential term as well. However, we can actually start from the model

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g}\, f(\mathscr{R}, \mathscr{N}\mathscr{Y}^*), \tag{6.1}$$

with f a general function of two variables. Now, if the following holds

$$\frac{\partial^2 F}{\partial \mathscr{R}^2} \frac{\partial^2 F}{\partial \mathscr{N}\mathscr{Y}^{*2}} - \frac{\partial^2 F}{\partial \mathscr{R} \partial \mathscr{N}\mathscr{Y}^*} \neq 0, \tag{6.2}$$

we can formulate the theory in the Jordan frame extending the procedure adopted in the previous sections to the case of a function with two variables, as it is commonly used in other contexts (see e.g. [208]). Let us introduce the scalar-tensor representation defined by

$$\phi \equiv \frac{\partial F}{\partial \mathscr{R}}, \tag{6.3}$$

$$\gamma \equiv \frac{\partial F}{\partial \mathscr{N}\mathscr{Y}^*}, \tag{6.4}$$

leading to the following action

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[\phi \mathscr{R} + \gamma \mathscr{N}\mathscr{Y}^* - W(\phi, \gamma)\right], \tag{6.5}$$

where the potential is given by

$$W(\phi,\gamma) \equiv \phi\mathscr{R}(\phi,\gamma) + \gamma\mathscr{N}\mathscr{Y}^*(\phi,\gamma) - f(\phi,\gamma). \tag{6.6}$$

In this way, we are able to generate in a natural way an Immirzi scalar field, as one of the scalar degrees of freedom emerging from the Jordan frame representation (6.5). This procedure, moreover, offers a viable mechanism able to produce the interaction term $W(\phi,\gamma)$ instead of introducing it by hand in the action as in (5.1). Now, varying (6.5) with respect to the connection we get

$$\begin{aligned}
&-\nabla_\lambda\left(\sqrt{-g}\phi g^{\mu\nu}\right) + \delta^\nu_\lambda\nabla_\rho\left(\sqrt{-g}\phi g^{\mu\rho}\right) + \sqrt{-g}\phi\left(g^{\mu\nu}T_\lambda - \delta^\nu{}_\lambda T^\mu + T^{\nu\mu}{}_\lambda\right)\\
&-\nabla_\rho\left(\sqrt{-g}\gamma\varepsilon_\lambda{}^{\rho\mu\nu}\right) + \sqrt{-g}\gamma\left[\varepsilon_\lambda{}^{\rho\mu\nu}T_\rho + \frac{1}{2}\varepsilon_\lambda{}^{\mu\rho\sigma}T^\nu{}_{\rho\sigma} - \frac{\lambda_2}{2}\varepsilon_\lambda{}^{\nu\rho\sigma}T^\mu{}_{\rho\sigma}\right.\\
&\left.+\varepsilon^{\mu\nu\rho\sigma}\left(\left(\lambda_1 - \frac{\lambda_2}{2}\right)T_{\lambda\rho\sigma} + \lambda_2\,Q_{\rho\sigma\lambda}\right)\right] = 0,
\end{aligned} \tag{6.7}$$

from which one can extract the equations for the four vector components, i.e.

$$\begin{cases}
(\lambda_1 - \lambda_2)S_\mu = 0\\
Q_\mu - 4P_\mu + \frac{\gamma}{\phi}(\lambda_1 - \lambda_2)S_\mu = 0\\
Q_\mu - P_\mu + 2T_\mu + \frac{\gamma}{2\phi}(1-\lambda_1)S_\mu = 3\bar{\nabla}_\mu \ln\phi\\
(1-\lambda_2)(Q_\mu - P_\mu) + 2(1-\lambda_1)T_\mu - \frac{\phi}{2\gamma}S_\mu = 3\bar{\nabla}_\mu\ln\gamma
\end{cases}. \tag{6.8}$$

From the first two conditions we see that the system is always characterized by Weyl geometry configurations, namely by $Q_\mu = 4P_\mu$. Therefore, we can rewrite the system as

$$\begin{cases}
(\lambda_1 - \lambda_2)S_\mu = 0\\
Q_\mu = 4P_\mu\\
3P_\mu + 2T_\mu + \frac{\gamma}{2\phi}(1-\lambda_1)S_\mu = 3\bar{\nabla}_\mu\ln\phi\\
3(1-\lambda_2)P_\mu + 2(1-\lambda_1)T_\mu - \frac{\phi}{2\gamma}S_\mu = 3\bar{\nabla}_\mu\ln\gamma
\end{cases}. \tag{6.9}$$

Now, before moving on to compute the solution of the system, let us take care of the tensorial components of torsion and nonmetricity. The corresponding equation is obtained from the connection equation using (6.9). After some manipulations we get

$$\begin{aligned}
&\frac{\phi}{\beta}\left(\Omega_{\lambda\mu\nu} - q_{\nu\mu\lambda}\right) = \frac{1}{2}\varepsilon^{\rho\sigma}{}_{\lambda\mu}q_{\nu\rho\sigma} - \frac{\lambda_2}{2}\varepsilon^{\rho\sigma}{}_{\lambda\nu}q_{\mu\rho\sigma} +\\
&+\varepsilon^{\rho\sigma}{}_{\mu\nu}\left(\left(\lambda_1 - \frac{\lambda_2}{2}\right)q_{\lambda\rho\sigma} + (\lambda_2 - 1)\Omega_{\rho\sigma\lambda}\right).
\end{aligned} \tag{6.10}$$

Then, symmetrizing on the indices μ, ν we can solve the nonmetricity 3-rank tensor in terms of the torsional analog as

$$\phi\,\Omega_{\lambda\mu\nu} = \phi\,q_{(\nu\mu)\lambda} + \frac{\beta(1-\lambda_2)}{2}\varepsilon^{\rho\sigma}{}_{\lambda(\mu}q_{\nu)\rho\sigma}. \tag{6.11}$$

The latter, inserted back in (6.10) leads to the trivial solution for the tensor modes $\Omega_{\lambda\mu\nu} = q_{\nu\mu\lambda} = 0$. In the following, therefore, we can only focus on the purely vector modes.

6.1 Projective symmetry breaking case

Going back to (6.9), we see that the structure of the solutions crucially depends on the values of the parameters λ_1 and λ_2. Indeed, when projective invariance is explicitly broken, as it occurs for $\lambda_1 \neq \lambda_2$, we are compelled to set $S_\mu = 0$ and the general solution is displayed by

$$\begin{cases} S_\mu = 0 \\ Q_\mu = 4P_\mu \\ P_\mu = \frac{1}{\lambda_1-\lambda_2}\bar{\nabla}_\mu \ln\beta + \frac{\lambda_1-1}{\lambda_1-\lambda_2}\bar{\nabla}_\mu \ln\phi \\ T_\mu = -\frac{3}{2}\frac{1}{\lambda_1-\lambda_2}\bar{\nabla}_\mu \ln\beta - \frac{3}{2}\frac{\lambda_2-1}{\lambda_1-\lambda_2}\bar{\nabla}_\mu \ln\phi \\ q_{\rho\mu\nu} = \Omega_{\rho\mu\nu} = 0 \end{cases}. \tag{6.12}$$

Let us now consider the expression of $\mathcal{N}\mathcal{Y}^*$ written in terms of its vector components, i.e.

$$\mathcal{N}\mathcal{Y}^* = -\frac{1}{2}\bar{\nabla}_\mu S^\mu - \frac{(1-\lambda_1)}{3}S_\mu T^\mu - \frac{(1-\lambda_2)}{2}S_\mu P^\mu. \tag{6.13}$$

We see that, even if the projective symmetry is broken, the solution $S_\mu = 0$ remarkably implies that the generalized Nieh-Yan term is identically vanishing on half-shell. In other words, the affine structure of the theory is such that terms violating projective invariance are harmless along the dynamics. This constitutes a counter-example to the results derived in [35] in the context of Ricci based gravity theories.
The absence of instabilities can be further appreciated by looking at the effective scalar tensor action stemming from (6.5), when (6.12) are plugged in it. Explicit calculations lead to

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[\phi R + \frac{3}{2\phi}\bar{\nabla}_\mu\phi\bar{\nabla}^\mu\phi - W(\phi,\beta)\right], \tag{6.14}$$

which resembles the form of a Palatini $f(R)$ theory, with a potential depending on two scalar fields. In particular, we see that in this case the equation for the Immirzi field is simply given by

$$\frac{\partial W(\phi,\beta)}{\partial\beta}=0, \tag{6.15}$$

which actually fixes the form of the Immirzi field in terms of the scalaron ϕ, i.e. $\beta=\beta(\phi)$. Then, including also matter contributions, it can be easily verified that the variation of (6.14) with respect to ϕ, combined with the trace of the equation for the metric field, results in the canonical structural equation featuring Palatini $f(R)$ theories [48], i.e.

$$\left[2W(\phi,\beta)-\phi\frac{\partial W(\phi,\beta)}{\partial\phi}\right]\bigg|_{\beta=\beta(\phi)}=\kappa^2 T, \tag{6.16}$$

which shows that the dynamics of the scalaron ϕ is frozen as well, and completely determined by the trace of the stress-energy tensor of matter. Conditions (6.15) and (6.16) then establish that the scalar fields ϕ and β are not truly propagating degrees of freedom, and reduce to constants in vacuum, where the theory is stable and the breaking of projective invariance does not lead to instabilities as in [35].

6.2 Projective invariant case

Let us go back to (6.9) and consider the projective invariant case, i.e. $\lambda_1=\lambda_2\equiv\lambda$. Now, we have at our disposal the projective invariance to get rid of one vector degree, which can be eliminated by properly setting the vector ξ_μ. We can decide, for instance, to set $\xi_\mu=-\frac{1}{2}P_\mu$, in order to deal in (6.9) only with torsion[1]. We obtain then:

$$\begin{cases} Q_\mu=4P_\mu=0\\ S_\mu=\frac{6\beta(1-\lambda)}{\beta^2(1-\lambda)^2+\phi^2}\bar\nabla_\mu\phi-\frac{6\phi}{\beta^2(1-\lambda)^2+\phi^2}\bar\nabla_\mu\beta\\ T_\mu=\frac{3}{2}\frac{\phi}{\beta^2(1-\lambda)^2+\phi^2}\bar\nabla_\mu\phi+\frac{3}{2}\frac{\beta(1-\lambda)}{\beta^2(1-\lambda)^2+\phi^2}\bar\nabla_\mu\beta\\ q_{\rho\mu\nu}=\Omega_{\rho\mu\nu}=0\end{cases}. \tag{6.17}$$

We remark that while in (6.12) the affine structure is strictly fixed, leading to the presence of torsion and nonmetricity, in (6.17), as a matter of fact, we could have

[1] For the sake of clarity we omit the tilde notation for transformed quantities.

chosen $\xi_\mu = \frac{T_\mu}{3}$ and retained the nonmetricity vector P_μ instead of the torsion trace. In this case we would obtain

$$\begin{cases} T_\mu = 0 \\ Q_\mu = 4P_\mu \\ S_\mu = \frac{6\beta(1-\lambda)}{\beta^2(1-\lambda)^2+\phi^2}\bar{\nabla}_\mu\phi - \frac{6\phi}{\beta^2(1-\lambda)^2+\phi^2}\bar{\nabla}_\mu\beta \\ P_\mu = \frac{\phi}{\beta^2(1-\lambda)^2+\phi^2}\bar{\nabla}_\mu\phi + \frac{\beta(1-\lambda)}{\beta^2(1-\lambda)^2+\phi^2}\bar{\nabla}_\mu\beta \\ q_{\rho\mu\nu} = \Omega_{\rho\mu\nu} = 0 \end{cases} . \tag{6.18}$$

Such a flexibility in the specific representation of the theory, however, does not reflect in a dynamical vagueness, and the proper degrees of freedom can be unambiguously identified.
Indeed, both choices lead to the same effective action, which reads

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[\phi R + \frac{3\phi}{2(\beta^2(1-\lambda)^2+\phi^2)}\left(\bar{\nabla}_\mu\phi\bar{\nabla}^\mu\phi - \bar{\nabla}_\mu\beta\bar{\nabla}^\mu\beta + \frac{2\beta(1-\lambda)}{\phi}\bar{\nabla}_\mu\phi\bar{\nabla}^\mu\beta\right) - W(\phi,\beta)\right], \tag{6.19}$$

where the mixing term $\bar{\nabla}_\mu\phi\bar{\nabla}^\mu\beta$ can be always reabsorbed by the transformation[2] $\psi \equiv \beta\phi^{\lambda-1}$, which puts the action in the diagonal form:

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[\phi R + \frac{3}{2\phi}\bar{\nabla}_\mu\phi\bar{\nabla}^\mu\phi - \frac{3\phi}{2}\frac{1}{\phi^{2\lambda}+(1-\lambda)^2\psi^2}\bar{\nabla}_\mu\psi\bar{\nabla}^\mu\psi - V(\phi,\psi)\right], \tag{6.20}$$

where we redefined $V(\phi,\psi) = W(\phi,\psi\phi^{1-\lambda})$. It is clear, therefore, that we expect in general the Immirzi field to be a well-behaved dynamical degree of freedom. To check that, one can perform the conformal transformation already introduced in the previous section, i.e. $\tilde{g}_{\mu\nu} = \phi\, g_{\mu\nu}$, leading to the Einstein frame. As in the Holst case the nonminimal coupling of ϕ with the Ricci scalar and its kinetic term are removed, leaving just the kinetic term for the Immirzi field, which reads:

$$-\frac{3}{2}\frac{\tilde{g}^{\mu\nu}\bar{\nabla}_\mu\psi\bar{\nabla}_\nu\psi}{\phi^{2\lambda}+(1-\lambda)^2\psi^2}. \tag{6.21}$$

Now, since the inequality $\phi^{2\lambda}+(1-\lambda)^2\psi^2 > 0$ holds irrespective of the values of ϕ, ψ and λ, (6.21) has always the correct sign and no ghost instability arises.

[2]We see that in the special case of $\lambda = 1$, when also topologicity is restored, no redefinition for the Immirzi field is required and his kinetic term simply boils down to $-\frac{3}{2\phi}(\nabla\beta)^2$.

6.3 Recovering Einstein-Cartan models

In this section we want to make contact with previous models studied in literature, where the original Nieh-Yan term (eq. (2.45)) has been analyzed in Einstein-Cartan spacetimes, i.e. in absence of nonmetricity. One might expect that the sector of the theory to be considered is the projectively violating one identified by $Q_{\mu\nu\rho} = 0$, $\lambda_1 = 1$ and $\lambda_2 = 0$. However, if we insert these conditions into (6.9), we find

$$\begin{cases} S_\mu = 0 \\ T_\mu = \frac{3}{2\phi}\bar{\nabla}_\mu\phi \\ \bar{\nabla}_\mu\beta = 0, \end{cases} \tag{6.22}$$

which is just a particular subset of solutions for the models violating projective invariance. Now, however, we are compelled to restrict to a constant Immirzi parameter, and by virtue of (6.15) the last one of the above conditions simply implies

$$\partial_\mu\beta = \frac{\partial\beta}{\partial\phi}\partial_\mu\phi(T) = 0, \tag{6.23}$$

which for a generic $T \neq 0$ is satisfied if $\partial\beta/\partial\phi = 0$, that is to say whenever the potential $W(\phi,\beta)$ does not depend on β. This requirement eliminates the Immirzi parameter from (6.14) and fully restores the equivalence of the theory with the Palatini $f(R)$ gravity.

Instead, if we want to properly replicate the Einstein-Cartan structure of [204, 168, 205, 206, 61, 169, 172, 171], it is necessary to implement the condition of vanishing nonmetricity directly into the action with a Lagrange multiplier, i.e.

$$S_g^{EC} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[F(\mathscr{R},\mathscr{N}\mathscr{Y}^*) + l^{\rho\mu\nu}Q_{\rho\mu\nu}\right], \tag{6.24}$$

where $l^{\rho\mu\nu} = l^{\rho\nu\mu}$. In so doing we are not forcing $S_\mu = 0$ because of the appearance of traces of $l^{\rho\mu\nu}$ in (6.9). Indeed, varying with respect to the Lagrange multiplier we get the condition of vanishing nonmetricity, $Q_{\rho\mu\nu} = 0$, which, substituted into (6.9) yields

$$\begin{cases} q_\mu = \frac{\beta}{2}(\lambda_1 - \lambda_2)S_\mu \\ p_\mu = -\frac{1}{2}q_\mu \\ S_\mu = \frac{6\beta(1-\lambda_1)}{\beta^2(1-\lambda_1)^2+\phi^2}\bar{\nabla}_\mu\phi - \frac{6\phi}{\beta^2(1-\lambda_1)^2+\phi^2}\bar{\nabla}_\mu\beta \\ T_\mu = \frac{3}{2}\frac{\phi}{\beta^2(1-\lambda_1)^2+\phi^2}\bar{\nabla}_\mu\phi + \frac{3}{2}\frac{\beta(1-\lambda_1)}{\beta^2(1-\lambda_1)^2+\phi^2}\bar{\nabla}_\mu\beta \end{cases}, \tag{6.25}$$

where the traces $q_\mu \equiv l_\mu{}^\rho{}_\rho$ and $p_\mu \equiv l^\rho{}_{\mu\rho}$ are completely solved in terms of the axial vector S_μ. Then, results of [204, 168, 205, 206, 61, 169, 172, 171] are simply obtained[3] by setting $\lambda_1 = 1$.

6.4 Dynamical Immirzi models and DHOST equivalence

In this section we will focus on the projective invariant case, in which the Immirzi field retains its dynamical character and it is not frozen by the dynamics. For $\lambda_1 = \lambda_2 = \lambda$, the metric and scalar fields equations read

$$\bar{G}_{\mu\nu} = \frac{\kappa^2}{\phi} T_{\mu\nu} + \frac{1}{\phi}\left(\bar{\nabla}_\mu \bar{\nabla}_\nu - g_{\mu\nu}\Box\right)\phi - \frac{3}{2\phi^2}\bar{\nabla}_\mu\phi\bar{\nabla}_\nu\phi + \frac{3}{2}\frac{\bar{\nabla}_\mu\psi\bar{\nabla}_\nu\psi}{\phi^{2\lambda} + (1-\lambda)^2\psi^2}$$
$$+ \frac{1}{2}g_{\mu\nu}\left(\frac{3(\bar{\nabla}\phi)^2}{2\phi^2} - \frac{3}{2}\frac{(\bar{\nabla}\psi)^2}{\phi^{2\lambda} + (1-\lambda)^2\psi^2} - \frac{V(\phi,\psi)}{\phi}\right), \tag{6.26}$$

$$2V(\phi,\psi) - \phi\frac{\partial V(\phi,\psi)}{\partial\phi} + \frac{3\lambda\phi^{2\lambda+1}}{\left(\phi^{2\lambda} + (1-\lambda)^2\psi^2\right)^2}(\bar{\nabla}\psi)^2 = \kappa^2 T, \tag{6.27}$$

$$\Box\psi - \frac{(1-\lambda)^2\psi}{\phi^{2\lambda} + (1-\lambda)^2\psi^2}(\bar{\nabla}\psi)^2 +$$
$$\left(1 - \frac{2\lambda\phi^{2\lambda}}{\phi^{2\lambda} + (1-\lambda)^2\psi^2}\right)\bar{\nabla}_\mu \ln\phi\bar{\nabla}^\mu\psi = \frac{\partial V(\phi,\psi)}{3\partial\psi}, \tag{6.28}$$

where (6.27) is obtained in analogy with the structural equation (2.23), featuring standard Palatini $f(R)$ gravity. From that equation we see that the scalaron ϕ can be algebraically solved in terms of the Immirzi field and its kinetic term $X \equiv (\bar{\nabla}\psi)^2$, i.e.

$$\phi = \phi(\psi, X, T), \tag{6.29}$$

so that we are left with an only propagating degree of freedom, the Immirzi field. Moreover, equation (6.29) suggests an intriguing analogy with the so-called Degenerate Higher-Order Scalar-Tensor (DHOST) theories [44, 209], where higher-order derivatives of the scalar field in the action do not actually lead to dynamical instabilities, by virtue of some degeneracy conditions on the kinetic matrix. An important subclass of DHOST theories is the one in agreement with the absence

[3] Obviously, the parameter λ_2 does not appear at all in the expressions for the vectors, since it is related to nonmetricity in the expression of $\mathcal{N}\mathcal{Y}^*$.

of graviton decay and the experimental constraint on the speed of gravitational waves [210], described by the action

$$S_{DHOST} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left[F_0 + F_1\Box\varphi + F_2 R + \frac{6F_{2X}^2}{F_2}\varphi^\mu\varphi^\nu\varphi_{\mu\lambda}\varphi^\lambda{}_\nu\right], \tag{6.30}$$

where $\varphi_\mu \equiv \bar{\nabla}_\mu\varphi$, $\varphi_{\mu\nu} \equiv \bar{\nabla}_\mu\bar{\nabla}_\nu\varphi$, and F_0, F_1, F_2 are functions of the kinetic term $X \equiv \varphi^\mu\varphi_\mu$. In this regard, then, consider the Nieh-Yan model in vacuum ($T = 0$), identified by $\lambda = 1$ and the condition $\partial V/\partial\psi = 0$. In this case equation (6.29) simply reads $\phi = \phi(X)$. Direct substitution of the latter into (6.20) yields the equivalence at the Lagrangian level with (6.30), upon identification of the DHOST scalar field φ with the Immirzi field and considering the following functional choices:

$$F_0 = -V(\phi(X)) - \frac{3X}{2\phi(X)}, \tag{6.31a}$$

$$F_1 = 0, \tag{6.31b}$$

$$F_2 = \phi(X). \tag{6.31c}$$

In addition, the requirement that the field equations stemming from (6.30) and (6.20) be equivalent, leads to the additional condition

$$X - \frac{\phi(X)}{\phi_X(X)} \neq 0, \tag{6.32}$$

which rules out the linear case $\phi(X) \propto X$. Moreover, we note that for the subclass (6.30), the degeneracy condition preventing the arising of Ostrogradsky instabilities simply reads $F_2(X) \neq 0$, which, in our case, is consistent with the requirement $\phi \neq 0$, generally holding in $f(R)$ frameworks.
Finally, as for the Holst term, also in Nieh-Yan models the Immirzi field allows for a close analogy between the theory just presented and the alternative $f(\mathscr{R} + \gamma\mathscr{N}\mathscr{Y}^*)$ possibility. This time, one has to consider the reparametrization

$$\tilde{\gamma} = \frac{\gamma}{\phi}, \tag{6.33}$$

which, applied to the effective scalar tensor action of the $f(\mathscr{R}) + \gamma\mathscr{N}\mathscr{Y}$ model, allows to recover the equations stemming from the $f(\mathscr{R} + \gamma\mathscr{N}\mathscr{Y})$ theory.

Chapter 7

Projective invariant metric-affine Chern-Simons gravity

In contrast to the Holst and Nieh-Yan terms, the Pontryagin density featuring the Chern-Simons term is quadratic in the curvature tensor. This constitutes the main difficulty in dealing with Chern-Simons gravity, especially in the metric-affine framework where the presence of affine contributions makes the theory even more involved. For this reason, we will consider a simpler implementation compared to the previous ones, without adopting any $f(R)$-like extensions. The simplest model featuring the generalization of Chern-Simons term presented in section 4 is obtained by including it in the Hilbert-Palatini first order action of General Relativity as

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left(\mathscr{R} + \frac{\alpha}{8}\theta(x)\varepsilon^{\mu\nu\rho\sigma}\left(\mathscr{R}^{\alpha}{}_{\beta\mu\nu}\mathscr{R}^{\beta}{}_{\alpha\rho\sigma} - \frac{1}{4}\hat{\mathscr{R}}_{\mu\nu}\hat{\mathscr{R}}_{\rho\sigma}\right) - \frac{\beta}{2}\nabla_{\mu}\theta\nabla^{\mu}\theta\right). \tag{7.1}$$

Given the topological character of the additional term, the coupling can be promoted to a (pseudo) scalar field $\theta(x)$ in order to include non-trivial modifications in the dynamics. In analogy with the metric case, two real parameters α and β feature the action. Dimensional analysis of (7.1) suggests that their dimension together with the one of the scalar field must be given by the following powers of a length:

$$[\alpha] = L^{A}, \qquad [\theta] = L^{2-A}, \qquad [\beta] = L^{2A-4}, \tag{7.2}$$

with A an arbitrary constant. Note that one of the two parameters is redundant. For instance, α can always be reabsorbed in the definition of θ by performing the rescaling $\theta \to \theta/\alpha$ and $\beta \to \alpha^2\beta$. This amounts to set $A = 0$ and work with $[\theta] = L^2$ and $[\beta] = L^{-4}$, as we will do in most of chapter 10.
In the action (7.1) we also included a standard kinetic term for the pseudoscalar field. However we want to stress that this is not strictly necessary. We remind

the reader to section 2.2.2 for a discussion on the difference between dynamical and non-dynamical Chern-Simons gravity in the metric formalism, where only the former is considered a viable theory in literature. Regarding the metric-affine formalism addressed here instead, both possibilities will be taken under consideration, allowing for arbitrary values of β, including also the $\beta = 0$ case. We will soon show that in this case the pseudoscalar field has dynamical character even without its kinetic term in the action, implying the absence of the shortcomings affecting the metric case.

7.1 Connection equation and dynamical instabilities

Let us now outline the role of the independent connection in this gravitational model. Varying the action with respect to the connection we obtain

$$\begin{aligned} &-\nabla_\lambda\left(\sqrt{-g}g^{\mu\nu}\right)+\delta^\nu_\lambda\nabla_\rho\left(\sqrt{-g}g^{\mu\rho}\right)+\sqrt{-g}\left(g^{\mu\nu}T^\tau{}_{\lambda\tau}-\delta^\nu{}_\lambda T^{\tau\mu}{}_\tau+T^{\nu\mu}{}_\lambda\right)= \\ &=\frac{\alpha}{2}\sqrt{-g}\,\varepsilon^{\alpha\beta\gamma\nu}\left(\mathscr{R}^\mu{}_{\lambda\beta\gamma}-\frac{1}{4}\delta^\mu{}_\lambda\hat{\mathscr{R}}_{\beta\gamma}\right)\nabla_\alpha\theta. \end{aligned} \tag{7.3}$$

We immediately see that the connection field equations are more complex compared to the other theories considered so far. In particular, now we are facing a non-linear partial differential equation in $\Gamma^\mu{}_{\nu\rho}$. For constant $\theta(x)$ the right hand side vanishes and we recover the standard equation (Cf. eq. (1.77)), according to the topological character of the Chern-Simons modification. The equations for the vector components of torsion and nonmetricity can be extracted with the usual contractions of (7.3), resulting in

$$3P_\mu-\frac{3}{2}Q_\mu-2T_\mu=\frac{\alpha}{2}\left(\varepsilon^{\alpha\beta\gamma\delta}\mathscr{R}_{\mu\beta\gamma\delta}+\frac{1}{4}\varepsilon_\mu{}^{\alpha\beta\gamma}\hat{\mathscr{R}}_{\beta\gamma}\right)\nabla_\alpha\theta, \tag{7.4}$$

$$P_\mu+\frac{1}{2}Q_\mu+2T_\mu=\frac{\alpha}{2}\left(\varepsilon^{\alpha\beta\gamma\delta}\mathscr{R}_{\beta\mu\gamma\delta}+\frac{1}{4}\varepsilon_\mu{}^{\alpha\beta\gamma}\hat{\mathscr{R}}_{\beta\gamma}\right)\nabla_\alpha\theta, \tag{7.5}$$

$$S_\mu=-\alpha\left(\delta^\rho{}_\mu\mathscr{R}-\mathscr{R}^\rho{}_\mu-\mathscr{R}^{\rho\sigma}{}_{\mu\sigma}\right)\nabla_\rho\theta\,. \tag{7.6}$$

The equations for the tensorial parts are instead given by

$$\begin{aligned} q_{\nu\mu\lambda}-\Omega_{\lambda\mu\nu}=&\frac{2}{3}\left(T_\mu g_{\nu\lambda}-T_\nu g_{\mu\lambda}\right)-\frac{1}{6}\varepsilon_{\nu\mu\lambda\sigma}S^\sigma+ \\ &-\frac{1}{9}\left(g_{\mu\nu}\left(2Q_\lambda+P_\lambda\right)-g_{\nu\lambda}\left(4Q_\mu-7P_\mu\right)-g_{\mu\lambda}\left(-\frac{1}{2}Q_\nu+2P_\nu\right)\right) \\ &+\frac{\alpha}{2}\varepsilon^{\alpha\beta\gamma}{}_\nu\left(\mathscr{R}_{\mu\lambda\beta\gamma}-\frac{1}{4}g_{\mu\lambda}\hat{\mathscr{R}}_{\beta\gamma}\right)\nabla_\alpha\theta. \end{aligned} \tag{7.7}$$

The contraction of (7.3) with δ^λ_μ yields the identity $0 = 0$ stemming from the projective invariance. The latter is present by construction and allows to simplify the equations by setting $T_\mu = 0$. Taking this into account, summing and subtracting the first two vectorial equations yields

$$4P_\mu - Q_\mu = \alpha\left(\varepsilon^{\alpha\beta\gamma\delta}\mathcal{R}_{(\mu\beta)\gamma\delta} + \frac{1}{4}\varepsilon_\mu{}^{\alpha\beta\gamma}\hat{\mathcal{R}}_{\beta\gamma}\right)\nabla_\alpha\theta, \tag{7.8}$$

$$P_\mu - Q_\mu = \alpha\varepsilon^{\alpha\beta\gamma\delta}\mathcal{R}_{[\mu\beta]\gamma\delta}\nabla_\alpha\theta, \tag{7.9}$$

$$S_\mu = \alpha\left(\mathcal{R}^\rho{}_\mu + \mathcal{R}^{\rho\sigma}{}_{\mu\sigma} - \delta^\rho{}_\mu\mathcal{R}\right)\nabla_\rho\theta, \tag{7.10}$$

and

$$\begin{aligned} q_{\nu\mu\lambda} - \Omega_{\lambda\mu\nu} = {}& \frac{\alpha}{2}\varepsilon^{\alpha\beta\gamma}{}_\nu\left(\mathcal{R}_{\mu\lambda\beta\gamma} - \frac{1}{4}g_{\mu\lambda}\hat{\mathcal{R}}_{\beta\gamma}\right)\nabla_\alpha\theta - \frac{1}{6}\varepsilon_{\nu\mu\lambda\sigma}S^\sigma + \\ & - \frac{1}{9}\left(g_{\mu\nu}\left(2Q_\lambda + P_\lambda\right) - g_{\nu\lambda}\left(4Q_\mu - 7P_\mu\right) + g_{\mu\lambda}\left(\frac{1}{2}Q_\nu - 2P_\nu\right)\right). \end{aligned} \tag{7.11}$$

Let us now address the topic of dynamical instabilities possibly sourced by the presence of derivatives of order higher than two in the field equations [117, 211, 212, 213, 214, 215, 216]. As already discussed in section 2.2.2, in metric Chern-Simons gravity their presence is due to third order derivatives of the metric contained in the C-tensor appearing in the metric field equations. In the metric-affine case instead, varying (7.1) with respect to the metric and including also the matter action (1.12), we obtain

$$\mathcal{R}_{(\mu\nu)} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = \kappa^2 T_{\mu\nu} + \frac{\beta}{2}\left(\nabla_\mu\theta\nabla_\nu\theta - \frac{1}{2}g_{\mu\nu}\nabla_\rho\theta\nabla^\rho\theta\right), \tag{7.12}$$

which can be written as

$$G_{\mu\nu} + C_{\mu\nu} = \kappa^2 T_{\mu\nu} + \frac{\beta}{2}\left(\nabla_\mu\theta\nabla_\nu\theta - \frac{1}{2}g_{\mu\nu}\nabla_\rho\theta\nabla^\rho\theta\right) \equiv \kappa^2 T^{TOT}_{\mu\nu}, \tag{7.13}$$

where we defined the metric-affine C-tensor as

$$\begin{aligned} C_{\mu\nu} \equiv {}& \bar\nabla_\rho N^\rho{}_{(\mu\nu)} - \bar\nabla_{(\nu}N^\rho{}_{\mu)\rho} + N^\rho{}_{\lambda\rho}N^\lambda{}_{(\mu\nu)} - N^\rho{}_{\lambda(\nu}N^\lambda{}_{\mu)\rho} + \\ & - \frac{1}{2}g_{\mu\nu}\left(\bar\nabla_\rho N^{\rho\sigma}{}_\sigma - \bar\nabla_\sigma N^{\rho\sigma}{}_\rho + N^\rho{}_{\lambda\rho}N^{\lambda\sigma}{}_\sigma - N^\rho{}_{\lambda\sigma}N^{\lambda\sigma}{}_\rho\right). \end{aligned} \tag{7.14}$$

It is worth mentioning that the C-tensor is not traceless in general:

$$C = C^\mu{}_\mu = -\bar\nabla_\rho N^{\rho\sigma}{}_\sigma + \bar\nabla_\sigma N^{\rho\sigma}{}_\rho - N^\rho{}_{\lambda\rho}N^{\lambda\sigma}{}_\sigma + N^\rho{}_{\lambda\sigma}N^{\lambda\sigma}{}_\rho. \tag{7.15}$$

This is a peculiarity of the metric-affine case, since in the purely metric version of the theory one has $C = 0$ (Cf. section 2.2.2). As a consequence, vacuum solutions of the theory are not necessarily characterized by a vanishing Ricci scalar as in GR. Rather, in general the relation

$$R = C - \kappa^2 g^{\mu\nu} T^{TOT}_{\mu\nu}, \tag{7.16}$$

must hold. Now, looking at the metric field equation one might conclude that no higher-order derivatives appear, since (7.12) only features up to second order derivatives of the metric and first order derivatives of torsion, nonmetricity and θ. This is the conclusion reached in [217]. However, a recurring feature of metric-affine models is that the affine connection behaves as an auxiliary variable in the sense that it is dynamically sourced by the metric and/or scalar fields present in the theory. Their relation to torsion and nonmetricity is not always a mere algebraic one and in general it may contain derivatives of the fields. For this reason we cannot simply conclude the absence of instabilities as in [217], since higher-order derivatives might appear on half-shell, at the effective level.
This calls for a deeper analysis of this topic. Considering the expression (7.14) the origin of possible dynamical instabilities can be identified in the dependence of the distorsion tensor on second order derivatives of the metric and pseudo-scalar fields. Regarding the latter, only first order derivatives are present in the connection equations, so that any dynamical instability will be sourced by derivatives of the metric field, namely by the metric Riemann tensors potentially appearing in (7.3). In particular, we will now show that the vector components are always safe from these kind of terms. In (7.8)-(7.9), indeed, one can use the Bianchi identity to remove any metric Riemann tensor contribution. The equation for the axial trace on the other hand, can be rewritten as

$$S_\mu = \alpha \left(2\kappa T^{TOT}_{\rho\mu} + \dots \right) \partial^\rho \theta, \tag{7.17}$$

where the dots represent terms proportional to the distorsion tensor and its metric covariant derivative. We can conclude that, in vacuum, where no additional interaction terms are introduced at the effective level by the axial trace, the vector parts of the affine connection cannot be responsible for the arising of higher-order derivatives and associated dynamical instabilities.
Actually, we note that a non vanishing stress-energy tensor does not necessarily imply the emergence of instabilities. Indeed, if the tensor $T_{\mu\nu}$ contains only first covariant derivatives of the field, as it occurs for example for the standard scalar or electromagnetic cases, the purely affine components of the Riemann tensor can generate in the equation of motions for $g_{\mu\nu}$ and θ at most second order derivatives. On the other hand, second derivatives of the matter fields featuring the $T_{\mu\nu}$ would already imply third order derivatives in the equations of those fields, representing an issue in itself, unrelated to the setting of metric-affine Chern-Simons gravity.

