On the Rozansky–Witten TQFT

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Abstract

This thesis studies a potential method for constructing the Rozansky–Witten TQFT as an extended $(1+1+1)$-TQFT. A monoidal 2-category consisting of schemes, complexes of sheaves and sheaf morphisms is constructed, and it is shown that there are $(1+1)$-TQFTs valued in the truncation of this category, whose state spaces agree with the Rozansky–Witten TQFT. However, it is also shown that if such a TQFT is based on a reduced Noetherian scheme, it cannot be extended upwards to a $(1+1+1)$-TQFT.

Statement of Originality

This thesis is entirely my own work except where material is cited from the literature or commonly known. It has not been submitted for a degree at another University, or for another degree at the University of Oxford.

A summary of the results of Chapters 3 and 4 has been submitted to the arXiv, and is under review for publication.

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Chapter 1

Introduction

The construction of manifold invariants has motivated many recent advances in topology. Some of the motivation for these invariants has come not from mathematics but from quantum theory. This work studies Rozansky–Witten theory, which gives rise to an invariant of 3-manifolds, approaching the construction from a mathematical viewpoint with the aim of finding a rigorous construction of the invariant.

This construction would take the form of a *topological quantum field theory*, henceforth “TQFT”. Introduced by Segal [43] and axiomatized by Atiyah [2], these are invariants of manifolds constructed as a functor Z between a bordism category and an algebraic category, often taken to be the category of vector spaces over a fixed ground field. The $(d+1)$-dimensional bordism category has as objects d-dimensional closed manifolds, and as morphisms diffeomorphism classes of $(d+1)$-bordisms: that is, compact $(d+1)$ manifolds with boundaries separated into a source component and a target component, considered up to diffeomorphisms preserving the boundaries. A bordism W also comes equipped with a parametrisation of a collar neighbourhood of its boundaries: this is a diffeomorphism between $\partial W \times [0, 1)$ and neighbourhood of ∂W, where $\partial W \times \{0\}$ is identified with ∂W. The composition of two bordisms is defined by gluing the target boundary of one bordism with the source boundary of the next; the collar neighbourhood allows the construction of a well-defined smooth structure on the result. Often, the bordism category is modified by requiring that all objects and bordisms come equipped with a particular structure. A common case is to require objects and bordisms to have an orientation, which gives rise to the oriented bordism category and oriented TQFTs. In this situation, the source boundary component is oriented with the boundary orientation, while the target component is oriented with the reverse of the boundary orientation (see Definition 1.12 for an explicit definition). All bordisms (and hence TQFTs) considered in this thesis will be assumed to be oriented.

The bordism category is naturally a symmetric monoidal category, with the monoidal product given by disjoint union. The target category for a TQFT is required to be a symmetric monoidal category, and the TQFT itself is required to be a symmetric monoidal functor. In the case of vector spaces, the monoidal product is the tensor product of vector spaces. As an immediate consequence, $Z(\emptyset) \cong k$, as the empty manifold and the base field serve as the monoidal unit for the two categories. This allows us to construct an invariant of $(d+1)$-dimensional manifolds from a $(d+1)$-dimensional TQFT as follows. First, the manifold is viewed as an endomorphism of the empty manifold in the $(d+1)$-dimensional bordism category. Applying the functor Z gives an endomorphism of $Z(\emptyset) \cong k$. Finally, the endomorphism group $\mathrm{End}_k(k)$ is identified with k itself, giving a value in k.

A benefit of this construction is that the invariant of a manifold can be calculated by splitting it into simpler pieces. Each of these submanifolds is then viewed as a bordism (unlike before, these bordisms may have non-empty boundary), and the manifold as a whole is the composition (in the bordism category) of these simpler manifolds with boundaries. The invariant for the original manifold can then be calculated by determining the value of the TQFT on these pieces, and composing the resulting morphisms.

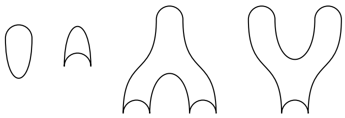
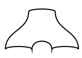
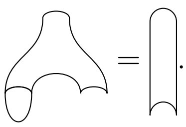
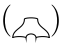


Figure 1.1: A generating set for the morphisms of the category $\mathbf{Bord}_{1+1}$.

This process naturally leads to a method for providing a classification of low-dimensional TQFTs. Since TQFTs are functors, a morphism between two TQFTs Z and Z' is defined to be a monoidal natural transformation. To classify a certain type of TQFT, we find an equivalence between the category of TQFTs and another category, often defined algebraically. A folklore result states that the category of oriented 2-dimensional TQFTs is equivalent to the category of commutative Frobenius k-algebras. The proof of this theorem can be found in many places (e.g. [10]), with the key idea of the proof being the identification of the 2-dimensional bordism category with the category generated as a symmetric monoidal category by a single object (representing a copy of S^1), along with the morphisms shown in Figure 1.1. The source component of the boundary of these manifolds is taken to be the boundary at the bottom of the picture, and the target boundary is at the top; hence the pair of pants  is a morphism $S^1 \sqcup S^1 \to S^1$. These morphisms are subject to the Morse relations arising from birth/death of critical points and handle slides; for instance, there is an equality

 (1.1)

Given an oriented $(1+1)$-TQFT, the vector space $A = Z(S^1)$ has the structure of a unital associative algebra, where the product is given by $Z($$)$ and the unit is given by $Z($$)$. The unit equality is then exactly the image of Equation (1.1) under Z. Next, A can be given a counital coassociative coproduct structure using the images of the copants and cap bordisms.

In a similar manner, $(2+1)$-TQFTs have been classified [21]. In this case, the objects of the bordism category are (diffeomorphic to) genus g-surfaces, for $g \geq 0$. Correspondingly, the classification is not in terms of a single vector space, but a *graded* vector space $A = \bigoplus A_g$, where Z sends a genus g surface to the vector space A_g. The bordisms from Figure 1.1 generalise to 3-dimensional bordisms in this category: the unit and counit become the solid ball D^3, viewed as either a bordism from the empty manifold to S^2 or vice versa; while the pants and copants become the twice-punctured ball $D^3 \setminus D^3 \sqcup D^3$. The images of these bordisms under Z give the graded vector space A an associative unital graded multiplication and a coassociative counital comultiplication. However, there is additional structure to consider. The trace of surgery along a pair of points on a genus g-surface gives a bordism between a surface of genus g and a surface of genus $g+1$. Similarly, the trace of a surgery along a longitudinal curve gives a degree-lowering map. These, along with the multiplication and comultiplication, give A its algebraic structure; such an object, satisfying the required relations, is known as a *graded nearly-Frobenius algebra*, or “GNF-algebra”. Full details of this construction are given in Definition 2.2.

However, this on its own is not enough to capture the full structure of the TQFT. Given a diffeomorphism of a genus g manifold $f \colon \Sigma_g \to \Sigma_g$, there is a bordism $c_f \colon \Sigma_g \to \Sigma_g$, known as the *mapping cylinder* which has as its underlying manifold the cylinder $\Sigma_g \times I$. The incoming boundary is $\Sigma_g \times \{0\}$; this has a collar neighbourhood $\Sigma_g \times [0, 1/3)$ which is identified with $\Sigma_g \times [0, 1)$ by stretching the second factor. Similarly, the neighbourhood $\Sigma_g \times (2/3, 1]$ of its outgoing boundary is identified with $\Sigma_g \times (-1, 0]$ by applying f to the first factor and stretching the second.

Applying the TQFT Z gives an element of $\mathrm{Aut}(A_g)$, and hence there is an action of $\mathrm{Aut}(\Sigma_g)$ on A_g. Since bordisms are considered only up to diffeomorphism, certain diffeomorphisms give the same mapping cylinder. It can be shown that pseudo-isotopic diffeomorphisms have the same mapping cylinder, and so the action factors through the *mapping-class group* $\mathrm{MCG}(\Sigma_g)$, formed from $\mathrm{Aut}(\Sigma_g)$ by identifying isotopic diffeomorphisms. Thus to classify 3-dimensional TQFTs, the GNF-algebra A must be augmented with a *mapping class group representation*, giving a J-algebra [21, Definition 5.1]. After defining a suitable notation of morphism of J-algebras, it is possible to construct a category of J-algebras. This category is equivalent to the category of 3-dimensional TQFTs.

Since the multiplication and comultiplication are graded, the vector space A_0 inherits a multiplicative and comultiplicative structure. These structures also satisfy the condition necessary for the object to be a Frobenius algebra, illustrating a particular case of the result that for a $(d+1)$-dimensional TQFT Z, $Z(S^d)$ has the structure of a Frobenius algebra. The GNF-algebra structure of the TQFT is partially determined by the algebraic structure of A_0: in Proposition 2.18, it is shown that if A_0 has dimension 1 (and so is isomorphic to k), then the GNF-algebra is in fact a free associative algebra.

From a physical point of view, TQFTs provide a method for giving a rigorous theory for path integrals. These formalise the notion of performing an integral over all possible fields on a d-dimensional manifold M valued within a fixed target space X. The target space determines the values that the fields can take; for example, if the target space is $\mathbb{R}$, then the fields are scalar-valued fields. Alternatively, to model quantum particles, the target space X is taken to be the spacetime in which the particle exists. The space of fields on a d-manifold M valued in a target space X is the space of maps $\mathrm{Hom}(M, X)$. Define an *action functional*

$$S\colon\ \mathrm{Hom}(M, X) \to \mathbb{R}\,.$$

Traditionally, the value of S evaluated on a morphism Φ is denoted $S[\Phi]$. This action is often given by an integral, and is additive in the sense that if $M = M_1 \sqcup_\partial M_2$ is a decomposition of M into submanifolds M_1 and M_2, which are disjoint apart from on their boundaries, then $S[\phi] = S[\phi\big|_{M_1}] + S[\phi\big|_{M_2}]$. Given an action, a *path integral* is an integral of the form

$$\int_{\Phi\in\mathrm{Hom}(M,X)} \exp(-S[\phi])D\Phi\,.$$

This is not a rigorous mathematical construction, but provides useful motivation for heuristic physical arguments.

It is possible to examine the properties that one would expect a path integral to obey. For example, for each closed d-dimensional manifold E there should be a Hilbert space $\mathscr{H}_E$ of states, consisting of functionals of fields $\psi\colon E \to X$. If M is a $(d+1)$-dimensional manifold with $\partial M = E$, then $Z(M) \in E$ should be given by

$$Z(M)(\psi) = \int_{\Phi\in\mathrm{Hom}(M,X),\Phi\big|_E=\psi} \exp(-S[\Phi])D\Phi.$$

For example, let M be a closed $(d+1)$-manifold and $U \subset M$ be a closed d-submanifold that splits M into two parts, M_1 and M_2, each with boundary U. Then

$$\begin{aligned}\int_{\Phi\in\mathrm{Hom}(M,X)} \exp(-S[\phi])D\Phi &= \int_{\psi\in\mathrm{Hom}(U,X)}\int_{\Phi\in\mathrm{Hom}(M,X),\Phi\big|_U=\psi} \exp(-S[\phi])D\Phi D\psi \\ &= \int_{\psi\in\mathrm{Hom}(U,X)}\int_{\Phi\in\mathrm{Hom}(M,X),\Phi\big|_U=\psi} \exp(-S[\phi\big|_{M_1}] - S[\phi\big|_{M_2}])D\Phi D\psi \\ &= \int_{\psi\in\mathrm{Hom}(U,X)} Z(M_1)(\psi)Z(M_2)(\psi)D\psi\,.\end{aligned}$$

where the second equality uses additivity of the action, and the third uses the fact that two fields defined on open subsets of M that agree on their overlap can be glued to give a field on all of M. Thus the invariant on the whole manifold is determined by the value of the invariant on the two submanifolds, in the same way that, given a $(d+1)$-TQFT, the invariant of a $(d+1)$-manifold can be determined by splitting it along d-manifolds. This motivation is expanded further in [10, Section 2.1].

A particular example of a path integral is Chern–Simons theory. Let G be a compact Lie group, and let the target space X be the moduli space $\mathbb{B}G_{\mathrm{conn}}$ of G-principal bundles with connections. Define the *Chern–Simons action functional* S by

$$S[A] = \gamma \int_M \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A),$$

where A is now a connection 1-form $A \in \Omega^1(M, \mathfrak{g})$, $\mathfrak{g}$ is the Lie algebra of a principle G-bundle $P \to M$, and $\gamma \in \mathbb{C} \setminus \{0\}$ is a non-zero constant.

This path integral can be extended to include *Wilson loop operators*. For an embedded knot $K \subset M$, pick an irreducible representation R of G and define $W_R(K)$ as the trace of the holonomy of A around K:

$$W_R(K) = \mathrm{Tr}_R\, P \exp \int_K A.$$

From a physical point of view, these define observables of the Chern–Simons field theory. Thus we compute an invariant of the pair (M, K) to be

$$Z_k(M; K, R) = \int_{A \in \mathrm{Hom}(M,X)} \exp(ikS[A]) W_R(K) DA$$

where k is an integer called the *level*. In the case that $G = \mathrm{SU}(2)$ and R is the fundamental 2-dimensional representation, then the invariant $Z_k(M; K, R)$ is the Jones polynomial [49] $J(q)$, evaluated at

$$q = \exp\left(\frac{2\pi i}{k+2}\right).$$

The Rozansky–Witten TQFT [40] is constructed from a similar path integral formula. A key feature is that the fields are valued in a holomorphic sympletic manifold X [1]. The choice of this manifold, and analysis of the resulting invariant,

[1]The original work of Rozansky and Witten considered hyperKähler manifolds, but subsequent work of Kontsevich [25] and Kapranov [22] showed that a holomorphic symplectic structure is sufficient

leads to some interesting geometric results. For example, it can be used to relate the $\hat{A}$-genus of a manifold (a topological property) to its curvature (a geometric property) [39]. The 3-manifold invariant itself is related to the LMO invariant [16], which is its the universal finite-type invariant of rational homology spheres. Its state spaces are conjectured [40] to be

$$Z(\Sigma_g) = \bigoplus_{q=0}^{\dim_{\mathbb{C}}(X)} H^q_{\bar{\partial}}\left(X, (\wedge T^{1,0}X)^{\otimes g}\right) . \tag{1.2}$$

However, being based on a path integral formulation, the Rozansky–Witten TQFT is not fully mathematically rigorous. Many attempts have been made to rectify this issue. Several avenues have been explored, but can be broadly separated into two main approaches: renormalizing the TQFT, or understanding the boundary conditions.

To understand the justification for analysing the boundary conditions of the Rozansky–Witten invariant, it is necessary to expand the definition of a TQFT given previously. The first generalisation is to change the target category. The only feature of the category of vector spaces that is required to give the definition of a TQFT is that it is a symmetric monoidal category. Instead of using the category of vector spaces, it is possible to study TQFTs valued in any symmetric monoidal category $\mathscr{C}$. Since the classification results mentioned above are based on giving a generators-and-relations presentation of the bordism category, they can be easily generalised to other target categories. The category of 2-dimensional TQFTs valued in a symmetric monoidal category $\mathscr{C}$ is equivalent to the category of symmetric Frobenius algebra objects in $\mathscr{C}$ (see Definition 4.1 for more details).

Another generalisation comes from changing the category of bordisms. There are two main ways of achieving this. The first is to allow decorations on bordisms, such as in the case of *defect TQFTs* [10]. In this case, $(d+1)$-bordisms are divided into submanifolds, which are labelled with labels from a set D_{d+1}. The boundaries of these submanifolds are labelled by elements of a set D_d, corners are labelled with elements of D_{d-1} and so on. These must meet in prescribed manners. Such a construction can be viewed as a collection of TQFTs which can be "glued together" along certain boundary conditions. In the case $d = 2$, these objects induce *pivotal* 2*-categories* [13]. They have been used to describe B-twisted sigma models (where the label sets are the set of complex manifolds and holomorphic vector bundles on them) and Landau-Ginzburg models [11].

An alternative generalisation comes from extending the bordism category to a higher category. Recall that a category (or, more explicitly, a 1-category) consists of a set of objects, and for any two objects a (possibly empty) set of morphisms between them, along with an associative composition of morphisms and unit morphisms. A 2-category introduces the idea of 2-morphisms: that is, morphisms between morphisms. An example of this is natural transformations, which give morphisms between functors, and allow the construction of a 2-category $\mathscr{C}at$ of categories, functors and natural transformations.

This extra level of morphisms introduces an alternative notion of equality. In the same manner that one may require two vector spaces not to be equal, but only isomorphic, it is now possible to talk of *morphisms* being isomorphic: that is, two 1-morphisms f and g are isomorphic if there is an invertible 2-morphism $\alpha\colon f \Rightarrow g$. This leads to a choice in the definition of a 2-category: given a composable tuple of 1-morphisms f, g, h, do we require $f \circ (g \circ h) = (f \circ g) \circ h$, or the weaker condition that $f \circ (g \circ h) \cong (f \circ g) \circ h$? The first condition leads to a *strict* 2-category, whereas the second condition gives a *weak* 2-category (also known as a bicategory [7]). An example of a bicategory is the *fundamental* 2*-groupoid* of a topological space: the objects are points, the 1-morphisms are paths between points, and the 2-morphisms are homotopies between paths. The composition of two paths requires reparameterization of the concatenation of the paths, which causes the composition to be only associative up to homotopy.

For the category of bordisms, this enables the addition of an extra layer of morphisms. The $(d+1+1)$-*bordism category* $\mathscr{B}\mathit{ord}_{d+1+1}$ (also known as the *once-extended* $(d+2)$-*bordism category*) has as objects d-manifolds, as 1-morphisms $(d+1)$-bordisms between manifolds, and as 2-morphisms $(d+2)$-bordisms between bordisms, considered up to diffeomorphism (for more details seeDefinition 1.15). A $(d+1+1)$-*TQFT* valued in a symmetric monoidal 2-category $\mathscr{C}$ is defined as a symmetric monoidal functor $Z\colon \mathscr{B}\mathit{ord}_{d+1+1} \to \mathscr{C}$.

The definition of $(d+1+1)$-TQFTs extend the idea that invariants can be computed by cutting and gluing into an extra dimension: the manifold may now be split into pieces which have corners. In the case where $d = 1$, these TQFTs have received much attention (e.g. by Bakalov and Kirillov [4] and Walker [47]); an overview of the categorical issues in the construction of the category $\mathscr{B}\mathit{ord}_{d+1+1}$ has been given by Lurie [31].

Once one has navigated the technical difficulties in constructing the bordism category, it may be viewed as a simpler object to work with than the non-extended bordism category. For example, the $(1+1+1)$ bordism category has a single generating object, whereas the $(2+1)$-bordism category, as noted above, does not have a finite generating set. A generators-and-relations presentation of this category was given by Bartlett, Douglas, Schommer-Pries and Vicary [6].

When considering $(d+1+1)$-TQFTs, there is a choice of target category. A typical choice is $2\mathscr{V}\mathit{ect}$, defined as follows. A *linear category* is a category enriched in **Vect**; that is, the Hom-sets are vector spaces and composition is bilinear. Such a category is *Cauchy-complete* [28] if it has finite coproducts and all idempotents split. Then $2\mathscr{V}\mathit{ect}$ is the symmetric monoidal category of Cauchy-complete linear categories, functors, and natural transformations. Bartlett, Douglas, Schommer-Pries and Vicary [6] showed that there is an equivalence between the category of once-extended 3-dimensional TQFTs valued in $2\mathscr{V}ect$ and modular tensor categories, a type of braided monoidal category with extra structure. From this construction, it is known that any $(1+1+1)$-TQFT can be written as a direct sum of TQFTs with $\dim(Z(S^2)) = 1$.

A well-known method for constructing $(2+1)$-TQFTs is the Reshetikhin–Turaev construction [37]. This construction takes as input a modular tensor category, and produces a (2+1)-TQFT valued in the category of vector spaces. In fact, the resulting TQFT can be shown to extend downwards to give a $(1+1+1)$-TQFT. All known examples of $(2+1)$-TQFTs can be extended to $(1+1+1)$-TQFTs. The following is an open question:

Question 1.1. Do all $(2+1)$-TQFTs valued in **Vect** extend to $(1+1+1)$-TQFTs valued in $2\mathscr{V}ect$?

The result that GNF algebras with $\dim A_0 = 1$ must be integral domains provides a small step towards this result, as the GNF-algebra induced by a $(1+1+1)$-dimensional TQFT must also be an integral domain. In contrast, the Rozansky–Witten TQFT has $A_0 \cong \frac{k[\alpha]}{\alpha^2}$ (where A_0 is considered as an algebra using the restriction of the graded product; see Definition 1.18). Thus it would give an example of a TQFT which cannot be extended in $2\mathscr{V}ect$.

On the other hand, it is believed that the Rozansky–Witten TQFT can be extended if the target category is chosen correctly. Roberts and Willerton [38] conjecture that the target 2-category should be constructed using the ideas of sheaves of modules. Recall [18] that given a topological space X, a *sheaf* $\mathscr{E}$ is a an assignment of a group $\mathscr{E}(U)$ for each open set $U \subset X$, satisfying a gluing and restriction identity. If $(X, \mathscr{O}_X)$ is a ringed space, then a *sheaf of modules* on X is a similar assignment, but now $\mathscr{E}(U)$ is an $\mathscr{O}_X(U)$-module. Morphisms between two sheaves are defined locally, giving a category of sheaves of modules $\mathscr{SH}_X$. Given two sheaves, their tensor product is defined by taking the tensor product locally. Given a morphism of ringed spaces

$f\colon (X,\mathscr{O}_X) \to (Y,\mathscr{O}_Y)$, there is a direct image functor $f_*\colon \mathscr{SH}_X \to \mathscr{SH}_Y$ defined by $f_*(\mathscr{E})(V) = \mathscr{E}(f^{-1}(V))$. However, this functor is not exact, in the sense that it does not preserve exact sequences [48]. This can be rectified by replacing the categories of sheaves of modules $\mathscr{SH}_X$ with the *derived category* $\mathscr{D}(X)$, and the functor f_* with its derived functor, also denoted f_* (these notions are recalled in Subsection 1.3.1). Similarly, the tensor product functor is replaced with its derived version.

Conjecture 1.2. *[38] There is a* $(1{+}1{+}1)$*-TQFT* Z*, given on objects by sending the disjoint union of* k *copies of* S^1 *to the derived category* $\mathscr{D}(X^k)$*. The* 1*-morphism* D^2*, viewed as a morphism* $S^1 \to \emptyset$*, is sent to the direct image functor* $\pi^X_{\{pt\}*}$*, where* $\{pt\}$ *is the manifold with a single point and* $\pi^X_{\{pt\}}$ *is the projection map. The pair-of-pants bordism is the functor given by taking the derived tensor product with the structure sheaf of the diagonal then pushing forward once.*

This conjecture does not give details of the 2-category that should serve as the target for the TQFT. A natural choice would be $2\mathscr{C}at$, the symmetric monoidal 2-category of categories, functors and natural transformations, with monoidal product given by the product of categories. There are two potential issues with using this as the target category. The first is that it forgets the "geometric" nature of the categories $\mathscr{D}(X)$. When considering functors between derived categories of sheaves, it is often more natural [9, 15] to consider functors which arise as *Fourier–Mukai* transforms. Recall [20] that for any $\mathscr{X} \in \mathscr{D}(X \times Y)$, the *Fourier–Mukai transform* $\Phi_{\mathscr{X}}\colon \mathscr{D}(X) \to \mathscr{D}(Y)$ is the functor

$$\Phi_{\mathscr{X}}(\mathscr{E}) = \pi^{XY}_{Y*}(\pi^{XY*}_{X}(\mathscr{E}) \boxtimes \mathscr{X}).$$

Given a morphism of sheaves $f \in \mathrm{Hom}_{\mathscr{D}(X\times Y)}(\mathscr{E},\mathscr{F})$, there is a natural transformation

$$\Phi_f\colon \Phi_{\mathscr{E}} \Rightarrow \Phi_{\mathscr{F}}$$

with components given by $\pi_{Y*}^{XY}(\pi_X^{XY*}(f) \boxtimes \mathscr{X})$, which gives a functor $\Phi\colon \mathscr{D}(X \times Y) \to \mathrm{Fun}(\mathscr{D}(X), \mathscr{D}(Y))$. Orlov's theorem [34, 33] shows that any fully faithful functor with left and right adjoints can be realised uniquely (up to isomorphism) as a Fourier–Mukai transform; Bondal and Bergh [8] later showed that the necessary adjoints exist, so that Φ is essentially surjective onto the full subcategory of fully faithful functors. Thus one often replaces the category $\mathrm{Fun}(\mathscr{D}(X), \mathscr{D}(Y))$ with the category $\mathscr{D}(X \times Y)$.

A second issue is with this choice of target category is that the functor Z would not be a strict monoidal functor. In general, $\mathscr{D}(X^2) \ncong \mathscr{D}(X) \times \mathscr{D}(X)$, so $Z(S^1 \sqcup S^1) \ncong Z(S^1) \otimes Z(S^1)$. The best one could hope for would be that Z is *lax monoidal*. Lax monoidal functors are defined in the same manner as strict monoidal functors, but without the condition that the comparison morphisms $\epsilon\colon \mathbb{1}_{\mathscr{D}} \to F(\mathbb{1}_{\mathscr{C}})$ and $\mu_{x,y}\colon F(c) \otimes F(c') \to F(c \otimes c')$ are isomorphisms.

These two issues can be resolved by using the 2-category $\mathscr{Var}$ in place of $2\mathscr{Cat}$ as the target category. This category has objects given by smooth quasi-coherent quasi-separated schemes and Hom-categories given by derived categories of sheaves of quasi-coherent modules (in the sense of [18]), immediately solving the first issue (see Definition 3.66). This category has been studied previously by e.g. Ganter and Kapranov [15] and Căldăraru and Willerton [12]. To give this 2-category a monoidal structure, we construct an isomorphic 2-category and give this 2-category a symmetric monoidal structure(see Chapter 3 and Proposition 3.69). This structure is such that the tensor product of schemes is given by their fibre product, which resolves the second issue.

The functor Φ constructed above using Fourier–Mukai transforms can be extended to give a 2-functor $\mathscr{Var} \to 2\mathscr{Cat}$ which sends an object X to $\mathscr{D}(X)$ and sends a sheaf to its Fourier–Mukai transform. This functor is a lax symmetric monoidal functor, with the functor $\mathscr{D}(X) \times \mathscr{D}(X) \to \mathscr{D}(X \times X)$ given by the *external product functor* (see Definition 3.25). Thus any strict functor $Z\colon \mathscr{B}ord_{1+1+1} \to \mathscr{V}ar$ induces a lax functor $Z\colon \mathscr{B}ord_{1+1+1} \to 2\mathscr{C}at$ by composition.

To analyse $(1+1+1)$-TQFTs valued in $\mathscr{V}ar$, we start by constructing $(1+1)$-TQFTs in the *truncated category* $\mathbf{H}\mathscr{V}ar$ (defined in Definition 4.2). Any $(1+1+1)$-TQFT induces a $(1+1)$-TQFT in the truncated category (Lemma 4.3). We show that it is possible to construct TQFTs which agree with the data given in Conjecture 1.2, and that in fact these are the unique geometric (in the sense of Definition 4.8) TQFTs (see Proposition 4.11). However, the only way it is possible for there to be an extended $(1+1+1)$-TQFT valued in $\mathscr{V}ar$ which induces such a $(1+1)$-TQFT is if the scheme X it is based on is trivial. Explicitly, this is the following theorem.

Theorem 1.3. *Let Z be a $(1+1+1)$-TQFT valued in $\mathscr{V}ar$ such that the induced $(1+1)$-TQFT is of the form Z_X defined in Proposition 4.7, where $X = Z(S^1)$. If X is of finite type and reduced, then it must be discrete. In this case, Z is isomorphic to a direct sum of extended TQFTs, each of which sends S^1 to a single point.*

The layout of the remainder of this thesis is as follows. Chapter 2 recalls the details of the GNF-algebra construction, and gives the proof that algebras based on 1-dimensional Frobenius algebras are integral domains. Chapter 3 recalls the traditional definition of the 2-category $\mathscr{V}ar$. It goes on to give this category a monoidal product by first constructing a double category whose underlying loose category is $\mathscr{V}ar$, before giving the double category a monoidal structure and lifting it to $\mathscr{V}ar$. Section 4.1 investigates Frobenius algebra objects valued in $\mathbf{H}\mathscr{V}ar$, the core truncation of $\mathscr{V}ar$, and shows that it is possible to construct Frobenius algebra objects using the functors described by Roberts and Willerton. Finally, Section 4.3 analyses the possibility of extending these Frobenius algebra objects upwards to $(1+1+1)$-TQFTs valued in $\mathscr{V}ar$. It first considers the simpler case of TQFTs valued in $\mathscr{A}ff\mathscr{V}ar$, the subcategory of affine schemes, before generalising the result to the whole category.

1.1 Category theory

1.1.1 Composition of natural transformations

Recall that natural transformations can be composed in various ways.

Definition 1.4. Let $\mathscr{C}, \mathscr{D}, \mathscr{E}$ be categories and $F_1, F_2 \colon \mathscr{C} \to \mathscr{D}$ and $G_1, G_2 \colon \mathscr{D} \to \mathscr{E}$ be functors. Let $\eta \colon F_1 \Rightarrow F_2$ and $\nu \colon G_1 \Rightarrow G_2$ be natural transformations. Then the *horizontal composition* of η and ν is given by the natural transformation

$$\nu * \eta \colon G_1 \circ F_1 \Rightarrow G_2 \circ F_2$$

with components

$$(\nu * \eta)_c = \nu_{F_2(c)} \circ G_1(\eta_c).$$

Definition 1.5. Let $\mathscr{C}, \mathscr{D}$ be categories and $F_1, F_2, F_3 \colon \mathscr{C} \to \mathscr{D}$ be functors. Let $\eta \colon F_1 \to F_2$ and $\nu \colon F_2 \to F_3$ be natural transformations. Then the *vertical composition* of η and ν is given by the natural transformation $\nu \circ \eta \colon F_1 \to F_3$ with components

$$(\nu \circ \eta)_c = \nu_c \circ \eta_c.$$

Definition 1.6. Let $\mathscr{C}, \mathscr{D}, \mathscr{E}, \mathscr{F}$ be categories and let $F \colon \mathscr{C} \to \mathscr{D}$, $G_1, G_2 \colon \mathscr{D} \to \mathscr{E}$ and $H \colon \mathscr{E} \to \mathscr{F}$ be functors. Let $\eta \colon G_1 \to G_2$ be a natural transformation. Then the *whiskering* of η by F is the natural transformation given by the horizontal composition

$$\eta * F = \eta * 1_F,$$

where 1_F is the identity natural transformation $F \Rightarrow F$. Similarly, the *whiskering* of η by H is the natural transformation

$$H * \eta = 1_H * \eta.$$

We abuse notation slightly and use the same symbol for the whiskering of a functor and a natural transformation as for the horizontal composition of two natural transformations; since the two operations act on different objects no confusion should arise.

The following result is trivial but will prove useful nonetheless. Intuitively, it states that the two orderings for applying two natural transformations to independent functors give equal transformations.

Lemma 1.7. *Let* $\xi\colon F_1 \Rightarrow F_2$ *and* $\eta\colon G_1 \to G_2$ *be natural transformations. Then there is a commutative diagram*

$$\begin{array}{ccc} G_1 \circ F_1 & \xrightarrow{G_1 * \xi} & G_1 \circ F_2 \\ \Big\downarrow{\scriptstyle \eta * F_1} & & \Big\downarrow{\scriptstyle \eta * F_2} \\ G_2 \circ F_1 & \xrightarrow{G_1 * \xi} & G_2 \circ F_2 \end{array}$$

where recall the composition of a functor with a natural transformation denotes whiskering (Definition 1.6)

Proof. Both compositions of transformations are equal to the horizontal composition $\eta * \xi$. □

1.1.2 Cartesian monoidal categories

We recall the following folklore proposition [3].

Proposition 1.8. *Let* $\mathscr{C}$ *be a category which has all products. Then* $\mathscr{C}$ *can be given a symmetric monoidal structure by taking the monoidal product to be the product of objects, and the braiding to the morphisms* $X \times Y \to Y \times X$ *induced by the universal property of the product.*

Definition 1.9. Let $\mathscr{C}$ be a category which has all products. Then $\mathscr{C}$, viewed as a symmetric monoidal category as in Proposition 1.8, is called a *Cartesian monoidal* *category.*

1.1.3 2-category theory

We use the term 2-category to refer to what is sometimes referred to as a weak 2-category; that is, composition is associative and unital only up to coherent 2-isomorphism. Leinster [29] gives an overview of the 2-category notions that will be used here (note Leinster uses the term "bicategory" for what we refer to as a 2-category, and the term 2-category for a strict 2-category); full details have been given by Schommer-Pries [42, Appendix A].

Definition 1.10. [42, Definition A.1] A 2-category $\mathscr{C}$ consists of:

- a collection of objects $\mathrm{Obj}(\mathscr{C})$;
- for any pair $c, d \in \mathscr{C}$, a category $\mathrm{Hom}_{\mathscr{C}}(c, d)$;
- composition functors $c_{cde} \colon \mathrm{Hom}_{\mathscr{C}}(d, e) \times \mathrm{Hom}_{\mathscr{C}}(c, d) \to \mathrm{Hom}_{\mathscr{C}}(c, e)$;
- identity 1-morphisms $I_c \in \mathrm{Hom}_{\mathscr{C}}(d, e) \times \mathrm{Hom}_{\mathscr{C}}(c, d) \to \mathrm{Hom}_{\mathscr{C}}(c, e)$;
- natural isomorphisms

  $$\alpha_{cdef} \colon c_{cdf} \circ (c_{def} \times \mathrm{Id}) \to c_{cef} \circ (\mathrm{Id} \times c_{cde});$$

  and
- a pair of 2-cells

  $$r_f \colon f \circ I_c \to f$$

  and

  $$l_f \colon I_d \circ f \to f$$

  for any $f \colon c \to d$,

such that the pentagon diagram

$$\begin{array}{ccccc}
((k \circ h) \circ g) \circ f & \longrightarrow & (k \circ (h \circ g)) \circ f & \longrightarrow & k \circ ((h \circ g) \circ f) \\
\downarrow & & & & \downarrow \\
(k \circ h) \circ (g \circ f) & & \longrightarrow & & k \circ (h \circ (g \circ f))
\end{array}$$

and the triangle diagram

$$\begin{array}{ccc}
(g \circ I) \circ f & \longrightarrow & g \circ (I \circ f) \\
& \searrow \quad \swarrow & \\
& g \circ f &
\end{array}$$

commute.

Definition 1.11. [42, Definition A.5] A *lax 2-functor* $F \colon \mathscr{C} \to \mathscr{D}$ between 2-categories is the data of:

- a function $\mathrm{Obj}(\mathscr{C}) \to \mathrm{Obj}(\mathscr{D})$;
- functors $F_{ab} \colon \mathrm{Hom}_{\mathscr{C}}(c, d) \to \mathrm{Hom}_{\mathscr{D}}(F(c), F(d))$;
- natural 2-morphisms

  $$\phi_{gf} \colon F(g) \circ F(f) \to F(g \circ f)$$

  and

  $$\phi_a \colon I_{F(a)} \to F(I_a)$$

  for any composable pair of morphisms $f, g \in \mathrm{Mor}(\mathscr{C})$ and any $a \in \mathscr{C}$;

such that the diagrams

$$\begin{array}{ccccc}
(F(h) \circ F(g)) \circ F(f) & \longrightarrow & F(h \circ g) \circ F(f) & \longrightarrow & F((h \circ g) \circ f) \\
\downarrow & & & & \downarrow \\
F(h) \circ (F(g) \circ F(f)) & \longrightarrow & F(h) \circ F(g \circ f) & \longrightarrow & F(h \circ (g \circ f))
\end{array},$$

$$\begin{array}{ccc} F(f) \circ (I_F(d)) & \longrightarrow & F(f) \\ \downarrow & & \uparrow \\ F(f) \circ F(I_d) & \longrightarrow & F(f \circ I_d) \end{array}$$

and

$$\begin{array}{ccc} I_{F(c)} \circ F(f) & \longrightarrow & F(I_c) \circ F(f) \\ \downarrow & & \downarrow \\ F(f) & \longleftarrow & F(I_c \circ f) \end{array}$$

commute.

For the definitions of symmetric monoidal 2-category and symmetric monoidal functors between such categories, see [42, Definitions 2.3 and 2.5].

1.2 TQFTs

Bordism categories

Recall that a $(d+1)$-dimensional TQFT is a functor from a category of bordisms to a symmetric monoidal category, and that we assume all manifolds (and hence bordisms and TQFTs) are oriented. Recall the definition of a $(d+1)$-bordism.

Definition 1.12. [32, Definition 1.5] An (oriented) *bordism* between two closed smooth (oriented) d-manifolds M_0 and M_1 is a tuple $(W; V_0, V_1; h_0, h_1)$ where W is a compact $(d+1)$-manifold with a decomposition of its boundary as $\partial W = V_0 \sqcup V_1$, and $h_0\colon M_i \times [0,1) \to W$ and $h_1\colon M_1 \times (-1,0]$ are orientation-preserving embeddings.

The boundary component V_0 is referred to as the *incoming* boundary component, and V_1 as the *outgoing* boundary component.

Definition 1.13. Let $(W; V_0, V_1; h_0, h_1)$ and $(W'; V_0', V_1'; h_0', h_1')$ be two bordisms. They are called *equivalent* if there is a diffeomorphism $g\colon W \to W'$ such that $g(V_i) = V_i'$ and $g \circ h_i = h_i'$ for $i \in \{0,1\}$.

Two bordisms $(W; V_0, V_1; h_0, h_1)$ and $(W'; V_1, V_1'; h_0', h_1')$ can be glued together to form a bordism from V_0 to V_1'. The smooth structure is well-defined due to the choice of collar neighbourhoods.

Definition 1.14. The $(d+1)$-dimensional bordism category $\mathbf{Bord}_{d+1}$ has as objects d-manifolds and as morphisms $(d+1)$-bordisms considered up to equivalence.

Bordism categories can also be constructed as higher categories.

Sketch Definition 1.15. [42] The $(d+1+1)$-dimensional bordism category $\mathscr{Bord}_{d+1+1}$ has as objects smooth oriented d-manifolds and as 1-morphisms $(d+1)$-bordisms. The 2-morphisms between two 1-morphisms B and B' are diffeomorphism classes of bordisms (possibly now with corners) between B and B' which are trivial along the boundary $\partial B \simeq \partial B'$.

Frobenius algebras

The introduction recalled the result that $(1+1)$-TQFTs valued in **Vect** are classified by commutative Frobenius algebras. This can be generalised by introducing the following notion.

Definition 1.16. [24] Let $\mathscr{C}$ be a monoidal category with unit I. A *Frobenius algebra* is a tuple $(A, \mu, \delta, \epsilon, \tau)$ such that:

1. the tuple (A, μ, ϵ) is a unital monoid with multiplication $\mu\colon A \otimes A \to A$ and unit $\epsilon\colon I \to A$; and
2. the tuple (A, δ, τ) is a counital comonoid with comultiplication $\delta\colon A \to A \otimes A$ and counit $\tau\colon A \to I$; and
3. the Frobenius identities hold:

$$(\mathrm{Id}_A \otimes \mu) \circ (\delta \otimes \mathrm{Id}_A) = \delta \circ \mu = (\mu \otimes \mathrm{Id}_A) \circ (\mathrm{Id}_A \otimes \delta)\,. \tag{1.3}$$

If $\mathscr{C}$ is a symmetric monoidal category, the algebra is *commutative* if $\mu \circ \sigma_{A,A} = \mu$, where $\sigma_{A,A} \colon A \otimes A \to A$ is the braiding.

Lemma 1.17. *Fix a dimension d, and let $Z \colon \mathbf{Bord}_{d+1} \to \mathbf{Vect}$ be a $(d+1)$-TQFT. Then $Z(S^d)$ has the structure of a commutative Frobenius algebra.*

Proof. Note that it is enough to show that S^d has the structure of a Frobenius algebra object in $\mathbf{Bord}_{d+1}$, as then the images of the structure morphisms give $Z(S^d)$ the structure of a Frobenius algebra object in **Vect**. In **Vect**, the definition in Definition 1.16 is equivalent to the usual definition in terms of Frobenius forms [45]. Let $\epsilon = D^{d+1}$ be the $(d+1)$-ball, viewed as a bordism $\emptyset \to S^d$, and let μ be the twice-punctured $(d+1)$-ball, viewed as a bordism from $S^d \sqcup S^d$ (formed of the boundaries of the removed balls) to S^d (viewed as the boundary of the original ball). It is clear that μ defines an associative multiplication, as $\mu \circ (\mu \times \mathrm{Id})$ and $\mu \circ (\mathrm{Id} \times \mu)$ both give thrice-punctured balls. Further, ϵ acts as a unit for μ, since inserting a ball into one of the punctures of μ gives the bordism $S^d \times I$, which is exactly the identity bordism for S^d.

The comultiplication and counit are taken to have the same underlying manifolds as the multiplication and unit, but with the incoming and outgoing boundaries exchanged. It is straightforward [5] to see that these satisfy the Frobenius identities. □

Definition 1.18. Let Z be a $(d+1)$-TQFT. Then $A = Z(S^d)$ has the structure of a Frobenius algebra object. The TQFT Z is said to be *based* on A.

Mapping cylinders and trace bordisms

Two important classes of bordisms are mapping cylinder and trace bordisms. Both of these are useful in the classification of $(2+1)$-dimensional TQFTs [21].

Definition 1.19. Let M be a d-dimensional manifold and $f \in \mathrm{Diffeo}(M)$ be a diffeomorphism of M. Associated to any such diffeomorphism is a *mapping cylinder* c_f, given by the tuple $(M \times I, M \times \{0\}, M \times \{1\}, h_0, h_1)$, where

$$\begin{aligned} h_0 \colon M \times [0,1) &\to M \times I, \\ (m,x) &\mapsto (m, x/3) \end{aligned}$$

and

$$\begin{aligned} h_1 \colon M \times (-1,0] &\to M \times I, \\ (m,x) &\mapsto (m, 1+x/3). \end{aligned}$$

Mapping cylinders give an action of the mapping class group $\mathrm{Aut}(M)$ on the space $Z(M)$, by the representation $f \mapsto Z(c_f)$. In the cases $d=0$ and $d=1$, there are no non-trivial diffeomorphisms, and so the action is trivial. The lowest dimensional interesting case is $d=2$, where they give mapping class group representations on GNF-algebras, and play an important role in the classification of 2-dimensional theories.

A second class of bordisms which also play an important role in this classification is the class of bordisms induced by the trace of surgery. This class is defined by handle attachment. We use the handle-attachment procedure described by Kosinski [26, Section VI.6], which has two benefits. The first is that the resulting manifold is smooth, so there is no need for smoothing out the corners that result from usual handle attachment. The second is that it is possible to attach the handle in such a way that the parametrisation of a collar neighbourhood of the boundary on the original manifold extends to a parametrisation of a collar neighbourhood of the boundary of the result of the attaching the handle.

Definition 1.20. Let M be a d-dimensional manifold, $d = p+q$, and let $\phi\colon S^p \times D^q \to M$ be an embedding. Let W be the result of attaching a handle (following the construction of Kosinski [26, Section VI.6]) to $M \times I$ along the embedding $i\colon x \mapsto (\phi(x), 1)$. Let $V_0 = M \times \{0\}$ and let $V_1 = (M \times I \setminus \phi(S^p \times \{0\})) \cup D^{p+1} \times S^{q-1}$ be the result of performing surgery on boundary components of W. Finally, let h_0 and h_1 be as for the mapping cylinder. Then the *trace* of the surgery is the cobordism $(W; V_0, V_1; h_0, \overline{h_1})$, where $\overline{h_1}$ is an extension of the parametrisation $h_1|_{U_1 \times I}$ to $V_1 \times I$.

1.3 Algebraic geometry

This section provides a summary of the algebraic geometry that is used later. More details on the definitions and results can be found in any textbook on algebraic geometry e.g. [18, 46]. Recall that the basic object in algebraic geometry is that of an *affine scheme* $\operatorname{Spec}(R)$, a ringed space constructed from a ring R by topologising the set of its prime ideals. A *sheaf of modules* on an affine scheme X is an assignment of an $\mathscr{O}_X(U)$-module to each open set $U \subset X$, satisfying a gluing and locality property. A sheaf of modules $\mathscr{E}$ on X is called *quasi-coherent* if it has a local presentation; that is, for any point of X there is an open neighbourhood U such that there is an exact sequence of modules

$$\mathscr{O}_X^{\oplus I}|_U \to \mathscr{O}_X^{\oplus J}|_U \to \mathscr{E} \to 0.$$

All sheaves considered here will in fact be quasi-coherent sheaves of modules, which will simply be called sheaves. Morphisms of sheaves are defined locally; that is, a morphism $f\colon \mathscr{E} \to \mathscr{F}$ is a collection of maps $f_U\colon \mathscr{E}(U) \to \mathscr{F}(U)$ that are compatible with restrictions. This gives a category $\mathscr{SH}_X$ of (quasi-coherent) sheaves over a fixed scheme X. If $X = \operatorname{Spec}(R)$ is affine, then $\mathscr{SH}_X$ is equivalent to the category of modules over R. This equivalence can be constructed by taking the global sections of a sheaf of modules.

These notions generalise to (not necessarily affine) *schemes*, which are locally ringed spaces that are locally affine. Many results involving such objects can be deduced by showing that they hold on affine schemes, and then deducing that if the result holds locally, it should hold globally. This is largely the approach taken here, so most of this section concerns itself with affine schemes.

Given a morphism of schemes $f\colon X \to Y$, there is an adjoint pair of functors (f^*, f_*) between $\mathscr{SH}_X$ and $\mathscr{SH}_Y$:

$$f_*\colon \mathscr{SH}_X \leftrightarrows \mathscr{SH}_Y\colon f^*.$$

If $X = \operatorname{Spec}(R)$, $Y = \operatorname{Spec}(S)$, and $f\colon X \to Y$ is induced by the ring morphism $f^\sharp\colon S \to R$, the direct image is given explicitly on global sections by

$$\begin{aligned} f_*^\sharp\colon S\textbf{-mod} &\to R\textbf{-mod} \\ M &\mapsto M_R \end{aligned}$$

where the notation M_R means the module M viewed as an R-module by $r \cdot m = f^\sharp(r) \cdot m$. The inverse image is given by

$$\begin{aligned} f^{\sharp,*}\colon R\textbf{-mod} &\to S\textbf{-mod} \\ N &\mapsto N \otimes_R S_R. \end{aligned}$$

1.3.1 Derived categories in algebraic geometry

We recall the construction of derived functors, which will provide a setting in which the projection and base-change formulas (Proposition 1.28 and Proposition 1.29) will hold. Many of these definitions are sketches; full details can be found in the books by Weibel [48] or Huybrechts [19], with the latter having more of a focus on algebraic geometry.

Definition 1.21. A category $\mathscr{C}$ is an *abelian category* if:

- it is enriched over the category **Ab** of abelian groups;
- it has a zero object;
- it has all kernels and cokernels; and
- every monomorphisms is the kernel of some morphism, and dually every epimorphisms is the cokernel of some morphism.

Definition 1.22. Let $F\colon \mathscr{C} \to \mathscr{D}$ be a functor between abelian categories. If for any exact sequence

$$0 \longrightarrow A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \longrightarrow 0 \tag{1.4}$$

in $\mathscr{C}$, the sequence

$$F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0$$

is exact, then F is *right exact.*

If instead for any exact sequence as in Equation (1.4), the sequence

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C)$$

is exact, then F is *left exact.*

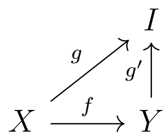
Definition 1.23. Let $\mathscr{C}$ be an abelian category. Then its *derived category* $\mathscr{D}(\mathscr{C})$ is formed from the category of chain complexes by formally inverting all quasi-isomorphisms.

