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Two-Dimensional Dispersionless Toda Lattice Hierarchy: Symmetry, New Extension, Hodograph Solutions, and Reduction

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Abstract: The symmetry for two-dimensional (2D) dispersionless Toda lattice hierarchy (dTLH) is firstly derived, and then the 2D dTLH is extended based on the symmetry constraint. The commutativity of two different flows for this new hierarchy is shown, which leads to the 2D dToda lattice equation with self-consistent sources (dTLESCSs) together with its conservation equation. The hodograph solutions to 2D dTLESCSs are also given. One dimensional reduction of extended 2D dTLH is finally investigated by finding the constraint, and a one-dimensional dTLESCS is shown.

Keywords: symmetry; extended 2D dispersionless toda lattice hierarchy; 2D dispersionless toda lattice equation with self-consistent sources; hodograph solutions; one-dimensional reduction

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MSC: 37K10; 35P30; 37K30

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1. Introduction

In recent years, dispersionless integrable systems (DISs) have been one of the hot topics in mathematical physics [1–23]. Dispersionless integrable hierarchies can be viewed as the dispersionless limit of ordinary integrable hierarchies and have important applications in conformal maps, hydrodynamics, and topological field theory [1–10]. The operators in the Lax equations are replaced by phase space functions for dispersionless hierarchies. In addition, the commutator is replaced by the Poisson bracket and the role of Lax pair equations by the conservation equations. Note that these dispersionless systems can be solved by using the hodograph reduction method [4], twistorial method [8,9], and the quasi-classical $\bar{\partial}$-method [10].

As an important DIS, dTLH has attracted much attention from many researchers. In 1993, dTLH was introduced by Takasaki and Takebe and it was represented in two equivalent ways: in the Lax–Sato form or in the Hirota form [11,12]. Some progresses have been made for the dTLH such as dispersionless Hirota equations for the extended dTLH [13], symmetric solutions [14], constrained reductions, Hamiltonian structure and interface dynamics [15], and so on. In 2006, the source generalizations of dispersionless KP (dKP) and dispersionless modified KP (dmKP) hierarchies were investigated by treating the constrained integrable hierarchy as the stationary system of the corresponding hierarchy. The hodograph solutions to dKP and dmKP equations with self-consistent sources were given as well [16,17]. However, up to now, the source generalization of dTLH still remains unsolved, and some integrable properties of the extended 2D dTLH such as hodograph solutions and one-dimensional reduction also deserve our further study.

In 2008, a systemic and unified method was proposed to obtain the generalization of integrable generalization [18,19]. It can be easily found that these new extended integrable systems are also the multi-component generalizations of original integrable systems. Quite recently, four-component integrable systems have been investigated by Ma (see [24–26]). Later, this method was used to extend other integrale systems [27]. In this paper, we will apply the method presented in [18,19] to explore the source generalization of dTLH . We think this paper will fill the gap of the work mentioned above and give an important supplement to dispersionless Sato theory.

The outline of this paper is as follows. In Section 2, the 2D Toda lattice hierarchy and dTLH are briefly reviewed. In Section 3, the symmetry for 2D dTLH is derived by taking dispersionless limit of the 2D Toda lattice hierarchy. In Section 4, based on the symmetry for 2D dTLH, 2D dTLH is extended, and the 2D dTLESCS is obtained as its first nontrivial equation. In Section 5, the hodograph solutions to the dTLESCS are given. In Section 6, one-dimensional reduction for extended 2D dTLH is constructed, which leads to a one-dimensional dTLESCS.

2. 2D Toda Lattice Hierarchy and 2D dToda Lattice Hierarchy

Here we briefly review 2D Toda lattice hierarchy and 2D dTLH. Let L, M be two Lax operators given by

$$\begin{aligned} L &= \Lambda + u_0 + u_1\Lambda^{-1} + u_2\Lambda^{-2} + \cdots, \\ M &= v_{-1}\Lambda^{-1} + v_0 + u_1\Lambda^{1} + u_2\Lambda^{2} + \cdots, \end{aligned} \tag{1}$$

where $\Lambda = \exp(\frac{\partial}{\partial s})$ is called shift operator satisfying $\Lambda f(n) = f(n+1)\Lambda$ [19,20]. All the coefficient functions u_i and v_i depend on x, y, in which $x = (x_1, x_2, x_3, \cdots)$ and $y = (y_1, y_2, y_3, \cdots)$ are two series of independent time variables.

2D Toda lattice hierarchy is defined by four infinite collections of Lax equation [18,19]

$$\begin{aligned} L_{x_n} &= [B_n, L] \ , \ L_{y_n} = [C_n, L], \\ M_{x_n} &= [B_n, M] \ , \ M_{y_n} = [C_n, M] \ , \ n = 1, 2, \cdots, \end{aligned} \tag{2}$$

where $B_n = (L^n)_{\geq 0} = L^n_+, C_n = (M^n)_{<0} = M^n_-$ denote the "non-negative" part of L^n with respect to the powers of Λ and the "negative" part of M^n, respectively. The bracket stands for the usual commutator of operators. L_{x_n}, L_{y_n} are the derivatives of L with respect to x_n, y_n, and M_{x_n}, M_{y_n} are defined similarly.

It is noticed that the commutativity of (2) gives rise to zero-curvature equations of 2dTLH, which leads to the 2D Toda lattice equation. The (2)D Toda lattice equation is an important integrable system and has attracted attention from all over the world. In recent years, the exact solutions to Toda-type integrable equations have been given using the Casoratian technique [28,29].