It remains to consider the equations for the tensor components. It is first convenient to separate (7.11) into its symmetric and antisymmetric parts in the indices μ, λ, in order to deal with the symmetric and antisymmetric part of the Riemann tensor in its first two indices, as in (7.8)-(7.9). Hence, we have

$$\begin{aligned}\Omega_{(\lambda\mu)\nu} = &-\frac{\alpha}{2}\,\varepsilon^{\alpha\beta\gamma}{}_{\nu}\left(\mathscr{R}_{(\mu\lambda)\beta\gamma} - \frac{1}{4}g_{\mu\lambda}\hat{\mathscr{R}}_{\beta\gamma}\right)\nabla_{\alpha}\theta + \\ &+ \frac{1}{9}\left(g_{\mu\nu}\left(4P_{\lambda} - Q_{\lambda}\right) + g_{\lambda\nu}\left(4P_{\mu} - Q_{\mu}\right) - \frac{1}{2}g_{\mu\lambda}\left(4P_{\nu} - Q_{\nu}\right)\right)\end{aligned} \tag{7.18}$$

and

$$\begin{aligned}q_{\nu\mu\lambda} - \Omega_{[\lambda\mu]\nu} = &\frac{\alpha}{2}\,\varepsilon^{\alpha\beta\gamma}{}_{\nu}\left(\mathscr{R}_{[\mu\lambda]\beta\gamma}\right)\nabla_{\alpha}\theta + \\ &-\frac{1}{6}\varepsilon_{\nu\mu\lambda\sigma}S^{\sigma} - \frac{1}{3}\left(g_{\mu\nu}\left(Q_{\lambda} - P_{\lambda}\right) - g_{\lambda\nu}\left(Q_{\mu} - P_{\mu}\right)\right).\end{aligned} \tag{7.19}$$

By virtue of the symmetries of the metric Riemann tensor, the latter only survives in (7.19), which is ultimately responsible for the presence or absence of dynamical instabilities.

This can help in designing strategies for ensuring the absence of higher-order derivatives in the field equations. This same requirement is needed in metric Chern-Simons gravity as well, as we mentioned in section 2.2.2. However, the crucial difference with respect to the metric version of the theory is that in that case higher-order derivatives directly appear in the metric field equation, while in the metric-affine theory they are introduced via the affine contributions only if the latter depend on them. This additional step allows for a new approach to this problem that is not available in metric Chern-Simons gravity, which consists on acting previously on the kinematic structure of the theory. This can be achieved, for instance, by considering from the very beginning simplified metric-affine geometries, where the rank-3 tensor part of torsion and nonmetricity is neglected, and we only deal with the vector components. Such an assumption, indeed, automatically gets rid of (7.19), leaving us only with the set of equations for the nonmetricity vectors (7.8) - (7.9), and the axial trace of torsion (7.10), which makes up a closed system. This represents the minimal prescription for preserving projective invariance, given that $q_{\rho\mu\nu}$ and $\Omega_{\rho\mu\nu}$ are unaffected by projective transformations, and the requirement that they be vanishing does not spoil the projective symmetry of (7.1). In this approach, higher-order derivatives would be absent at the full unperturbed and background independent level, without imposing restrictions on the parameters of the theory.

If instead the tensorial parts are considered, one is forced to resort to approaches analogues to the ones adopted in the metric theory. For instance, one can consider metric-affine Chern-Simons gravity only as an effective theory in the small

coupling limit $\alpha \ll 1$ (or $\alpha\theta \ll 1$, after the rescalings performed in section 10), thus making the first term on the right-hand side of (7.19) negligible, ensuring the absence of higher-order derivatives at the effective level. The same outcome without imposing restrictions on the parameters can also be obtained in the linearized setting introduced in the next section that will be adopted in deriving the results of section 10.1, where the perturbative expansion of the scalar field is performed on a constant background.
Finally, a viable strategy consists in seeking for spacetime configurations where the components of the metric Riemann tensor containing second order derivatives of the metric field are prevented to appear in (7.19) by the symmetries of the problem, which restricting the possible dependence of $\theta(x)$ on specific spacetime coordinates, also selects the components of the Riemann tensor. The equations describing the evolution of the connection reduce then to a highly coupled system of first order differential equations for the metric-affine components, which contain at most first derivatives of the pseudo-scalar field and of the metric tensor, by means of the Levi-Civita connection appearing in the metric covariant derivatives.
Let us end this section with a remark on matter couplings. The previous conclusions rely on the assumption that the connection does not couple with matter. In fact, in the general case of a non vanishing hypermomentum [67, 68]

$$\Delta_\lambda{}^{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta\mathscr{L}_m}{\delta\Gamma^\lambda{}_{\mu\nu}} \neq 0, \tag{7.20}$$

where $\mathscr{L}_m$ denotes collectively the Lagrangian for the matter fields, we expect contributions analogous to the one in (7.17), in terms of the hypermomentum components, also in the equations for the other parts of $\Gamma^\lambda{}_{\mu\nu}$, so that no conclusive arguments can be drawn in this situation.

7.2 Perturbative solution for the connection

After outlining the general picture of the theory presented in the previous section, we would like to investigate observable effects ultimately related to the non-trivial affine structure. To do this it would be convenient to reformulate the theory on half-shell, in the form of a scalar-tensor theory, where torsion and nonmetricity degrees of freedom are embodied in the non-trivial interactions between the metric tensor and the pseudoscalar field. However, the situation is more involved compared to the Holst and Nieh-Yan models previously considered. The technical difficulties preventing the application of that same procedure are rooted in the complex structure of the connection field equation (7.3).

In this case, providing an exact, non perturbative and background independent solution for the affine sector in terms of $g_{\mu\nu}$ and θ is an hopeless task and we are forced to resort to alternative approaches.
There are basically two possible paths to be followed, each of which involves giving up either the non perturbative or the background independent aspect. Here we will discuss the former, while the latter will be the subject of section 10.2.
The main idea consists in performing a perturbative expansion of the metric and pseudoscalar fields:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \tag{7.21}$$

$$\theta = \bar{\theta} + \delta\theta. \tag{7.22}$$

Here, a bar denotes the background configuration of which $h_{\mu\nu}$ and $\delta\theta$ represent small deviations from. For the time being we will keep the background metric as general as possible, while we will assume the background scalar field $\bar{\theta}$ to be constant. The reason for this is twofold. On one hand this amounts to consider small deviations from General Relativity, since the latter is recovered for a constant θ in (7.3), and it is consistent with the idea that the affine structure is generated by a non-trivial behavior of the scalar field. On the other hand, from a more technical point of view, since θ only appears with its derivative, this allows to simplify the perturbed connection equations preventing the presence of couplings between $h_{\mu\nu}$ and $\bar{\theta}$.
Concretely, since $\nabla\bar{\theta} = 0$, substituting the metric and scalar field expansions into (7.8), (7.9), (7.10) one can set the right hand sides to zero and the background (zeroth order) configuration for the affine vectors vanishes. Substituting into (7.11) yields also the vanishing of the tensorial part. As a consequence, at first order one has that (7.8) and (7.9) yield

$$4\delta P_\mu - \delta Q_\mu = \alpha\varepsilon^{\alpha\beta\gamma\delta}\bar{R}_{(\mu\beta)\gamma\delta}\nabla_\alpha\delta\theta = 0, \tag{7.23}$$

$$\delta P_\mu - \delta Q_\mu = \alpha\varepsilon^{\alpha\beta\gamma\delta}\bar{R}_{[\mu\beta]\gamma\delta}\nabla_\alpha\delta\theta = 0, \tag{7.24}$$

where $\bar{R}_{\mu\nu\rho\sigma}$ denotes the metric Riemann tensor of the background metric $\bar{g}_{\mu\nu}$ and we used its symmetries and the Bianchi identities to set to zero the right hand side of the above equations. It follows that

$$\delta P_\mu = \delta Q_\mu = 0. \tag{7.25}$$

From (7.10) instead, we obtain the axial trace of torsion at first order

$$\delta S_\mu = 2\alpha\,\bar{G}_{\rho\mu}\nabla^\rho\delta\theta, \tag{7.26}$$

where $\bar{G}_{\rho\mu}$ is the Einstein tensor of $\bar{g}_{\rho\mu}$. The perturbations of the rank-3 components can be computed from (7.3), which gives

$$\delta q_{\nu\mu\lambda} - \delta\Omega_{\lambda\mu\nu} = \frac{\alpha}{2}\varepsilon^{\alpha\beta\gamma}{}_{\nu}\bar{R}_{\mu\lambda\beta\gamma}\nabla_{\alpha}\delta\theta - \frac{\alpha}{3}\varepsilon_{\nu\mu\lambda\rho}\bar{G}^{\sigma\rho}\nabla_{\sigma}\delta\theta. \tag{7.27}$$

Symmetrizing on μ, λ we obtain the antisymmetry of the $\Omega_{\mu\nu\rho}$ perturbation on the first two indices:

$$\delta\Omega_{(\mu\lambda)\nu} = 0. \tag{7.28}$$

This, together with the symmetry on the last two indices, i.e. $\delta\Omega_{\mu[\lambda\nu]} = 0$, can be used to prove that

$$\delta\Omega_{\mu\nu\rho} = 0, \tag{7.29}$$

resulting in the vanishing of the full nonmetricity perturbation $\delta Q_{\mu\nu\rho} = 0$. Going back to (7.27), we are left with

$$\delta q_{\nu\mu\lambda} = \frac{\alpha}{2}\varepsilon^{\alpha\beta\gamma}{}_{\nu}\bar{R}_{\mu\lambda\beta\gamma}\nabla_{\alpha}\delta\theta - \frac{\alpha}{3}\varepsilon_{\nu\mu\lambda\rho}\bar{G}^{\sigma\rho}\nabla_{\sigma}\delta\theta, \tag{7.30}$$

which is consistent with the property $\varepsilon^{\nu\mu\lambda\tau}\delta q_{\nu\mu\lambda} = 0$. Summing up, at first order the nonmetricity is vanishing and the torsion is given by

$$\delta T_{\rho\mu\nu} = \frac{\alpha}{2}\varepsilon^{\alpha\beta\gamma}{}_{\rho}\bar{R}_{\mu\nu\beta\gamma}\nabla_{\alpha}\delta\theta. \tag{7.31}$$

This perturbative expression for torsion is consistent with previous results derived in the Einstein-Cartan framework, where nonmetricity was a priori neglected [113]. Therefore, in this particular case the inclusion of nonmetricity does not play any role. We stress however, that this is true only in the perturbative approach we are employing here, and it is not a general feature of metric-affine Chern-Simons gravity. As we will see in section 10.2 and 10.3, relaxing the assumption $\bar{\theta} = \text{const}$ is enough to generate non-trivial nonmetricity contributions, both at the background and perturbative levels.

We are now in a position to reformulate the theory on-half shell, even if just at the perturbative level. To do this it is sufficient to substitute the results just obtained in the metric and scalar field equations. At the lowest order, equation (7.12) simply reduce to Einstein equations for the background metric, i.e.

$$\bar{G}_{\mu\nu} = \kappa^2\bar{T}_{\mu\nu}, \tag{7.32}$$

from which we conclude that, in vacuum where $\bar{T}_{\mu\nu} = 0$, the axial trace of torsion vanishes by virtue of (7.26) and the only affine contribution consists of the perturbation of the rank-3 torsion tensor. Next, the metric equations at first order are given by

$$\delta G_{\mu\nu} + \delta C_{\mu\nu} = \kappa^2\delta T_{\mu\nu}, \tag{7.33}$$

where $\delta T_{\mu\nu}$ only includes matter perturbations, since the scalar field stress-energy tensor is at least quadratic in $\delta\theta$. In the above equation, the first order Einstein and Cotton tensors are given by

$$\delta G_{\mu\nu} \equiv 2\bar{\nabla}_\alpha \bar{\nabla}_{(\mu} h_{\nu)}{}^\alpha - \bar{g}_{\mu\nu}\left(\bar{\nabla}_\alpha\bar{\nabla}_\beta - \bar{R}_{\alpha\beta}\right)h^{\alpha\beta} - \left(\Box + \bar{R}\right)h_{\mu\nu} + \left(\bar{g}_{\mu\nu}\Box - \bar{\nabla}_\mu\bar{\nabla}_\nu\right)h \tag{7.34}$$

and

$$\delta C_{\mu\nu} \equiv -\bar{\nabla}^\rho \delta T_{(\mu|\rho|\nu)} = -\alpha\varepsilon_{(\mu|\alpha\gamma\delta}\left(\bar{R}_{|\nu)\beta}{}^{\gamma\delta}\bar{\nabla}^\beta\bar{\nabla}^\alpha\delta\theta + \bar{\nabla}_\beta\bar{R}_{|\nu)}{}^{\beta\gamma\delta}\bar{\nabla}^\alpha\delta\theta\right), \tag{7.35}$$

respectively. Let us turn the attention to the scalar field equation which is obtained varying (7.1) with respect to $\theta(x)$:

$$\beta\Box\theta + \frac{\alpha}{8}\varepsilon^{\mu\nu\rho\sigma}\left(\mathscr{R}^\alpha{}_{\beta\mu\nu}\mathscr{R}^\beta{}_{\alpha\rho\sigma} - \frac{1}{4}\hat{\mathscr{R}}_{\mu\nu}\hat{\mathscr{R}}_{\rho\sigma}\right) = 0. \tag{7.36}$$

At order zero this gives

$$\varepsilon^{\mu\nu\rho\sigma}\bar{R}^\alpha{}_{\beta\mu\nu}\bar{R}^\beta{}_{\alpha\rho\sigma} = 0, \tag{7.37}$$

which formally coincides with the Pontryagin constraint present in the metric Chern-Simons theory but it actually binds only the background metric $\bar{g}_{\mu\nu}$ and not the full metric tensor $g_{\mu\nu}$. At this point we see that the background metric cannot be completely arbitrary but has to satisfy both (7.32) and (7.44). Without imposing any other restriction on $\bar{g}_{\mu\nu}$, we can write the linearized equation for $\delta\theta(x)$ as

$$\left(\mathscr{D}_1(\alpha,\beta;\bar{g}_{\rho\sigma}) + \mathscr{D}_2(\alpha;\bar{g}_{\rho\sigma}) + \mathscr{D}_3(\alpha;\bar{g}_{\rho\sigma})\right)\delta\theta + \mathscr{D}_4^{\mu\nu}(\alpha;\bar{g}_{\rho\sigma})h_{\mu\nu} + C(\alpha;\bar{g}_{\rho\sigma},h_{\rho\sigma}) = 0, \tag{7.38}$$

where (up to second order) differential operators acting on the field perturbations are denoted by $\mathscr{D}_i$ while $C(\alpha;\bar{g}_{\rho\sigma},h_{\rho\sigma})$ is a function of both the background metric and the metric perturbation. Explicitly, they are defined by

$$\mathscr{D}_1(\alpha,\beta;\bar{g}_{\rho\sigma}) \equiv \left(\beta + \alpha^2\left(\frac{3}{4}\bar{R}^{\mu\nu\rho\sigma}\bar{R}_{\mu\nu\rho\sigma} - \bar{R}_{\alpha\beta}\bar{R}^{\alpha\beta}\right)\right)\Box \equiv \left(\beta + \alpha^2 K\right)\Box, \tag{7.39}$$

$$\mathscr{D}_2(\alpha;\bar{g}_{\rho\sigma}) \equiv \alpha^2\left(\bar{R}^\mu{}_\sigma\bar{R}_{\mu\tau} + 2\bar{R}^{\mu\nu}\bar{R}_{\sigma\mu\tau\nu} - 2\bar{R}_{\sigma\mu\nu\rho}\bar{R}_\tau{}^{\mu\nu\rho}\right)\bar{\nabla}^\sigma\bar{\nabla}^\tau, \tag{7.40}$$

$$\begin{aligned}\mathscr{D}_3(\alpha;\bar{g}_{\rho\sigma}) \equiv \frac{\alpha^2}{2}\Big(&\bar{\nabla}_\sigma K + \bar{R}^{\mu\nu\rho\tau}\bar{\nabla}_\nu\bar{R}_{\sigma\mu\rho\tau} - 3\bar{R}_\sigma{}^{\tau\mu\rho}\bar{\nabla}_\nu\bar{R}^\nu{}_{\tau\mu\rho} + 2\bar{R}_\sigma{}^\beta\bar{\nabla}_\mu\bar{R}_\beta{}^\mu \\ &+2\bar{\nabla}_\tau\left(\bar{R}^{\mu\nu}\bar{R}_{\mu\sigma\nu}{}^\tau\right)\Big)\bar{\nabla}^\sigma,\end{aligned} \tag{7.41}$$

$$\mathscr{D}_4^{\mu\nu}(\alpha;\bar{g}_{\rho\sigma}) \equiv \alpha\,{}^*\bar{R}^\mu{}_\beta{}^\nu{}_\delta\bar{\nabla}^\delta\bar{\nabla}^\beta, \tag{7.42}$$

$$C(\alpha;\bar{g}_{\rho\sigma},h_{\rho\sigma}) \equiv \frac{\alpha}{2}\,{}^*\bar{R}_{\mu\nu\beta\delta}\left(h^{\alpha\beta}\bar{R}_\alpha{}^{\delta\mu\nu} + h^{\alpha\mu}\bar{R}_\alpha{}^{\nu\beta\delta} - \frac{1}{4}h^\alpha{}_\alpha\bar{R}^{\beta\delta\mu\nu}\right). \tag{7.43}$$

Now, some comments are in order. First, we can compare the perturbative expression of the C-tensor in equation (7.35) with the nonperturbative C-tensor characterizing the purely metric theory [142] (Cf. eq. (2.68)). At first sight it may seem that they have a different structure, given the Riemann tensor replacing the Ricci one in the first term. However, by using the contracted Bianchi identities one can show that they actually coincide, although just at the formal level (in one case we deal with the full unperturbed metric and scalar field, the other features the metric of the background spacetime and the scalar field perturbation).
Another aspect which is worth discussing is the behavior of the scalar field in the $\beta = 0$ case. We see that the limit $\beta \to 0$ does not necessarily deprive the scalar perturbation of a proper dynamical character, in contrast to metric Chern-Simons gravity. The reason for this can be traced back to (7.31), which introduces scalar field derivatives via the Riemann contractions of (7.36). In particular, the box term in (7.38) survives even setting $\beta = 0$. This implies that the condition

$$\varepsilon^{\mu\nu\rho\sigma} R^{\alpha}{}_{\beta\mu\nu} R^{\beta}{}_{\alpha\rho\sigma} = 0, \tag{7.44}$$

affecting the metric theory and preventing some geometric configuration to be actually feasible, does not hold in this case. Instead, the metric-affine version of (7.44) results in a larger variety of possible dynamical solutions, and for $\beta = 0$ the theory is still well-behaved.

7.2.1 Projective symmetry breaking case

In the previous section we considered the projective invariant Chern-Simons term, identified by $\lambda_2 = -1/4$ (Cf. section 3.2). As we saw, at the linearized level the nonmetricity perturbation is trivial and we have $\delta Q_{\mu\nu\rho} = 0$. The same is true for the background configuration $\bar{Q}_{\mu\nu\rho} = 0$, as a consequence of the assumption of constant background scalar field. Now, since the homotetic curvature tensor is proportional to the Weyl vector derivative, it may seem that the additional term included in the generalized Chern-Simons invariant plays no role in the stability of the theory at the linearized level.
However, we remark that when the projective symmetry is broken, i.e. in the $\lambda_2 \neq -1/4$ case, the contraction of the connection field equation with $\delta^{\lambda}{}_{\mu}$ does not lead to the trivial identity $0 = 0$, but rather to the additional constraint:

$$\epsilon^{\alpha\mu\beta\gamma} \hat{\mathscr{R}}_{\beta\gamma} \nabla_{\alpha}\theta = \epsilon^{\alpha\mu\beta\gamma} \partial_{\alpha}\theta \partial_{\beta} Q_{\gamma} = 0, \tag{7.45}$$

allowing for a more general form of the Weyl vector: $Q_{\mu} = \partial_{\mu}\lambda$, where λ is an undetermined scalar field. This arbitrariness can be traced back to the absence of projective invariance, which prevents us to set to zero the torsion trace T_μ. Thus, one has to deal directly with (7.4)-(7.5), while the equation for the axial part of torsion is insensitive. In this case computations similar to the ones performed in the previous section yield the following linearized solution:

$$\delta P_\mu = \frac{1}{4}\partial_\mu \lambda, \tag{7.46}$$

$$\delta T_\mu = -\frac{\alpha}{4}\varepsilon^{\alpha\beta\gamma\delta}\bar{R}_{\mu\beta\gamma\delta}\nabla_\alpha \delta\theta - \frac{1}{2}\partial_\mu \lambda. \tag{7.47}$$

This in turn would imply a more complicated form of the on half-shell metric and scalar field equations. Then the relevant question arises of whether the breaking of projective symmetry leads to instabilities, as in [35], or it is actually harmless, as in the Nieh-Yan case discussed in section 6. However, given the more complicated picture and the absence of works dedicated to observational effects of the metric-affine Chern-Simons gravity, in the next chapters we will focus on the $\lambda_2 = -1/4$ case, leaving the analysis of the projective breaking case for future works.

Part III

Phenomenological aspects of metric-affine gravity models

From the more theoretical-oriented discussion characterizing the second part of the thesis, we now move to some concrete applications of the proposed models. On one hand, this will shed further light on some fundamental properties of the models. On the other hand, it will allow to derive relevant observational effects that can be used to put the theories to the test, comparing with observations. Both cosmological and astrophysical settings will be considered, specializing to relevant and widely studied symmetric configurations.

Among them, black hole spacetimes received increasing attention in literature both from a theoretical perspective and regarding their astrophysical properties. We will investigate the former in the Holst model presented in the previous sections. In particular, we are interested in the two alternative possibilities of having a constant Immirzi parameter or a theory featuring a dynamical Immirzi field, and how (and if) the presence of the latter can influence black hole solutions and their well-known thermodynamic properties. The possibility of deriving this kind of results is subject to the existence of black hole solutions endowed with a non-trivial profile for the scalar field. This is known to be particularly difficult because of the presence of no-hair theorems [218] which prevent this possibility and for this reason we decide to pick the simplest among the models considered in this thesis. Regarding their astrophysical properties instead, black holes in metric-affine Chern-Simons gravity will be studied, with special emphasis on linear perturbations of Schwarzschild black holes [B219].

Cosmological scenarios will be investigated in the Nieh-Yan model, discussing the presence and severity of singularities in the evolution of the universe, which is again influenced by the scalar fields present in the theory. In this regard, the resolution of the big-bang singularity has already been observed in some modified gravity models and we will check if the same scenario can be induced by the presence of a Nieh-Yan term in anisotropic cosmological models, extending previous results derived in the homogeneous and isotropic FLRW solution (see e.g. [170]).

Finally, peculiar parity violating effects in the propagation of gravitational waves will be derived in the context of metric-affine Chern-Simons theory. These include the phenomenon of gravitational birefringence [185] and the gravitational analogue of the electromagnetic Landau damping [220]. These observational signatures affecting the gravitational waves phenomenology are particularly interesting since they will allow to constrain the theory in the future.

Chapter 8

Black hole solutions in Holst models and their thermodynamics

From the literature on the Immirzi field cited in section 2.2.1 we saw that it has been investigated in several different contexts, including gravitational waves, cosmology and also from a particle physics perspective. However, regarding the static, spherically symmetric sector of the theory, no investigations have been carried out in literature, except for the results presented in the appendix of [221], where the authors provide the expressions for the components of the spin connection in a spherically symmetric spacetime. However, they eventually rely on no-hair theorems results [218] to conclude that there are no black holes with a non-trivial configuration for the Immirzi field. With the aim of filling this gap in the literature, here we will consider the possibility of going beyond the restrictions imposed by no-hair theorems. Such theorems establish in quite general settings and under specific assumptions that the only possible radial profile for additional scalar fields is the trivial constant one. One of the hypothesis at the ground of these theorems is asymptotic flatness, which can be violated looking at more general solutions in which the Immirzi field can exhibit a non-trivial behavior. Actually, these theorems can be extended to asymptotically de Sitter spacetimes as well, while the Anti-de Sitter scenario offers a way to overcome them. Although the de Sitter case offers a better physical interpretation as a black hole immersed in an expanding universe, it can be interesting to pursue the analysis also in the Anti-de Sitter scenario. Motivations in support of the latter can be found in different contexts such as the AdS/CFT correspondence and the interplay of loop quantum gravity and string theories [222, 223, 224, 225]. Moreover, asymptotically Anti-de Sitter spacetimes allow for a greater variety of solutions with non-trivial horizon topology [226, 227], as well as a wide spectrum of suggestive thermodynamic phenomena [228].

We will start by focusing on static spherically symmetric spacetimes described by the line element

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\sigma^2. \tag{8.1}$$

For the time being we are not going to assume a specific topology for the 2-dimensional sub-manifold with line element $d\sigma^2$. Rather, it can describe a surface with either spherical or hyperbolic topology[1], namely

$$d\sigma^2 = d\theta^2 + c\sinh\left(\sqrt{d}\theta\right)d\varphi^2, \tag{8.2}$$

with $c = d = -1$ or $c = d = 1$, respectively. In particular, if we consider Schwarzschild-(Anti)de Sitter spacetimes, we have

$$A(r) = \frac{1}{B(r)} = b - \frac{2m}{r} - \frac{\Lambda}{3}r^2, \tag{8.3}$$

where $b = 1$ in the spherical case and $b = -1$ in the hyperbolic one. It turns out that only the spherical topology is compatible with both positive and negative (as well as vanishing) asymptotic curvature, while the hyperbolic topology can only be realized for asymptotically Anti-de Sitter spacetimes [229].
Before reporting the actual solutions, we need to specify a particular model. The metric and scalar field expressions presented in the next section are solution to the Holst theory (5.1) for a specific choice of the function f and potential W. The function is chosen to be a generalization of the Starobinsky model [85] with the inclusion of a cosmological constant term[2]:

$$f(\chi) = \frac{1}{1+8\alpha\Lambda}\left(\chi + \alpha\chi^2 - 2\Lambda\right), \tag{8.4}$$

for a general argument χ. The Jordan frame potential can be computed using its definition (5.3) in terms of f and it turns out to be given by

$$V(\phi) = \frac{(\phi-1)^2}{4\alpha} + 2\Lambda\phi^2. \tag{8.5}$$

The Immirzi field potential is set as

$$W(\psi) = \frac{4\Lambda}{\operatorname{csch}^2\left(\frac{\psi-\psi_0}{\sqrt{12}}\right) - 16\alpha\Lambda}. \tag{8.6}$$

Although in the $\Lambda < 0$ case it is characterized by a negative mass term, i.e. $\frac{d^2W}{d\psi^2}\Big|_{\psi_0} = -2/l^2$, it satisfies the so-called Breitenlohner-Freedman bound which guarantees Anti-de Sitter space stability [230, 231]. We will now focus on the case of a negative cosmological constant and postpone the de Sitter case to the end of this section.

[1]One can also consider toroidal topology, see [229] for details

[2]The prefactor $1 + 8\alpha\Lambda$ is chosen for later convenience

8.1 Black holes with Immirzi scalar hair

We will first present a solution for the $\Lambda < 0$ case and then discuss some of its properties. The line element is given by

$$ds^2 = \Omega(r)\left[-h(r)dt^2 + h^{-1}(r)dr^2 + r^2 d\sigma^2\right], \tag{8.7}$$

where the metric function and the conformal factor read

$$h(r) = -\left(1+\frac{m}{r}\right)^2 + \frac{r^2}{l^2}, \tag{8.8}$$

$$\Omega(r) = \frac{r(r+2m) + 48\alpha m^2/l^2}{(r+m)^2}, \tag{8.9}$$

respectively. The parameter l is related to the cosmological constant by $\Lambda = -3/l^2 < 0$. The radial profile for the Immirzi field is given by

$$\psi(r) = \psi_0 + \sqrt{12}\,\text{arctanh}\left(\frac{m}{r+m}\right), \tag{8.10}$$

where ψ_0 is a constant. The expression for the scalaron instead, is obtained by solving the structural equation (5.15) and it reads

$$\phi = 1 + 4\alpha W(\psi), \tag{8.11}$$

namely

$$\phi(r) = \frac{r(2m+r)}{2mr + r^2 - 16m^2\alpha\Lambda}. \tag{8.12}$$

This configuration is a generalization of the previously known MTZ black hole solution, found in [232]. The latter can be obtained as a special case setting $\alpha = 0$ and $\psi_0 = 0$. Notice that the the potential $V(\phi)$ is singular in $\alpha = 0$. However, one can safely take the limit $\alpha \to 0$ a priori in (8.4), yielding $f(\chi) = \chi - 2\Lambda$, i.e. $V = 2\Lambda$.

Let us now dwell for a while on some properties of this solution. The most peculiar feature is the non-trivial topology of the 2-dimensional sub-manifold Σ which is a 2-surface of negative constant curvature. Its line element $d\sigma$ appearing in (8.7) describes a space with hyperbolic topology and genus $g \geqslant 2$, with area $\sigma = 4\pi(g-1)$ [226, 227, 229].

The other relevant feature, necessary to overcome no-hair theorems as we have discussed above, is the presence of a negative cosmological constant and its associated Anti-de Sitter asymptotics. Even if Λ is not introduced in the theory in the usual way, namely as a constant term added into the action, it can be shown that the quantity associated to the Anti-de Sitter behavior is indeed Λ, while the actual constant term in the Lagrangian, which comes from the potential (8.5) and is given by $1/(4\alpha)$, does not play any essential role. To see this one can look at the asymptotic behavior at radial infinity for the Ricci scalar, which reads

$$R \sim 4\Lambda + O\left(\frac{1}{r^2}\right), \tag{8.13}$$

identifying the spacetime as asymptotically Anti-de Sitter space with radius $l = \sqrt{-3/\Lambda}$.