Definition 1.24. Let $\mathscr{C}$ be an abelian category. An object P is *projective* if for any epimorphism $e\colon E \to X$ and any morphism $f\colon P \to X$, there is a morphism $f'\colon P \to E$ such that the diagram

$$\begin{array}{ccc} & & E \\ & \overset{f'}{\nearrow} & \big\downarrow{\scriptstyle e} \\ P & \xrightarrow{\ f\ } & X \end{array}$$

commutes.

Dually, any object I is *injective* if for any monomorphism $f\colon X \to Y$ and morphism $g\colon X \to I$, there is a morphism $g'\colon Y \to I$ such that the diagram



commutes.

Definition 1.25. An abelian category $\mathscr{C}$ is said to have *enough projectives* if for any object M there is a projective object P and an epimorphism $P \to M$. It has *enough injectives* if for any M there is an injective object I and a monomorphism $M \to I$.

Definition 1.26. A *projective resolution* of $A \in \mathscr{C}$ is an exact sequence

$$\cdots \longrightarrow P_i \longrightarrow \cdots \longrightarrow P_1 \longrightarrow A \longrightarrow 0$$

where each P_i is a projective module.

Definition 1.27. Let $F\colon \mathscr{C} \to \mathscr{D}$ be a right exact functor between abelian categories, where $\mathscr{C}$ has enough projectives. Then its *left derived functor* $LF\colon \mathscr{D}(\mathscr{C}) \to \mathscr{D}(\mathscr{D})$ is defined by sending an object A to the chain complex formed by applying F termwise to a projective resolution of A.

A similar process can be used to define a *right derived functor*, by applying a left exact functor termwise to an injective resolution of an object.

Since $f^* \dashv f_*$, it follows that f_* preserves monomorphisms and so is f_*, and hence we can form its derived functor. Similarly, f^* is right exact, so we can form its left derived functor. In the remainder of this thesis, f_* and f^* will refer to the derived functors (unless otherwise noted). More details on these constructions can be found in e.g. [48].

Proposition 1.28. *[30, Proposition 3.9.4] Let $f\colon X \to Y$ be a quasi-compact quasi-separated morphisms of schemes. Then there is an isomorphism of quasi-coherent sheaves*

$$f_*(f^*(\mathscr{E}) \otimes \mathscr{F}) \cong \mathscr{E} \otimes f_*(\mathscr{F})$$

which is natural in both $\mathscr{E}$ and $\mathscr{F}$.

Proposition 1.29. *[30, Proposition 3.9.5] Let*

$$\begin{array}{ccc} A & \xrightarrow{g'} & B \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{g} & V \end{array}$$

be a Cartesian diagram in the category of schemes. Suppose the map g is flat, and f is quasi-compact and quasi-separated. Then the base-change morphism, induced by the adjunction of direct images and inverse images, is an isomorphism $g^* \circ f_* \cong f'_* \circ g'^*$.

Chapter 2

GNF algebras

This chapter recalls the definitions of a GNF-algebra from [21]. This definition will be used to show that if A is a GNF-algebra based on a 1-dimensional Frobenius algebra, then the algebra must be an integral domain. This provides some progress to showing that any (2+1)-TQFT is extendable in $2\mathscr{V}\!ect$ (giving a positive answer to Question 1.1) since any (1+1+1)-TQFT valued in $2\mathscr{V}\!ect$ must induce a GNF-algebra which is an integral domain.

2.1 GNF algebras

In a similar manner to $(1+1)$-dimensional TQFTs, $(2+1)$-dimensional TQFTs have been classified [21]. Let Σ_g be the connect sum $\#^g(S^1 \times S^1)$ of g tori; these are called *standard surfaces*. Any object of the bordism category $\mathbf{Bord}_{2+1}$ is diffeomorphic to a disjoint union of such surfaces. Consequently, the classification includes a choice of countably many vector spaces $A_g = Z(\Sigma_g)$, which are combined into a single graded vector space $A = \bigoplus A_g$. This can be given a graded product structure by taking the images of the bordisms $\Sigma_i \sqcup \Sigma_j \to \Sigma_{i+j}$ given by the trace of performing surgery on two points, one in each component (the bordism $\mu_{i,j}$ in Figure 2.1). This product is unital, with the unit given by D^3, viewed as a bordism $\emptyset \to S^2$ (the bordism ϵ in

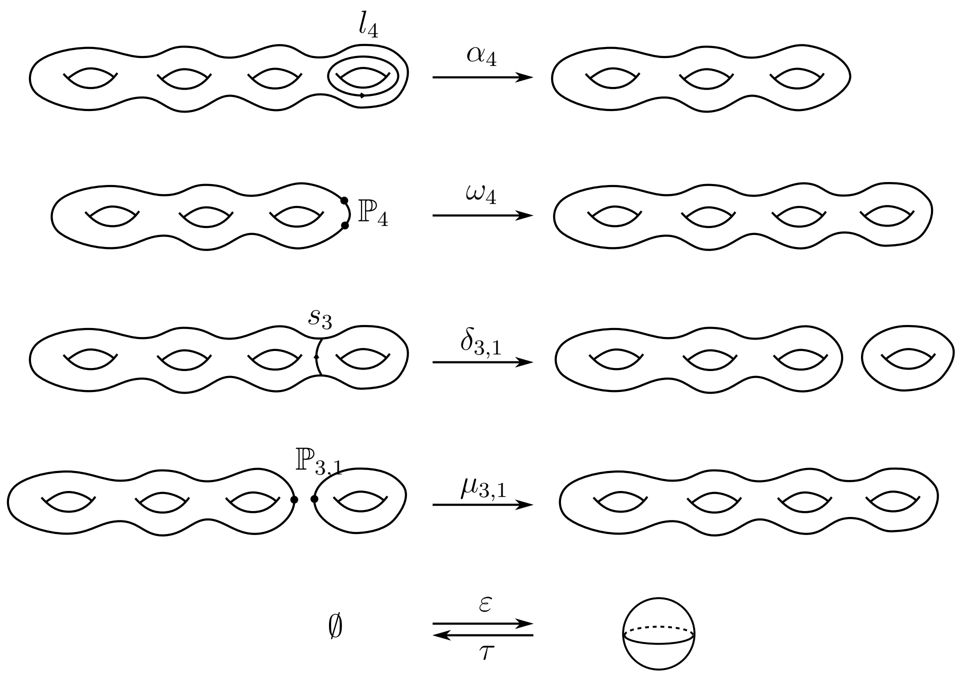


Figure 2.1: The bordisms that give rise to the structure of a GNF*-algebra. The bordisms α_i, ω_j, $\delta_{i,j}$ and $\mu_{i,j}$ are given by the trace of surgery along the marked curve in the left-hand diagram; the bordisms $\mathscr{E}$ and τ are given by the ball D^3, viewed either as a bordism $\emptyset \to S^2$ or vice versa. Reproduced from [21] with permission.

Figure 2.1). The algebra A is also a coalgebra, with coproduct given by summing the contributions from the morphisms $A_{i+j} \to A_i \otimes A_j$ given by the image of the bordism $\delta_{i,j}$ in Figure 2.1. Further, for any g there is a automorphism of Σ_g given by rotating Σ_g about the z-axis by π radians. Collating these data, along with the conditions that they satisfy, gives a GNF*-algebra.

Definition 2.1. Let $A = \oplus_{i \geq 0} A_i$ be a graded coalgebra with graded coproduct δ. Let $\pi_i \colon A \to A_i$ be the map projecting to the i-th homogeneous component, and let $\delta_{i,j} = (\pi_i \otimes \pi_j) \circ \delta$. A *partial left counit* for δ is a map $\tau \colon A_0 \to k$ such that $(\tau \otimes \mathrm{Id}_{A_j}) \circ \delta_{0,j} = \mathrm{Id}_{A_j}$.

Definition 2.2. [21, Definition 5.1] A *graded involutive nearly Frobenius algebra* $\mathscr{A}$ is a graded algebra $A = \oplus_{i \geq 0} A_i$, with unit ϵ, along with a coassociative coproduct δ, with a partial left counit τ and a grading-preserving involution $*$. Let $\mu_{i,j} \colon A_i \otimes A_j \to A_{i+j}$ be the multiplication map on graded components and similarly let $\delta_{i,j} = (\pi_i \otimes \pi_j) \circ \delta$

be the components of the coproduct map. These data must satisfy the following:

1. each homogeneous component of $\mathscr{A}$ is finite-dimensional;

2. the involution is the identity for elements of degree 0 and 1, and is an anti-automorphism (so $* \circ \mu_{i,j} = \mu_{j,i} \circ T_{i,j}$ and $\delta_{i,j} \circ * = T_{j,i} \circ \delta_{j,i}$, where $T_{i,j}(x \otimes y) = y^* \otimes x^*$ for $x \in A_i, y \in A_j$); and

3. the algebra satisfies the *Frobenius condition*

$$\begin{array}{ccc} A_i \otimes A_{j+k} & \xrightarrow{\mathrm{Id}_{A_i} \otimes \delta_{j,k}} & A_i \otimes A_j \otimes A_k \\ \Big\downarrow{\scriptstyle \mu_{i,j+k}} & & \Big\downarrow{\scriptstyle \mu_{i,j} \otimes \mathrm{Id}_{A_k}} \\ A_{i+j+k} & \xrightarrow{\delta_{i+j,k}} & A_{i+j} \otimes A_k \end{array} . \tag{2.1}$$

The Frobenius condition in this definition appears to be asymmetric in the following sense. Let $z = \mu_{r,s}(x \otimes y)$ be an element of A_{r+s} that can be written as a product. Then Equation (2.1) allows the computation of $\delta_{r+t,s-t}(z)$ in terms of $\delta_{t,s-t}(y)$ but says nothing about how to compute $\delta_{r-t,s+t}(z)$; one would expect that this could be expressed in terms of $\delta_{r-t,t}(x)$. This is in fact the case; a "mirrored" version of the Frobenius identity also holds due to the involution.

Lemma 2.3. *[21, Lemma 5.6] If $\mathscr{A}$ is a GNF*-algebra, there is a commutative diagram*

$$\begin{array}{ccc} A_k \otimes A_{j+i} & \xrightarrow{\delta_{k,j} \otimes \mathrm{Id}_{A_i}} & A_k \otimes A_j \otimes A_i \\ \Big\downarrow{\scriptstyle \mu_{k,j+i}} & & \Big\downarrow{\scriptstyle \mathrm{Id}_{A_k} \otimes \mu_{j,i}} \\ A_{i+j+k} & \xrightarrow{\delta_{k,i+j}} & A_k \otimes A_{i+j} \end{array} . \tag{2.2}$$

The GNF*-algebra on its own does not capture the full structure of a TQFT. Each of the maps defined above have degree 0. In terms of the bordisms to which they correspond, the genus of the incoming boundary is the same as that of the outgoing boundary. However clearly not all bordisms are of this form: for example, the solid torus minus a ball. The image of this map should have degree 1. This construction can be generalised to surfaces of arbitrary genus: starting from a 2-manifold Σ, add a 1-handle to $\Sigma \times I$ along two (framed) points chosen in $\Sigma \times \{1\}$. Let V_0 (the incoming boundary) be $\Sigma \times \{0\}$ and let V_1 (the outgoing boundary) be the remaining boundary component. This component is Σ with a 1-handle added; that is, a surface with genus one greater than Σ. Hence the image of this bordism should be a degree 1 map. Adding the handle to points chosen in $\Sigma \times \{0\}$ would give a bordism whose image is a degree -1 map. This motivates the following definition.

Definition 2.4. [21, Definition 5.2] A *modular splitting* of a GNF*-algebra $\mathscr{A}$ consists of a pair (ω, α) of endomorphisms of degree 1 and -1 respectively, such that they are both module homomorphisms and

$$\begin{aligned} \delta \circ \alpha &= (\mathrm{Id} \otimes \alpha) \circ \delta, \\ \delta \circ \omega &= (\mathrm{Id} \otimes \omega) \circ \delta, \text{ and} \\ \alpha \circ \omega &= \mathrm{Id}_A . \end{aligned}$$

This is still not enough to capture the entire structure: none of the information above captures the action of mapping cylinders. As such, it is natural to expect to have an action of the mapping class group $\mathrm{MCG}(X) = \mathrm{Diffeo}(X)/\sim$ on the vector space associated to X, where $\sim$ identifies isotopic maps[1]. This is formalised with the idea of a *J-algebra*: a GNF*-algebra with a modular splitting and an action of the mapping class group. The details of how to make this precise are given in [21, Definition 5.13].

Definition 2.5. A morphism of graded vector spaces $\phi\colon \mathscr{A} \to \mathscr{B}$ is said to *intertwine*

[1] Recall that pseudo-isotopic maps will have the same action. However, in keeping with the definition of the mapping class group the relation used here is isotopy; as a result, some of the information of this action is superfluous.

the maps

$$f_{\mathscr{A}} \colon \mathscr{A}^{\otimes n_1} \to \mathscr{A}^{\otimes n_2}$$

and

$$f_{\mathscr{B}} \colon \mathscr{B}^{\otimes n_1} \to \mathscr{B}^{\otimes n_2}$$

if

$$\phi^{\otimes n_2} \circ f_{\mathscr{A}} = f_{\mathscr{B}} \circ \phi^{n_1}.$$

Definition 2.6. A *morphism* of GNF*-algebras is a morphism of graded vector spaces $\phi \colon \mathscr{A} \to \mathscr{B}$ that intertwines the product, coproduct, unit, counit and involution maps.

A morphism of J-algebras is a morphism of GNF*-algebras that also interwines the modular splitting and mapping class group actions.

Definition 2.7. The category **J-Alg** has J-algebras as objects and J-algebra morphisms as morphisms.

Theorem 2.8. *[21] There is an equivalence between the symmetric monoidal category of* $(2+1)$*-dimensional TQFTs and* **J-Alg***.*

2.2 Canonical simple GNF*-algebra

This section gives a construction of a particular family of GNF*-algebras which have $\dim A_0 = 1$. Proposition 2.18 will show that in fact all GNF*-algebras satisfying this condition are isomorphic to a GNF*-algebra of this form.

Definition 2.9. Suppose $\operatorname{char} k = 0$. Let $\lambda \in k^\times$ be a unit of k and I and J be disjoint sets of formal variables with each variable having an associated strictly positive degree, and J having no elements of degree 1. Let $\mathscr{B} = I \cup J$ and $\mathscr{B}_n = \{x \in \mathscr{B} : \deg x = n\}$ and suppose that $\mathscr{B}_n$ is finite. Let $A = k\langle \mathscr{B} \rangle$ be the graded free algebra generated by $\mathscr{B}$.

Define a coproduct map $\delta\colon A \to A \otimes A$ as follows. If $x \in \mathscr{B}_n$, then $\delta(x) = \lambda \otimes x + x \otimes \lambda$ (viewing λ as an element of $\mathscr{A}_0$). Given this definition, extend δ using linearity and the Frobenius condition to A. Define the counit $\tau\colon A_0 \to k$ by $1 \mapsto \lambda^{-1}$.

Define the involution $*$ by requiring that elements of I (resp. J) are eigenvectors with eigenvalue 1 (resp. -1). This extends using linearity to sums of elements of $\mathscr{B}$. It can be extended to A by requiring that it satisfies the anti-automorphism condition $\mu(x \otimes y)^* = \mu(y^* \otimes x^*)$.

The tuple $(A, \mu, \epsilon, \delta, \tau, *)$ defines a *canonical simple GNF*-algebra.*

Before giving a proof that this construction does indeed give a GNF*-algebra, we record a straightforward lemma:

Lemma 2.10. *If* $\delta_{0,k} = \lambda \otimes \mathrm{Id}$, *then the map* $\nu_{i,k} = \delta_{i,k} \circ \mu_{i,k}$ *is given by* $\nu_{i,k} = \lambda\,\mathrm{Id}_{A_i} \otimes \mathrm{Id}_{A_k}$.

Proof. This follows by applying the Frobenius identity (Equation (2.1)) with $j = 0$. □

Corollary 2.11. *In the setting of Lemma 2.10, the map* $\mu_{i,k}$ *is injective.*

Proposition 2.12. *The tuple constructed in Definition 2.9 gives a GNF*-algebra.*

Proof. Each vector space A_n is finite dimensional as there are only finitely many formal variables of degree at most n. By construction the product map is graded linear and associative, and the unit is indeed a left unit. Similarly the coproduct is graded linear and the counit is a partial left counit. The Frobenius condition holds as the coproduct was chosen to be precisely the map such that it does hold. As J has no elements of degree less than 2, the involution is the identity on A_0 and A_1. Thus all that is left is to check that the coproduct is coassociative and that the involution is an anti-automorphism.

Fix i, j, k and let $i+j+k=n$. Let

$$D_1 = (\mathrm{Id}_i \otimes \delta_{j,k}) \circ \delta_{i,j+k}$$

$$D_2 = (\delta_{i,j} \otimes \mathrm{Id}_k) \circ \delta_{i+j,k}$$

Let $\mu_{i,j,k} = \mu_{i,j+k} \circ (\mathrm{Id}_{A_i} \otimes \mu_{j,k})$. Consider the map $S = \lambda^{-2}\mu_{i,j,k} \circ D_1$. Note that by Lemma 2.10, $\delta_{i,k} \circ \mu_{i,k} = \lambda^{-1} \mathrm{Id}_{A_i} \otimes \mathrm{Id}_{A_k}$. Thus its conclusion is valid, and applying it gives $S^2 = S$ so S is an idempotent. As such, there is a decomposition $A_n = \ker S \oplus \mathrm{Im}\, S$. On $\mathrm{Im}\, S$, coassociativity holds since Lemma 2.10 implies $D_1 \circ \mu_{i,j,k} = D_2 \circ \mu_{i,j,k} = \lambda^2 \mathrm{Id}_{A_i \otimes A_j \otimes A_k}$. As P is an integral domain, $\ker S = \ker D_1$, so δ is coassociative if $\ker D_1 \subset \ker D_2$. Say $v \in \ker D_1 \setminus \{0\}$; without loss of generality assume v is a product of elements of $\mathscr{B}$. Suppose that v has the form $v = \mu_{i,j+k}(x \otimes y)$. Then

$$0 = D_1(v) = \lambda x \otimes \delta_{j,k}(y)$$

so $\delta_{j,k}(y) = 0$ since μ is injective. Applying the Frobenius identity

$$\delta_{i+j,k} \circ \mu_{i,j+k}(x \otimes y) = (\mu_{i,j} \otimes \mathrm{Id}_k) \circ (\mathrm{Id}_i \otimes \delta_{j,k})(x \otimes y) = 0$$

so $D_2(v) = 0$. If v is not of this form, then it must be of the form $v = \mu_{r,s,t}(x \otimes y \otimes z)$ where $r < i$, $y \in B_s$. Say initially $r+s > i+j$; then applying $\delta_{i+j,k}$ and the Frobenius identity gives

$$\mu_{i,i-r,r+s-i,t}(x \otimes \delta_{i-r,r+s-i}(y) \otimes z) = 0$$

so $D_2(v) = 0$. Otherwise, applying D_2 and the Frobenius identity gives

$$\mu_{r,i-r,r+s-i,i+j-r-s,k}(x \otimes \delta_{i-r,r+s-i}(y) \otimes \delta_{i+j-r-s,k}(z)) = 0$$

and so again $D_2(v) = 0$.

The final condition to check is that the involution is an anti-automorphism. It obeys the rule for products by definition, so consider the coproduct. For $x \in \mathscr{B}_n$, $\delta_{i,n-i}(x) = 0 = \delta_{n-i,i}(x*)$ for $i \notin \{0, n\}$. For $i = 0$ or $i = n$ the result follows from the definition of δ and that $*$ fixes A_0 (in particular $\mathbb{1}^* = \mathbb{1}$). On products, the result follows by the Frobenius identity. $\square$

Lemma 2.13. *Let $\mathscr{A} = \mathscr{A}^{\lambda}_{I,J}$ be a canonical simple GNF* algebra. Let $\mathscr{B}$ be any GNF* algebra. If the $\tau_B \circ \epsilon_B(1) \neq \lambda^{-1}$ then there are no maps between the two. Otherwise, there is a 1-1 correspondence*

$$\{\Phi\colon \mathscr{A} \to \mathscr{B}\} \longleftrightarrow \left\{ \phi : I \sqcup J \to \mathscr{B} : \begin{array}{ll} \phi(x)^* = \phi(x) & x \in I \\ \phi(y)^* = -\phi(y) & y \in J \\ \delta_{i,n-i}(\phi(x)) = 0 & x \in \mathscr{B}_n, i = 1, \ldots, n-1 \end{array} \right\}$$

Proof. Say $\Phi\colon \mathscr{A} \to \mathscr{B}$ is a morphism. Then there is a map $\Phi_0\colon A_0 \to B_0$ such that $\Phi_0 \epsilon_A = \epsilon_B$; and $\tau_B \circ \Phi_0 = \tau_A$, so

$$\tau_B \circ \epsilon_B = \tau_B \circ \Phi_0 \circ \epsilon_A = \tau_A \circ \epsilon_A.$$

In particular, $\lambda^{-1} = \tau_A \circ \epsilon_A(1) = \tau_B \circ \epsilon_B(1)$.

Given a homomorphism of GNF*-algebras, the map ϕ is constructed by restriction. Thus it remains to give an inverse to this process.

Suppose ϕ is such a map. Now $\langle \mathscr{B} \rangle = \mathscr{A}$, so this can be extended algebraically to $\mathscr{A} \setminus A_0$. Define $\Phi_0\colon A_0 \to B_0$ by $\mathbb{1}_A \mapsto \mathbb{1}_B$. This map will be graded as the original map ϕ respects the grading.

By construction Φ intertwines μ and ϵ. As $\tau_A \circ \epsilon_A(1) = \lambda^{-1} = \tau_B \circ \epsilon_B(1)$, Φ intertwines τ. It also respects involution by definition of ϕ, leaving only the map δ to check. Now for any generator $x \in I \sqcup J$, $\delta(x) = \lambda \otimes x + x \otimes \lambda$, so $\Phi \circ \delta(x) = \Phi(x) \otimes \lambda + \lambda \otimes \Phi(x)$ as required (since Φ preserves the unit).

It is easy to see that these constructions are inverses. □

Lemma 2.14. *The GNF*-algebras* $\mathscr{A}^{\lambda}_{I,J}$ *and* $\mathscr{A}^{\lambda'}_{I',J'}$ *are isomorphic if and only if* $\lambda' = \lambda$ *and both* I *and* I' *and* J *and* J' *have the same number of elements of each degree.*

Proof. The "if" part of the statement is clear.

Suppose $\Phi\colon \mathscr{A}^{\lambda}_{I,J} \to \mathscr{A}^{\lambda'}_{I',J'}$ is an isomorphism. Then by Lemma 2.13 $\lambda = \lambda'$.

Let $U_n \subset A_n$ be the subspace spanned by non-trivial products of elements of A. Consider A_n/U_n; this vector space has basis $\{x + U_n : x \in \mathscr{B}_n\}$. As $*$ fixes U_n it descends to this vector space, which can be used to split A_n/U_n into a direct sum with the summands given by its eigenspaces. The dimensions of these summands is given by the number of elements of I (resp. J) of degree n.

It is clear that $\Phi_n U_n = U'_n$, where U'_n is the equivalent construction in A'_n. As $\Phi_n \circ * = *' \circ \Phi_n$, Φ also preserves this direct sum. Hence I' and J' have the same number of elements of degree n for every n. □

2.2.1 Simple GNF*-algebras

Recall that for any $(d+1)$-dimensional TQFT Z, the vector space $Z(S^d)$ has the structure of a Frobenius algebra, and Z is said to be *based* on this algebra. Sawin [41] gave the following result regarding direct sum decompositions of TQFTs.

Proposition 2.15. *A TQFT* Z *is based on a direct sum* $A = A_1 \oplus A_2$ *of Frobenius algebras if and only if* Z *splits as a direct sum* $Z \cong Z_1 \oplus Z_2$ *of TQFTs.*

Sawin also gave a classification of simple commutative Frobenius algebras over an algebraically closed field k [41, Proposition 2]. He defines the following families of Frobenius algebras:

1. For $\lambda \in k^\times$, let S_λ be the algebra k with counit $\tau(x) = \lambda^{-1}x$.

2. Let A be a commutative algebra generated by its unit and at least one nilpotent. Recall its socle is defined as

$$\mathrm{soc}(A) = \{x \in A : ax = 0 \text{ for any nilpotent } a\}.$$

Suppose this is one-dimensional (as a linear subspace) and let $\tau : A \to k$ be a linear functional which is non-zero on the socle. Let $N_{A,\tau}$ be the Frobenius algebra A with the functional τ.

Proposition 2.16. *S_λ and $N_{A,\tau}$ are indecomposable Frobenius algebras. Further, every commutative indecomposable Frobenius algebra is isomorphic to one of these, and these are nonisomorphic up to algebra isomorphism.*

The statement that these algebras are nonisomorphic up to algebra isomorphism means that distinct $\lambda \in k^\times$ give non-isomorphic Frobenius algebras S_λ; S_λ is never isomorphic to $N_{A,\tau}$; and N_{A,τ_A} and N_{B,τ_B} are isomorphic if and only if there is an isomorphism $\phi\colon A \to B$ such that $\tau_B \circ \phi = \tau_A$.

We now focus our attention on a TQFT based on the simplest possible Frobenius algebra S_λ. The corresponding J-algebra then has $A_0 = S_\lambda$. We identify A_0 with k, and write 1 and λ to mean the corresponding elements of A_0.

Lemma 2.17. *If A is a GNF*-algebra with $A_0 = S_\lambda$, then*

$$\delta_{0,k} = \lambda \otimes \mathrm{Id}_k \ .$$

Proof. Let $x \in A_k$. Then $\delta_{0,k}(x) = 1 \otimes y$ for some $y \in A_k$. As τ is a counit, applying $\tau \otimes \mathrm{Id}$ gives $x = \lambda^{-1} y$, so $y = \lambda x$ as required. □

Proposition 2.18. *If A is a GNF*-algebra with* $\dim A_0 = 1$*, then A is a free algebra.*

Proof. Start by writing A as a quotient of graded algebras

$$A = k\langle X\rangle / R$$

for some set of generators X and some homogeneous ideal of relations R. Suppose this presentation has the fewest possible number of generators; then in particular X cannot contain any elements of degree 0. Let $\pi\colon k\langle X\rangle \to A$ be the quotient map.

Suppose R is non-empty and let $r \in R$ be a homogeneous element of lowest degree. Then $\pi_d\colon k\langle X\rangle_d \to A_d$ is an isomorphism of vector spaces for $d < \deg r$. Write $r = \sum_l x_l y_l, x_l \in X, y_l \in k\langle X\rangle \setminus \{0\}$, where x_l are distinct, and let

$$i = \max_l \{\deg x_l\}, \quad j = \deg r - i.$$

Renumbering if needed, assume that $\deg x_0 = i$.

Suppose for contradiction that $i = \deg r$. Then $\deg y_0 = j = 0$, so $y_0 \in k\langle X\rangle_0 = k$ is a constant. Thus

$$x_0 - \sum_{l \neq 0} x_l y_l \in R$$

and hence x_0 can be removed from the generating set X, which contradicts the assumption of minimality. Thus $i < \deg r$.

The map $\delta_{i,j}$ can be lifted to a map $\delta_{i,j}\colon k\langle X\rangle_{i+j} \to k\langle X\rangle_i \otimes k\langle X\rangle_j$. This lift fits into a commutative diagram

$$\begin{array}{ccc} k\langle X\rangle_{i+j} & \xrightarrow{\delta_{i,j}} & k\langle X\rangle_i \otimes k\langle X\rangle_j \\ {\scriptstyle \pi_{i+j}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_i \otimes \pi_j} \\ A_{i+j} & \xrightarrow{\delta_{i,j}} & A_i \otimes A_j \end{array}$$

so $(\pi_i \otimes \pi_j) \circ \delta_{i,j}(R) = 0$. Now π_i is an isomorphism so $(\mathrm{Id}_{A_i} \otimes \pi_j) \circ \delta_{i,j}(R) = 0$, which implies $\delta_{i,j}(r) \in k\langle X\rangle_i \otimes R_j$. Now since i is the maximal degree of any x_l, r can be written as

$$r = \sum_{d=1}^{i} \sum_{\deg x_l = d} x_l y_l.$$

By assumption $\dim(A_0) = 1$, so by Proposition 2.16 there is an isomorphism of Frobenius algebras $A_0 \cong S_\lambda$ for some λ. Thus by Lemma 2.17 and Lemma 2.10 there is an equality

$$\delta_{i,j} \circ \mu_{i,j} = \lambda \operatorname{Id}_{A_i} \otimes \operatorname{Id}_{A_j}.$$

Next, by the Frobenius identity, for $d < i$ there is an equality

$$\delta_{i,j} \circ \mu_{d,i+j-d} = (\mu_{d,i-d} \otimes \operatorname{Id}_{A_j}) \circ (\operatorname{Id}_{A_d} \otimes \delta_{i-d,j}).$$

Hence $\delta_{i,j}(r)$ is given by

$$\begin{aligned}\delta_{i,j}(r) &= \lambda \sum_{\deg x_l = i} x_l \otimes y_l + \sum_{d=1}^{i-1} \sum_{\deg x_l = ds} \delta_{i,j} \circ \mu_{d,i+j-d}(x_l \otimes y_l) \\ &= \lambda \sum_{\deg x_l = i} x_l \otimes y_l + \sum_{d=0}^{i-1} \sum_{\deg x_l = d} (\mu_{d,i-d} \otimes \operatorname{Id}_{A_j})(\operatorname{Id}_{A_d} \otimes \delta_{i-d,j})(x_l \otimes y_l).\end{aligned}$$

Now the set $\{x_l\}$ is linearly independent (otherwise X is not a minimal generating set). Since $\delta_{i,j}(r) \in k\langle X\rangle_i \otimes R_j$, it follows that $y_0 \in R_j$ is a homogeneous element of R of degree strictly less than r, which contradicts the assumption that r has minimal degree. □

Theorem 2.19. *Every GNF*-algebra with* $\dim A_0 = 1$ *is isomorphic to a canonical simple GNF*-algebra.*

Proof. Let $\mathscr{A}$ be a GNF*-algebra with $\dim A_0 = 1$. Then $\mathscr{A}$ is a free algebra with some generating set X. Let $\lambda = (\tau(\mathbb{1}_A))^{-1}$. We modify the elements of X to ensure that $\delta(x) = \lambda \otimes x + x \otimes \lambda$ for any $x \in X$. Suppose we have already achieved this goal for all elements of at most some degree k. Pick $x \in X$ of degree $k+1$. Suppose

$\delta_{i,j}(x) \neq 0$ for some $0 < i < \deg x$; take i to be the small such (non-zero) value. Then replace x by $x' = \lambda x - \mu_{i,j}\delta_{i,j}(x)$. Now $\delta_{i,j}(x') = 0$. Applying $\delta_{k,l}$ for $k < i$ gives

$$\begin{aligned}\delta_{k,l}(x') &= \delta_{k,i+j-k} \circ \mu_{i,j} \circ \delta_{i,j}(x) \\ &= (\mathrm{Id}_k \otimes \mu_{i-k,j}) \circ (\delta_{k,i-k} \otimes \mathrm{Id}_j)\delta_{i,j}(x) \\ &= (\mathrm{Id}_k \otimes \mu_{i-k,j}) \circ (\mathrm{Id}_k \otimes \delta_{i-k,j})\delta_{k,i+j-k}(x)\end{aligned}$$

where the second equality follows from the Frobenius identity in Equation (2.2), and the third equality follows from the coassociativity of the coproduct. From the assumption that $k < i$ and that i is the smallest positive integer with $\delta_{i,j}(x) \neq 0$, it follows that $\delta_{k,i+j-k} = 0$, and hence $\delta_{k,l}(x') = 0$ as desired.

Hence x can be adjusted so that $\delta(x)$ has one fewer term. This process can be repeated to eliminate all unwanted terms.

As $*$ satisfies $*^2 = \mathrm{Id}$, it can be diagonalised $*$ on the linear space spanned by X. Let I (resp. J) denote a basis of eigenvectors with eigenvalue 1 (resp. -1). By linearity, $\delta = \lambda \otimes \mathrm{Id} + \mathrm{Id} \otimes \lambda$ on this vector space, and hence this property is preserved for the basis. Then it is clear that $\mathscr{A} \cong A^{\lambda}_{I,J}$. $\square$

Chapter 3

Constructing symmetric monoidal double categories from geofibred categories

The goal of this chapter is to give full details of the construction of $\mathscr{V}\!ar$ as a symmetric monoidal 2-category. An explicit definition of the data needed to construct a symmetric monoidal 2-category was given by Schommer-Pries [42]. Although many of the conditions are natural conditions that one would expect to hold, a large number of auxiliary natural transformations must be given, and many conditions shown to hold.

Instead of constructing this 2-category directly, a symmetric monoidal double category is constructed as an intermediate step. A *double category* [14] $\mathbb{D}$ is a form of 2-category where the 1-morphisms are separated into two classes, one for which composition is strictly associative (tight or "vertical" morphisms) and one where the composition is only weakly so (loose or "horizontal" morphisms). An example of such a double category (which has strong links to the category $\mathscr{V}\!ar$) is the category $\mathscr{CAlg}_k$, where the objects are algebras over a fixed base field k, the tight morphisms between two algebras A and B are morphisms of algebras, the loose morphisms between two algebras A and B are A-B-bimodules, and the 2-cells are morphisms of bimodules (Definition 4.19). For any such category, there is an associated 2-category called its underlying loose 2-category $\mathscr{L}(\mathbb{D})$ (see Definition 3.30). If the double category is monoidal, and the monoidal constraints have loosely strong companions (Definition 3.32), then the underlying loose 2-category will also be monoidal. Since many of the conditions required to be a monoidal category involve tight morphisms, which compose strictly, less information is required to construct a symmetric monoidal double category.

The construction of the double category in this chapter does not use the fact that the objects are schemes explicitly. In fact, the construction is valid for any symmetric monoidal *geofibred category*. A geofibred category, as introduced by Reich [36], consists of a pair of categories, known as the *category of shapes* **Sh** and the *category of spaces* **Sp** (where **Sp** is required to have all finite fibred products), a functor $F\colon \mathbf{Sh} \to \mathbf{Sp}$, and functors as follows (full details of the conditions these data must satisfy are given in Definition 3.3). For any $X \in \mathbf{Sp}$, let $\mathbf{Sh}_X \subset \mathbf{Sh}$ be the subcategory consisting of all objects $\mathscr{E}$ such that $F(\mathscr{E}) = X$, and all morphisms f such that $F(f) = \mathrm{Id}_X$. We require that for any $f \in \mathrm{Hom}_{\mathbf{Sp}}(A, B)$, there is a pair of functors $f_*\colon \mathbf{Sh}_A \to \mathbf{Sh}_B$ and $f^*\colon \mathbf{Sh}_B \to \mathbf{Sh}_A$ such that f^* is left adjoint to f_*.

For any choice of suitable geofibred category (see Proposition 3.1 for a complete list of conditions), we construct a double category. The structure of this category is inspired by the 2-category $\mathscr{V}\!ar$; in fact, when the construction is applied to the geofibred category of derived categories of sheaves of quasi-coherent modules, the resulting underlying loose 2-category is exactly $\mathscr{V}\!ar$. Thus the objects are objects in **Sp**, and the tight morphisms are morphisms in **Sp**. The loose morphisms are objects of **Sh**, with composition inspired by the Fourier–Mukai composition of sheaves. Recall that a sheaf $\mathscr{X} \in \mathscr{D}(X \times Y)$ induces a functor called the *Fourier–Mukai transform*

$\Phi_{\mathscr{X}} \colon \mathscr{D}(X) \to \mathscr{D}(Y)$ by

$$\Phi_{\mathscr{X}}(\mathscr{E}) = \pi^{XY}_{Y*}(\pi^{XY*}_{X}(\mathscr{E}) \otimes \mathscr{X}).$$

It is possible to define a composition of sheaves $\odot$ such that $\Phi_{\mathscr{F} \odot \mathscr{E}} = \Phi_{\mathscr{F}} \circ \Phi_{\mathscr{E}}$ [20]. To generalise this composition to geofibred categories, we need to define a symmetric monoidal structure on a geofibred category. This is the content of Definition 3.27. Part of this monoidal structure is a family of functors

$$\boxtimes_{AB} \colon \mathbf{Sh}_A \times \mathbf{Sh}_B \to \mathbf{Sh}_{AB}$$

which generalise the external product of sheaves as defined in Definition 3.25. This is used to define composition in Proposition 3.1.

Proposition 3.1. *Let $F \colon \mathbf{Sh} \to \mathbf{Sp}$ be a symmetric monoidal geofibred category, where $\mathbf{Sp}$ has a terminal object, and the class of projection morphisms is pull-geolocalizing (as defined in Definition 3.9). Then there is a double category $\mathbb{D}(F)$ with:*

- *objects given by objects in $\mathbf{Sp}$;*
- *tight morphisms given by diagrams of the form* $\begin{array}{c} A \\ \downarrow f \\ U \end{array}$*, where $f \in \mathrm{Hom}_{\mathbf{Sp}}(A, U)$;*
- *loose morphisms given by diagrams* $A \xrightarrow{\mathscr{E}} B$ *, where $\mathscr{E} \in \mathbf{Sh}_{A \times B}$; and*
- *2-cells given by diagrams*

$$\begin{array}{ccc} A & \xrightarrow{\mathscr{E}} & B \\ {\scriptstyle f}\downarrow & \Uparrow \alpha & \downarrow{\scriptstyle g} \\ U & \xrightarrow{\mathscr{F}} & V \end{array}$$

  where:

  - *$A, B, U, V \in \mathbf{Sp}$;*

– *$f \in \mathrm{Hom}(A, U)$, $g \in \mathrm{Hom}(B, V)$;*

– *$\mathscr{E} \in \mathbf{Sh}_{A\times B}$, $\mathscr{F} \in \mathbf{Sh}_{U\times V}$; and*

– *$\alpha \in \mathrm{Hom}(\mathscr{F}, (f \times g)_*(\mathscr{E}))$.*

The double category is such that in its loose 2-category, composition of 1- and 2-morphisms are given by the Fourier–Mukai composition, defined for $\mathscr{E} \in \mathscr{SH}_{AB}, \mathscr{F} \in \mathscr{SH}_{CD}$ by

$$\mathscr{E} \odot \mathscr{F} = \pi^{ABC}_{AC} \circ i^{124*}_{1234}(\mathscr{E} \boxtimes \mathscr{F})\,.$$

Furthermore, $\mathbb{D}(F)$ can be given the structure of a symmetric monoidal double category, with monoidal product given on objects by

$$X \otimes Y = X \times Y$$

and on loose morphisms by

$$\mathscr{E} \otimes \mathscr{F} = \mathscr{E} \boxtimes \mathscr{F}\,.$$

The monoidal product also satisfies that the product of globular 2-cells is given by

$$\begin{array}{ccc} A & \xrightarrow{\mathscr{E}} & B \\ {\scriptstyle \mathrm{Id}_A}\big\downarrow & {\scriptstyle\alpha}\Uparrow & \big\downarrow{\scriptstyle \mathrm{Id}_B} \\ A & \xrightarrow[\mathscr{P}]{} & B \end{array} \otimes \begin{array}{ccc} C & \xrightarrow{\mathscr{F}} & D \\ {\scriptstyle \mathrm{Id}_C}\big\downarrow & {\scriptstyle\beta}\Uparrow & \big\downarrow{\scriptstyle \mathrm{Id}_D} \\ C & \xrightarrow[\mathscr{Q}]{} & D \end{array} = \begin{array}{ccc} A \times C & \xrightarrow{\mathscr{E}\boxtimes\mathscr{F}} & B \times D \\ {\scriptstyle \mathrm{Id}_{A\times C}}\big\downarrow & {\scriptstyle\alpha\boxtimes\beta}\Uparrow & \big\downarrow{\scriptstyle \mathrm{Id}_{B\times D}} \\ A \times C & \xrightarrow[\mathscr{P}\boxtimes\mathscr{Q}]{} & B \times D \end{array}\,.$$

The proof of this proposition is the content of Section 3.5, Section 3.6 and Section 3.7.

Furthermore, the monoidal product on the double category $\mathbb{D}(F)$ will induce a monoidal product on the 2-category $\mathscr{L}(\mathbb{D}(F))$; this is the content of Corollary 3.65.

The layout of this chapter is as follows. The definition of a geofibred category is recalled in Subsection 3.1.1, and the new notion of a monoidal structure on a geofibred category is introduce in Subsection 3.1.4. Notation that will be used throughout the rest of the chapter is introduced in Subsection 3.1.3. The definition of objects, morphisms and cells that are used for Proposition 3.1 are given in Section 3.3. The definition of loose composition and the natural isomorphisms required for the associativity and unital constraints are delayed until after Section 3.4, where a method of constructing functors valued in this double category is given, as well as natural transformations between such functors. Section 3.5 gives the definition for the loose composition functor. Finally, these functors are shown to satisfy the necessary conditions to give a double category.

Section 3.6 gives a construction of a monoidal structure on the double category. Subsection 3.6.2 and Subsection 3.6.3 give the auxiliary data for this structure, and Lemma 3.6.4 shows that these satisfy the conditions required of a monoidal double category structure.

3.1 Geofibred categories

Geometrically fibred or "geofibred" categories, as introduced by Reich [36], provide an abstract model for the fibred category of sheaves. Their motivation lies in the desire to show that certain classes of diagrams are automatically commutative, much in the same way that Mac Lane [27] showed that if an associator satisfies the pentagon identity, then all possibly compositions of associators between two fixed objects must in fact give the same morphism. Rather than associators, the diagrams in question are compositions of natural transformations between functors between categories of geometric objects, typically sheaves over a scheme.

As a concrete motivational example, consider the categories of sheaves and schemes. Let X be a scheme, and let $f\colon X \to Y$ be a functor. Let $\mathscr{SH}_X$ be the category of sheaves of modules over X. Recall f induces a direct image functor

$$f_*\colon \mathscr{SH}_X \to \mathscr{SH}_Y$$

defined by

$$f_*(\mathscr{E})(V) = \mathscr{E}(f^{-1}(V))$$

for any sheaf $\mathscr{E} \in \mathscr{SH}_X$ and any open set $V \subset Y$. Similarly, the inverse image functor $f^*\colon \mathscr{SH}_Y \to \mathscr{SH}_X$ is defined by first constructing the presheaf

$$f^*_{\text{pre}}(\mathscr{F})(U) = \lim_{V \supset U} (\mathscr{F}(V) \otimes_{\mathscr{O}_Y(V)} \mathscr{O}_X(U))$$

for any sheaf $\mathscr{F} \in \mathscr{SH}_Y$ and any open set $U \subset X$. The inverse image $f^*(\mathscr{F})$ is then the sheafification of this presheaf.

It is clear from this construction that there are certain natural isomorphisms between functors constructed from composing these direct image and inverse image functors. For example, the direct image functor $\mathrm{Id}_{X*}\colon \mathscr{SH}_X \to \mathscr{SH}_X$ is in fact the identity functor $\mathrm{Id}_{\mathscr{SH}_X}$, while the inverse image functor Id_X^* is naturally isomorphic to the identity functor Id_Y. Similarly, there are natural isomorphism

$$\mathrm{comp}_*\colon g_* \circ f_* \to (g \circ f)_* \quad \text{and} \quad \mathrm{comp}^*\colon f^* \circ g^* \to (g \circ f)^*$$

for any composable pair of morphisms $f\colon X \to Y$ and $g\colon Y \to Z$. One can show that these composition operations are associative, in the sense that there is a commutative diagram

$$\begin{array}{ccc} f_* \circ g_* \circ h_* & \longrightarrow & f_* \circ (g \circ h)_* \\ \downarrow & & \downarrow \\ (f \circ g)_* \circ h_* & \longrightarrow & (f \circ g \circ h)_* \end{array}$$

Since these operations are strictly associative, any sequence of compositions of comp between two objects will give the same morphism. Geofibred categories show that a larger class of diagrams are automatically commutative; see Theorem 3.17 for more details.

3.1.1 Basic definitions

This section recalls from Reich [36] the definitions needed for the rest of this chapter. To introduce the definition of a geofibred category, first recall from Definition 1.11 the notion of a lax 2-functor.

Definition 3.2. Let $F\colon \mathscr{C} \to \mathscr{D}$ be a functor. Then for any $d \in \mathscr{D}$, there is a subcategory $\mathscr{C}_d$ consisting of objects $c \in \mathscr{C}$ such that $F(c) = D$, and morphisms f such that $F(f) = \mathrm{Id}_d$. This is called the *fibre category* over d.

Definition 3.3. A *geofibred category* consists of a pair of categories $(\mathbf{Sh}, \mathbf{Sp})$, known as the categories of shapes and spaces respectively, and a functor $F\colon \mathbf{Sh} \to \mathbf{Sp}$, such that:

- the category $\mathbf{Sp}$ has all finite fibred products;
- for every morphism $f \in \mathrm{Hom}_{\mathbf{Sp}}(A, B)$, there are functors

  $$f_*\colon \mathbf{Sh}_A \rightleftarrows \mathbf{Sh}_B \colon f^*$$

  between the fibre categories over A and B respectively, which form an adjoint pair $f^* \dashv f_*$;
- the assignment $A \mapsto \mathbf{Sh}_A$, $f \mapsto f^*$ gives a lax 2-functor $\mathbf{Sp} \to \mathscr{Cat}$, where $\mathbf{Sp}$ is viewed as a 2-category with only identity 2-morphisms;
- the assignment $A \mapsto \mathbf{Sh}_A$, $f \mapsto f_*$ gives a lax 2-functor $\mathbf{Sp} \to \mathscr{Cat}^{\mathrm{op}}$; and
- the functor data satisfy the compatibility conditions Equation (3.3) and Equation (3.4).

Definition 3.4. Let $f\colon X \to Y$ be a morphism of spaces. The functors f_* and f^* are called *basic standard geometric functors*.

A *standard geometric functor* is any composition of basic standard geometric functors.

Standard geometric natural transformations

Definition 3.5. Between pairs of SGFs there are canonical natural transformations of the following three types that are called the *basic standard geometric natural transformations*:

$$\left.\begin{aligned} \mathrm{unit}(f)\colon\ \mathrm{Id} &\Rightarrow f_*f^* \\ \mathrm{counit}(f)\colon\ f^*f_* &\Rightarrow \mathrm{Id} \end{aligned}\right\} \textit{Adjunctions}, \tag{3.1a}$$

$$\left.\begin{aligned} \mathrm{comp}^*(f,g)\colon\ f^*g^* &\Rightarrow (gf)^* \\ \mathrm{comp}_*(f,g)\colon\ g_*f_* &\Rightarrow (gf)_* \end{aligned}\right\} \textit{Compositions}, \text{ and} \tag{3.1b}$$

$$\left.\begin{aligned} \mathrm{triv}_*(f)\colon\ \mathrm{Id}_* &\Rightarrow \mathrm{Id} \\ \mathrm{triv}^*(f)\colon\ \mathrm{Id}^* &\Rightarrow \mathrm{Id} \end{aligned}\right\} \textit{Trivializations}. \tag{3.1c}$$

For the latter two types, $^{-1}$ will denote their inverses.

The natural transformation comp_* (resp. comp^*) is specified by the data showing that $f \mapsto f_*$ (resp. $f \mapsto f_*$) is a lax 2-functor: in the notation of Definition 1.11, it is the natural transformation γ. Similarly, the transformation triv_* (resp. triv^*) is the natural transformation ι. The unit natural transformations is specified by the data showing that f_* and f^* are adjoint.

It will be useful to talk about natural transformations that are formed from the above by whiskering with standard geometric functors. The next definition gives definitions of classes of transformations constructed in this manner.