Introducing a Planck constant $\hbar$ into the 2D Toda lattice hierarchy (2) and taking $X = \hbar x, Y = \hbar y$, 2D dTLH is obtained by the dispersionless limit of (2), as follows [11]:

$$\begin{aligned} \partial_{X_n}\mathcal{L} &= \{\mathcal{B}_n, \mathcal{L}\} \ , \ \partial_{Y_n}\mathcal{L} = \{\mathcal{C}_n, \mathcal{L}\}, \\ \partial_{X_n}\mathcal{M} &= \{\mathcal{B}_n, \mathcal{M}\} \ , \ \partial_{Y_n}\mathcal{M} = \{\mathcal{C}_n, \mathcal{M}\}. \end{aligned} \tag{3}$$

where$X = (X_1, X_2, X_3, \cdots)$, $Y = (Y_1, Y_2, Y_3, \cdots)$. The Sato functions $\mathcal{L}$ and $\mathcal{M}$ are defined by

$$\begin{aligned} \mathcal{L} &= p + \sum_{n=0}^{\infty} U_n(X, Y, S)p^{-n}, \\ \mathcal{M} &= V_{-1}(X, Y, S)p^{-1} + \sum_{n=0}^{\infty} V_n(X, Y, S)p^{n}, \end{aligned} \tag{4}$$

where S is a new spatial variable. The two sets of flows in (3) commute, which follows from

the zero-curvature representation
for $m \neq n$

$$\begin{aligned}\mathcal{B}_{m,x_n} - \mathcal{B}_{n,x_m} + \{\mathcal{B}_m, \mathcal{B}_n\} = 0,\\ \mathcal{C}_{m,Y_n} - \mathcal{C}_{n,Y_m} + \{\mathcal{C}_m, \mathcal{C}_n\} = 0,\\ \mathcal{B}_{m,Y_n} - \mathcal{C}_{n,X_m} + \{\mathcal{B}_m, \mathcal{C}_n\} = 0,\end{aligned} \quad (5)$$

and for $m = n$,

$$\mathcal{B}_{m,Y_m} - \mathcal{C}_{m,X_m} + \{\mathcal{B}_m, \mathcal{C}_m\} = 0, \quad (2.6)$$

in which $\mathcal{B}_n = \mathcal{L}_+^n$ and $\mathcal{C}_n = \mathcal{M}_-^n$ denote as functions of p and the Poisson bracket is defined as [11]

$$\{A(p,S), B(p,S)\} = p \cdot \frac{\partial A}{\partial p}\frac{\partial B}{\partial S} - p \cdot \frac{\partial A}{\partial S}\frac{\partial B}{\partial p}.$$

The conservation equation associated with (6) reads as

$$p_{X_m} = p[\mathcal{B}_m(p)]_S \ , \ p_{Y_m} = p[\mathcal{C}_m(p)]_S. \quad (6)$$

When $m = n = 1$, (6) yields 2D dToda lattice equation [21]

$$V_S + [\partial_S^{-1}(\frac{V_X}{V})]_Y = 0, \quad (7)$$

where $U = U_0$, $V = V_{-1}$, $X = X_1$, $Y = Y_1$ and $U_{0S}V_{-1} = V_{-1X}$.
The conservation equation of (7) reads as

$$p_X = p[\partial_S^{-1}(\frac{V_X}{V}) + p]_S \ , \ p_Y = p[\frac{V}{p}]_S.$$

3. The Symmetry of 2D dToda Lattice Hierarchy

In order to construct the extended 2D Toda lattice hierarchy, the authors in [18] made full use of the symmetry for 2D Toda lattice hierarchy to introduce a new time-flow $\overline{y}_m$. A new evolution equation was obtained as follows [18]:

$$L_{\overline{y}_m} = [\overline{C}_m, L] \ , \ M_{\overline{y}_m} = [\overline{C}_m, M], \quad (8a)$$

$$\overline{C}_m = C_m + \sum_{i=1}^{N} w_i \Delta_-^{-1} w_i^* \ , \ C_m = M_-^m \ , \ m \geq 1, \quad (8b)$$

where $\Delta_-^{-1} = \sum_{i=1}^{N} \Lambda^{-i}$, w_i, and w_i^* satisfy

$$\begin{aligned}w_{i,x_n} = B_n(w_i) \ , \ w^*_{i,x_n} = -B_n^*(w_i^*),\\ w_{i,y_n} = C_n(w_i) \ , \ w^*_{i,y_n} = -C_n^*(w_i^*) \ , \ i = 1,2,\cdots,N,\end{aligned} \quad (9)$$

where w_i and w_i^* are called wave and adjoint wave functions, respectively. B_n^* and C_n^* are the adjoint operators of B_n and C_n.
It can be easily found that the compatibility of (1) and (8) leads to the extended 2D Toda lattice hierarchy given in [18,19]. In addition, we also find that $\sum_{i=1}^{N} w_i \Delta_-^{-1} w_i^*$ is the symmetry for 2D Toda lattice hierarchy.
Next, we will construct the symmetry of 2D dTLH by taking the dispersionless limit of Equation (8b). Defining $\hbar$: $\exp(n\hbar\frac{\partial}{\partial S})f(S) = f(S + n\hbar)$ as [11] and considering the limit $\hbar \longrightarrow 0$, L and M in (2) are changed into

$$L_{\hbar} = e^{\hbar\frac{\partial}{\partial S}} + \sum_{n=0}^{\infty} u_n(\hbar, x, y, s)e^{-n\hbar\frac{\partial}{\partial S}},$$
$$M_{\hbar} = v_{-1}(\hbar, x, y, s)e^{-\hbar\frac{\partial}{\partial S}} + \sum_{n=0}^{\infty} v_n(\hbar, x, y, s)e^{n\hbar\frac{\partial}{\partial S}}. \tag{10}$$