Let us now discuss the causal structure of the spacetime described by (8.7). Coordinate singularities are located where the metric function $h(r)$ vanishes. It has four roots, one of which is always negative, while the others are

$$r_e = \frac{l}{2}\left(1 + \sqrt{1 + \frac{4m}{l}}\right), \tag{8.14}$$

$$r_+ = \frac{l}{2}\left(1 - \sqrt{1 + \frac{4m}{l}}\right), \tag{8.15}$$

$$r_- = \frac{l}{2}\left(-1 + \sqrt{1 - \frac{4m}{l}}\right). \tag{8.16}$$

Although the parameter m will later be related to the black hole mass it is interesting to classify all possible cases considering also $m < 0$. For positive values of the mass parameter, r_e is the only positive real root. For $m < 0$ instead, we can distinguish two different cases. If $-l/4 < m < 0$ there are three positive real roots, namely $0 < r_- < r_+ < r_e$, and for $m < -l/4$, the only positive real root is r_-. In each case these may be identified as black hole, cosmological or inner horizons, depending on their relative position with respect to the curvature singularities, located at $r = 0$ and at the roots of the conformal factor $\Omega(r)$, namely $r_\pm^\Omega = -m \pm \sqrt{m^2\left(1 - 48\alpha/l^2\right)}$. Both at the origin and in $r_\pm^\Omega$, the scalars of curvature diverge.

Now, the physical interpretation of the solution depends on the values of the mass and the parameter α. All possible cases can be summarized as follows:

- For $\alpha \geqslant l^2/48$ the solution represents a black hole for every value of m. Indeed, the roots $r_\pm^\Omega$ become complex and the event horizons at r_e or r_- are always hiding the remaining singularity at the origin;

- For $0 \leqslant \alpha < l^2/48$, the three curvature singularities are hidden only if $m > -l/4$ holds;

- For $\alpha < 0$ instead, in order for the singularities to be hidden one must have

$m_- < m < m_+$, with

$$m_- = -\frac{l^2\sqrt{l^2-48\alpha}}{2l\left(\sqrt{l^2-48\alpha}+l\right)-48\alpha} \geqslant -\frac{l}{4}, \tag{8.17}$$

$$m_+ = \frac{l^2\sqrt{l^2-48\alpha}}{\left(l-\sqrt{l^2-48\alpha}\right)^2}. \tag{8.18}$$

Finally, a limiting case that will be relevant in the following is given by $m = m_c \equiv -l/4$. In this case, the metric function $h(r)$ has two positive roots, the greater one being at $r = r_c \equiv l/2$.

Although m is just a parameter now, we will later show that it is related to the mass-energy of the black hole and that it has to satisfy $m > m_c$. This has two consequences. On one hand, we can discard the third case scenario, where the presence of the upper bound m_+ would imply that an increase in the black hole mass be associated to the development of a naked singularity. In the following, therefore, we will restrict to the $\alpha > 0$ case. On the other hand, the fact that $m > m_c$ implies that the outer event horizon is always located at r_e.

Let us now discuss the behaviour of the scalar fields. If the mass parameter m is restricted to be in the aforementioned range, both the scalaron and the Immirzi field are regular on and outside the event horizon. In particular, ϕ has two poles at the radii $r^{\Omega}_{\pm}$, which are either complex or hidden, depending on α. Its radial profile increases monotonically starting from $\phi(r_e)$ and reaching 1 asymptotically as $r \to \infty$. Regarding the Immirzi field instead, taking into account (5.11), it is given by

$$\gamma(r) = \frac{e^{\psi_0/\sqrt{3}}(r+2m)^2 - e^{-\psi_0/\sqrt{3}}r^2}{2r(r+2m)}. \tag{8.19}$$

It asymptotically relaxes to the constant value $\gamma_0 \equiv \sinh(\frac{\psi_0}{\sqrt{3}})$ at infinity. Therefore, since at infinity we have $\gamma \to \gamma_0$ and $\phi \to 1$, together with $W \to 0$ and $V \to 2\Lambda$, in the asymptotic region the theory reduces to General Relativity with a constant Immirzi parameter, namely to the usual formulation of loop quantum gravity with a cosmological constant Λ. Moreover, in this limit the bare cosmological constant introduced in the action via the potential $V(\phi)$ cancels with the $-1/(4\alpha)$ term coming from in $W(\psi)$.

Asymptotically de Sitter solutions

We conclude this section reporting some solutions with positive cosmological constant. As we mentioned, no-hair theorems can be extended to asymptotically

de Sitter spacetimes. Indeed, even if the following solutions are characterized by non-trivial scalar fields, they exhibit singularities which are not hidden by horizons.
For $\Lambda > 0$ an asymptotically de Sitter solution to the field equations (5.12)-(5.13), with the same potentials as in the previous section, is given by

$$ds^2 = \Omega(r)\left[-h(r)dt^2 + \frac{1}{h(r)}dr^2 + r^2 d\omega^2\right], \tag{8.20}$$

where now

$$h(r) = \left(1 - \frac{m}{r}\right)^2 - \frac{\Lambda}{3}r^2, \tag{8.21}$$

$$\Omega(r) = \frac{r(r-2m) - 16m^2\alpha\Lambda}{(r-m)^2}, \tag{8.22}$$

and the topology is spherical, i.e. $d\omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$. From the expression of the conformal factor, similar to [233], we see that the spacetime has two asymptotic regions for $r \to \infty$ and $r \to m$, due to the pole of $\Omega(r)$. The scalar fields profile is given by

$$\psi(r) = \psi_0 + \sqrt{12}\,\text{arctanh}\left(\frac{m}{r-m}\right), \tag{8.23}$$

$$\phi(r) = \frac{r(r-2m)}{r(r-2m) - 16\alpha m^2\Lambda}. \tag{8.24}$$

Curvature singularities are located at $r = 0$ and at the roots of $\Omega(r)$, i.e. $r_\pm^\Omega = m(1 \pm \sqrt{1 + 48\alpha/l^2})$, where we have defined $l^2 = 3/\Lambda$. Coordinate singularities correspond to the roots of $h(r)$ whose characterization depends on the value of the mass parameter m. They are located at

$$r_i = \frac{l}{2}\left(-1 + \sqrt{1 + \frac{4m}{l}}\right), \tag{8.25a}$$

$$r_e = \frac{l}{2}\left(1 - \sqrt{1 - \frac{4m}{l}}\right), \tag{8.25b}$$

$$r_c = \frac{l}{2}\left(1 + \sqrt{1 - \frac{4m}{l}}\right), \tag{8.25c}$$

and we can summarize all possibilities as follows:

- For $0 < m < l/4$, the roots are all positive and they satisfy $0 < r_i < m < r_e < l/2 < r_c < l$. Therefore:

  - In the region $0 < r < m$, there is only a cosmological horizon at r_i and the singularity at the origin is naked;
  - When $m < r < \infty$, there is an event horizon at r_e and a cosmological horizon at r_c. This situation is similar to the Schwarzschild-de Sitter black hole with the singularity replaced by the new asymptotic region;

- For $m > l/4$, the only real root is r_i.
  - The region $0 < r < m$ has a cosmological horizon there and the singularity at the origin is naked;
  - The region $m < r < \infty$ has no horizons at all and it represents a inhomogeneous bouncing cosmology [233];

- For $m < 0$ there is only one asymptotic region at $r \to \infty$ and only one cosmological horizon at r_c, leaving the singularity at $r = 0$ naked.

Therefore, there are only two cases potentially describing solutions without naked singularities, namely the region $m < r < \infty$ for either $0 < m < l/4$ or $m > l/4$. Now, one must take into account the singularities at $r^{\Omega}_{\pm}$. The situation is similar to the one in the previous section. For $\alpha \leqslant -l^2/48$, the roots of $\Omega(r)$ are complex and there are no additional singularities. For $\alpha > -l^2/48$ the requirement that the singularity at $r^{\Omega}_{\pm}$ be hidden would impose restrictions on the values of the parameter m. However, even if this aspect is taken into account, the solution always presents a singularity in terms of the behaviour of the scalar fields. Indeed, the Immirzi field, which is given by

$$\gamma(r) = \frac{e^{-\frac{\psi_0}{\sqrt{3}}}}{2}\left(\frac{re^{\frac{2\psi_0}{\sqrt{3}}}}{r-2m} + \frac{2m}{r} - 1\right) \tag{8.26}$$

is singular at $r = 2m$. Moreover, the scalar field ϕ vanishes there, which is inconsistent within the scalar-tensor representation of $f(R)$ theories. The surface $r = 2m$ belongs to the region $m < r < \infty$ and it is always outside the event horizon r_e. Therefore, this singularity is naked in both potentially regular solutions mentioned above and we conclude that this class of de Sitter solutions is nonphysical.

Despite this, it is still interesting to analyse the region $m < r < \infty$, for $m > l/4$, where we have that the function $h(r)$ becomes negative and the metric changes signature. The r and t coordinates become time-like and space-like, respectively. After the exchange $r \leftrightarrow t$ the metric can be written as a Kantowski–Sachs spacetime, namely

$$ds^2 = -N(t)dt^2 + a(t)^2dr^2 + b(t)^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right), \tag{8.27}$$

where

$$a(t)^2 \equiv \Omega(t)|h(t)|, \tag{8.28}$$
$$b(t)^2 \equiv t^2\Omega(t), \tag{8.29}$$

while the lapse function is $N(t) \equiv \Omega(t)/|h(t)|$. It is interesting to check for the existence of big-bounces in the universe evolution. To this aim, let us define the Hubble functions

$$A(t) \equiv \frac{\dot{a}(t)}{a(t)}, \qquad B(t) \equiv \frac{\dot{b}(t)}{b(t)}. \tag{8.30}$$

The condition for the existence of a bounce [234] in the scale factor a at time t_a and in b at time t_b is, respectively

$$A(t_a) = 0, \qquad \dot{A}(t_a) > 0; \tag{8.31}$$
$$B(t_b) = 0, \qquad \dot{B}(t_b) > 0. \tag{8.32}$$

It is clear that, although it may be possible to have a bounce in one of the scale factors but not the other, this does not lead to a new expanding universe region. We therefore require that a bounce occurs in both scale factors, even though they may in general occur at different times [234]. Indeed, for $\alpha < -l^2/48$, a bounce in $b(t)$ occurs at

$$t_b = m\left(1 + \sqrt[3]{-\left(1 + \frac{48\alpha}{l^2}\right)}\right), \tag{8.33}$$

in which conditions (8.32) are satisfied. It can be shown that there exists a time t_a for which conditions (8.31) are satisfied too, although it is not possible to express t_a analytically. In this cosmological interpretation, the Immirzi field $\phi(t)$ diverges when $t = 2m$. This time happens to be before both bounces, for a large range of values of the parameters. It is also interesting to notice that, when the Immirzi field diverges, one has that

$$A(2m) = B(2m), \tag{8.34}$$

for every values of the parameters.

8.2 Thermodynamics and black hole chemistry

Although the asymptotically Anti-de Sitter behavior is not really prone to a direct astrophysical interpretation of the solution, it allows to deepen the analysis in the semiclassical setting in which the thermodynamics of AdS black hole solutions is

well defined and rich of interesting phenomena in which it is worth to investigate the effects of the Immirzi field.
Among the many existing approaches to black hole thermodynamics, Euclidean path integral methods are particularly handy and straightforward to apply [112, 111]. They are based on the thermodynamic interpretation of the gravitational partition function[3]

$$Z = \int d[g] e^{iI[g]}. \tag{8.35}$$

The usual procedure consists in performing a Wick rotation to imaginary time $t \to i\tau$ on which one imposes periodic boundary conditions. In this way, τ becomes an angular coordinate with period β which can be identified with the inverse temperature of the black hole [112]. Since the latter is a solution to the field equations, it extremizes the action, allowing to perform a saddle point approximation around the classical solution, yielding

$$Z(\beta) \approx e^{-I_E(\beta)}, \tag{8.36}$$

where $I_E(\beta)$ is the on-shell Euclidean action. Then, usual thermodynamic relations hold, like

$$I_E = S - \beta M, \tag{8.37}$$

which where M and S are the mass-energy and entropy of the black hole, respectively. These thermodynamic quantities can then be computed via

$$M = -\frac{\partial I_E}{\partial \beta}, \tag{8.38a}$$

$$S = I_E - \beta \frac{\partial I_E}{\partial \beta}, \tag{8.38b}$$

once the Euclidean on-shell action is known. However, this is not necessary for the computation of the black hole temperature, to which we now turn the attention. After the Wick rotation, the metric signature changes from Lorentzian to Euclidean, yielding

$$ds_E^2 = \Omega(r) \left[h(r) d\tau^2 + h^{-1}(r) dr^2 + r^2 d\sigma^2 \right]. \tag{8.39}$$

In order to appreciate how the Euclidean time is actually an angular coordinate let us look at the Euclidean metric near the horizon, namely for $r \approx r_e$. We have

$$ds_E^2 = d\tilde{r}^2 + \frac{h'(r_e)^2}{4} \tilde{r}^2 d\tau^2 + \Omega(r_e) r_e^2 d\sigma^2, \tag{8.40}$$

[3] In order to avoid confusion with the black hole entropy S, in this section we will use I to represent the action. Moreover, we set $\kappa^2 = 1$.

where a prime denotes a derivative with respect to r and we have defined a new radial coordinate as

$$\tilde{r} \equiv 2\sqrt{\frac{\Omega(r_e)(r-r_e)}{h'(r_e)}}. \tag{8.41}$$

Now, the 2-dimensional subspace $\tilde{r}-\tau$, is nothing but flat space in polar coordinates, with metric

$$ds^2_{\tilde{r}-\tau} = d\tilde{r}^2 + \tilde{r}^2 d\tilde{\vartheta}^2, \tag{8.42}$$

where the polar angle is defined in terms of τ as:

$$\tilde{\vartheta} \equiv \frac{h'(r_e)}{2}\tau. \tag{8.43}$$

However, there is a conical singularity at the origin unless the period of $\tilde{\vartheta}$ is precisely 2π. This implies that τ must have period β given by

$$\beta = \frac{4\pi}{h'(r_e)} = \frac{2\pi l^2}{2r_e - l}. \tag{8.44}$$

With the usual identification of the inverse temperature with the period we obtain the expression of the black hole temperature as follows:

$$T = \frac{1}{2\pi l}\left(\frac{2r_e}{l} - 1\right). \tag{8.45}$$

Some remarks are in order. First, the temperature is not always positive definite. To guarantee its positivity, we must restrict to the branch of solutions above the critical configuration introduced in the previous section, namely to the case $r_e > r_c$, which, in terms of the mass parameter, reads $m > m_c$. In this regard, we can draw some comparisons with the results of [226, 227], where a similar scenario occurs. There, the critical configuration has also the feature of possessing the minimum value of the mass parameter which still allows for a black hole interpretation of the solution, since for smaller masses, a naked singularity develops. In the present case however, the same situation holds only if no restrictions on the model are imposed. Indeed, the curvature singularities can always be concealed restricting α to be greater than $l^2/48$, allowing for a black hole interpretation for every value of m.
Let us now turn to the computation of the Euclidean action. There are two important aspects that need to be taken care of. One is that the variational principle must be well posed, as discussed in 2.2. In the present case we actually started from a first order action of the form (5.1). Since it does not contain second order derivatives its variation does not produce any unwanted boundary term and one might be tempted to conclude that no additional boundary terms are required.

However, we will compute the Euclidean action working on half-shell, starting from (5.10). The correct equivalent second order action is the one yielding an equivalent set of field equations via a well-posed (second order) variational principle. This is not simply given by (5.10), whose variation would give rise to unwanted non vanishing boundary terms arising from the ϕR term. Rather, it must be completed with the inclusion of the appropriate Gibbons-Hawking-York-like boundary term. In this case, it is given by [235]

$$I_{GHY} = \frac{1}{8\pi}\int_{\partial\mathcal{M}} d^3x\sqrt{|^{(3)}g|}\phi K, \tag{8.46}$$

where $^{(3)}g$ is the determinant of the induced metric on the boundary $\partial\mathcal{M}$ and K the trace of its extrinsic curvature. The variation of I_{GHY} precisely cancels non vanishing boundary contributions arising from ϕR.
The second aspect that must be faced is the fact that the on-shell Euclidean action for asymptotically AdS spacetimes is divergent. Actually, the simplest example, i.e. the asymptotically flat Schwarzschild solution, already poses the same problem. In that case, one usually adopts a subtraction method, where the contribution from a reference background solution is subtracted from the divergent on-shell action, regularizing the result. In the Schwarzschild case the background is flat space, with the GHY boundary term computed via the extrinsic curvature of a boundary surface of identical intrinsic geometry, embedded in flat spacetime. However, in the present case the identification of the correct background is not as straightforward and different choices may lead to different results [226, 227]. Therefore, we prefer to follow a different path which we will refer to as the counter-terms method [236, 237]. It consists in adding counter-terms to the action which are still surface integrals depending on the induced metric on the boundary and, in the present case, on the scalar fields as well. The advantage is that one can apply the method without specifying the explicit expression of the counter-terms, as done in [232]. We will first generalize the treatment of [232] to the case of non-minimal coupling and then we will also provide the explicit covariant expression of the counter-terms. Let us first rewrite the Euclidean metric as

$$ds_E^2 = N^2(r)f^2(r)d\tau^2 + f^{-2}(r)dr^2 + \rho^2(r)d\sigma^2, \tag{8.47}$$

where the old metric functions are related to the new ones by

$$N = \Omega, \qquad f^2 = \frac{h}{\Omega}, \qquad \rho^2 = \Omega r^2. \tag{8.48}$$

Then, the Euclidean action can be written in Hamiltonian formalism as

$$I = -\frac{\beta\sigma}{8\pi}\int_{r_e}^{\infty} dr N H + B. \tag{8.49}$$

In the expression above we have performed the integrations over τ and the base manifold Σ. The boundary term B, which is for now unspecified, plays both roles mentioned above, namely rendering the variational principle well-posed and the action finite. In addition, since we moved to the Hamiltonian formulation, it actually contains also boundary terms arising during the spacetime splitting procedure from the Gauss-Codazzi relation [B18]. The Hamiltonian reads

$$
\begin{aligned}
H =&\rho^2\left\{\phi\left[\frac{f^{2\prime}\rho'}{\rho}+\frac{2f^2\rho''}{\rho}+\frac{(1+f^2\rho'^2)}{\rho^2}\right]-\frac{3}{4\phi}f^2\phi'^2+\frac{\phi}{4}f^2\psi'^2+\frac{V(\phi)+W(\psi)}{2}\right.\\
&\left.+\frac{\rho^{2\prime}f^2\phi'}{\rho^2}+\frac{f^{2\prime}\phi'}{2}+f^2\phi''\right\},
\end{aligned}
\tag{8.50}
$$

where the second line features new terms that are absent [232], arising from the non-minimal coupling. Moreover, we neglected terms involving the momenta and the shift vector since they are vanishing due to the static and spherically symmetric character of the solution. In principle, one should also include an additional term proportional to the structural equation which is known to manifest itself as a secondary constraint in Hamiltonian formalism [48]. However, this term would not contain derivatives of the fields with respect to r and thus it is irrelevant in the following calculations. Now, in order to compute (8.49) we first note that the Hamiltonian is vanishing on-shell, so that the only relevant contribution comes from the boundary term, i.e.

$$
I = B\big|_{\infty} - B\big|_{r_e}. \tag{8.51}
$$

To compute the contributions of B at infinity and on the horizon we can vary the action with respect to the metric functions and scalar fields as

$$
\delta I = -\frac{\beta\sigma}{8\pi}\int_{r_e}^{\infty} drN\delta H + \delta B. \tag{8.52}
$$

Then, we must impose the variation of B to cancel the non vanishing boundary terms coming from the variation of the Hamiltonian. This gives

$$
\delta B = \delta B_g + \delta B_\phi + \delta B_\psi, \tag{8.53}
$$

where

$$\delta B_g = \frac{\beta\sigma}{8\pi}\left[\left(N\rho\phi\rho' + \frac{1}{2}N\rho^2\phi'\right)\delta f^2 - \left(2N'\rho\phi f^2 + N\rho\phi f^{2\prime}\right)\delta\rho\right.$$
$$\left. + 2N\rho\phi f^2\delta\rho'\right]_{r_e}^{\infty}, \tag{8.54}$$

$$\delta B_\phi = \frac{\beta\sigma}{8\pi}\left[-\left(\frac{3}{2\phi}N\rho^2 f^2\phi' + N'\rho^2 f^2 + \frac{1}{2}N\rho^2 f^{2\prime}\right)\delta\phi + N\rho^2 f^2\delta\phi'\right]_{r_e}^{\infty}, \tag{8.55}$$

$$\delta B_\psi = \frac{\beta\sigma}{8\pi}\left[\frac{1}{2}\phi N\rho^2 f^2\psi'\delta\psi\right]_{r_e}^{\infty}. \tag{8.56}$$

From the exact solution (8.20), (8.23) and (8.24) we can compute the asymptotic expression at infinity for the variation of the fields with respect to the black hole mass:

$$\delta f^2\big|_\infty = \left[\frac{2m\left(l^2 - 48\alpha\right)}{l^4} + \frac{6m^2\left(48\alpha - l^2\right) - 2l^4}{l^4 r} + O\left(\frac{1}{r^2}\right)\right]\delta m, \tag{8.57}$$

$$\delta\rho\big|_\infty = \left[\frac{m\left(48\alpha - l^2\right)}{l^2 r} + \frac{3m^2\left(l^2 - 48\alpha\right)}{l^2 r^2} + O\left(\frac{1}{r^3}\right)\right]\delta m, \tag{8.58}$$

$$\delta\phi\big|_\infty = \left[-\frac{96\alpha m}{l^2 r^2} + \frac{288\alpha m^2}{l^2 r^3} + O\left(\frac{1}{r^4}\right)\right]\delta m, \tag{8.59}$$

$$\delta\psi\big|_\infty = \left[\frac{2\sqrt{3}}{r} - \frac{4\sqrt{3}m}{r^2} + O\left(\frac{1}{r^3}\right)\right]\delta m. \tag{8.60}$$

Summing all contributions we obtain

$$\delta B\big|_\infty = -\frac{\beta\sigma}{4\pi}\delta m + O\left(\frac{1}{r^2}\right), \tag{8.61}$$

from which the boundary term at infinity can be read off to be

$$B\big|_\infty = -\frac{\beta\sigma}{4\pi}m. \tag{8.62}$$

To compute $B\big|_{r_e}$ instead, let us first notice that $f^2(r_e) = h(r_e)/\Omega(r_e) = 0$, which implies $\delta B_\psi\big|_{r_e} = 0$ and allows to simplify the terms in (8.54) and (8.55) that are evaluated at r_e. Then, using the following relations

$$\delta\rho\big|_{r_e} = \delta\rho(r_e) - \rho'\big|_{r_e}\delta r_e, \tag{8.63}$$

$$\delta f^2\big|_{r_e} = -f^{2\prime}\big|_{r_e}\delta r_e, \tag{8.64}$$

$$\delta\phi\big|_{r_e} = \delta\phi(r_e) - \phi'\big|_{r_e}\delta r_e, \tag{8.65}$$

one has

$$\delta B\big|_{r_e} = -\frac{\beta\sigma}{16\pi}\left[N\phi {f^2}'\delta\rho^2(r_e) + N{f^2}'\rho^2\delta\phi(r_e)\right]$$
$$= -\frac{\beta\sigma}{16\pi}N{f^2}'\big|_{r_e}\delta\left(\phi(r_e)\rho^2(r_e)\right). \tag{8.66}$$

Recalling the definition of the Euclidean time period,

$$N{f^2}'\big|_{r_e} = h'\big|_{r_e} = \frac{4\pi}{\beta}, \tag{8.67}$$

the result can be written as

$$\delta B\big|_{r_e} = -\frac{\sigma}{4}\delta\left(\phi(r_e)\rho^2(r_e)\right), \tag{8.68}$$

which leads to the boundary term at the horizon

$$B\big|_{r_e} = -\frac{\sigma}{4}\phi(r_e)\rho^2(r_e). \tag{8.69}$$

The addition of the two contributions finally gives the Euclidean action:

$$I = -\frac{\beta\sigma}{4\pi}m + \phi(r_e)\frac{\sigma\rho^2(r_e)}{4}. \tag{8.70}$$

This procedure allows to obtain the above result without knowing the full expression of B, only its value at the boundary. However, we provide here the explicit covariant expression for the counter-terms:

$$I^1_{ct} = -\frac{1}{8\pi}\int_{\partial\mathcal{M}} d^3x\sqrt{|^{(3)}g|}\frac{2}{l}\phi\sqrt{\phi}, \tag{8.71}$$

$$I^2_{ct} = -\frac{1}{8\pi}\int_{\partial\mathcal{M}} d^3x\sqrt{|^{(3)}g|}\frac{l}{2}\sqrt{\phi}\ {}^3R, \tag{8.72}$$

$$I^\psi_{ct} = \frac{1}{16\pi}\int_{\partial\mathcal{M}} d^3x\sqrt{|^{(3)}g|}\frac{\phi\sqrt{\phi}}{6l}\left[\frac{2l(\psi-\psi_0)}{\sqrt{\phi}}n^\mu\partial_\mu\psi - (\psi-\psi_0)^2\right], \tag{8.73}$$

where ${}^3R$ is the three dimensional Ricci curvature of the boundary metric and n^μ the unit normal to the boundary. They must be included at the Lagrangian level. The role of I^1_{ct}, I^2_{ct} and I^ψ_{ct} is solely the regularization of the divergence of the action. For this reason, and because we introduced B at the Hamiltonian level, a direct comparison between their asymptotic expansion at infinity and (8.61) is not possible. However, it is easy to show that the original action (5.10) completed with (8.71), (8.72), (8.73) and (8.46) leads to the same result as in (8.70).

In the limit $\phi \to 1$, these counter-terms correctly reproduce the ones reported in [237] for a minimally coupled scalar field and they are a generalization of them to the non-minimal coupling case. Contrary to what happens in the absence of additional scalar fields or for localized distributions of matter [236] with radial fall off $\sim r^{-3/2+\varepsilon}$ at infinity, the above counter-terms explicitly depend on the scalar fields. The reason is that the fall off of at infinity is given by $\psi \sim \psi_0 + 2\sqrt{3}mr^{-1} + O(r^{-2})$ and it is slower with respect to localized distributions of matter. This results in a back-reaction on the metric and requires the counter-terms to depend on the scalar fields as well, in order to cancel the divergences.
We can now use (8.70) to compute the energy and entropy of the black hole. Applying (8.38) we get

$$M = \frac{\sigma m}{4\pi}, \tag{8.74}$$

$$S = \phi(r_e)\frac{A}{4}, \tag{8.75}$$

where the area of the horizon is defined by

$$A = \sigma\rho^2(r_e) = \sigma\Omega(r_e)r_e^2. \tag{8.76}$$

Now, since the scalaron is actually an auxiliary field in this theory, being its dynamics sourced by the Immirzi field via the structural equation, one can use (8.11) to write the entropy as

$$S = \left[1 + 4\alpha W(\psi_e)\right]\frac{A}{4}, \tag{8.77}$$

where we have denoted with ψ_e the value of the Immirzi field at the black hole event horizon. We see a deviation from the well-known Bekenstein-Hawking entropy formula, i.e. $S = A/4$. Moreover, we note here that in the $m = 0$ case, although the mass of the black hole is vanishing, the entropy is not, having instead the residual contribution $S(m = 0) = \sigma l^2/4$. This property is consistent with AdS topological black holes with no hair studied in literature [226, 227].
A further check of our result for the expression of the entropy comes from the non-minimal coupling of the scalar field, which has been shown to lead to similar modifications in other contexts [238, 239].
Finally, we find agreement with the outcome obtained by applying Wald's entropy formula [240] starting from the original first order action (1.73), notwithstanding the presence of torsion in the theory (see [241] for a discussion on Wald's entropy in models with torsion).
Now, as mentioned in the introduction of this section, asymptotically AdS black holes offer an interesting arena to investigate thermodynamic properties that go even beyond the traditional laws of black hole thermodynamics [228]. This is

due to the possibility of enlarging the thermodynamic phase space with the inclusion of an effective pressure that is generated by the cosmological constant as

$$P = -\frac{\Lambda}{8\pi}, \tag{8.78}$$

which is positive definite for negative Λ. Together with pressure, we have its conjugate quantity, the thermodynamic volume V. Its expression is determined by imposing the first law of thermodynamics, i.e.

$$dM = TdS + VdP. \tag{8.79}$$

In the context of the extended phase space approach, M in the formula above is interpreted as the enthalpy rather than the internal energy of the system. We will now make a brief detour to present the literature regarding some known results on the properties of the thermodynamic volume.
In [242] a conjecture was proposed, stating that for every asymptotically AdS black hole, the so-called reverse isoperimetric inequality (RII) holds. Here the term *reverse* stands for the fact that the behavior is actually inverted if compared to the isoperimetric inequality that relates area and the volume enclosed in Euclidean geometry. The RII states that $\mathscr{I} \geqslant 1$, where

$$\mathscr{I} = \left(\frac{(d-1)V}{\omega_{d-2}^{(k)}}\right)^{\frac{1}{d-1}} \left(\frac{\omega_{d-2}^{(k)}}{A}\right)^{\frac{1}{d-2}}, \tag{8.80}$$

for arbitrary dimension d and generalized unit volume $\omega_{d-2}^{(k)}$ of the $d-2$ dimensional base manifold of constant curvature k. Originally, the motivation in support of this conjecture was simply the observation that all known solutions satisfied the inequality. However, after a while some counterexamples were found for which the conjecture is violated.
The physical interpretation behind the inequality is the following. Since the inequality is saturated by the Schwarzschild-Anti de Sitter (SAdS) black hole, which settles on the lower bound $\mathscr{I} = 1$, then, according to the conjecture, this solution is the one maximising the entropy for a given thermodynamic volume. This is the reason why solutions violating the conjecture have been dubbed *superentropic* black holes: for a given volume they allow for a greater area (and therefore[4] a greater entropy) than the simple SAdS case.
Examples of superentropic black holes include black holes with non-compact

[4]This implication is trivially true when the entropy is given by $S = A/4$. However, for more complex cases, as the one considered here, it should be verified explicitly as we do in the following.

horizons, Lifshitz black holes, three-dimensional black holes [243, 244, 245, 246]. Violations for hairy black holes with planar ($k = 0$) horizons in four dimensions were also observed in [247].
Further investigations in literature were devoted to analyse the thermodynamic stability of superentropic solutions. In [248] the superentropic behaviour was shown to be related to a thermodynamic instability caused by a negative specific heat at constant volume c_V. Subsequently, in [249] exotic BTZ black holes were shown to lead to RII violations even for $c_V > 0$. However, the authors also proved that, whenever the condition $c_V > 0$ holds, the specific heat at constant pressure c_P becomes negative, leading again to a thermodynamic instability.
Now we want to understand how the solution presented in this section fits in this landscape. First, we can compute the volume substituting the expression for the other thermodynamic variables into the first law. The outcome is

$$V = \frac{\sigma l^2}{3}(l + 3m) = \frac{\sigma}{3}\left(3lr_e^2 - 3l^2 r_e + l^3\right). \tag{8.81}$$

The first thing to notice is that it does not coincide with the geometric volume, i.e.

$$V_{geom} = \frac{\sigma(r_e\sqrt{\Omega(r_e)})^3}{3}. \tag{8.82}$$

The same happens also for other solutions that are more complex compared to the Schwarzschild-AdS case [250, 246, 251]. Despite this discrepancy, it shares some common properties with V_{geom}: it is a positive definite increasing monotonic function of r_e (for $r > r_c$) and it is proportional to the genus g of the horizon. A minimum value $V_{min} = \sigma l^3/12$ exists and it is attained at $r_e = r_c$. One can also verify that the Smarr formula holds:

$$M = 2(TS - PV). \tag{8.83}$$

We can now compute the isoperimetric ratio $\mathcal{I}$ for our black hole. Setting $d = 4$, $k = -1$ and $\omega_2^{(-1)} = \sigma$ in (8.80) and substituting the expressions for A and V yields

$$\mathcal{I} = \frac{\left(3lr_e^2 - 3l^2 r_e + l^3\right)^{\frac{1}{3}}}{\sqrt{l(2r_e - l) + \frac{48\alpha}{l^2}(r_e - l)^2}}. \tag{8.84}$$

It is trivial to show that the RII is violated in almost all parameter space. In particular we have that:

- For $\alpha \geqslant l^2/(24\sqrt[3]{2})$ the black hole is always super-entropic. The violation of the RII occurs for every $r_e > r_c$, namely for every $T > 0$ and $V > V_{min}$;

- For $0 < \alpha < l^2/(24\sqrt[3]{2})$ the RII is violated only for $r_e > \bar{r}$, where $\bar{r}$ is one of the roots of the fourth order polynomial

$$\begin{aligned}\mathscr{P}(r) = &\ 110592\alpha^3 r^4 + \left(13824\alpha^2 l^3 - 442368\alpha^3 l\right) r^3 \\ &+ \left(-9l^8 + 576\alpha l^6 - 34560\alpha^2 l^4 + 663552\alpha^3 l^2\right) r^2 \\ &+ \left(8l^9 - 576\alpha l^7 + 27648\alpha^2 l^5 - 442368\alpha^3 l^3\right) r \\ &- 2l^{10} + 144\alpha l^8 - 6912\alpha^2 l^6 + 110592\alpha^3 l^4. \end{aligned} \tag{8.85}$$

  The root $\bar{r}$ satisfies $\bar{r} > r_c$ and it is related to a specific temperature $\bar{T} \equiv T(\bar{r})$ via (8.45). In this case the black hole is superentropic only above this temperature, i.e. for $T > \bar{T}$;

- Finally, for $\alpha = 0$, i.e. for the MTZ solution [232], the black hole is subentropic since $\mathscr{I} \geqslant 1$ always holds. The inequality is saturated by $r_e = l$, for which $\mathscr{I} = 1$, which corresponds to pure AdS space.