Definition 3.6. In this definition, F and G represent any SGFs; f and g represent any maps of spaces. Define:

$$\mathrm{unit} = \{F * \mathrm{unit}(f) * G\} \qquad \mathrm{counit} = \{F * \mathrm{counit}(f) * G\}$$

$$\mathrm{comp}_0 = \{F * \mathrm{comp}_*(f, g) * G\} \cup \{F * \mathrm{comp}^*(f, g) * G\} \qquad \mathrm{comp} = \mathrm{comp}_0 \cup \mathrm{comp}_0^{-1}$$

$$\mathrm{triv}_0 = \{F * \mathrm{triv}_* * G\} \cup \{F * \mathrm{triv}^* *, G\} \qquad \mathrm{triv} = \mathrm{triv}_0 \cup \mathrm{triv}_0^{-1}$$

Definition 3.7. A *standard geometric natural transformation* (SGNT) is an element of the class (in which the notation $\langle S\rangle$ denotes the class generated by S via composition)

$$\mathrm{SGNT} = \langle \mathrm{unit} \cup \mathrm{counit} \cup \mathrm{comp} \cup \mathrm{triv}\rangle. \tag{3.2}$$

For any SGNT $\phi\colon F \to G$ between two SGFs, write $F = \mathrm{dom}(\phi)$ and $G = \mathrm{cod}(\phi)$.

Using these definitions, it is possible to state the compatibility conditions required for a geofibred category.

The first condition is that the adjunctions and compositions are compatible in the following sense. For any f, the functors f^* and f_* are adjoint, giving an isomorphism $\mathrm{Hom}(\mathscr{E}, f_*(\mathscr{F})) \to \mathrm{Hom}(f^*(\mathscr{E}), \mathscr{F})$. The following diagram of morphisms is required to commute:

$$\begin{array}{ccc}
\mathrm{Hom}((fg)^*(\mathscr{E}), \mathscr{F}) & \xrightarrow{\mathrm{Hom}(\mathrm{comp}^*(f,g)_{\mathscr{E}}, \mathscr{F})} & \mathrm{Hom}(g^* \circ f^*(\mathscr{E}), \mathscr{F}) \\
 & & \downarrow{\scriptstyle \mathrm{adjunction}} \\
\downarrow{\scriptstyle \mathrm{adjunction}} & & \mathrm{Hom}(f^*(\mathscr{E}), g_*(\mathscr{F})) \\
 & & \downarrow{\scriptstyle \mathrm{adjunction}} \\
\mathrm{Hom}(\mathscr{E}, (fg)_*(\mathscr{F})) & \xrightarrow{\mathrm{Hom}(\mathscr{E}, \mathrm{comp}_*(f,g)_F)} & \mathrm{Hom}(\mathscr{E}, f_* \circ g_*(\mathscr{F}))
\end{array} \tag{3.3}$$

The second condition is that trivialisations are compatible with adjunctions in a similar manner:

$$\begin{array}{ccc} \mathrm{Hom}(\mathrm{Id}^*(\mathscr{E}),\mathscr{F}) & \xrightarrow{\mathrm{Hom}(\mathrm{triv}^*_{\mathscr{E}},\mathscr{F})} & \\ \Big\downarrow{\scriptstyle \text{adjunction}} & & \mathrm{Hom}(\mathscr{E},\mathscr{F}) \\ \mathrm{Hom}(\mathscr{E},\mathrm{Id}_*(\mathscr{F})) & \xrightarrow{\mathrm{Hom}(\mathscr{E},\mathrm{triv}_{*,\mathscr{F}})} & \end{array} \tag{3.4}$$

Base change diagrams

Consider a commutative diagram in the category of spaces:

$$\begin{array}{ccc} A & \xrightarrow{f'} & B \\ \Big\downarrow{\scriptstyle g'} & & \Big\downarrow{\scriptstyle g} \\ U & \xrightarrow{f} & V \end{array} . \tag{3.5}$$

Since this is a commutative diagram, there is an equality of morphisms $f \circ g' = g \circ f'$. Hence there is a natural isomorphism

$$\eta\colon f_* \circ g'_* \xrightarrow{\mathrm{comp}} (f \circ g')_* = (g \circ f')_* \xrightarrow{\mathrm{comp}^{-1}} g_* \circ f'_* .$$

This natural transformation can be composed with the unit and counit SGNFs, giving another natural transformation

$$\mathrm{c.d.}(f,g;f',g')\colon g^* \circ f_* \xrightarrow{g^* \circ f_* \circ \mathrm{counit}(g')} g^* \circ f_* \circ g'_* \circ g'^* \xrightarrow{\eta} g^* \circ g_* \circ f'_* \xrightarrow{\mathrm{unit}(g) \circ f'_* \circ g'^*} f'_* \circ g'^* .$$

Definition 3.8. Let $f\colon U \to V$ and $g\colon B \to V$ be morphisms of spaces. Let

$$\begin{array}{ccc} A & \xrightarrow{f'} & B \\ \Big\downarrow{\scriptstyle g'} & & \Big\downarrow{\scriptstyle g} \\ U & \xrightarrow{f} & V \end{array}$$

be a Cartesian diagram. Then define the *base-change natural transformation*

$$\text{b.c.}(f, g) = \text{c.d.}(f, g; f', g')\,.$$

Furthermore, let b.c. $= \langle F \text{ b.c. } G \rangle$ be the class of natural transformations generated by base-changes.

Definition 3.9. A class P of morphisms in $\mathbf{Sp}$ is *pull-geolocalizing* if it contains every isomorphism and is closed under composition, and for each $g \colon B \to V$ in P and $f \in \mathrm{Hom}_{\mathbf{Sp}}(U, V)$, the base-change transformation b.c.(f, g) is an isomorphism and $g' \colon B \times_V U \to U$ is in P.

A class Q of morphisms in $\mathbf{Sp}$ is *push-geolocalizing* if it satisfies the same condition with the roles of f and g exchanged: explicitly, if it contains all isomorphisms and is closed under composition, and for $f \colon U \to V$ in Q and $g \in \mathrm{Hom}_{\mathbf{Sp}}(B, V)$, the base-change transformation b.c.(f, g) is an isomorphism and $f' \colon B \times_V U \to B$ is in Q.

Diagrams

As in [36], standard geometric functors will often be represented by directed graphs which are topologically linear.

Definition 3.10. Let Γ be a diagram in $\mathbf{Sp}$ which is topologically linear. After choosing one of the leaves of the diagram as the source, this determines the *functor associated to* Γ by taking direct image functors (when travelling in the direction of the edge) and inverse image functors (when travelling against the direction of the edge). Conversely, given a SGF F, a graph Γ *represents* F if F is the functor associated to Γ.

For example, the functor associated to the diagram

$$X \xleftarrow[g]{} Y \xrightarrow{f} Z$$

is $f_* \circ g^*$ (when read left-to-right).

These diagrams can be stacked vertically, with labelled arrows between each layer, to represent natural transformations and compositions thereof. For example, the base-change morphism corresponding to the pullback diagram

$$\begin{array}{ccc} U & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & Z \end{array}$$

can be depicted as

$$\begin{array}{ccccc} Y & \xrightarrow{f} & Z & \xleftarrow[g]{} & X \\ & & \Downarrow{\scriptstyle \text{b.c.}} & & \\ Y & \xleftarrow[g']{} & U & \xrightarrow{f'} & X \end{array}$$

Recall that for any geofibred category $F\colon \mathbf{Sh} \to \mathbf{Sp}$, the category $\mathbf{Sp}$ has all finite fibred products. If $\mathbf{Sp}$ has a terminal object T, let $X \times Y = X \times_T Y$ denote the product over this terminal object.

Alternating functors and roofs

Let F be a standard geometric functor; recall this means that it is a composition of direct image and inverse image functors. If F has at any point in its construction the composition of two direct image functors consecutively, then F is naturally isomorphic to a functor consisting of the composition of fewer functors, using the composition natural transformations comp_*. Similarly, the composition of two inverse image functors can be simplified. This gives a canonical form for F, known as an *alternating functor*.

Definition 3.11. A standard geometric functor is *alternating* if it is of the form $f_* \circ g^* \circ h_* \ldots$ or $g^* \circ h_*$, where no morphism is the identity map. An *alternating reduction* of a standard geometric functor F is the alternating standard geometric functor F' admitting a SGNT in $\langle \mathrm{comp} \cup \mathrm{triv}_0 \rangle$ from F to F'.

The alternating reduction of a functor is in fact unique [36, Proposition 4.11].

Although alternating functors are canonical, it is not immediately apparent when two such functors are naturally isomorphic. A further reduction to a simpler form can be defined as follows.

Definition 3.12. For a SGF $F\colon \mathbf{Sh}_X \to \mathbf{Sh}_Y$, its *source* $S(F)$ is the space X, and its *target* $T(F)$ is the space Y.

Definition 3.13. Let F be a SGF, and consider a diagram in $\mathbf{Sp}$ which represents F (as defined in Definition 3.10). Define the *roof* of F, denoted $\mathrm{roof}(F)$, to be the final object in the category of spaces with maps to this diagram (this exists and is unique by the universal property of the fibre product). This also gives two morphisms $a_F\colon \mathrm{roof}(F) \to T(F)$ and $b_F\colon \mathrm{roof}(F) \to S(F)$. The functor $a_{F*} \circ b_F^*$ will also be referred to as $\mathrm{roof}(F)$.

The terminology "roof" is used as the space $\mathrm{roof}(F)$, along with the morphisms a_F and b_F, form a diagram in the shape of a sloped roof:

$$\begin{array}{ccccccccc} & & & & \mathrm{roof}(F) & & & & \\ & \swarrow^{b'_F} & & & & & & \searrow^{a_F} & \\ A_1 & \longleftarrow & A_2 & \longrightarrow & \cdots & \longleftarrow & A_{n-1} & \longrightarrow & A_n \end{array}$$

Definition 3.14. A functor F is *weakly admissible* if the pair morphism (a_F, b_F) defined in Definition 3.13 is a universal monomorphism; that is, for any Cartesian diagram

$$\begin{array}{ccc} \mathrm{roof}(F) \times_{S(F)\times T(F)} C & \xrightarrow{\;g\;} & C \\ \downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{roof}(F) & \xrightarrow{(b_F, a_F)} & S(F) \times T(F) \end{array}$$

the morphism g is a monomorphism.

Using base-change morphisms, it is possible to construct a natural transformation between any standard geometric functor and its roof [36, Proposition 1.13].

Definition 3.15. An alternating functor F is said to be *good* if either every functor f^* in F is pull-geolocalizing, or every functor of the form f_* is push-geolocalizing. An SGF F is said to be good if it has a good alternating reduction.

Proposition 3.16. *[36, Corollary 1.14] If F is a good SGF, then the SGF* $\operatorname{roof}(F)$ *is good, and the SGNT* $\operatorname{roof}(F)\colon F \to \operatorname{roof}(F)$ *is a natural isomorphism.*

3.1.2 Coherence results

A key result regarding standard geometric natural transformations is the following theorem. The result here is stated for a smaller class of natural transformations than the original statement, as this is sufficient for the proofs in the remainder of this thesis.

Theorem 3.17. *[36, Theorem 2.4] Let $\phi\colon F \to G$ be a natural transformation of standard geometric functors.*

Write $S(F) = S(G) = X$ and $T(F) = T(G) = Y$; denote $Z = X \times Y$ and let $b\colon \operatorname{roof}(F) \times_Z \operatorname{roof}(G)$ be the projection map, where $\operatorname{roof}(F) \to Z$ and $\operatorname{roof}(G) \to Z$ are the pair maps (b_F, a_F) and (b_G, a_G). Suppose that the unit map $a_{G*} \operatorname{unit}(b) b_G^*$ is an isomorphism.

Suppose further that the map $\operatorname{roof}(G)\colon G \to \operatorname{roof}(G)$ is an isomorphism. If $\phi \in \langle SGNT \cup bc^{-1} \cup Unit^{-1} \rangle$, then it is the unique map in that class.

The conditions in Theorem 3.17 can be verified more easily using the following result.

Lemma 3.18. *[36, Lemma 2.5] The map $\operatorname{roof}(G)$ is an isomorphism if G is good. The condition that $a_{G*} \operatorname{unit}(b) b_G^*$ is an isomorphism holds if F is weakly admissible and (b_G, a_G) factors through (b_F, a_F).*

3.1.3 Notation

In the remainder of this chapter, the following conventions regarding notation are used:

1. Capital Roman letters $A, B, \ldots$ are used to denote objects in **Sp**; the exceptions are F, G, S and T which are reserved for functors;
2. Minuscule Roman letters starting from f are used to denote morphisms in **Sp**;
3. Script letters $\mathscr{E}, \mathscr{F}, \ldots$ are used to denote objects in **Sh**;
4. Minuscule Greek letters $\alpha, \beta, \ldots$ are used to denote morphisms in **Sh**;
5. The minuscule Greek letters η and ξ are used to denote natural transformations; and
6. Fraktur letters $\mathfrak{a}, \mathfrak{l}, \mathfrak{r}$ etc. are used to denote the structural natural isomorphisms used to define a double category.

Furthermore, the fibre product of spaces will be denoted by concatenation; that is, XY will be used for the product $X \times Y$.

Product of SGFs

Definition 3.19. Let $f\colon A \to B$ and $G\colon C \to D$ be morphisms of spaces. Define the product $f_* \times g_*\colon \mathbf{Sh}_{AC} \to \mathbf{Sh}_{BD}$ to be the functor $(f \times g)_*$. Define the product of inverse image functors likewise.

Families of morphisms

Definition 3.20. Let $\{X_i : i \in I\}$ be a finite collection of spaces. Define X_I to be the fibre product of all the objects in this collection.

Definition 3.21. Let $\{X_i : i \in I\}$ be a finite collection of spaces. Let $J \subset I$. Define a morphism $\pi^I_J \colon X_I \to X_J$ to be projection on to the factors in the set J.

Definition 3.22. Let $\{X_i : i \in I\}$ be a finite collection of spaces. Let $\sim$ be an equivalence relation on the set I such that if $i \sim j$, then $X_i = X_j$. Pick a set of representatives of equivalence classes $J \subset I$. Define a morphism $i^J_\sim \colon X_J \to X_I$ to be such that for any $i \in I$, there is an equality

$$\pi^I_{\{i\}} \circ i^J_\sim = \pi^J_{j_i}$$

where j_i is the unique element of J such that $j_i \sim i$.

For P a partition of I, define $i^I_P = i^I_{\sim_P}$, where $\sim_P$ is the equivalence relation $i \sim_P i'$ if and only if i and i' lie in the same part of P.

As an example, if $X_1 = X_2 = X$, then the morphism $i^1_{12} \colon X_1 \to X \times X$ is the diagonal morphism.

Definition 3.23. Let $\{X_i : i \in I\}$ be a finite collection of spaces, where I is considered to be an ordered set. Let J be a re-ordering of the set I; that is, J has the same elements as I but potentially in a different order. Define a morphism

$$j^I_J \colon X_I \to X_J$$

to be the isomorphism such that

$$\pi^J_j \circ j^I_J = \pi^I_i.$$

Thus for example if $X_1 = X, X_2 = Y$ then $j^{12}_{21} \colon X \times Y \to Y \times X$ is the map which exchanges the two factors.

A morphism between two spaces, where the domain is a product of distinct spaces, and the codomain is a product of (some subset of) those spaces, possibly with repetition, may be denoted with an unlabelled arrow, and will stand for the composition of the implied permutation and inclusion morphisms; for example (recalling that XY is used to denote the product space $X \times Y$),

$$\begin{array}{c} XY \\ \big\downarrow \\ YX \end{array}$$

is the isomorphism which gives the braiding structure on the product of spaces, and

$$\begin{array}{c} XYZ \\ \big\downarrow \\ XZ \end{array}$$

is the projection map.

Lemma 3.24. *Let $\{X_i : i \in I\}$ be a finite collection of spaces. Let $\sim_1$ and $\sim_2$ be two equivalence relations on I, with equivalence class representatives J_1 and J_2. Let $\sim$ be the equivalence relation generated by the transitive closure of the relation $i \sim j$ if $i \sim_1 j$ or $i \sim_2 j$. Let J be a set of representatives of equivalence classes of $\sim$. Define an equivalence relation $\sim'_i$ on J_i by $j_1 \sim'_i j_2$ if and only if $j_1 \sim j_2$. Then there is a Cartesian diagram*

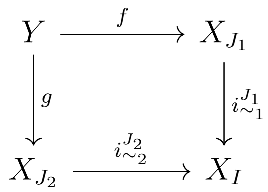
$$\begin{array}{ccc} X_J & \xrightarrow{i^J_{\sim'_2}} & X_{J_1} \\ \Big\downarrow{\scriptstyle i^J_{\sim'_1}} & & \Big\downarrow{\scriptstyle i^{J_1}_{\sim_1}} \\ X_{J_2} & \xrightarrow{i^{J_2}_{\sim_2}} & X_I \end{array} \tag{3.6}$$

Proof. For $i \in I$, there is an equality

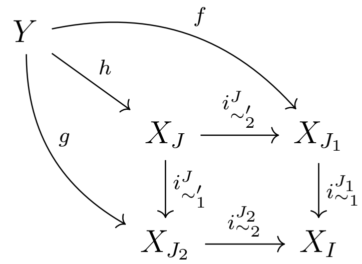
$$\begin{aligned} \pi^I_i \circ i^{J_2}_{\sim_2} \circ i^J_{\sim'_2} &= \pi^{J_2}_{j_{2,i}} \circ i^J_{\sim'_2} \\ &= \pi^J_j \end{aligned} \tag{3.7}$$

where $j_{2,i}$ is the unique element of J_2 such that $j_{2,i} \sim_2 i$, and j is the unique element of J such that $j \sim i$. The other direction in the diagram gives the same result, so the diagram commutes.

Now suppose



is another commutative diagram. To show that Equation (3.6) is a Cartesian square, it is necessary to show that there is a unique morphism $h\colon Y \to X_J$ such that the diagram



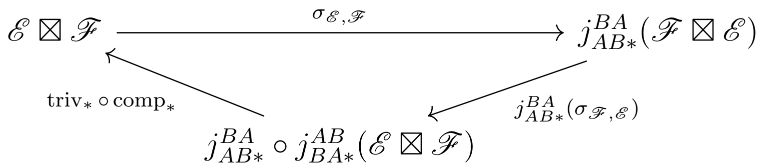
commutes. By Equation (3.7), the map h must satisfy

$$\begin{aligned} \pi^J_j \circ h &= \pi^I_i \circ i^{J_2}_{\sim_2} \circ i^J_{\sim_2'} \circ h \\ &= \pi^I_i \circ i^{J_2}_{\sim_2} \circ g \end{aligned}$$

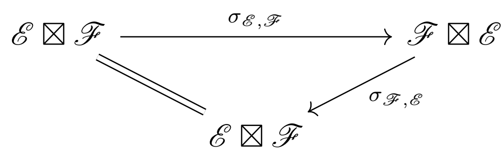
where i is any element of I such that $j \sim i$. Thus h (if it exists) is uniquely determined. Now if $j_1 \sim_2 j_2$, then $\pi^I_{j_2} \circ i^{J_2}_{\sim_2} \circ g = \pi^I_{j_1} \circ i^{J_2}_{\sim_2} \circ g$ by definition of $i^{J_2}_{\sim_2}$. Similarly, if $j_1 \sim_2 j_2$, then $\pi^I_{j_2} \circ i^{J_1}_{\sim_1} \circ f = \pi^I_{j_1} \circ i^{J_1}_{\sim_1} \circ f$. Hence h is independent of the choice of i in its definition, so is well-defined. □

Implicit permutation functors

When verifying the conditions for a monoidal double category, there will be many commutative diagrams to check. For brevity, it will sometimes be necessary to not include direct images by the permutation morphisms introduced in Definition 3.23. For example, consider the braiding morphism (which will be defined in Definition 3.27, along with the external product functor $\boxtimes\colon \mathbf{Sh}_A \times \mathbf{Sh}_B \to \mathbf{Sh}_{AB}$) $\sigma_{\mathscr{E},\mathscr{F}}\colon \mathscr{E} \boxtimes \mathscr{F} \to j^{BA}_{AB*}(\mathscr{F} \boxtimes \mathscr{E})$. This may be denoted as a morphism $\mathscr{E} \boxtimes \mathscr{F} \to \mathscr{F} \boxtimes \mathscr{E}$, with the direct image j^{BA}_{AB*} left implicit. Since all permutation morphisms are invertible, and pushing forward by the identity permutation is naturally isomorphic to the identity, the choice of where to insert the direct image functor has no effect on whether a given diagram commutes. Objects which are isomorphic by the isomorphisms of comp_* and triv_* may also be identified implicitly. For example, the symmetry constraint could be explicitly written as



or, using this implicit direct image notation, as



Similarly, for these diagrams, sub- and superscripts may be dropped for conciseness.

3.1.4 Monoidal structure

This sections introduces a notion for a monoidal structure on a geofibred category, inspired by the external product of sheaves.

Definition 3.25. Let $\mathscr{E} \in \mathscr{SH}_A$ and $\mathscr{F} \in \mathscr{SH}_B$ be sheaves. Then the *external product* of $\mathscr{E}$ and $\mathscr{F}$, denoted $\mathscr{E} \boxtimes \mathscr{F}$, is defined by

$$\mathscr{E} \boxtimes \mathscr{F} = \pi_A^{AB*}(\mathscr{E}) \otimes \pi_B^{AB*}(\mathscr{F}) \in \mathcal{SH}_{A\times B}.$$

The external product of sheaves is used as the basis for the monoidal structure, rather than the usual tensor product of sheaves, as it is a more natural setting to use the results of Theorem 3.17. As noted in [36], the tensor product can be recovered from the external tensor product by

$$\mathscr{E} \otimes \mathscr{F} = i_{12}^{1*}(\mathscr{E} \boxtimes \mathscr{F}).$$

The braiding on the tensor product then follows from applying the functor j_{AB}^{BA*}, and the associator from the equality

$$i_{13}^{1} \circ i_{12,3}^{1,3} = i_{12}^{1} \circ i_{1,23}^{1,2}.$$

Definition 3.26. Let $F\colon \mathbf{Sh} \to \mathbf{Sp}$ be a geofibred category. A *monoidal structure* on F consists of a family of functors

$$\boxtimes_{AB}\colon \mathbf{Sh}_A \times \mathbf{Sh}_B \to \mathbf{Sh}_{A\times B}$$

along with families of natural isomorphisms:

- the associators

$$\alpha^{\boxtimes}_{A,B,C}\colon\ \boxtimes_{A,BC} \circ(\mathrm{Id}_{\mathbf{Sh}_A} \times \boxtimes_{B,C}) \to \boxtimes_{AB,C} \circ (\boxtimes_{A,B} \times \mathrm{Id}_{\mathbf{Sh}_C});$$

- the unitors

$$\rho^{\boxtimes}\colon \mathscr{E} \boxtimes \mathscr{O}_U \to \pi_A^{AU*}(\mathscr{E}) \quad \text{and} \quad \lambda\colon \mathscr{O}_V \boxtimes \mathscr{E} \to \pi_A^{VA*}(\mathscr{E});$$

and

- the distributors

$$\xi_*\colon f_* \boxtimes_{U,V} g_* \to (f\times g)_* \circ \boxtimes_{A,B} \tag{3.8}$$

and

$$\xi^*\colon f^* \boxtimes_{A,B} g^* \to (f\times g)^* \circ \boxtimes_{U,V}. \tag{3.9}$$

These isomorphisms must be compatible, in that the following are commutative diagrams:

- The pentagon identity

$$\begin{array}{ccccc} \mathscr{E}\boxtimes(\mathscr{F}\boxtimes(\mathscr{G}\boxtimes\mathscr{H})) & \longrightarrow & (\mathscr{E}\boxtimes\mathscr{F})\boxtimes(\mathscr{G}\boxtimes\mathscr{H}) & \longrightarrow & ((\mathscr{E}\boxtimes\mathscr{F})\boxtimes\mathscr{G})\boxtimes\mathscr{H} \\ \downarrow & & & & \downarrow \\ \mathscr{E}\boxtimes((\mathscr{F}\boxtimes\mathscr{G})\boxtimes\mathscr{H}) & & \longrightarrow & & (\mathscr{E}\boxtimes(\mathscr{F}\boxtimes\mathscr{G}))\boxtimes\mathscr{H} \end{array} \tag{3.10}$$

- Distributivity commutes with functor compositions:

$$\begin{array}{ccc} f_*\circ g_*(\mathscr{E})\boxtimes h_*\circ k_*(\mathscr{F}) & \xrightarrow{\text{comp}_*\boxtimes\text{comp}_*} & (f\circ g)_*(\mathscr{E})\boxtimes(h\circ k)_*(\mathscr{F}) \\ \downarrow{\scriptstyle \xi_*} & & \\ (f\times h)_*(g_*(\mathscr{E})\boxtimes k_*(\mathscr{F})) & & \downarrow{\scriptstyle \xi_*} \\ \downarrow{\scriptstyle \xi_*} & & \\ (f\times h)_*(g\times k)_*(\mathscr{E}\boxtimes\mathscr{F}) & \xrightarrow{\text{comp}_*} & ((f\times h)\circ(g\times k))_*(\mathscr{E}\boxtimes\mathscr{F}) \end{array} \tag{3.11}$$

and likewise for ξ^*.

- Distributivity commutes with associators

$$
\begin{array}{ccc}
(f_*(\mathscr{E}) \boxtimes g_*(\mathscr{F})) \boxtimes h_*(\mathscr{G}) & \xrightarrow{\alpha^{\boxtimes}} & f_*(\mathscr{E}) \boxtimes (g_*(\mathscr{F}) \boxtimes h_*(\mathscr{G})) \\
\Big\downarrow \xi_* & & \Big\downarrow \xi_* \\
(f \times g)_*(\mathscr{E} \boxtimes \mathscr{F}) \boxtimes h_*(\mathscr{G}) & & f_*(\mathscr{E}) \boxtimes (g \times h)_*(\mathscr{F} \boxtimes \mathscr{G}) \\
\Big\downarrow \xi_* & & \Big\downarrow \xi_* \\
(f \times g \times h)_*((\mathscr{E} \boxtimes \mathscr{F}) \boxtimes \mathscr{G}) & \xrightarrow{\alpha^{\boxtimes}} & (f \times g \times h)_*(\mathscr{E} \boxtimes (\mathscr{F} \boxtimes \mathscr{G}))
\end{array}
\tag{3.12}
$$

- Distributivity commutes with base-change morphisms

$$
\begin{array}{ccc}
f^* \circ g_*(\mathscr{E}) \boxtimes h^* \circ k_*(\mathscr{F}) & \xrightarrow{\text{b.c.}(f,g) \boxtimes \text{b.c.}(h,k)} & g'_* \circ f'_*(\mathscr{E}) \boxtimes k'_* h'^*(\mathscr{F}) \\
\Big\downarrow \xi_* \circ \xi^* & & \Big\downarrow \xi_* \circ \xi^* \\
(f \times h)^* \circ (g \times k)_*(\mathscr{E} \boxtimes \mathscr{F}) & \xrightarrow{\text{b.c.}(f \times h, g \times k)} & (g' \circ k')_* \circ (f' \times h')^*(\mathscr{E} \boxtimes \mathscr{F})
\end{array}
\tag{3.13}
$$

- Distributivity preserves trivialisations

$$
\begin{array}{ccc}
\mathscr{E} \boxtimes \mathscr{F} & \xleftarrow[\text{triv}_*]{} & \mathrm{Id}_{AB*}(\mathscr{E} \boxtimes \mathscr{F}) \\
{\scriptstyle \text{triv}_* \boxtimes \text{triv}_*} \nwarrow & & \nearrow {\scriptstyle \xi_*} \\
& \mathrm{Id}_{A*}(\mathscr{E}) \boxtimes \mathrm{Id}_{B*}(\mathscr{F}) &
\end{array}
\tag{3.14}
$$

- Distributivity is compatible with unitors

$$
\begin{array}{ccc}
f_*(\mathscr{E}) \boxtimes \mathscr{O}_B & \xrightarrow{\rho_{f_*(\mathscr{E})}} & \pi_A^{AB*} \circ f_*(\mathscr{E}) \\
\Big\downarrow \xi_* & & \Big\downarrow \text{b.c.} \\
(f \times \mathrm{Id}_B)_*(\mathscr{E} \boxtimes \mathscr{O}_B) & \xrightarrow{(f \times \mathrm{Id}_B)_*(\rho_{\mathscr{E}})} & (f \times \mathrm{Id}_B)_* \circ \pi_A^{AB*}(\mathscr{E})
\end{array}
\tag{3.15}
$$

- Unitors commute with the unit and counit natural transformations,

$$
\begin{array}{ccccc}
(f \times g)_*(\mathscr{O}_{A \times B}) & & \xleftarrow[\text{unit}]{} & & \mathscr{O}_{U \times V} \\
\rho^{\boxtimes} \Big\uparrow & & & & \rho^{\boxtimes} \Big\uparrow \\
(f \times g)_*(\mathscr{O}_A \boxtimes \mathscr{O}_B) & \xleftarrow[\xi_*]{} & f_*(\mathscr{O}_A) \boxtimes f_*(\mathscr{O}_B) & \xleftarrow[\text{unit} \boxtimes \text{unit}]{} & \mathscr{O}_U \boxtimes \mathscr{O}_V
\end{array}
\tag{3.16}
$$

and likewise for counits.

- Left- and right-unitors are compatible with each other

$$\begin{array}{ccc} \mathscr{E} \boxtimes \mathscr{O}_B \boxtimes \mathscr{F} & \xrightarrow{\rho^{\boxtimes}_{\mathscr{E}}} & \pi_A^{AB*}(\mathscr{E}) \boxtimes \mathscr{F} \\ \downarrow{\scriptstyle \lambda_{\mathscr{F}}} & & \downarrow{\scriptstyle \xi^*} \\ \mathscr{E} \boxtimes \pi_C^{BC*}(\mathscr{F}) & \xrightarrow{\xi^*} & \pi_{AC}^{ABC*}(\mathscr{E} \boxtimes \mathscr{F}) \end{array} \tag{3.17}$$

Definition 3.27. A *braided monoidal geofibred category* is a monoidal geofibred category, along with natural isomorphisms

$$\sigma_{A,B} \colon \mathscr{E} \boxtimes_{A,B} \mathscr{F} \to j^{B,A}_{A,B*}(\mathscr{F} \boxtimes_{A,B} \mathscr{E})\,,$$

where $j^{A,B}_{B,A} \colon A \times B \to B \times A$ is the braiding morphism induced from the fibre product on **Sp**. These isomorphisms must satisfy the usual identities, modified to account for the direct image functor in the definition; that is, the following diagrams must commute:

$$\begin{array}{ccccc} \mathscr{E} \boxtimes \mathscr{F} \boxtimes \mathscr{G} & & \xrightarrow{\sigma_{A,BC}} & & j_*(\mathscr{F} \boxtimes \mathscr{G} \boxtimes \mathscr{E}) \\ \downarrow{\scriptstyle \sigma_{A,B} \boxtimes C} & & & & \\ j_*(\mathscr{F} \boxtimes \mathscr{E}) \boxtimes \mathscr{G} & & & & \uparrow{\scriptstyle \mathrm{comp}_*} \\ \downarrow{\scriptstyle \xi_*} & & & & \\ j_*(\mathscr{F} \boxtimes \mathscr{E} \boxtimes \mathscr{G}) & \xrightarrow{B \boxtimes \sigma_{A,C}} & j_*(\mathscr{F} \boxtimes j_*(\mathscr{G} \boxtimes \mathscr{E})) & \xrightarrow{\xi_*} & j_* \circ j_*(\mathscr{F} \boxtimes \mathscr{G} \boxtimes \mathscr{E}) \end{array} \tag{3.18}$$

and

$$\begin{array}{ccccc} \mathscr{E} \boxtimes \mathscr{F} \boxtimes \mathscr{G} & & \xrightarrow{\sigma_{AB,C}} & & j_*(\mathscr{G} \boxtimes \mathscr{E} \boxtimes \mathscr{F}) \\ \downarrow{\scriptstyle A \boxtimes \sigma_{B,C}} & & & & \\ \mathscr{E} \boxtimes j_*(\mathscr{G} \boxtimes \mathscr{F}) & & & & \uparrow{\scriptstyle \mathrm{comp}_*} \\ \downarrow{\scriptstyle \xi_*} & & & & \\ j_*(\mathscr{E} \boxtimes \mathscr{G} \boxtimes \mathscr{F}) & \xrightarrow{\sigma_{A,C} \boxtimes B} & j_*(j_*(\mathscr{G} \boxtimes \mathscr{E}) \boxtimes \mathscr{F})) & \xrightarrow{\xi_*} & j_* \circ j_*(\mathscr{G} \boxtimes \mathscr{E} \boxtimes \mathscr{F}) \end{array} \;. \tag{3.19}$$

They must also be compatible with distributivity, units, and natural transformations, in that the following diagrams commute:

$$\begin{array}{ccc} f_*(\mathscr{E}) \boxtimes g_*(\mathscr{F}) & \xrightarrow{\sigma_{f_*(\mathscr{E}),g_*(\mathscr{F})}} & j^{AB}_{BA*}(g_*(\mathscr{F}) \boxtimes f_*(\mathscr{E})) \\ \Big\downarrow \xi_* & & \Big\downarrow \mathrm{comp}_* \circ \xi_* \\ (f \times g)_*(\mathscr{E} \boxtimes \mathscr{F}) & \xrightarrow{(f\times g)_*(\sigma_{\mathscr{E},\mathscr{F}})} & j^{AB}_{BA*}(f \times g)_* \circ j^{AB}_{BA*}(\mathscr{F} \boxtimes \mathscr{E}) \end{array} , \tag{3.20}$$

$$\begin{array}{ccccc} j_* \circ \pi^{ABCD*}_{AB}(\mathscr{E} \boxtimes \mathscr{F}) & \xleftarrow[\rho^{\boxtimes}]{} & j_*(\mathscr{E} \boxtimes \mathscr{F} \boxtimes \mathscr{O}_{CD}) & \xleftarrow[\rho^{\boxtimes}]{} & j_*(\mathscr{E} \boxtimes \mathscr{F} \boxtimes \mathscr{O}_C \boxtimes \mathscr{O}_D) \\ \mathrm{b.c.}\circ\xi^* \Big\uparrow & & & & s_{\mathscr{O}_C,\mathscr{F}} \Big\uparrow \\ \pi^{AC*}_A(\mathscr{E}) \boxtimes \pi^{BC*}_B(\mathscr{F}) & & \xrightarrow{\rho^{\boxtimes}\boxtimes\rho^{\boxtimes}} & & \mathscr{E} \boxtimes \mathscr{O}_C \boxtimes \mathscr{F} \boxtimes \mathscr{O}_D \end{array} , \tag{3.21}$$

and

$$\begin{array}{ccc} \mathscr{O}_{BA} & \xrightarrow{\text{unit}} & s_{A,B*}(\mathscr{O}_{AB}) \\ \rho^{\boxtimes} \Big\uparrow & & \rho^{\boxtimes} \Big\uparrow \\ \mathscr{O}_B \boxtimes \mathscr{O}_A & \xrightarrow{s_{\mathscr{O}_B,\mathscr{O}_A}} & s_{A,B*}(\mathscr{O}_A \boxtimes \mathscr{O}_B) \end{array} . \tag{3.22}$$

Definition 3.28. A braided monoidal structure is called symmetric if $\sigma^2_{AB} = \mathrm{Id}$.

3.1.5 Examples of geofibred categories

Sets

A straightforward example of a braided monoidal geofibred category consists of sets and subsets. Let **Sp** be $\mathbf{Set}^{\mathrm{op}}$, the category of sets and functions, and let **Sh** be the category with objects given by pairs of sets (A, X) with $A \subset X$. The Hom-set $\mathrm{Hom}_{\mathbf{Sh}}((A, X), (B, Y))$ has a single element, written $A \leq B$, if $X = Y$ and $B \subset A$;

it is empty otherwise. Define a functor

$$F_{\mathbf{Set}} \colon \mathbf{Sh} \to \mathbf{Sp},$$
$$(A, X) \mapsto X,$$
$$f \colon (A, X) \to (B, X) \mapsto \mathrm{Id}_X .$$

Given a morphism $f \in \mathrm{Hom}_{\mathbf{Sp}}(X, Y) = \mathrm{Hom}_{\mathbf{Set}}(Y, X)$, the direct image functor f_* is given by

$$f_*(A, X) = (f^{-1}(A), Y)$$

and the inverse image functor f^* is given by

$$f^*(B, Y) = (f(B), X).$$

To show these are adjoint, note that

$$f_* \circ f^*(B, Y) = (f^{-1}(f(B)), Y).$$

Then $B \subset f^{-1}(f(B))$ so $B \leq f^{-1}(f(B)) \in \mathrm{Hom}_{\mathbf{Sh}}((B, Y), (f^{-1}(f(B), Y))$. Thus there is a natural transformation $\eta \colon \mathrm{Id}_{\mathbf{Sh}_Y} \to f_* \circ f^*$ with these morphisms as components. Similarly,

$$f^* \circ f_*(A, X) = (f(f^{-1}(A)), X)$$

and $f(f^{-1}(A)) \subset A$ so $f(f^{-1}(A)) \leq A \in \mathrm{Hom}_{\mathbf{Sh}}((f(f^{-1}(A)), X), (A, X))$. Thus there is a natural transformation $\epsilon \colon f^* \circ f_* \to \mathrm{Id}_{\mathbf{Sh}_X}$ with these morphisms as components. These natural transformations satisfy the conditions to be a unit and counit of the adjunction $f^* \dashv f_*$.

Note that for $A, B \subset X$, $A \leq B \in \mathrm{Hom}_{\mathbf{Sh}}((A, X), (B, X))$ is invertible if and only if $A = B$. Thus f_* and f^* form an adjoint equivalence if and only if $f^{-1}(f(A)) = A$

for all A; in other words, if f is a bijection.

Since composition is strictly associative, the natural transformations $\mathrm{comp}_{*,f,g}$ and $\mathrm{comp}^*_{f,g}$ are all identities. Similarly, the trivialisations are also identity maps. Consequently, the compatibility conditions are satisfied trivially.

Limits in $\mathbf{Set}^{\mathrm{op}}$ are colimits in $\mathbf{Set}$, so the terminal object of $\mathbf{Set}^{\mathrm{op}}$ is the empty set, and products are given by disjoint union. The braiding is given by the function $\sigma_{X,Y} \colon X \sqcup Y \to Y \sqcup X$ which exchanges factors. Then define

$$(A, X) \boxtimes (B, Y) = (A \sqcup B, X \sqcup Y).$$

Note that $j^{YX}_{XY*}(B \sqcup A, Y \sqcup X) = (A \sqcup B, X \sqcup Y)$ and so we take the braiding morphisms $(A \sqcup B, X \sqcup Y) \to j^{YX}_{XY*}(B \sqcup A, Y \sqcup)$ to be the unique morphism between these objects.

Bimodules

Let $\mathbf{Sp} = \mathbf{CRing}^{\mathrm{op}}$ be the category of commutative rings. Let $\mathbf{Sh}_R = R\text{-}\mathbf{mod}$ be the category of R-modules. For a ring morphism $f \in \mathrm{Hom}_{\mathbf{Sp}}(S, R) = \mathrm{Hom}_{\mathbf{CRing}}(R, S)$, define functors

$$\begin{aligned} f^* \colon \mathbf{Sh}_R &\to \mathbf{Sh}_S \\ M &\mapsto S \otimes_R M \end{aligned}$$

and

$$\begin{aligned} f_* \colon \mathbf{Sh}_S &\to \mathbf{Sh}_R \\ M &\mapsto M_R. \end{aligned}$$

Then for any $M \in \mathbf{Sh}_R$, there are morphisms

$$M \to (S \otimes_R M)_R$$
$$m \mapsto (1 \otimes m)$$

and

$$S \otimes_R M_R \to N$$
$$(s, n) \mapsto s \cdot n.$$

These give natural transformations which form a unit and counit for the adjunction $f^* \dashv f_*$.

Given $f \in \mathrm{Hom}_{\mathbf{Sh}}(R, S)$ and $g \in \mathrm{Hom}_{\mathbf{Sh}}(S, T)$, there is an equality of functors $g_* \circ f_* = (g \circ f)_*$; similarly, $\mathrm{Id}_{R*} = \mathrm{Id}_{\mathbf{Sh}_R}$, so comp_* and triv_* can be taken as the identity natural transformations. The natural isomorphism $\mathrm{comp}^*(f, g)\colon f^* \circ g^* \cong (g \circ f)^*$ has components given by R-module isomorphism

$$R \otimes_S (S \otimes_T M) \to R \otimes_T M,$$
$$r \otimes s \otimes m \mapsto rf(s) \otimes m.$$

The natural transformation triv^* has components given by the isomorphisms $R \otimes_R M \cong M$.

The terminal object of $\mathbf{Ring}^{\mathrm{op}}$ is Z, with the product given by tensor products. The external product functor $\boxtimes$ can be defined as

$$\boxtimes_{R,S}\colon \mathbf{Sh}_R \times \mathbf{Sh}_S \to \mathbf{Sh}_{R \otimes S}$$
$$(M, N) \mapsto M \otimes_Z N$$

with the obvious choice of braiding.

Geometric categories

Some geometric examples of geofibred categories are given in [36, Section 2]. These include the main example for the remainder of this work: the category **Sp** being the category of schemes, and the category **Sh** having objects as pairs $(X, \mathscr{E})$, where $X \in \mathbf{Sp}$ and $\mathscr{E} \in \mathscr{D}(X)$, and with no morphisms between $(X, \mathscr{E})$ and $(Y, \mathscr{F})$ unless $X = Y$, in which case the morphisms are exactly $\mathrm{Hom}_{\mathscr{D}(X)}(\mathscr{E}, \mathscr{F})$. This is considered in more details in Definition 3.66.

3.2 Double categories

This section recalls the definition of a double category. It also recalls a necessary condition to be satisfied for a monoidal structure on a double category to give a monoidal structure on its loose 2-category (as defined in Definition 3.30). For the definition of a monoidal structure on a double category, see Wester Hansen and Shulman [17, Definition 2.10], who provide an explicit list of conditions necessary to define a monoidal structure. This chapter follows the notation of this paper.

Definition 3.29. A *double category* consists of a category of objects $\mathbb{D}_0$ and a category of arrows $\mathbb{D}_1$, along with structure functors

$$
\begin{aligned}
U &\colon \mathbb{D}_0 \to \mathbb{D}_1, \\
S, T &\colon \mathbb{D}_1 \to \mathbb{D}_0, \text{ and} \\
\odot &\colon \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \to \mathbb{D}_1
\end{aligned}
$$

such that

$$S(U_A) = A, \quad S(M \odot N) = S(N),$$
$$T(U_B) = B, \quad T(M \odot N) = T(M)\,,$$

and equipped with natural isomorphisms

$$\mathfrak{a}\colon (M \odot N) \odot P \to M \odot (N \odot P),$$
$$\mathfrak{l}\colon U_B \odot M \to M, \text{ and}$$
$$\mathfrak{r}\colon M \odot U_A \to M,$$

such that $S(\mathfrak{a}), T(\mathfrak{a}), S(\mathfrak{l}), T(\mathfrak{l}), S(\mathfrak{r})$, and $T(\mathfrak{r})$ are all identities, and such that the pentagon diagram

$$\begin{array}{ccccc} M \odot (N \odot (P \odot Q)) & \longrightarrow & (M \odot N) \odot (P \odot Q) & \longrightarrow & ((M \odot N) \odot P) \odot Q \\ \downarrow & & & & \uparrow \\ M \odot ((N \odot P) \odot Q) & & \longrightarrow & & (M \odot (N \odot P)) \odot Q \end{array}$$

and triangle diagram

$$\begin{array}{ccc} N \odot (U_B \odot M) & \xrightarrow{\alpha_{N,U_B,M}} & (N \odot U_B) \odot M \\ & {\scriptstyle N \odot \mathfrak{l}_B} \searrow \quad \swarrow {\scriptstyle \mathfrak{r} \odot M} & \\ & A \odot B & \end{array}$$

commute.

The objects of $\mathbb{D}_0$ will be referred to as objects, the morphisms of $\mathbb{D}_0$ as tight morphisms, the objects of $\mathbb{D}_1$ as loose morphisms, and the morphisms of $\mathbb{D}_1$ as 2-cells.

Definition 3.30. Let $\mathbb{D}$ be a double category, with categories of arrows $\mathbb{D}_1$ and category of objects $\mathbb{D}_0$. Then there is a 2-category $\mathscr{L}(\mathbb{D})$, called the *loose 2-category*, with objects given by $\mathrm{Obj}(\mathbb{D}_0)$, with 1-morphisms given by loose morphisms in $\mathbb{D}$, and 2-cells given by 2-cells of $\mathbb{D}$ where the tight morphisms are both identities.

Definition 3.31. Let $\mathbb{D}$ be a double category and $f\colon A \to B$ be a tight 1-morphism. A *companion* of f is a loose morphism $\hat{f}\colon A \to B$ together with 2-morphisms $\epsilon_{\hat{f}} \in \mathrm{Hom}_{\mathbb{D}_1}(\hat{f}, U_B)$ and $\eta_{\hat{f}} \in \mathrm{Hom}(U_A, \hat{f})$ such that $S(\epsilon_{\hat{f}}) = T(\eta_{\hat{f}}) = f$, $T(\epsilon_{\hat{f}}) = \mathrm{Id}_B$, $S(\eta_{\hat{f}}) = \mathrm{Id}_A$, and

$$\epsilon_{\hat{f}} \circ \eta_{\hat{f}} = U_f$$

and

$$\epsilon_{\hat{f}} \odot \eta = \mathrm{Id}_{\hat{f}}\,.$$

Definition 3.32. A transformation [17, Definition 2.8] α between functors $F, G\colon \mathbb{D} \to \mathbb{E}$ between double categories has loosely strong companions if each component α_A has a loose companion, and for any loose 1-cell $A \xrightarrow{\mathscr{E}} B \in \mathbb{D}_1$, the 2-cell

$$\begin{array}{ccccccc}
FA & \xrightarrow{U_{FA}} & FA & \xrightarrow{F\mathscr{E}} & FB & \xrightarrow{\widehat{\alpha_B}} & GB \\
\Big\| & & \Big\downarrow{\scriptstyle \alpha_A} & {\scriptstyle \alpha_{\mathscr{E}}}\Uparrow & \Big\downarrow{\scriptstyle \alpha_B} & {\scriptstyle \epsilon_{\widehat{\alpha_B}}}\Uparrow & \Big\| \\
FA & \xrightarrow{\widehat{\alpha_A}} & GA & \xrightarrow{G\mathscr{E}} & GB & \xrightarrow{U_{GB}} & GB
\end{array}$$

is invertible.

3.3 Objects, tight and loose morphisms and 2-cells

This section defines the categories of objects and arrows for a double category constructed from a geofibred category. In the remainder of the chapter, the geofibred category F is considered fixed, and so the dependence on this choice will not be made explicit in the notation.

Definition 3.33. Let $F\colon \mathbf{Sh} \to \mathbf{Sp}$ be a geofibred category. Define the category of objects $\mathbb{D}_0$ associated to F to be the category $\mathbf{Sp}$, with morphisms denoted by vertical diagrams

$$\begin{array}{c} A \\ \Big\downarrow f \\ B \end{array} .$$

Define the category of arrows associated to F to have as objects diagrams of the form

$$A \xrightarrow{\mathscr{E}} B \ ,$$

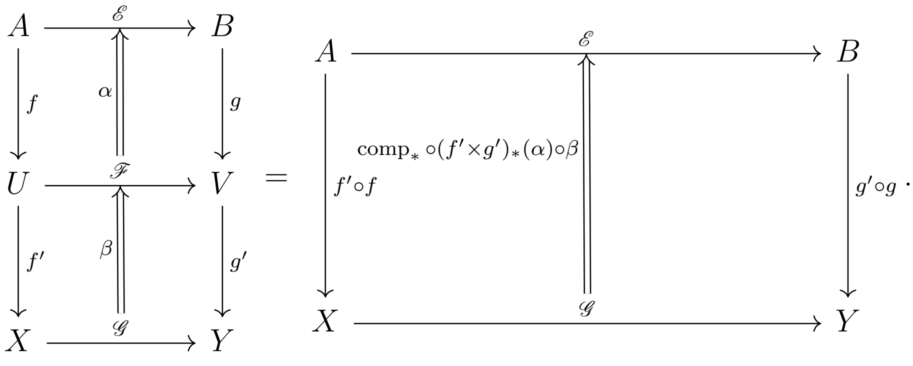
where $A, B \in \mathbb{D}_0$ and $\mathscr{E} \in \mathbf{Sp}_{A\times B}$. The Hom-set $\mathrm{Hom}_{\mathbb{D}_1}(\ A \xrightarrow{\mathscr{E}} B \ , \ U \xrightarrow{\mathscr{F}} V \ )$ is given by all diagrams of the form

$$\begin{array}{ccc} A & \xrightarrow{\mathscr{E}} & B \\ \Big\downarrow f & \alpha \Uparrow & \Big\downarrow g \\ U & \xrightarrow{\mathscr{F}} & V \end{array}$$

where $f \in \mathrm{Hom}_{\mathrm{Sp}}(A, U)$, $g \in \mathrm{Hom}_{\mathrm{Sp}}(B, V)$, and $\alpha\colon \mathscr{F} \to (f \times g)_*(\mathscr{E})$.