Thinking of $u_n(\hbar, x, y, s) = U_n(X, Y, S) + O(\hbar)$ and $v_n(\hbar, x, y, s) = V_n(X, Y, S) + O(\hbar)$ as $\hbar \longrightarrow 0$, then Equation (8b) becomes

$$\overline{C}_{\hbar m} = C_{\hbar m} + \sum_{i=1}^{N} w_i(\hbar, x, y)(e^{-\hbar\frac{\partial}{\partial S}} + \cdots + e^{-n\hbar\frac{\partial}{\partial S}} + \cdots)w_i^*(\hbar, x, y),$$
$$C_{\hbar m} = (M_{\hbar}^m)_- \ , \ m \geq 1, \ i = 1, 2, \cdots , \tag{11}$$

where $w_i(\hbar, x, y, s)$ and $w_i^*(\hbar, x, y, s)$ satisfy

$$\hbar[w_i(\hbar, x, y, s)]_{X_n} = B_{\hbar n}(w_i(\hbar, x, y, s)) \ , \ \hbar[w_i^*(\hbar, x, y, s)]_{X_n} = -B_{\hbar n}^*(w_i^*(\hbar, x, y, s)),$$
$$\hbar[w_i(\hbar, x, y, s)]_{Y_n} = C_{\hbar n}(w_i(\hbar, x, y)) \ , \ \hbar[w_i^*(\hbar, x, y, s)]_{Y_n} = -C_{\hbar n}^*(w_i^*(\hbar, x, y, s)). \tag{12}$$

It was shown in [11] that

$$\mathcal{L} = \sigma^{\hbar}(L_{\hbar}) = p + \sum_{n=0}^{\infty} U_n(X, Y, S)p^{-n},$$
$$\mathcal{M} = \sigma^{\hbar}(M_{\hbar}) = V_{-1}(X, Y, S)p^{-1} + \sum_{n=0}^{\infty} V_n(X, Y, S)p^n, \tag{13}$$

is a solution of 2D dTLH, and satisfies

$$\partial_{Y_n}\mathcal{L} = \{\mathcal{B}_n, \mathcal{L}\} \ , \ \partial_{Y_n}\mathcal{M} = \{\mathcal{C}_n, \mathcal{M}\},$$

where $\sigma^{\hbar}$ denotes the principal symbol.
Regarding

$$w_i(\hbar, x, y, s) \sim \exp[\frac{\mathcal{S}(X, Y, S, \lambda_i)}{\hbar} + \alpha_{i1} + O(\hbar)] \ , \ \hbar \longrightarrow 0,$$
$$w_i^*(\hbar, x, y, s) \sim \exp[-\frac{\mathcal{S}(X, Y, S, \lambda_i)}{\hbar} + \alpha_{i2} + O(\hbar)] \ , \ i = 1, 2, \cdots , N, \tag{14}$$

we find that when $\hbar \longrightarrow 0$

$$w_i(\hbar, x, y, s)(e^{-\hbar\frac{\partial}{\partial S}} + \cdots + e^{-n\hbar\frac{\partial}{\partial S}} + \cdots)w_i^*(\hbar, x, y, s)$$
$$= \exp(\alpha_{i1} + \alpha_{i2})[\exp(\frac{\partial\mathcal{S}(X, Y, S, \lambda_i)}{\partial S})e^{-\hbar\frac{\partial}{\partial S}} + \cdots$$
$$+ \exp(\frac{\partial\mathcal{S}(X, Y, S, \lambda_i)}{\partial S})e^{-n\hbar\frac{\partial}{\partial S}} + \cdots]. \tag{15}$$

Setting

$$a_i = \exp(\alpha_{i1} + \alpha_{i2}) \ , \ p_i = \exp(\frac{\partial\mathcal{S}(X, Y, S, \lambda_i)}{\partial X}) \ , \ p = \exp(\frac{\partial\mathcal{S}(X, Y, S, \lambda)}{\partial X}).$$

and substituting (15) into (11), and then, taking the principal symbol of both sides of (11), we have

$$\begin{aligned}\overline{\mathcal{C}}_m &= \mathcal{C}_m + \sum_{i=1}^{N} \exp(\alpha_{i1} + \alpha_{i2})(p_i p^{-1} + p_i^2 p^{-2} + \cdots) \\ &= \mathcal{C}_m + \sum_{i=1}^{N} \frac{a_i p_i}{p - p_i},\end{aligned} \tag{16}$$

where $\mathcal{C}_m = \mathcal{M}^m_-$.

We find that $\sum\limits_{i=1}^{N} \frac{a_i p_i}{p - p_i}$ is the symmetry of 2D dTLH. In addition, it was also shown in [11] that ψ , $\overline{\psi}$ have the following WKB asymptotic expansions as $\hbar \longrightarrow 0$

$$\begin{aligned}\psi &= \exp[\frac{\mathcal{S}(X, Y, S, \lambda)}{\hbar} + O(\hbar^0)], \\ \overline{\psi} &= \exp[\frac{\overline{\mathcal{S}}(X, Y, S, \overline{\lambda})}{\hbar} + O(\hbar^0)],\end{aligned} \tag{17}$$

and they satisfy the following linear equation,

$$\hbar \partial_{X_n} \psi = B_n \psi \ , \ \hbar \partial_{Y_n} \overline{\psi} = C_n \overline{\psi}. \tag{18}$$

Combining (17) with (18), we obtain a hierarchy of conservation equations for the momentum function $p = \exp \frac{\partial \mathcal{S}(X,Y,S,\lambda)}{\partial S} = \exp \frac{\partial \overline{\mathcal{S}}(X,Y,S,\overline{\lambda})}{\partial S}$,

$$p_{X_n} = p[\mathcal{B}_n(p)]_S \ , \ p_{Y_n} = p[\mathcal{C}_n(p)]_S.$$

In the same way as [16,17], we obtain the equations of hydrodynamical type from (12), (14), and (16).