Now, there is a caveat in this case and for every solution in which the relation between area and entropy is not the trivial Bekenstein-Hawking law. From the three cases above we can conclude the violation (or the validity) of the RII. Can we directly relate these results to the super(sub)-entropic behavior of the black hole? In other words, when $\mathscr{I} < 1$ the black hole can have a larger area than that of a SAdS one. Thus a larger area also imply a larger entropy? Only if the entropy is a monotonically increasing function of the area. We can check that by solving (8.76) for $r_e(A)$ (choosing the positive branch, for which $r_e > 0$ and $A > 0$) and then substituting it in the definition of the entropy. This yields

$$S(A) = \frac{\sigma l^2}{96\alpha}\left(24\alpha - l^2 + \sqrt{l^2(l^2 - 48\alpha) + \frac{48\alpha}{\sigma}A}\right), \tag{8.86}$$

which is a monotonically increasing function of A. Therefore, a violation of the RII still implies that the black hole is super-entropic.
Having established that, let us now discuss the thermodynamic stability of the black hole and in particular of its superentropic configurations. The solution at hand is halfway between the two solutions of [248] and [249], since c_P is always positive, as in [248], but there are configurations characterized by $c_V > 0$ and for which the RII is violated, namely there are superentropic black holes with both specific heats positive. In spite of that, we conclude that superentropic black holes are always thermodynamically unstable.
To see this, we first note that, comparing (8.45) and (8.77), we can relate entropy and temperature as

$$S = \frac{\sigma\pi}{2}\left(\frac{3}{8\pi P}\right)^{\frac{3}{2}} T. \tag{8.87}$$

Then, the specific heat at constant pressure can be computed, yielding

$$c_P = T\frac{\partial S}{\partial T}\bigg|_P = \frac{\sigma\pi}{2}\left(\frac{3}{8\pi P}\right)^{\frac{3}{2}} T > 0, \tag{8.88}$$

which is manifestly positive.
Let us now turn to the more involved computation of c_V. It can be carried out using the following relations:

$$c_P - c_V = TV\alpha_P^2 k_T, \tag{8.89}$$

$$\frac{c_V}{c_P} = k_T\beta_S, \tag{8.90}$$

where the isobaric thermal expansion coefficient, the isothermal bulk modulus and the adiabatic compressibility are defined by, respectively

$$\alpha_P \equiv \frac{1}{V}\frac{\partial V}{\partial T}\bigg|_P, \tag{8.91}$$

$$k_T \equiv -V\frac{\partial P}{\partial V}\bigg|_T, \tag{8.92}$$

$$\beta_S \equiv -\frac{1}{V}\frac{\partial V}{\partial P}\bigg|_S. \tag{8.93}$$

Eliminating k_T from (8.89) and (8.90) and using (8.88) yields

$$c_V = \frac{S^2\beta_S}{S\beta_S + TV\alpha_P^2}. \tag{8.94}$$

Now, the volume (8.81) can be express in terms of P and M and then, by virtue of the Smarr formula (8.83), we can write

$$V = \frac{\sigma}{12}\left(\frac{3}{8\pi P}\right)^{\frac{3}{2}}\left(1 + \frac{9\pi T^2}{2P}\right), \tag{8.95}$$

which easily yields β_S and α_P. Substituting them in (8.94) and using (8.87) results in

$$c_V = \frac{3\sqrt{\frac{3\pi}{2}}T\left(2P - 3\pi T^2\right)}{8P^{3/2}\left(2P + 15\pi T^2\right)}, \tag{8.96}$$

which is independent on the parameter α. We now want to study the dependence of this function on T at fixed volume. In order to do this we must first express

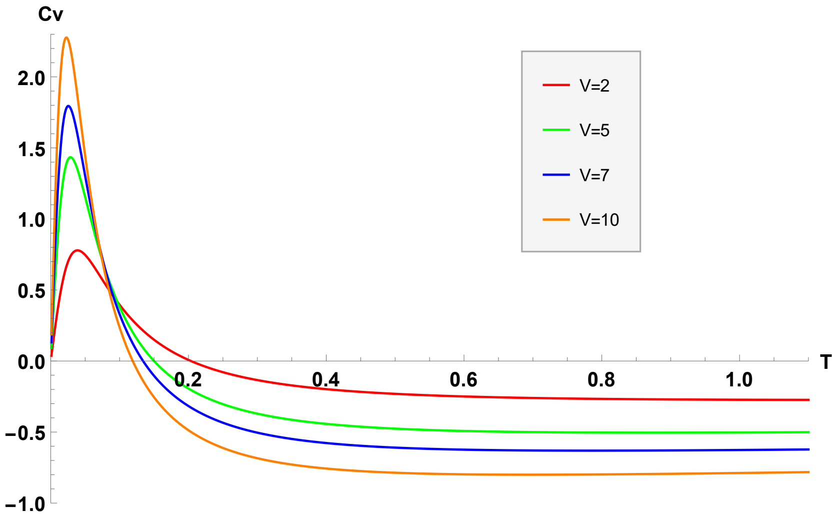


Figure 8.1: Specific heat at constant volume c_V as a function of T for different values of V.

pressure in terms of T and V inverting (8.95). We do this numerically for different values of T and V, choosing positive values of T and checking that $V > V_{min}$ for every value of V. The outcome presented in Fig. 8.1, shows that for every value of the volume there exists a temperature T^* below which the specific heat becomes positive. Now, for $\alpha \geqslant l^2/(24\sqrt[3]{2})$ the black hole is always superentropic and it is possible to simultaneously have $\mathcal{I} < 1$ and $c_V > 0$, together with $c_P > 0$, which always holds. For $0 < \alpha < l^2/(24\sqrt[3]{2})$ instead, the black hole is superentropic only for $T > \bar{T}$. However, even if the value $\bar{T}$ can be either above or below the turning point T^*, depending on the specific value of α, we found that for every α there is always a thermodynamic configuration, namely values of V and T, such that $\bar{T} < T^*$. Therefore, in all cases there are superentropic black holes with both specific heats positive.

We observe a situation similar to the one in [249]. There, the behaviour of c_V is inverted, being positive at large temperatures. The crucial difference is that here c_P is positive definite. This seems to suggest that thermodynamic equilibrium is possible for superentropic black holes since there are configurations characterized by positive specific heats. However, for every value of V there is always a temperature above which c_V becomes negative. This region could be excluded if there existed two separate branches of black hole solutions, as it happens for the SAdS solution [228]. However, such separation does not occur here as there is only one connected branch. As argued in [249], it is sufficient to have $c_V < 0$ for at least some part of the branch to make the whole branch thermodynamically unstable. Therefore, we conclude that the black hole solution we found satisfies the broader conjecture, proposed in [249], that black holes violating the reverse isoperimetric inequality are thermodynamically unstable.

Chapter 9

Homogeneous cosmological models with generalized Nieh-Yan term

From a theoretical consistency point of view, the presence of singularities in the solutions of General Relativity constitutes perhaps one of the greatest shortcoming of the theory. Beside black hole spacetimes, singularities are also present in cosmological solutions, to which this section will be devoted. The issue is already present in the relevant homogeneous and isotropic case, described by the FLRW metric, which present a big-bang singularity at the origin of coordinate time. Although the FLRW metric offers a great description of most stages of the universe evolution, it is widely believed that one cannot extrapolate the solution to arbitrarily small times, and some corrections must be implemented in the most primordial epochs. These corrections may be offered by quantum gravity effects, see e.g. [252], but they can also be provided by completely classical mechanisms, as it happens in modified gravity models [253, 254, 255, 256, 257]. In this section we will focus on the second option, with the aim of checking the feasibility of these mechanisms in the generalized Nieh-Yan model introduced before.
Homogeneity and isotropy are other features that may need to be abandoned in the primordial stages. Indeed, many cosmological models attempting to described the very early stages of the universe exist, in which one or both assumptions are discarded [B258]. The straightforward generalization of the FLRW solution consists in keeping the homogeneity condition and dropping the assumption of isotropy, leading to the Bianchi homogeneous cosmological models [259, B258]. Among them, the Bianchi I solution constitutes the simplest model, in which three different scale factors take part in the universe evolution, one for each spatial direction.
When considered in modified gravity theories, both the FLRW and Bianchi I mod-

els have been shown to allow for a resolution of the big-bang singularity, which is replaced by a big-bounce scenario. The latter is characterized by a contraction of the universe volume down to a minimum value and a subsequent re-expansion of the universe.
Among the many gravitational models in which this behavior has been observed, those featuring an Immirzi field are given by [169, 170]. In particular, in [170] the existence of a big-bounce solution was established in the presence of the standard Nieh-Yan term. Here we will generalize this result to the model introduced in section 6.
In order to proceed further, we will consider as a specific theory the topological case of the Nieh-Yan model in (6.20), which is identified by $\lambda = 1$. Moreover, we will also assume the potential to satisfy $V(\phi, \psi) = V(\phi)$. As a concrete example we will consider for the function $F(\mathscr{R}, \mathscr{N}\mathscr{Y}^*)$ the effective form $F(\mathscr{R}, \mathscr{N}\mathscr{Y}^*) \simeq \mathscr{R} + \alpha\mathscr{R}^2 + \mathscr{N}\mathscr{Y}^*$, which corresponds to the Starobinsky quadratic potential [85]:

$$V(\phi) = \frac{1}{\alpha}\left(\frac{\phi - 1}{2}\right)^2 . \tag{9.1}$$

These are necessary simplifications for the achievement of the following results. We will rewrite here the field equations for convenience:

$$\begin{aligned}\bar{G}_{\mu\nu} = {}& \frac{\kappa^2}{\phi}T_{\mu\nu} + \frac{1}{\phi}\left(\bar{\nabla}_\mu\bar{\nabla}_\nu - g_{\mu\nu}\Box\right)\phi - \frac{3}{2\phi^2}\bar{\nabla}_\mu\phi\bar{\nabla}_\nu\phi + \frac{3}{2\phi^2}\bar{\nabla}_\mu\psi\bar{\nabla}_\nu\psi + \\ & + \frac{1}{2}g_{\mu\nu}\left(\frac{3}{2\phi^2}(\bar{\nabla}\phi)^2 - \frac{3}{2\phi^2}(\bar{\nabla}\psi)^2 - \frac{V(\phi)}{\phi}\right),\end{aligned} \tag{9.2}$$

and

$$2V(\phi) - \phi\frac{dV(\phi)}{d\phi} = \kappa^2 T - \frac{3(\bar{\nabla}\psi)^2}{\phi}, \tag{9.3}$$

$$\Box\psi - \bar{\nabla}_\mu \ln\phi\bar{\nabla}^\mu\psi = 0. \tag{9.4}$$

We will now provide a semi-analytical solution to these equations in the anisotropic cosmological setting.

9.1 Big-Bounce in Bianchi I anisotropic models

Let us therefore consider the line element of a Bianchi I flat metric, given by

$$ds^2 = -dt^2 + a(t)^2dx^2 + b(t)^2dy^2 + c(t)^2dz^2. \tag{9.5}$$

The anisotropic character of the spacetime is encoded in the presence of three different scale factors, $a(t), b(t), c(t)$, constituting the simplest homogeneous cosmology beyond the isotropic FLRW spacetime. The matter content is included as a perfect fluid with stress-energy tensor given by

$$T_{\mu\nu} = \mathrm{diag}(\rho, a^2 p, b^2 p, c^2 p), \tag{9.6}$$

where ρ is the energy density and p the pressure. From (9.2) it is easy to show that the covariant conservation law

$$\bar{\nabla}_\mu T^{\mu\nu} = 0 \tag{9.7}$$

holds, from which the continuity equation is derived as

$$\dot{\rho} + \left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c}\right)(\rho + p) = 0, \tag{9.8}$$

where a dot represents derivatives with respect to the coordinate time t. For the usual equation of state $p = w\rho$ we can then write

$$\rho(t) = \frac{\mu^2}{(abc)^{w+1}}, \tag{9.9}$$

for constant μ^2. Now, the first great simplification comes from the fact that the Immirzi field derivative $\dot{\psi}$ and the scalaron ϕ can both be expressed in term of the volume-like variable defined by $v = abc$. Indeed, equation (9.4) gives

$$\dot{\psi} = \frac{k_0 \phi}{abc}, \tag{9.10}$$

which substituted into (9.3) yields

$$\phi = \frac{v^2 f(v)}{6\alpha k_0^2 + v^2}, \tag{9.11}$$

where we defined the function

$$f(v) \equiv 1 - 2\alpha\kappa^2(3w - 1)\rho(v). \tag{9.12}$$

Regarding the metric field equations instead, the relevant components are the tt, xx, yy and zz ones, given by

$$\frac{\dot{a}\dot{b}}{ab}+\frac{\dot{a}\dot{c}}{ac}+\frac{\dot{b}\dot{c}}{bc}=\frac{\kappa^2\rho}{\phi}+\frac{3k_0^2}{4v^2}-\left(\frac{\dot{a}}{a}+\frac{\dot{b}}{b}+\frac{\dot{c}}{c}\right)\frac{\dot{\phi}}{\phi}-\frac{3\dot{\phi}^2}{4\phi^2}+\frac{V(\phi)}{2\phi}, \tag{9.13}$$

$$\frac{\ddot{b}}{b}+\frac{\ddot{c}}{c}+\frac{\dot{b}\dot{c}}{bc}=-\left(\frac{\dot{b}}{b}+\frac{\dot{c}}{c}\right)\frac{\dot{\phi}}{\phi}+\Phi, \tag{9.14}$$

$$\frac{\ddot{a}}{a}+\frac{\ddot{c}}{c}+\frac{\dot{a}\dot{c}}{ac}=-\left(\frac{\dot{a}}{a}+\frac{\dot{c}}{c}\right)\frac{\dot{\phi}}{\phi}+\Phi, \tag{9.15}$$

$$\frac{\ddot{a}}{a}+\frac{\ddot{b}}{b}+\frac{\dot{a}\dot{b}}{ab}=-\left(\frac{\dot{a}}{a}+\frac{\dot{b}}{b}\right)\frac{\dot{\phi}}{\phi}+\Phi, \tag{9.16}$$

respectively, where

$$\Phi\equiv-\frac{\kappa^2 p}{\phi}-\frac{3k_0^2}{4v^2}-\frac{\ddot{\phi}}{\phi}+\frac{3\dot{\phi}^2}{4\phi^2}+\frac{V(\phi)}{2\phi}. \tag{9.17}$$

Now, in order to get some insight on the physical consequences of this system, it will be useful to perform some manipulations in order to rewrite the first equation in terms of the volume v alone. This equation reduces to the Friedmann equation in the isotropic limit $a=b=c$, and will allow to determine the presence of singularities, big-bounce or turning points in the volume evolution. Combining equations (9.15) and (9.16), we obtain

$$\frac{\ddot{b}}{b}-\frac{\ddot{c}}{c}+\left(\frac{\dot{b}}{b}-\frac{\dot{c}}{c}\right)\frac{\dot{a}}{a}=-\left(\frac{\dot{b}}{b}-\frac{\dot{c}}{c}\right)\frac{\dot{\phi}}{\phi}, \tag{9.18}$$

which can be integrated for a yielding

$$a=\frac{k_1}{\phi(\dot{b}c-b\dot{c})}, \tag{9.19}$$

where k_1 is an integration constant. Similar relations can be derived for the other scale factors as

$$b=\frac{k_2}{\phi(\dot{a}c-a\dot{c})}, \tag{9.20}$$

$$c=\frac{k_3}{\phi(\dot{a}b-a\dot{b})}. \tag{9.21}$$

It is easy to check that the following constraint must hold:

$$k_1-k_2+k_3=0. \tag{9.22}$$

It is now convenient to introduce Hubble-like functions defined by

$$H_A = \frac{\dot{a}}{a}, \tag{9.23}$$

$$H_B = \frac{\dot{b}}{b}, \tag{9.24}$$

$$H_C = \frac{\dot{c}}{c}, \tag{9.25}$$

in terms of which we can express (9.19)-(9.21) as

$$H_B - H_C = \frac{k_1}{\phi v}, \tag{9.26}$$

$$H_A - H_C = \frac{k_2}{\phi v}, \tag{9.27}$$

$$H_A - H_B = \frac{k_2}{\phi v}, \tag{9.28}$$

which combined give

$$H_A H_B + H_A H_C + H_B H_C = H_A^2 + H_B^2 + H_C^2 - \frac{3\mu_A^2}{\phi^2 v^2}. \tag{9.29}$$

Here we have defined the anisotropy density parameter as

$$\mu_A^2 \equiv \frac{k_1^2 + k_2^2 + k_3^2}{6\kappa^2}. \tag{9.30}$$

Next, noting that

$$\frac{\dot{v}}{v} = H_A + H_B + H_C, \tag{9.31}$$

and changing time derivatives of ϕ in (9.13) into derivatives with respect to the volume via

$$\frac{\dot{\phi}}{\phi} = \frac{\dot{v}}{\phi}\frac{d\phi}{dv}, \tag{9.32}$$

we can write

$$H_A^2 + H_B^2 + H_C^2 = \left(\frac{\dot{v}}{v}\right)^2 - 2(H_A H_B + H_A H_C + H_B H_C). \tag{9.33}$$

Then, taking into account (9.29), (9.32) and (9.33), from (9.13) we finally obtain

$$H^2 \equiv \left(\frac{\dot{v}}{3v}\right)^2 = \frac{\frac{\kappa^2}{3}\left(\frac{\mu_I^2}{v^2} + \frac{\rho}{\phi} + \frac{\mu_{AN}^2}{\phi^2 v^2}\right) + \frac{V(\phi)}{6\phi}}{\left(1 + \frac{3v}{2}\frac{d}{dv}\ln\phi\right)^2}, \tag{9.34}$$

where the energy density parameter for the Immirzi field has been defined as

$$\mu_I^2 \equiv \frac{3k_0^2}{4\kappa^2}. \tag{9.35}$$

The merit of these manipulations is that now the l.h.s of equation (9.34) is manifestly positive and the r.h.s. is a rational function of v. It follows that zeros and poles of the function on the r.h.s. signal the presence of singularities, big-bounce and turning points, allowing to determine the qualitative behavior of the volume evolution by means of simple algebraic considerations.
In particular, the sign of the r.h.s can be affected by different matter contents and different values of the parameter α. For configurations for which the r.h.s is always positive, the volume v can span all the positive values, i.e. $v \in \mathbb{R}^+$, and the dynamics will still be singular in $v = 0$. On the other hand, the r.h.s. may change sign for some values of v where $H^2 = 0$. Then, one has to select those intervals in which $H^2 > 0$. Lower bounds of such intervals correspond to big-bounces and upper bounds to turning points.

9.1.1 Vacuum case

Let us perform a preliminary investigation in the simplest case provided by the vacuum configuration, where $\rho = 0$ and $f(v) = 1$. In that case, we can rearrange (9.34) as

$$H^2(v) = \frac{\kappa^2(v^2+\eta_I)\left(P_A(v)\mu_A^2 + P_I(v)\mu_I^2\right)}{6v^6\left(v^2+4\eta_I\right)^2}, \tag{9.36}$$

where

$$P_A(v) = 2v^6 + 6\eta_I v^4 + 6\eta_I^2 v^2 + 2\eta_I^3, \tag{9.37}$$

$$P_I(v) = 2v^4(v^2+2\eta_I) \tag{9.38}$$

and $\eta_I \equiv 6\alpha\kappa^2\mu_I^2$. Now, for $\alpha > 0$ (i.e. $\eta_I > 0$) it is clear that the r.h.s is always positive, so that the singularity at $v = 0$ is not resolved. For the $\alpha < 0$ case instead, the outcome is determined by the inequality

$$(v^2+\eta_I)\left(P_A(v)\mu_A^2 + P_I(v)\mu_I^2\right) \geq 0, \tag{9.39}$$

which is solved by

$$\begin{aligned} &0 < v^2 < v_T^2 \equiv -\eta, \\ &v_B^2 < v^2, \end{aligned} \tag{9.40}$$

where v_B^2 is the only real root of the third order polynomial in $x = v^2$ featuring (9.39). Its explicit expression reads

$$v_B^2 = -\frac{1}{3(1+\lambda_{AI}^2)\eta_I}\left[\frac{2^{1/3}(4+3\lambda_{AI}^2)}{Q_{\frac{4}{3}}(\lambda_{AI})} + (2+3\lambda_{AI}^2)\eta_I^2 + \frac{Q_{\frac{4}{3}}(\lambda_{AI})}{2^{1/3}}\eta_I^4\right], \tag{9.41}$$

where

$$Q_{\frac{4}{3}}(\lambda_{AI}) \equiv \sqrt[3]{16 - Q_2(\lambda_{AI})\lambda_{AI} + 45\lambda_{AI}^2 + 27\lambda_{AI}^4}, \tag{9.42}$$

$$Q_2(\lambda_{AI}) \equiv 3\sqrt{3}\sqrt{32 + 91\lambda_{AI}^2 + 86\lambda_{AI}^3 + 27\lambda_{AI}^4}, \tag{9.43}$$

From (9.40) we can outline two disconnected branches for the volume evolution. The first one describes a closed universe where the singularity is not removed. In addition, one cannot recover General Relativity since the scalaron ϕ diverges in the limit $v \to -\eta$. The second case looks more promising, describing an open universe where the big-bang singularity is classically resolved and replaced by a big-bounce in $v = v_B$. Moreover, the General Relativity scenario is correctly recovered in the limit $v \to +\infty$, in which $\phi \to 1$. In the same limit one is also able to reproduce the ordinary loop quantum gravity picture with a constant Immirzi parameter, since the Immirzi field derivative (9.10) vanishes in the late-time region.

We confirm these qualitative results via numerical solutions obtained integrating equation (9.36) for $v(t)$. In order to work with dimensionless variables all quantities are rescaled by the appropriate power of the Planck time t_{Pl}. The big-bounce in the volume evolution is shown in Fig. 9.2a, where another feature stands out: there is a future finite-time singularity (see [260, 261] for details concerning their classification) caused by the pole of equation (9.36) in $v_c = -4\eta_I$, where the Hubble function diverges. From the viewpoint of the numerical techniques used, this causes a breakdown in the integration which requires taking care of the two regions adjacent to the singularity separately, matching the two solutions across $v_c = -4\eta_I$.

In general, quantum effects of particle creation in the presence of cosmological horizons [262, 263, 264, 265] can lead to additional terms in the Friedman equation, able to stabilize such singular behaviours of the Hubble parameter.

However, the physical viability of the solutions presented here must be tested in order to assure that their extension across the singularities be without ambiguities or inconsistencies. In particular, in the next section we will analyze the behavior of null geodesics and scalar perturbations close to the singular points, requiring them to be free of pathologies in order to accept the solution or discard it in the negative case.

Beside the behavior of the volume, one can also track the time evolution of the scalar fields via (9.10) and (9.11). Results in Fig. 9.2c show the asymptotic relaxation of the scalaron and the Immirzi field derivative to 1 and zero, respectively. The time evolution of each scale factor is reported in Fig. 9.2b while Fig. 9.2d displays the minimum value of the volume attained at the bounce v_B for varying values of the parameter α.

9.1.2 Radiation and dust

The simple vacuum scenario just outlined can be extended to the inclusion of matter in the form of radiation and pressureless dust, respectively identified by $w = 1/3$ and $w = 0$. In this case (9.11) becomes

$$\phi(v) = \frac{v(v + 2\eta_D)}{v^2 + \eta_I}, \tag{9.44}$$

from which we see that there is an additional zero in $v_P = -2\eta_D \equiv -2\alpha\kappa^2\mu_D^2$, caused by the presence of dust. This value of the volume can be lesser or greater than the value at the bounce v_B. In the former case, it is excluded from the domain of the values of v and it is never reached. In the latter case, it actually affects the evolution of the scale factors $a(t)$, $b(t)$ and $c(t)$, since a zero of ϕ corresponds to a pole in (9.19)-(9.21) and the physical feasibility of the corresponding singularity must be analyzed.
In the presence of matter, the Hubble rate (9.34) can be written as

$$H^2 = \frac{\kappa^2(v^2 + \eta_I)\sum_j P_j(v)\mu_j^2}{6v^4\left(v^3 - \eta_D v^2 + 4\eta_I v + 5\eta_I\eta_D\right)^2}, \tag{9.45}$$

where $j = D, R, A, I$ and

$$P_D(v) = 2v^7 + 5\eta_D v^6 + 2(\eta_D^2 + \eta_I)v^5 + 7\eta_I\eta_D v^4 + \frac{7}{2}\eta_I^2 v^3 + 5\eta_I^2\eta_D v^2, \tag{9.46}$$

$$P_R(v) = 2v^{5/3}(v^5 + 2\eta_D v^4 + 2\eta_I v^3 + 4\eta_I\eta_D v^2 + \eta_I^2 v + 2\eta_I^2\eta_D). \tag{9.47}$$

The main feature present in the vacuum case, namely the big-bang regularization via a big-bounce scenario, is present also in this case. However, the solutions now present some different properties, both in the early stages of the universe evolution and in the late-time asymptotic region. Regarding the latter, the presence of matter affects the behavior of the degree of anisotropy of the universe, which can be defined as

$$A(t) = \frac{\left(H_A^2 + H_B^2 + H_C^2\right)}{3H^2} - 1, \tag{9.48}$$

with $A(t) = 1$ representing a perfectly isotropic universe. We can compute the function $A(t)$ from the time evolution of the single scale factors, obtained integrating directly (9.19), (9.20) and (9.21), once $v(t)$ and $\phi(t)$ are known. Results are shown in Fig. 9.1b. We see that the vacuum case is an oversimplification which does not allow for the universe isotropization at late times, which is instead possible whenever the presence of dust and/or radiation is taken into account.
In the early phase instead, two different scenarios settle in presence of matter, depending on the value of α:

- In the parameter space region identified by $\bar{\alpha} < \alpha < 0$, with

$$\bar{\alpha} = -2\mu_I^2/\mu_D^4, \tag{9.49}$$

  the volume and scalar fields behave qualitatively in the same way as in the vacuum case and the results are similar to the ones already discussed. However, now the value of the volume at the bounce cannot be analytically computed and it depends on the additional dust and radiation energy densities. The finite-time singularity is now located at the real root of the cubic featuring the denominator of (9.45):

$$v_r = \frac{1}{3}\left(\eta_D - \frac{2^{1/3}(12\eta_I - \eta_D^2)}{P_1(\eta_I, \eta_D)} + \frac{P_1(\eta_I, \eta_D)}{2^{1/3}}\right), \tag{9.50}$$

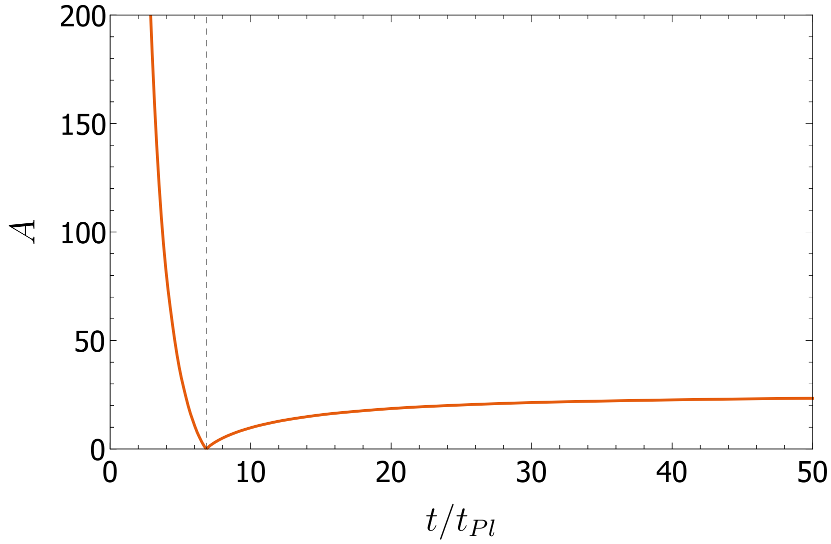
  where we introduced

$$P_1(\eta_I, \eta_D) \equiv \left(2\eta_D^3 - 17\eta_D\eta_I + P_{\frac{5}{3}}(\eta_I, \eta_D)\right)^{5/3}, \tag{9.51}$$

$$P_{\frac{5}{3}}(\eta_I, \eta_D) \equiv 48\sqrt{3}\sqrt{\eta_I(\eta_I + 4\eta_D^2)\left(\eta_I - \frac{5\eta_D^2}{256}\right)}. \tag{9.52}$$

- For $\alpha < \bar{\alpha}$ instead, the solutions present fundamentally different properties. In particular, we note the absence of any future finite-time singularity for v, since the pole of (9.45) is always lesser than the value of the volume at the bounce. On the other hand, now the scalaron vanishes after the bounce, implying zeros and singularities in the scale factors, which can remarkably combine giving a regular behavior for v. The Immirzi field derivative is now negative close to the bounce and it goes to zero at large times, still reproducing a constant Immirzi parameter. The results obtained in this case are presented in Fig. 9.3.

(a) Behavior near the finite-time singularity (dashed line) for $\mu_R = 0$, $\mu_D = 0$.



(b) Asymptotic behavior for various values of μ_R, μ_D after the finite-time singularity.

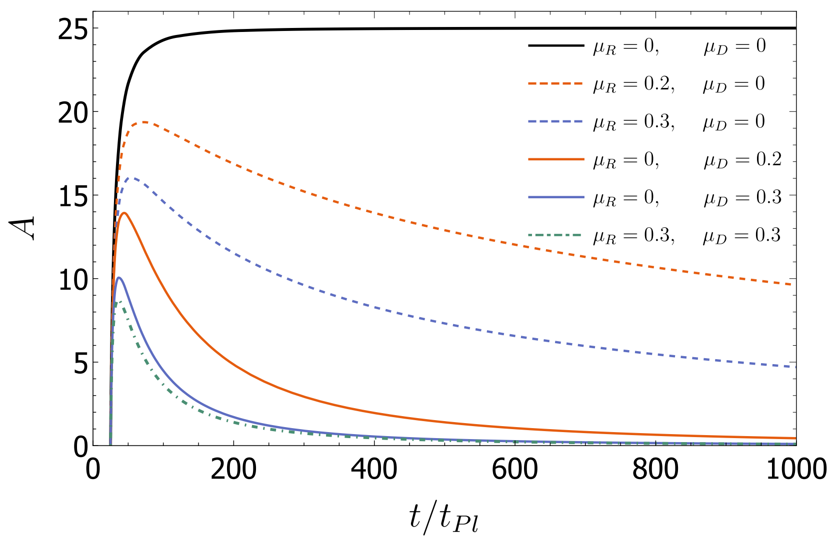


Figure 9.1: Anisotropy degree A as a function of t/t_{Pl} for $\alpha = -5/3$, $\mu_I = \sqrt{3}$, $\mu_A = 0.2\mu_I$.