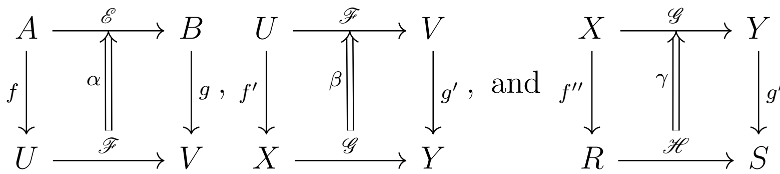
Recall that morphisms in $\mathbb{D}_0$ are considered as tight morphisms in the double category, while objects of $\mathbb{D}_1$ are considered as loose morphisms. The morphisms of $\mathbb{D}_1$ form the 2-cells; this gives justification to the format of the diagrams in this definition. Note that Wester Hansen and Shulman [17] denote 2-cells with downward pointing arrows. Here, the arrows are drawn pointing upwards in keeping with the fact that the morphism α has as its domain the object labelling the bottom horizontal arrow, and its codomain is the object labelling the top horizontal arrow. The 2-cell itself is still considered a morphism from the top line to the bottom line.

Composition within the category $\mathbb{D}_1$ corresponds to the tight composition of 2-cells; diagramatically, this is depicted by stacking 2-cells vertically. The composition is defined by:

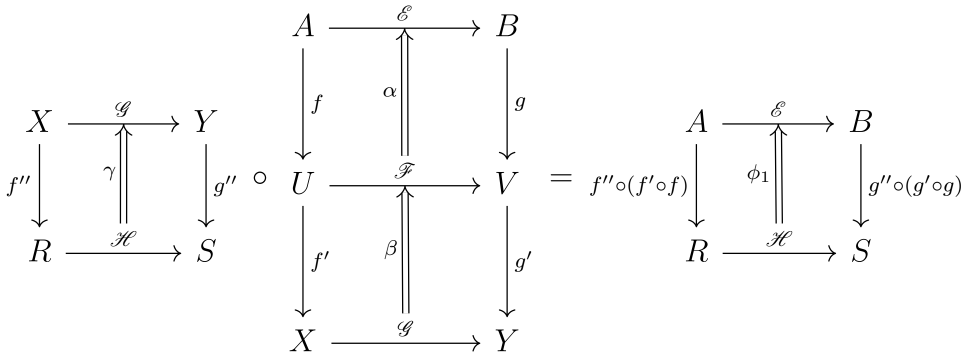


Lemma 3.34. *Composition in* $\mathbb{D}_1$ *is associative.*

Proof. Consider a composable triple of morphisms



in $\mathbb{D}_1$. Let $h = f \times g, h' = f' \times g', h'' = f'' \times g''$. One order of composition for this triple gives



(3.23)

where ϕ_1 is given by

$$
\begin{aligned}
\phi_1 &= \operatorname{comp}_*(h'', h' \circ h) \circ h''_* \left(\operatorname{comp}_*(h', h) \circ h'_*(\alpha) \circ \beta\right)\right) \circ \gamma \\
&= \operatorname{comp}_*(h'', h' \circ h) \circ h''_*(\operatorname{comp}_*(h', h)) \circ h''_*(h'_*(\alpha)) \circ h''_*(\beta) \circ \gamma.
\end{aligned}
$$

The other order of composition gives

$$
\begin{array}{ccc}
U & \xrightarrow{\mathscr{F}} & V \\
\Big\downarrow{\scriptstyle f'} & \Uparrow{\scriptstyle \beta} & \Big\downarrow{\scriptstyle g'} \\
X & \xrightarrow{\mathscr{G}} & Y \\
\Big\downarrow{\scriptstyle f''} & \Uparrow{\scriptstyle \gamma} & \Big\downarrow{\scriptstyle g''} \\
R & \xrightarrow{\mathscr{H}} & S
\end{array}
\circ
\begin{array}{ccc}
A & \xrightarrow{\mathscr{E}} & B \\
{\scriptstyle f}\Big\downarrow & \Uparrow{\scriptstyle \alpha} & \Big\downarrow{\scriptstyle g} \\
U & \xrightarrow{\mathscr{F}} & V
\end{array}
=
\begin{array}{ccc}
A & \xrightarrow{\mathscr{E}} & B \\
{\scriptstyle (f''\circ f')\circ f}\Big\downarrow & \Uparrow{\scriptstyle \phi_2} & \Big\downarrow{\scriptstyle (g''\circ g')\circ g} \\
R & \xrightarrow{\mathscr{H}} & S
\end{array}
\tag{3.24}
$$

where

$$
\phi_2 = \operatorname{comp}_*(h'' \circ h', h) \circ (h'' \circ h')_*(\alpha) \circ \operatorname{comp}_*(h'', h') \circ h''_*(\beta) \circ \gamma.
$$

Since composition of morphisms in **Sp** is strictly associative, $(f'' \circ f') \circ f = f'' \circ (f' \circ f)$ (and likewise for g) so the cells in Equation (3.23) and Equation (3.24) are the same if and only if $\phi_1 = \phi_2$. Now since $\operatorname{comp}_*(h'', h')\colon h''_* \circ h'_* \to (h'' \circ h')_*$ is a natural transformation, there is an equality of morphisms $(h'' \circ h')_*(\alpha) \circ \operatorname{comp}_*(h'', h') = \operatorname{comp}_*(h'', h') \circ h''_*(h'_*(\alpha))$. Thus ϕ_2 becomes

$$
\phi_2 = \operatorname{comp}_*(h'' \circ h', h) \circ \operatorname{comp}_*(h'', h') \circ h''_*(h'_*(\alpha)) \circ h''_*(\beta) \circ \gamma.
$$

Comparing this to ϕ_1, we see that $\phi_1 = \phi_2$ if

$$\mathrm{comp}_*(h'', h' \circ h) \circ (h''_* * \mathrm{comp}_*(h', h)) = \mathrm{comp}_*(h'' \circ h', h) \circ (\mathrm{comp}_*(h'', h') * h_*), \quad (3.25)$$

where the expressions have been written using whiskering to emphasise that the domain of natural transformation is $h''_* \circ h'_* \circ h_*$. Recall from Definition 3.5 that comp_* is part of the coherence data for the lax 2-functor $f \mapsto f_*$. Consequently, it obeys the coherence law expressing that comp_* is associative, which is exactly the condition in Equation (3.25). □

The unit morphism for the object $A \xrightarrow{\mathscr{E}} B$ is the 2-cell

$$\begin{array}{ccc} A & \xrightarrow{\mathscr{E}} & B \\ {\scriptstyle \mathrm{Id}_A}\downarrow & \Uparrow {\scriptstyle \mathrm{Id}_{\mathscr{E}}} & \downarrow{\scriptstyle \mathrm{Id}_B} \\ A & \xrightarrow{\mathscr{E}} & B \end{array} .$$

Definition 3.35. Define a pair of functors

$$S, T \colon \mathbb{D}_1 \to \mathbb{D}_0$$

by taking the left (resp. right) side of the diagrams in Definition 3.33; that is,

$$S\left(\begin{array}{ccc} A & \xrightarrow{\mathscr{E}} & B \\ {\scriptstyle f}\downarrow & \Uparrow {\scriptstyle \alpha} & \downarrow{\scriptstyle g} \\ C & \xrightarrow{\mathscr{F}} & D \end{array}\right) = \begin{array}{c} A \\ \downarrow{\scriptstyle f} \\ C \end{array}$$

and

$$T\left(\begin{array}{ccc} A & \xrightarrow{\mathscr{E}} & B \\ {\scriptstyle f}\downarrow & \Uparrow {\scriptstyle \alpha} & \downarrow{\scriptstyle g} \\ C & \xrightarrow{\mathscr{F}} & D \end{array}\right) = \begin{array}{c} B \\ \downarrow{\scriptstyle g} \\ D \end{array} .$$

Lemma 3.36. *The fibre of* $(S, T)\colon \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$ *over* (A, B)*, denoted* $(\mathbb{D}_1)_{A,B}$*, is isomorphic to the category* $\mathbf{Sh}^{op}_{A\times B}$*.*

Proof. Define a functor

$$F\colon \mathbf{Sh}^{\mathrm{op}}_{A\times B} \to (\mathbb{D}_1)_{A,B},$$

$$\mathscr{E} \mapsto \; A \xrightarrow{\;\mathscr{E}\;} B \;,$$

$$\alpha \in \mathrm{Hom}_{\mathbf{Sh}^{\mathrm{op}}_{AB}}(\mathscr{E}, \mathscr{F}) \mapsto \begin{array}{ccc} A & \xrightarrow{\;\mathscr{E}\;} & B \\ {\scriptstyle \mathrm{Id}_A}\downarrow & \Uparrow{\scriptstyle \alpha} & \downarrow{\scriptstyle \mathrm{Id}_B} \\ A & \xrightarrow[\;\mathscr{F}\;]{} & B \end{array} .$$

Note that α is in the opposite category $\mathbf{Sh}^{\mathrm{op}}_{AB}$, and so is a morphism $\alpha \in \mathrm{Hom}_{\mathbf{Sh}_{AB}}(\mathscr{F}, \mathscr{E})$.

The functor F is a bijection on objects and Hom-sets, so defines an isomorphism of the categories. □

3.4 Constructing functors and natural transformations

To define the double category structure, it is necessary to define functors valued in $\mathbb{D}_1$. These are often constructed on object by compositions of direct image and inverse image functors and the external product functor. However, the morphisms of $\mathbb{D}_1$ will often need a "correction" term added to account for the fact that morphisms have codomain given by the direct image of an object.

For example, consider defining a functor $F\colon \mathbb{D}_1 \to \mathbb{D}_1$ which on objects is given by taking the external product with a fixed shape $\mathscr{X} \in \mathbf{Sh}_{XY}$; that is, such that

$$F\left(A \xrightarrow{\;\mathscr{E}\;} B \right) = AX \xrightarrow{\;\mathscr{E}\boxtimes\mathscr{X}\;} BY$$

It still remains to define F on morphisms. Let

$$\phi = \begin{array}{ccc} A & \xrightarrow{\mathscr{E}} & B \\ {\scriptstyle f}\downarrow & \overset{\alpha}{\Uparrow} & \downarrow{\scriptstyle g} \\ U & \xrightarrow{\mathscr{F}} & V \end{array}$$

be a morphism in $\mathbb{D}_1$. Then $\alpha \in \mathrm{Hom}(\mathscr{F}, (f \times g)_*(\mathscr{E}))$. The natural starting point for the construction $F(\phi)$ is the morphism

$$\alpha \boxtimes \mathscr{X} : \mathscr{F} \boxtimes \mathscr{X} \to (f \times g)_*(\mathscr{E}) \boxtimes \mathscr{X}.$$

However, although the domain of this morphism is correct, its codomain is not.

To rectify this, recall there is a natural isomorphism $\xi_* \colon f_* \boxtimes \mathrm{Id}_* \to (f \times \mathrm{Id})_*$. The component at $(\mathscr{E}, \mathscr{X})$ is a morphism

$$\xi_{*\mathscr{E},\mathscr{X}} \colon (f \times g)_*(\mathscr{E}) \boxtimes \mathscr{X} \to (f \times g \times \mathrm{Id}_{XY})_*(\mathscr{E} \boxtimes \mathscr{X}).$$

Composing this with the morphism $\alpha \boxtimes \mathscr{X}$ gives a definition for $F(\phi)$:

$$F(\phi) = \begin{array}{ccc} AX & \xrightarrow{\mathscr{E}\boxtimes\mathscr{X}} & BY \\ {\scriptstyle f\times\mathrm{Id}_X}\downarrow & \overset{\xi_{*\mathscr{E},\mathscr{X}}\circ(\alpha\boxtimes\mathscr{X})}{\Uparrow} & \downarrow{\scriptstyle g\times\mathrm{Id}_Y} \\ UX & \xrightarrow{\mathscr{F}\boxtimes\mathscr{X}} & VY \end{array}$$

This process is formalised in the Lemma 3.37. Similarly, Lemma 3.38 shows that given two functors constructed in this manner, it is possible to specify a natural transformation between the two by constructing a natural transformation between the "obvious" functors.

3.4.1 Functors

Lemma 3.37. *Let $\mathscr{C}$ be a category and let $F_S, F_T\colon \mathscr{C} \to \mathbb{D}_0$ be functors. This gives a fibration $(F_S, F_T)\colon \mathscr{C} \to \mathbb{D}_0 \times \mathbb{D}_0$. Let*

$$\{\mathscr{C}_i\}_{i\in I}$$

be a collection of subcategories for some index set I, and let $F_S^I, F_T^I\colon I \to \mathrm{Obj}(\mathbf{Sh})$ be morphisms of sets such that $\mathscr{C}_i$ is contained in the fibre of $\mathscr{C}$ over $(F_S^I(i), F_T^I(i))$. Suppose

$$F_i\colon \mathscr{C}_i \to \mathbf{Sh}^{opp}_{F_S^I(i),F_T^I(i)}$$

is a collection of functors, and for $c \in \mathscr{C}_i$, $d \in \mathscr{C}_j$ and $\phi \in \mathrm{Hom}_{\mathscr{C}}(c, d)$, the following data is specified:

1. *a functor $\Phi_\phi\colon \mathscr{C}_i \to \mathscr{C}_j$;*
2. *a morphism $\tilde{\phi} \in \mathrm{Hom}_{\mathscr{C}_j}(\Phi_\phi(c), d)$; and*
3. *a natural transformation $\xi^{F,\phi}\colon F_j \circ \Phi_\phi \to (F_S(\phi) \times F_T(\phi))_* \circ F_i$.*

Suppose these satisfy the following conditions:

1. *the assignments $\phi \mapsto \Phi_\phi$ and $\phi \mapsto \xi^{F,\phi}$ respect identities; that is, $\Phi_{\mathrm{Id}_c} = \mathrm{Id}_{c_i}$ and $\xi^{F,\mathrm{Id}_c} = \mathrm{Id}_{F_i}$;*
2. *the assignment $\phi \mapsto \Phi_\phi$ is functorial in the sense that $\Phi_{\psi\circ\phi} = \Phi_\psi \circ \Phi_\phi$;*
3. *the assignment $\phi \mapsto \xi^{F,\phi}$ is functorial in the sense that $\xi^{F,\psi\circ\phi} = (F_S(\psi) \times F_T(\psi))_*(\xi^{F,\phi}) \circ (\xi^{F,\psi} \circ \Phi_\phi)$; and*
4. *the assignment $\phi \mapsto \tilde{\phi}$ satisfies the condition $\widetilde{\psi \circ \phi} = \Phi_\psi(\tilde{\phi}) \circ \tilde{\psi}$ (and hence $\tilde{\mathrm{Id}} = \mathrm{Id}$).*

Then there is a functor $F \colon \mathscr{C} \to \mathbb{D}_1$ given on objects by

$$F(c) = \quad X \xrightarrow{F_i(c)} Y$$

where $c \in \mathscr{C}_i$, and on morphisms by

$$F(\phi) = \begin{array}{ccc} F^I_S(i) & \xrightarrow{\quad F_i(c) \quad} & F^I_T(i) \\ {\scriptstyle F_S(\phi)}\big\downarrow & \xi^{F,\phi}_c \circ F_j(\tilde{\phi}) \Uparrow & \big\downarrow{\scriptstyle F_T(\phi)} \\ F^I_S(j) & \xrightarrow[\quad F_j(d) \quad]{} & F^I_T(j) \end{array} .$$

Proof. The first condition to check is that F preserves identities. For $c \in \mathscr{C}_i$, let $A = F^I_S(i)$ and $B = F^I_T(i)$. Then

$$F(\mathrm{Id}_c) = \begin{array}{ccc} A & \xrightarrow{\quad F_i(c) \quad} & B \\ {\scriptstyle \mathrm{Id}_A}\big\downarrow & \xi^{F,\mathrm{Id}_c}_c \circ F_i(\tilde{\mathrm{Id}_c}) \Uparrow & \big\downarrow{\scriptstyle \mathrm{Id}_B} \\ A & \xrightarrow[\quad F_i(c) \quad]{} & B \end{array}$$

Thus since $\tilde{\mathrm{Id}_c} = \mathrm{Id}_c$ and $\xi^{F,\mathrm{Id}_c} = \mathrm{Id}_{\mathscr{F}_i}$, this gives the identity morphism.

Next it is necessary to show that F respects composition. Let $\phi \in \mathrm{Hom}_{\mathscr{C}}(c, d)$ and $\psi \in \mathrm{Hom}_{\mathscr{C}}(d, e)$, where $c \in \mathscr{C}_i$, $d \in \mathscr{C}_j$, $e \in \mathscr{C}_k$. Let $U = F^I_S(j)$, $W = F^I_T(j)$, $X = F^I_S(k)$ and $Z = F^I_T(k)$. Then

$$F(\psi \circ \phi) = \begin{array}{ccc} A & \xrightarrow{\quad F_i(c) \quad} & B \\ {\scriptstyle F_S(\psi\circ\phi)}\big\downarrow & \xi^{F,\psi\circ\phi}_c \circ F_k(\tilde{\psi\circ\phi}) \Uparrow & \big\downarrow{\scriptstyle F_T(\psi\circ\phi)} \\ X & \xrightarrow[\quad F_k(e) \quad]{} & Z \end{array} \tag{3.26}$$

and

$$F(\psi) \circ F(\phi) = \begin{CD} A @>{F_i(c)}>{\Uparrow\, \xi_c^{F,\phi} \circ F_j(\tilde{\phi})}> B \\ @V{F_S(\phi)}VV @VV{F_T(\phi)}V \\ U @>{F_j(d)}>{\Uparrow\, \xi_d^{F,\psi} \circ F_k(\tilde{\psi})}> V \\ @V{F_S(\psi)}VV @VV{F_T(\psi)}V \\ X @>{F_k(e)}>> Z \end{CD}$$

$$= \begin{CD} A @>{F_i(c)}>{\Uparrow\, \beta}> B \\ @V{F_S(\psi) \circ F_S(\phi)}VV @VV{F_T(\psi) \circ F_T(\phi)}V \\ X @>{F_k(e)}>> Z \end{CD} \quad ,$$

where the morphism β is given by

$$\beta = (F_S(\psi) \times F_T(\psi))_*(\xi_c^{F,\phi} \circ F_j(\tilde{\phi})) \circ \xi_d^{F,\psi} \circ F_k(\tilde{\psi})$$

For conciseness, let $\psi_{st} = (F_S(\psi) \times F_T(\psi))$. Then

$$\begin{aligned} \beta &= \psi_{st*}(\xi_c^{F,\phi}) \circ \psi_{st*}(F_j(\tilde{\phi})) \circ \xi_d^{F,\psi} \circ F_k(\tilde{\psi}) \\ &= \psi_{st*}(\xi_c^{F,\phi}) \circ \xi_{\Phi_\phi(c)}^{F,\psi} \circ (F_k \circ \Phi_\psi)(\tilde{\phi}) \circ F_k(\tilde{\psi}) \end{aligned}$$

where the first line uses the functoriality of ψ_{st*}, and the second the naturality of the transformation $\xi^{F,\psi}$. Recalling the definition of whiskering of natural transformations (Definition 1.6), this can be rewritten as

$$\beta = (\psi_{st*} * \xi^{F,\phi})_c \circ (\xi^{F,\psi} * \Phi_\phi)_c \circ (F_k * \Phi_\psi)(\tilde{\phi}) \circ F_k(\tilde{\psi}).$$

Using condition 3, functoriality of F_k and finally condition 2 gives the sequence of equalities

$$\begin{aligned}\beta &= \xi^{F,\psi\circ\phi} \circ F_k \left(\Phi_\psi(\tilde{\phi}) \circ \tilde{\psi}\right) \\ &= \xi^{F,\psi\circ\phi} \circ F_k \left(\widetilde{\psi \circ \phi}\right)\end{aligned}$$

which is exactly the morphism in the cell in Equation (3.26). □

3.4.2 Natural transformations

Lemma 3.38. *Let $F, G \colon \mathscr{C} \to \mathbb{D}_1$ be functors constructed from the data described in Lemma 3.37, with both using the same index set I. Using the same notation as in the lemma, let*

$$f_i \colon F_S^I(i) \to G_S^I(i)$$

and

$$h_i \colon F_T^I(i) \to G_T^I(i)$$

be morphisms in **Sp** *such that for any $\phi \in \mathrm{Hom}_{\mathscr{C}}(c, d)$, there is an equality of morphisms*

$$F_S(\phi) \circ f_i = f_j \circ G(\phi) \tag{3.27}$$

(and likewise for g_i).

Let

$$\eta^i \colon G_i \to (f_i \times h_i)_* \circ F_i$$

be a collection of natural transformations. For any $c \in \mathscr{C}_i$, define a morphism $\eta_c \colon F(c) \to G(c)$ by

$$\eta_c = \begin{array}{ccc} A & \xrightarrow{F(c)} & C \\ {\scriptstyle f_i}\downarrow & \Uparrow {\scriptstyle \eta_c^i} & \downarrow{\scriptstyle h_i} \\ U & \xrightarrow[G(c)]{} & V \end{array}$$

where $A = F^I_S(i)$, $C = F^I_T(i)$, $U = G^I_S(i)$ *and* $V = G^I_T(i)$.

Then η *is a natural transformation if for all* $c \in \mathscr{C}_i, d \in \mathscr{C}_j$ *and* $\phi \in \mathrm{Hom}_{\mathscr{C}}(c, d)$ *there is an equality of natural transformations*

$$((f_j \times h_j)_* * \xi^{F,\phi}) \circ \left(\eta^j * \Phi_\phi\right) = \left((F_S(\phi) \times F_T(\phi))_* * \eta^i\right) \circ \xi^{G,\phi}, \tag{3.28}$$

where the composition of a natural transformation with a functor denotes whiskering.

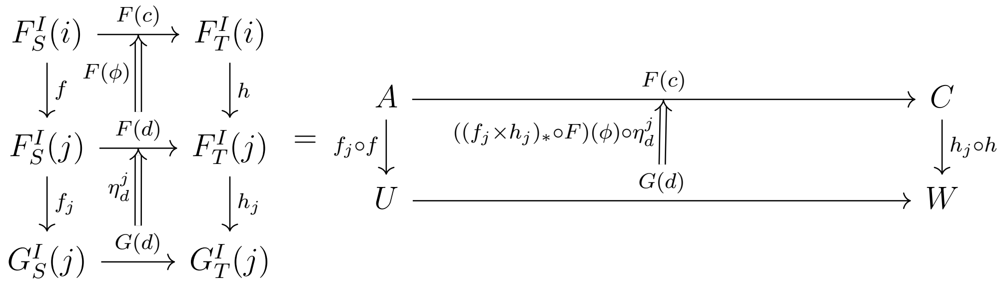
Proof. We need to prove that the collection of morphisms defined in the statement of the lemma is natural. Let $c \in \mathscr{C}_i, d \in \mathscr{C}_j$ and $\phi \in \mathrm{Hom}_{\mathscr{C}}(c, d)$. The condition that the morphisms define a natural transformation becomes the statement that there is a commutative diagram

$$\begin{array}{ccc} F(c) & \xrightarrow{F(\phi)} & F(d) \\ \big\downarrow{\scriptstyle \eta_c} & & \big\downarrow{\scriptstyle \eta_d} \\ G(c) & \xrightarrow{G(\phi)} & G(d) \end{array} .$$

Let $f = F_S(\phi)$ and $h = F_T(\phi)$. The clockwise route around the diagram gives the 2-cell

$$\begin{array}{ccc} F^I_S(i) & \xrightarrow{F(c)} & F^I_T(i) \\ \big\downarrow{\scriptstyle f_i} & {\scriptstyle \eta^i_c}\Uparrow & \big\downarrow{\scriptstyle g_i} \\ G^I_S(i) & \xrightarrow{G(c)} & G^I_T(i) \\ \big\downarrow{\scriptstyle f} & {\scriptstyle G(\phi)}\Uparrow & \big\downarrow{\scriptstyle h} \\ G^I_S(j) & \xrightarrow{G(d)} & G^I_T(j) \end{array} \quad = \quad \begin{array}{ccc} F^I_S(i) & \xrightarrow{\qquad\qquad F(c) \qquad\qquad} & F^I_T(i) \\ {\scriptstyle f\circ f_i}\big\downarrow & {\scriptstyle (f\times h)_*(\eta^i_c)\circ G(\phi)}\Uparrow & \big\downarrow{\scriptstyle h\circ h_i} \\ G^I_S(j) & \xrightarrow[\qquad\qquad G(d) \qquad\qquad]{} & G^I_T(j) \end{array} \tag{3.29}$$

while the anticlockwise route gives



(3.30)

The tight morphisms of spaces on both of these diagrams agree by the assumption in Equation (3.27).

Now $F(\phi)$ can be written as

$$F(\phi) = \xi^{F,\phi} \circ F_j(\tilde{\phi}),$$

and so the morphism inside the cell in Equation (3.30) can be written (using the naturality of $\eta^{U\vec{V}W}$) as

$$\begin{aligned} ((f_j \times h_j)_* \circ F)(\phi) \circ \eta_d &= ((f_j \times h_j)_* \circ \xi^{F,\phi}) \circ ((f_j \times h_j)_* \circ F_j)(\tilde{\phi}) \circ \eta^j_d \\ &= ((f_j \times h_j)_* \circ \xi^{F,\phi}) \circ \eta^j_{\Phi_\phi(c)} \circ G_j(\tilde{\phi}) \\ &= ((f_j \times h_j)_* \circ \xi^{F,\phi}) \circ (\eta^j \circ \Phi_\phi)_c \circ G_j(\tilde{\phi})\,. \end{aligned}$$

The condition in Equation (3.27) gives

$$((f_j \times h_j)_* \circ F)(\phi) \circ \eta_d = (f_j \times h_j)_*(\eta^i) \circ \xi^{G,\phi} \circ G_j(\tilde{\phi})\,.$$

Now by the construction of the functor G from Lemma 3.37, the functor G is given on morphisms by $G(\phi) = \xi^{G,\phi} \circ G_{U,\vec{V},W}(\tilde{\phi})$, so

$$((f_j \times h_j)_* \circ F)(\phi) \circ \eta_d = (f \times h)_*(\eta^i) \circ G(\phi)$$

and the right-hand side of this equation is the morphism in the cell in Equation (3.29) as required. □

A common use case of Lemma 3.38 is when the source category $\mathscr{C}$ is $\mathbb{D}_1$ itself, or possibly a (repeated) fibre product of copies of $\mathbb{D}_1$. To continue with the example of taking the external tensor product with a fixed sheaf $\mathscr{X}$ (as introduced in the opening paragraphs of this section), take $\mathscr{C} = \mathbb{D}_1$ and let

$$\phi = \begin{array}{ccc} X & \xrightarrow{\mathscr{E}_0} & Y \\ {\scriptstyle r}\downarrow & \Uparrow{\scriptstyle \alpha_0} & \downarrow{\scriptstyle h} \\ U & \xrightarrow[\mathscr{F}_0]{} & V \end{array} .$$

Define a functor Φ_ϕ by applying the direct image functor $(r \times h)_*$ where necessary; explicitly,

$$\Phi_f \left(\begin{array}{ccc} X & \xrightarrow{\mathscr{E}} & Y \\ {\scriptstyle \mathrm{Id}_X}\downarrow & \Uparrow{\scriptstyle \alpha} & \downarrow{\scriptstyle \mathrm{Id}_Y} \\ X & \xrightarrow[\mathscr{F}]{} & Y \end{array} \right) = \begin{array}{ccc} U & \xrightarrow{(r\times h)_*(\mathscr{E})} & V \\ {\scriptstyle \mathrm{Id}_U}\downarrow & \Uparrow{\scriptstyle (r\times h)_*(\alpha)} & \downarrow{\scriptstyle \mathrm{Id}_V} \\ U & \xrightarrow[(r\times h)_*(\mathscr{F})]{} & V \end{array} .$$

The morphism $\tilde{\phi}$ is then taken to be the cell

$$\tilde{\phi} = \begin{array}{ccc} U & \xrightarrow{(r\times h)_*(\mathscr{E}_0)} & V \\ {\scriptstyle \mathrm{Id}_U}\downarrow & \Uparrow{\scriptstyle \alpha_0} & \downarrow{\scriptstyle \mathrm{Id}_V} \\ U & \xrightarrow[\mathscr{F}_0]{} & V \end{array}$$

which clearly satisfies $F_\alpha(f) = F(\alpha_0) = F_{UV}(\tilde{\phi})$.

3.5 Double category structure

This section will equip the categories of objects and arrows constructed in Definition 3.33 with the structure of a double category. The functors will be constructed using Lemma 3.37 and the natural transformations exhibiting compatibility will be constructed from Lemma 3.38. Frequently, Theorem 3.17 will be used to show that the natural transformations satisfy coherence conditions, by showing that there is a unique SGNT between two functors.

3.5.1 Loose composition

The loose composition of 2-cells is inspired by the composition of Fourier–Mukai kernels. Recall that for any $\mathscr{E} \in \mathscr{D}(X \times Y), \mathscr{F} \in \mathscr{D}(Y \times Z)$, their composition as Fourier–Mukai kernels is equal to

$$\begin{aligned} \mathscr{F} \odot \mathscr{E} &= \pi^{XYZ}_{XZ*} \left(\pi^{XYZ*}_{YZ}(\mathscr{F}) \otimes \pi^{XYZ*}_{XY}(\mathscr{E})\right) \\ &= \pi^{XYZ}_{XZ*} \circ i^{124*}_{1234} \left(\mathscr{E} \boxtimes \mathscr{F}\right) . \end{aligned}$$

This can be generalised to give the definition of the loose composition functor $\odot$.

This functor can be constructed using Lemma 3.37. First, consider $\mathscr{C} = \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1$ as a category fibred by

$$S \times (S,T) \colon \mathscr{C} \to \mathbb{D}_0 \times \mathbb{D}_0 \times \mathbb{D}_0 .$$

Partition $\mathscr{C}$ into subcategories $\mathscr{C}_{ABC}$, where each of these subcategories is the fibre over the object (A, B, C). Explicitly, the objects of $\mathscr{C}_{ABC}$ are of the form

$$A \xrightarrow{\mathscr{E}} B \xrightarrow{\mathscr{F}} C$$

and the morphisms are globular cells

$$\begin{array}{ccccc} A & \xrightarrow{\mathscr{E}} & B & \xrightarrow{\mathscr{F}} & C \\ \downarrow & \alpha \Uparrow & \downarrow & \beta \Uparrow & \downarrow \\ A & \xrightarrow{\mathscr{P}} & B & \xrightarrow{\mathscr{Q}} & C . \end{array}$$

This gives an obvious isomorphism of categories $\mathscr{C}_{ABC} \cong (\mathbb{D}_1)_{AB} \times (\mathbb{D}_1)_{BC} \cong \mathbf{Sh}^{\mathrm{opp}}_{AB} \times \mathbf{Sh}^{\mathrm{opp}}_{BC}$. For the remainder of the construction, this identification will be used implicitly when defining functors.

Recall there is an external product functor $\boxtimes\colon \mathbf{Sh}^{\mathrm{opp}}_{AB} \times \mathbf{Sh}^{\mathrm{opp}}_{BC} \to \mathbf{Sh}^{\mathrm{opp}}_{ABBC}$. Define

$$\begin{aligned} \odot_{ABC}\colon \mathscr{C}_{ABC} &\to \mathbf{Sh}^{\mathrm{opp}}_{AC} \\ (\mathscr{E}, \mathscr{F}) &\mapsto \pi^{ABC}_{AC*} \circ i^{124*}_{1234}(\mathscr{E} \boxtimes \mathscr{F})\,. \end{aligned}$$

Let $\phi \in \mathrm{Hom}_{\mathscr{C}}(c, d)$ for some $c, d \in \mathscr{C}$. Then ϕ has the form

$$\phi = \begin{array}{ccccc} A & \xrightarrow{\mathscr{E}} & B & \xrightarrow{\mathscr{F}} & C \\ \downarrow{\scriptstyle f} & \alpha\Uparrow & \downarrow{\scriptstyle g} & \beta\Uparrow & \downarrow{\scriptstyle h} \\ U & \xrightarrow[\mathscr{P}]{} & V & \xrightarrow[\mathscr{Q}]{} & W \end{array}\,.$$

Define a functor

$$\Phi_\phi : \mathbf{Sh}^{\mathrm{opp}}_{AB} \times \mathbf{Sh}^{\mathrm{opp}}_{BC} \to \mathbf{Sh}^{\mathrm{opp}}_{UV} \times \mathbf{Sh}^{\mathrm{opp}}_{VW}$$

by $\Phi_\phi = (f \times g)_* \times (g \times h)_*$. Define

$$\widetilde{\phi} \in \mathrm{Hom}_{\mathbf{Sh}^{\mathrm{opp}}_{UV} \times \mathbf{Sh}^{\mathrm{opp}}_{VW}}(\Phi_\phi(c), d)$$

to be the pair of morphisms (α, β). The final piece of data needed to construct the loose composition functor is the natural transformation

$$\xi^{\odot,\phi}\colon\ \odot_{UVW} \circ \Phi_\phi \to (f \times h)_* \circ \odot_{ABC}\,. \tag{3.31}$$

Note that

$$\odot_{UVW} \circ \Phi_\phi\colon \mathbf{Sh}^{\mathrm{opp}}_{AB} \times \mathbf{Sh}^{\mathrm{opp}}_{BC} \to \mathbf{Sh}^{\mathrm{opp}}_{UW}$$

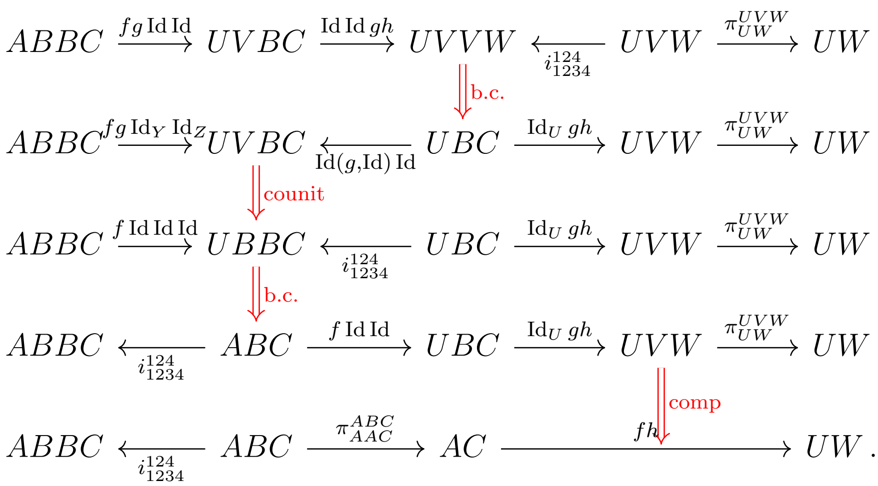
is given by the composition of functors

$$\odot_{UVW} \circ \Phi_\phi = \pi^{UVW}_{UW*} \circ i^{124*}_{1234} \circ (fggh)_* \,.$$

Using the diagramatic notation for functors introduced in Subsection 3.1.3, this can be expressed as

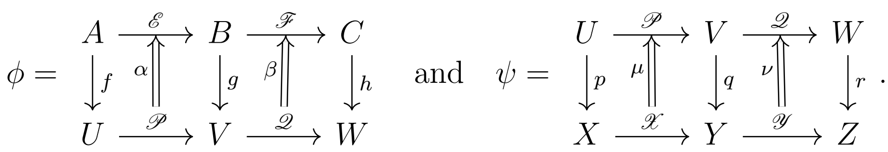
$$ABBC \xrightarrow{fg\,\mathrm{Id}\,\mathrm{Id}} UVBC \xrightarrow{\mathrm{Id}\,\mathrm{Id}\,gh} UVVW \xleftarrow[i^{124}_{1234}]{} UVW \xrightarrow{\pi^{X'Y'Z'}_{X'Z'}} UW \;\;.$$

There is a natural transformation between this functor and $(f \times h)_* \circ \odot_{ABC}$, given by



Lemma 3.39. *The data $(\odot_{ABC}, \Phi, \xi^{\odot})$ satisfy the conditions in Lemma 3.37.*

Proof. It is immediately the case that $\Phi_{\mathrm{Id}_c} = \mathrm{Id}_{\mathscr{C}_{ABC}}$ and $\xi^{\odot,\mathrm{Id}_c} = \mathrm{Id}_{\odot_{ABC}}$. Now let



The next condition to check is that

$$\xi^{\odot,\psi\circ\phi} = \left((p \times r)_* \circ \xi^{\odot,\phi}\right) \circ \left(\xi^{\odot,\psi} \circ \Phi_\phi\right) \,. \tag{3.32}$$

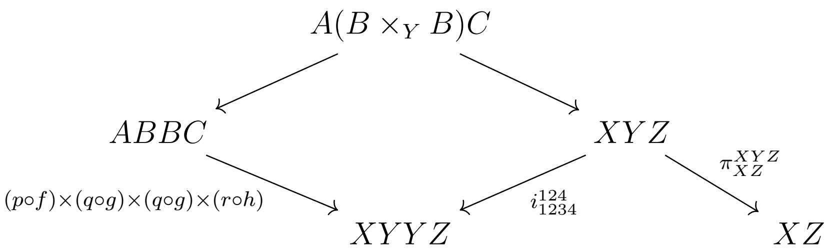
These are natural transformations between the functors

$$F_1 = \odot_{UVW} \circ \Phi_\psi \circ \Phi_\phi = \pi^{XYZ}_{XZ*} \circ i^{124*}_{1234} \circ ((p \circ f) \times (r \circ h))_*$$

and

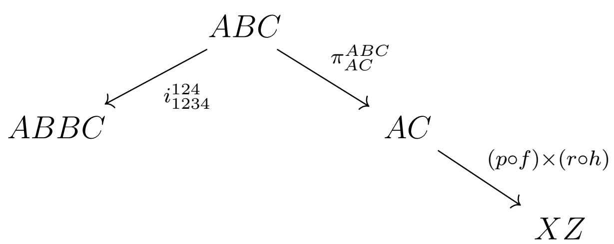
$$F_2 = ((p \circ f) \times (r \circ h))_* \circ \odot_{ABC} = ((p \circ f) \times (r \circ h))_* \circ \pi^{ABC}_{AC*} \circ i^{124*}_{1234} \, .$$

These can be shown to be equal using Theorem 3.17 as follows. The functor F_2 is (after composing direct and inverse image functors) of the form $f_* g^*$, and so is certainly isomorphic to its roof. The roof of F_1 is given by



,

whilst the roof of F_2 is



.

It is clear that the maps (a_{F_2}, b_{F_2}) factor through (a_{F_1}, b_{F_1}). This latter pair is a universal monomorphism, so by Theorem 3.17 there is a unique transformation to this functor, and hence the two natural transformations in Equation (3.32) are equal.

The final condition to check is that

$$\widetilde{\psi \circ \phi} = \Phi_\psi(\tilde{\phi}) \circ \tilde{\psi} \, .$$

Note that

$$\psi \circ \phi = \begin{array}{ccccc} A & \xrightarrow{\mathscr{E}} & B & \xrightarrow{\mathscr{F}} & C \\ {\scriptstyle p\circ f}\downarrow & \Uparrow{\scriptstyle (p\times q)_*(\alpha)\circ\mu} & {\scriptstyle q\circ g}\downarrow & \Uparrow{\scriptstyle (q\times r)_*(\beta)\circ\nu} & \downarrow{\scriptstyle r\circ h} \\ X & \xrightarrow[\mathscr{X}]{} & Y & \xrightarrow[\mathscr{Y}]{} & Z\,. \end{array} .$$

Then

$$\begin{aligned} \widetilde{\psi \circ \phi} &= ((p \times q)_*(\alpha) \circ \mu, (q \times r)_*(\beta) \circ \nu) \\ &= ((p \times q)_*(\alpha), (q \times r)_*(\beta)) \circ (\mu, \nu) \\ &= \Phi_\psi(\tilde{\phi}) \circ \tilde{\psi} \end{aligned}$$

as required. □

Definition 3.40. The loose composition functor $\odot\colon \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \to \mathbb{D}_1$ associated to a geofibred category is defined as the functor constructed from Lemma 3.37 using the data $(\odot_{ABC}, \Phi, \xi^{\odot})$.

3.5.2 Loose unit

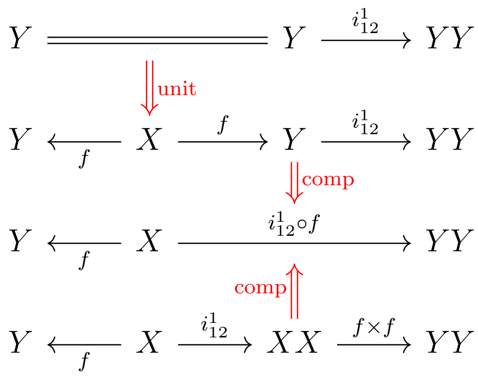
Let $X \in \mathbf{Sp}$. Recall from Definition 3.22 that $i^1_{12}\colon X \to X \times X$ is the diagonal map, and that $\mathscr{O}_{\Delta_X} = i^1_{12*}(\mathscr{O}_X)$ is the diagonal sheaf.

Definition 3.41. The loose unit functor $U\colon \mathbb{D}_0 \to \mathbb{D}_1$ is defined by

$$U(X) = \quad X \xrightarrow{\mathscr{O}_{\Delta_X}} X \quad ,$$

$$U\left(\begin{array}{c} X \\ \downarrow{\scriptstyle f} \\ Y \end{array}\right) = \begin{array}{ccc} X & \xrightarrow{\mathscr{O}_{\Delta_X}} & X \\ {\scriptstyle f}\downarrow & \Uparrow{\scriptstyle \nu_f} & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[\mathscr{O}_{\Delta_Y}]{} & Y \end{array} .$$

The map ν_f is constructed as the natural transformation



evaluated at $\mathscr{O}_Y$.

That this is indeed a functor follows from Theorem 3.17.

3.5.3 Loose unitor natural isomorphism

We need to define left and right unitors for loose composition; that is, natural isomorphisms

$$\mathfrak{l}^{\odot} \colon U_B \odot M \Rightarrow M \,,$$
$$\mathfrak{r}^{\odot} \colon M \odot U_A \Rightarrow M \,.$$

The definition for $\mathfrak{r}^{\odot}$ is given below; $\mathfrak{l}^{\odot}$ can be defined similarly.

This natural isomorphism will be constructed using Lemma 3.38. This requires that the source and target functors are of the form constructed from Lemma 3.37. Partition the category $\mathbb{D}_1$ based on the fibration

$$(S, T) \colon \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0 \,.$$

Recall by Lemma 3.36 that $(\mathbb{D}_1)_{XY} \cong \mathbf{Sh}^{\mathrm{op}}_{X\times Y}$. This identification will be used without further mention in this section.

The target functor of $\mathfrak{r}^{\odot}$ is

$$G = \mathrm{Id}_{\mathbb{D}_1} \colon \mathbb{D}_1 \to \mathbb{D}_1,$$

which is clearly of the form constructed in Lemma 3.37, with

$$G_{AB} = \mathrm{Id}_{\mathbf{Sh}^{\mathrm{op}}_{A\times B}} \colon \mathbf{Sh}^{\mathrm{op}}_{A\times B} \to \mathbf{Sh}^{\mathrm{op}}_{A\times B}$$

and

$$\xi^{G,\phi} = \mathrm{Id}_{f\times g} \colon (f\times g)_* \Rightarrow (f\times g)_* \,.$$

We will now show that the domain of $\mathfrak{r}^{\odot}$, denoted F, also arises from this construction. Let

$$\phi = \begin{array}{ccc} A & \xrightarrow{\mathscr{E}} & B \\ {\scriptstyle f}\big\downarrow & \overset{\alpha}{\Uparrow} & \big\downarrow{\scriptstyle g} \\ U & \xrightarrow[\mathscr{F}]{} & V \end{array} \in \mathrm{Hom}_{\mathscr{D}_1}\left(A \xrightarrow{\mathscr{E}} B \;,\; U \xrightarrow{\mathscr{F}} V \right).$$

and let

$$\Phi_\phi = (f\times g)_* \colon \mathbf{Sh}^{\mathrm{op}}_{A\times B} \to \mathbf{Sh}^{\mathrm{op}}_{U\times V} \,.$$

Define

$$\tilde{\phi} = \alpha \in \mathrm{Hom}_{\mathbf{Sh}^{\mathrm{opp}}_{UV}}(\Phi_f(\mathscr{E}), \mathscr{F})$$

(as when constructing this morphism in the definition of loose composition, the fact that f is a morphism in $\mathbf{Sh}^{\mathrm{op}}_{UV}$ accounts for the swapping of domain and codomain of α).