$$\begin{aligned}&p_{i,X_n} = p_i[\mathcal{B}_n(p_i)]_S \ , \ a_{i,X_n} = [a_i p_i(\frac{\partial \mathcal{B}_n(p_i)}{\partial p_i})]_S, \\ &p_{i,Y_n} = p_i[\mathcal{C}_n(p_i)]_S \ , \ a_{i,Y_n} = [a_i p_i(\frac{\partial \mathcal{C}_n(p_i)}{\partial p_i})]_S \ , \ i = 1, 2, \cdots, N.\end{aligned} \tag{19}$$

4. New Extension of 2D dToda Lattice Hierarchy

In this section, inspired by [18,19], we will use Equation (16), in which a_i and p_i are defined by (18), to investigate the new extension of 2D dTLH. It is found that a particular Y_m -flow will be extended to $\bar{Y}_m$-flow given by

$$\mathcal{L}_{\overline{Y}_m} = \{\overline{\mathcal{C}}_m, \mathcal{L}\} \ , \ \mathcal{M}_{\overline{Y}_m} = \{\overline{\mathcal{C}}_m, \mathcal{M}\}.$$

where $\overline{\mathcal{C}}_m$ are defined by (16) and (18). Then, we have the following extended 2D dTLH.

Definition 1. *For a fix $m \in \mathbb{N}$, the extended 2D dTLH is defined by*

$$\mathcal{L}_{X_n} = \{\mathcal{B}_n, \mathcal{L}\} \ , \ \mathcal{M}_{X_n} = \{\mathcal{B}_n, \mathcal{M}\} \tag{20a}$$

$$\mathcal{L}_{Y_n} = \{\mathcal{C}_n, \mathcal{L}\} \ , \ \mathcal{M}_{Y_n} = \{\mathcal{C}_n, \mathcal{M}\} \tag{20b}$$

$$\mathcal{L}_{\overline{Y}_m} = \{\overline{\mathcal{C}}_m, \mathcal{L}\} \ , \ \mathcal{M}_{\overline{Y}_m} = \{\overline{\mathcal{C}}_m, \mathcal{M}\} \ , \ n \neq m, \tag{20c}$$

$$p_{i,X_n} = p_i[\mathcal{B}_n(p_i)]_S \ , \ a_{i,X_n} = [a_i p_i(\frac{\partial \mathcal{B}_n(p_i)}{\partial p_i})]_S) \tag{21a}$$

$$p_{i,Y_n}=p_i[\mathcal{C}_n(p_i)]_S\ ,\ a_{i,Y_n}=[a_ip_i(\frac{\partial\mathcal{C}_n(p_i)}{\partial p_i})]_S\ ,\ i=1,2,\cdots,N. \tag{21b}$$

We can show that flows defined by (20) commute when Equations (21a) and (21b) are satisfied, in which the following Lemma is needed. However, we find that the proof of Lemma 1 is similar to [22,23], so we omit it here.

Lemma 1. *There holds the identity*

$$(\frac{a_ip_i}{p-p_i})_{X_n}=\{\mathcal{B}_n,\frac{a_ip_i}{p-p_i}\}_-, \tag{22}$$

$$(\frac{a_ip_i}{p-p_i})_{Y_n}=\{\frac{a_ip_i}{p-p_i},\mathcal{M}_+^n,\}_-\ ,\ i=1,2,\cdots,N. \tag{23}$$

Next, we will use Lemma 1 to show the following theorem. We can easily find from Theorem 1 that the zero-curvature representation of the extended 2D dTLH (20) is 2D dTLH with self-consistent sources.

Theorem 1. *Under (21a,b), the commutativity of (20a)–(20c) led to the zero-curvature representation of the extended 2D dTLH (20), which should be written in two cases, $n\neq m$ or $m=k$.*
For $n\neq m$, the zero-curvature form for the extended 2D dTLH (20) is

$$\mathcal{B}_{n,X_m}-\mathcal{B}_{m,X_n}+\{\mathcal{B}_n,\mathcal{B}_m\}=0, \tag{24a}$$

$$mathcalC_{n,\overline{Y}_m}-\overline{\mathcal{C}}_{m,Y_n}+\{\mathcal{C}_n,\overline{\mathcal{C}}_m\}=0, \tag{24b}$$

$$\mathcal{B}_{n,\overline{Y}_m}-\overline{\mathcal{C}}_{m,X_n}+\{\mathcal{B}_n,\overline{\mathcal{C}}_m\}=0, \tag{24c}$$

$$\mathcal{B}_{m,Y_n}-\mathcal{C}_{n,X_m}+\{\mathcal{B}_m,\mathcal{C}_n\}=0, \tag{24d}$$

$$p_{i,X_n}=p_i[\mathcal{B}_n(p_i)]_S\ ,\ a_{i,X_n}=[a_ip_i(\frac{\partial\mathcal{B}_n(p_i)}{\partial p_i})]_S, \tag{25a}$$

$$p_{i,Y_n}=p_i[\mathcal{C}_n(p_i)]_S\ ,\ a_{i,Y_n}=[a_ip_i(\frac{\partial\mathcal{C}_n(p_i)}{\partial p_i})]_S\ ,\ i=1,2,\cdots,N. \tag{25b}$$

and for $n=m$, the zero-curvature form for the extended 2D dTLH (20) is

$$\mathcal{B}_{m,\overline{Y}_m}-\overline{\mathcal{C}}_{m,X_m}+\{\mathcal{B}_m,\overline{\mathcal{C}}_m\}=0, \tag{26a}$$

$$p_{i,X_m}=p_i[\mathcal{B}_m(p_i)]_S\ ,\ a_{i,X_m}=[a_ip_i(\frac{\partial\mathcal{B}_m(p_i)}{\partial p_i})]_S\ ,\ i=1,2,\cdots,N, \tag{26b}$$

where $\overline{\mathcal{C}}_m=\mathcal{C}_m+\sum_{i=1}^N\frac{a_ip_i}{p-p_i}$, $\mathcal{C}_m=\mathcal{M}_-^m$.
Under (25b), the conservation equation of (25a)

$$p_{X_m}=p[\mathcal{B}_m(p)]_S\ ,\ p_{\overline{Y}_m}=p[\mathcal{C}_m(p)+\sum_{i=1}^N\frac{a_ip_i}{p-p_i}]_S. \tag{27}$$