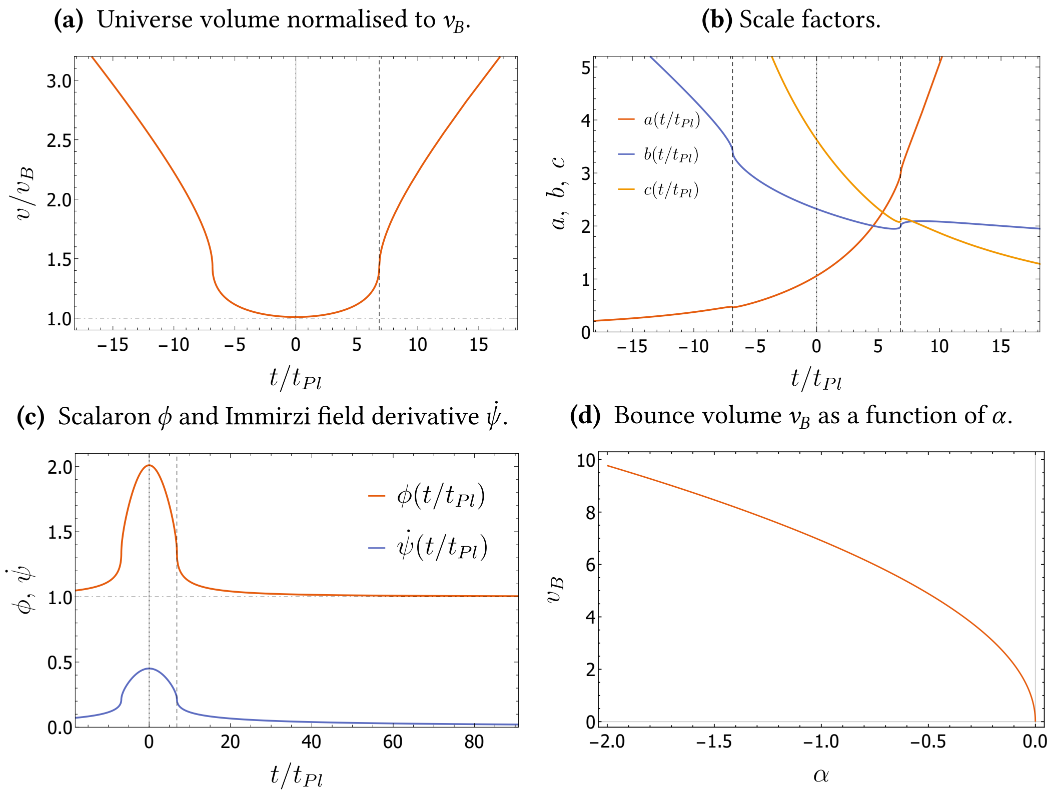
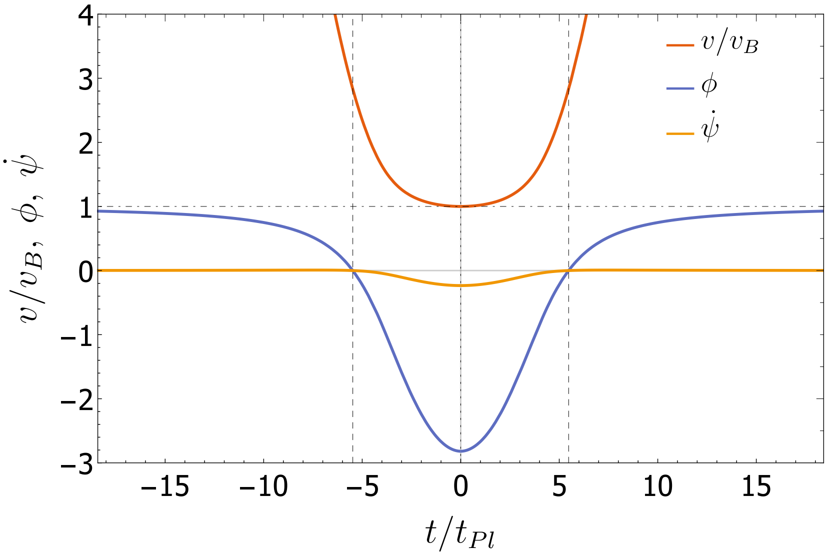


Figure 9.2: Numerical solutions for $\alpha = -5/3$, $\mu_I = \sqrt{3}$, $\mu_A = 0.2\mu_I$ as a function of t/t_{Pl}. Dotted and dashed lines represent where bounce and future time singularity happen, respectively. The bounce is centered at the origin of time for convenience, and the values of the parameters are chosen in order to yield graphs that display features in a clear fashion.

9.2 Future finite-time singularities resolution

The solutions presented in the previous section are characterized by singularities located at a given instant t_c, in proximity of which the numerical integrations break down. These pathologies affect either the evolution of the volume of the universe (Fig. 9.2a) or the behavior of the individual scale factors (Fig. 9.3b) and are related to the divergence of curvature invariants that involve the Hubble function H and its derivatives. The situation demands for a more detailed analysis of the physical implications of such singularities in order to understand how severe they are and to which extent they can be accepted. For concreteness, the discussion will be focused on the behavior of null geodesics and scalar perturbations close to the singularities. Let us start from the former, considering null geodesics with affine parameter s and tangent vector $u^\alpha = dx^\alpha/ds$. The geodesic equation

(a) Volume normalised to v_B, ϕ and derivative of the Immirzi field.



(b) Scale factors.

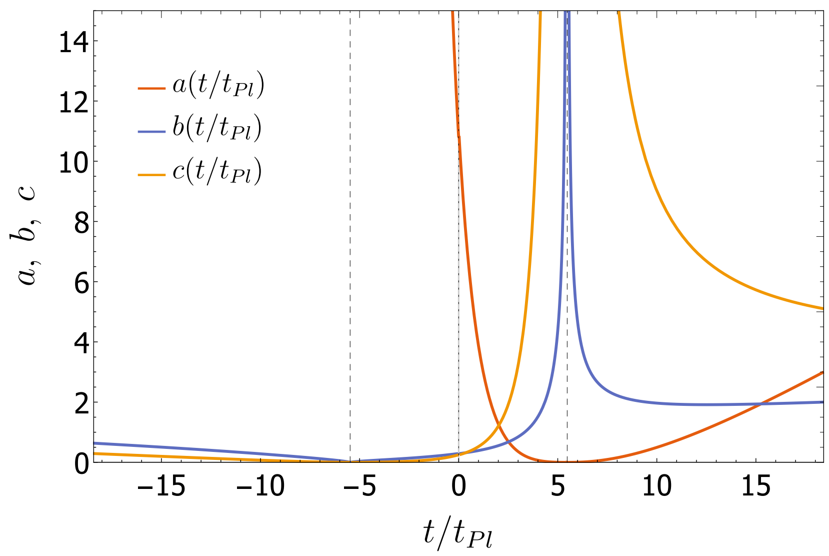


Figure 9.3: Numerical solutions as a function of t/t_{Pl} for $\mu_I = 0.057$, $\mu_A = 2.4$, $\mu_D = 0.365$, $\mu_R = 1.56$ and $\alpha = -8.42 < \bar{\alpha}$. The dashed lines represent where the scalaron vanishes (color online).

leads to [266, 267]

$$x'' = -2x't'H_A, \tag{9.53}$$
$$y'' = -2y't'H_B, \tag{9.54}$$
$$z'' = -2z't'H_C, \tag{9.55}$$
$$t'' = -a^2 H_A x'^2 - b^2 H_A y'^2 - c^2 H_C z'^2, \tag{9.56}$$

where a prime denotes derivative with respect to s. A first integral of these equations is provided by

$$x' = \frac{k_a}{a^2}, \tag{9.57}$$
$$y' = \frac{k_b}{b^2}, \tag{9.58}$$
$$z' = \frac{k_c}{c^2}, \tag{9.59}$$
$$t' = \left(\frac{k_a^2}{a^2} + \frac{k_b^2}{b^2} + \frac{k_c^2}{c^2}\right)^{1/2} + C_0, \tag{9.60}$$

where k_a, k_b, k_c and C_0 are integration constants. From basic arguments of first-order differential equations it follows that, whenever the functions $a(t)$, $b(t)$, and $c(t)$ are continuous and non vanishing, the tangent vector will be unique and well defined. If that is the case, the geodesics are non-singular and the spacetime is geodesically complete, a result that holds both in the anisotropic and in the isotropic case [268]. It is clear that the first kind of solutions, characterized by finite-time singularities but non vanishing scale factors, belong to this class (both in vacuum and with matter).
Regarding the solutions with $\alpha < \bar{\alpha}$ instead, we have that the volume and its derivative are always regular but the scale factors either collapse to zero or diverge at t_c. While their divergence is not a problem for the geodesics, their vanishing may cause issues in the geodesics behavior implying the impossibility of a unique extension across the singularity. In particular, let us consider the case in which one of the scale factors vanishes at some affine parameter s_c. Assuming that

$$a(s) = a_0 (s - s_c)^{\gamma}, \tag{9.61}$$

with $\gamma > 0$, by virtue of (9.60), the relevant equations would be

$$x' = \frac{k_a}{a_0^2 (s - s_c)^{2\gamma}}, \tag{9.62}$$
$$t' = C_0 + \frac{k_a}{a_0 (s - s_c)^{\gamma}}. \tag{9.63}$$

Integrating them leads to

$$x(s) = x_c + \frac{k_a (s - s_c)^{1-2\gamma}}{a_0^2 (1 - 2\gamma)}, \tag{9.64}$$

$$t(s) = t_c + C_0 (s - s_c) + \frac{k_a (s - s_c)^{1-\gamma}}{a_0 (1 - \gamma)}, \tag{9.65}$$

which are smooth provided that $0 < \gamma < 1/2$ and $0 < \gamma < 1$, respectively. In particular, if $1/2 < \gamma < 1$ then one has that $x(s) \xrightarrow{s \to s_c} \pm\infty$, which amounts to travel to infinity in finite coordinate time. On the other hand, if we restrict to the case in which $0 < \gamma < 1/2$, then the geodesics will cover the whole range $\{t, x\} \in (-\infty, \infty)$, resulting in a geodesically complete spacetime, despite the vanishing of the scale factor at some instant in time. Hence, in order to assess the geodesic behavior for the solutions in Fig. 9.3b we must understand how rapidly the zero is reached by the scale factor, i.e. we need an estimate for γ. We may proceed in the following way: for different values of γ, we can invert (9.65) for $s(t)$ which, substituted in (9.61), gives the scale factor $a(s(t))$ as a function of t. The latter can then be compared to the numerical solution in Fig. 9.3b. The outcome of such procedure is shown in Fig.9.4. We see that the solution is characterized by a value of γ larger than $1/2$, between 0.72 and 0.75. Therefore, the zero is approached too rapidly and we can conclude that the second kind of solutions represents a geodesically incomplete spacetime and must be discarded. Let us now turn to the analysis of scalar perturbations. This is somehow complementary to the null geodesics test just performed, since the latter can be considered as high-frequency (or infinite frequency) modes while the analysis of scalar perturbations allows to test the behavior of finite frequency modes. The following results are valid for inhomogeneous perturbations of a generic scalar field around the given homogeneous background solution but also in the case in which the scalar field is identified with the Immirzi field ψ itself. In any case, writing the scalar perturbation as $\sigma_{\vec{k}}(t, \vec{x}) = \Theta(t) e^{i\vec{k}\cdot\vec{x}}$, its dynamics is given by

$$\ddot{\Theta} + h(v) H \dot{\Theta} + \left(\frac{k_x^2}{a^2} + \frac{k_y^2}{b^2} + \frac{k_z^2}{c^2} \right) \Theta = 0 \,, \tag{9.66}$$

where $h(v)$ is a regular function of the volume v and $\vec{k} = (k_x, k_y, k_z)$ a set of constants. By inspection we see that both kind of pathologies, either in the scale factors or in the Hubble function, may affect the propagation of scalar perturbations. Let us begin by considering the first kind of solution, for which the scale factors are non vanishing everywhere. In this case any potential problem must come from the second term involving the Hubble function $H = \dot{v}/3v$ which

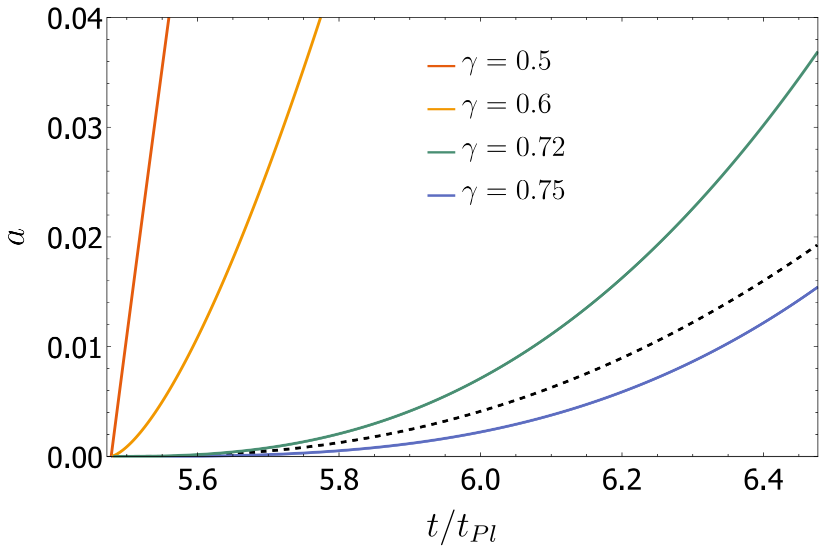


Figure 9.4: Outcomes of null geodesics test for $\alpha < \bar{\alpha}$. Scale factor $a(s(t))$ for different values of γ and $k_a = C_0 = a_0 = 1$. The dashed-black line represent the numerical solution $a(t)$ reported in Fig. 9.3b.

diverges at t_c. In the vacuum case, this point is reached when the volume approaches $v^2 \to 4|\eta_I| \equiv v_c^2$. In this limit one can approximate the Hubble function as

$$H^2 \approx \frac{\kappa^2\mu_I^2}{2^{10}}\frac{(32+27\lambda_{AI}^2)}{(v-v_c)^2}, \tag{9.67}$$

leading to the following approximation for the divergent term in (9.66):

$$\frac{\dot{v}}{3v} \approx \pm\sqrt{\frac{\kappa^2\mu_I^2}{2^{10}}\frac{(32+27\lambda_{AI}^2)}{(v-v_c)^2}} \equiv \pm\frac{C_1}{|v-v_c|}\,, \tag{9.68}$$

where the $\pm$ sign corresponds to the expanding/contracting phase. This allows to write

$$\dot{v} \approx \pm\frac{3v_cC_1}{|v-v_c|}, \tag{9.69}$$

$$|v-v_c| \approx \sqrt{6v_cC_1}|t-t_c|^{1/2}\,. \tag{9.70}$$

Therefore, in this case the differential equation (9.66) has an avoidable singular point at $t = t_c$. The dominant contribution in its neighbourhood is given by

$$\ddot{\Theta} \pm \frac{\tilde{h}_c}{|t-t_c|^{1/2}}\dot{\Theta} \approx 0\,, \tag{9.71}$$

where $\tilde{h}_c \equiv h(v_c)\sqrt{\frac{3C_1}{2v_c}}$. The approximate solution in this limit is provided by

$$\Theta(t) \approx \Theta_c + \frac{\dot{\Theta}_c}{2\tilde{h}_c^2} e^{\mp 2\tilde{h}_c |t-t_c|^{1/2}} \left(1 \pm 2\tilde{h}_c |t-t_c|^{1/2}\right) , \tag{9.72}$$

with $\Theta_c, \dot{\Theta}_c$ integration constants. We see that the scalar field perturbations remain bounded around the singularity at t_c, notwithstanding the divergence in the Hubble function. This holds regardless of the sign of the parameter $\tilde{h}_c$. We can easily extend the analysis to the presence of matter, where the Hubble function is given by (9.45). Let us consider the worst case scenario, namely the presence of a triple root in the denominator: $H^2 \approx C_2/(v-v_c)^6$. This would imply $|v - v_c| \sim |t - t_c|^{1/4}$ and $\dot{v}/v \sim \pm 1/|t-t_c|^{3/4}$, from which one has

$$\ddot{\Theta} \pm \frac{\tilde{h}_c}{|t-t_c|^{3/4}} \dot{\Theta} \approx 0, \tag{9.73}$$

where now $\tilde{h}_c = h(v_c)(3C_2/v_c^3)^{1/4}$. The correspondent solution is given by

$$\Theta \approx \Theta_c - \frac{\dot{\Theta}_c e^{\mp 4\tilde{h}_c |t-t_c|^{1/4}}}{32\tilde{h}_c^4} \left(3 \pm 12\tilde{h}_c |t-t_c|^{1/4} + 24\tilde{h}_c^2 |t-t_c|^{1/2} \pm 32\tilde{h}_c^3 |t-t_c|^{3/4}\right), \tag{9.74}$$

which is again bounded. Therefore, we conclude that for the first kind of solutions, scalar perturbations are well-behaved in the proximity of the singular points, both in vacuum and with matter. Together with the geodesic completeness this guarantees the physical viability of such solutions.

Regarding the second kind of solutions, the geodesic analysis would already be sufficient to discard them but we can still investigate the behavior of scalar perturbations. In this case equation (9.66) describes a harmonic oscillator with a time dependent frequency,

$$\ddot{\Theta}(t) + \frac{k_x^2}{a^2(t)} \Theta(t) \approx 0, \tag{9.75}$$

which diverges as $a(t) \to 0$. Although there might be cases in which the equation can be integrated giving a finite result, for the profile of $a(t)$ obtained numerically here it can be shown that the solution is not well-behaved, confirming the conclusions reached by the null geodesic test.

Chapter 10

Gravitational perturbations in metric-affine Chern-Simons gravity

As we already pointed out, the more involved structure of the metric-affine Chern-Simons field equations calls for alternative lines of investigation in order to perform computations and derive concrete predictions. The study of linear perturbations is particularly suited in this regard. Expanding the fields around a background configuration and studying the dynamics of perturbations by solving the linearized version of the equations introduces a great simplification allowing to derive results both in a numerical and analytical way. Moreover, linear perturbations may represent several physical scenarios suitable for a direct comparison with experimental observations. In this section, the discussion will be focused on black hole perturbations and gravitational waves propagation.
Regarding the former, the physical setting to have in mind is the following. Linear perturbations describe small deviations from a background configuration which may be identified with some known exact black hole solution of the theory. Some event like the infalling of matter in the black hole or a merger event causes a deviation from the equilibrium configuration and provides non-trivial initial conditions for the dynamics of the linear perturbations. These constitute a gravitational signal which, after an initial transient regime, is characterized by exponentially damped oscillations, called quasinormal modes, ending with a power law tail in the late-time region.
The specific event causing the perturbation determines the initial conditions and only affects the initial transient regime. On the other hand, the quasinormal modes and the power law tails are completely independent on that, being instead a property of the system itself (very much like vibrational normal modes in ordinary matter systems). It is therefore meaningful to just focus on the middle and final part of the signal and, in a given theory, a black hole can be characterized by

the spectrum of the (complex valued) frequencies of the quasinormal modes and the exponents of the power law tails. Eventually, at late times the perturbations go to zero and the spacetime relaxes again to a new equilibrium configuration. In the last years an increasing number of works in literature have been devoted to both theoretical and experimental investigations of such phenomena. In particular, given the recent developments in the field of gravitational waves astronomy, the computation of quasinormal modes seems very timely and may offer a way to test the theory with observations in the near future.

10.1 Perturbations of Schwarzschild black holes

The equations ruling the evolution of metric and scalar perturbations on an arbitrary background have already been derived in section 7.2. By arbitrary we actually mean that the only requirements are that the background scalar field must be constant and that the background metric must be a solution of Einstein's equations satisfying the Pontryagin constraint. The simplest example is provided by the Schwarzschild metric

$$\bar{g}_{\mu\nu}dx^\mu dx^\nu = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2, \tag{10.1}$$

where $f(r) = 1 - 2m/r$ and m represents the black hole mass. This, together with $\bar{\theta}$ = const and vanishing torsion and nonmetricity is a vacuum solution of the theory. Since we now have $\bar{R} = 0 = \bar{R}_{\mu\nu}$, the metric equation retains the same structure as in (7.33) with a vanishing right hand side, i.e.

$$\delta G_{\mu\nu} + \delta C_{\mu\nu} = 0, \tag{10.2}$$

and a simplified first order Einstein tensor

$$\delta G_{\mu\nu} \equiv 2\bar{\nabla}_\alpha \bar{\nabla}_{(\mu} h_{\nu)}{}^\alpha - \bar{g}_{\mu\nu}\bar{\nabla}_\alpha\bar{\nabla}_\beta h^{\alpha\beta} - \Box h_{\mu\nu} + \left(\bar{g}_{\mu\nu}\Box - \bar{\nabla}_\mu\bar{\nabla}_\nu\right) h. \tag{10.3}$$

The scalar field equation (7.38) reduces to

$$\left(\mathcal{S}_1(\alpha,\beta;\bar{g}_{\rho\sigma}) + \mathcal{S}_2(\alpha;\bar{g}_{\rho\sigma}) + \mathcal{S}_3(\alpha;\bar{g}_{\rho\sigma})\right)\delta\theta + \mathcal{S}_4^{\mu\nu}(\alpha;\bar{g}_{\rho\sigma})h_{\mu\nu} = 0, \tag{10.4}$$

where

$$\mathcal{S}_1(\alpha,\beta;\bar{g}_{\rho\sigma}) \equiv \left(\beta + \alpha^2\left(\frac{3}{4}\bar{R}^{\mu\nu\rho\sigma}\bar{R}_{\mu\nu\rho\sigma}\right)\right)\Box, \tag{10.5}$$

$$\mathcal{S}_2(\alpha;\bar{g}_{\rho\sigma}) \equiv \frac{\alpha^2}{4}\left(3\bar{R}^{\mu\nu\rho\sigma}\bar{\nabla}_\alpha\bar{R}_{\mu\nu\rho\sigma} + 2\bar{R}^{\mu\nu\rho\sigma}\bar{\nabla}_\nu\bar{R}_{\alpha\mu\rho\sigma}\right)\bar{\nabla}^\alpha, \tag{10.6}$$

$$\mathcal{S}_3(\alpha;\bar{g}_{\rho\sigma}) \equiv -2\alpha^2\bar{R}_{\sigma\mu\nu\rho}\bar{R}_\tau{}^{\mu\nu\rho}\bar{\nabla}^\sigma\bar{\nabla}^\tau, \tag{10.7}$$

$$\mathcal{S}_4^{\mu\nu}(\alpha;\bar{g}_{\rho\sigma}) \equiv \alpha\, {}^*\bar{R}^\mu{}_\beta{}^\nu{}_\delta\bar{\nabla}^\delta\bar{\nabla}^\beta. \tag{10.8}$$

Now, in order to solve the equations and derive the evolution of the perturbations it is convenient to take advantage of the spherical symmetry of the problem, adopting a harmonic decomposition of the perturbations. Regarding the scalar field, we can write its perturbation as

$$\delta\theta = \frac{\Theta(r)}{r} Y^{\ell\ell'}(\vartheta,\varphi) e^{-i\omega t}, \tag{10.9}$$

where $Y^{\ell\ell'}(\vartheta,\varphi)$ are the standard spherical harmonics, with angular momentum numbers ℓ and ℓ'. The treatment of tensor perturbations is more involved and requires some additional care. We will adopt the same decomposition developed by Regge, Wheeler and Zerilli [B219], to which we remind the reader for further details. In the so-called Regge-Wheeler gauge, the metric perturbation can be written as

$$h_{\mu\nu} = \begin{pmatrix} H_0^{\ell\ell'} Y^{\ell\ell'} & H_1^{\ell\ell'} Y^{\ell\ell'} & h_0^{\ell\ell'} S_\vartheta^{\ell\ell'} & h_0^{\ell\ell'} S_\varphi^{\ell\ell'} \\ * & H_2^{\ell\ell'} Y^{\ell\ell'} & h_1^{\ell\ell'} S_\vartheta^{\ell\ell'} & h_1^{\ell\ell'} S_\varphi^{\ell\ell'} \\ * & * & r^2 K^{\ell\ell'} Y^{\ell\ell'} & 0 \\ * & * & 0 & r^2 \sin^2\vartheta K^{\ell\ell'} Y^{\ell\ell'} \end{pmatrix} e^{-i\omega t}, \tag{10.10}$$

where asterisks denote components obtained by symmetry and we have defined

$$S_\vartheta^{\ell\ell'} = -\csc\vartheta\,\partial_\varphi Y^{\ell\ell'}, \tag{10.11}$$

$$S_\varphi^{\ell\ell'} = \sin\vartheta\,\partial_\vartheta Y^{\ell\ell'}. \tag{10.12}$$

The functions $H_0, H_1, H_2, K, h_0, h_1$ depend on the radial coordinate alone. Among the benefits of this decomposition, an important feature is that the metric components are clearly separated according to their behavior under parity transformations. In particular, the capital letter functions H_0, H_1, H_2, K describe (even parity) polar modes, while the (odd parity) axial modes are encoded in h_0, h_1.

When the purely metric version of Chern-Simons gravity is considered, a decoupling of polar and axial modes is observed [140, 141, 142]. Regarding the metric-affine case, substituting (10.9) and (10.10) into (10.2), we find complete agreement with the analogue equations in the metric Chern-Simons gravity case, as expected. Differences arise instead in the scalar field equation (10.4). However, only terms involving the scalar perturbation are modified with respect to the purely metric case, while the last term in (10.4) retains the same expression of [140, 141, 142]. These considerations are enough to conclude that the result derived in [140, 141, 142], regarding the decoupling of polar and axial modes, holds also in the present case. In particular, polar perturbations do not couple to the additional scalar field and they are not modified with respect to General Relativity. Therefore, from now on we focus on the axial modes which are the only ones affected.

This result is expected and consistent with the parity violating character of the Chern-Simons modification characterizing the theory.
We can derive the equations for the axial perturbations from the $t\varphi$, $r\varphi$ and $\vartheta\varphi$ components of (10.2), which can be written as

$$E_1 \equiv h_0'' + i\omega\left(\partial_r + \frac{2}{r}\right)h_1 + \left(\frac{2f'}{rf} - \frac{\ell(\ell+1)}{r^2 f}\right)h_0 - \frac{6\alpha m}{r^4}\left(\Theta' - \frac{2}{r}\Theta\right) = 0, \quad (10.13)$$

$$E_2 \equiv -\omega^2 h_1 + i\omega\left(\partial_r - \frac{2}{r}\right)h_0 + \frac{f(\ell+2)(\ell-1)}{r^2}h_1 - \frac{6\alpha m i\omega}{r^4}\Theta = 0, \quad (10.14)$$

$$E_3 \equiv \frac{i\omega}{f}h_0 + \partial_r(fh_1) = 0, \quad (10.15)$$

having used the following well-known property of spherical harmonics:

$$\left[\frac{1}{\sin\vartheta}\partial_\vartheta\left(\sin\vartheta\,\partial_\vartheta\right) + \frac{1}{\sin^2\vartheta}\partial_\varphi^2\right]Y^{\ell\ell'} = -\ell(\ell+1)Y^{\ell\ell'}. \quad (10.16)$$

Despite having three equations for the two functions h_0 and h_1 the system is not over-constrained since the following relation holds [140]

$$E_1 + \frac{fr^4}{i\omega}(E_2/r^2)' - \frac{(\ell+2)(\ell-1)r}{i\omega}E_3 = 0. \quad (10.17)$$

Moreover, the equation $E_3 = 0$ can be solved for h_0 and, substituting the result into E_2, an equation for the remaining function h_1 is obtained. Redefining h_1 in terms of a new function Q as

$$Q \equiv fh_1/r \quad (10.18)$$

and employing the tortoise coordinate defined by

$$r_* = r + 2m\ln(r/2m - 1), \quad (10.19)$$

the equation takes the following form

$$\frac{d^2Q}{dr_*^2} + \left[\omega^2 - f\left(\frac{\ell(\ell+1)}{r^2} - \frac{6m}{r^3}\right)\right]Q = -\frac{6\alpha m i\omega}{r^5}f\Theta. \quad (10.20)$$

Finally, using $E_2 = 0$ to express $(\partial_r - 2/r)h_0$ in terms of h_1 and Θ, the equation for the scalar perturbation is eventually obtained as

$$\left(\beta + \frac{12\alpha^2 m^2}{r^6}\right)\left(\frac{d^2\Theta}{dr_*^2} + \left[\omega^2 - f\left(\frac{\ell(\ell+1)}{r^2} + \frac{2m}{r^3}\right)\right]\Theta\right) - \frac{72\alpha^2 m^2}{r^7}f\frac{d\Theta}{dr_*}$$
$$+ \frac{36\alpha^2 m^2}{r^8}f(2f - \ell(\ell+1))\Theta = \frac{6\alpha m}{-i\omega r^5}\frac{(\ell+2)!}{(\ell-2)!}fQ. \quad (10.21)$$

It will also be convenient to introduce yet another redefinition for Q, i.e.

$$\Psi = iQ/\omega, \tag{10.22}$$

in terms of which we have

$$\frac{d^2\Psi}{dr_*^2} + \left[\omega^2 - f\left(\frac{\ell(\ell+1)}{r^2} - \frac{6m}{r^3}\right)\right]\Psi = \frac{6\alpha m\omega}{r^5} f\Theta, \tag{10.23}$$

$$\left(\beta + \frac{12\alpha^2 m^2}{r^6}\right)\left(\frac{d^2\Theta}{dr_*^2} + \left[\omega^2 - f\left(\frac{\ell(\ell+1)}{r^2} + \frac{2m}{r^3}\right)\right]\Theta\right) - \frac{72\alpha^2 m^2}{r^7} f \frac{d\Theta}{dr_*}$$
$$+ \frac{36\alpha^2 m^2}{r^8} f(2f - \ell(\ell+1))\Theta = \frac{6\alpha m}{r^5}\frac{(\ell+2)!}{(\ell-2)!} f\Psi. \tag{10.24}$$

The last two equations form a set of coupled second order differential equations for the axial metric and scalar perturbations. Numerically solving this system of equations one can extract the time evolution of the perturbations at some fix radius from the black hole. Then, the quasinormal modes frequencies and the exponents of the power law tails can be estimated from the time series $\Psi(t)$ and $\Theta(t)$. We postpone the discussion on the details of the numerical integration to the end of this section, focusing now on the main results. As already discussed in section 2.2.2, we can set $\alpha = 1$, leaving β as the only free parameter of the theory. For different values of the latter, we obtained solutions characterized by the same qualitative behavior observed in the General Relativity case, namely damped oscillations ending with a late-time power law tail, as expected. Some examples are shown in Fig. 10.1.

10.1.1 Quasinormal modes frequencies

From an observational point of view, the most relevant quasinormal frequencies characterizing the first part of the signal are those associated to the lowest lying $l = 2$ modes. In this regard, the results obtained can be classified into three different region in parameter space.

- $\beta \to \infty$: in the limit of large β the system is approximated by

$$\frac{d^2\Psi}{dr_*^2} + \left[\omega^2 - f\left(\frac{l(l+1)}{r^2} - \frac{6m}{r^3}\right)\right]\Psi = \frac{6m}{r^5} f\Theta, \tag{10.25}$$

$$\frac{d^2\Theta}{dr_*^2} + \left[\omega^2 - f\left(\frac{l(l+1)}{r^2} + \frac{2m}{r^3}\right)\right]\Theta = 0. \tag{10.26}$$

  Thus, in this limit a decoupling of the scalar equation from the metric perturbation Ψ is observed, resulting in equation (10.26) which is nothing

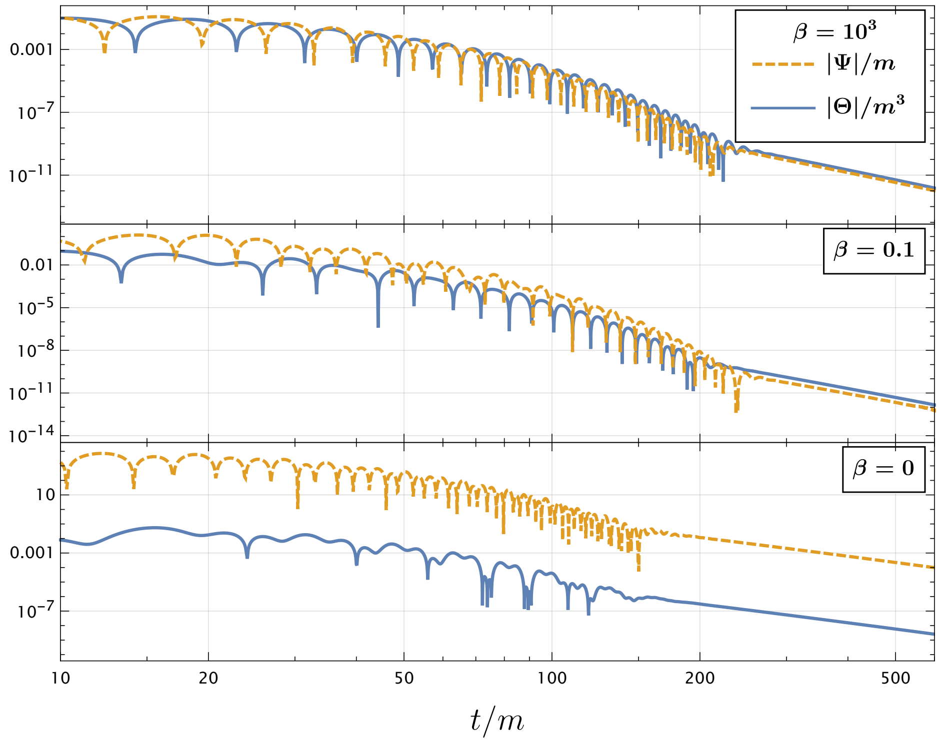


Figure 10.1: Evolution of metric (continuous) and scalar (dashed) perturbations as a function of t/m for $\beta = 10^3$ (top), $\beta = 0.1$ (center) and $\beta = 0$ (bottom). Straight lines represent a power law behavior.

but the perturbed Klein-Gordon equation on a Schwarzschild background. The metric equation instead, still retains a modification induced by the scalar field perturbation sourcing the non vanishing right hand side in (10.25). From this considerations we can already foresee some features in the perturbations evolution. Regarding the scalar perturbation, no deviations from the standard General Relativity case are expected. Therefore, $\Theta(t)$ should be characterized by oscillations dominated by one single mode which will be referred to as *scalar mode* in the following. Its frequency should coincide with the $\ell = 2$ mode frequency of scalar perturbations in General Relativity. On the other hand, the metric perturbation should be characterized by a superposition of two fundamental modes: the scalar mode already mentioned and another one which we will call *tensor mode*. The latter is the one characterizing metric perturbations in the General Relativity case, namely the homogeneous equation associated to (10.25). The superposition with the scalar mode is instead due to the non vanishing source term in (10.25). These predictions are confirmed by the outcomes of numerical integrations performed with $\beta = 100, 1000$. Indeed, the time series obtained for the scalar perturbation is characterized by single mode oscillations with frequency

$$\omega_s = 0.48 - i\,0.097, \tag{10.27}$$

while the metric perturbation is compatible with a two modes fit, superposition of ω_s and

$$\omega_g = 0.37 - i\,0.089. \tag{10.28}$$

Note that the above values coincide with the frequencies of the $\ell = 2$ modes found in the General Relativity case [B219] for scalar and tensor perturbations, respectively.