Consider the functor F; this can be given explicitly as

$$
F\colon \mathbb{D}_1 \to \mathbb{D}_1
$$

$$
\begin{array}{ccc}
A & \xrightarrow{\mathscr{E}} & B \\
{\scriptstyle f}\downarrow & {\scriptstyle \alpha}\Uparrow & \downarrow{\scriptstyle g} \\
U & \xrightarrow[\mathscr{F}]{} & V
\end{array}
\mapsto
\begin{array}{ccccc}
A & \xrightarrow{\mathscr{O}_{\Delta_A}} & A & \xrightarrow{\mathscr{E}} & B \\
\downarrow{\scriptstyle f} & {\scriptstyle \xi^{U,f}}\Uparrow & \downarrow{\scriptstyle f} & {\scriptstyle \alpha}\Uparrow & \downarrow{\scriptstyle g} \\
U & \xrightarrow[\mathscr{O}_{\Delta_U}]{} & U & \xrightarrow[\mathscr{F}]{} & V
\end{array}
$$

$$
= \begin{array}{ccc}
A & \xrightarrow{\qquad\qquad \mathscr{E}\circ\mathscr{O}_{\Delta_A} \qquad\qquad} & B \\
{\scriptstyle f}\downarrow & {\scriptstyle \xi^{\odot,ffg}\circ(\alpha\odot_{UUV}\xi^{U,f})}\Uparrow & \downarrow{\scriptstyle g} \\
U & \xrightarrow[\qquad\qquad F\circ\mathscr{O}_{\Delta_U} \qquad\qquad]{} & V
\end{array}
\tag{3.33}
$$

This can be constructed using Lemma 3.37 by starting from the family of functors

$$
\begin{aligned}
F_{XY}\colon \mathbf{Sh}^{\mathrm{opp}}_{X\times Y} &\to \mathbf{Sh}^{\mathrm{opp}}_{X\times Y} \\
\mathscr{E} &\mapsto \mathscr{E} \odot_{XXY} \mathscr{O}_{\Delta_X} \\
\alpha &\mapsto \alpha \odot_{XXY} \mathrm{Id}_{\mathscr{O}_{\Delta_X}} \,.
\end{aligned}
$$

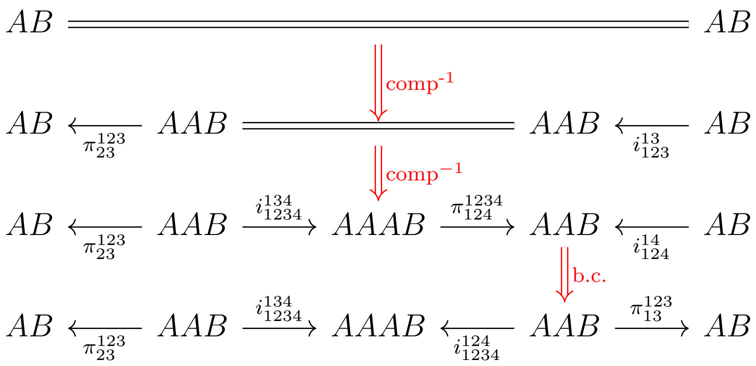
Now the morphism inside the cell in Equation (3.33) can be rewritten as

$$
\begin{aligned}
\xi^{\odot,ffg} \circ (\alpha \odot_{UUV} \xi^{U,f}) &= \xi^{\odot,ffg} \circ (\mathrm{Id}_{\mathscr{E}} \odot_{UUV} \xi^{U,f}) \circ (\alpha \odot_{UUV} \mathrm{Id}_{\mathscr{O}_{\Delta_X}}) \\
&= \xi^{\odot,ffg} \circ (\mathrm{Id}_{\mathscr{E}} \odot_{UUV} \xi^{U,f}) \circ F_{XY}(\alpha)\,,
\end{aligned}
$$

and so F is of the form constructed from Lemma 3.37, with

$$
\xi^{F,\phi} = \xi^{\odot,ffg} \circ (\mathrm{Id}_{\mathscr{E}} \odot_{UUV} \xi^{U,f})\colon (fg)_*(-) \odot \mathscr{O}_{\Delta_U} \Rightarrow (fg)_*(- \odot \mathscr{O}_{\Delta_A})\,.
$$

Hence a natural transformation $F \Rightarrow G$ can be constructed by giving a collection of natural transformations $\rho_{AB}\colon G_{AB} \Rightarrow F_{AB}$ which satisfy Equation (3.27). Consider the natural transformation $\eta^{\rho,\odot}$ given by the diagram



Evaluating this natural transformation on a sheaf $\mathscr{E}$ gives a morphism

$$\mathscr{E} \to \pi^{123}_{13*} \circ i^{124*}_{1234} \circ i^{134}_{1234*} \circ \pi^{123*}_{23}(\mathscr{E}) \,.$$

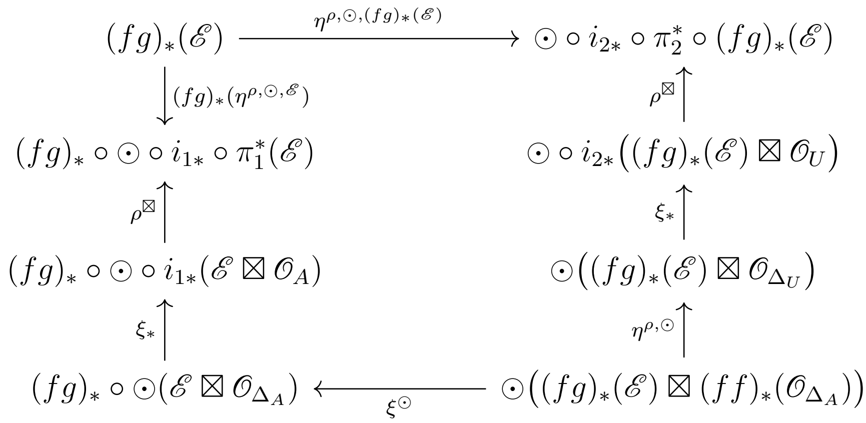
Now $\pi^{123*}_{23}(\mathscr{E}) \cong \mathscr{O}_X \boxtimes \mathscr{E}$. Define ρ to be the composition of morphisms

$$\begin{array}{c} \mathscr{E} \\ \big\downarrow \eta^{\rho,\odot} \\ \pi^{123}_{13*} \circ i^{124*}_{1234} \circ i^{134}_{1234*} \circ \pi^{123*}_{23}(\mathscr{E}) \\ \big\downarrow \rho^{\boxtimes -1} \\ \pi^{123}_{13*} \circ i^{124*}_{1234} \circ i^{134}_{1234*}(\mathscr{O}_A \boxtimes \mathscr{E}) \\ \big\downarrow \xi_*^{-1} \\ \pi^{123}_{13*} \circ i^{124*}_{1234}(\mathscr{O}_{\Delta_A} \boxtimes \mathscr{E}) = F_{AB}(\mathscr{E}) \end{array} .$$

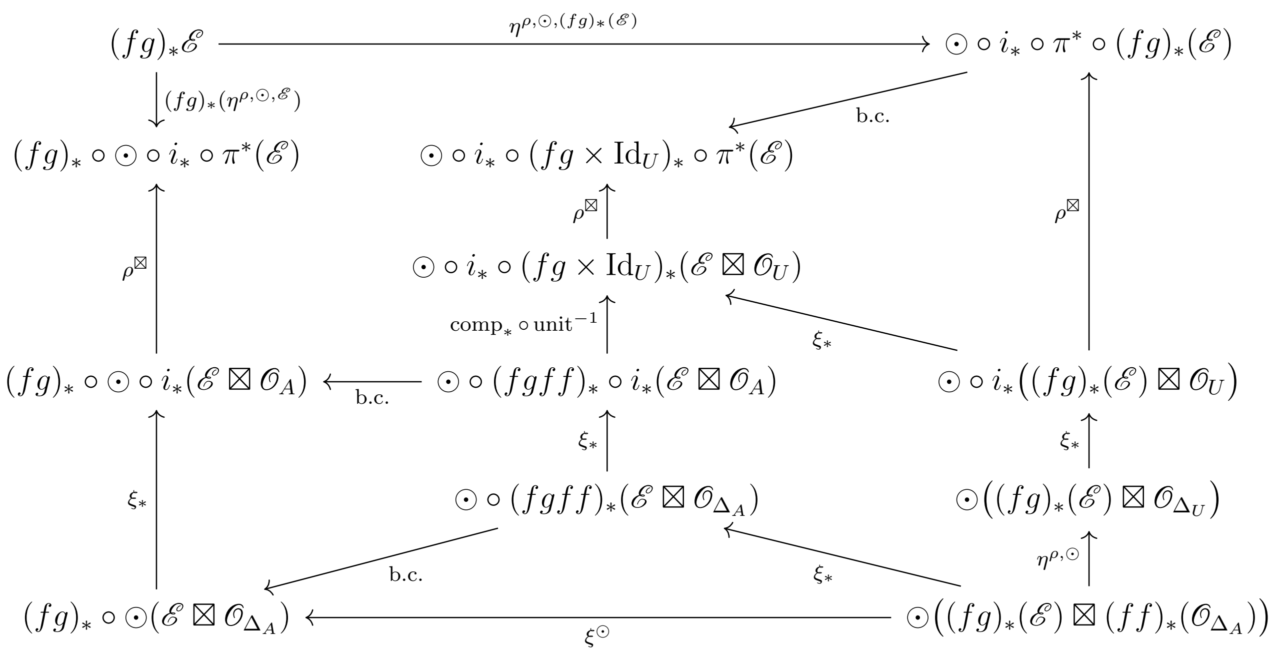
For this family of natural transformations to give a natural transformation $F \Rightarrow G$, they must satisfy

$$\xi^{F,f} \circ (\rho^{UV} * \Phi_f) = ((f \times g)_* * \rho^{XY}) \circ \xi^{G,f} \,. \tag{3.34}$$

Let $\pi_1 = \pi^{ABA}_{AB}, \pi_2 = \pi^{UVU}_{UV}, i_1 = i^{ABA}_{ABAA}$ and $i_2 = i^{UVU}_{UVUU}$. Then evaluating Equation (3.34) on an object $\mathscr{E} \in \mathbf{Sh}_{AB}$ gives the commutative diagram



where the left-hand column is equal to the right-hand side of Equation (3.34), and the rest of the diagram gives the left-hand side. This diagram can be simplified by adding the following objects and morphisms



.

For brevity, some subscripts have been omitted from this diagram. All regions apart from the upper-left commute for the following reasons:

- the lower triangle is the definition of the natural transformation $\xi^{\odot}$;

- the trapezium in the lower left corner commutes as the base-change natural transformation and ξ_* act independently, see Lemma 1.7;
- the lower-right parallelogram follows from Equation (3.11), since $\eta^{\rho,\odot}$ is formed from applying composition and adjunction morphisms; and
- the upper-right trapezium is the unit-direct image coherence condition in Equation (3.15).

Hence the original diagram is commutative if and only if the upper-left pentagonal region is. This region is the outer part of the diagram

$$\begin{array}{ccccc}
(fg)_*(\mathscr{E}) & \xrightarrow{\eta^{\rho,\odot,(fg)_*(\mathscr{E})}} & & & \odot\circ i_*\circ\pi^*\circ(fg)_*(\mathscr{E}) \\
\downarrow {\scriptstyle (fg)_*(\eta^{\rho,\odot,\mathscr{E}})} & & & & \downarrow {\scriptstyle \text{b.c.}} \\
 & & & & \odot\circ i_*\circ(fg\times \mathrm{Id}_U)_*\circ\pi^*(\mathscr{E})\\
 & & \nearrow {\scriptstyle \text{b.c.}} & & {\scriptstyle \rho^{\boxtimes}}\uparrow\\
(fg)_*\circ\odot\circ i_*\circ\pi^*(\mathscr{E}) & \xleftarrow[\text{b.c.}]{} & \odot\circ(fgff)_*\circ i_*\circ\pi^*(\mathscr{E}) & & \odot\circ i_*\circ(fg\times\mathrm{Id}_U)_*(\mathscr{E}\boxtimes\mathscr{O}_U)\\
{\scriptstyle \rho^\boxtimes}\uparrow & & {\scriptstyle \rho^\boxtimes}\uparrow & & \downarrow {\scriptstyle \text{unit}}\\
(fg)_*\circ\odot\circ i_*(\mathscr{E}\boxtimes\mathscr{O}_A) & \xleftarrow[\text{b.c.}]{} & \odot\circ(fgff)_*\circ i_*(\mathscr{E}\boxtimes\mathscr{O}_A) & \xleftarrow[\xi_*]{} & \odot\circ(fg\times\mathrm{Id}_{U^2})_*\circ i_*(\mathscr{E}\boxtimes f_*(\mathscr{O}_A))
\end{array}$$

The natural transformations in the lower-left square act independently and so commute (Lemma 1.7). The upper-left pentagon commutes by Theorem 3.17. After using comp_* to swap the order of i and ff in the lower-right pentagon, the diagram becomes equal to the result of applying $\odot \circ i_* \circ (fg \times \mathrm{Id}_U)_*$ to the diagram

$$\begin{array}{ccc}
(\mathrm{Id}_{AB}\times f)_*\circ\pi_{AB}^{ABA*}(\mathscr{E}) & \xleftarrow[\text{unit}]{} & \pi_{AB}^{ABU}*(\mathscr{E})\\
 & & {\scriptstyle \rho^\boxtimes}\uparrow\\
{\scriptstyle \rho^\boxtimes}\uparrow & & \mathscr{E}\boxtimes\mathscr{O}_U\\
 & & \downarrow{\scriptstyle \text{unit}}\\
(\mathrm{Id}_{AB}\times f)_*(\mathscr{E}\boxtimes\mathscr{O}_A) & \xleftarrow[\xi_*]{} & \mathscr{E}\boxtimes f_*(\mathscr{O}_A)
\end{array}.$$

This commutes as the unitor commutes with the unit natural transformation (Equation (3.16)).

3.5.4 Associator natural isomorphism

The associator $\mathfrak{a}^{\odot}$ is a natural isomorphism

$$\mathfrak{a}^{\odot}\colon (M \odot N) \odot P \to M \odot (N \odot P)\,.$$

This transformation will again be constructed using Lemma 3.38. Partition $\mathscr{C} = \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1$ into parts $\mathscr{C}_{ABCD}$ given by the fibre over (A, B, C, D) from the fibration $S \times S \times (S, T)\colon \mathscr{C} \to {\mathbb{D}_0}^4$. Then

$$\begin{aligned}\mathscr{C}_{ABCD} = (\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1)_{ABCD} &= (\mathbb{D}_1)_{AB} \times (\mathbb{D}_1)_{BC} \times (\mathbb{D}_1)_{CD} \\ &\cong \mathbf{Sh}^{\mathrm{op}}_{AB} \times \mathbf{Sh}^{\mathrm{op}}_{BC} \times \mathbf{Sh}^{\mathrm{op}}_{CD}.\end{aligned}$$

Let

$$F\colon \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \to \mathbb{D}_1$$

be the functor $(M \odot N) \odot P$. If $f = \mathrm{Id}_A, g = \mathrm{Id}_B, h = \mathrm{Id}_C, i = \mathrm{Id}_D$, then the right-hand side becomes $(\alpha \odot \beta) \odot \gamma$. Thus

$$F_{ABCD}(\alpha, \beta, \gamma) = (\alpha \odot \beta) \odot \gamma.$$

Now if

$$\phi = \left(\begin{array}{ccc} A & \xrightarrow{\mathscr{E}} & B \\ {\scriptstyle f}\downarrow & \overset{\alpha}{\Uparrow} & \downarrow{\scriptstyle g} \\ U & \xrightarrow[\mathscr{P}]{} & V \end{array} , \begin{array}{ccc} B & \xrightarrow{\mathscr{F}} & C \\ {\scriptstyle g}\downarrow & \overset{\beta}{\Uparrow} & \downarrow{\scriptstyle h} \\ V & \xrightarrow[\mathscr{Q}]{} & W \end{array} , \begin{array}{ccc} C & \xrightarrow{\mathscr{G}} & D \\ {\scriptstyle h}\downarrow & \overset{\gamma}{\Uparrow} & \downarrow{\scriptstyle i} \\ W & \xrightarrow[\mathscr{R}]{} & X \end{array} \right)$$

is a morphism in $\mathscr{C}$, then

$$F(\phi) = \begin{array}{ccc} A & \xrightarrow{(\mathscr{E}\odot\mathscr{F})\odot\mathscr{G}} & D \\ {\scriptstyle f}\Big\downarrow & {\scriptstyle \delta}\Uparrow & \Big\downarrow{\scriptstyle i} \\ U & \xrightarrow{(\mathscr{P}\odot\mathscr{Q})\odot\mathscr{R}} & X \end{array}$$

where δ is given by

$$\begin{aligned} \delta &= \eta^{\odot,fhi} \circ \big((\eta^{\odot,fgh} \circ (\alpha \odot \beta)) \odot \gamma\big) \\ &= \eta^{\odot,fhi} \circ (\eta^{\odot,fgh} \odot \mathrm{Id}) \circ ((\alpha \odot \beta) \odot \gamma) \\ &= \eta^{\odot,fhi} \circ (\eta^{\odot,fgh} \odot \mathrm{Id}) \circ F_{UVWX}(\alpha, \beta, \gamma) \end{aligned}$$

where in the second line the equality follows from the functoriality of $\odot$. Thus define $\xi^{F,\phi} = \eta^{\odot,fhi} \circ (\eta^{\odot,fgh} \odot \mathrm{Id})\colon F_{UVWX} \circ \Phi_f \to (f \times i)_* \circ F_{ABCD}$.

A similar calculation shows that G can be constructed by taking

$$\begin{aligned} G_{UVWX}(\alpha, \beta, \gamma) &= \alpha \odot (\beta \odot \gamma) \\ \xi^{G,\phi} &= \eta^{\odot,fgi} \circ (\mathrm{Id} \odot \eta^{\odot,ghi}) \,. \end{aligned}$$

Hence F and G are functors given by the construction in Lemma 3.37. A natural transformation between them can thus be constructed using Lemma 3.38 by giving a compatible collection of natural transformation $G_{ABCD} \Rightarrow F_{ABCD}$. Note that F_{ABCD} and G_{ABCD} both factor through the functor

$$\begin{aligned} \boxtimes\colon \mathbf{Sh}^{\mathrm{op}}_{AB} \times \mathbf{Sh}^{\mathrm{op}}_{BC} \times \mathbf{Sh}^{\mathrm{op}}_{CD} &\to \mathbf{Sh}^{\mathrm{op}}_{ABBCCD} \\ (\mathscr{E}, \mathscr{F}, \mathscr{G}) &\mapsto \mathscr{E} \boxtimes \mathscr{F} \boxtimes \mathscr{G} \end{aligned}$$

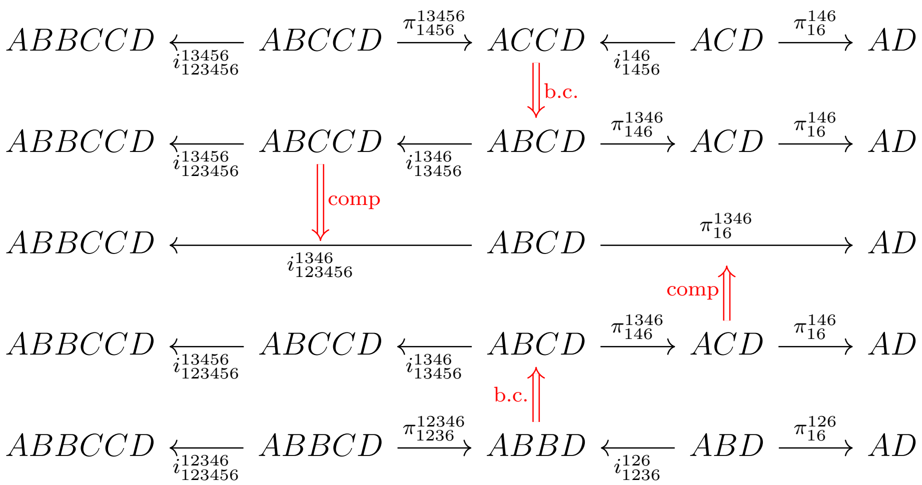
Using the diagram notation introduced in Definition 3.10, $F_{ABCD} = F'_{ABCD} \circ \boxtimes$, where

$$F'_{ABCD} = \quad ABBCCD \xleftarrow[i^{13456}_{123456}]{} ABCCD \xrightarrow{\pi^{13456}_{1456}} ACCD \xleftarrow[i^{146}_{1456}]{} ACD \xrightarrow{\pi^{146}_{16}} AD \quad .$$

Similarly, $G_{ABCD} = G'_{ABCD} \circ \boxtimes$, where

$$G'_{ABCD} = \quad ABBCCD \xleftarrow[i^{12346}_{123456}]{} ABBCD \xrightarrow{\pi^{12346}_{1236}} ABBD \xleftarrow[i^{126}_{1236}]{} ABD \xrightarrow{\pi^{126}_{16}} AD.$$

Now there is a natural transformation $\eta' \colon F'_{ABCD} \Rightarrow G'_{ABCD}$ given by the diagram



Note that the base change in the final line is invertible, since π^{12346}_{1236} is pull-geolocalizing, so its inverse can be used to construct the natural transformation. Let $\eta_{ABCD} = \eta'_{ABCD} * \boxtimes$.

We now verify that these indeed give a natural transformation, by showing that the condition in Lemma 3.38 is satisfied. That condition reads

$$\xi^{F,f} \circ (\eta^{UVWX'} * (fgghhi)_*) = ((f \times i)_* * \eta^{XYZW'}) \circ \xi^{G,f} \, .$$

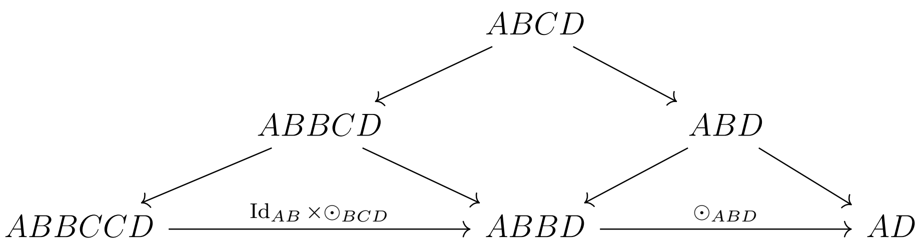
Both of these are natural transformations from

$$G_{fghi} = G_{UVWX} \circ (fgghhi)_*$$

to

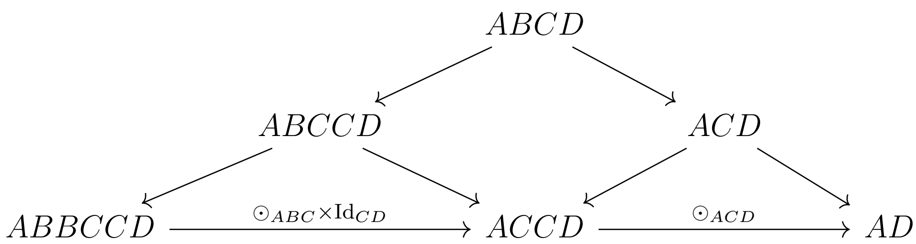
$$F_{fghi} = (f \times i)_* \circ F_{ABCD} \,.$$

The desired equality will follow from an application of Theorem 3.17. For this, it is necessary to calculate the roofs of the functors F_{fghi} and G_{fghi}. As an intermediate step, the roofs of F'_{ABCD} and G'_{ABCD} can be calculated as



(3.35)

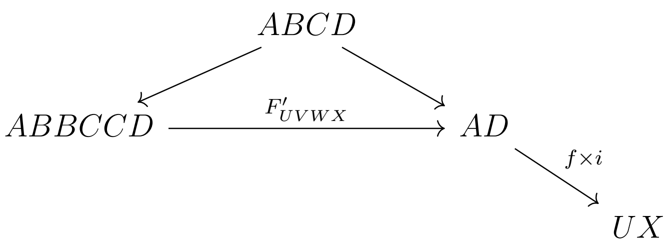
and



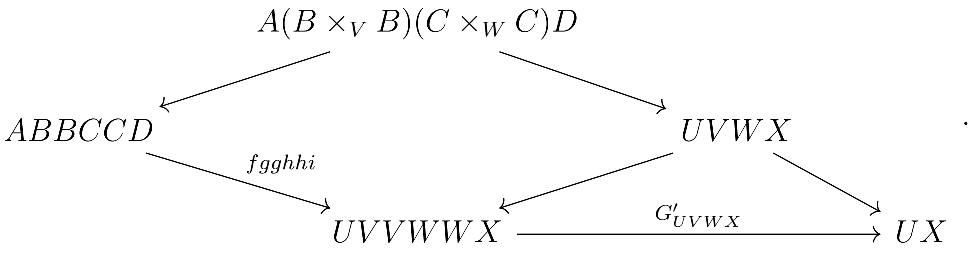
(3.36)

respectively. Note that both F'_{UVWX} and G'_{UVWX} have the same roof.

Next, the roof of $(f \times i)_* F'_{UVWX}$ is given by



while the roof of $G'_{UVWX} \circ (fgghhi)_*$ is



It is clear, then, that F_{fghi} is isomorphic to its roof, that the roof maps (a_F, b_F) factor through (a_G, b_G), and that these are a universal monomorphism. Hence there is a unique natural transformation, and so the above data does indeed give a natural transformation.

3.5.5 Coherence axioms

We need to verify the standard coherence axioms for these data. We will use Theorem 3.17.

Proposition 3.42. *The categories of objects $\mathbb{D}_0$ and arrows $\mathbb{D}_1$, along with the functors $S, T, U, \odot$ and natural transformations $\mathfrak{a}^\odot, \mathfrak{l}^\odot$, and $\mathfrak{r}^\odot$ form a double category.*

Proof. Recall the conditions conditions needed for these data to form a bicategory are given in Definition 3.29. It is immediately clear that $S(\mathfrak{a}^\odot), T(\mathfrak{a}^\odot), S(\mathfrak{l}^\odot), T(\mathfrak{l}^\odot), S(\mathfrak{r}^\odot)$, and $T(\mathfrak{r}^\odot)$ are all identities. Thus it remains to show that the unitors are compatible and that the pentagon diagram commutes. These are shown in Lemma 3.43 and Lemma 3.44 respectively. $\square$

Unitor compatibility

Lemma 3.43. *The triangle diagram*

$$\begin{array}{ccc} d\odot(U(B)\odot c) & \xrightarrow{\mathfrak{a}_{d,U(B),c}} & (d\odot U(B))\odot c \\ & {\scriptstyle \mathrm{Id}_d\odot\mathfrak{l}_c}\searrow \qquad \swarrow{\scriptstyle \mathfrak{l}_d\odot\mathrm{Id}_c} & \\ & d\odot c & \end{array} \tag{3.37}$$

commutes.

Proof. Let $(c,d)\in\mathbb{D}_1\times_{\mathbb{D}_0}\mathbb{D}_1$ be loosely composable objects, with $S(d)=T(c)=B$ for some object $B\in\mathbb{D}_0$.

The two morphisms corresponding to the two different compositions in Equation (3.37) can be written out using their definitions. These become

$$\begin{array}{ccc} A & \xrightarrow{\mathscr{E}\odot(\mathscr{O}_{\Delta_B}\odot\mathscr{F})} & C \\ \downarrow & {\scriptstyle \mathscr{E}\odot\lambda_{\mathscr{F}}}\Uparrow & \downarrow \\ A & \xrightarrow{\mathscr{E}\odot\mathscr{F}} & C \end{array} \tag{3.38}$$

and

$$\begin{array}{ccc} A & \xrightarrow{\mathscr{E}\odot(\mathscr{O}_{\Delta_B}\odot\mathscr{F})} & C \\ \downarrow & {\scriptstyle \alpha^{\odot}}\Uparrow & \downarrow \\ A & \xrightarrow{(\mathscr{E}\odot\mathscr{O}_{\Delta_B})\odot\mathscr{F}} & C \\ \downarrow & {\scriptstyle \rho_{\mathscr{E}}\odot\mathscr{F}}\Uparrow & \downarrow \\ A & \xrightarrow{\mathscr{E}\odot\mathscr{F}} & C \end{array} = \begin{array}{ccc} A & \xrightarrow{\odot_{A,B,C}(\mathscr{E}\boxtimes\odot_{B,B,C}(\mathscr{O}_{\Delta_B}\boxtimes\mathscr{F}))} & C \\ \downarrow & {\scriptstyle \alpha^{\odot}}\Uparrow & \downarrow \\ A & \xrightarrow{\odot_{A,B,C}(\odot_{A,B,B}(\mathscr{E}\boxtimes\mathscr{O}_{\Delta_B})\boxtimes\mathscr{F})} & C \\ \downarrow & {\scriptstyle \odot_{A,B,C}(\rho_{\mathscr{E}}\boxtimes\mathscr{F})}\Uparrow & \downarrow \\ A & \xrightarrow{\odot_{A,B,C}(\mathscr{E}\boxtimes\mathscr{F})} & C \end{array} \tag{3.39}$$

We wish to transform this diagram into one where the objects are given by applying functors to the composition of external products of sheaves. Note that the cell

$$\begin{array}{ccc} A & \xrightarrow{\odot_{A,B,C}\circ\pi_*\circ i^*(\mathscr{E}\boxtimes(\mathscr{O}_{\Delta_B}\boxtimes\mathscr{F}))} & C \\ \downarrow & {\scriptstyle \xi_*}\Uparrow & \downarrow \\ A & \xrightarrow{\mathscr{E}\odot(\mathscr{O}_{\Delta_B}\odot\mathscr{F})} & C \end{array} \tag{3.40}$$

is invertible, so the equality in Equation (3.39) holds if and only if the equation pre-composed with this cell holds. Further, there is an equality

$$
\begin{array}{ccc}
A & \xrightarrow{\odot_{A,B,C}(\odot_{A,B,B}(\mathscr{E}\boxtimes\mathscr{O}_{\Delta_B})\boxtimes\mathscr{F})} & C \\
\downarrow & \Uparrow \mathrm{Id} & \downarrow \\
A & \xrightarrow{\odot_{A,B,C}(\odot_{A,B,B}(\mathscr{E}\boxtimes\mathscr{O}_{\Delta_B})\boxtimes\mathscr{F})} & C
\end{array}
\quad = \quad
\begin{array}{ccc}
A & \xrightarrow{\odot_{A,B,C}(\odot_{A,B,B}(\mathscr{E}\boxtimes\mathscr{O}_{\Delta_B})\boxtimes\mathscr{F})} & C \\
\| & \uparrow \xi_*^{-1}\circ\xi^{*-1} & \| \\
A & \xrightarrow{\odot_{A,B,C}\circ\pi_*\circ i^*(\mathscr{E}\boxtimes(\mathscr{O}_{\Delta_B}\boxtimes\mathscr{F}))} & C \\
\| & \uparrow \xi^*\circ\xi_* & \| \\
A & \xrightarrow{\odot_{A,B,C}(\odot_{A,B,B}(\mathscr{E}\boxtimes\mathscr{O}_{\Delta_B})\boxtimes\mathscr{F})} & C
\end{array}
$$

Inserting this composition in the right-hand side of Equation (3.39), and pre-composing with the cell Equation (3.40), gives

$$
\begin{array}{ccc}
A & \xrightarrow{\odot_{A,B,C}\circ\pi_*\circ i^*(\mathscr{E}\boxtimes(\mathscr{O}_{\Delta_B}\boxtimes\mathscr{F}))} & C \\
\downarrow & \odot_{A,B,C}\circ\pi_*\circ i^*(\alpha^{\odot}) \Uparrow & \downarrow \\
A & \xrightarrow{\odot_{A,B,C}\circ\pi_*\circ i^*((\mathscr{E}\boxtimes\mathscr{O}_{\Delta_B})\boxtimes\mathscr{F})} & C \\
\downarrow & \odot_{A,B,C}*\xi_* \Uparrow & \downarrow \\
A & \xrightarrow{\odot_{A,B,C}(\odot_{A,B,B}(\mathscr{E}\boxtimes\mathscr{O}_{\Delta_B})\boxtimes\mathscr{F})} & C \\
\downarrow & \odot_{A,B,C}*(\rho_{\mathscr{E}}\boxtimes\mathscr{F}) \Uparrow & \downarrow \\
A & \xrightarrow{\odot_{A,B,C}(\mathscr{E}\boxtimes\mathscr{F})} & C
\end{array}
$$

Meanwhile, pre-composing Equation (3.38) with the same cell, the diagram becomes

$$
\begin{array}{ccc}
A & \xrightarrow{\odot\circ\pi_*\circ i^*(\mathscr{E}\boxtimes(\mathscr{O}_{\Delta_B}\boxtimes\mathscr{F}))} & C \\
\downarrow & \xi_*\circ\odot(\mathscr{E}\boxtimes\lambda_{\mathscr{F}}) \Uparrow & \downarrow \\
A & \xrightarrow{\mathscr{E}\odot\mathscr{F}} & C
\end{array}
$$

Now every sheaf and morphism in these diagrams factors through $\odot_{A,B,C}$, so it is sufficient to show that the compositions commute when all applications of the functor are removed. Hence it is sufficient to show that the diagram

$$
\begin{array}{ccc}
\mathscr{E} \boxtimes \mathscr{F} & = & \mathscr{E} \boxtimes \mathscr{F} \\
\downarrow {\scriptstyle \mathscr{E} \boxtimes \lambda_{\mathscr{F}}} & & \downarrow {\scriptstyle \rho_{\mathscr{E}} \boxtimes \mathscr{F}} \\
\mathscr{E} \boxtimes \odot(\mathscr{O}_{\Delta_B} \boxtimes \mathscr{F}) & & \odot(\mathscr{E} \boxtimes \mathscr{O}_{\Delta_B}) \boxtimes \mathscr{F} \\
\downarrow {\scriptstyle \xi^* \circ \xi_*} & & \downarrow {\scriptstyle \xi^* \circ \xi_*} \\
(\mathrm{Id} \times \odot)(\mathscr{E} \boxtimes \mathscr{O}_{\Delta_B} \boxtimes \mathscr{F}) & \xrightarrow{\text{b.c.}} & (\odot \times \mathrm{Id})(\mathscr{E} \boxtimes \mathscr{O}_{\Delta_B} \boxtimes \mathscr{F})
\end{array}
$$

commutes, where the associator $\alpha^{\boxtimes}$ has been dropped for brevity. Substituting in the definitions for $\lambda_{\mathscr{F}}$ and $\rho_{\mathscr{E}}$ gives

$$
\begin{array}{ccc}
\mathscr{E} \boxtimes \mathscr{F} & = & \mathscr{E} \boxtimes \mathscr{F} \\
{\scriptstyle \text{b.c.}} \uparrow & & {\scriptstyle \text{b.c.}} \uparrow \\
\mathscr{E} \boxtimes \odot \circ i_{1*} \circ \pi_1^*(\mathscr{F}) & & \odot \circ i_{2*} \circ \pi_2^*(\mathscr{E}) \boxtimes \mathscr{F} \\
{\scriptstyle \lambda_{\mathscr{F}}} \uparrow & & {\scriptstyle \rho_{\mathscr{F}}} \uparrow \\
\mathscr{E} \boxtimes \odot \circ i_{1*}(\mathscr{O}_B \boxtimes \mathscr{F}) & & \odot \circ i_{2*}(\mathscr{E} \boxtimes \mathscr{O}_B) \boxtimes \mathscr{F} \\
{\scriptstyle \xi_*} \uparrow & & {\scriptstyle \xi_*} \uparrow \\
\mathscr{E} \boxtimes \odot(\mathscr{O}_{\Delta_B} \boxtimes \mathscr{F}) & & \odot(\mathscr{E} \boxtimes \mathscr{O}_{\Delta_B}) \boxtimes \mathscr{F} \\
\downarrow {\scriptstyle \xi^* \circ \xi_*} & & \downarrow {\scriptstyle \xi^* \circ \xi_*} \\
(\mathrm{Id} \times \odot)(\mathscr{E} \boxtimes \mathscr{O}_{\Delta_B} \boxtimes \mathscr{F}) & \xrightarrow{\text{b.c.}} & (\odot \times \mathrm{Id})(\mathscr{E} \boxtimes \mathscr{O}_{\Delta_B} \boxtimes \mathscr{F})
\end{array}
\tag{3.41}
$$

where $\pi_1 = \pi_{BC}^{BBC}, \pi_2 = \pi_{AB}^{ABB}, i_1 = i_{BB,B,C}^{B,B,C}$, and $i_2 = i_{A,B,B,B}^{A,B,B}$. The left side of this diagram can be simplified by noting that (after composition) it is of the form in

Lemma 1.7, and so fits into a commutative diagram

$$\begin{array}{ccc}
\odot \circ i_{3*} \circ \pi_3^*(\mathscr{E} \boxtimes \mathscr{F}) & \xrightarrow{\text{b.c.}} & \mathscr{E} \boxtimes \mathscr{F} \\
\uparrow{\scriptstyle \xi^*} & & \uparrow{\scriptstyle \text{b.c.}} \\
\odot \circ i_{3*}(\mathscr{E} \boxtimes \pi_1^*(\mathscr{F})) & \xleftarrow[\xi_* \circ \xi^* \circ \xi_*]{} & \mathscr{E} \boxtimes \odot \circ i_{1*} \circ \pi_1^*(\mathscr{F}) \\
\uparrow & & \uparrow{\scriptstyle \lambda_{\mathscr{F}}} \\
 & & \mathscr{E} \boxtimes \odot \circ i_{1*}(\mathscr{O}_B \boxtimes \mathscr{F}) \\
{\scriptstyle \lambda_{\mathscr{F}}} & & \uparrow{\scriptstyle \xi_*} \\
 & & \mathscr{E} \boxtimes \odot(\mathscr{O}_{\Delta_B} \boxtimes \mathscr{F}) \\
\vert & & \downarrow{\scriptstyle \xi^* \circ \xi_*} \\
\odot \circ i_{3*}(\mathscr{E} \boxtimes \mathscr{O}_B \boxtimes \mathscr{F}) & \xleftarrow[\xi_*]{} & (\mathrm{Id} \times \odot)(\mathscr{E} \boxtimes \mathscr{O}_{\Delta_B} \boxtimes \mathscr{F})
\end{array}$$

where $\pi_3 = \pi^{AB,B,BC}_{AB,BC}$ and $i_3 = i^{AB,B,BC}_{AB,BB,BC}$. Using the same construction on the right-hand side, the diagram in Equation (3.41) is equivalent to

$$\begin{array}{ccc}
\odot \circ i_{3*} \circ \pi_3^*(\mathscr{E} \boxtimes \mathscr{F}) & = & \odot \circ i_{3*} \circ \pi_3^*(\mathscr{E} \boxtimes \mathscr{F}) \\
\uparrow{\scriptstyle \xi^*} & & \uparrow{\scriptstyle \xi^*} \\
\odot \circ i_{3*}(\mathscr{E} \boxtimes \pi^*(\mathscr{F})) & & \odot \circ i_{3*}(\pi_2^*(\mathscr{E}) \boxtimes \mathscr{F}) \\
\uparrow{\scriptstyle \rho_{\mathscr{E}}} & & \uparrow{\scriptstyle \lambda_{\mathscr{F}}} \\
\odot \circ i_{3*}(\mathscr{E} \boxtimes \mathscr{O}_B \boxtimes \mathscr{F}) & = & \odot \circ i_{3*}(\mathscr{E} \boxtimes \mathscr{O}_B \boxtimes \mathscr{F})
\end{array}$$

This is (after cancelling the functor $\odot \circ i_{3*}$) exactly the diagram in Equation (3.17). □

Pentagon diagram

Lemma 3.44. *The pentagon diagram*

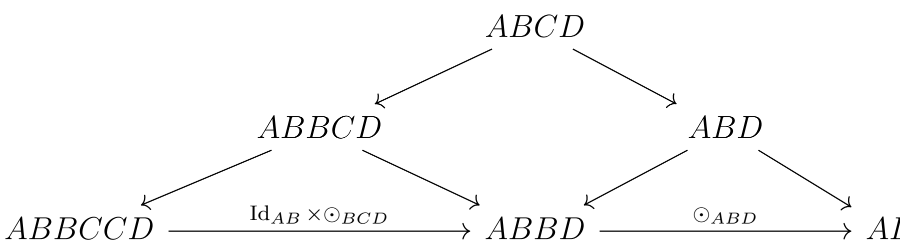
$$\begin{array}{ccccc}
\mathscr{H} \odot (\mathscr{G} \odot (\mathscr{F} \odot \mathscr{E})) & \longrightarrow & (\mathscr{H} \odot \mathscr{G}) \odot (\mathscr{F} \odot \mathscr{E}) & \longrightarrow & ((\mathscr{H} \odot \mathscr{G}) \odot \mathscr{F}) \odot \mathscr{E} \\
\downarrow & & & & \uparrow \\
\mathscr{H} \odot ((\mathscr{G} \odot \mathscr{F}) \odot \mathscr{E}) & & \longrightarrow & & (\mathscr{H} \odot (\mathscr{G} \odot \mathscr{F})) \odot \mathscr{E}
\end{array} \tag{3.42}$$

commutes.

Proof. Consider first the object in the upper-left corner. This is isomorphic to

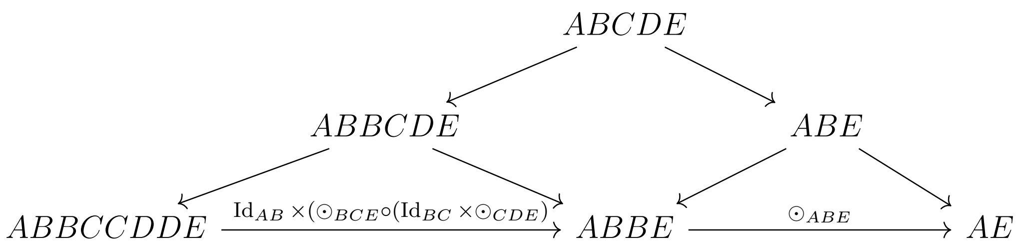
$$\odot_{ABE} \circ (\mathrm{Id}_{AB} \times \odot_{BCE}) \circ (\mathrm{Id}_{AB} \times \mathrm{Id}_{BC} \times \odot_{CDE})(\mathscr{H} \boxtimes (\mathscr{G} \boxtimes (\mathscr{F} \boxtimes \mathscr{E})))$$

by applying the isomorphisms ξ_* and ξ^* repeatedly. This functor can be simplified as follows. Recall from Equation (3.35) that the roof of $\odot_{ABC} \circ (\mathrm{Id}_{AB} \times \odot_{BCD})$ is given by the diagram



(3.43)

Using this result, the roof of the functor from the upper-left vertex of Equation (3.42) is



Both of the natural transformations are isomorphisms, since π^{ABBCD}_{ABBD} and π^{ABBCDE}_{ABBE} are pull-geolocalizing. A similar calculation shows that all the other objects in the diagram are isomorphic to the result of applying the functor

$$ABBCCDDE \longleftarrow ABCDE \longrightarrow AE \tag{3.44}$$

to external tensor products of $\mathscr{H}, \mathscr{G}, \mathscr{F},$ and $\mathscr{E}$, parenthesised in the order indicated in the diagram.

Since the morphism $\alpha^{\odot}$ is constructed from the associator $\alpha^{\boxtimes}$, and the base-change morphisms commute with the morphisms ξ_* and ξ^*, the diagram Equation (3.42) is isomorphic to the result of applying the functor from Equation (3.44) to the diagram

$$\begin{array}{ccccc}
\mathscr{H} \boxtimes (\mathscr{G} \boxtimes (\mathscr{F} \boxtimes \mathscr{E})) & \longrightarrow & (\mathscr{H} \boxtimes \mathscr{G}) \boxtimes (\mathscr{F} \boxtimes \mathscr{E}) & \longrightarrow & ((\mathscr{H} \boxtimes \mathscr{G}) \boxtimes \mathscr{F}) \boxtimes \mathscr{E} \\
\downarrow & & & & \uparrow \\
\mathscr{H} \boxtimes ((\mathscr{G} \boxtimes \mathscr{F}) \boxtimes \mathscr{E}) & & \longrightarrow & & (\mathscr{H} \boxtimes (\mathscr{G} \boxtimes \mathscr{F})) \boxtimes \mathscr{E}
\end{array} .$$

This is exactly the pentagon diagram for the associator $\alpha^{\boxtimes}$ (Equation (3.10)) and so commutes as required. □

3.6 Monoidal structure

This section defines a monoidal structure for the double category. Wester Hansen and Shulman [17, Definition 2.10] give a full list of data that must be provided to define such a structure, along with a list of the conditions that they must satisfy.

3.6.1 Monoidal structure of categories of arrows and objects

To give the double category a monoidal structure, the categories $\mathbb{D}_0$ and $\mathbb{D}_1$ must first be given a monoidal structure. For $A, B \in \mathbb{D}_0$ let the monoidal product be the fibre product in **Sp**; that is,

$$A \otimes B = A \times B \,.$$

The monoidal unit is the terminal object of **Sp**, denoted $\{\mathrm{pt}\}$. The associator and unitor isomorphisms follow from the associator and unitor morphisms of the fibre product.

Monoidal product of cells

We construct the monoidal product on $\mathbb{D}_1$ using Lemma 3.37. Partition the category

$$\mathscr{C} = \mathbb{D}_1 \times \mathbb{D}_1$$

into parts $\mathscr{C}_{ABCD}$ by taking the fibre over (A, B, C, D) of the functor $(S, T) \times (S, T)$. As previously, this can be identified as

$$\mathscr{C}_{ABCD} \cong \mathbf{Sh}^{\mathrm{op}}_{AB} \times \mathbf{Sh}^{\mathrm{op}}_{CD}.$$

For a morphism $\phi \in \mathscr{C}$, given by

$$\phi = \left(\begin{array}{ccc} A & \xrightarrow{\mathscr{E}} & B \\ {\scriptstyle f}\downarrow & \Uparrow{\scriptstyle \alpha} & \downarrow{\scriptstyle g} \\ U & \xrightarrow{\mathscr{P}} & V \end{array} , \begin{array}{ccc} C & \xrightarrow{\mathscr{F}} & D \\ {\scriptstyle h}\downarrow & \Uparrow{\scriptstyle \beta} & \downarrow{\scriptstyle i} \\ W & \xrightarrow{\mathscr{Q}} & X \end{array} \right),$$

define

$$\Phi_\phi \colon \mathbf{Sh}^{\mathrm{op}}_{AB} \times \mathbf{Sh}^{\mathrm{op}}_{CD} \to \mathbf{Sh}^{\mathrm{op}}_{UV} \times \mathbf{Sh}^{\mathrm{op}}_{WX}$$

by $\Phi_\phi = ((f \times g)_*, (h \times i)_*)$. Also define

$$\tilde{\phi} = (\alpha, \beta) \in \mathrm{Hom}_{\mathbf{Sh}^{\mathrm{op}}_{UV} \times \mathbf{Sh}^{\mathrm{op}}_{WX}} \big(\Phi_\phi(\mathscr{E}, \mathscr{F}), (\mathscr{P}, \mathscr{Q}) \big) .$$

Next, define a functor $\otimes_{ABCD}$ by

$$\begin{aligned} \otimes_{ABCD} \colon \mathbf{Sh}^{\mathrm{op}}_{AB} \times \mathbf{Sh}^{\mathrm{op}}_{CD} &\to \mathbf{Sh}^{\mathrm{op}}_{ACBD} \\ (\mathscr{E}, \mathscr{F}) &\mapsto j^{1234}_{1324*}(\mathscr{E} \boxtimes \mathscr{F}) \\ (\alpha, \beta) &\mapsto j^{1234}_{1324*}(\alpha \boxtimes \beta) . \end{aligned}$$

where $j^{1234}_{1324}\colon ABCD \to ACBD$ permutes the order of the factors of the fibre product.

Finally, let the natural transformation

$$\xi^{\otimes}\colon \otimes_{UVWX} \circ ((f\times g)_*, (h\times i)_*) \to (fhgi)_* \circ \otimes_{ABCD}$$

be the natural transformation from Equation (3.8).

Lemma 3.45. *The data* $(\otimes_{ABCD}, \Phi, \xi^{\otimes})$ *satisfies the conditions from Lemma 3.37 and hence defines a functor.*

Proof. First, note that $\Phi_{\mathrm{Id}_c} = \mathrm{Id}_{\mathscr{C}_{ABCD}}$ (by construction) and $\xi^{\otimes,\mathrm{Id}_c} = \mathrm{Id}_{F_i}$ (by Equation (3.14)). Secondly, there is an equality

$$\begin{aligned}\Phi_{\psi\circ\phi} &= (((p\circ f)\times(q\circ g))_*, ((r\circ h)\times(s\circ i))_*)\\ &= ((p\times q)_*, (r\times s)_*)\circ(f\times g, r\times i)_*)\,.\end{aligned}$$

The third condition is the equality

$$\xi^{\otimes,\psi\circ\phi} = (fhgi)_*(\xi^{\otimes,\phi})\circ(\xi^{\otimes,\psi}\circ(fg,hi)_*)$$

which is exactly the statement that distributivity commutes with compositions (Equation (3.20)). The fourth and final condition is that

$$\widetilde{\psi\circ\phi} = \Phi_{\psi}(\tilde{\phi})\circ\tilde{\psi}$$

which follows immediately from the definition. □

Definition 3.46. The monoidal product $\otimes\colon \mathbb{D}_1\times\mathbb{D}_1 \to \mathbb{D}_1$ is defined to be the functor constructed in Lemma 3.45.

Unit and unitor

The unit object for the monoidal structure is the loose morphism

$$I_{\mathbb{D}_1} = \{\mathrm{pt}\} \xrightarrow{\mathscr{O}_{\Delta_{\{\mathrm{pt}\}}}} \{\mathrm{pt}\} \ .$$

Let

$$\begin{aligned} F\colon \mathbb{D}_1 &\to \mathbb{D}_1 \\ c &\mapsto c \otimes I \end{aligned}$$

and $G = \mathrm{Id}_{\mathbb{D}_1}$. Then the right unitor is a natural transformation $\lambda\colon F \Rightarrow G$. This can be constructed using Lemma 3.38; to use this, F and G must be constructed using Lemma 3.37.