Example 1. *A 2+1 dimensional dToda lattice equation with self-consistent sources.*

When $n=m=1$ in (25), we determine the 2D dTLESCS as follows

$$(V+\sum_{i=1}^N a_ip_i)_S+[\partial_S^{-1}(\frac{V_X}{V})]_Y=0, \tag{28a}$$

$$p_{i,X} = p_i(\partial_S^{-1}(\frac{V_X}{V}) + p_i)_S \ , \ a_{i,X} = (a_i p_i)_S, \tag{28b}$$

where $U = U_0$, $V = V_{-1}$, $X = X_1$, $Y = \overline{Y}_1$ and $U_{0S}V_{-1} = V_{-1X}$.
The associated conservation equation of (28) reads as

$$p_X = p[\partial_S^{-1}(\frac{V_X}{V}) + p]_S \ , \ p_Y = p[\frac{V}{p} + \sum_{i=1}^{N} \frac{a_i p_i}{p - p_i}]_S. \tag{29}$$

Remark 1. *We can also replace X_m-flow in (20) by $\bar{X}_m$-flow as*

$$\mathcal{L}_{\overline{X}_m} = \{\overline{\mathcal{B}}_m, \mathcal{L}\} \ , \ \mathcal{M}_{\overline{X}_m} = \{\overline{\mathcal{B}}_m, \mathcal{M}\},$$

where $\overline{\mathcal{B}}_m = \mathcal{B}_m + \sum_{i=1}^{N} \frac{a_i p_i}{p - p_i}$.
Then, Lax-type equations and the zero-curvature representation of the resulting extended 2D dTLH can be given likewise; however, we omit it here.

5. The Hodograph Solutions for the 2D dToda Lattice Equation with Self-Consistent Sources

In this section, using M-reduction method together with the hodograph transformation, we derive the hodograph solutions to the 2D dTLESCSs (28). Following [4], one can consider the M-reduction of the conservation Equation (29) so that the momentum function p, the auxiliary potentials a_i, and p_i , $i = 1 \cdots N$ only depend on a set of functions $W = (W_1, \cdots, W_M)$ with $W_1 = V$, and $(W_1, \cdots, W_M)$ satisfies commuting flows

$$\frac{\partial W}{\partial T_n} = A_n(W) \frac{\partial W}{\partial X} \ , \ n \geq 2, \tag{30}$$

where the $N \times N$ matrice A_n are only the functions of $(W_1 \cdots W_M)$. In the following, we will take the 2D dTLESCS (28) as an example and show its hodograph solutions in the case of $M = 1$ and $M = 2$.

1. $M = 1$

In this case, we will get

$$p = p(V) \ , \ a_i = a_i(V) \ , \ p_i = p_i(V), \tag{31}$$

and

$$V_X = A(V)V_S \ , \ V_Y = B(V)V_S. \tag{32}$$

(31), which, together with (25b) and (27), imply that

$$\begin{aligned}
&(\frac{V}{p_i} - \frac{V}{A})\frac{dp_i}{dV} = 1, \\
&(\frac{V}{p} - \frac{V}{A})\frac{dp}{dV} = 1, \\
&(A - p_i)\frac{da_i}{dV} = a_i \frac{dp_i}{dV}, \\
&B\frac{dp}{dV} = 1 - \frac{V}{p}\frac{dp}{dV} + \sum_{i=1}^{N} \frac{pp_i}{p - p_i}\frac{da_i}{dV} + \sum_{i=1}^{N} \frac{a_i p}{p - p_i}\frac{dp_i}{dV} - \sum_{i=1}^{N} \frac{a_i p p_i}{(p - p_i)^2}(\frac{dp}{dV} - \frac{dp_i}{dV}).
\end{aligned} \tag{33}$$

Equations (32) implies that

$$B = -\frac{V}{A} - V\sum_{i=1}^{N} \frac{da_i}{dV} \ , \ A = V\frac{dU}{dV}. \tag{34}$$

It is very easy to verify that with (33) and (32), (31) are compatible, making the hodograph transformation with the change of variables $(S,X,Y) \to (V,X,Y)$ with $S = S(V,X,Y)$. The hodograph equations for S are given by

$$\frac{\partial S}{\partial X} = -A \ , \ \frac{\partial S}{\partial Y} = -B = \frac{V}{A} + V\sum_{i=1}^{N}\frac{da_i}{dV}. \tag{35}$$

which can be easily integrated as

$$S + A(V)X - (\frac{V}{A} + V\sum_{i=1}^{N}\frac{da_i}{dV})Y = F(V), \tag{36}$$

where $F(V)$ is an arbitrary function of V.
If we chose $A(V) = 2V^2$, $F(V) = 0$, $a_i = c_iV^2$, c_i , $i = 1, \cdots, N$, as constants, using the Carl Dan formula, we get an explicit solution for the 2D dTLESCS (28)

$$V = \sqrt[3]{-\xi + \sqrt{\xi^2 + \eta^3}} + \sqrt[3]{-\xi - \sqrt{\xi^2 + \eta^3}}, \tag{37a}$$

$$a_i = c_iV^2 = c_i(\sqrt[3]{-\xi + \sqrt{\xi^2 + \eta^3}} + \sqrt[3]{-\xi - \sqrt{\xi^2 + \eta^3}})^2, \tag{37b}$$

$$p_i = V^2 = (\sqrt[3]{-\xi + \sqrt{\xi^2 + \eta^3}} + \sqrt[3]{-\xi - \sqrt{\xi^2 + \eta^3}})^2 \ , \ i = 1, \cdots, N. \tag{37c}$$

where $\xi = \frac{Y}{8Y\sum_{i=1}^{N} c_i - 8X}$, $\eta = \frac{S}{6X - 6Y\sum_{i=1}^{N} c_i}$.
We find when $\sum_{i=1}^{N} c_i = 0$, (37) is degenerated to the hodograph solution for the 2D dToda lattice equation.