- $10^{-1} \lesssim \beta \lesssim 10$: In this range the numerical values of the frequencies present in the perturbations evolution is unchanged and they are still given by ω_s and ω_g. However, now also the scalar perturbation starts oscillating with a superposition of the scalar and tensor mode, as the Ψ sourcing term in (10.21) starts to become relevant.

- $\beta \lesssim 10^{-1}$: in the small β limit, we still observe a superposition of the tensor and scalar modes but the numerical values of the associated frequencies start deviating from the General Relativity values.

These results are summarized in Fig. 10.2. When two modes fits are used to extract the frequencies, one actually gets two numerical values for each frequency, obtained from the scalar and metric perturbation, respectively. As long as the

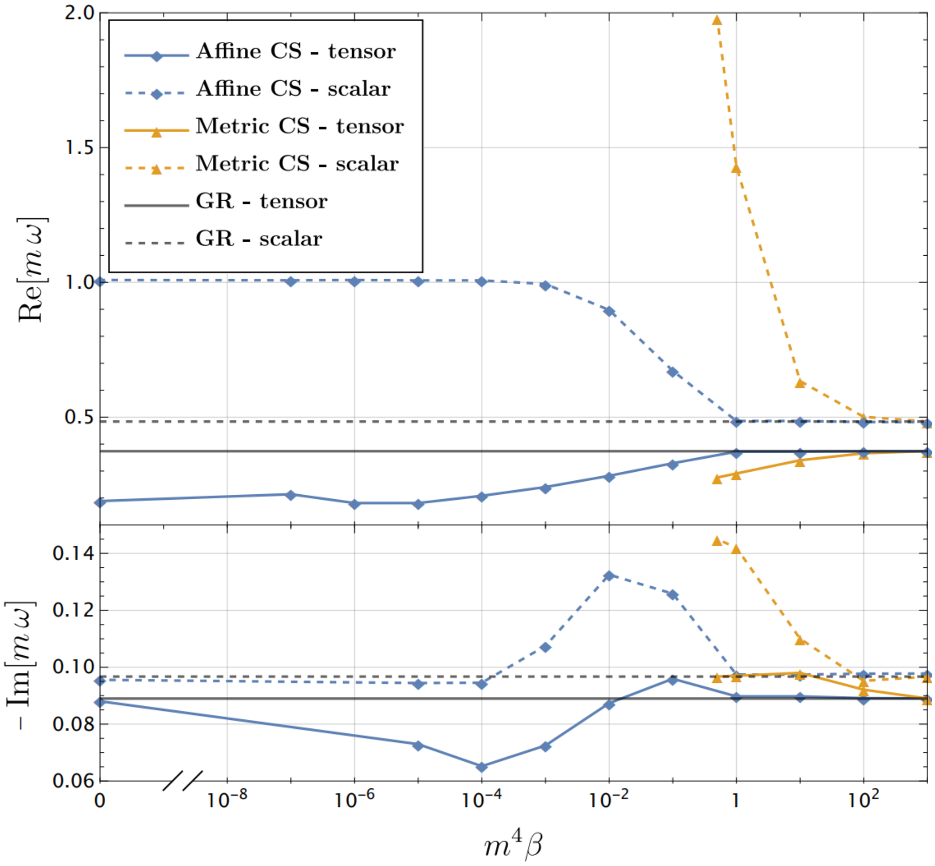


Figure 10.2: Real (top) and imaginary (bottom) part of quasinormal frequencies of the fundamental $\ell = 2$ tensor (continuous) and scalar (dashed) modes as a function of β, for metric (data taken from [142]) and metric-affine CSMG.

real parts of the frequency are considered, the two values obtained in this way are always consistent with each other. On the other hand, the computation of the imaginary part cannot always be carried out without ambiguities and we are not able to compute it for every value of β.

10.1.2 Power-law tails

After the damped oscillatory regime, a power law decay settles down, characterizing the signal in the late-time region. This time there is no difference between metric and scalar perturbations and they both behave as $\sim t^{-\mu}$ for large t. Again, this is qualitatively the same as in General Relativity but there are some quantitative differences. In order to highlight them it is interesting to look at the values of the power law exponents μ for different angular momentum number ℓ. In particular, we performed integrations for $2 \leq l \leq 12$. The results obtained are insensible to the specific value of β and only two cases can be distinguished:

- $\beta \neq 0$: for non vanishing β there are no deviations from General Relativity and the exponents depend on the angular momentum number ℓ via the relation
$$\mu = 2l + 3, \tag{10.29}$$
valid for both scalar and tensor modes and for every β.

- $\beta = 0$: in absence of the kinetic term for the scalar field we observe a departure from General Relativity. The solutions obtained in this case are shown in Fig. 10.3. Linear fits performed on the last section of the signals yield the exponents in Fig. 10.4 and Table 10.1. In particular, we still observe a linear relation but consistent with
$$\mu = 0.884\, l + 2.78. \tag{10.30}$$

10.1.3 Comparison with metric Chern-Simons gravity

Beside the deviations from General Relativity discussed above, the metric-affine framework also introduces modifications with respect to the purely metric version of Chern-Simons gravity. In order to compare the two models one must consider the effective theory at the perturbative level, characterized by the equations (7.33) and (10.4).
In particular, despite the scalar field equation (7.36) being formally equivalent to its metric analogue (eq. (2.70)), it actually features the affine Riemann tensor, including all non-Riemannian contributions. This results in a more complex differential structure of the equation at the effective level, since the affine connection ultimately depends on the scalar field derivatives. Indeed, the additional $\delta\theta$ derivative terms appear explicitly once equation (7.36) is recast as (7.38). Even if some of these differences may disappear once some specific background configuration is singled out, when the Schwarzschild case is considered the scalar field equation retains the additional terms present in (10.4), eventually resulting in the modified structure of (10.21), which is responsible for the deviations with

l	2	3	4	5	6	7	8	9	10	11	12
μ	4.75	5.40	6.81	7.36	7.74	8.53	9.37	10.2	11.1	14.6	12.8

Table 10.1: Exponents characterizing power law tails $\sim t^{-\mu}$ for different values of l in the $\beta = 0$ case.

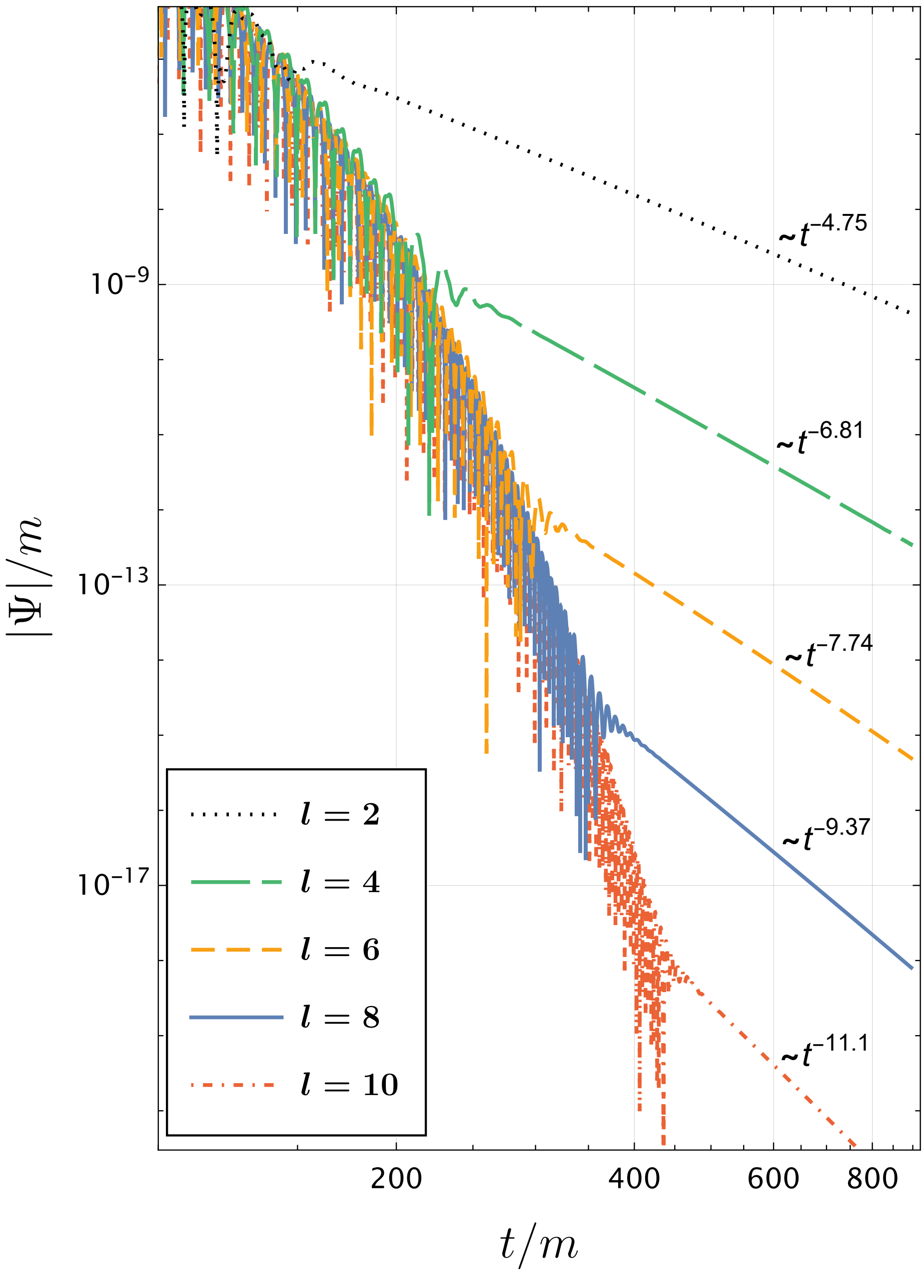


Figure 10.3: Time evolution of the metric perturbation in the $\beta = 0$ case, for different values of l. The behavior of scalar perturbations is qualitatively the same.

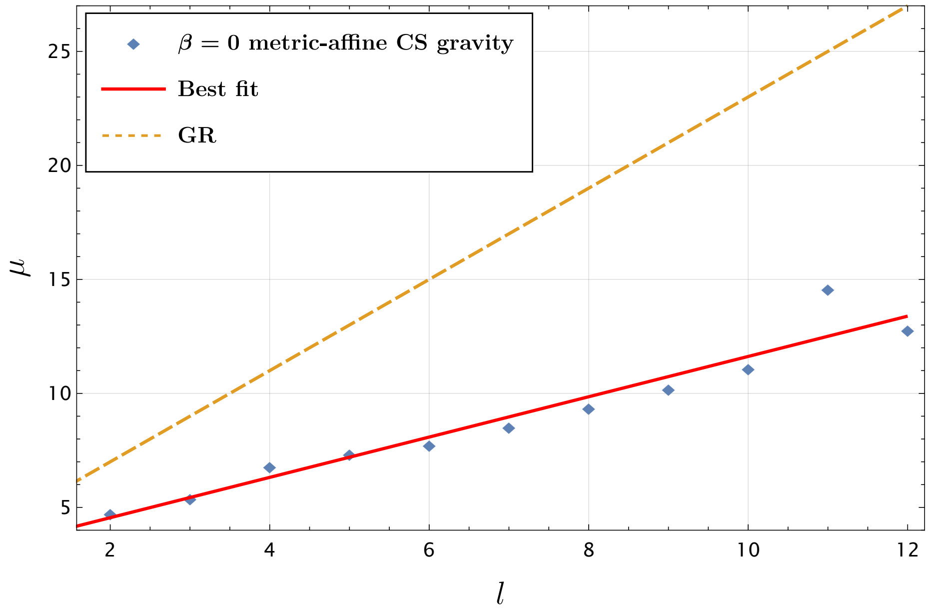


Figure 10.4: Exponents characterizing the power law tails in the $\beta = 0$ case as a function of l. The best fit (10.30) (continuous) and General Relativity case (dashed) are also shown.

respect to the metric case frequencies reported in Fig. 10.2.
Another difference between the predictions of the two theories is the absence of purely decaying modes in the metric-affine case. These are characterized by $\Re[\omega] = 0$ and $\Im[\omega] < 0$ and are present in metric Chern-Simons gravity for $\beta \lesssim 0.5$, as shown in [142]. In our case, we checked the absence of such solutions for β down to 10^{-7}.
The last distinction regards the late-time tails. In the metric Chern-Simons theory, these are indistinguishable from General Relativity and a direct observation of the last part of the gravitational signal would not allow to falsify the theory. While this is true also in the metric-affine case for $\beta \neq 0$, the $\beta = 0$ would show a distinctive signature in the exponents of the power law tails, since they satisfy the modified relation (10.30).
Moreover, the $\beta = 0$ case, turns out to be a viable theory endowed with an additional stable scalar degree of freedom with a proper dynamical character, contrary to what happens in the metric version of the theory,which is affected by the issue of over-constrained black hole perturbations.

10.1.4 Details on the numerical integration

Numerical techniques for the integration of equations of the form (10.23) and (10.24) have been developed in several works. Here we follow the lines of [269, 270, 271] with the aim of obtaining the perturbations as a function of time at some fixed radius. As a first step we can get rid of one of the free parameter via the rescalings $\Theta \to \Theta/\alpha$ and $\beta \to \alpha^2\beta$. This amounts to set $\alpha = 1$ and leaves only β as the free parameter of the theory. Then, employing the light-cones variables $u = t - r_*$ and $v = t + r_*$, the equations become

$$4\frac{\partial^2\Psi}{\partial u\partial v} + V_1(r)\Psi = V_2(r)\Theta, \tag{10.31}$$

$$4W_1(r)\frac{\partial^2\Theta}{\partial u\partial v} + W_2(r)\left(\frac{\partial\Theta}{\partial u} - \frac{\partial\Theta}{\partial v}\right) + W_3(r)\Theta = W_4(r)\Psi, \tag{10.32}$$

where the effective potentials have been defined as

$$V_1(r) = f(r)\left(\frac{\ell(\ell+1)}{r^2} - \frac{6m}{r^3}\right), \tag{10.33}$$

$$V_2(r) = -\frac{6m}{r^5}f(r), \tag{10.34}$$

$$W_1(r) = \beta + \frac{12m^2}{r^6}, \tag{10.35}$$

$$W_2(r) = -\frac{72m^2}{r^7}f(r), \tag{10.36}$$

$$W_3(r) = f(r)\left(\beta + \frac{12m^2}{r^6}\right)\left(\frac{\ell(\ell+1)}{r^2} + \frac{2m}{r^3}\right) - \frac{36m^2 f(r)}{r^8}\left(2f(r) - \ell(\ell+1)\right), \tag{10.37}$$

$$W_4(r) = -\frac{6m}{r^5}\frac{(\ell+2)!}{(\ell-2)!}f(r). \tag{10.38}$$

Note that the function W_1 multiplies the second derivative in the scalar equation. Whenever W_1 vanishes, the differential order of the equation is lowered and one must expect some kind of singularity in the solutions. However, if $\beta \geq 0$ then W_1 is non vanishing (and positive). Since the parameter β is associated to the kinetic term of the scalar field in the action, the arising of ghost instabilities is expected in the $\beta < 0$ case. Therefore, we can exclude the negative branch and focus on the $\beta \geq 0$ case, so that the condition $W_1 \neq 0$ is also assured.
Then, the set up of the numerical integration consists in performing a discretization of the $u - v$ plane by a lattice spacing Δ. By approximating the derivatives

with finite differences, the discretized version of the equations reads

$$\Psi_N = \Psi_W + \Psi_E - \Psi_S + \frac{\Delta^2}{2}\left[V_1(r_c)\left(\Psi_W + \Psi_E\right) - V_2(r_c)\left(\Theta_W + \Theta_E\right)\right], \tag{10.39}$$

$$\begin{aligned}\Theta_N = {} & \Theta_W + \Theta_E - \Theta_S + \Delta\frac{W_2(r_c)}{W_1(r_c)}\left(\Theta_E - \Theta_W\right) \\ & + \frac{\Delta^2}{2}\left[\frac{W_3(r_c)}{W_1(r_c)}\left(\Theta_E + \Theta_W\right) - \frac{W_4(r_c)}{W_1(r_c)}\left(\Psi_E + \Psi_W\right)\right].\end{aligned} \tag{10.40}$$

Here we used the subscript notation to denote fields evaluated at the lattice points $S = (u, v)$, $W = (u + \Delta, v)$, $E = (u, v + \Delta)$ and $N = (u + \Delta, v + \Delta)$, while the potentials are computed at the off-grid point $r_c = (u + \Delta/2, v + \Delta/2)$. The domain of integration consists of a portion of the $u - v$ plane ranging from zero to (u_{max}, v_{max}) and the boundary conditions are assigned on the two axes $u = 0$ and $v = 0$. Following [142], we can set the perturbations to zero on the u axis, i.e. $\Psi(u, 0) = \Theta(u, 0) = 0$, while Gaussian initial data are used for their values on the v axis: $\Psi(0, v) = \Theta(0, v) = \exp[-(v - v_c)^2/2\sigma]$.
Then, the integration is performed in the following way. Using the initial data for the fields at the three points S, W, and E, one can compute the value of the fields at N by solving (10.39)-(10.40). Then, one proceeds increasing the value of v at fixed u, completing the row. The process is repeated until the full grid is completed row by row, increasing u each time. The most computationally expensive part of the algorithm is due to the fact that at each step one has to compute the value of r_c in which the potentials must be evaluated. This must be done inverting the tortoise coordinate expression for r. To speed up the computation we used the approximation $r \approx 2m(1 + \exp[(r_* - 2m)/2m])$ for $r \approx 2m$, and the inversion is performed numerically only for larger values of r. Another difficulty comes from the fact that for $\ell > 2$ the evolution leads to very small amplitudes for the perturbations (Cf. figure 10.3), resulting in an insufficient precision for the appreciation of the late-time section of the signals, which is unacceptable when the amplitude is of the order of$\sim 10^{-14}$ and below. This problem can be overcome increasing the accuracy of the numerical computations using the *mpmath* Python library [272].
Once the integration is complete, the discretized values of the functions $\Psi(u, v)$ and $\Theta(u, v)$ is known. Hence, the last step consists in extracting the functions $\Psi(t) = \Psi(t - r_*, t + r_*)$ and $\Theta(t) = \Theta(t - r_*, t + r_*)$, for some constant r_*. The results presented in this thesis are obtained with the following choices for the parameters of the integration: $u_{max} = v_{max} = 1000$, $\Delta = 0.1$, $v_c = 10$, $\sigma = 1$. The fields are extracted at $r_* = 50m$.
Once the time series $\Psi(t)$ and $\Theta(t)$ are known, the characteristic frequencies of oscillation are extracted fitting the data. We use either a single mode oscillation

or a superposition of two modes, with fitting function

$$y(t) = \sum_{j=1}^{n} A_j e^{\Im[\omega_j]t} \cos(\Re[\omega_j]t + c_j), \tag{10.41}$$

for $n = 1$ or $n = 2$, respectively.

10.2 Homogeneous and isotropic cosmological solutions

As we will see in the next section, in order to give rise to non-trivial effects in the propagation of gravitational waves, it is not enough to considered the flat Minkowski spacetime as a background and more general solutions are needed. To this aim, exact cosmological solutions will be derived in this chapter, providing the expressions for the metric, scalar field and for the affine sector as well.
In section 7 we adopted a perturbative approach to tackle the problem of solving the connection in terms of the metric and scalar field. However, this is not the only available solution. A nonperturbative scheme consists in taking advantage of the symmetries of specific spacetime configurations. These are known to constrain the most general form of the affine connection when the vanishing of its Lie derivative along the Killing vector fields of the spacetime is imposed. This amounts to require the affine connection to possess the same symmetries of the metric under consideration.
For instance, we may specialize to homogeneous and isotropic spacially flat spacetimes described by the FLRW metric:

$$ds^2 = -dt^2 + a^2(t) \left(dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)\right), \tag{10.42}$$

where $a(t)$ is the scale factor of the universe. As shown in [273, 68], the most general affine connection respecting the same symmetries only depends on five free functions of time $\{K_1(t), ..., K_5(t)\}$. In particular, the non vanishing components of torsion and nonmetricity are given by

$$T^t = 3(K_3 - K_4), \tag{10.43}$$

$$S^t = -\frac{12K_5}{a}, \tag{10.44}$$

$$Q^t = 2K_1 + 6K_4 - 6H, \tag{10.45}$$

$$P^t = 2K_1 - 3\frac{K_2}{a^2} + 3K_3, \tag{10.46}$$

$$\Omega_{ttt} = -K_1 + K_3 + K_4 - \frac{K_2}{a^2} - H, \tag{10.47}$$

$$\Omega_{t\psi\psi} = \Omega_{\psi\psi t} = r^2\Omega_{trr}, \tag{10.48}$$

$$\Omega_{t\varphi\varphi} = \Omega_{\varphi\varphi t} = r^2\sin^2\psi\,\Omega_{trr}, \tag{10.49}$$

$$\Omega_{trr} = \Omega_{rrt} = \frac{a^2}{3}\Omega_{ttt}, \tag{10.50}$$

$$q_{\mu\nu\rho} = 0, \tag{10.51}$$

where $H = \dot{a}/a$ is the Hubble function. From these components the full affine connection can be reconstructed via the decomposition (1.23). Then, with the help of the xAct Mathematica package, using (10.51) several equations in the unknown functions $\{K_1(t), ..., K_5(t)\}$ are derived from the components of (7.3). The main advantage of this approach is that we are not dealing with differential equations anymore but with algebraic ones. Moreover, as usual projective invariance allows to set $T_\mu = 0$, implying $K_4 = K_3$. The remaining functions are determined by solving the equations and they are given by

$$K_1(t) = -\frac{HK_5\dot{\theta}}{a + K_5\dot{\theta}}, \tag{10.52}$$

$$K_2(t) = \frac{a^3H}{a + K_5\dot{\theta}}, \tag{10.53}$$

$$K_3(t) = K_4 = \frac{aH}{a + K_5\dot{\theta}}, \tag{10.54}$$

$$K_5(t) = \frac{a}{\dot{\theta}}\left(-1 + \epsilon\sqrt{\frac{1 + \sqrt{1 + 4H^2\dot{\theta}^2}}{2}}\right), \tag{10.55}$$

where we introduced a parameter $\epsilon = \pm 1$. Moreover, we assumed the scalar field to respect the same spacetime symmetries, and in particular we have $\theta = \theta(t)$. In the derivation of this solution we assumed $a + K_5\dot{\theta} \neq 0$. The case $a + K_5\dot{\theta} = 0$ can be analysed separately and it leads to the trivial case of a static universe with $H(t) \equiv 0$.

Eventually, substituting the expressions for the functions $\{K_1(t), ..., K_5(t)\}$, the affine sector turns out to be determined only by the vector components of torsion and nonmetricity and the geometry is of the Weyl type, i.e. $Q_{\rho\mu\nu} = P_\rho g_{\mu\nu}$. The only non vanishing components are given by

$$S^t = \frac{12K_5}{a} = \frac{12}{\dot{\theta}}\left(-1 + \epsilon\sqrt{\frac{1+\sqrt{1+4H^2\dot{\theta}^2}}{2}}\right), \tag{10.56}$$

$$Q^t = 4P^t = -8K_1 = 8H\left(1 - \epsilon\sqrt{\frac{2}{1+\sqrt{1+4H^2\dot{\theta}^2}}}\right). \tag{10.57}$$

Hence, the affine degrees of freedom have been re-expressed in terms of the metric (the scale factor) and the scalar field, allowing to work on half-shell in a nonperturbative way, although a particular background has been singled out. At the effective level the metric field equation is given by

$$G_{\mu\nu} + C_{\mu\nu} = \frac{\beta}{2}\left(\nabla_\mu\theta\nabla_\nu\theta - \frac{1}{2}g_{\mu\nu}\nabla_\rho\theta\nabla^\rho\theta\right) + \kappa^2 T_{\mu\nu}, \tag{10.58}$$

where here and in the following we will consider matter in the form of a perfect fluid with stress-energy tensor given by

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}. \tag{10.59}$$

The C-tensor assumes the quite simple expression

$$C_{\mu\nu} = -\bar{\nabla}_{(\mu}P_{\nu)} + g_{\mu\nu}\bar{\nabla}_\rho P^\rho + \frac{1}{2}\left(P_\mu P_\nu + \frac{1}{2}g_{\mu\nu}P_\rho P^\rho\right) - \frac{1}{72}\left(S_\mu S_\nu + \frac{1}{2}g_{\mu\nu}S_\rho S^\rho\right), \tag{10.60}$$

of which we can also provide the individual components in terms of the function $K_5(t)$:

$$C^t{}_t = 3\left(\frac{K_5}{a}\right)^2 + 3H^2\left(1 - \left(\frac{a}{a + K_5\dot{\theta}}\right)^2\right), \tag{10.61}$$

$$C^r{}_r = \left(\frac{K_5}{a}\right)^2 + (2\dot{H} + 3H^2)\left(\frac{K_5\dot{\theta}}{a + K_5\dot{\theta}}\right)^2 + \frac{2\frac{d}{dt}(aH\dot{\theta}K_5)}{a^2\left(a + K_5\dot{\theta}\right)^2}, \tag{10.62}$$

$$C^\theta{}_\theta = C^\varphi{}_\varphi = C^r{}_r. \tag{10.63}$$

In order to gain some physical insights on the solutions we will present in the following, it is helpful to recast the C-tensor contributions in the form of an effective stress-energy tensor with effective energy density ρ_{eff} and pressure p_{eff}, i.e.

$$T^{eff}_{\mu\nu} = -\frac{C_{\mu\nu}}{\kappa^2} \equiv (\rho_{eff} + p_{eff})u_\mu u_\nu + p_{eff} g_{\mu\nu}, \tag{10.64}$$

where the following identifications have been made

$$\rho_{eff} = -\frac{1}{\kappa^2}\left(\frac{3}{4}(P^t)^2 - \frac{1}{48}(S^t)^2 - 3HP^t\right), \tag{10.65}$$

$$p_{eff} = -\frac{1}{\kappa^2}\left(\frac{1}{144}(S^t)^2 - \frac{1}{4}(P^t)^2 + 2HP^t + \dot{P}^t\right), \tag{10.66}$$

These definitions allow to write the relevant metric equations in the comoving frame defined by $u^\mu = (1, 0, 0, 0)$, as

$$3H^2 = \kappa^2\tilde{\rho} + \frac{\beta}{4}\dot{\theta}^2, \tag{10.67}$$

$$3H^2 + 2\dot{H} = -\kappa^2\tilde{p} - \frac{\beta}{4}\dot{\theta}^2, \tag{10.68}$$

where we defined

$$\tilde{\rho} \equiv \rho + \rho_{eff}, \qquad \tilde{p} \equiv p + p_{eff}. \tag{10.69}$$

We see that torsion and nonmetricity provide a contribution to the total amount of energy which ultimately determines the evolution of the scale factor. Therefore, it is reasonable to require the positiveness of $\tilde{\rho}$, rather than merely ρ.
Regarding the scalar field equation, starting from (7.36), we obtain

$$\beta\left(\ddot{\theta} + 3H\dot{\theta}\right) + \mathscr{D}_1(a, \dot{a}, \dot{\theta}, \ddot{\theta}) + \mathscr{D}_2(a, \dot{a}, \ddot{a}, \dot{\theta}) = 0, \tag{10.70}$$

where the functions $\mathscr{D}_1$, $\mathscr{D}_2$ are given by

$$\mathscr{D}_1(a, \dot{a}, \dot{\theta}, \ddot{\theta}) \equiv 6\dot{K}_5\left(\frac{2a^2H^2}{(a + K_5\dot{\theta})^3} - \frac{aH^2}{(a + K_5\dot{\theta})^2} - \frac{K_5^2}{a^3}\right) - \frac{12aH^2K_5^2}{(a + K_5\dot{\theta})^3}\ddot{\theta}, \tag{10.71}$$

$$\mathscr{D}_2(a, \dot{a}, \ddot{a}, \dot{\theta}) \equiv 12aHK_5\left(\frac{2H^2 + \dot{H}}{(a + K_5\dot{\theta})^2} - \frac{aH^2}{(a + K_5\dot{\theta})^3}\right). \tag{10.72}$$

As in the general case and in the static spherically symmetric setting, also here we can appreciate how the limit $\beta \to 0$ still implies a dynamical equation for $\theta(t)$, as opposed to the metric formulation of Chern-Simons gravity.
We will now present some analytical solutions in the theory identified by $\beta = 0$. We will assume the usual equation of state $p = w\rho$ for the perfect fluid, defining the analogue of the polytropic index w also for the effective stress-energy tensor,

i.e. $w_{eff} = p_{eff}/\rho_{eff}$. Moreover, since $\bar{\nabla}_\mu C^{\mu\nu} = 0$, the stress-energy tensor is still conserved, allowing to write the energy density as

$$\rho = \frac{\rho_0}{a^{3(1+w)}}, \tag{10.73}$$

where ρ_0 is a constant. It is possible to show, moreover, that the relation $a + K_5\dot{\theta} \neq 0$ is preserved by the dynamics. Then, the first one of the metric equations can be rewritten as

$$H^2 = \frac{\left(2\left(1 + \frac{\dot{\theta}^2}{6}\left(\kappa^2\rho + \frac{\beta}{4}\dot{\theta}^2\right)\right)^2 - 1\right)^2 - 1}{4\dot{\theta}^2}, \tag{10.74}$$

representing a modified Friedmann equation. Now, setting $\beta = 0$ it is easy to show that the following configurations are solutions of the theory.

10.2.1 de Sitter phase of expansion

For the following choices of the parameters

$$w = -1, \tag{10.75}$$
$$\epsilon = -1, \tag{10.76}$$

a couple of solutions is given by

$$a(t) = a_0 e^{t\sqrt{\Lambda/3}}, \tag{10.77}$$
$$\theta(t) = \pm\frac{6}{\sqrt{\Lambda}}t + \theta_0, \tag{10.78}$$
$$S^t(t) = \mp 6\sqrt{\Lambda}, \tag{10.79}$$
$$Q^t(t) = 4P^t(t) = 4\sqrt{3\Lambda}. \tag{10.80}$$

where $\Lambda > 0$, while a_0 and θ_0 are arbitrary constants. Since $w = -1$, the energy density is just constant, i.e. $\rho = \rho_0 = \text{const}$, ant its value is determined by

$$\rho_0 = -\Lambda/2\kappa^2. \tag{10.81}$$

The solution describes a de Sitter expansion phase of the universe. However, since $\Lambda > 0$ the bare energy density ρ_0 is actually negative. In the General Relativity case, where the C-tensor is absent, the same solution would be obtained setting $\rho_0 = \frac{\Lambda}{\kappa^2} > 0$. In the present case instead, there is an additional contribution to the energy density coming from the affine components (10.79) and (10.80), which results in

$$\rho_{eff} = \frac{3\Lambda}{2\kappa^2}, \qquad p_{eff} = -\frac{3\Lambda}{2\kappa^2}, \tag{10.82}$$

corresponding to $w_{eff} = -1$.

10.2.2 Power law solutions

Setting again $\epsilon = -1$ a set of solutions is derived as

$$a(t) = a_0 t^m, \tag{10.83}$$

$$\theta(t) = \pm\frac{\sqrt{3}}{m}t^2 + \theta_0, \tag{10.84}$$

$$\rho(t) = -\frac{3m^2}{2\kappa^2}\frac{1}{t^2}, \tag{10.85}$$

$$S^t(t) = \mp\frac{6\sqrt{3}m}{t}, \tag{10.86}$$

$$Q^t(t) = 4P^t(t) = \frac{12m}{t}, \tag{10.87}$$

where a_0, θ_0 are arbitrary constants and the parameter m is related to the polytropic index by

$$w = \frac{2}{3m} - 1. \tag{10.88}$$

These are power law solutions with the affine vectors decaying linearly in time. In particular, known General Relativity solutions can be obtained setting m to be $m_{rad} = 1/2$ ($w = 1/3$) or $m_{mat} = 2/3$ ($w = 0$), corresponding to a radiation and matter dominated universe, respectively. Also in this case, the bare energy density is negative but the missing contribution is again given by the affine contributions, i.e.

$$\rho_{eff} = \frac{9m^2}{2\kappa^2 t^2}, \qquad p_{eff} = -\frac{m}{\kappa^2 t^2}\left(3 - \frac{9m}{2}\right), \tag{10.89}$$

resulting in the effective polytropic index $w_{eff} = \frac{2}{3m} - 1$, in agreement with (10.88).