Let

$$\phi = \begin{array}{ccc} A & \xrightarrow{\mathscr{E}} & B \\ {\scriptstyle f}\downarrow & \overset{\alpha}{\Uparrow} & \downarrow{\scriptstyle g} \\ U & \xrightarrow[\mathscr{F}]{} & V \end{array}$$

and define

$$\Phi_\phi = (f \times g)_*\colon \mathbf{Sh}^{\mathrm{op}}_{AB} \to \mathbf{Sh}^{\mathrm{op}}_{UV}$$

and

$$\tilde{\phi} = \alpha\,.$$

Next, define

$$F_{AB}\colon \mathbf{Sh}^{\mathrm{op}}_{AB} \to \mathbf{Sh}^{\mathrm{op}}_{\{\mathrm{pt}\}A\{\mathrm{pt}\}B}$$

by $F_{AB}(\mathscr{E}) = \mathscr{E} \boxtimes \mathscr{O}_{\Delta_{\{\mathrm{pt}\}}}$. To complete the construction of F, it is necessary to give a natural transformation

$$\xi^{F,\phi}\colon F_{UV} \circ (f \times g)_* \Rightarrow (\mathrm{Id}_{\{\mathrm{pt}\}} \times f \times \mathrm{Id}_{\{\mathrm{pt}\}} \times g)_* \circ F_{AB}\,,$$

which is taken to be the isomorphism

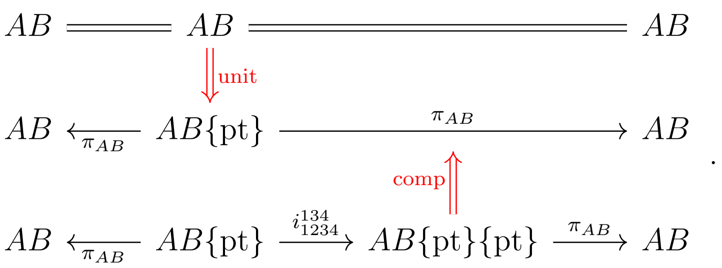
$$\xi_*\colon (f \times g)_*(\mathscr{E}) \boxtimes \mathscr{O}_{\Delta_{\{\mathrm{pt}\}}} \cong (\mathrm{Id} \times \mathrm{Id} \times f \times g)_*(\mathscr{O}_{\Delta_{\{\mathrm{pt}\}}} \boxtimes \mathscr{E})$$

from the definition of the monoidal structure on a geofibred category.

The functor G can be given by the same construction, taking $G_{XY} = \mathrm{Id}_{\mathbf{Sh}^{\mathrm{op}}_{AB}}$ and $\xi^{G,\phi} = \mathrm{Id}$. Then the unitor natural transformation $F \Rightarrow G$ is constructed by giving natural transformations

$$\rho_{AB}\colon G_{AB} \to \pi^{\{\mathrm{pt}\}A\{\mathrm{pt}\}B}_{AB*} \circ F_{AB}\,.$$

There is a natural transformation, which will be denoted $\eta^{\rho,\otimes}$, given by



Now recall $\rho^{\boxtimes}\colon \mathscr{E} \boxtimes \mathscr{O}_{\{\mathrm{pt}\}} \xrightarrow{\sim} \pi^{AB\{\mathrm{pt}\}*}_{AB}(\mathscr{E})$. Thus the composition $\rho_{AB} = \rho^{\boxtimes^{-1}} \circ \eta^{\rho,\otimes}$ gives the required natural transformation.

To show that this is indeed a natural transformation, the condition in Lemma 3.38 must hold; this condition is the equality

$$((\pi^{\{\mathrm{pt}\}A\{\mathrm{pt}\}B}_{AB})_* \circ \xi^{F,\phi}) \circ (\rho_{CD} \circ (fg)_*) = ((fg)_* \circ \rho_{AB})\,.$$

This is equivalent to equality of the outer paths of the diagram

$$
\begin{array}{ccccc}
(fg)_*(\mathscr{E}) & & \xrightarrow{\eta^{\lambda,\otimes}_{(fg)_*\mathscr{E}}} & & (fg)_* \circ \pi_* \circ i_* \circ \pi^*(\mathscr{E}) \\
\Big\downarrow \eta_{(fg)_*(\eta^{\lambda,\otimes})} & & & \swarrow \text{comp}_* & \Big\uparrow \rho^{\boxtimes} \\
\pi_* \circ i_* \circ \pi^* \circ (fg)_*(\mathscr{E}) & \xrightarrow{\text{b.c.}} & \pi_* \circ i_* \circ (fg \times \mathrm{Id}_{\{\mathrm{pt}\}^2})_* \circ \pi^*(\mathscr{E}) & & \\
\Big\uparrow \rho^{\boxtimes} & & \Big\uparrow \rho^{\boxtimes} & & \\
\pi_* \circ i_*((fg)_*(\mathscr{E}) \boxtimes \mathscr{O}_{\{\mathrm{pt}\}}) & \xrightarrow{\xi_*} & \pi_* \circ i_* \circ (fg \times \mathrm{Id}_{\{\mathrm{pt}\}^2})(\mathscr{E} \boxtimes \mathscr{O}_{\{\mathrm{pt}\}}) & \xleftarrow{\text{comp}_*} & (fg)_* \circ \pi_* \circ i_*(\mathscr{E} \boxtimes \mathscr{O}_{\{\mathrm{pt}\}}) \\
\Big\uparrow \xi_* & & & & \Big\uparrow \xi_* \\
 & & & & (fg)_* \circ \pi_*(\mathscr{E} \boxtimes \mathscr{O}_{\Delta_{\{\mathrm{pt}\}}}) \\
 & & & & \Big\downarrow \text{comp}_* \\
\pi_*((fg)_*(\mathscr{E}) \boxtimes \mathscr{O}_{\Delta_{\{\mathrm{pt}\}}}) & & \xrightarrow{\xi_*} & & \pi_* \circ (fg \times \mathrm{Id}_{\{\mathrm{pt}\}^2})_*(\mathscr{E} \boxtimes \mathscr{O}_{\Delta_{\{\mathrm{pt}\}}})
\end{array}
$$

where subscripts have been dropped for brevity. Now

- the top trapezium commutes by Theorem 3.17;
- the upper right trapezium commutes as the natural transformations are independent;
- the left square commutes by Equation (3.15); and
- the lower rectangle commutes as ξ_* and comp_* satisfy the coherence condition Equation (3.11).

Consequently, the two outside paths are equal as required.

Associator

Next, we construct the associator for the monoidal structure on $\mathbb{D}_1$. This will follow from the associativity of the external product $\boxtimes$.

The associator is a natural transformation

$$\mathfrak{a}^{\otimes} \colon \otimes \circ (\mathrm{Id} \times \otimes) \Rightarrow \otimes \circ (\otimes \times \mathrm{Id}) \, .$$

Partitioning $\mathbb{D}_1 \times \mathbb{D}_1 \times \mathbb{D}_1$ based on its fibration $(S, T) \times (S, T) \times (S, T)$, these functors become

$$F_{ABCDEF} \colon \mathbf{Sh}^{\mathrm{op}}_{AB} \times \mathbf{Sh}^{\mathrm{op}}_{CD} \times \mathbf{Sh}^{\mathrm{op}}_{EF} \to \mathbf{Sh}^{\mathrm{op}}_{ACEBDF},$$
$$(\mathscr{E}, \mathscr{F}, \mathscr{G}) \mapsto j_* \left(\mathscr{E} \boxtimes j_*(\mathscr{F} \boxtimes \mathscr{G})\right),$$

where j_* are the relabelling maps. The functor G is given similarly.

Using the tensor-direct image isomorphism ξ_*, there is an isomorphism

$$F_{ABCDEF}(\mathscr{E}, \mathscr{F}, \mathscr{G}) \cong j^{ABCDEF}_{ACEBDF*}(\mathscr{E} \boxtimes (\mathscr{F} \boxtimes \mathscr{G}))$$

and similarly an isomorphism

$$G_{ABCDEF}(\mathscr{E}, \mathscr{F}, \mathscr{G}) \cong j^{ABCDEF}_{ACEBDF*}((\mathscr{E} \boxtimes \mathscr{F}) \boxtimes \mathscr{G})$$

Recall that the associator for the external tensor product is a morphism

$$\alpha^{\boxtimes} \colon \mathscr{E} \boxtimes (\mathscr{F} \boxtimes \mathscr{G}) \to (\mathscr{E} \boxtimes \mathscr{F}) \boxtimes \mathscr{G} \, .$$

Composing this natural transformation on the left with the functor $j^{ABCDEF}_{ACEBDF*}$ gives a natural transformation

$$j^{ABCDEF}_{ACEBDF*} * \alpha^{\boxtimes} \colon F_{ABCDEF} \to G_{ABCDEF}.$$

The verification that this is in fact a natural transformation is straightforward, as all functors involved are simple direct image functors.

Lemma 3.47. *The monoidal product $\otimes$ defined in Definition 3.46, along with the unitor and associator defined in this section, give $\mathbb{D}_1$ the structure of a monoidal category.*

Proof. The two conditions to check are that unitors are compatible (Lemma 3.48) and that the pentagon diagram is satisfied (Lemma 3.49). □

Monoidal unitor coherence

Lemma 3.48. *The diagram*

$$\begin{array}{ccc} \mathscr{E}\otimes(\mathscr{O}_{\Delta_{\{pt\}}}\otimes\mathscr{F}) & \xrightarrow{\quad\alpha\quad} & (\mathscr{E}\otimes\mathscr{O}_{\Delta_{\{pt\}}})\otimes\mathscr{F} \\ & {\scriptstyle \mathscr{E}\otimes\mathfrak{r}^{\otimes,\mathscr{F}}}\searrow \qquad \swarrow{\scriptstyle \mathfrak{r}^{\otimes,\mathscr{E}}\otimes\mathscr{F}} & \\ & \mathscr{E}\otimes\mathscr{F} & \end{array}$$

commutes.

Proof. The diagram can be written more explicitly as

$$\begin{array}{ccc} \pi^{AC\{\mathrm{pt}\}\{\mathrm{pt}\}BD}_{ACBD*}(\mathscr{E}\boxtimes\mathscr{O}_{\Delta_{\{\mathrm{pt}\}}}\boxtimes\mathscr{F}) & = & \pi^{AC\{\mathrm{pt}\}\{\mathrm{pt}\}BD}_{ACBD*}(\mathscr{E}\boxtimes\mathscr{O}_{\Delta_{\{\mathrm{pt}\}}}\boxtimes\mathscr{F}) \\ \uparrow{\scriptstyle \xi_*} & & \uparrow{\scriptstyle \xi_*} \\ \mathscr{E}\boxtimes\pi^{\{\mathrm{pt}\}BD}_{BD*}\left(\pi^{\{\mathrm{pt}\}\{\mathrm{pt}\}}_{\{\mathrm{pt}\}*}\circ i_{1*}(\mathscr{O}_{\{\mathrm{pt}\}})\boxtimes\mathscr{F}\right) & & \pi^{AC\{\mathrm{pt}\}}_{AC*}(\mathscr{E}\boxtimes\pi^{\{\mathrm{pt}\}\{\mathrm{pt}\}}_{\{\mathrm{pt}\}*}\circ i_{1*}(\mathscr{O}_{\{\mathrm{pt}\}}))\boxtimes\mathscr{F} \\ \downarrow{\scriptstyle \xi_*} & & \downarrow{\scriptstyle \xi_*} \\ \mathscr{E}\boxtimes\pi^{\{\mathrm{pt}\}\{\mathrm{pt}\}BD}_{BD*}\circ i_{2*}(\mathscr{O}_{\{\mathrm{pt}\}}\boxtimes\mathscr{F}) & & \pi^{AC\{\mathrm{pt}\}\{\mathrm{pt}\}}_{AC*}\circ i_{3*}(\mathscr{E}\boxtimes\mathscr{O}_{\{\mathrm{pt}\}})\boxtimes\mathscr{F} \\ \downarrow{\scriptstyle \rho^{\boxtimes}} & & \downarrow{\scriptstyle \rho^{\boxtimes}} \\ \mathscr{E}\boxtimes\pi^{\{\mathrm{pt}\}\{\mathrm{pt}\}BD}_{BD*}\circ i_{2*}\circ\pi^{\{\mathrm{pt}\}BD}_{BD}(\mathscr{F}) & & \pi^{AC\{\mathrm{pt}\}\{\mathrm{pt}\}}_{AC*}\circ i_{3*}\circ\pi^{AC\{\mathrm{pt}\}*}_{AC}(\mathscr{E})\boxtimes\mathscr{F} \\ \uparrow{\scriptstyle \eta^{\lambda,\otimes}} & & \uparrow{\scriptstyle \eta^{\rho,\otimes}} \\ \mathscr{E}\boxtimes\mathscr{F} & = & \mathscr{E}\boxtimes\mathscr{F} \end{array}$$

where $i_1 = i^{\{\mathrm{pt}\}}_{\{\mathrm{pt}\}\{\mathrm{pt}\}}$, $i_2 = i^{\{\mathrm{pt}\}BD}_{\{\mathrm{pt}\}\{\mathrm{pt}\}BD}$, and $i_3 = i^{AC\{\mathrm{pt}\}}_{AC\{\mathrm{pt}\}\{\mathrm{pt}\}}$. This can be simplified by splitting into smaller parts as follows:

$$
\begin{array}{ccccc}
\pi_*(\mathscr{E} \boxtimes \mathscr{O}_{\Delta_{\{\mathrm{pt}\}}} \boxtimes \mathscr{F}) & & = = = = = = & & \pi_*(\mathscr{E} \boxtimes \mathscr{O}_{\Delta_{\{\mathrm{pt}\}}} \boxtimes \mathscr{F}) \\
\uparrow{\scriptstyle \xi_*} & & & & \uparrow{\scriptstyle \xi_*} \\
\mathscr{E} \boxtimes \pi_*(\pi_* \circ i_*(\mathscr{O}_{\{\mathrm{pt}\}}) \boxtimes \mathscr{F}) & & & & \pi_*(\mathscr{E} \boxtimes \pi_* \circ i_*(\mathscr{O}_{\{\mathrm{pt}\}})) \boxtimes \mathscr{F} \\
\downarrow{\scriptstyle \xi_*} & & & & \downarrow{\scriptstyle \xi_*} \\
\mathscr{E} \boxtimes \pi_* \circ i_*(\mathscr{O}_{\{\mathrm{pt}\}} \boxtimes \mathscr{F}) & \xrightarrow{\xi_*} & \pi_* \circ i_*(\mathscr{E} \boxtimes \mathscr{O}_{\{\mathrm{pt}\}} \boxtimes \mathscr{F}) & \xleftarrow[\xi_*]{} & \pi_* \circ i_*(\mathscr{E} \boxtimes \mathscr{O}_{\{\mathrm{pt}\}}) \boxtimes \mathscr{F} \\
\downarrow{\scriptstyle \rho^{\boxtimes}} & & & & \downarrow{\scriptstyle \rho^{\boxtimes}} \\
\mathscr{E} \boxtimes \pi_* \circ i_* \circ \pi^*(\mathscr{F}) & \xrightarrow{\xi^* \circ \xi_*} & \pi_* \circ i_* \circ \pi^*(\mathscr{E} \boxtimes \mathscr{F}) & \xleftarrow[\xi^* \circ \xi_*]{} & \pi_* \circ i_* \circ \pi^*(\mathscr{E}) \boxtimes \mathscr{F} \\
\uparrow{\scriptstyle \eta^{\lambda,\otimes}} & \nearrow{\scriptstyle \mathrm{Id} \times \eta^{\lambda,\otimes}} & & \nwarrow{\scriptstyle \mathrm{Id} \times \eta^{\rho,\otimes}} & \uparrow{\scriptstyle \eta^{\rho,\otimes}} \\
\mathscr{E} \boxtimes \mathscr{F} & & = = = = = = & & \mathscr{E} \boxtimes \mathscr{F}
\end{array}
$$

where for brevity sub- and superscripts have been omitted. The top rectangle commutes since ξ_* commutes with comp_*. The lower two side triangles commute as ξ_* commutes with base-changes, Equation (3.13). The bottom middle triangle commutes by Theorem 3.17. In the middle rectangle, the applications of the natural transformations ξ_* commute with $\rho^{\boxtimes}$ as they act on different functors (Lemma 1.7), so the whole diagram commutes if and only if the diagram

$$
\begin{array}{ccc}
\pi_* \circ i_*(\mathscr{E} \boxtimes \mathscr{O}_{\{\mathrm{pt}\}} \boxtimes \mathscr{F}) & \xrightarrow{\rho^{\boxtimes}} & \pi_* \circ i_*(\pi^*(\mathscr{E}) \boxtimes \mathscr{F}) \\
\downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \xi^*} \\
\pi_* \circ i_*(\mathscr{E} \boxtimes \pi^*(\mathscr{F})) & \xrightarrow{\xi^*} & \pi_* \circ i_*(\pi^*(\mathscr{E} \boxtimes \mathscr{F}))
\end{array}
$$

commutes. This is the result of applying the functor $\pi_* \circ i_*$ to the diagram expressing the coherence of left- and right-unitors, Equation (3.17). □

Monoidal pentagon identity

Lemma 3.49. *The pentagon identity holds for the monoidal product $\otimes$ on $\mathbb{D}_1$.*

Proof. Similarly to the pentagon identity for the associator for loose composition, the pentagon identity for the associator for the monoidal product follows from the same identity for the external tensor product. □

3.6.2 Interchange isomorphism

We next need to define the globular isomorphism

$$(\mathscr{E} \otimes \mathscr{P}) \odot (\mathscr{F} \otimes \mathscr{Q}) \to (\mathscr{E} \odot \mathscr{F}) \otimes (\mathscr{P} \odot \mathscr{Q})\,.$$

Let

$$F, G\colon (\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1) \times (\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1) \to \mathbb{D}_1$$

be the source and target functors of this transformation. Partition the category

$$\mathscr{C} = (\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1) \times (\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1)$$

into its fibres over the fibration

$$(S \times (S, T)) \times (S \times (S, T)) : \mathscr{C} \to {\mathbb{D}_0}^6\,.$$

Then

$$F_{ABCUVW}\colon \mathbf{Sh}^{\mathrm{op}}_{AB} \times \mathbf{Sh}^{\mathrm{op}}_{UV} \times \mathbf{Sh}^{\mathrm{op}}_{BC} \times \mathbf{Sh}^{\mathrm{op}}_{VW} \to \mathbf{Sh}^{\mathrm{op}}_{ACUW}$$

can be written

$$\begin{aligned} F_{ABCUVW}(\mathscr{E}, \mathscr{P}, \mathscr{F}, \mathscr{Q}) &= \odot_{AU,BV,CW}(j^{ABUV}_{AUBV*}(\mathscr{E} \boxtimes \mathscr{P}) \boxtimes j^{BCVW}_{BVCW*}(\mathscr{F} \boxtimes \mathscr{Q})) \\ &\cong \odot_{AU,BV,CW} \circ j^{ABUVBCVW}_{AUBVBVCW*}(\mathscr{E} \boxtimes \mathscr{P} \boxtimes \mathscr{F} \boxtimes \mathscr{Q})\,. \end{aligned}$$

Similarly,

$$G_{ABCUVW}(\mathscr{E},\mathscr{P},\mathscr{F},\mathscr{Q}) = j^{ACUW}_{AUCW*}(\odot_{A,B,C}(\mathscr{E}\boxtimes\mathscr{F})\boxtimes\odot_{U,V,W}(\mathscr{P}\boxtimes\mathscr{Q}))$$
$$\cong j^{ACUW}_{AUVW*}\circ\pi^{ABCUVW}_{ACUW*}\circ i^{ABCUVW*}_{ABBCUVVW}(\mathscr{E}\boxtimes\mathscr{F}\boxtimes\mathscr{P}\boxtimes\mathscr{Q}).$$

Thus we can construct a morphism between the two objects by applying the braiding $\mathscr{E}\boxtimes s_{\mathscr{P},\mathscr{F}}\boxtimes\mathscr{F}$.

To verify that this is indeed a natural transformation, the condition Equation (3.27) must hold. The natural transformations ξ^F and ξ^G are constructed solely from the natural transformations ξ_* and ξ^*; since these commute with base-change and composition, Equation (3.27) becomes equivalent to showing that the diagram

$$\begin{array}{ccc} f_*(\mathscr{E})\boxtimes g_*(\mathscr{F})\boxtimes h_*(\mathscr{P})\boxtimes k_*(\mathscr{Q}) & \xrightarrow{\mathfrak{s}_{g_*(\mathscr{F}),h_*(\mathscr{P})}} & f_*(\mathscr{E})\boxtimes h_*(\mathscr{P})\boxtimes g_*(\mathscr{F})\boxtimes k_*(\mathscr{Q}) \\ \downarrow{\scriptstyle \xi_*} & & \downarrow{\scriptstyle \xi_*} \\ (fghk)_*(\mathscr{E}\boxtimes\mathscr{F}\boxtimes\mathscr{P}\boxtimes\mathscr{Q}) & \xrightarrow{\mathfrak{s}_{\mathscr{F}\boxtimes\mathscr{P}}} & (fghk)_*(\mathscr{E}\boxtimes\mathscr{P}\boxtimes\mathscr{F}\boxtimes\mathscr{Q}) \end{array}$$

commutes. This diagram is equivalent to

$$\begin{array}{ccc} g_*(\mathscr{F})\boxtimes h_*(\mathscr{P}) & \xrightarrow{\mathfrak{s}_{g_*(\mathscr{F}),h_*(\mathscr{P})}} & h_*(\mathscr{F})\boxtimes g_*(\mathscr{P}) \\ \downarrow{\scriptstyle \xi_*} & & \downarrow{\scriptstyle \xi_*} \\ (gh)_*(\mathscr{F}\boxtimes\mathscr{P}) & \xrightarrow{s_{\mathscr{F},\mathscr{P}}} & (gh)_*(\mathscr{P}\boxtimes\mathscr{F}) \end{array}$$

which is exactly the condition that ξ_* commutes with the braiding, Equation (3.20).

3.6.3 Unit-tensor distributor

As well as the interchange isomorphism, we need to define a distributor isomorphism

$$\mathfrak{u}\colon U_{A\otimes B}\to U_A\otimes U_B\,.$$

Define the component of this transformation over the object (A, B) to be the morphism

$$\mathfrak{u}_{A,B} = \begin{array}{ccc} AB & \xrightarrow{\mathscr{O}_{\Delta_{AB}}} & AB \\ \downarrow & \Uparrow \upsilon_{A,B} & \downarrow \\ AB & \xrightarrow{\mathscr{O}_{\Delta_A} \boxtimes \mathscr{O}_{\Delta_B}} & AB \end{array},$$

where $\upsilon_{A,B}$ is the morphism given by the composition

$$\upsilon_{A,B} \colon \; \mathscr{O}_{\Delta_A} \boxtimes \mathscr{O}_{\Delta_B} \xrightarrow{\xi_*} i^{AB}_{AABB*}(\mathscr{O}_A \boxtimes \mathscr{O}_B) \xrightarrow{\rho^{\boxtimes}} i^{AB}_{AABB*} \circ \pi^{AB*}_B(\mathscr{O}_B) = \mathscr{O}_{\Delta_{AB}} .$$

That this is indeed a natural transformation can be verified explicitly. Let $f \colon A \to U, g \colon B \to V$ be morphisms in **Sp**. This is a natural transformation if the diagram

$$\begin{array}{ccc} \mathscr{O}_{\Delta_{A\times B}} & \xrightarrow{\mathfrak{u}} & \mathscr{O}_{\Delta_A} \otimes \mathscr{O}_{\Delta_B} \\ \downarrow U_{f\times g} & & \downarrow U_f \otimes U_g \\ \mathscr{O}_{\Delta_{U\times V}} & \xrightarrow{\mathfrak{u}} & \mathscr{O}_{\Delta_U} \otimes \mathscr{O}_{\Delta_V} \end{array}$$

is commutative. Writing this diagram explicitly, it is equivalent to

$$\begin{array}{ccccc} (f\times g)_*(\mathscr{O}_{\Delta_{A\times B}}) & & \xleftarrow{\quad U_{f\times g} \quad} & & \mathscr{O}_{\Delta_{U\times V}} \\ \rho^{\boxtimes}\uparrow & & & & \rho^{\boxtimes}\uparrow \\ (f\times g)_* \circ i_*(\mathscr{O}_A \boxtimes \mathscr{O}_B) & & & & i_*(\mathscr{O}_U \boxtimes \mathscr{O}_V) \\ (f\times g)_*(\xi_*)\uparrow & & & & \xi_*\uparrow \\ (f\times g)_*(\mathscr{O}_{\Delta_A} \boxtimes \mathscr{O}_{\Delta_B}) & \xleftarrow{\xi_*} & (ff)_*(\mathscr{O}_{\Delta_A}) \boxtimes (gg)_*(\mathscr{O}_{\Delta_B}) & \xleftarrow{U_f \boxtimes U_g} & \mathscr{O}_{\Delta_U} \boxtimes \mathscr{O}_{\Delta_V} \end{array} .$$

This diagram can be split into smaller sections to give

$$\begin{array}{ccccc} (f\times g)_*(\mathscr{O}_{\Delta_{A\times B}}) & & \xleftarrow{\quad U_{f\times g} \quad} & & \mathscr{O}_{\Delta_{U\times V}} \\ \rho^{\boxtimes}\uparrow & & & & \rho^{\boxtimes}\uparrow \\ (f\times g)_* \circ i_*(\mathscr{O}_A \boxtimes \mathscr{O}_B) & \xleftarrow{\xi_*} & i_*(f_*(\mathscr{O}_A) \boxtimes g_*(\mathscr{O}_B)) & \xleftarrow{i_*(\mathrm{unit} \boxtimes \mathrm{unit})} & i_*(\mathscr{O}_U \boxtimes \mathscr{O}_V) \\ (f\times g)_*(\xi_*)\uparrow & & \mathrm{comp}\uparrow & & \xi_*\uparrow \\ (f\times g)_*(\mathscr{O}_{\Delta_A} \boxtimes \mathscr{O}_{\Delta_B}) & \xleftarrow{\xi_*} & (ff)_*(\mathscr{O}_{\Delta_A}) \boxtimes (gg)_*(\mathscr{O}_{\Delta_B}) & \xleftarrow{U_f \boxtimes U_g} & \mathscr{O}_{\Delta_U} \boxtimes \mathscr{O}_{\Delta_V} \end{array}$$

where the lower-right square commutes by definition of U, and the lower-left square commutes by the compatibility of ξ_* with comp (Equation (3.11)). This leaves the upper rectangle. Note that (after applying comp_*) the elements all factor through i_*. Hence the diagram commutes if the following diagram is commutative:

$$
\begin{array}{ccccc}
(f \times g)_*(\mathscr{O}_{A\times B}) & & \xleftarrow{\quad\quad\quad\quad \text{unit} \quad\quad\quad\quad} & & \mathscr{O}_{U\times V} \\
\uparrow{\scriptstyle \rho^{\boxtimes}} & & & & \uparrow{\scriptstyle \rho^{\boxtimes}} \\
(f \times g)_*(\mathscr{O}_A \boxtimes \mathscr{O}_B) & \xleftarrow{\ \xi_*\ } & f_*(\mathscr{O}_U) \boxtimes f_*(\mathscr{O}_V) & \xleftarrow{\text{unit} \boxtimes \text{unit}} & \mathscr{O}_U \boxtimes \mathscr{O}_V
\end{array}
.
$$

This diagram is exactly the condition that the unitor morphisms are compatible with adjoints (Equation (3.16)).

3.6.4 Coherence

Lemma 3.50. *The monoidal structure on $\mathbb{D}_0$ and $\mathbb{D}_1$, along with the natural transformations defined previously, give the double category a monoidal structure.*

Proof. Wester Hansen and Shulman [17, Definition 2.10] gives an explicit list of conditions needed to give a double category a monoidal structure. Condition (i) is that $\mathbb{D}_0$ and $\mathbb{D}_1$ are monoidal categories. The monoidal structure on $\mathbb{D}_0$ arises from the fact that its monoidal product is a fibre product. The monoidal structure on $\mathbb{D}_1$ is given in Lemma 3.47. Conditions (ii) and (iii), that U_I is the monoidal unit of $\mathbb{D}_1$ and that S and T are strict monoidal, are easily seen to be true. Conditions (iv), (v) and (vi) consist of showing that certain diagrams commute. These diagrams are considered in turn in the following subsections. □

Interchange-associator compatibility 1

Lemma 3.51. *The following diagram commutes:*

$$\begin{array}{ccc}
((\mathscr{E} \otimes \mathscr{P}) \odot (\mathscr{F} \otimes \mathscr{Q})) \odot (\mathscr{G} \otimes \mathscr{R})) & \longrightarrow & ((\mathscr{E} \odot \mathscr{F}) \otimes (\mathscr{P} \odot \mathscr{Q})) \odot (\mathscr{G} \otimes \mathscr{R}) \\
\downarrow & & \downarrow \\
(\mathscr{E} \otimes \mathscr{P}) \odot ((\mathscr{F} \otimes \mathscr{Q}) \odot (\mathscr{G} \otimes \mathscr{R})) & & ((\mathscr{E} \odot \mathscr{F}) \odot \mathscr{G}) \otimes ((\mathscr{P} \odot \mathscr{Q}) \odot \mathscr{R}) \\
\downarrow & & \downarrow \\
(\mathscr{E} \otimes \mathscr{P}) \odot ((\mathscr{F} \odot \mathscr{G}) \otimes (\mathscr{Q} \odot \mathscr{R})) & \longrightarrow & (\mathscr{E} \odot (\mathscr{F} \odot \mathscr{G})) \otimes (\mathscr{P} \odot (\mathscr{Q} \odot \mathscr{R}))
\end{array}$$

Proof. Writing this in terms of morphisms of shapes, this is equivalent to the commutativity of the following diagram

$$\begin{array}{ccc}
\mathscr{E} \boxtimes \mathscr{F} \boxtimes \mathscr{G} \boxtimes \mathscr{P} \boxtimes \mathscr{Q} \boxtimes \mathscr{R} & \xrightarrow{\xi_* \circ s_{\mathscr{F} \boxtimes \mathscr{G}, \mathscr{P}}} & j_*(\mathscr{E} \boxtimes \mathscr{P} \boxtimes \mathscr{F} \boxtimes \mathscr{G} \boxtimes \mathscr{Q} \boxtimes \mathscr{R}) \\
\downarrow{\scriptstyle \xi_* \circ s_{\mathscr{G}, \mathscr{P} \boxtimes \mathscr{Q}}} & & \downarrow{\scriptstyle \xi_* \circ s_{\mathscr{G}, \mathscr{Q}}} \\
j_*(\mathscr{E} \boxtimes \mathscr{F} \boxtimes \mathscr{P} \boxtimes \mathscr{Q} \boxtimes \mathscr{G} \boxtimes \mathscr{R}) & \xrightarrow{\xi_* \circ s_{\mathscr{F}, \mathscr{P}}} & j_*(\mathscr{E} \boxtimes \mathscr{P} \boxtimes \mathscr{F} \boxtimes \mathscr{Q} \boxtimes \mathscr{G} \boxtimes \mathscr{R})
\end{array}$$

This commutes as the two braids represented by applying the swap morphisms are equal, and since the action of s factors through the braid group. □

Loose and monoidal unitors compatibility

Lemma 3.52. *The diagram*

$$\begin{array}{ccc}
(\mathscr{E} \otimes \mathscr{F}) \odot U_{A \times C} & \xrightarrow{\mathrm{Id} \odot \mathfrak{u}} & (\mathscr{E} \otimes \mathscr{F}) \odot (U_A \otimes U_C) \\
\downarrow{\scriptstyle \mathfrak{r}} & & \downarrow{\scriptstyle \mathfrak{x}} \\
\mathscr{E} \otimes \mathscr{F} & \xleftarrow[\mathfrak{r} \otimes \mathfrak{r}]{} & (\mathscr{E} \odot U_A) \otimes (\mathscr{F} \odot U_C)
\end{array} \tag{3.45}$$

commutes.

Proof. Explicitly, this diagram is

$$
\begin{array}{ccc}
\odot_{AB,AB,CD}(\mathscr{E} \boxtimes \mathscr{F} \boxtimes \mathscr{O}_{\Delta_{AB}}) & \xleftarrow[\rho^{\boxtimes}]{} & \odot_{AB,AB,CD}(\mathscr{E} \boxtimes \mathscr{F} \boxtimes i_*(\mathscr{O}_A \boxtimes \mathscr{O}_C)) \\
\big\downarrow \xi_* & & \xi_* \big\uparrow \\
\odot_{AB,AB,CD} \circ i_*(\mathscr{E} \boxtimes \mathscr{F} \boxtimes \mathscr{O}_{AC}) & & \odot_{AB,AB,CD}(\mathscr{E} \boxtimes \mathscr{F} \boxtimes \mathscr{O}_{\Delta_A} \boxtimes \mathscr{O}_{\Delta_C}) \\
\big\downarrow \rho^{\boxtimes} & & \text{b.c.} \big\uparrow \\
\odot_{AB,AB,CD} \circ i_* \circ \pi^*(\mathscr{E} \boxtimes \mathscr{F}) & & (\odot_{A,A,B} \times \odot_{C,C,D}) \circ j_*(\mathscr{E} \boxtimes \mathscr{F} \boxtimes \mathscr{O}_{\Delta_A} \boxtimes \mathscr{O}_{\Delta_C}) \\
\eta^{\mathfrak{r}} \big\uparrow & & \xi_* \big\uparrow \\
\mathscr{E} \boxtimes \mathscr{F} & & (\odot_{A,A,B} \times \odot_{C,C,D})(\mathscr{E} \boxtimes j_*(\mathscr{F} \boxtimes \mathscr{O}_{\Delta_A}) \boxtimes \mathscr{O}_{\Delta_B}) \\
\big\downarrow \eta^{\mathfrak{r}} \boxtimes \eta^{\mathfrak{r}} & & s_{\mathscr{O}_{\Delta_A},\mathscr{F}} \big\uparrow \\
\odot_{A,A,B} \circ i_* \circ \pi^*(\mathscr{E}) \boxtimes \odot_{C,C,D} \circ i_* \circ \pi^*(\mathscr{F}) & & (\odot_{A,A,B} \times \odot_{C,C,D})(\mathscr{E} \boxtimes \mathscr{O}_{\Delta_A} \boxtimes \mathscr{F} \boxtimes \mathscr{O}_{\Delta_B}) \\
\rho^{\boxtimes} \big\uparrow & & \xi^* \circ \xi_* \big\uparrow \\
\odot_{A,A,B} \circ i_*(\mathscr{E} \boxtimes \mathscr{O}_A) \boxtimes \odot_{C,C,D} \circ i_*(\mathscr{F} \boxtimes \mathscr{O}_C) & \xleftarrow[\xi_*]{} & \odot_{A,A,B}(\mathscr{E} \boxtimes \mathscr{O}_{\Delta_A}) \boxtimes \odot_{C,C,D}(\mathscr{F} \boxtimes \mathscr{O}_{\Delta_C})
\end{array}
\tag{3.46}
$$

The right-hand side fits into a commutative diagram

$$
\begin{array}{ccc}
\odot_{AB,AB,CD}(\mathscr{E} \boxtimes \mathscr{F} \boxtimes i_*(\mathscr{O}_A \boxtimes \mathscr{O}_C)) & \xrightarrow{\xi_*} & \odot_{AB,AB,CD} \circ i_*(\mathscr{E} \boxtimes \mathscr{F} \boxtimes \mathscr{O}_A \boxtimes \mathscr{O}_C) \\
\xi_* \big\uparrow & & \big\| \\
\odot_{AB,AB,CD}(\mathscr{E} \boxtimes \mathscr{F} \boxtimes \mathscr{O}_{\Delta_A} \boxtimes \mathscr{O}_{\Delta_C}) & \xrightarrow{\xi_*} & \odot_{AB,AB,CD} \circ i_*(\mathscr{E} \boxtimes \mathscr{F} \boxtimes \mathscr{O}_A \boxtimes \mathscr{O}_C) \\
\text{b.c.} \big\uparrow & & \text{b.c.} \big\uparrow \\
(\odot_{A,A,B} \times \odot_{C,C,D}) \circ j_*(\mathscr{E} \boxtimes \mathscr{F} \boxtimes \mathscr{O}_{\Delta_A} \boxtimes \mathscr{O}_{\Delta_C}) & \xrightarrow{\xi_*} & (\odot_{A,A,B} \times \odot_{C,C,D}) \circ j_* \circ i_*(\mathscr{E} \boxtimes \mathscr{F} \boxtimes \mathscr{O}_A \boxtimes \mathscr{O}_C) \\
\xi_* \big\uparrow & & \big\| \\
(\odot_{A,A,B} \times \odot_{C,C,D})(\mathscr{E} \boxtimes j_*(\mathscr{F} \boxtimes \mathscr{O}_{\Delta_A}) \boxtimes \mathscr{O}_{\Delta_B}) & \xrightarrow{\xi_*} & (\odot_{A,A,B} \times \odot_{C,C,D}) \circ j_* \circ i_*(\mathscr{E} \boxtimes \mathscr{F} \boxtimes \mathscr{O}_A \boxtimes \mathscr{O}_C) \\
\sigma_{\mathscr{O}_{\Delta_A},\mathscr{F}} \big\uparrow & & \xi_* \circ \sigma_{\mathscr{O}_A,\mathscr{F}} \big\uparrow \\
(\odot_{A,A,B} \times \odot_{C,C,D})(\mathscr{E} \boxtimes \mathscr{O}_{\Delta_A} \boxtimes \mathscr{F} \boxtimes \mathscr{O}_{\Delta_B}) & \xrightarrow{\xi_*} & (\odot_{A,A,B} \times \odot_{C,C,D}) \circ i_*(\mathscr{E} \boxtimes \mathscr{O}_A \boxtimes \mathscr{F} \boxtimes \mathscr{O}_B) \\
\xi^* \circ \xi_* \big\uparrow & & \\
\odot_{A,A,B}(\mathscr{E} \boxtimes \mathscr{O}_{\Delta_A}) \boxtimes \odot_{C,C,D}(\mathscr{F} \boxtimes \mathscr{O}_{\Delta_C}) & &
\end{array}
$$

The lower half of the left-hand side also fits into a commutative diagram,

$$
\begin{array}{ccc}
 & & \mathscr{E} \boxtimes \mathscr{F} \\
 & & \big\downarrow{\scriptstyle \eta^{\mathfrak{r}} \boxtimes \eta^{\mathfrak{r}}} \\
(\odot \times \odot) \circ (i \times i)_* \circ \pi^*(\mathscr{E} \boxtimes \mathscr{F}) & \xleftarrow[\xi_* \circ \xi^* \circ \xi_*]{} & \odot_{A,A,B} \circ i_* \circ \pi^*(\mathscr{E}) \boxtimes \odot_{C,C,D} \circ i_* \circ \pi^*(\mathscr{F}) \\
{\scriptstyle \xi^* \circ (\rho \boxtimes \rho)}\big\uparrow & & {\scriptstyle \rho^{\boxtimes}}\big\uparrow \\
(\odot \times \odot) \circ (i \times i)_*(\mathscr{E} \boxtimes \mathscr{O}_A \boxtimes \mathscr{F} \boxtimes \mathscr{O}_C) & \xleftarrow[\xi_* \circ \xi^* \circ \xi_*]{} & \odot_{A,A,B} \circ i_*(\mathscr{E} \boxtimes \mathscr{O}_A) \boxtimes \odot_{C,C,D} \circ i_*(\mathscr{F} \boxtimes \mathscr{O}_C)
\end{array}
$$

Joining these together, the diagram in Equation (3.46) is commutative if the following diagram is commutative:

$$
\begin{array}{ccc}
\odot_{AB,AB,CD} \circ i_*(\mathscr{E} \boxtimes \mathscr{F} \boxtimes \mathscr{O}_{AC}) & = & \odot_{AB,AB,CD} \circ i_*(\mathscr{E} \boxtimes \mathscr{F} \boxtimes \mathscr{O}_A \boxtimes \mathscr{O}_C) \\
\big\downarrow{\scriptstyle \rho^{\boxtimes}} & & {\scriptstyle \text{b.c.}}\big\uparrow \\
\odot_{AB,AB,CD} \circ i_* \circ \pi^*(\mathscr{E} \boxtimes \mathscr{F}) & & (\odot_{A,A,B} \times \odot_{C,C,D}) \circ j_* \circ i_*(\mathscr{E} \boxtimes \mathscr{F} \boxtimes \mathscr{O}_A \boxtimes \mathscr{O}_C) \\
{\scriptstyle \eta^{\mathfrak{r}}}\big\uparrow & & \big\| \\
\mathscr{E} \boxtimes \mathscr{F} & & (\odot_{A,A,B} \times \odot_{C,C,D}) \circ j_* \circ i_*(\mathscr{E} \boxtimes \mathscr{F} \boxtimes \mathscr{O}_A \boxtimes \mathscr{O}_C) \\
\big\downarrow{\scriptstyle \xi_* \circ \xi^* \circ \xi_* \circ \xi^* \circ \eta^{\mathfrak{r}} \boxtimes \eta^{\mathfrak{r}}} & & {\scriptstyle \xi_* \circ \sigma_{\mathscr{O}_A, \mathscr{F}}}\big\uparrow \\
(\odot \times \odot) \circ (i \times i)_* \circ \pi^*(\mathscr{E} \boxtimes \mathscr{F}) & & (\odot_{A,A,B} \times \odot_{C,C,D}) \circ i_*(\mathscr{E} \boxtimes \mathscr{O}_A \boxtimes \mathscr{F} \boxtimes \mathscr{O}_B) \\
{\scriptstyle \xi^* \circ (\rho \boxtimes \rho)}\big\uparrow & \swarrow\!\!\!= & \\
(\odot \times \odot) \circ (i \times i)_*(\mathscr{E} \boxtimes \mathscr{O}_A \boxtimes \mathscr{F} \boxtimes \mathscr{O}_C) & &
\end{array}
$$

Since the base-change and unitor morphisms at the top of the diagram act independently, they can be exchanged, giving

$$
\begin{array}{ccc}
(\odot_{A,A,B} \times \odot_{C,C,D}) \circ j_* \circ i_* \circ \pi^*(\mathscr{E} \boxtimes \mathscr{F}) & \xleftarrow[\rho^{\boxtimes}]{} & (\odot_{A,A,B} \times \odot_{C,C,D}) \circ j_* \circ i_*(\mathscr{E} \boxtimes \mathscr{F} \boxtimes \mathscr{O}_A \boxtimes \mathscr{O}_C) \\
\big\downarrow{\scriptstyle \text{b.c.}} & \nwarrow & {\scriptstyle \xi_* \circ \sigma_{\mathscr{O}_A, \mathscr{F}}}\big\uparrow \\
\odot_{AB,AB,CD} \circ i_* \circ \pi^*(\mathscr{E} \boxtimes \mathscr{F}) & \text{b.c.} & (\odot_{A,A,B} \times \odot_{C,C,D}) \circ i_*(\mathscr{E} \boxtimes \mathscr{O}_A \boxtimes \mathscr{F} \boxtimes \mathscr{O}_B) \\
{\scriptstyle \eta^{\mathfrak{r}}}\big\uparrow & & \big\downarrow{\scriptstyle \xi^* \circ (\rho^{\boxtimes} \boxtimes \rho^{\boxtimes})} \\
\mathscr{E} \boxtimes \mathscr{F} & \xrightarrow{\xi_* \circ \xi^* \circ \xi_* \circ \xi^* \circ \eta^{\mathfrak{r}} \boxtimes \eta^{\mathfrak{r}}} & (\odot_{A,A,B} \times \odot_{C,C,D}) \circ i_* \circ \pi^*(\mathscr{E} \boxtimes \mathscr{F})
\end{array}
$$

The upper-right half of the diagram commutes as $\rho^{\boxtimes}$ commutes with σ (Equation (3.21)), while the lower-left corner commutes since ξ_* and ξ^* commute with unit and comp. $\square$

Interchange-associativity coherence 2

Lemma 3.53. *The diagram*

$$\begin{array}{ccc}
((\mathscr{E} \boxtimes \mathscr{F}) \boxtimes \mathscr{G}) \odot ((\mathscr{P} \boxtimes \mathscr{Q}) \boxtimes \mathscr{R}) & \xrightarrow{\mathfrak{a}^{\otimes} \odot \mathfrak{a}^{\otimes}} & (\mathscr{E} \boxtimes (\mathscr{F} \boxtimes \mathscr{G})) \odot (\mathscr{P} \boxtimes (\mathscr{Q} \boxtimes \mathscr{R})) \\
\downarrow{\scriptstyle \mathfrak{x}} & & \downarrow{\scriptstyle \mathfrak{x}} \\
((\mathscr{E} \boxtimes \mathscr{F}) \odot (\mathscr{P} \boxtimes \mathscr{Q})) \boxtimes (\mathscr{G} \boxtimes \mathscr{R}) & & (e \odot \mathscr{P}) \boxtimes ((\mathscr{F} \boxtimes \mathscr{G}) \odot (\mathscr{Q} \boxtimes \mathscr{R})) \\
\downarrow{\scriptstyle \mathfrak{x} \odot \mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Id} \odot \mathfrak{x}} \\
((\mathscr{E} \odot \mathscr{P}) \boxtimes (\mathscr{F} \odot \mathscr{Q})) \boxtimes (\mathscr{G} \odot \mathscr{R}) & \xrightarrow{\mathfrak{a}^{\otimes}} & (\mathscr{E} \odot \mathscr{P}) \boxtimes ((\mathscr{F} \odot \mathscr{Q}) \boxtimes (\mathscr{G} \odot \mathscr{R}))
\end{array}$$

commutes.

Proof. The diagram simplifies to

$$\begin{array}{ccc}
j_*(\mathscr{E} \boxtimes \mathscr{F} \boxtimes \mathscr{G} \boxtimes \mathscr{P} \boxtimes \mathscr{Q} \boxtimes \mathscr{R}) & \xleftarrow[\xi_* \circ s_{\mathscr{P}, \mathscr{F} \boxtimes \mathscr{G}}]{} & j_*(\mathscr{E} \boxtimes \mathscr{P} \boxtimes \mathscr{F} \boxtimes \mathscr{G} \boxtimes \mathscr{Q} \boxtimes \mathscr{R}) \\
{\scriptstyle \xi_* \circ s_{\mathscr{P} \boxtimes \mathscr{Q}, \mathscr{G}}} \uparrow & & {\scriptstyle \xi_* \circ \sigma_{\mathscr{Q}, \mathscr{G}}} \uparrow \\
j_*(\mathscr{E} \boxtimes \mathscr{F} \boxtimes \mathscr{P} \boxtimes \mathscr{Q} \boxtimes \mathscr{G} \boxtimes \mathscr{R}) & \xleftarrow[\xi_* \circ \sigma_{\mathscr{P}, \mathscr{F}}]{} & \mathscr{E} \boxtimes \mathscr{P} \boxtimes \mathscr{F} \boxtimes \mathscr{Q} \boxtimes \mathscr{G} \boxtimes \mathscr{R}
\end{array}$$

which commutes as the family of morphisms σ form a braiding, and the two braids representing the two compositions in the diagram are equal. □

Unit-unitor compatibility

Lemma 3.54. *The diagram*

$$\begin{array}{ccc}
\mathscr{O}_{\Delta_{(A \times B) \times C}} & \xrightarrow{U_{\mathfrak{a}^{\otimes}}} & \mathscr{O}_{\Delta_{A \times (B \times C)}} \\
\downarrow{\scriptstyle \mathfrak{u}} & & \downarrow{\scriptstyle \mathfrak{u}} \\
\mathscr{O}_{\Delta_{A \times B}} \otimes \mathscr{O}_{\Delta_C} & & \mathscr{O}_{\Delta_A} \boxtimes \mathscr{O}_{\Delta_{B \times C}} \\
\downarrow{\scriptstyle \mathfrak{u} \otimes \mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Id} \otimes \mathfrak{u}} \\
(\mathscr{O}_{\Delta_A} \boxtimes \mathscr{O}_{\Delta_B}) \boxtimes \mathscr{O}_{\Delta_C} & \xrightarrow{\mathfrak{a}^{\otimes}} & \mathscr{O}_{\Delta_A} \boxtimes (\mathscr{O}_{\Delta_B} \boxtimes \mathscr{O}_{\Delta_C})
\end{array}$$

commutes.