2. $M = 2$

In this case, we denote $W_1 = V$, $W_2 = W$, $a_i = a_i(V,W)$, $p_i = p_i(V,W)$, $p = p(V,W)$, in which (V,W) satisfies the following commuting flow

$$\begin{pmatrix} V \\ W \end{pmatrix}_X = A\begin{pmatrix} V \\ W \end{pmatrix}_S \ , \ \begin{pmatrix} V \\ W \end{pmatrix}_Y = B\begin{pmatrix} V \\ W \end{pmatrix}_S, \tag{38}$$

where $A = (A)_{ij}$ and $B = (B)_{ij}$ are 2×2 matrix functions of V and W, and A is invertible. By requiring that V_X and W_X are independent, (25b) and (27) give rise to the following equations for $a_i(V,W)$, $p_i(V,W)$ and $p(V,W)$,

$$\begin{aligned}
(\frac{\partial p}{\partial V}, \frac{\partial p}{\partial W})A =& p(\frac{\partial p}{\partial V}, \frac{\partial p}{\partial W}) + \frac{p}{V}(A_{11}, A_{12}), \\
(\frac{\partial p_i}{\partial V}, \frac{\partial p_i}{\partial W})A =& p_i(\frac{\partial p_i}{\partial V}, \frac{\partial p_i}{\partial W}) + \frac{p_i}{V}(A_{11}, A_{12}), \\
(\frac{\partial a_i}{\partial V}, \frac{\partial a_i}{\partial W})A =& p_i(\frac{\partial a_i}{\partial V}, \frac{\partial a_i}{\partial W}) + a_i(\frac{\partial p_i}{\partial V}, \frac{\partial p_i}{\partial W}), \\
(\frac{\partial p}{\partial V}, \frac{\partial p}{\partial W})B =& (1,0) - \frac{V}{p}(\frac{\partial p}{\partial V}, \frac{\partial p}{\partial W}) + \sum_{i=1}^{N}\frac{pp_i}{p - p_i}(\frac{\partial a_i}{\partial V}, \frac{\partial a_i}{\partial W}) + \sum_{i=1}^{N}\frac{a_ip}{p - p_i}(\frac{\partial p_i}{\partial V}, \frac{\partial p_i}{\partial W}) \\
&+ \sum_{i=1}^{N}\frac{a_ipp_i}{(p - p_i)^2}(\frac{\partial (p - p_i)}{\partial V}, \frac{\partial (p - p_i)}{\partial W}).
\end{aligned} \tag{39}$$

We can easily find from (39) that $A(V,W)$ and $B(V,W)$ must satisfy

$$B = -VA^{-1} - V\sum_{i=1}^{N}\frac{\partial a_i}{\partial V}I - \begin{pmatrix} 0 & \frac{\partial a_i}{\partial W} \\ \frac{A_{21}}{A_{12}}\frac{\partial a_i}{\partial W} & \frac{A_{22}-A_{11}}{A_{12}}\frac{\partial a_i}{\partial W} \end{pmatrix}, \tag{40}$$

where A^{-1} denotes the inverse matrix of A, and I is the 2×2 identity matrix, and $A_{11} = V\frac{\partial U}{\partial V}$ and $A_{12} = V\frac{\partial U}{\partial W}$. For simplicity, we assume $\frac{\partial a_i}{\partial W} = 0$, $i = 1, \cdots, N$, by using the formula

$$A^2 = (trA)A - (detA)I \Longrightarrow A^{-1} = \frac{trA}{detA}I - \frac{A}{detA}, \tag{41}$$

and we have

$$B = \frac{VA}{detA} - (\frac{VtrA}{detA} + V\sum_{i=1}^{N}\frac{\partial a_i}{\partial V})I, \tag{42}$$

where $detA = A_{11}A_{22} - A_{12}A_{21}$ and $trA = A_{11} + A_{22}$.
With (42), the compatibility condition for (38) requires A to satisfy

$$\begin{pmatrix} -\frac{\partial}{\partial W}(\frac{VtrA}{detA} + V\sum_{i=1}^{N}\frac{\partial a_i}{\partial V}) \\ \frac{\partial}{\partial V}(\frac{VtrA}{detA} + V\sum_{i=1}^{N}\frac{\partial a_i}{\partial V}) \end{pmatrix} = A\begin{pmatrix} -\frac{\partial}{\partial W}(\frac{V}{detA}) \\ \frac{\partial}{\partial V}(\frac{V}{detA}) \end{pmatrix}. \tag{43}$$

To solve (38), we use hodograph transformation by changing the independent variables (S,X,Y) to (V,W,Y) with the dependent variables $S = S(V,W,Y)$, $X = X(V,W,Y)$. In terms of the new variables, (38) becomes

$$\begin{pmatrix} -S_W \\ S_V \end{pmatrix} = A\begin{pmatrix} X_W \\ -X_V \end{pmatrix} , \begin{pmatrix} \frac{\partial(S,X)}{\partial(W,Y)} \\ -\frac{\partial(S,X)}{\partial(V,Y)} \end{pmatrix} = B\begin{pmatrix} X_W \\ -X_V \end{pmatrix}, \tag{44}$$

where $\frac{\partial(S,X)}{\partial(W,Y)} = S_W X_Y - S_Y X_W$.
It can be easily found that (55) has solutions in the form

$$\begin{aligned} &S - (\frac{VtrA}{detA} + V\sum_{i=1}^{N}\frac{\partial a_i}{\partial V})Y = F(V,W), \\ &X + \frac{V}{detA}Y = G(V,W), \end{aligned} \tag{45}$$

where X_W and X_V are required to be independent, and F and G are two arbitrary functions satisfying the linear equations

$$\begin{pmatrix} -F_W \\ F_V \end{pmatrix} = A\begin{pmatrix} G_W \\ -G_V \end{pmatrix}, \tag{46}$$