10.2.3 Solution reproducing linear growth of the scale factor

Finally, for

$$\epsilon = 1, \tag{10.90}$$

$$p = -\rho/3, \tag{10.91}$$

another solution exists:

$$a(t) = a_0 t, \tag{10.92}$$

$$\theta(t) = \theta_1 t^2 + \theta_0, \tag{10.93}$$

$$\rho(t) = \frac{3}{2\kappa^2\theta_1^2}\left(-1 + \frac{1}{\sqrt{2}}\sqrt{1+\sqrt{1+16\theta_1^2}}\right)\frac{1}{t^2}, \tag{10.94}$$

$$S^t(t) = \frac{6}{\theta_1}\left(-1 + \frac{1}{\sqrt{2}}\sqrt{1+\sqrt{1+16\theta_1^2}}\right)\frac{1}{t}, \tag{10.95}$$

$$Q^t(t) = 4P^t(t) = 8\left(1 - \frac{\sqrt{2}}{\sqrt{1+\sqrt{1+16\theta_1^2}}}\right)\frac{1}{t}. \tag{10.96}$$

with a_0, θ_0 and θ_1 arbitrary constants. The scale factor grows linearly in time and in this case we have $\rho > 0$.

10.3 Gravitational waves propagation

In this section we will finally derive some predictions of the theory regarding the propagation of gravitational waves. Given the violation of parity the theory is endowed with, we expect parity breaking effects in the results, similarly to the black hole perturbations scenario, where only the evolution of axial modes is affected. In particular we will probe the existence of two effects. Gravitational birefringence, consisting in a difference in the speed of propagation of left and right-handed polarizations, and gravitational Landau damping, i.e. a kinematic damping of the wave's amplitude due to the interaction with matter. In the era of gravitational wave astronomy, looking for effects of this kind in the propagation of spacetime perturbations seems very promising.
In order to proceed in this direction, a perturbative expansion of the fields must be performed, much like as in section 7.2. While the Landau damping effect can already be appreciated on the simplest background possible, namely flat Minkowski spacetime, the latter is not enough to give rise to the phenomenon of gravitational birefringence, when the propagation of perturbations is considered to happen in vacuum. Rather, a non-flat background on which to consider the evolution of perturbations needs to be selected. Among the solutions presented above, the first one describing a de Sitter phase is certainly one of the most interesting from the cosmological standpoint. Moreover, it is also the simplest background on which perturbative computations can be performed analytically, yielding the results presented in the next section.

10.3.1 Gravitational birefringence in late-time cosmology

The starting point is again the expansions

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \qquad \theta = \bar{\theta} + \delta\theta, \tag{10.97}$$

where now a bar denotes the exact solution (10.77)-(10.78). The main discussion will be focused on the propagation of tensor modes, thus setting $\delta\theta = 0$, while we postpone the analysis of scalar modes to the end of this section. In order to consider gravitational waves propagating along the $z-$axis, the gauge condition $h_{t\mu} = 0$ can be enforced and the perturbations can be written as

$$\delta\theta = 0, \qquad h_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{10.98}$$

where h_+ and $h_\times$ are functions of t and z alone. We now come to the perturbation of the affine connection. In general, it can be expanded as

$$\Gamma^{\rho}{}_{\mu\nu} = \bar{\Gamma}^{\rho}{}_{\mu\nu} + \delta\Gamma^{\rho}{}_{\mu\nu}, \tag{10.99}$$

where again the bar denotes the affine connection obtained from (10.79)-(10.80). Regarding the perturbation $\delta\Gamma^{\rho}{}_{\mu\nu}$, it includes perturbations of all components of torsion and nonmetricity, namely the vectors δS^μ, δP^μ, δQ^μ and the tensors $\delta q_{\mu\nu\rho}$ and $\delta\Omega_{\mu\nu\rho}$, considered as functions of t and z alone. Moreover, we will require the tensors $\delta q_{\mu\nu\rho}$ and $\delta\Omega_{\mu\nu\rho}$ to possess the same symmetries as their complete unperturbed versions.

Then, we use the xAct Mathematica package to perform the linearization of the connection field equations on the given background and to obtain a partial solution for the perturbations in the following way. First, the vanishing of the affine vectorial perturbations results as a consequence of the equations. This is expected since only the tensor modes of the metric perturbation have been selected and there are no vector modes available for the coupling with the affine vectors. Regarding the affine tensor perturbations, several equations can be algebraically solved, resulting in the following non vanishing components:

$$\delta q_{110} = \delta q_{202}, \qquad \delta q_{102} = \delta q_{201}, \qquad \delta q_{232} = \delta q_{113}, \qquad \delta q_{123} = \delta q_{213}, \tag{10.100}$$

$$\delta\Omega_{111} = -\delta\Omega_{122} = -\delta\Omega_{212} = \frac{3}{2a\sqrt{\Lambda}}\delta\Omega'_{112}, \qquad \delta\Omega_{211} = -\delta\Omega_{222} = \delta\Omega_{112}, \tag{10.101}$$

where temporal and spacial indices are here represented by 0 and 1, 2, 3, respectively, while primes denote derivatives with respect to z. The nonmetricity tensor perturbation turns out to represent non propagating degrees, since it ultimately satisfies the spatial harmonic oscillator equation

$$\delta\Omega''_{112} + \frac{4\Lambda}{9}a^2\delta\Omega_{112} = 0, \tag{10.102}$$

which is solved by

$$\delta\Omega_{112} = C_1(t)\cos\left(\frac{2\sqrt{\Lambda}}{3}a(t)z\right) + C_2(t)\sin\left(\frac{2\sqrt{\Lambda}}{3}a(t)z\right), \tag{10.103}$$

where C_1 and C_2 are arbitrary functions of time. Using these results it is easy to check that the linearized scalar equation is automatically satisfied. At this point, there are six equations left for the two metric functions h_+, $h_\times$ and for q_{201}, q_{113}, q_{202}, q_{213}. They read

$$\frac{6}{\sqrt{\Lambda}}\delta q'_{201} + \left(2a\delta q_{202} - \sqrt{3}\delta q_{213}\right) = \left(\frac{3}{\sqrt{\Lambda}}\dot{h}'_\times - 2\sqrt{3}h'_\times - \frac{3}{2}a\dot{h}_+ + \sqrt{3\Lambda}ah_+\right), \tag{10.104}$$

$$\frac{6}{\sqrt{\Lambda}}\delta q'_{113} + \left(\sqrt{3}a^2\delta q_{202} + 2a\delta q_{213}\right) = \left(-\frac{3}{\sqrt{\Lambda}}h''_+ - \frac{\sqrt{3}}{2}a^2\dot{h}_+ + \sqrt{\Lambda}a^2h_+\right), \tag{10.105}$$

$$\frac{6}{\sqrt{\Lambda}}\delta q'_{202} + \left(\sqrt{3}\delta q_{113} - 2a\delta q_{201}\right) = \left(-\frac{3}{\sqrt{\Lambda}}\dot{h}'_+ + 2\sqrt{3}h'_+ - \frac{3}{2}a\dot{h}_\times + \sqrt{3\Lambda}ah_\times\right), \tag{10.106}$$

$$\frac{6}{\sqrt{\Lambda}}\delta q'_{213} - \left(2a\delta q_{113} + \sqrt{3}a^2\delta q_{201}\right) = \left(-\frac{3}{\sqrt{\Lambda}}h''_\times - \frac{\sqrt{3}}{2}a^2\dot{h}_\times + \sqrt{\Lambda}a^2h_\times\right), \tag{10.107}$$

and

$$\ddot{h}_+ - \frac{1}{a^2}h''_+ - 4\sqrt{\frac{\Lambda}{3}}\dot{h}_+ + \frac{4\Lambda}{3}h_+ = 2\left(\frac{1}{a^2}\delta q'_{113} - \delta\dot{q}_{202}\right) + \sqrt{\Lambda}\left(\sqrt{3}\delta q_{202} + \frac{1}{a}\delta q_{213}\right), \tag{10.108}$$

$$\ddot{h}_\times - \frac{1}{a^2}h''_\times - 4\sqrt{\frac{\Lambda}{3}}\dot{h}_\times + \frac{4\Lambda}{3}h_\times = 2\left(\frac{1}{a^2}\delta q'_{213} + \delta\dot{q}_{201}\right) - \sqrt{\Lambda}\left(\sqrt{3}\delta q_{201} + \frac{1}{a}\delta q_{113}\right), \tag{10.109}$$

having used $\dot{a} = a\sqrt{\Lambda/3}$ and $\ddot{a} = a\Lambda/3$. In order to proceed further the adiabatic approximation will be assumed, consisting in considering the scale factor nearly constant, i.e. $a(t) \approx 1$, during the gravitational wave propagation. Hence, in

Fourier space the equations become

$$ik\delta q_{201} + \frac{\sqrt{\Lambda}}{6}\left(2\delta q_{202} - \sqrt{3}\delta q_{213}\right) = \frac{\sqrt{\Lambda}}{12}\left[\left(3i\omega + 2\sqrt{3\Lambda}\right)h_+ + k\left(\frac{6}{\sqrt{\Lambda}}\omega - 4i\sqrt{3}\right)h_\times\right], \tag{10.110}$$

$$ik\delta q_{113} + \frac{\sqrt{\Lambda}}{6}\left(\sqrt{3}\delta q_{202} + 2\delta q_{213}\right) = \frac{1}{2}\left(k^2 + i\frac{\omega}{2}\sqrt{\frac{\Lambda}{3}} + \frac{\Lambda}{3}\right)h_+, \tag{10.111}$$

$$ik\delta q_{202} + \frac{\sqrt{\Lambda}}{6}\left(\sqrt{3}\delta q_{113} - 2\delta q_{201}\right) = \frac{\sqrt{\Lambda}}{12}\left[\left(3i\omega + 2\sqrt{3\Lambda}\right)h_\times - k\left(\frac{6}{\sqrt{\Lambda}}\omega - 4i\sqrt{3}\right)h_+\right], \tag{10.112}$$

$$ik\delta q_{213} + \frac{\sqrt{\Lambda}}{6}\left(-2\delta q_{113} - \sqrt{3}\delta q_{201}\right) = \frac{1}{2}\left(k^2 + i\frac{\omega}{2}\sqrt{\frac{\Lambda}{3}} + \frac{\Lambda}{3}\right)h_\times, \tag{10.113}$$

and

$$\left(\omega^2 - k^2 - \frac{4\Lambda}{3} - 4i\sqrt{\frac{\Lambda}{3}}\omega\right)h_+ = -2ik\,\delta q_{113} - (2i\omega + \sqrt{3\Lambda})\delta q_{202} - \sqrt{\Lambda}\,\delta q_{213}, \tag{10.114}$$

$$\left(\omega^2 - k^2 - \frac{4\Lambda}{3} - 4i\sqrt{\frac{\Lambda}{3}}\omega\right)h_\times = -2ik\,\delta q_{213} + (2i\omega + \sqrt{3\Lambda})\delta q_{201} + \sqrt{\Lambda}\,\delta q_{113}. \tag{10.115}$$

The system is reduced to an algebraic one and the torsion perturbations can now be solved in terms of the metric ones as

$$\delta q_{201} = \frac{p(k,\omega,\Lambda)h_+ - q(k,\omega,\Lambda)h_\times}{\Delta(k,\Lambda)}, \tag{10.116}$$

$$\delta q_{113} = \frac{-m(k,\omega,\Lambda)h_+ + n(k,\omega,\Lambda)h_\times}{\Delta(k,\Lambda)}, \tag{10.117}$$

$$\delta q_{202} = \frac{q(k,\omega,\Lambda)h_+ + p(k,\omega,\Lambda)h_\times}{\Delta(k,\Lambda)}, \tag{10.118}$$

$$\delta q_{213} = -\frac{n(k,\omega,\Lambda)h_+ + m(k,\omega,\Lambda)h_\times}{\Delta(k,\Lambda)}, \tag{10.119}$$

where we introduced

$$\Delta(k,\Lambda) \equiv 2(1296k^4 - 72\Lambda k^2 + 49\Lambda^2), \tag{10.120}$$

$$p(k,\omega,\Lambda) \equiv 2k\sqrt{\Lambda}\left(108k^2\omega + 2i\sqrt{3}\Lambda^{3/2} - 3\Lambda\omega\right), \tag{10.121}$$

$$q(k,\omega,\Lambda) \equiv 3\left(72k^4\left(5\sqrt{3}\sqrt{\Lambda} + 6i\omega\right) - 6k^2\left(3\sqrt{3}\Lambda^{3/2} + 8i\Lambda\omega\right) + 7\Lambda^2\left(2\sqrt{3}\sqrt{\Lambda} + 3i\omega\right)\right), \tag{10.122}$$

$$m(k,\omega,\Lambda) \equiv 12ik\left(108k^4 - 3k^2\Lambda + 2i\sqrt{3}\Lambda^{3/2}\omega + 4\Lambda^2\right), \tag{10.123}$$

$$n(k,\omega,\Lambda) \equiv \sqrt{\Lambda}\left(432k^4 - 12k^2\left(\Lambda - 3i\sqrt{3}\sqrt{\Lambda}\omega\right) + 7i\sqrt{3}\Lambda^{3/2}\omega + 14\Lambda^2\right). \tag{10.124}$$

The expressions for the δq's can also be written in terms of the vectors

$$\mathbf{Q} = \begin{pmatrix} \delta q_{201} \\ \delta q_{113} \\ \delta q_{202} \\ \delta q_{213} \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} h_+ \\ h_\times \end{pmatrix} \tag{10.125}$$

and the matrix

$$M = \frac{1}{\Delta}\begin{pmatrix} p & -q \\ -m & n \\ q & p \\ -n & -m \end{pmatrix}, \tag{10.126}$$

as

$$\mathbf{Q} = M\mathbf{H}. \tag{10.127}$$

Using these quantities the remaining two metric equations read

$$d(k,\omega,\Lambda)\mathbf{H} = N\mathbf{Q}, \tag{10.128}$$

where we defined

$$N \equiv \begin{pmatrix} 0 & -2ik & -(2i\omega + \sqrt{3\Lambda}) & -\sqrt{\Lambda} \\ 2i\omega + \sqrt{3\Lambda} & \sqrt{\Lambda} & 0 & -2ik \end{pmatrix} \tag{10.129}$$

and

$$d(k,\omega,\Lambda) \equiv \omega^2 - k^2 - \frac{4\Lambda}{3} - 4i\sqrt{\frac{\Lambda}{3}}\omega. \tag{10.130}$$

Then, combining (10.127) and (10.128), we obtain

$$\left(d(k,\omega,\Lambda)\, I - P\right)\mathbf{H} = 0, \tag{10.131}$$

where I is the 2×2 identity matrix and $P = NM$ is explicitly given by

$$P = \begin{pmatrix} p_{11} & -p_{12} \\ p_{12} & p_{11} \end{pmatrix}, \tag{10.132}$$

where

$$p_{11} \equiv \frac{2ikm - (2i\omega + \sqrt{3\Lambda})q + \sqrt{\Lambda}n}{\Delta}, \qquad p_{12} \equiv \frac{2ikn + (2i\omega + \sqrt{3\Lambda})p - \sqrt{\Lambda}m}{\Delta}. \tag{10.133}$$

In the particular case in which $p_{12} = 0$ we have that the plus and cross modes do not mix and the usual dispersion relation is recovered for both of them:

$$\mathscr{D}(k,\omega) \equiv d(k,\omega,\Lambda) - p_{11} = 0. \tag{10.134}$$

In the $p_{12} \neq 0$ case instead, the two polarization are mixed. Introducing the left and right handed polarization states defined by

$$h_L = \frac{1}{\sqrt{2}}(h_+ - ih_\times), \quad h_R = \frac{1}{\sqrt{2}}(h_+ + ih_\times), \tag{10.135}$$

they can be decoupled, yielding

$$(d(k,\omega,\Lambda) - p_{11} + ip_{12})h_L = 0, \tag{10.136}$$

$$(d(k,\omega,\Lambda) - p_{11} - ip_{12})h_R = 0. \tag{10.137}$$

This results in two dispersion relations for the right and left handed modes

$$\mathscr{D}_{L,R}(k,\omega) \equiv d(k,\omega,\Lambda) - p_{11} \pm ip_{12} = 0, \tag{10.138}$$

that can be solved for the two frequencies

$$\omega_L = \pm\frac{\sqrt{36k^4 - 12k^3\sqrt{\Lambda} - 35k^2\Lambda - 18k\Lambda^{3/2} - 3\Lambda^2}}{2\left(3k + \sqrt{\Lambda}\right)} + i\sqrt{\frac{\Lambda}{12}}, \tag{10.139}$$

$$\omega_R = \pm\frac{\sqrt{36k^4 + 12k^3\sqrt{\Lambda} - 35k^2\Lambda + 18k\Lambda^{3/2} - 3\Lambda^2}}{2\left(3k - \sqrt{\Lambda}\right)} + i\sqrt{\frac{\Lambda}{12}}. \tag{10.140}$$

At this point the condition

$$36k^4 \pm 12\sqrt{\Lambda}k^3 - 35\Lambda k^2 \pm 18\Lambda^{3/2}k - 3\Lambda^2 > 0 \tag{10.141}$$

must be imposed in order to ensure the wave's propagation. This condition implies

$$k < 0,\ 0 < \Lambda < 9k^2; \qquad k > 0,\ 0 < \Lambda < \gamma k^2, \tag{10.142}$$

for the left mode and

$$k > 0,\ 0 < \Lambda < 9k^2; \qquad k < 0,\ 0 < \Lambda < \gamma k^2, \tag{10.143}$$

for the right mode, where the parameter γ is

$$\gamma = \frac{1}{9}\left(\sqrt[3]{2592\sqrt{62}-6697} - \frac{719}{\sqrt[3]{2592\sqrt{62}-6697}} + 11\right) \sim 0.543988. \qquad (10.144)$$

In the adiabatic limit we can consider modes with wavelength much smaller than the curvature radius of the background spacetime, i.e. $|k| \gg \sqrt{\frac{\Lambda}{3}}$. Hence, from (10.142)-(10.143) the propagation occurs for all wave vectors under consideration and no further condition must be imposed. The group and phase velocities can be computed as

$$v^g_{L,R} \equiv \frac{d\omega_{L,R}}{dk} = \frac{108k^4 \pm 54k^3\sqrt{\Lambda} - 18k^2\Lambda \mp 8k\Lambda^{\frac{3}{2}}}{2\left(3k \pm \sqrt{\Lambda}\right)^2 \sqrt{36k^4 \mp 12k^3\sqrt{\Lambda} - 35k^2\Lambda \mp 18k\Lambda^{\frac{3}{2}} - 3\Lambda^2}}, \qquad (10.145)$$

$$v^p_{L,R} \equiv \frac{\omega_{L,R}}{k} = \frac{\sqrt{36k^4 \mp 12k^3\sqrt{\Lambda} - 35k^2\Lambda \mp 18k\Lambda^{3/2} - 3\Lambda^2}}{2k\left(3k \pm \sqrt{\Lambda}\right)}. \qquad (10.146)$$

We observe a difference in the velocity of propagation between the left and right handed mode, a phenomenon already observed in the metric formulation of CSMG (see [184, 185, 186] for a comparison) and known as velocity birefringence. Regarding the wave's amplitude instead, there is no discrepancy between the two modes and the friction term responsible for the amplitude damping is common to both modes and it is the usual one given by the cosmological constant, i.e. the $i\sqrt{\Lambda/12}$ terms in (10.139) and (10.140).
In order to give some estimates of the effects just derived it is convenient to expand the velocities in a power series in the parameter $\epsilon \equiv \sqrt{\Lambda}/k$ which satisfies $|\epsilon| \ll 1$ in the adiabatic limit. The outcome is

$$v^g_{L,R} = 1 + \frac{\epsilon^2}{3} \pm \frac{4\epsilon^3}{9} + \mathcal{O}\left(\epsilon^4\right), \qquad (10.147)$$

$$v^p_{L,R} = 1 \mp \frac{\epsilon}{2} - \frac{\epsilon^2}{3} \mp \frac{2\epsilon^3}{9} + \mathcal{O}\left(\epsilon^4\right). \qquad (10.148)$$

We find some common features with the results obtained in [156]. In particular, the group velocity turns out to be super-luminal and polarization independent at order $\mathcal{O}(\epsilon^2)$. We conclude that the most efficient way to detect parity violating effects consists in measuring the phase velocity, since $|v^p_L - v^p_R| \sim \mathcal{O}(\epsilon)$, whereas $|v^g_L - v^g_R| \sim \mathcal{O}(\epsilon^3)$.
Nevertheless, we can compare the deviation in the group velocities from the

speed of light in vacuum with the current experimental bound on the gravitational waves speed [274], i.e.

$$-3\times 10^{-15} \le v_g - 1 \le 7\times 10^{-16}. \tag{10.149}$$

If we assume a magnitude of the wavenumber $k = 10^{-7}\,\mathrm{m}^{-1}$, corresponding to a frequency $\nu \approx 50\,\mathrm{Hz}$ well inside the sensitivity curves of ground-based interferometers, and the measured value of the cosmological constant $\Lambda_{\mathrm{exp}} \approx 10^{-52}\,\mathrm{m}^{-2}$ we calculate a theoretically expected deviation $\mathcal{O}\left(\epsilon^2\right) = 10^{-38}$. Therefore the bound (10.149) is not sufficiently tight in order to falsify this model of modified gravity. However, the dependence of the deviation parameter with respect to the wavenumber, i.e. $\epsilon^2 \propto k^{-2}$, indicates that for detections in the low-frequencies domain the expected deviation would result significantly larger. By considering a signal in the mHz band, which will become accessible with the space interferometer LISA, we obtain a much greater expected deviation $\mathcal{O}\left(\epsilon^2\right) = 10^{-29}$. The maximum magnitude for the deviation parameter is reached in the case of a nHz gravitational wave, detectable with pulsar timing arrays, which we calculate to be $\mathcal{O}\left(\epsilon^2\right) = 10^{-17}$.

Scalar modes

We will conclude this section briefly discussing the propagation of scalar perturbations. In this case we assume $\delta\theta = \delta\theta(t,z) \neq 0$ and we adopt the same gauge choice for the metric perturbation, which now has the following form

$$h_{ij} = \frac{1}{3}\bar{g}_{ij}h + \partial_i\partial_j B - \frac{1}{3}\bar{g}_{ij}\partial^k\partial_k B, \tag{10.150}$$

in terms of the trace $h = h^i_i$ and a traceless part depending on the gradients of the scalar function $B(t,z)$. Regarding the affine sector, we assume that only the components behaving as scalars under spatial rotations couple to the metric and scalar perturbations. This leaves us with $\delta\Omega_{000}$ and the time components of the vectors δS^t, δP^t and δQ^t. In principle, also contributions of the form $\delta\Omega_{0ij} = \psi(t,z)\delta_{ij}$ should be included but they are eventually ruled out by the traceless character of $\delta\Omega_{\mu\nu\rho}$ and $\delta q_{\mu\nu\rho}$. Solving the linearized equations we obtain

$$h_{xx} = h_{yy} = a^2\left(-\frac{c_1}{3\sqrt{3\Lambda}}e^{-\sqrt{3\Lambda}t} + \frac{c_2}{3}\right), \tag{10.151}$$

$$h_{zz} = a^2\left(-\frac{c_1}{3\sqrt{3\Lambda}}e^{-\sqrt{3\Lambda}t} + \frac{c_2}{3} + f(z)\right), \tag{10.152}$$

so that

$$h = -\frac{c_1}{\sqrt{3\Lambda}}e^{-\sqrt{3\Lambda}t} + c_2 + f(z), \tag{10.153}$$

while the perturbation of the pseudo-scalar field is given by

$$\delta\theta = -\frac{c_1}{6\Lambda^{3/2}}e^{-\sqrt{3\Lambda}t} + \frac{c_2}{2\sqrt{3}\Lambda} + c_0. \tag{10.154}$$

The solution for the affine perturbations is characterized by $\delta\Omega_{000} = 0$ and

$$\delta S^t = \frac{c_1}{2\sqrt{3}}e^{-\sqrt{3\Lambda}t}, \tag{10.155}$$

$$\delta Q^t = 4\delta P^t = -\frac{5c_1}{3}e^{-\sqrt{3\Lambda}t}. \tag{10.156}$$

In the above expressions c_0, c_1, c_2 are constants of integration and $f(z)$ is an arbitrary function of z.
We conclude that there are no perturbations in the scalar sector propagating as waves, a result which is in agreement with the purely metric formulation of Chern-Simons gravity. Rather, all scalar perturbations suffer an exponentially decay over a time scale of order $t_D \sim 1/3H_0$. Therefore, in the adiabatic approximation, they can be considered nearly constant with respect to the tensor modes.

10.3.2 Gravitational Landau damping

Landau damping is a well-known effect characterizing electromagnetic waves travelling in a plasma, whose particles can exchange energy with the wave. As a result, when the particles acquire energy from the wave, the amplitude of the latter is affected by an exponential damping. The possibility of an analogous effects in gravitational settings was first investigated in [275] (see also [276, 277, 275, 278, 279, 280, 281, 282]). The idea is that the interaction of gravitational waves with non-collisional matter media can yield a damping or enhancement of the metric perturbation. However, in order for this effect to be present in General Relativity, it is not enough to consider the simplest scenario of gravitational waves propagating on a flat Minkowski background and interacting with a matter medium characterized by an isotropic background configuration. Indeed, in this case the phase velocity of gravitational waves turns out to be superluminal, which is enough to prevent the arising of Landau damping, as we will see in the following. The latter appears in General Relativity only considering more general assumptions, such as the introduction of anisotropies in the medium or considering the propagation of gravitational waves on a FLRW background.
Going beyond General Relativity, in [220] Horndeski theories of gravity were considered and gravitational Landau damping of the additional scalar mode characterizing the theory was established, even on Minkowski background and with

an isotropic matter medium. The authors also show that the same effect is forbidden for tensor modes, as in General Relativity.
Hence the question of whether also tensor modes can suffer Landau damping in extended theories of gravity is still open. In this regard, parity violating theories offer an interesting possibility. Indeed, the gravitational birefringence effects just discussed can yield a suppression of the phase velocity of one of the two chiral polarizations of the wave, thus allowing for a subluminal phase velocity. In turn, this makes the arising of Landau damping possible.
In the following, in order to derive the equations for the perturbations we will proceed analogously to the previous chapters but including a non-vanishing stress-energy tensor and its perturbation $\delta T_{\mu\nu}$. Moreover, we will make some assumptions needed to obtain the results that will be derived. These are:

- Adiabatic approximation: the background scalar field configuration is assumed to depend only on the cosmological time and to be slowly varying in the time interval considered. This amounts to consider its first derivative constant ($\dot{\theta} \sim \dot{\theta}_B$) and to neglect higher derivatives.

- Relativistic medium: the matter medium in which the wave is propagating is assumed to be composed by relativistic particles of mass m. Before the passage of the wave, it will be considered in an equilibrium configuration at temperature Θ, described by a distribution function f_0.

- Small wavelength approximation: the wavelength of the gravitational waves is assumed to be much smaller than the characteristic length scale of variation of the thermodynamic properties of the medium, such as temperature and pressure.

- Homogeneity and isotropy: the previous condition allows to choose as an equilibrium function f_0 a homogeneous and isotropic distribution $f_0(p)$, where p is the flat three momentum, i.e. $p \equiv \sqrt{\delta^{ij} p_i p_j}$.

- Weak damping: the imaginary part of the frequency is assumed to be much smaller than the real part, thus avoiding signals that are too rapidly damped to be detected.

- Wave and medium velocities approximation: the phase velocity of the gravitational wave is assumed to be much greater than the thermal velocity of the medium, a condition that holds in weak field regimes.

Finally, we will later introduce a parameter $k_{CS} \equiv 1/(8\alpha\dot{\theta}_B)$, which is assumed to be positive. The results derived below can be straightforwardly adapted to the $k_{CS} < 0$ case (see [5]).

Now, the starting point of the analysis is the set of linearized equations describing the evolution of metric and affine perturbations on a flat Minkowski background. One could be tempted to derive the equations taking the $\Lambda \to 0$ limit of the system (10.104)-(10.107) and (10.108)-(10.109). However, the de Sitter background solution employed there is singular in $\Lambda = 0$ (see (10.78) for instance) and the case of vanishing cosmological constant must be addressed separately, starting with $a(t) = 1$ from the very beginning. Following similar steps but assuming the background metric to be the flat Minkowski one, the outcome is

$$\delta q_{102} + \frac{\alpha\dot{\theta}_B}{2}\left(2\alpha\dot{\theta}_B \delta q''_{102} + \dot{h}'_+ - \alpha\dot{\theta}_B \dot{h}''_\times\right) = 0, \tag{10.157}$$

$$\delta q_{213} + \frac{\alpha\dot{\theta}_B}{2}\left(2\alpha\dot{\theta}_B \delta q''_{213} - h''_+ + \alpha\dot{\theta}_B h'''_\times\right) = 0, \tag{10.158}$$

for the torsion perturbations, and

$$\ddot{h}_+ - h''_+ + \alpha\dot{\theta}_B\left(h'''_\times - \ddot{h}'_\times + 2\delta q''_{213} + 2\delta\dot{q}'_{102}\right) = 2\kappa^2 \delta T_{11}, \tag{10.159}$$

$$\ddot{h}_\times - h''_\times - 2\delta q'_{213} - 2\delta\dot{q}_{102} = 2\kappa^2 \delta T_{12}, \tag{10.160}$$

for the metric ones, where δT_{11} and δT_{12} are the only non vanishing components of the stress-energy tensor perturbation. We assumed again a propagation along the z-axis and a metric perturbation of the form (10.98). Regarding the affine perturbations, only the perturbations of the rank-3 tensor part of torsion and nonmetricity were considered, i.e. $\delta q_{\mu\nu\rho}$ and $\delta\Omega_{\mu\nu\rho}$. Only some of their components are actually propagating and taking part in the dynamics. These are all related to δq_{102} and δq_{213} by

$$\delta q_{201} = \delta q_{102}, \tag{10.161}$$

$$\delta q_{123} = \delta q_{213}, \tag{10.162}$$

$$\delta q_{110} = \delta q_{202} = \alpha\dot{\theta}_B\left(\delta q'_{102} - \frac{1}{2}\dot{h}'_{12}\right), \tag{10.163}$$

$$\delta q_{223} = \delta q_{131} = \alpha\dot{\theta}_B\left(\delta q'_{213} + \frac{1}{2}h''_{12}\right). \tag{10.164}$$

All other components are either vanishing or not propagating. The non vanishing ones are

$$\delta q_{010}, \quad \delta q_{313}, \quad \delta q_{020}, \quad \delta q_{323}, \quad \delta q_{031}, \quad \delta q_{301} \quad \delta q_{032}, \quad \delta q_{302}, \tag{10.165}$$

for the torsion, and

$$\delta\Omega_{n00}, \quad \delta\Omega_{n11}, \quad \delta\Omega_{n22}, \quad \delta\Omega_{n03}, \quad \delta\Omega_{n12}, \quad \delta\Omega_{n33}, \tag{10.166}$$

with $n = 1, 2$ and

$$\delta\Omega_{n01}, \quad \delta\Omega_{n02}, \quad \delta\Omega_{n13}, \quad \delta\Omega_{n23}, \tag{10.167}$$

with $n = 0, 3$, for the nonmetricity, and they all satisfy the same harmonic oscillator equation, reading

$$u(t,z) + \frac{\alpha^2 \dot{\theta}_B^2}{4} u''(t,z) = 0, \tag{10.168}$$

whose solution reads

$$u(t,z) = C_1(t) \cos\left(\frac{2z}{\alpha \dot{\theta}_B}\right) + C_2(t) \sin\left(\frac{2z}{\alpha \dot{\theta}_B}\right), \tag{10.169}$$

where C_1 and C_2 are arbitrary functions of time.
The system of equations is closed by including the equation for the matter distribution. The latter is perturbed by the passage of the gravitational wave, deviating from its equilibrium configuration by a variation δf of order h, so that we can write

$$f(\vec{x}, \vec{p}, t) = f(\vec{x}, \vec{p}, 0) + \delta f(\vec{x}, \vec{p}, t), \tag{10.170}$$

where $\vec{x}$ and $\vec{p}$ denotes the spatial components of coordinates and momenta of the particles. Moreover, $f(\vec{x}, \vec{p}, 0)$ can be shown to be related to the equilibrium distribution $f_0(p)$ by [220]

$$f(\vec{x}, \vec{p}, 0) = f_0(p) - \frac{f_0'(p)}{2} \frac{p_i p_j}{p} h_{ij}(\vec{x}, 0). \tag{10.171}$$

The equation for the evolution of δf is obtained linearizing the Vlasov equation and yields [220]

$$\frac{\partial \delta f}{\partial t} + \frac{p^m}{p^0} \frac{\partial \delta f}{\partial x^m} - \frac{f_0'(p)}{2p} \frac{\partial h_{ij}}{\partial t} p_i p_j = 0. \tag{10.172}$$

The last step we need to perform before solving the equations consists in relating δf to the stress-energy tensor perturbations appearing in (10.159) and (10.160). As shown in [220] the correct relation is

$$T_{ij}(\vec{x}, t) = \int d^3p \, \frac{p_i p_j}{p^0} \delta f(\vec{x}, \vec{p}, t), \tag{10.173}$$