Proof. The diagram here becomes

$$\begin{CD}
\mathscr{O}_{\Delta_{ABC}} @<{\rho^{\boxtimes}}<< i_*(\mathscr{O}_A \boxtimes \mathscr{O}_{BC}) @<{\xi_*}<< \mathscr{O}_{\Delta_A} \boxtimes \mathscr{O}_{\Delta_{BC}} \\
@A{\rho^{\boxtimes}}AA @. @A{\rho^{\boxtimes}}AA \\
i_*(\mathscr{O}_{AB} \boxtimes \mathscr{O}_C) @. @. \mathscr{O}_{\Delta_A} \boxtimes i_*(\mathscr{O}_B \boxtimes \mathscr{O}_C) \\
@A{\xi_*}AA @. @A{\xi_*}AA \\
\mathscr{O}_{\Delta_{AB}} \boxtimes \mathscr{O}_{\Delta_C} @<{\rho^{\boxtimes}}<< i_*(\mathscr{O}_A \boxtimes \mathscr{O}_B) \boxtimes \mathscr{O}_{\Delta_C} @<{\xi_*}<< \mathscr{O}_{\Delta_A} \boxtimes \mathscr{O}_{\Delta_B} \boxtimes \mathscr{O}_{\Delta_C}
\end{CD}$$

which fits into a larger diagram

$$\begin{CD}
\mathscr{O}_{\Delta_{ABC}} @<{\rho^{\boxtimes}}<< i_*(\mathscr{O}_A \boxtimes \mathscr{O}_{BC}) @<{\xi_*}<< \mathscr{O}_{\Delta_A} \boxtimes \mathscr{O}_{\Delta_{BC}} \\
@A{\rho^{\boxtimes}}AA @A{\rho^{\boxtimes}}AA @A{\rho^{\boxtimes}}AA \\
i_*(\mathscr{O}_{AB} \boxtimes \mathscr{O}_C) @<{\rho^{\boxtimes}}<< i_*(\mathscr{O}_A \boxtimes \mathscr{O}_B \boxtimes \mathscr{O}_C) @<{\xi_*}<< \mathscr{O}_{\Delta_A} \boxtimes i_*(\mathscr{O}_B \boxtimes \mathscr{O}_C) \\
@A{\xi_*}AA @A{\xi_*}AA @A{\xi_*}AA \\
\mathscr{O}_{\Delta_{AB}} \boxtimes \mathscr{O}_{\Delta_C} @<{\rho^{\boxtimes}}<< i_*(\mathscr{O}_A \boxtimes \mathscr{O}_B) \boxtimes \mathscr{O}_{\Delta_C} @<{\xi_*}<< \mathscr{O}_{\Delta_A} \boxtimes \mathscr{O}_{\Delta_B} \boxtimes \mathscr{O}_{\Delta_C}
\end{CD}$$

The top-right and lower-left squares commute by Equation (3.15), while the lower-right square commutes by Equation (3.12). The upper-left square is the result of applying i_* to the diagram

$$\begin{CD}
\mathscr{O}_A \boxtimes \mathscr{O}_B \boxtimes \mathscr{O}_C @>{\rho^{\boxtimes}}>> \mathscr{O}_A \boxtimes \pi^*(\mathscr{O}_B) \\
@V{\rho^{\boxtimes}}VV @| \\
\pi^*(\mathscr{O}_A) \boxtimes \mathscr{O}_C @. \mathscr{O}_A \boxtimes \mathscr{O}_{BC} \\
@V{\rho}VV @V{\rho^{\boxtimes}}VV \\
\pi^*(\mathscr{O}_A) @= \pi^*(\mathscr{O}_A)
\end{CD} .$$

This diagram commutes by Mac Lane's coherence theorem, since it consists solely of unitor morphisms. □

Interchange-tensor unitor coherence

Lemma 3.55. *The diagram*

$$\begin{array}{ccc} (\mathscr{E} \boxtimes \mathscr{O}_{\Delta_{\{pt\}}}) \odot (\mathscr{F} \boxtimes \mathscr{O}_{\Delta_{\{pt\}}}) & \xrightarrow{\mathfrak{x}} & (\mathscr{E} \odot \mathscr{F}) \boxtimes (\mathscr{O}_{\Delta_{\{pt\}}} \odot \mathscr{O}_{\Delta_{\{pt\}}}) \\ \Big\downarrow {\scriptstyle \mathfrak{r}^{\otimes} \odot \mathfrak{r}^{\otimes}} & & \Big\downarrow {\scriptstyle (\mathscr{E} \odot \mathscr{F}) \boxtimes \mathfrak{r}^{\otimes}_{\mathscr{O}_{\Delta_{\{pt\}}}}} \\ \mathscr{E} \odot \mathscr{F} & \xleftarrow[\mathfrak{r}^{\otimes}_{\mathscr{E} \boxtimes \mathscr{F}}]{} & (\mathscr{E} \odot \mathscr{F}) \boxtimes \mathscr{O}_{\Delta_{\{pt\}}} \end{array}$$

commutes.

Proof. Writing out the diagram explicitly, then using the distributors ξ_* and ξ^* to simplify each term to a functor applied to a single external tensor product, gives

$$\begin{array}{ccc} \pi^*(\mathscr{E}) \boxtimes \pi^*(\mathscr{F}) & \xrightarrow{\xi^*} & \pi^*(\mathscr{E} \boxtimes \mathscr{F}) \\ & & \Big\uparrow {\scriptstyle \rho_{\pi^*(\mathscr{E} \boxtimes \mathscr{F})}} \\ \Big\uparrow {\scriptstyle \rho_{\mathscr{E}} \boxtimes \rho_{\mathscr{F}}} & & \pi^*(\mathscr{E} \boxtimes \mathscr{F}) \boxtimes \mathscr{O}_{\{\mathrm{pt}\}} \\ & & \Big\uparrow {\scriptstyle \rho_{\mathscr{E} \boxtimes \mathscr{F}}} \\ (\mathscr{E} \boxtimes \mathscr{O}_{\{\mathrm{pt}\}}) \boxtimes (\mathscr{F} \boxtimes \mathscr{O}_{\{\mathrm{pt}\}}) & \xleftarrow[\sigma_{\mathscr{F}, \mathscr{O}_{\{\mathrm{pt}\}}}]{} & \mathscr{E} \boxtimes \mathscr{F} \boxtimes \mathscr{O}_{\{\mathrm{pt}\}} \boxtimes \mathscr{O}_{\{\mathrm{pt}\}} \end{array}$$

which is the condition that unitors are compatible with braiding (Equation (3.21)).

□

Unitor coherence

Lemma 3.56. *The diagram*

$$\begin{array}{ccc} \mathscr{O}_{\Delta_{A \times \{pt\}}} & \xrightarrow{\mathfrak{u}} & \mathscr{O}_{\Delta_A} \boxtimes \mathscr{O}_{\Delta_{\{pt\}}} \\ & {\scriptstyle U_{i^A_{A \times \{pt\}}}} \searrow & \Big\downarrow {\scriptstyle \mathfrak{r}^{\otimes}} \\ & & \mathscr{O}_{\Delta_A} \end{array}$$

commutes.

Proof. Explicitly, this becomes

$$
\begin{array}{ccc}
\pi^{A\{\mathrm{pt}\}A\{\mathrm{pt}\}}_{AA*}(\mathscr{O}_{\Delta_{A\times\{\mathrm{pt}\}}}) & = & \pi^{A\{\mathrm{pt}\}A\{\mathrm{pt}\}}_{AA*} \circ i^{A\{\mathrm{pt}\}}_{A\{\mathrm{pt}\}A\{\mathrm{pt}\}*} \circ \pi^{A\{\mathrm{pt}\}*}_{A}(\mathscr{O}_A) \\
\| & & \uparrow{\scriptstyle \rho^{\boxtimes}} \\
\pi^{A\{\mathrm{pt}\}A\{\mathrm{pt}\}}_{AA*} \circ i^{A\{\mathrm{pt}\}}_{A\{\mathrm{pt}\}A\{\mathrm{pt}\}*}(\mathscr{O}_{A\times\{\mathrm{pt}\}}) & & \pi^{A\{\mathrm{pt}\}A\{\mathrm{pt}\}}_{AA*} \circ i^{A\{\mathrm{pt}\}}_{A\{\mathrm{pt}\}A\{\mathrm{pt}\}*}(\mathscr{O}_A \boxtimes \mathscr{O}_{\{\mathrm{pt}\}}) \\
\downarrow{\scriptstyle \mathrm{unit}} & & \uparrow{\scriptstyle \xi_*} \\
\pi^{A\{\mathrm{pt}\}A\{\mathrm{pt}\}}_{AA*} \circ i^{A\{\mathrm{pt}\}}_{A\{\mathrm{pt}\}A\{\mathrm{pt}\}*} \circ i^{A}_{A\{\mathrm{pt}\}*} \circ i^{A*}_{A\{\mathrm{pt}\}}(\mathscr{O}_{A\times\{\mathrm{pt}\}}) & & \pi^{A\{\mathrm{pt}\}A\{\mathrm{pt}\}}_{AA*} \circ i^{AA\{\mathrm{pt}\}}_{AA\{\mathrm{pt}\}\{\mathrm{pt}\}*}(\mathscr{O}_{\Delta_A} \boxtimes \mathscr{O}_{\{\mathrm{pt}\}}) \\
\downarrow{\scriptstyle \mathrm{comp}_*} & & \downarrow{\scriptstyle \rho} \\
i^{A}_{AA*} \circ i^{A*}_{A\{\mathrm{pt}\}}(\mathscr{O}_{A\times\{\mathrm{pt}\}}) & & \pi^{A\{\mathrm{pt}\}A\{\mathrm{pt}\}}_{AA*} \circ i^{AA\{\mathrm{pt}\}}_{AA\{\mathrm{pt}\}\{\mathrm{pt}\}*} \circ \pi^{AA\{\mathrm{pt}\}*}_{AA}(\mathscr{O}_{\Delta_A}) \\
\| & & \uparrow{\scriptstyle \eta^{\rho,\otimes}} \\
i^{A}_{AA*}(\mathscr{O}_A) & = & \mathscr{O}_{\Delta_A}
\end{array}
$$

Since the unitor ρ commutes with distributors, the right-hand side can be simplified to give

$$
\begin{array}{ccc}
\pi^{A\{\mathrm{pt}\}A\{\mathrm{pt}\}}_{AA*}(\mathscr{O}_{\Delta_{A\times\{\mathrm{pt}\}}}) & = & \pi^{A\{\mathrm{pt}\}A\{\mathrm{pt}\}}_{AA*} \circ i^{A\{\mathrm{pt}\}}_{A\{\mathrm{pt}\}A\{\mathrm{pt}\}*} \circ \pi^{A\{\mathrm{pt}\}*}_{A}(\mathscr{O}_A) \\
\| & & \uparrow \\
\pi^{A\{\mathrm{pt}\}A\{\mathrm{pt}\}}_{AA*} \circ i^{A\{\mathrm{pt}\}}_{A\{\mathrm{pt}\}A\{\mathrm{pt}\}*}(\mathscr{O}_{A\times\{\mathrm{pt}\}}) & & | \\
\downarrow{\scriptstyle \mathrm{unit}} & & {\scriptstyle \mathrm{b.c.}} \\
\pi^{A\{\mathrm{pt}\}A\{\mathrm{pt}\}}_{AA*} \circ i^{A\{\mathrm{pt}\}}_{A\{\mathrm{pt}\}A\{\mathrm{pt}\}*} \circ i^{A}_{A\{\mathrm{pt}\}*} \circ i^{A*}_{A\{\mathrm{pt}\}}(\mathscr{O}_{A\times\{\mathrm{pt}\}}) & & | \\
\downarrow{\scriptstyle \mathrm{comp}_*} & & | \\
i^{A}_{AA*} \circ i^{A*}_{A\{\mathrm{pt}\}}(\mathscr{O}_{A\times\{\mathrm{pt}\}}) & & \pi^{A\{\mathrm{pt}\}A\{\mathrm{pt}\}}_{AA*} \circ i^{AA\{\mathrm{pt}\}}_{AA\{\mathrm{pt}\}\{\mathrm{pt}\}*} \circ \pi^{AA\{\mathrm{pt}\}*}_{AA}(\mathscr{O}_{\Delta_A}) \\
\| & & \uparrow{\scriptstyle \eta^{\rho,\otimes}} \\
i^{A}_{AA*}(\mathscr{O}_A) & = & \mathscr{O}_{\Delta_A}
\end{array}
$$

This diagram commutes by Theorem 3.17. □

3.7 Braiding

This section defines the braiding on the double category constructed in Proposition 3.1. This construction requires that the geofibred category used for this construction is a *symmetric* monoidal geofibred category. The structure of a braiding on its own is insufficient, as the assumption of symmetry is used in the proof of Lemma 3.59.

3.7.1 Braiding transformation

A braided double category is a double category in which the categories of objects and morphisms are braided, and the braiding is compatible with the interchange and units.

Definition 3.57. Since $\mathbb{D}_0$ has all products, it can be given the structure of a Cartesian monoidal category (as in Definition 1.9) and hence it is a symmetric monoidal category. The braiding morphisms $s_{AB}\colon A \times B \to B \times A$ are the morphisms from the universal property of the fibre product structure.

Define another family of morphisms in $\mathbb{D}_1$, parameterised by pairs of objects $\mathscr{E}, \mathscr{F} \in \mathbb{D}_1$, by

$$\mathfrak{s}_{\mathscr{E},\mathscr{P}} = \begin{array}{ccc} A \times U & \xrightarrow{\mathscr{E}\boxtimes\mathscr{P}} & B \times V \\ {\scriptstyle s_{AU}}\downarrow & \overset{\sigma_{AB}}{\Uparrow} & \downarrow{\scriptstyle s_{BV}} \\ U \times A & \xrightarrow[\mathscr{P}\boxtimes\mathscr{E}]{} & V \times B \end{array}$$

where σ_{AB} is the braiding of the external tensor product $\boxtimes$ introduced in Definition 3.27.

Lemma 3.58. *The families of morphisms in Definition 3.57 define a braiding on* $\mathbb{D}_0$ *and* $\mathbb{D}_1$.

Proof. For these families to define braidings for the monoidal structure, they need to satisfy the two hexagonal identities. For $\mathbb{D}_0$, this follows by the construction of the fibre product.

Let $A \xrightarrow{\mathscr{E}} B$, $U \xrightarrow{\mathscr{F}} V$ and $X \xrightarrow{\mathscr{G}} Y$ be objects in $\mathbb{D}_1$. Then one of the hexagon identities becomes (after dropping the associators) the commutative diagram

$$\begin{array}{ccc} \mathscr{E} \boxtimes \mathscr{F} \boxtimes \mathscr{G} & \xrightarrow{\mathscr{E} \boxtimes \mathfrak{s}_{\mathscr{F},\mathscr{G}}} & \mathscr{E} \boxtimes \mathscr{G} \boxtimes \mathscr{F} \\ & {\scriptstyle \mathfrak{s}_{\mathscr{E}\boxtimes\mathscr{F},\mathscr{G}}} \searrow \quad \swarrow {\scriptstyle \mathfrak{s}_{\mathscr{E},\mathscr{G}} \boxtimes \mathrm{Id}_{\mathscr{F}}} & \\ & \mathscr{G} \boxtimes \mathscr{E} \boxtimes \mathscr{F} & \end{array} .$$

Recalling the construction of the monoidal product from Subsection 3.6.1, this becomes equivalent to showing that there is an equality of 2-cells

$$\begin{array}{ccc} XAU & \xrightarrow{\mathscr{G}\boxtimes\mathscr{E}\boxtimes\mathscr{F}} & YBV \\ {\scriptstyle s_{X,AU}} \downarrow & \Uparrow \sigma_{\mathscr{E}\boxtimes\mathscr{F},\mathscr{G}} & \downarrow {\scriptstyle s_{Y,BV}} \\ AUX & \xrightarrow{\mathscr{E}\boxtimes\mathscr{F}\boxtimes\mathscr{G}} & BVY \end{array} = \begin{array}{ccc} XAU & \xrightarrow{\mathscr{G}\boxtimes\mathscr{E}\boxtimes\mathscr{F}} & YBV \\ \downarrow & \Uparrow \xi_* \circ \sigma_{\mathscr{E},\mathscr{G}} \boxtimes \mathscr{F} & \downarrow \\ AXU & \xrightarrow{\mathscr{E}\boxtimes\mathscr{G}\boxtimes\mathscr{F}} & BYV \\ \downarrow & \Uparrow \xi_* \circ \mathscr{E} \boxtimes \sigma_{\mathscr{F},\mathscr{G}} & \downarrow \\ AUX & \xrightarrow{\mathscr{E}\boxtimes\mathscr{F}\boxtimes\mathscr{G}} & BVY \end{array} .$$

This equality is exactly the condition in Equation (3.18) from the definition of a braiding of a monoidal structure on a geofibred category (Definition 3.27). The other hexagon identity likewise becomes Equation (3.19) □

3.7.2 Coherence

Lemma 3.59. *The braiding on $\mathbb{D}_0$ and $\mathbb{D}_1$ in Definition 3.57 give the double category the structure of a braided double category.*

Proof. Recall from Wester Hansen and Shulman [17, Definition 2.10] the conditions for a double category to be braided monoidal. Condition (vii)[1] is that $\mathbb{D}_1$ and $\mathbb{D}_0$ are

[1]Note that [17] continues the numbering from the definition of monoidal category, so that condition (vii) is the first condition for a braided monoidal category

braided monoidal. Condition (viii) is that the functors S and T are strict braided monoidal, which is true in this case. Condition (ix) is that the diagrams in the following subsections commute. □

Braiding-interchange compatibility

Lemma 3.60. *The diagram*

$$\begin{array}{ccc} (\mathscr{E} \odot \mathscr{F}) \boxtimes (\mathscr{P} \odot \mathscr{Q}) & \xrightarrow{\quad \mathfrak{s} \quad} & (\mathscr{P} \odot \mathscr{Q}) \boxtimes (\mathscr{E} \odot \mathscr{F}) \\ \downarrow{\scriptstyle \mathfrak{x}} & & \downarrow{\scriptstyle \mathfrak{x}} \\ (\mathscr{E} \boxtimes \mathscr{P}) \odot (\mathscr{F} \boxtimes \mathscr{Q}) & \xrightarrow{\quad \mathfrak{s} \odot \mathfrak{s} \quad} & (\mathscr{P} \boxtimes \mathscr{E}) \odot (\mathscr{Q} \boxtimes \mathscr{F}) \end{array} \tag{3.47}$$

commutes.

Proof. The result of following this diagram clockwise is the composition

$$\begin{array}{ccc} AU & \xrightarrow{(\mathscr{E}\odot\mathscr{F})\boxtimes(\mathscr{P}\odot\mathscr{Q})} & CW \\ {\scriptstyle s_{AU}}\downarrow & \Uparrow{\scriptstyle \sigma} & \downarrow{\scriptstyle s_{CW}} \\ UA & \xrightarrow{(\mathscr{P}\odot\mathscr{Q})\boxtimes(\mathscr{E}\odot\mathscr{F})} & WC \\ \downarrow & \Uparrow{\scriptstyle \chi} & \downarrow \\ UA & \xrightarrow{(\mathscr{P}\boxtimes\mathscr{E})\odot(\mathscr{Q}\boxtimes\mathscr{F})} & WC \end{array}$$

where ξ is the interchange transformation constructed in Subsection 3.6.2. The anti-clockwise direction gives

$$\begin{array}{ccc} AU & \xrightarrow{(\mathscr{E}\odot\mathscr{F})\boxtimes(\mathscr{P}\odot\mathscr{Q})} & CW \\ \downarrow & \Uparrow{\scriptstyle \chi} & \downarrow \\ AU & \xrightarrow{(\mathscr{E}\boxtimes\mathscr{P})\odot(\mathscr{F}\boxtimes\mathscr{Q})} & CW \\ {\scriptstyle s_{AU}}\downarrow & \Uparrow{\scriptstyle \sigma\odot\sigma} & \downarrow{\scriptstyle s_{CW}} \\ UA & \xrightarrow{(\mathscr{P}\boxtimes\mathscr{E})\odot(\mathscr{Q}\boxtimes\mathscr{F})} & WC \end{array}$$

For conciseness, let $L = AB, M = BC, N = UV, P = VW$. Then Equation (3.47) becomes the outer paths of the diagram

$$
\begin{array}{ccc}
\odot((\mathscr{P} \boxtimes \mathscr{E}) \boxtimes (\mathscr{Q} \boxtimes \mathscr{F})) & \xrightarrow{s_{\mathscr{P},\mathscr{E}} \boxtimes s_{\mathscr{Q},\mathscr{F}}} & \odot(j^{LN}_{NL*}(\mathscr{E} \boxtimes \mathscr{P}) \boxtimes j^{MP}_{PM*}(\mathscr{F} \boxtimes \mathscr{Q})) \\
\Big\downarrow{\scriptstyle \xi_* \circ s_{\mathscr{E},\mathscr{Q}}} & & \Big\downarrow{\scriptstyle \xi_*} \\
\odot \circ j^{NPLM}_{NLPM*}(\mathscr{P} \boxtimes \mathscr{Q} \boxtimes \mathscr{E} \boxtimes \mathscr{F}) & & \odot \circ j^{LNMP}_{NLPM*}(\mathscr{E} \boxtimes \mathscr{P} \boxtimes \mathscr{F} \boxtimes \mathscr{Q}) \\
{\scriptstyle \text{b.c.}}\Big\uparrow & & \Big\downarrow{\scriptstyle \text{b.c.}} \\
(\odot \times \odot)(\mathscr{P} \boxtimes \mathscr{Q} \boxtimes \mathscr{E} \boxtimes \mathscr{F}) & & (s_{AU} \times s_{CW})_* \circ \odot(\mathscr{E} \boxtimes \mathscr{P} \boxtimes \mathscr{F} \boxtimes \mathscr{Q}) \\
{\scriptstyle \xi^* \circ \xi_*}\Big\uparrow & & \Big\downarrow{\scriptstyle \xi_* \circ s_{\mathscr{P},\mathscr{F}}} \\
\odot(\mathscr{P} \boxtimes \mathscr{Q}) \boxtimes \odot(\mathscr{E} \boxtimes \mathscr{F}) & & (s_{AU} \times s_{CW})_* \circ \odot \circ j_*(\mathscr{E} \boxtimes \mathscr{F} \boxtimes \mathscr{P} \boxtimes \mathscr{Q}) \\
\Big\downarrow{\scriptstyle \sigma} & & {\scriptstyle \text{b.c.}}\Big\uparrow \\
(s_{AU} \times s_{CW})_*(\odot(\mathscr{E} \boxtimes \mathscr{F}) \boxtimes \odot(\mathscr{P} \boxtimes \mathscr{Q})) & \xrightarrow{\xi^* \circ \xi_*} & (s_{AU} \times s_{CW})_* \circ (\odot \times \odot)((\mathscr{E} \boxtimes \mathscr{F}) \boxtimes (\mathscr{P} \boxtimes \mathscr{Q}))
\end{array}
$$

This diagram can be simplified by using the fact that independent transformations commute to group together morphisms which act on the external tensor product and morphisms which act on the functor applied to this product. The diagram then becomes

$$
\begin{array}{ccc}
\odot(\mathscr{P} \boxtimes \mathscr{E} \boxtimes \mathscr{Q} \boxtimes \mathscr{F}) & \xrightarrow{\xi_* \circ (\sigma_{\mathscr{P},\mathscr{E}} \boxtimes \sigma_{\mathscr{Q},\mathscr{F}})} & \odot \circ j^{LNMP}_{NLPM*}(\mathscr{E} \boxtimes \mathscr{P} \boxtimes \mathscr{F} \boxtimes \mathscr{Q}) \\
\Big\downarrow{\scriptstyle \xi_* \circ \sigma_{e,\mathscr{Q}}} & & \Big\downarrow{\scriptstyle \xi_* \circ \sigma_{\mathscr{P},\mathscr{F}}} \\
\odot \circ j^{NPLM}_{NLPM*}(\mathscr{P} \boxtimes \mathscr{Q} \boxtimes \mathscr{E} \boxtimes \mathscr{F}) & & \odot \circ j^{LNMP}_{NLPM*} \circ j^{LMNP}_{LNMP*}(\mathscr{E} \boxtimes \mathscr{F} \boxtimes \mathscr{P} \boxtimes \mathscr{Q}) \\
\Big\downarrow{\scriptstyle \xi_* \circ \sigma_{\mathscr{P} \boxtimes \mathscr{Q}, \mathscr{E} \boxtimes \mathscr{F}}} & \nearrow{\scriptstyle \text{comp}_*} & {\scriptstyle \text{b.c.}}\Big\uparrow \\
\odot \circ j^{NPLM}_{NLPM*} \circ j^{LMNP}_{NPLM*}(\mathscr{E} \boxtimes \mathscr{F} \boxtimes \mathscr{P} \boxtimes \mathscr{Q}) & \xleftarrow[\text{b.c.}]{} & (s_{AU} \times s_{CW})_* \circ (\odot \times \odot)(\mathscr{E} \boxtimes \mathscr{F} \boxtimes \mathscr{P} \boxtimes \mathscr{Q})
\end{array}
$$

The lower-right triangle commutes by Theorem 3.17. The upper pentagon commutes if and only if the braiding σ is symmetric, since in this case the braiding gives a representation which factors through the symmetric group. $\square$

Braiding-unit compatibility

Lemma 3.61. *The diagram*

$$\begin{array}{ccc} \mathscr{O}_{\Delta_{A\times B}} & \xrightarrow{\mathfrak{u}_{A,B}} & \mathscr{O}_{\Delta_A} \otimes \mathscr{O}_{\Delta_B} \\ \Big\downarrow{\scriptstyle U_{s_{AB}}} & & \Big\downarrow{\scriptstyle \mathfrak{s}_{\mathscr{O}_{\Delta_A},\mathscr{O}_{\Delta_B}}} \\ \mathscr{O}_{\Delta_{B\times A}} & \xrightarrow{\mathfrak{u}_{A,B}} & \mathscr{O}_{\Delta_B} \otimes \mathscr{O}_{\Delta_A} \end{array} \tag{3.48}$$

commutes.

Proof. Recall from Subsection 3.6.3 the definition of $\mathfrak{u}_{A,B}$ as the 2-cell with morphism $\upsilon_{A,B}$, which is defined to be the morphism

$$\mathscr{O}_{\Delta_A} \boxtimes \mathscr{O}_{\Delta_B} \xrightarrow{\xi_*} i^{AB}_{AABB*}(\mathscr{O}_A \boxtimes \mathscr{O}_B) \xrightarrow{\rho^{\boxtimes}} i^{AB}_{AABB*} \circ \pi^{AB*}_B(\mathscr{O}_B) =\!=\!=\!= \mathscr{O}_{\Delta_{AB}} .$$

The clockwise direction of Equation (3.48) is the composition

$$\begin{array}{ccc} AB & \xrightarrow{\qquad \mathscr{O}_{\Delta_{A\times B}} \qquad} & AB \\ \| & \Uparrow{\scriptstyle \upsilon_{BA}} & \| \\ AB & \xrightarrow{j^{AABB}_{ABAB*}(\mathscr{O}_{\Delta_A} \boxtimes \mathscr{O}_{\Delta_B})} & AB \\ \Big\downarrow{\scriptstyle s_{AB}} & \Uparrow{\scriptstyle \sigma_{AB}} & \Big\downarrow{\scriptstyle s_{AB}} \\ BA & \xrightarrow{j^{BBAA}_{BABA*}(\mathscr{O}_{\Delta_B} \boxtimes \mathscr{O}_{\Delta_A})} & BA \end{array}$$

while the anticlockwise direction gives

$$\begin{array}{ccc} AB & \xrightarrow{\qquad \mathscr{O}_{\Delta_{A\times B}} \qquad} & AB \\ \Big\downarrow{\scriptstyle s_{AB}} & \Uparrow{\scriptstyle U_{s_{A,B}}} & \Big\downarrow{\scriptstyle s_{AB}} \\ BA & \xrightarrow{\qquad \mathscr{O}_{\Delta_{B\times A}} \qquad} & BA \\ \| & \Uparrow{\scriptstyle \upsilon} & \| \\ BA & \xrightarrow{j^{BBAA}_{BABA*}(\mathscr{O}_{\Delta_B} \boxtimes \mathscr{O}_{\Delta_A})} & BA \end{array}$$

Hence the commutativity of Equation (3.48) is equivalent to commutativity of the following diagram:

$$\begin{array}{ccc}
\mathscr{O}_{\Delta_{BA}} & \xrightarrow{\text{unit}} & i_* \circ s_{A,B*}(\mathscr{O}_{AB}) \\
 & & \downarrow{\scriptstyle \text{comp}_*} \\
\uparrow{\scriptstyle \rho^{\boxtimes}} & & (s_{A,B} \times s_{A,B})_* \circ i_*(\mathscr{O}_{AB}) \\
 & & \uparrow{\scriptstyle \rho^{\boxtimes}} \\
i_*(\mathscr{O}_B \boxtimes \mathscr{O}_A) & & (s_{A,B} \times s_{A,B})_* \circ i_*(\mathscr{O}_A \boxtimes \mathscr{O}_B) \\
\uparrow{\scriptstyle \xi_*} & & \uparrow{\scriptstyle \xi_*} \\
\mathscr{O}_{\Delta_B} \boxtimes \mathscr{O}_{\Delta_A} & \xrightarrow{\sigma_{\mathscr{O}_{\Delta_B},\mathscr{O}_{\Delta_A}}} & (s_{A,B} \times s_{A,B})_*(\mathscr{O}_{\Delta_A} \boxtimes \mathscr{O}_{\Delta_B})
\end{array}$$

This can be split into simpler diagrams

$$\begin{array}{ccc}
\mathscr{O}_{\Delta_{BA}} & \xrightarrow{\text{unit}} & i_* \circ s_{A,B*}(\mathscr{O}_{AB}) \\
 & & \downarrow{\scriptstyle \text{comp}} \\
\uparrow{\scriptstyle \rho^{\boxtimes}} & & (s_{A,B} \times s_{A,B})_* \circ i_*(\mathscr{O}_{AB}) \\
 & & \uparrow{\scriptstyle \rho^{\boxtimes}} \\
i_*(\mathscr{O}_B \boxtimes \mathscr{O}_A) & \xrightarrow{s_{\mathscr{O}_B,\mathscr{O}_A}} & (s_{A,B} \times s_{A,B})_* \circ i_*(\mathscr{O}_A \boxtimes \mathscr{O}_B) \\
\uparrow{\scriptstyle \xi_*} & & \uparrow{\scriptstyle \xi_*} \\
\mathscr{O}_{\Delta_B} \boxtimes \mathscr{O}_{\Delta_A} & \xrightarrow{s_{\mathscr{O}_{\Delta_B},\mathscr{O}_{\Delta_A}}} & (s_{A,B} \times s_{A,B})_*(\mathscr{O}_{\Delta_A} \boxtimes \mathscr{O}_{\Delta_B})
\end{array}$$

where the lower rectangle is commutative by Equation (3.20). After re-ordering the direct image functors, the upper half of the diagram is isomorphic to applying i_* to the diagram

$$\begin{array}{ccc}
\mathscr{O}_{BA} & \xrightarrow{\text{unit}} & s_{A,B*}(\mathscr{O}_{AB}) \\
\uparrow{\scriptstyle \rho^{\boxtimes}} & & \uparrow{\scriptstyle \rho^{\boxtimes}} \\
\mathscr{O}_B \boxtimes \mathscr{O}_A & \xrightarrow{s_{\mathscr{O}_B,\mathscr{O}_A}} & s_{A,B*}(\mathscr{O}_A \boxtimes \mathscr{O}_B)
\end{array}$$

which is exactly Equation (3.22). □

3.8 Loosely strong companions

Recall Definition 3.31 defines the notion of a companion for a tight morphism. This notion allows the translation of tight morphisms into loose morphisms, and hence can be used to show that the monoidal constraints (which are written in terms of tight morphisms) correspond to loose morphisms which are retained when passing to the underlying loose 2-category.

In the double category constructed from Definition 3.29, all tight morphisms have companions (Lemma 3.62). A natural isomorphism α between functors valued in such a double category has loosely strong companions (Lemma 3.63, first stated as [17, Remark 4.11]).

Lemma 3.62. *Let* $\begin{array}{c} A \\ \downarrow{\scriptstyle f} \\ B \end{array}$ *be a tight morphism. It has a companion given by*

$$\hat{f} = \; A \xrightarrow{(\mathrm{Id}_A, f)_*(\mathscr{O}_A)} B \; .$$

Proof. Recall the morphism $\nu_f \colon \mathscr{O}_{\Delta_B} \to (f \times f)_*(\mathscr{O}_{\Delta_A})$ from Definition 3.41. There is a natural isomorphism

$$\gamma_1 \colon (f \times f)_* \circ i^1_{12*} \xrightarrow{\mathrm{comp}_*} (f, f)_* \xrightarrow{\mathrm{comp}_*^{-1}} (f \times \mathrm{Id}_B)(\mathrm{Id}_A, f)$$

which forms a cell

$$\begin{array}{ccc} A & \xrightarrow{\hat{f}} & B \\ {\scriptstyle f}\downarrow & \overset{\gamma_1 \circ \nu_f}{\Uparrow} & \downarrow{\scriptstyle \mathrm{Id}_B} \\ A & \xrightarrow{\mathscr{O}_{\Delta_B}} & B \end{array} .$$

There is also a natural transformation

$$\gamma_2 \colon (\mathrm{Id}, f)_* \xrightarrow{\mathrm{comp}_*^{-1}} (\mathrm{Id} \times f)_* \circ i^1_{12*}$$

which gives a cell

$$\begin{array}{ccc} A & \xrightarrow{\mathscr{O}_{\Delta_A}} & A \\ {\scriptstyle \mathrm{Id}_A}\downarrow & \overset{\gamma_2}{\Uparrow} & \downarrow{\scriptstyle f} \\ A & \xrightarrow[\hat{f}]{} & B \end{array} .$$

The tight composition of these cells gives the cell $U(f)$, since ν_f is exactly the morphism used to construct $U(f)$; the composition morphisms cancel by Theorem 3.17. Similarly, the loose composition gives $\mathrm{Id}_{\hat{f}}$, and hence the pair exhibit $\hat{f}$ as a companion for f. □

We restate a result [17, Remark 4.11] and give a sketch of its proof.

Lemma 3.63. *Let $\alpha\colon F \Rightarrow G$ be a natural isomorphism, where $F, G\colon \mathbb{D} \to \mathbb{E}$ are functors between double categories. If $\mathbb{E}$ has companions for all tight morphisms, then α has loosely strong companions.*

Proof. Let $A \in \mathbb{D}$. Since α is an isomorphism, α_A has an inverse α_A^{-1}. By [17, Lemmas 3.15 and 3.16], there is an adjoint equivalence (internal to the category $\mathscr{L}(\mathbb{E})$) $\widehat{\alpha_A} \dashv \widehat{\alpha_A^{-1}}$. These adjoint equivalence give an adjoint equivalence $\widehat{\alpha} \dashv \widehat{\alpha^{-1}}$, internal to the category of functors $\mathscr{L}(\mathbb{D}) \to \mathscr{L}(\mathbb{E})$, colax transformations, and modifications. By doctrinal adjunction [23], a colax structure on a right adjoint gives a lax structure on the left adjoint, giving $\widehat{\alpha}$ a lax structure. Any transformation which has a lax and colax structure necessarily has each α_f invertible, which gives the desired result. □

Corollary 3.64. *The monoidal constraints (consisting of the associator and the left and right unitors) of $\mathbb{D}(F)$ have loosely strong companions.*

Corollary 3.65. *The 2-category $\mathscr{L}(\mathbb{D}(F))$ has a monoidal structure.*

3.9 Geofibred category of sheaves

This section shows that the geofibred category of sheaves of quasi-coherent modules over schemes satisfies the conditions of Proposition 3.1, and hence that it determines a double category. The underlying loose 2-category of this double category is $\mathscr{V}\!ar$, the 2-category of schemes, sheaves and morphisms of sheaves.

Definition 3.66. Let $\mathbf{Sp}_{\mathscr{V}ar}$ be the category with objects quasi-compact quasi-separated smooth schemes over a fixed field k, and morphisms scheme morphisms. Let $\mathbf{Sh}_{\mathscr{V}ar}$ be the category with objects tuples $(X, \mathscr{E})$, where $X \in \mathbf{Sp}_{\mathscr{V}ar}$ and $\mathscr{E} \in \mathscr{D}(X)$ is an object in the derived category of sheaves of quasi-coherent modules. A morphism from $(X, \mathscr{E})$ to $(X, \mathscr{F})$ in $\mathbf{Sh}_{\mathscr{V}ar}$ is a morphism $\alpha \in \operatorname{Hom}_{\mathscr{D}(X)}(\mathscr{E}, \mathscr{F})$; if $X \neq Y$, there are no morphisms between $(X, \mathscr{E})$ and $(Y, \mathscr{F})$. Define a functor $F_{\mathscr{V}ar} \colon \mathbf{Sh}_{\mathscr{V}ar} \to \mathbf{Sp}_{\mathscr{V}ar}$ by

$$\begin{aligned} F_{\mathscr{V}ar} \colon \mathbf{Sh}_{\mathscr{V}ar} &\to \mathbf{Sp}_{\mathscr{V}ar}, \\ (X, \mathscr{E}) &\mapsto X, \\ \alpha &\mapsto \operatorname{Id}_X . \end{aligned}$$

Further, define a family of functors

$$\boxtimes_{AB} \colon (\mathbf{Sh}_{\mathscr{V}ar})_A \times (\mathbf{Sh}_{\mathscr{V}ar})_B \to (\mathbf{Sh}_{\mathscr{V}ar})_{A \times B}$$

to be the external product of sheaves,

$$\mathscr{E} \boxtimes_{AB} \mathscr{F} = \pi_A^{AB*}(\mathscr{E}) \otimes \pi_B^{AB*}(\mathscr{F}),$$

where $\otimes$ is the derived tensor product of sheaves. Finally, define braiding morphisms

$$\sigma_{A,B} \colon \mathscr{E} \boxtimes_{A,B} \mathscr{F} \to j^{BA}_{AB*}(\mathscr{F} \boxtimes_{A,B} \mathscr{E})$$

to be the maps corresponding under adjunction to the morphism

$$j^{BA*}_{AB}(\pi^{AB*}_{A}(\mathscr{E}) \otimes \pi^{AB*}_{B}(\mathscr{F})) \cong \pi^{BA*}_{A}(\mathscr{E}) \boxtimes \pi^{BA*}_{B}(\mathscr{F}) \cong \pi^{BA*}_{B}(\mathscr{F}) \otimes \pi^{BA*}_{A}(\mathscr{E})$$

where the first morphism is the distributivity of inverse images over tensor products, and the second is braiding of the tensor product.

Lemma 3.67. *The external product functor $\boxtimes_{AB}$ is an exact functor. Further, on affine schemes, it is given on global sections by the tensor product over k.*

Proof. Let

$$0 \to \mathscr{E} \xrightarrow{f} \mathscr{F} \xrightarrow{g} \mathscr{G} \to 0$$

and

$$0 \to \mathscr{E}' \xrightarrow{f'} \mathscr{F}' \xrightarrow{g'} \mathscr{G}' \to 0$$

be exact sequences of quasi-coherent sheaves of modules over two schemes A and B. Applying $\boxtimes_{AB}$ gives the (not necessarily exact) sequence

$$0 \to \mathscr{E} \boxtimes \mathscr{E}' \to \mathscr{F} \boxtimes \mathscr{F}' \to \mathscr{G} \boxtimes \mathscr{G}' \to 0. \tag{3.49}$$

Recall a sequence of quasi-coherent sheaves is exact if and only if it is exact on stalks. Points in the fibre product $A \times B$ are given by [44, Tag 01JT] tuples $(a, b, \mathfrak{p})$, where $a \in A$, $b \in B$, and $\mathfrak{p}$ is a prime ideal of $\mathscr{O}_{A,a} \otimes_k \mathscr{O}_{B,b}$. Let $R_{ab} = \mathscr{O}_{A,a} \otimes_k \mathscr{O}_{B,b}$; then the stalk of the structure sheaf $\mathscr{O}_{AB}$ at the point $(a, b, \mathfrak{p})$ is given by the localisation

$$R_{\mathfrak{p}} = (R_{ab})_{\mathfrak{p}}.$$

Let $\mathscr{E}_\mathfrak{p}$ be the $\mathscr{O}_{A,a}$-module $\mathscr{E}_a$ viewed as an R_{ab}-module and then localised at the prime ideal $\mathfrak{p}$. Define $\mathscr{F}_\mathfrak{p}$ and $\mathscr{G}_\mathfrak{p}$ likewise. Similarly, let $\mathscr{E}'_\mathfrak{p}$ be the $\mathscr{O}_{B,b}$-module $\mathscr{E}'_b$ viewed as an R_{ab}-module and localised at $\mathfrak{p}$. Then the stalk of the sequence Equation (3.49) is

$$0 \to \mathscr{E}_\mathfrak{p} \otimes_{R_\mathfrak{p}} \mathscr{E}'_\mathfrak{p} \to \mathscr{F}_\mathfrak{p} \otimes_{R_\mathfrak{p}} \mathscr{F}'_\mathfrak{p} \to \mathscr{G}_\mathfrak{p} \otimes_{R_\mathfrak{p}} \mathscr{G}'_\mathfrak{p} \to 0. \tag{3.50}$$

Since localisation commutes with tensor products, this can be written as the localisation at $\mathfrak{p}$ of the sequence

$$0 \to \mathscr{E}_a \otimes_{R_{ab}} \mathscr{E}'_b \to \mathscr{F}_a \otimes_{R_{ab}} \mathscr{F}'_b \to \mathscr{G}_a \otimes_{R_{ab}} \mathscr{G}'_b \to 0. \tag{3.51}$$

Since localisation is an exact functor, Equation (3.50) is exact if Equation (3.51) is exact. By assumption, the sequences

$$0 \to \mathscr{E}_a \xrightarrow{f_a} \mathscr{F}_a \xrightarrow{g_a} \mathscr{G}_a \to 0 \quad \text{and} \quad 0 \to \mathscr{E}'_b \xrightarrow{f'_b} \mathscr{F}'_b \xrightarrow{g'_b} \mathscr{G}'_b \to 0$$

are exact. Since the tensor product is a right-exact functor, it is sufficient to show that the first morphism in Equation (3.51) is injective. Since R_{ab} is formed by taking a tensor product over a field, the kernel of $f_a \otimes f'_b$ is

$$\ker(f_a \otimes f'_b) = \ker(f_a) \otimes \mathscr{E}'_b + e_a \otimes \ker(f'_b) = \{0\}$$

as required. □

Proposition 3.68. *The functor $F_{\mathscr{V}ar}$ constructed in Definition 3.66, along with the derived direct image and inverse image functors, defines a geofibred category satisfying the conditions of Proposition 3.1.*

Moreover, the functors $\boxtimes_{AB}$ give this a monoidal structure. Define morphisms

$$\sigma_{\mathscr{E},\mathscr{F}} \colon \mathscr{E} \boxtimes \mathscr{F} \to s_{BA*}(\mathscr{F} \boxtimes \mathscr{E})$$

by the adjoint to the map

$$s_{BA}^{*}(\mathscr{E} \boxtimes \mathscr{F}) \xrightarrow{\xi^{*-1}} \pi_A^{BA*}(\mathscr{E}) \otimes \pi_B^{BA*}(\mathscr{F}) \cong \pi_B^{BA*}(\mathscr{F}) \otimes \pi_B^{BA*}(\mathscr{F}),$$

where the final morphism is the braiding of the tensor product. These give the monoidal geofibred category a symmetric braiding.

Proof. The key part of the proof is the fact that the non-derived external tensor functor $\boxtimes_{AB}$ is exact, and consequently constructions can be computed locally without having to take resolutions of objects.

Reich [36] showed that $F_{\mathscr{Var}}$ is a geofibred category. It is clear that **Sp** has all products and a terminal object $\{\mathrm{pt}\} = \mathrm{Spec}(k)$. By Proposition 1.29, the class of projection morphisms is pull-geolocalizing.

The associators, unitors and distributors can be defined by working locally on an affine patch. As an example, we give an explicit construction of the associator. Let $\mathscr{E}_\bullet \in \mathscr{D}(X), \mathscr{F}_\bullet \in \mathscr{D}(Y), \mathscr{G}_\bullet \in \mathscr{D}(Z)$ be elements of $\mathbf{Sh}_{\mathscr{Var}}$. Then, by exactness,

$$(\mathscr{E} \boxtimes (\mathscr{F} \boxtimes \mathscr{G}))_i = \mathscr{E}_i \boxtimes (\mathscr{F}_i \boxtimes \mathscr{G}_i)$$

and similarly

$$((\mathscr{E} \boxtimes \mathscr{F}) \boxtimes \mathscr{G})_i = (\mathscr{E}_i \boxtimes \mathscr{F}_i) \boxtimes \mathscr{G}_i.$$

These are both elements of $\mathscr{D}(X \times Y \times Z)$. Take an affine open cover of $X \times Y \times Z$ formed from sets $U \times V \times W$, where U, V and W are open affine neighbourhoods of

X, Y and Z respectively. Then

$$(\mathscr{E} \boxtimes (\mathscr{F} \boxtimes \mathscr{G}))_i(U \times V \times W) = \mathscr{E}_i(U) \otimes \mathscr{F}_i(V) \otimes \mathscr{G}_i(W).$$

Since U is an open affine neighbourhood, $U \cong \operatorname{Spec}(R)$ for some k-algebra R and $\mathscr{E}_i(U)$ can be identified with $\widetilde{M}$ for some R-module M. Likewise, $\mathscr{F}_i(V)$ and $\mathscr{G}_i(W)$ can be identified with modules N and P over some rings S and T respectively. Then there is a morphism

$$M \otimes_k (N \otimes_k P) \to (M \otimes_k N) \otimes_k P$$

given by the associator of the tensor product of vector spaces. Gluing together these morphisms gives the associator for $\boxtimes$.

The unitor can be constructed similarly, using the identification $M \otimes_R (R \otimes_k S) \cong M \otimes_k S$. The distributors can be constructed from the distributivity of base-change of modules over the tensor product over a field; that is, the identification

$$M_R \otimes_k N_S \cong (M \otimes_k N)_{R \otimes_k S}$$

which is equality on the underlying vector spaces. Since this is an equality, it commutes with the associator.

Continuing to work locally, the composition natural transformation comp_* is the identification

$$(M_{S'})_{S''} = M_{S''}$$

for a S-module M and morphisms $f\colon S'' \to S'$, $g\colon S' \to S$. From this, the commutativity of distributors with composition is clear. The composition of inverse images comp^* is the isomorphism

$$(M \otimes_{S''} S') \otimes_{S'} S \cong M \otimes_{S''} S$$

given by $(m, s_1, s_2) \mapsto (m, g(s_1)s_2)$. Again, that this commutes with distributivity is clear.

Similarly, distributivity commutes with base-change morphisms, trivialisations and units.