An example of a solution is given by

$$A = \begin{bmatrix} 2V^2 & 0 \\ -3VW & V^2 \end{bmatrix}, \tag{47a}$$

and

$$p_i = V^2 \ , \ a_i = c_i V^2 \ , \ U = V^2 \ , i = 1, \cdots, N, \tag{47b}$$

where c_i , $i = 1, \cdots, N$ are constants.
Then, (45) becomes

$$S - (\frac{3}{2V} + 2V^2 \sum_{i=1}^{N} c_i)Y = F(V, W),$$

$$X + \frac{Y}{2V^3} = G(V, W).$$

From (46) and $F_{UW} = F_{WU}$, G must satisfy

$$VG_W + V^2 G_{WV} - 3VWG_{WW} = 0. \tag{48}$$

We notice that $G = \frac{W}{V}$ is a particular solution of (5.19). From (46), we get $F = -2VW$, and via the Carl Dan formula, we can get an explicit hodograph solution for the 2D dTLESCS (28)

$$V = \sqrt[3]{-\xi + \sqrt{\xi^2 + \eta^3}} + \sqrt[3]{-\xi - \sqrt{\xi^2 + \eta^3}}, \tag{49a}$$

$$a_i = c_i V^2 = c_i (\sqrt[3]{-\xi + \sqrt{\xi^2 + \eta^3}} + \sqrt[3]{-\xi - \sqrt{\xi^2 + \eta^3}})^2, \tag{49b}$$

$$p_i = V^2 = (\sqrt[3]{-\xi + \sqrt{\xi^2 + \eta^3}} + \sqrt[3]{-\xi - \sqrt{\xi^2 + \eta^3}})^2 \ , \ i = 1, \cdots, N. \tag{49c}$$

where $\xi = \frac{Y}{8Y\sum_{i=1}^{N} c_i - 8X}$, $\eta = \frac{S}{6X - 6Y\sum_{i=1}^{N} c_i}$.

6. One-Dimensional Reduction for Extended 2D dToda Lattice Hierarchy

In this section, we will consider one-dimensional reduction for the extended 2D dTLH (20). The constraint is given by

$$\mathcal{L} + \mathcal{L}^{-1} = \mathcal{M} + \mathcal{M}^{-1}, \tag{50a}$$

$$(\mathcal{L} + \mathcal{L}^{-1})|_{p=p_i} = \mu_i + \mu_i^{-1}, \tag{50b}$$

$$[a_i p_i (\frac{\partial(\mathcal{L} + \mathcal{L}^{-1})}{\partial p}|_{p=p_i})]_S = 0. \tag{50c}$$

where μ_i are constants. It was noticed that this constraint (50) has never been obtained before, which will reduce the 2D extended dTLH to one-dimensional dTLH with self-consistent sources (dTLHSCSs). Before we show this result, we need to prove two lemmas as follows.

Lemma 2. *The constraint (50) is compatible with extended 2D dTLH (20).*

Proof. **(1)** We firstly prove that the constraint (50a) is compatible with the ∂_{X_n} , ∂_{Y_n} , $\partial_{\overline{Y}_m}$ flows, respectively, noting that

$$(\mathcal{L} + \mathcal{L}^{-1})_{X_n} = \mathcal{L}_{X_n} + (\mathcal{L}^{-1})_{X_n} = \mathcal{L}_{X_n} - (\mathcal{L}^{-2})\mathcal{L}_{X_n} = \{\mathcal{B}_n, \mathcal{L}\} - \mathcal{L}^{-2}\{\mathcal{B}_n, \mathcal{L}\},$$

then, one has

$$(\mathcal{L} + \mathcal{L}^{-1})_{X_n} = \{\mathcal{B}_n, \mathcal{L}\} + \{\mathcal{B}_n, \mathcal{L}^{-1}\} = \{\mathcal{B}_n, \mathcal{L} + \mathcal{L}^{-1}\}.$$

From the constraint (50a), we obtain

$$(\mathcal{L} + \mathcal{L}^{-1})_{X_n} = \{\mathcal{B}_n, \mathcal{M} + \mathcal{M}^{-1}\} = (\mathcal{M} + \mathcal{M}^{-1})_{X_n}.$$

Therefore, we conclude that the ∂_{X_n}-flow is compatible with constraint (50a). In the same way, we can prove that ∂_{Y_n} , $\partial_{\overline{Y}_m}$-flows are also compatible with the constraint (50a).

(2) Secondly, we prove that ∂_{X_n} , ∂_{Y_n} flows are both compatible with (50b), noting that

$$[(\mathcal{L}+\mathcal{L}^{-1})|_{p=p_i}]_{X_n} = \{\mathcal{B}_n, \mathcal{L}+\mathcal{L}^{-1}\}|_{p=p_i} + \frac{\partial(\mathcal{L}+\mathcal{L}^{-1})}{\partial p}|_{p=p_i} \cdot \frac{\partial p_i}{\partial X_n}.$$

By using the definition of the Poisson bracket, we have

$$\begin{aligned}[(\mathcal{L}+\mathcal{L}^{-1})|_{p=p_i}]_{X_n} =&[p\frac{\partial \mathcal{B}_n}{\partial p}\frac{\partial(\mathcal{L}+\mathcal{L}^{-1})}{\partial S} - p\frac{\partial \mathcal{B}_n}{\partial S}\frac{\partial(\mathcal{L}+\mathcal{L}^{-1})}{\partial p}]|_{p=p_i} \\ &+ \frac{\partial(\mathcal{L}+\mathcal{L}^{-1})}{\partial p}|_{p=p_i} \cdot \frac{\partial p_i}{\partial X_n}.\end{aligned} \tag{51}$$