Now we have a closed system of five differential equations for the two metric perturbations, the two torsion perturbations and δf. By performing a Fourier transform on the spatial coordinates and a Laplace transform on the time coordinate the equations reduce to algebraic ones, drastically simplifying the problem. In particular, the linearized Vlasov equation is readily solved by

$$\delta f^{(k,s)}(\vec{p}) = \frac{\frac{f_0'(p)}{2p}\left(s\,h_{ij}^{(k,s)} - h_{ij}^{(k)}(0)\right)p_i p_j}{s + ik\frac{p_3}{p^0}}, \tag{10.174}$$

where the Fourier and Fourier-Laplace components of a generic field are displayed as $\phi^{(k)}(t)$ and $\phi^{(k,s)}$, respectively. Then, the torsion perturbations are solved in terms of the metric ones as

$$\delta q_{102}^{(k,s)} = \frac{i\alpha\dot{\theta}_B k}{2\sqrt{2}}\left(\frac{sh_L^{(k,s)} - h_L^{(k)}(0)}{1-\alpha\dot{\theta}_B k} + \frac{sh_R^{(k,s)} - h_R^{(k)}(0)}{1+\alpha\dot{\theta}_B k}\right), \tag{10.175}$$

$$\delta q_{213}^{(k,s)} = -\frac{\alpha\dot{\theta}_B k^2}{2\sqrt{2}}\left(\frac{h_L^{(k,s)}}{1-\alpha\dot{\theta}_B k} + \frac{h_R^{(k,s)}}{1+\alpha\dot{\theta}_B k}\right), \tag{10.176}$$

which, once substituted into the metric equations yield

$$\frac{(s^2+k^2)h_L^{(k,s)} - sh_L^{(k)}(0)}{1-\alpha\dot{\theta}_B k} = 2\chi T_L, \tag{10.177}$$

$$\frac{(s^2+k^2)h_R^{(k,s)} - sh_R^{(k)}(0)}{1+\alpha\dot{\theta}_B k} = 2\chi T_R, \tag{10.178}$$

where, using (10.173) and (10.174), the left and right handed components of the stress-energy tensor can be written as [220]

$$T_L = \frac{\pi}{4}\int_0^\infty d\rho \int_{-\infty}^{+\infty} dp_3\, \frac{f_0'(p)\rho^5}{p\left(p^0 s + ikp_3\right)}\left(sh_L^{(k,s)} - h_L^k(0)\right), \tag{10.179}$$

$$T_R = \frac{\pi}{4}\int_0^\infty d\rho \int_{-\infty}^{+\infty} dp_3\, \frac{f_0'(p)\rho^5}{p\left(p^0 s + ikp_3\right)}\left(sh_R^{(k,s)} - h_R^k(0)\right), \tag{10.180}$$

having introduced cylindrical coordinates in the momentum space, i.e.

$$p_1 = \rho\cos\varphi, \quad p_2 = \rho\sin\varphi, \quad p_3 = p_3. \tag{10.181}$$

Eventually, the left and right handed metric perturbations are obtained as

$$h_{L,R}^{(k,s)} = \frac{\left(s - \frac{\pi\chi(1\mp\alpha\dot{\theta}_B k)}{2}\int_0^\infty d\rho\int_{-\infty}^{+\infty} dp_3\,\frac{f_0'(p)\rho^5}{p(p^0 s + ikp_3)}\right)h_{L,R}^{(k)}(0)}{\left(s^2+k^2\right)\epsilon_{L,R}(k,s)}, \tag{10.182}$$

where the chiral dielectric functions are defined by

$$\epsilon_{L,R}(k,s) = 1 - \frac{\pi\chi s\, B_{L,R}(k)}{2\left(s^2+k^2\right)} \int d\rho dp_3 \frac{f_0'(p)}{p} \frac{\rho^5}{p^0 s + ikp_3}, \tag{10.183}$$

and the birefringence factor is given by

$$B_{L,R}(k) = 1 \mp \alpha\dot{\theta}_B k = 1 \mp \frac{k}{8k_{CS}}, \tag{10.184}$$

where $k_{CS} \equiv 1/(8\alpha\dot{\theta}_B)$ has dimensions of momentum. Now, the standard procedure is the following. The dispersion relation $\omega_r = \omega_r(k)$ is computed solving the condition

$$\Re(\epsilon_{L,R})(k,\omega_r) = 0. \tag{10.185}$$

Then, the damping coefficient is obtained from

$$\omega_i = -\left.\frac{\Im(\epsilon_{L,R})}{\frac{\partial\Re(\epsilon_{L,R})}{\partial\omega}}\right|_{\omega=\omega_r}. \tag{10.186}$$

To proceed further a specific background distribution must be chosen. We choose to deal with a Jüttner-Maxwell distribution, i.e.

$$f_0(p) = \frac{n}{4\pi m^2\Theta K_2(x)} e^{-\frac{\sqrt{m^2+p^2}}{\Theta}}, \tag{10.187}$$

where n is the density of particles, m is their mass and $K_\nu(x)$ are the modified Bessel functions of the second kind with real index ν, evaluated at $x \equiv m/\Theta$. For the Jüttner-Maxwell distribution we can rewrite the dielectric functions as

$$\epsilon_{L,R}(k,\omega_r) = 1 - \frac{n\chi B_{L,R}(k)}{4k^2m^2\Theta^2K_2(x)} \left(\frac{v_p}{1-v_p^2}\right)^2 \int d\rho dp_3 \frac{\rho^5 e^{-\frac{\sqrt{m^2+\rho^2+p_3^2}}{\Theta}}}{p_3^2 - \frac{v_p^2}{1-v_p^2}\left(m^2+\rho^2\right)}, \tag{10.188}$$

where the phase velocity of the wave is $v_p \equiv \omega_r/k$. From this we see that, in order for the Landau damping effect to arise, the phase velocity of the wave must be subluminal. Indeed, a non vanishing damping coefficient ω_i requires a non vanishing imaginary part of the dielectric functions. The latter can only be produced by poles in the denominator of the integral in (10.188), which exist only for $v_p < 1$ and are given by

$$p_3 = \pm\sqrt{\frac{v_p^2}{1-v_p^2}\left(m^2+\rho^2\right)}. \tag{10.189}$$

In order to check the subluminal condition we first need to compute the dispersion relation. Expanding the denominator up to second order in p_3 we get the real part of the dielectric function:

$$\Re(\epsilon_{L,R}) = 1 + \frac{2\omega_0^2 B_{L,R}(k)}{x^2k^2}\left(\frac{x}{1-\frac{\omega^2}{k^2}} + \frac{\gamma(x)}{\frac{\omega^2}{k^2}}\right), \tag{10.190}$$

where $\omega_0^2 = \chi nm$ is the proper frequency of the medium and $\gamma(x) \equiv K_1(x)/K_2(x)$. The p_3 expansion we employed amounts to assuming

$$\frac{p_3}{v_p}\sqrt{\frac{1-v_p^2}{m^2+\rho^2}} \ll 1, \tag{10.191}$$

which is consistent with the last assumption in the list at the beginning of this chapter. Now, imposing the vanishing of the real part of the dielectric functions yields two fourth order equations which are solved by

$$\omega_{L,R}^2 = \frac{k^2}{2}\left(1 + \frac{2\omega_0^2\,(x-\gamma)\,B_{L,R}(k)}{x^2k^2} \pm \sqrt{\left(1+\frac{2\omega_0^2\,(x-\gamma)\,B_{L,R}(k)}{x^2k^2}\right)^2 + \frac{8\omega_0^2\gamma B_{L,R}(k)}{x^2k^2}}\right). \tag{10.192}$$

Hence, we have 8 frequencies in total, 4 per each polarization, which is identified by the subscripts of the function $B_{L,R}(k)$, belonging to two branches, identified by the $\pm$ sign explicitly appearing in the expression above. However, the reality condition $\omega^2 > 0$ must be required. This, together with the subluminal condition explained above, will select one of the two branches of the frequencies and identify ranges in the momentum space in which the wave is allowed to propagate within the medium and the damping effect is present. In the case at hand, we can distinguish two cases, depending on the sign of $B_{L,R}(k)$. If $B_{L,R}(k) > 0$ then the reality of the frequency is assured only for the plus sign in (10.192). However, in this case one always has $v_p > 1$. Hence, the wave propagates throughout the medium without any damping.
For $B_{L,R}(k) < 0$ instead, the reality condition is satisfied by both signs in (10.192), provided that the following condition holds

$$k^2 + \delta^2\left(1 \mp \frac{k}{8k_{CS}}\right) > 0, \tag{10.193}$$

where we defined

$$\delta^2 \equiv \frac{2\omega_0^2}{x^2}\left(\sqrt{x}+\sqrt{\gamma(x)}\right)^2. \tag{10.194}$$

The outcome depends on the sign of $\Delta \equiv \delta^2 - 256k_{CS}^2$. If $\Delta < 0$ the inequality (10.193) is satisfied for every value of the momentum. If instead $\Delta > 0$ then (10.193) is valid only for the following wavenumbers

$$\text{Left mode: } k < k^0_{-,L} \,\cup\, k > k^0_{+,L} \qquad \text{Right mode: } k < k^0_{+,R} \,\cup\, k > k^0_{-,R}, \tag{10.195}$$

where the boundaries of the intervals are given by

$$k^0_{\pm,L} = \frac{\delta^2}{16k_{CS}}\left(1 \pm \sqrt{1 - \frac{256k_{CS}^2}{\delta^2}}\right), \tag{10.196}$$

$$k^0_{\pm,R} = -\frac{\delta^2}{16k_{CS}}\left(1 \pm \sqrt{1 - \frac{256k_{CS}^2}{\delta^2}}\right). \tag{10.197}$$

Now, Δ relates the parameters of the theory to the thermodynamic properties of the medium, encoded in the definition of δ. Therefore, a positive sign of Δ corresponds to thermodynamic configurations of the matter medium in which the propagation of gravitational waves is not allowed for any wavenumber k, and a range of wavenumbers exists for which the wave is totally reflected before entering the medium [B283].
Taking also into account the subluminal condition, we have that propagation with subluminal phase velocity is allowed in

$$\text{Left mode: } k > 8k_{CS} \qquad \text{Right mode: } k < -8k_{CS}, \tag{10.198}$$

when $\Delta < 0$, and in

$$\begin{aligned} &\text{Left mode:} && 8k_{CS} < k < k^0_{-,L} && \cup && k > k^0_{+,L} \\ &\text{Right mode:} && k < k^0_{+,R} && \cup && k^0_{-,R} < k < -8k_{CS} \end{aligned} \tag{10.199}$$

for $\Delta > 0$. For these modes we can compute the damping coefficient via (10.186), resulting in

$$\omega_i = \frac{\pi e^{-\frac{x}{\sqrt{1-v_p^2}}}\left(3\left(1 - v_p^2\right) + 3x\sqrt{1 - v_p^2} + x^2\right)}{4xK_2(x)\left(\frac{x}{\left(1-v_p^2\right)^2} - \frac{\gamma(x)}{v_p^4}\right)}k, \tag{10.200}$$

where the contribution of $B_{L,R}(k)$ is implicit in the phase velocity. The ranges of the momentum for which propagation is allowed and damped are summarized in figure 10.5 and 10.6. Finally, we can require the absence of instabilities induced

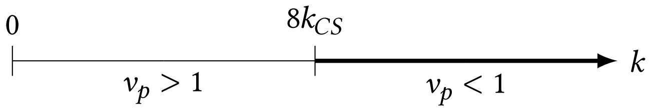


Figure 10.5: Metric-affine Chern-Simons model ($\Delta < 0$)

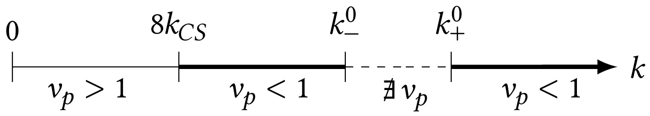


Figure 10.6: Metric-affine Chern-Simons model ($\Delta > 0$)

by a positive damping coefficient, by imposing $\omega_i < 0$, which amounts to set

$$\frac{x}{\left(1-v_p^2\right)^2} - \frac{\gamma(x)}{v_p^4} < 0. \tag{10.201}$$

The resulting bound on the velocity, i.e.

$$v_p < \bar{v} \equiv \left(1 + \sqrt{\frac{x}{\gamma(x)}}\right)^{-\frac{1}{2}}, \tag{10.202}$$

still allows for a wide range of phase velocities much greater than the average velocity of the particles $\langle v \rangle$, obtained averaging over the Jüttner-Maxwell distribution. Indeed, it can be shown that the ratio $\bar{v}/\langle v \rangle$ always grows with x, being greater than 2 as soon as $x \gtrsim 15$.

Concluding remarks

Summary of the results

The previous pages have been devoted to the analysis of three gravitational models formulated in the metric-affine framework. Existing models characterized by topological terms have been extended to their most general formulation, including all possible affine structures such as torsion and nonmetricity. During this procedure, we kept track of the topological character and projective symmetry of such terms.
Among the many existing modified gravity models it is important to discard the ones which are not physically viable. One way to proceed in this direction is to check for the internal theoretical consistency of the theory considered. In particular, the physical degrees of freedom must be well-behaved and one must check for the absence of dynamical instabilities. To this aim the presence of ghost degrees of freedom and Ostrogradski instabilities introduced by higher-order derivatives in the field equations has been investigated. Moreover, the relation of such pathologies with the invariance under projective transformations of the models has been taken into account.
On the other hand, the derivation of concrete predictions of gravitational models is foundational. In this regard, results encompassing black hole thermodynamics, primordial cosmology, black hole perturbations and gravitational waves propagation have been derived.
The main results are summarized below, concluding with some comments and possible perspectives for future investigations.

Holst term and hairy black holes

The most straightforward generalization of the Holst term to the case of non vanishing nonmetricity is simply obtained via the usual contraction of the Levi-Civita tensor with the affine Riemann tensor, but considering the latter as built with the most general asymmetric and not metric compatible connection. The

resulting expression is invariant under projective transformations and it is still vanishing on half-shell, when the term is included in the first order action of General Relativity.
Deviations from Einstein gravity can instead be introduced by promoting the Immirzi parameter to a dynamical field and implementing the Holst term in the Palatini $f(R)$ theory of gravity. The Jordan frame formulation of the model is characterized by two scalar fields: the Immirzi field and the $f(R)$ scalaron. However, there is only one additional degree of freedom, since a modified structural equation allows to algebraically solve the scalaron in terms of the Immirzi field. The theory is devoid of ghost-like instabilities since the additional scalar degree of freedom is well-behaved, as it is apparent from the canonical form of its kinetic term in the Einstein frame.
Another feature of the Holst model, which is also present in the Nieh-Yan case, is that the presence of the Immirzi field offers a resolution of ambiguities in the way the theory can be formulated. In particular, under a suitable rescaling of the Immirzi field, the theory obtained including the Holst (or Nieh-Yan) term inside the argument of the function f is equivalent to the model featuring the same term added directly to the function. This is only possible if the Immirzi parameter is promoted to a dynamical field, explaining the reason why previous works in literature considered models obtained including the terms inside or outside the function f as completely inequivalent.

In this theoretical framework, new analytical hairy black hole solutions have been derived and their thermodynamics has been investigated. Once a quadratic form for the function f is singled out and a specific expression for the Immirzi field potential is chosen, there are solutions with non-trivial scalar hair provided by the Immirzi field with both de Sitter and Anti-de Sitter asymptotics. While the former presents some singularities, in the latter the fields are regular on and outside the black hole event horizon. The solution describes an asymptotically Anti-de Sitter hairy black hole whose horizon has a hyperbolic topology. The Immirzi field relaxes to the constant configuration at infinity, recovering the standard picture featuring a constant Immirzi parameter.
The thermodynamic framework presents interesting features as well. The computations have been performed with Euclidean path integral methods, where a regularization procedure has to be taken into account in order to remove singularities in the on-shell Euclidean action. The necessary counter-terms to be added to the variational principle were known for the case of a minimally coupled scalar field. Here we provided their correct generalization to the case of non-minimal coupling. The analysis reveals a modified expression for the black hole entropy which deviates from the usual Bekenstein-Hawking formula, where it is just proportional to the horizon area. In the present case instead, it acquires

a correction depending on the value of the Immirzi field computed at the event horizon. Although far from being an observational signature, this is a specific effect of the dynamical nature of the Immirzi field. Indeed, one can show that if the latter is treated a priori as a constant Immirzi parameter then the entropy retains its standard expression.
Enlarging the thermodynamic phase space to include a pressure-volume pair of conjugate variables, reveals peculiar outcomes regarding the properties of the thermodynamic volume and the associated reverse isoperimetric inequality. A violation of the latter is observed, so that the solution adds to the list of super-entropic black hole solutions known in literature. We confirm the conjecture on their thermodynamic instability by computing the specific heats at constant pressure and volume, showing that unstable thermodynamic configurations where the latter is negative exist whenever the reverse isoperimetric inequality is violated.

Bouncing cosmologies with projective-invariant Nieh-Yan term

As in the Holst case, one may generalize the Nieh-Yan term by retaining its formal expression but allowing for the connection to break metric compatibility. However, this leads to a non topological and projective-breaking term. The two properties can instead be accommodated by including an additional contribution which couples the torsion and the nonmetricity tensors. By introducing two real parameters λ_1 and λ_2 in the definition of this generalized Nieh-Yan term, one can impose projective symmetry by setting $\lambda_1 = \lambda_2 = \lambda$, still violating topologicity as long as $\lambda \neq 1$. If in addition the condition $\lambda = 1$ is imposed, the resulting term is endowed with both projective symmetry and topologicity. The latter guarantees the preservation of the classical theory when the term is considered in a first order formulation of General Relativity.
Instead of promoting the Immirzi parameter to a scalar field by hand in the action, a model of modified gravity can be obtained by considering a general function of two arguments, the Ricci scalar and the generalized Nieh-Yan term. In this way, two scalar degrees of freedom are introduced in the Jordan frame, one of which can be identified with the Immirzi field, which acquires a dynamical character in a more natural way, as well as a potential term (the same approach can be followed also in the Holst case). Depending on the values of the parameters, we can identify two alternative models with different dynamical content.
For $\lambda_1 \neq \lambda_2$ the model is dynamically equivalent to Palatini $f(R)$ gravity, as a consequence of the on half-shell vanishing of the generalized Nieh-Yan term. Hence, there are no additional degrees of freedom and the theory is free of instabilities,

despite projective symmetry being broken by the choice of the parameters.
On the other hand, the model endowed with projective symmetry ($\lambda_1 = \lambda_2 = \lambda$) offers new dynamical features with respect to Palatini $f(R)$ gravity, being on half-shell equivalent to a scalar tensor theory characterized again by a modified structural equation, which provides an algebraic expression for the scalaron in terms of the Immirzi field and its kinetic term. By inspecting the action in the Einstein frame, the absence of ghost instabilities is established also in this case. Beside them, Ostrogradski instabilities could in principle be present, given the dependence of the scalaron on the Immirzi field kinetic term. Since second derivatives of the scalaron appear in the metric field equations, third derivatives of the Immirzi field are generated. However, they do not necessarily imply Ostrogradski instabilities. Indeed, we could prove that in the $\lambda = 1$ case (and under the assumption of negligible Immirzi field potential) the theory is free of instabilities. This comes from the fact that the theory is equivalent to a subclass of the experimental compatible sector of degenerate higher-order scalar-tensor (DHOST) theories, in which specific degeneracy conditions in the Hamiltonian formulation prevent higher derivatives to give rise to unstable modes.

Also in this case, a quadratic model for the function f is specified, but we need to neglect the Immirzi field potential to derive semi-analytical solutions. These are cosmological solutions where a Bianchi I metric describes a possible anisotropic behavior of the primordial universe. In particular we focus on the region of parameter space which allows for a regularization of the Big Bang singularity in favour of a big bounce scenario, where the universe volume reaches a minimum value and then expands back in another branch. The Immirzi and scalaron fields have non-trivial behavior near the bounce and relax to constant configurations in the late time region, where the isotropic FLRW setting is recovered through an isotropization mechanism offered by the inclusion of matter content in the form of radiation and/or pressure-less dust.
Despite the singularity at the origin of cosmological time being regularized, the cosmic evolution presents other future finite-time singularities. Numerical integrations reveal two possibilities. In the first kind of solutions, the singularities affect the Hubble function, while the scale factors and the scalar fields are always finite and non vanishing. The second kind of solutions is instead characterized by a regular behavior for the Hubble function and zeros or poles in the scale factors, as well as the vanishing of the scalaron field. To probe the physical relevance of such singularities, we proceed investigating the behavior of null geodesics and scalar perturbations near the singular events. We conclude that only the first kind of solutions allow for geodesic completeness and bounded scalar perturbations, while the second kind must be discarded. Beside such considerations, we stress that a further regularization of this kind of singularities can be provided

by considering quantum effects of particle creation, a possibility that we leave for future works.

Projective-invariant Chern-Simons term: black holes and gravitational waves phenomenology

Contrary to the Nieh-Yan case, the simplest extension of the Chern-Simons term to generic metric-affine spacetimes, obtained simply promoting the connection to a completely unconstrained independent variable, is enough to preserve its topological character. However, the projective symmetry is broken. We propose a generalized Chern-Simons term featuring three parameters λ_1, λ_2 and λ_3, with the following properties. It is projective-invariant but not topological if $\lambda_2 = (\lambda_1 - 4)/16$ and $\lambda_3 = -\lambda_1/2$ but $\lambda_1 \neq 0$ and/or $\lambda_3 \neq 0$. Instead, the topological character can be recovered setting $\lambda_1 = \lambda_3 = 0$ and at the same time one can break the projective symmetry by considering $\lambda_2 \neq -1/4$. In particular, $\lambda_2 = 0$ yields an expression formally equivalent to the original Chern-Simons term. Finally, topologicity and projective invariance can be simultaneously implemented by imposing $\lambda_1 = \lambda_3 = 0$ and $\lambda_2 = -1/4$. Focusing on the last possibility we introduced the generalized Chern-Simons term in the Hilbert-Palatini action of General Relativity, promoting the coupling to a pseudoscalar field. Two different theories exist, depending on whether a standard kinetic term for the scalar field is included in the action or not. We multiplied the latter by a free parameter β and considered both the $\beta = 0$ and $\beta \neq 0$ case.
The theory is more involved than the Nieh-Yan and Holst models and it is not possible to express it as an effective second order scalar-tensor theory at the full unperturbed and background independent level. Therefore, conclusions regarding the stability of the additional scalar degree of freedom and the absence of ghosts can not be reached as in the other models. However, for $\beta \geq 0$ the scalar field is well behaved in relevant physical scenarios, as we demonstrated studying the linearized theory and the evolution of scalar perturbations on spherically symmetric and cosmological backgrounds.
Instead, the possibility of pathologies affecting the theory resides in the presence of higher-order derivatives in the field equations, which can give rise to Ostrogradski instabilities. Although in the first order framework these are apparently absent, they can arise on half-shell, once the affine sector is solved in terms of the metric and scalar field. This is indeed the case for the projective-invariant metric-affine Chern-Simons theory at hand, which in general presents derivatives of the metric tensor higher than the second order. The same is true for the metric version of the theory and common methods exist to remove higher-order derivatives, e.g. linearizing the equations on a constant scalar field background

or imposing specific spacetime symmetries. However, the novelty typical of the metric-affine formulation consists in restricting to specific subclass of metric-affine manifolds, where the rank-3 tensor parts of torsion and nonmetricity are absent. Such requirement is enough to avoid the presence of Ostrogradski instabilities.
Another peculiarity of the metric-affine version of Chern-Simons gravity regards the dynamical nature of the pseudoscalar field in the $\beta = 0$ theory. Indeed, the scalar field has a proper dynamical character even in absence of its kinetic term in the action. This can be appreciated at linear level, where derivatives of the scalar field are produced on half-shell by the torsion perturbation, and at the background level, where solutions with non-trivial θ exist for $\beta = 0$. Moreover, contrary to metric Chern-Simons gravity, the $\beta = 0$ theory is viable and does not lead to the shortcomings known to affect its analogue metric version.

By studying black hole perturbations and gravitational waves propagation we provided for the first time in literature the derivation of potentially observable effects for the metric-affine version of Chern-Simons gravity. Peculiar signatures arise in the dynamics of linear perturbations of Schwarzschild black holes. The quasinormal modes characterizing the evolution of axial perturbations exhibit modifications with respect to General Relativity and metric Chern-Simons gravity. Depending on the value of the parameter β, the values of the quasinormal frequencies are either coinciding with the General Relativity outcome ($\beta \to \infty$) or deviating from it ($\beta \lesssim 10^{-1}$). The prediction of the $\beta = 0$ theory are smoothly reached by the $\beta \neq 0$ theory, as we checked decreasing β down to 10^{-7}. Moreover, the scalar field and metric perturbations are coupled, resulting in a superposition of the scalar and tensor modes, leaving an imprint in the signal even in the $\beta \to \infty$ limit. Similar (but quantitatively different) effects are produced by the metric version of the theory as well. The latter also predicts purely decaying, non-oscillatory modes with imaginary frequency, which are instead absent in the metric-affine framework. A smoking gun for the metric-affine Chern-Simons gravity is instead provided by the late time part of the signal which is characterized by power law tails, as in General Relativity. The dependence of the exponents of the tails on the angular momentum number of the perturbations is exactly the same in the $\beta \neq 0$ metric-affine theory, the metric version of Chern-Simons theory and General Relativity. However, the $\beta = 0$ metric-affine Chern-Simons theory predicts a modified relation, potentially observable in the last part of the signal.
The propagation of gravitational waves is also affected by parity violating effects induced by the Chern-Simons modification. In particular, we established for the first time the existence of velocity birefringence in metric-affine Chern-Simons gravity. The propagation of the wave must be considered on background

spacetimes more complex than the flat Minkowski one, which can only yield a trivial dispersion relation. To this aim, we were able to find exact homogeneous and isotropic cosmological solutions in the $\beta = 0$ case. They exhibit known behaviors for the scale factor, including a de-Sitter expansion phase as well as power laws describing radiation and matter dominated eras. Gravitational waves propagating on the de-Sitter background are characterized by different dispersion relations for the left/right-handed polarizations, yielding the gravitational birefringence effect. The latter causes deviations both in the group and phase velocity of the wave. However, current experimental constraints on the group velocity are not tight enough to test the theory. Experimental constraints on the birefringence effect can also be derived for the phase velocity. However, the peculiar dependence of the correction on the wavenumber requires a generalization of analysis existing in literature before applying them to the present case. Finally, the propagation of gravitational waves in matter has been addressed. We first proved the existence of ranges of wavenumbers for which the propagation is not allowed and the wave is reflected before entering the matter medium. Hence, the latter can act as an inverse band-pass filter but, although theoretically detectable, the window of non allowed modes turns out to be extremely narrow. In regions of the momentum space where the propagation is allowed, one has a subluminal (superluminal) phase velocity, for wavenumbers greater (smaller) than a specific threshold identified by the parameters of the theory. When the phase velocity is subluminal, the amplitude of the wave suffers the kinematic damping known as gravitational Landau damping, which is instead absent for superluminal phase velocity. The existence of gravitational Landau damping for tensor perturbations in modified gravity theories is hereby established for the first time. The threshold separating damped from undamped modes in momentum space is in principle observable, although the magnitude of the relative absorption is well below the sensitivity of current observations. For explicit numerical estimates on both the window of forbidden wavenumbers and the Landau damping effect we remind the reader to [5].

Discussion and future perspectives

The investigations performed revealed aspects of projective-invariant models that add new perspectives to the existing literature on the topic. In particular, a discussion on the relation between projective symmetry and dynamical instabilities is in order. We start recalling again the role of projective symmetry in Ricci based gravity, where instabilities are present if the symmetry is broken and they can be removed either imposing the symmetry or setting torsion to zero [34]. The results presented in this thesis reveal a picture which is not always consist-

ent with these conclusions.
The Holst case seems to be in line with the situation outlined in [34]. Indeed, the theory is projective-invariant and devoid of both ghost and Ostrogradski kind of instabilities. However, being the Holst term already projective invariant, there is no room to probe any projective-breaking scenario, unless arbitrary additional terms that are not invariant under projective transformations are introduced by hand in the action. Therefore, one can not conclude that the absence of projective symmetry would imply instabilities.
Regarding the generalized Nieh-Yan term instead, a peculiar situation emerges. First, the breaking of projective symmetry does not lead to instabilities, constituting a counter-example to the analysis performed in [34]. Actually, one *needs* to break projective symmetry in order to avoid higher-order derivatives in the field equations. However, the absence of instabilities in that case is trivial, since it is just related to the fact that there are no new degrees of freedom at all, being the theory equivalent to Palatini $f(R)$ gravity. If one considers the richer framework offered by the dynamical models instead, with additional degrees of freedom and higher-order derivatives in the field equations, than the projective symmetry is a necessary condition for proving the absence of Ostrogradski instabilities (the DHOST equivalence holds for $\lambda_1 = \lambda_2 = \lambda = 1$). We remark that the same could also be true for $\lambda_1 = \lambda_2 = \lambda \neq 1$. However, in that case one can not rely on the equivalence with DHOST theories and a full analysis of the Hamiltonian system would be required, checking if degeneracy conditions similar to the ones present in DHOST theories hold.
The most involved landscape arises in the Chern-Simons case. Imposing projective symmetry is not enough to avoid instabilities. Indeed, higher-order derivatives of the metric tensor appear in the field equations. Although we proved that one of the two methods used in [34] to eliminate them works for the Chern-Simons theory as well, i.e. a priori setting to zero some affine components ($q_{\mu\nu\rho} = \Omega_{\mu\nu\rho} = 0$ instead of $T^{\mu}{}_{\nu\rho} = 0$), it seems that projective symmetry has no role in the presence or absence of Ostrogradski-like instabilities. However, unlike the Nieh-Yan case, we could not investigate the projective breaking sector of the theory in great detail (see section 7.2.1). It is in principle possible that the symmetry could be hiding even worse instabilities that could be unveiled studying the projective-breaking case, offered by $\lambda_2 \neq -1/4$, which deserves further investigations.
Another topic which would be worth investigating is the initial value formulation of the theory. In the metric Chern-Simons theory, this is known to be the setting in which higher-order derivatives give rise to shortcomings which are overcome adopting a small coupling limit. It could be interesting to extend the analysis of [193] to the metric-affine case considering the generalized Chern-Simons term.

Regarding the scalar sector of the theory, the condition $\beta \geq 0$ is necessary in order to avoid a wrong sign for the kinetic term. Therefore it seems that one needs to impose $\beta \geq 0$ to avoid ghosts. However, additional contributions to the kinetic term comes from the affine sector and at the effective level the relevant quantity seems to be $\beta + \alpha K$ rather than just β (Cf. eq. (7.38)). Even if the condition $\beta < 0$ is enough to yield unstable modes on the Schwarzschild background, a wider range of β values could be allowed if those modes were absent on more general and physically more relevant backgrounds such as Kerr-like spacetimes, where the αK term could behave differently and possibly dominate over β.
Future works could also be devoted to the extension of the analysis of black holes quasinormal modes and parity violating effects in gravitational waves propagation to the Holst and Nieh-Yan models. No further technical or conceptual difficulties are expected in this direction. However, the opposite is not true, in the sense that the results regarding hairy black holes and big-bounce solutions rely on the relatively simple structure of the Holst and Nieh-Yan models, and adapting the same treatment to the Chern-Simons theory seems not that straightforward.
Finally, let us stress that all the results regarding projective symmetry and its relation to ghosts, higher-derivatives and Ostrogradski instabilities are not inherent properties of the terms themselves. Indeed, the conclusions reported above depend on the gravitational models in which these terms are introduced and must be contextualized accordingly. Their implementation within different settings could yield other outcomes.
In any case, the discrepancies with respect to the results of [34] could be related to the violation of parity which is present in all the models we considered. Indeed, parity violation is reflected in the presence of the Levi-Civita completely antisymmetric tensor used to build the Holst, Nieh-Yan and Chern-Simons terms, as well as their generalizations. Since the shift in the independent connection (and related curvature objects) produced by projective transformations is proportional to a delta, contractions with a completely antisymmetric tensor makes the implementation of the projective symmetry easier. In absence of the Levi-Civita tensor, more terms produced by the transformation would survive, resulting in more severe constraints on the model and its parameters, that could also hinder the presence of instabilities. Therefore, the interplay between projective symmetry and parity constitutes another intriguing line of investigation.
This thesis is rather heterogeneous and considers multiple models applied to several different physical settings. It is therefore difficult to conclude with a simple take-home message. On one hand, we can assert that the behavior of gravitational metric-affine models under projective transformations reveals important dynamical aspects and must be taken into consideration. However, one must be aware that projective symmetry is not always a necessary and sufficient condition for the absence of instabilities in metric-affine theories, especially in pres-

ence of parity violation. On the other hand, the expectation is that upcoming improvements in experimental observations will allow to test, and possibly falsify, the theories considered, hopefully allowing to reject as many as possible of them, thus narrowing down the number of current available possibilities.

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