Noting that $p_{i,X_n} = p_i[B_n(p_i)]_S$, we get from (51)

$$\begin{aligned}[(\mathcal{L}+\mathcal{L}^{-1})|_{p=p_i}]_{X_n} =&p_i\frac{\partial \mathcal{B}_n(p_i)}{\partial p_i}\frac{\partial(\mathcal{L}+\mathcal{L}^{-1})}{\partial S}|_{p=p_i} - p_i\frac{\partial \mathcal{B}_n(p_i)}{\partial S}\frac{\partial(\mathcal{L}+\mathcal{L}^{-1})}{\partial p}|_{p=p_i}+ \\ &p_i[B_n(p_i)]_S\frac{\partial(\mathcal{L}+\mathcal{L}^{-1})}{\partial p}|_{p=p_i} \\ =&p_i\frac{\partial \mathcal{B}_n(p_i)}{\partial p_i}\frac{\partial(\mathcal{L}+\mathcal{L}^{-1})}{\partial S}|_{p=p_i}.\end{aligned}$$

By means of the constraint (50b), one has

$$[(\mathcal{L}+\mathcal{L}^{-1})|_{p=p_i}]_{X_n} = p_i\frac{\partial \mathcal{B}_n(p_i)}{\partial p_i}\frac{\partial(\mu_i+\mu_i^{-1})}{\partial S}|_{p=p_i} = 0.$$

With the help of (24b), ∂_{Y_n}-flow is similarly shown to be compatible with the constraint (50b). It can be shown from (50b) that (50c) holds for $\mathcal{L}, a_i$ and p_i. We noticed that (50b) is compatible with $\partial_{Y_n}, \partial_{X_n}$ flows, respectively, and so is (50c). □

Lemma 3. *The constraint (50a) implies that*

$$\mathcal{B}_n + \mathcal{C}_n = \mathcal{L}^n + \mathcal{L}^{-n} = \mathcal{M}^n + \mathcal{M}^{-n},$$

$$\mathcal{B}_n - \mathcal{C}_n = (\mathcal{L}^n + \mathcal{L}^{-n})_+ - (\mathcal{L}^n + \mathcal{L}^{-n})_- , \ (n \in \mathbb{N}).$$

Proof. Noting that the proof of Lemma 3 is similar to [19], we omit the details here. □

We change the "X" and "Y" time variables of the extended 2D dTLH (20) to the "τ" and "T" time variables by skewing the coordinates as

$$\tau_n = X_n \ , \ \tau_m = X_m \ , \ T_n = X_n - Y_n \ , \ \overline{T}_m = \overline{X}_m - \overline{Y}_m \ , \ n \neq m.$$

Next, we will use Lemma 2 and Lemma 3 to show the following theorem. We will find that the constraint (50) will reduce the extended 2D dTLH (20) to a one-dimensional dTLHSCS.

Theorem 2. *Under the constraint (50), the Lax operators $\mathcal{L}$ and $\mathcal{M}$ will not depend on the time variables τ_n. The extended 2D dTLH can be reduced to the following one-dimensional dTLHSCS.*

$$\mathfrak{L}_{\overline{T}_m} = \{\mathfrak{B}_m - \sum_{i=1}^{N}\frac{a_i p_i}{p-p_i}, \mathfrak{L}\}, \tag{52a}$$

$$(\mathfrak{L})|_{p=p_i} = \mu_i + \mu_i^{-1}, \tag{52b}$$

$$[a_i p_i (\frac{\partial \mathfrak{L}|_{p=p_i}}{\partial p_i})]_S = 0. \tag{52c}$$

where $\mathfrak{L} = \mathcal{B}_1 + \mathcal{C}_1$, $\mathfrak{B}_m = (\mathfrak{L}^m + \sum_{i=0}^{m-1} c_{mi} \mathfrak{L}^i)_+$, *and the coefficients* c_{mi} *are integers determined by the recurrence relation*

$$c_{m+1,i} = c_{m,i-1} - c_{m-1,i} \ , \ \ (m > 1 \ , \ 0 \leq i \leq m), \tag{53}$$

with the initial and boundary conditions: $c_{mm} = 1$, $c_{m,0} = 2\cos(\frac{m\pi}{2})$ *and* $c_{mi} = 0$ *when* $i < 0$ *or* $i > m$*.*

Proof. Noting that the proof of Theorem 2 is similar to [19], we omitted the details here. Especially, when $m = 1$, we obtain the one-dimensional dTLESCS as follows

$$(\partial_S^{-1} U_T) U_S - \partial_S^{-1} U_{TT} + (\sum_{i=1}^{N} a_i p_i)_T + (\sum_{i=1}^{N} a_i p_i^2)_S = 0, \tag{54a}$$

$$p_i + U + \frac{V}{p_i} = \mu_i + \mu_i^{-1}, \tag{54b}$$

$$[a_i p_i (U - \frac{V}{p_i^2})]_S = 0, \tag{54c}$$

where $U = U_0$, $V = V_{-1}$, $T = \overline{T}_1$, $V = \partial_S^{-1} U_T - \sum_{i=1}^{N} a_i p_i$. □

7. Summary

In this article, the symmetry for 2D dTLH is derived by taking the dispersionless limit of that for 2D Toda lattice hierarchy. In addition, the new extension of the dTLH is considered. We can easily find that the new extended 2D dTLH is Lax integrable, and that the zero-curvature equation contains a 2D dTLESCS. The hodograph solutions to the 2D dTLESCS are obtained by the reduction method together with hodograph transformation. One-dimensional reduction of the extended 2D dTLH is finally constructed, which leads to a one-dimensional dTLESCS. Our results give a supplement to the previous studies about the dToda hierarchy.

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