Let $f\colon R \to S$ be a morphism of rings. Then for any R-module M we have $f_* \circ f^*(M) = (M \otimes_R S)_R$. The unit natural transformation unit: $\mathrm{Id} \to f_* \circ f^*$ has as its components

$$\begin{aligned} \mathrm{unit}(f)_M \colon M &\to (M \otimes_R S)_R, \\ m &\mapsto m \otimes_R 1_S. \end{aligned}$$

For morphisms $f\colon R \to S$ and $g\colon R' \to S'$, the unit-unitor diagram condition Equation (3.16) becomes

$$\begin{array}{ccccc} (S \otimes_k S')_{R \otimes_k R'} & & \xleftarrow{\quad r_1 \otimes r_2 \mapsto r_1 \otimes r_2 \quad} & & R \otimes_k R' \\ \| & & & & \| \\ (S \otimes_k S')_{R \times R'} & \xleftarrow{s_1 \otimes s_2 \mapsto s_1 \otimes s_2} & S_R \otimes_k S'_{R'} & \xleftarrow{r_1 \otimes r_2 \mapsto r_1 \otimes r_2} & R \otimes_k R' \end{array}$$

which clearly commutes. Left- and right-unitors are likewise compatible.

The monoidal compatibility conditions can likewise be checked locally on affine patches, where they are trivial.

The braiding maps can be given explicitly on an affine patch by

$$\begin{aligned} M \otimes_k N &\to N \otimes_k M \\ (m, n) &\mapsto (n, m). \end{aligned}$$

From this definition, the identities follow immediately. □

Proposition 3.69. *The category $\mathscr{V\!ar}$ is isomorphic to the category $\mathscr{L}(\mathbb{D}(F_{\mathscr{V\!ar}}))$.*

Proof. The objects of the two categories are both quasi-compact quasi-separated smooth schemes over k. The 1-morphisms of $\mathscr{L}(\mathbb{D}(F_{\mathscr{V}ar}))$ are diagrams

$$A \xrightarrow{\mathscr{E}} B$$

which can be identified with the sheaf $\mathscr{E} \in \mathrm{Hom}_{\mathscr{V}ar}(A, B)$. Finally, the 2-morphisms of $\mathscr{L}(\mathbb{D}(F_{\mathscr{V}ar}))$ are diagrams

$$\begin{array}{ccc} A & \xrightarrow{\mathscr{E}} & B \\ \downarrow & \Uparrow \alpha & \downarrow \\ A & \xrightarrow{\mathscr{F}} & B \end{array}$$

where $\alpha \in \mathrm{Hom}_{\mathscr{V}ar}(\mathscr{E}, (\mathrm{Id}_A \times \mathrm{Id}_B)_*(\mathscr{F}))$. This can be identified with the morphism $\mathrm{triv}_* \circ \alpha \in \mathrm{Hom}_{\mathscr{V}ar}(\mathscr{E}, \mathscr{F})$. $\square$

Chapter 4

TQFTs valued in $\mathscr{V}ar$

4.1 Frobenius algebra objects

It is well-known [24] that oriented $(1+1)$-TQFTs valued in the category of vector spaces are classified by commutative Frobenius algebras. This proof immediately generalises to the case when the target category is any symmetric monoidal category. In this case, TQFTs are classified by *Frobenius algebra objects*. Recall from Definition 1.16 the definition of a Frobenius algebra object.

Definition 4.1. [24] Let $\mathscr{C}$ be a monoidal category with unit I. A Frobenius algebra object is a tuple $(A, \mu, \delta, \epsilon, \tau)$ such that:

1. the tuple (A, μ, ϵ) is a unital monoid with multiplication $\mu \colon A \otimes A \to A$ and unit $\epsilon \colon I \to A$; and

2. the tuple (A, δ, τ) is a counital comonoid with comultiplication $\delta \colon A \to A \otimes A$ and counit $\tau \colon A \to I$; and

3. the Frobenius identities hold:

$$(\mathrm{Id}_A \otimes \mu) \circ (\delta \otimes \mathrm{Id}_A) = \delta \circ \mu = (\mu \otimes \mathrm{Id}_A) \circ (\mathrm{Id}_A \otimes \delta) \,.$$

A $(1+1+1)$-TQFT induces a $(1+1)$-TQFT by truncation; that is, by considering 2-manifolds only up to diffeomorphism, and taking the isomorphism class of the image of this object. We formalise this folklore process below as a first step to investigating the possible construction of a $(1+1+1)$-TQFT.

Definition 4.2. Let $2\text{-}\mathbf{Cat}$ be the 1-category of (weak) 2-categories and (lax) 2-functors, and let $\mathbf{Cat}$ be the 1-category of 1-categories and functors. For $\mathscr{C} \in 2\text{-}\mathbf{Cat}$, let $\mathbf{H}\mathscr{C}$ be the 1-category with the same objects as $\mathscr{C}$, but with 1-morphisms given by isomorphism classes of 1-morphisms in $\mathscr{C}$ (this is sometimes referred to as the *classifying category* of $\mathscr{C}$). For a 2-functor $\mathscr{F} : \mathscr{C} \to \mathscr{D} \in \mathrm{Hom}_{2\text{-}\mathbf{Cat}}(\mathscr{C}, \mathscr{D})$, define

$$\begin{aligned} \mathbf{H}\mathscr{F} : \mathbf{H}\mathscr{C} &\to \mathbf{H}\mathscr{D}\,, \\ x &\mapsto \mathscr{F}(x)\,, \\ [f] &\mapsto [\mathscr{F}(f)]\,, \end{aligned}$$

where $[f]$ denotes the isomorphism class of the 1-morphism f. This defines the *1-truncation functor*

$$\mathbf{H} \colon 2\text{-}\mathbf{Cat} \to \mathbf{Cat}\,.$$

Lemma 4.3. *Let $\mathscr{C}$ be a 2-category and $\mathbf{H}\mathscr{C}$ be its truncation. A $(1+1+1)$-TQFT $Z \colon \mathscr{B}ord_{1+1+1} \to \mathscr{C}$ induces a $(1+1)$-TQFT valued in the category $\mathbf{H}\mathscr{C}$.*

Proof. The category $\mathbf{HBord}_{1+1+1}$ is a quotient category of $\mathbf{Bord}_{1+1}$ (since 1-morphisms in this latter category are bordisms considered up to diffeomorphism, and the mapping cylinder associated to this diffeomorphism is an invertible 2-morphism in $\mathscr{B}ord_{1+1+1}$). The composition

$$\mathbf{Bord}_{1+1} \to \mathbf{HBord}_{1+1+1} \xrightarrow{\mathbf{H}Z} \mathbf{H}\mathscr{C}$$

gives the induced $(1+1)$-TQFT. □

4.1.1 Frobenius algebra objects in $\mathbf{H}\mathscr{L}(\mathbb{D}(F))$

Lemma 4.4. *Let $F\colon \mathbf{Sh} \to \mathbf{Sp}$ be a symmetric monoidal geofibred category satisfying the conditions of Proposition 3.1 and let $\mathscr{L}(\mathbb{D}(F))$ be its associated 2-category. Let $X \in \mathbf{Sp}$, and recall $i^1_{123}\colon X \to X^3$ is the triagonal map. Let $\mathscr{O}_{\Delta_3} = i^1_{123*}(\mathscr{O}_X)$. Then $X \in \mathbf{H}\mathscr{L}(\mathbb{D}(F))$ can be given the structure of a commutative monoid object by taking $\mathscr{O}_{\Delta_3}$ as the multiplication map and $\mathscr{O}_{\{pt\}\times X}$ to be the unit.*

Remark 4.5. It is worth noting the intuition behind choosing the $\mathscr{O}_{\Delta_3}$ to represent the multiplication. Consider $F = F_{\mathscr{V}ar}\colon \mathbf{Sh}_{\mathscr{V}ar} \to \mathbf{Sp}_{\mathscr{V}ar}$. Recall Huybrechts [20] showed that if $f\colon X \to Y$ and $\Gamma_f = (\mathrm{Id}_X, f)\colon X \to X \times Y$, then the Fourier–Mukai transform with kernel $\Gamma_{f*}(\mathscr{O}_X) \in \mathscr{D}(Y \times X)$ is the inverse image functor f^*. In particular, the Fourier–Mukai transform associated to $\mathscr{O}_{\Delta_3}$ is the inverse image functor i^{1*}_{12}.

If $\mathscr{E}, \mathscr{F} \in \mathscr{D}(X)$ then $i^{1*}_{12}(\mathscr{E} \boxtimes \mathscr{F}) = \mathscr{E} \otimes \mathscr{F}$, so this choice of multiplication recovers the tensor product on the subcategory $\mathscr{D}(X) \otimes \mathscr{D}(X) \subset \mathscr{D}(X \times X)$. A similar calculation shows that the Fourier–Mukai transform corresponding to the unit is the inverse image functor $\pi^{1*}_{\emptyset}\colon \mathscr{D}(\{\mathrm{pt}\}) \to \mathscr{D}(X)$.

Proof. We first prove that the multiplication is unital; that is, that

$$\mathscr{O}_{\Delta_3} \odot (\mathscr{O}_\Delta \boxtimes \mathscr{O}_X) \cong \mathscr{O}_\Delta. \tag{4.1}$$

By Definition 3.26, there are isomorphism

$$i^1_{12*}(\mathscr{O}_X) \boxtimes \mathscr{O}_X \xrightarrow[\xi_*]{\sim} i^{13}_{12,3*}(\mathscr{O}_X \boxtimes \mathscr{O}_X) \xrightarrow[\rho^{\boxtimes}]{\sim} i^{13}_{12,3*}(\mathscr{O}_{X\times X}) \ .$$

Hence the left-hand side of Equation (4.1) can be written as

$$i^4_{456*}(\mathscr{O}_X) \odot i^{13}_{12,3*}(\mathscr{O}_{X\times X}) = \pi^{1236}_{16*} \circ i^{1236*}_{1,24,35,6} \left(i^4_{456*}(\mathscr{O}_X) \boxtimes i^{13}_{123*}(\mathscr{O}_{X\times X}) \right) \ .$$

Now

$$
\begin{aligned}
i^{4}_{456*}(\mathscr{O}_X) \boxtimes i^{13}_{123*}(\mathscr{O}_{X\times X}) &\cong (i^{4}_{456} \times i^{13}_{123})_*(\mathscr{O}_X \boxtimes \mathscr{O}_{X\times X}) \\
&= i^{134}_{123456*}(\mathscr{O}_{X^3})
\end{aligned}
$$

(where the isomorphism in the first line is from ξ_*), so

$$
\begin{aligned}
i^{4}_{456*}(\mathscr{O}_X) \odot i^{13}_{12,3*}(\mathscr{O}_{X\times X}) &\cong \pi^{1236}_{16*} \circ i^{1236*}_{1,24,35,6} \circ i^{134}_{12,3,456*}(\mathscr{O}_{X^3}) \\
&\cong \pi^{1236}_{16*} \circ i^{1}_{123456*} \circ i^{1*}_{134}(\mathscr{O}_{X^3}) \\
&\cong i^{1}_{16*}(\mathscr{O}_X)
\end{aligned}
$$

where the second line follows by applying a base-change isomorphism, and the third by the composition of direct image functors. Thus in $\mathbf{H}\mathscr{L}(\mathbb{D}(F))$, the equality $[\mathscr{O}_{\Delta_3}] \odot [\mathscr{O}_\Delta \boxtimes \mathscr{O}_X] = [\mathscr{O}_\Delta]$ holds as required.

The right unital identity follows similarly, as does associativity.

Finally, for the structure to be commutative, the following diagram must commute:

$$
\begin{array}{ccc}
X \times X & \xrightarrow{\;i^{12}_{21*}(\mathscr{O}_{X\times X})\;} & X \times X \\
 & {\scriptstyle \mathscr{O}_{\Delta_3}} \searrow \quad \swarrow {\scriptstyle \mathscr{O}_{\Delta_3}} & \\
 & X &
\end{array}
\quad .
$$

This commutes if $\mathscr{O}_{\Delta_3} \odot i^{1,2}_{14,23*}(\mathscr{O}_{X\times X}) \cong \mathscr{O}_{\Delta_3}$. The left-hand side is

$$
\begin{aligned}
\mathscr{O}_{\Delta_3} \odot i^{14,23*}(\mathscr{O}_{X\times X}) &= \pi^{12347}_{127*} \circ i^{1,2,3,4,7*}_{1,2,35,46,7}(i^{1,2}_{14,23*}(\mathscr{O}_X) \boxtimes i^{5}_{567*}(\mathscr{O}_X)) \\
&\cong \pi^{12347}_{127*} \circ i^{1,2,3,4,7*}_{1,2,35,46,7}(i^{1,2,5}_{14,23,567*}(\mathscr{O}_{X^3})) \\
&\cong \pi^{12347}_{127*} \circ i^{1}_{1234567*}(\mathscr{O}_X) \\
&\cong i^{1}_{127*}(\mathscr{O}_X)
\end{aligned}
$$

where the second line uses the distributor isomorphism ξ_*, the third line is a base-change isomorphism and the final line follows by composing the two functors. □

Lemma 4.6. *Let $F\colon \mathbf{Sh} \to \mathbf{Sp}$ be a symmetric monoidal geofibred category satisfying the conditions of Proposition 3.1. Let $X \in \mathbf{Sp}$ and let $\mathscr{O}_{\Delta_3} = i^1_{123*}(\mathscr{O}_X)$. Then $X \in \mathbf{H}\mathscr{L}(\mathbb{D}(F))$ can be given the structure of a comonoid object by taking $\mathscr{O}_{\Delta_3}$ to be the comultiplication map and $\mathscr{O}_{X\times\{pt\}}$ to be the counit.*

Proof. These calculations can be performed in a manner similar to those in Lemma 4.4. □

Proposition 4.7. *Let $F\colon \mathbf{Sh} \to \mathbf{Sp}$ be a symmetric monoidal geofibred category satisfying the conditions of Proposition 3.1, and let $X \in \mathbf{Sp}$ and $\mathscr{O}_{\Delta_3} = i^1_{123*}(\mathscr{O}_X)$. Then the tuple $(X, \mathscr{O}_{\Delta_3}, \mathscr{O}_{\Delta_3}, \mathscr{O}_{\{pt\}\times X}, \mathscr{O}_{X\times\{pt\}})$ is a commutative Frobenius algebra object in $\mathbf{H}\mathscr{L}(\mathbb{D}(F))$.*

As mentioned in the introduction to Section 4.1, commutative Frobenius objects in a given category classify oriented $(1+1)$-TQFTs valued in that category. Consequently, given a smooth scheme over k, there is a $(1+1)$-TQFT corresponding to the commutative Frobenius object constructed in this proposition, which will be denoted $Z_X\colon \mathbf{Bord}_{1+1} \to \mathbf{H}\mathscr{L}(\mathbb{D}(F))$.

Recall there is a functor $\mathscr{L}(\mathbb{D}(F)) \to 2\mathscr{C}\!at$ sending X to its derived category $\mathscr{D}(X)$ and a sheaf $\mathscr{E}$ to its Fourier–Mukai transform $\Phi_{\mathscr{E}}$. Thus given any oriented $(1+1)$-TQFT valued in $\mathbf{H}\mathscr{V}\!ar$, we can form a functor $Z'_X\colon \mathbf{Bord}_{1+1} \to \mathbf{H}2\mathscr{C}\!at$. Then

$$Z'_X(\ \ ) = \Phi_{\mathscr{O}_{\Delta_3}} = i^{1*}_{12}$$

where the second equality follows from Remark 4.5. This is the same functor that Roberts and Willerton [38] conjectured should be assigned to the pair of pants (see Conjecture 1.2. They expressed the functor as taking the tensor product with the diagonal sheaf and pushing forward; this can be seen to be equivalent to applying i^{1*}_{12}

since

$$\pi^{12}_{2*}(\mathscr{E}\otimes\mathscr{O}_\Delta)=\pi^{12}_{2*}(\mathscr{E}\otimes i^1_{12*}(\mathscr{O}_X))\cong\pi^{12}_{2*}\circ i^1_{12*}\circ i^{1*}_{12}(\mathscr{E})=i^{1*}_{12}(\mathscr{E})\,,$$

where the isomorphism follows from the projection formula.

Proof. Lemma 4.4 and Lemma 4.6 show that this tuple gives rise to a commutative monoid and a cocommutative comonoid, so it remains to show that the Frobenius identities hold.

Consider first the composition $\mathscr{O}_{\Delta_3}\odot\mathscr{O}_{\Delta_3}=i^1_{123*}(\mathscr{O}_X)\odot i^4_{456*}(\mathscr{O}_X)$. There is an isomorphism

$$i^1_{123*}(\mathscr{O}_X)\boxtimes i^4_{456*}(\mathscr{O}_X)\cong i^{14}_{123,456*}(\mathscr{O}_{X\times X})$$

so the composition is given by

$$\begin{aligned}i^1_{123*}(\mathscr{O}_X)\odot i^4_{456*}(\mathscr{O}_X)&\cong\pi^{13456}_{1346*}\circ i^{12356*}_{1,2,34,5,6}\circ i^{14}_{123,456*}(\mathscr{O}_{X\times X})\\&\cong\pi^{13456}_{1346*}\circ i^1_{13456*}\circ i^{1*}_{14}(\mathscr{O}_{X\times X})\\&\cong i^1_{1346*}(\mathscr{O}_X),\end{aligned}$$

where the isomorphism in the second line is a base-change isomorphism.

We now simplify $(\mathscr{O}_\Delta\boxtimes\mathscr{O}_{\Delta_3})\odot(\mathscr{O}_{\Delta_3}\boxtimes\mathscr{O}_\Delta)$ to the same form. The two sheaves that are being composed are given by

$$\mathscr{O}_\Delta\boxtimes\mathscr{O}_{\Delta_3}\cong i^{01}_{023,14*}(\mathscr{O}_{X\times X})$$

and

$$\mathscr{O}_{\Delta_3}\boxtimes\mathscr{O}_\Delta\cong i^{89}_{58,679*}(\mathscr{O}_{X\times X})\,.$$

The external product of these is given by

$$(\mathscr{O}_\Delta \boxtimes \mathscr{O}_{\Delta_3}) \boxtimes (\mathscr{O}_{\Delta_3} \boxtimes \mathscr{O}_\Delta) \cong i^{0189}_{023,14,58,679*}(\mathscr{O}_{X^4}).$$

Hence the composition is given by

$$\begin{aligned}(\mathscr{O}_\Delta \boxtimes \mathscr{O}_{\Delta_3}) \odot (\mathscr{O}_{\Delta_3} \boxtimes \mathscr{O}_\Delta) &\cong \pi^{0123489}_{0189*} \circ i^{0123489*}_{0,1,25,36,47,8,9} \circ i^{0189}_{023,14,58,679*}(\mathscr{O}_{X^4}) \\ &\cong \pi^{0123489}_{0189*} \circ i^{0}_{0123489*} \circ i^{0*}_{0189}(\mathscr{O}_{X^4}) \\ &\cong i^{0}_{0189*}(\mathscr{O}_X)\end{aligned}$$

where the isomorphism in the second line is a base-change isomorphism. Hence the left Frobenius identity holds; the right identity follows similarly. □

We now show that under a mild condition on the geofibred category F, this is the unique "geometric" Frobenius algebra object; that is, one where the Fourier–Mukai functors associated to the product and coproduct maps are inverse and direct image functors respectively.

Definition 4.8. A monoid (resp. comonoid) in $\mathbf{H}\mathscr{L}(\mathbb{D}(F))$ is *geometric* if the Fourier–Mukai transform associated to its product (resp. coproduct) is of the form f^* (resp. f_*) for some morphism $f \in \mathrm{Hom}_{\mathbf{Sp}}(X, X \times X)$. A Frobenius algebra object is geometric if it is geometric as a monoid and comonid.

Lemma 4.9. *Let* $F\colon \mathbf{Sh} \to \mathbf{Sp}$ *be a symmetric monoidal geofibred category satisfying the conditions of Proposition 3.1. Let* $(X, \mathscr{M}, \mathscr{E})$ *be a geometric monoid in* $\mathscr{L}(\mathbb{D}(F))$*. Then* $\mathscr{E} \cong \mathscr{O}_X$*; and* $f = (f_1, f_2)$ *for some* $f_1, f_2 \in \mathrm{Hom}_{\mathbf{Sp}}(X, X)$ *such that for any* $\mathscr{X} \in \mathbf{Sh}_X$ *there are isomorphisms* $f_1^*(\mathscr{X}) \cong \mathscr{X} \cong f_2^*(\mathscr{X})$*.*

Proof. The morphism f can be written as $f = (f_1 \times f_2) \circ i^1_{12}$, for some $f_1, f_2 \in$

$\mathrm{Hom}_{\mathbf{Sp}}(X, X)$. Then the unital identity becomes

$$\mathscr{X} \cong i_{12}^{1*} \circ (f_1 \times f_2)^*(\mathscr{E} \boxtimes \mathscr{X})$$

for any $\mathscr{X} \in \mathbf{Sp}_X$. Taking $\mathscr{X} = \mathscr{O}_X$ and using the isomorphism $\mathscr{E} \boxtimes \mathscr{O}_X \cong \pi_1^{12*}(\mathscr{E})$ gives $\mathscr{O}_X \cong f_1^*(\mathscr{E})$. Then

$$\begin{aligned}\mathscr{X} &\cong i_{12}^{1*} \circ (f_1 \times f_2)^*(\mathscr{E} \boxtimes \mathscr{X}) \\ &\cong i_{12}^{1*} (f_1^*(\mathscr{E}) \boxtimes f_2^*(\mathscr{X})) \\ &\cong i_{12}^{1*} \circ \pi_2^{12*} \circ f_2^*(\mathscr{X}) \\ &\cong f_2^*(\mathscr{X}).\end{aligned}$$

By symmetry, we also find that $\mathscr{O}_X \cong f_2^*(\mathscr{E})$ and $\mathscr{X} \cong f_1^*(\mathscr{X})$. Taking $\mathscr{X} = \mathscr{E}$ gives $\mathscr{E} \cong f_1^*(\mathscr{E}) \cong \mathscr{O}_X$. □

Lemma 4.10. *Let $F \colon \mathbf{Sh} \to \mathbf{Sp}$ be a symmetric monoidal geofibred category satisfying the conditions of Proposition 3.1. Let $(X, \mathscr{M}, \mathscr{E})$ be a comonoid in $\mathscr{L}(\mathbb{D}(F))$. Suppose that $\Phi_{\mathscr{M}} = f_*$ for some $f \in \mathrm{Hom}_{\mathbf{Sp}}(X, X \times X)$. Then $\mathscr{E} \cong \mathscr{O}_X$; and $f = (f_1, f_2)$ for some $f_1, f_2 \in \mathrm{Hom}_{\mathbf{Sp}}(X, X)$ such that for any $\mathscr{X} \in \mathbf{Sh}_X$ there are isomorphisms $f_{1*}(\mathscr{X}) \cong \mathscr{X} \cong f_{2*}(\mathscr{X})$.*

Proof. This follows in the same manner as Lemma 4.9. □

Proposition 4.11. *Let $F \colon \mathbf{Sh} \to \mathbf{Sp}$ be a symmetric monoidal geofibred category satisfying the conditions of Proposition 3.1. Suppose that this satisfies the additional conditions that the only morphism $f \in \mathrm{Hom}_{\mathbf{Sp}}(X, X)$ such that $f^*(\mathscr{X}) \cong \mathscr{X}$ for all $\mathscr{X} \in X$ is the identity, and likewise for f^*. Then the commutative Frobenius algebra constructed in Proposition 4.7 is the unique geometric Frobenius algebra structure on X.*

Proof. This follows by Lemma 4.9 and Lemma 4.10. □

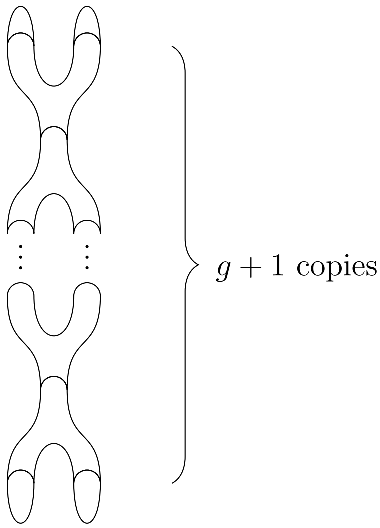


Figure 4.1: A decomposition of the genus g surface

4.1.2 Frobenius algebra objects in $\mathbf{H}\mathscr{V}\!ar$

The results of Subsection 4.1.1 applied to the 2-category associated to a double category constructed from any geofibred category using Proposition 3.1; from now on, we will use results specific to the case when $F = F_{\mathscr{V}ar} \colon \mathbf{Sh}_{\mathscr{V}ar} \to \mathbf{Sp}_{\mathscr{V}ar}$, and hence $\mathscr{L}(\mathbb{D}(F)) \cong \mathscr{V}ar$.

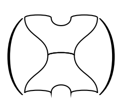
Lemma 4.12. *Let Σ_g be a genus g surface. Then*

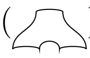
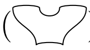
$$Z_X(\Sigma_g) \cong H^*\left(\mathscr{O}_\Delta^{\otimes(g+1)}\right) ,$$

where the right-hand side is viewed as a chain complex

$$\cdots \xrightarrow{0} H^i(\mathscr{O}_\Delta^{\otimes(g+1)}) \xrightarrow{0} H^{i-1}(\mathscr{O}_\Delta^{\otimes(g+1)}) \xrightarrow{0} \cdots \xrightarrow{0} H^0(\mathscr{O}_\Delta^{\otimes(g+1)}) \ .$$

Proof. Consider the decomposition of a genus g surface as shown in Figure 4.1. Applying Z_X gives

$Z_X(\Sigma_g) = Z_X\,($  $) \circ Z_X\,($  $)^{g+1} \circ Z_X\,($  $)$.

Now consider the Fourier–Mukai transform $\Phi_{Z_X(\Sigma_g)}$ associated to the kernel $Z_X(\Sigma_g)$. Note that $Z_X(\Sigma_g) \cong \Phi_{Z_X(\Sigma_g)}(k)$, so to determine $Z_X(\Sigma_g)$ it is sufficient to compute the functor $\Phi_{Z_X(\Sigma_g)}$. Let $\mathscr{C}_1 = Z_X($$), \mathscr{D}_1 = Z_X($$), \mathscr{D}_2 = Z_X($$)$ and $\mathscr{C}_2 = Z_X($$)$. Now

$$\mathscr{C}_1 = \mathscr{O}_X \boxtimes \mathscr{O}_X = \mathscr{O}_{X\times X}$$

and likewise

$$\mathscr{C}_2 = \mathscr{O}_X \boxtimes \mathscr{O}_X = \mathscr{O}_{X\times X}\,;$$

hence the corresponding Fourier–Mukai transforms are

$$\begin{aligned}
\Phi_{\mathscr{C}_1}(\mathscr{E}) &= \pi^{12}_{\emptyset *}(\pi^{12*}_{12}(\mathscr{E}) \otimes \mathscr{O}_{X\times X}) = \pi^{12}_{\emptyset *}(\mathscr{E})\,,\\
\Phi_{\mathscr{C}_2}(\mathscr{E}) &= \pi^{12}_{12*}(\pi^{12*}_{\emptyset}(\mathscr{E}) \otimes \mathscr{O}_{X\times X}) = \pi^{12*}_{\emptyset}(\mathscr{E})\,.
\end{aligned}$$

By Remark 4.5, the Fourier–Mukai transform $\Phi_{\mathscr{D}_1}$ is $\Phi_{\mathscr{D}_1} = i^1_{12*}$ and similarly $\Phi_{\mathscr{D}_2} = i^{1*}_{12}$. By the projection formula,

$$i^1_{12*} \circ i^{1*}_{12}(\mathscr{E}) \cong i^{1*}_{12}(\mathscr{E} \boxtimes i^1_{12*}(\mathscr{O}_X)) \cong \mathscr{E} \otimes i^1_{12*}(\mathscr{O}_X)\,,$$

so $\Phi_{\mathscr{D}_1} \circ \Phi_{\mathscr{D}_2}(\mathscr{E}) = \mathscr{E} \otimes i^1_{12*}(\mathscr{O}_X) = \mathscr{E} \otimes \mathscr{O}_\Delta$. Combining these results gives

$$\begin{aligned}
\Phi_{Z(\Sigma_g)}(k) &= \Phi_{\mathscr{C}_1} \circ (\Phi_{\mathscr{D}_1} \circ \Phi_{\mathscr{D}_2})^{g+1} \circ \Phi_{\mathscr{C}_2}(k)\\
&= \pi^{12}_{\emptyset *}(\pi^{12*}_{\emptyset}(k) \otimes \mathscr{O}_\Delta^{\otimes(g+1)})\\
&= \pi^{12}_{\emptyset *}(\mathscr{O}_\Delta^{\otimes(g+1)})\,.
\end{aligned}$$

Since the direct image functor with codomain a single point equals the global sections functor, and sheaf cohomology is the derived global sections functor, this expression is exactly $H^*\left(\mathscr{O}_\Delta^{\otimes(g+1)}\right)$. □

We now look to give an explicit computation of the derived tensor product of the diagonal sheaf. To do this, we will find a flat resolution of the diagonal. Pragacz, Srinivas and Pati [35] give such a resolution in the case where X is a smooth variety which satisfies a certain property.

Definition 4.13. [35] A smooth variety X is said to have property (D) when there exists a vector bundle $\mathscr{E}$ of rank equal to $\dim(X)$ on $X \times X$ and a section s of $\mathscr{E}$, such that the zero scheme of s is the diagonal subscheme $\Delta \subset X \times X$.

In particular, if X is a projective surface which is birational to a $K3$ surface with two disjoint rational curves, then X satisfies property (D). The property is also closed under taking fibre products.

Proposition 4.14. *Suppose that X is a scheme which has property (D). Then*

$$Z_X(\Sigma_g) \cong H^*\left(\left(\bigwedge^* \Omega\right)^{\otimes g}\right), \tag{4.2}$$

where Ω denotes the cotangent bundle and $\bigwedge^* \Omega$ is the chain complex with the sheaf $\wedge^i \Omega$ in degree i and zero differential.

Proof. For $g = 0$, the state space is given by

$$Z_X(\Sigma_0) = \pi^{12}_{\emptyset *}(i^1_{12*}(\mathscr{O}_X)) = \pi^1_{\emptyset *}(\mathscr{O}_X) \cong H^*(\mathscr{O}_X)\,.$$

Now consider the case $g \geq 1$. Under the assumption that X has property (D), there is a Koszul resolution [1, Remark 1.23] of the diagonal sheaf of the form

$$\mathscr{O}_\Delta \cong C_\bullet = 0 \to \det(\mathscr{E}^*) \to \cdots \to \mathscr{E}^* \xrightarrow{s^*} \mathscr{O}_{X\times X} \to 0\,, \tag{4.3}$$

where $\mathscr{E}$ is a locally free sheaf and s is a section of $\mathscr{E}$ which vanishes on the diagonal of $X \times X$. Here, $\mathscr{E}^*$ is the dual sheaf and $s^* \colon \mathscr{E}^* \to \mathscr{O}_{X\times X}$ is the dual map given by evaluating at the section s.

Since $C_\bullet$ is a complex of free modules, the derived functor Δ_X^* is the same as applying the non-derived functor Δ_X^* termwise. Now [35] $\Delta_X^*(\mathscr{E}^*) = \Omega_X$, so

$$\Delta_X^*(C_\bullet) = 0 \to \det(\Omega_X) \to \cdots \to \Omega_X \xrightarrow{0} \mathscr{O}_X \to 0 = \bigwedge^* \Omega\,. \tag{4.4}$$

Thus

$$\Delta_X^*(\mathscr{O}_\Delta^{\otimes g}) \cong (\Delta_X^*(\mathscr{O}_\Delta))^{\otimes g} \cong \left(\bigwedge^* \Omega\right)^{\otimes g}.$$

By the projection formula, there is an isomorphism

$$\mathscr{O}_\Delta^{\otimes g} \otimes \mathscr{O}_\Delta \cong \Delta_{X*} \circ \Delta_X^*(\mathscr{O}_\Delta^{\otimes g}),$$

so

$$\begin{aligned} \mathscr{O}_\Delta^{\otimes(g+1)} &\cong \Delta_{X*} \circ \Delta_X^*(\mathscr{O}_\Delta^{\otimes g}) \\ &\cong \Delta_{X*}\left(\left(\bigwedge^* \Omega\right)^{\otimes g}\right), \end{aligned}$$

and hence

$$Z(\Sigma_g) = \pi_{\emptyset *}^{12}(\mathscr{O}_\Delta^{\otimes(g+1)}) = \pi_{\emptyset *}^{1}\left(\left(\bigwedge^* \Omega\right)^{\otimes g}\right) = H^*\left(\left(\bigwedge^* \Omega\right)^{\otimes g}\right)$$

as required. □

Corollary 4.15. *Under the hypothesis of Proposition 4.14, the state spaces of the TQFT Z_X are isomorphic to those of the Rozansky–Witten TQFT.*

Proof. Recall from Equation (1.2) that the state spaces of the Rozansky-Witten TQFT are given by the formula

$$Z(\Sigma_g) = \bigoplus_{q=0}^{\dim_{\mathbb{C}}(X)} H_{\bar\partial}^q\left(X, (\wedge T^{1,0}X)^{\otimes g}\right).$$

which can be seen to be agree with Equation (4.2) by identifying $T^{1,0}$ with the cotangent sheaf Ω. □

4.2 The affine subcategory $\mathscr{AffVar}$

Consider the full subcategory $\mathscr{AffVar} \subset \mathscr{Var}$ whose objects are affine schemes over k. Recall that there is a simple description of quasi-coherent sheaves of modules over an affine scheme $\mathrm{Spec}(R)$: they are determined by their global sections, which are exactly R-modules [18]. We use this to give an equivalence between the categories $\mathscr{AffVar}$ and the category $\mathscr{CAlg}_k^d$ formed of commutative algebras, complexes of bimodules and chain maps (see Definition 4.20 for the full definition).

4.2.1 Non-derived equivalence

Let R be a k-algebra. Recall the global sections functor

$$\Gamma\colon \mathscr{SH}(\mathrm{Spec}(R)) \to R\text{-}\mathbf{mod}$$

sends a sheaf to its global sections. This functor has a quasi-inverse functor

$$\widetilde{}\colon R\text{-}\mathbf{mod} \to \mathscr{SH}(\mathrm{Spec}(R))$$

which sends a module M to the $\mathscr{O}_{\mathrm{Spec}(R)}$-module $\widetilde{M}$. This gives an equivalence of categories [18]

$$R\text{-}\mathbf{mod} \underset{\Gamma}{\overset{\sim}{\rightleftarrows}} \mathscr{SH}(\mathrm{Spec}(R))\,, \tag{4.5}$$

which also gives an equivalence of the derived categories.

It will be useful to translate the tensor product functor and the direct and inverse image functors to this algebraic setting.

Lemma 4.16. *The global sections functor distributes over the tensor product; that is,* $\Gamma(\mathscr{E} \otimes \mathscr{F}) \cong \Gamma(\mathscr{E}) \otimes \Gamma(\mathscr{F})$. *Similarly,* $\widetilde{M \otimes N} \cong \widetilde{M} \otimes \widetilde{N}$.

Definition 4.17. Let $f^{\#}\colon R \to S$ be a ring morphism, and let $f\colon \mathrm{Spec}(S) \to \mathrm{Spec}(R)$

be the induced morphism of schemes. Recall from Section 1.3 the definitions

$$f_*^{\#} \colon S\text{-}\mathbf{mod} \to R\text{-}\mathbf{mod}\,,$$
$$M \mapsto M_R\,,$$

where M_R denotes the S-module M considered as an R-module by $rs = f(r)s$, and

$$f^{\#*} \colon R\text{-}\mathbf{mod} \to S\text{-}\mathbf{mod}\,,$$
$$N \mapsto N \otimes_R S\,,$$

where R acts on S by $rs = f^{\#}(r)s$.

Lemma 4.18. *[18] The functors $f^{\#*}$ and $f_*^{\#}$ correspond to the inverse image and direct image functors respectively; explicitly, there are commutative diagrams of functors*

$$\begin{array}{ccc} \mathscr{SH}(\mathrm{Spec}(S)) & \xrightarrow{f_*} & \mathscr{SH}(\mathrm{Spec}(R)) \\ \downarrow{\scriptstyle \Gamma} & & \downarrow{\scriptstyle \Gamma} \\ S\text{-}\mathbf{mod} & \xrightarrow{f_*^{\#}} & R\text{-}\mathbf{mod} \end{array}$$

and

$$\begin{array}{ccc} \mathscr{SH}(\mathrm{Spec}(R)) & \xrightarrow{f^*} & \mathscr{SH}(\mathrm{Spec}(S)) \\ \downarrow{\scriptstyle \Gamma} & & \downarrow{\scriptstyle \Gamma} \\ R\text{-}\mathbf{mod} & \xrightarrow{f^{\#*}} & S\text{-}\mathbf{mod} \end{array} .$$

4.2.2 2-categorical equivalence

Definition 4.19. The 2-*category of commutative algebras over* k $\mathscr{CAlg}_k$ is a symmetric monoidal 2-category, with objects commutative k-algebras and categories of 1-morphisms given by

$$\mathrm{Hom}_{\mathscr{CAlg}_k}(A, B) = A\text{-}B\text{-}\mathbf{bimod},$$

where A-B-**bimod** is the category of A-B-bimodules. The composition in this category is given by relative tensor product: given $M \in \mathrm{Hom}_{\mathscr{CAlg}_k}(A, B)$, $N \in \mathrm{Hom}_{\mathscr{CAlg}_k}(B, C)$ their composition is $M \circ N = M \otimes_B N$. The monoidal structure is the tensor product over k.

Since we are working with derived categories, we want to consider a derived version of this category. Since A-B-**bimod** is equivalent to $(A \otimes_k B^{\mathrm{op}})$-**mod**, which in turn (since B is commutative) is isomorphic to the category $(A \otimes_k B)$-**mod**, we replace A-B-**bimod** with $\mathscr{D}((A \otimes_k B)\text{-}\mathbf{mod})$.

Definition 4.20. Let $\mathscr{CAlg}_k^d$ be the symmetric monoidal 2-category with the same objects as $\mathscr{CAlg}_k$, and with categories of 1-morphisms

$$\mathrm{Hom}_{\mathscr{CAlg}_k^d}(A, B) = \mathscr{D}((A \otimes_k B)\text{-}\mathbf{mod})\,.$$

Vertical composition is the usual composition of morphisms in $\mathscr{D}((A \otimes_k B)\text{-}\mathbf{mod})$. The composition functor is given by the derived tensor product

$$\otimes_B \colon \mathscr{D}((B \otimes_k C)\text{-}\mathbf{mod}) \times \mathscr{D}((A \otimes_k B)\text{-}\mathbf{mod}) \to \mathscr{D}((A \otimes_k C)\text{-}\mathbf{mod})\,,$$

where an $(A \otimes_k B)$-module is viewed as a B-module by the inclusion of algebras $B \hookrightarrow A \otimes_k B$ (and likewise for $(B \otimes_k C)$-modules). The monoidal structure is given by taking the tensor product over k. The braiding $S_{AB} \colon A \otimes_k B \to B \otimes_k A$ is given by the module S_{AB}, which has underlying abelian group $A \otimes_k B$ and an action of $(A \otimes_k B) \otimes_k (B \otimes_k A)$ given by

$$(a_1 \otimes b_1 \otimes b_2 \otimes a_2) \cdot (a_3 \otimes b_3) = a_1 a_2 a_3 \otimes b_1 b_2 b_3.$$

We want to use the equivalence of 1-categories in Equation (4.5) to construct an equivalence of 2-categories between $\mathscr{CAlg}_k^d$ and $\mathscr{AffVar}$. Let $\Phi \colon \mathscr{CAlg}_k^d \to \mathscr{AffVar}$

be defined on objects by $\Phi(A) = \mathrm{Spec}(A)$. On Hom-categories, define

$$\Phi\colon \mathrm{Hom}_{\mathscr{CAlg}^d_k}(A,B) \to \mathrm{Hom}_{\mathscr{AffVar}}(\Phi(A),\Phi(B))$$

to be the derived functor of the (exact) functor $\widetilde{\ }$.

Lemma 4.21. *The map Φ can be given the structure of a symmetric monoidal 2-functor.*

Proof. We give an overview of the transformations needed to show that Φ is a symmetric monoidal functor, and leave it to the reader to verify that they satisfy the necessary axioms.

First, we need to show that Φ is a functor of 2-categories; that is, that it respects identity 1-morphisms and composition of 1- and 2-morphisms up to coherent natural 2-isomorphism. Let $A \in \mathscr{CAlg}^d_k$; then note that

$$\Gamma(\mathscr{O}_\Delta) = A\,,$$

where A has the structure of an $(A \otimes_k A)$-bimodule where the left and right actions are both multiplication. This is exactly the identity 1-morphism $\mathrm{Id}_A \in \mathrm{Hom}_{\mathscr{CAlg}^d_k}(A,A)$. Hence there is an isomorphism $\mathscr{O}_\Delta \cong \widetilde{\Gamma(\mathscr{O}_\Delta)} = \widetilde{\mathrm{Id}_A}$.

Now let $M \in \mathrm{Hom}_{\mathscr{CAlg}^d_k}(A,B)$, $N \in \mathrm{Hom}_{\mathscr{CAlg}^d_k}(B,C)$. For brevity of notation, let $AB = A \otimes_k B$ and likewise for AC, BC and ABC. Let $X_1 = \mathrm{Spec}(A), X_2 = \mathrm{Spec}(B), X_3 = \mathrm{Spec}(C)$. Following the notation of Definition 4.17, the map π^{123}_{12} is induced by the mophism of rings

$$\pi^{123\#}_{12}\colon AB \to ABC\,,$$
$$a \otimes b \mapsto a \otimes b \otimes 1_C\,,$$

and $\pi_{13}^{123\#}$ and $\pi_{23}^{123\#}$ can be calculated similarly. Then

$$\begin{aligned}\Gamma\left(\Phi(N)\circ\Phi(M)\right) &= \Gamma\circ\pi_{13*}^{123}\left(\pi_{23}^{123*}(\Phi(N))\otimes\pi_{12}^{123*}(M)\right)\\ &= \pi_{13*}^{123\#}\left(\pi_{23}^{123\#*}(\Gamma\circ\Phi(N))\otimes\pi_{12}^{123\#*}(\Gamma\circ\Phi(M))\right)\\ &= \left((N\otimes_{BC}ABC)\otimes_{ABC}(M\otimes_{AB}ABC)\right)_{AC},\end{aligned}$$

where the equality in the second line follows from Lemma 4.18 and Lemma 4.16.

Let $N_\bullet$ be a free resolution of N and $M_\bullet$ be a free resolution of M as ABC-modules. Then

$$N_\bullet\otimes_{BC}ABC = N_\bullet\otimes_k A$$

and

$$M_\bullet\otimes_{AB}ABC = M_\bullet\otimes_k C\,,$$

so

$$\begin{aligned}\Gamma(\Phi(N)\circ\Phi(M)) &\cong \left((N_\bullet\otimes_{BC}ABC)\otimes_{ABC}(M_\bullet\otimes_{AB}ABC)\right)_{AC}\\ &\cong \left((N_\bullet\otimes_k A)\otimes_{ABC}(M_\bullet\otimes_k C)\right)_{AC}\\ &= \left(N_\bullet\otimes_B M_\bullet\right)_{AC}.\end{aligned}$$

On the other hand,

$$\begin{aligned}N\circ M &= (N\otimes_B M)_{AC}\\ &\cong (N_\bullet\otimes_B M_\bullet)_{AC}\,.\end{aligned}$$

Thus $\Gamma(\Phi(N)\circ\Phi(M))\cong N\circ M$. Applying Φ to both sides and using the fact that Φ and ψ are quasi-inverse gives $\Phi(N)\circ\Phi(M)\cong\Phi(N\circ M)$ as required.

Next, we need to show that Φ respects the monoidal structure; that is, we need to give invertible 1-morphisms

$$\chi\colon \Phi(A) \otimes \Phi(B) \to \Phi(A \otimes B)$$

and

$$\iota\colon \{\mathrm{pt}\} \to \Phi(k).$$

By definition, $\{\mathrm{pt}\} = \mathrm{Spec}(k)$ and so ι can be taken to be the identity $\mathscr{O}_{\Delta_{\{\mathrm{pt}\}}} \in \mathrm{Hom}_{\mathscr{AffVar}}(\{\mathrm{pt}\}, \{\mathrm{pt}\})$. Next, for any $A, B \in \mathscr{CAlg}_k^d$, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(A \otimes_k B) & \longrightarrow & \mathrm{Spec}(A) \\ \downarrow{\scriptstyle \mathrm{Spec}(i_A)} & & \downarrow{\scriptstyle \mathrm{Spec}(i_B)} \\ \mathrm{Spec}(B) & \longrightarrow & \mathrm{Spec}(k) \end{array}$$

where

$$i_A\colon A \to A \otimes_k B$$
$$a \mapsto a \otimes_k 1_B$$

and i_B is defined similarly. This diagram satisfies the universal property of limits, and hence there is a canonical isomorphism $\chi_0\colon \mathrm{Spec}(A) \otimes \mathrm{Spec}(B) \to \mathrm{Spec}(A \otimes_k B)$. Let $\chi = \hat{\chi_0}$ be the loose companion to this tight morphism (as defined in Lemma 3.62); explicitly, this is given by

$$\chi = (\mathrm{Id}, \chi_0)_*(\mathscr{O}_{\mathrm{Spec}(A)\otimes\mathrm{Spec}(B)}) \in \mathrm{Hom}_{\mathscr{AffVar}}(\mathrm{Spec}(A) \otimes \mathrm{Spec}(B), \mathrm{Spec}(A \otimes_k B)).$$

Finally, we give the 2-morphisms

$$u_{AB}\colon \Phi(S_{AB}) \circ \chi_{A,B} \to \chi_{BA} \circ \widehat{s_{\Phi(A),\Phi(B)}} \tag{4.6}$$

which shows that Φ respects the symmetric structure. Recall [17, Lemma 3.10] that the composition two companions gives a companion of the composition. Since $\chi_{BA} = \widehat{(\chi_{0,BA})}$, the right-hand side is a companion of $\chi_{0,BA} \circ s_{AB}$. Now note

$$\begin{aligned}\Gamma(\widehat{\Phi(s_{AB})}) &= \Gamma\left((\mathrm{Id}, \Phi(s_{AB}))_*(\mathscr{O}_{AB})\right) \\ &\cong (\mathrm{Id}_{A\otimes_k B} \otimes_k s_{A\otimes_k B})_*(AB) \\ &= S_{AB}.\end{aligned}$$

Thus $\Phi(S_{AB})$ is a companion of $\Phi(s_{AB})$. Thus the left-hand side of Equation (4.6) is isomorphic to $\widehat{\Phi(s_{AB})} \circ \widehat{\chi_{0,A,B}}$. The composition of two companions is a companion for the composition, so this is a companion for $\Phi(s_{AB}) \circ \chi_{0,A,B}$. Companions are unique up to isomorphism, and we take this isomorphism to be u_{AB}. ☐

Remark 4.22. The proof of Lemma 4.21 makes use of the ideas of companions from double categories. Both $\mathscr{AffVar}$ and $\mathscr{CAlg}_k^d$ can be constructed as underlying loose categories of double categories, and the functor Φ can be constructed from a functor between the two double categories.

We now construct a quasi-inverse to Φ. Let $\Psi\colon \mathscr{AffVar} \to \mathscr{CAlg}_k^d$ take a scheme to the global sections of its structure sheaf. On 1- and 2-morphisms, define this to be the derived functor of the (exact) functor Γ.

On objects, $\Phi\circ\Psi(X) = \mathrm{Spec}(\Gamma(X)) \cong X$, and $\Psi\circ\Phi(A) = A$. Further, the functors Φ_{AB} and $\Psi_{\Phi(A),\Phi(B)}$ define equivalences of categories. This, along with the fact that Φ is a 2-functor, is enough for $\mathscr{C}$ and $\mathscr{D}$ to be equivalent 2-categories. Formally, we have the following folklore result [42, Theorem A.16]:

Lemma 4.23. *Let $\mathscr{C}, \mathscr{D}$ be 2-categories, with $\Phi\colon \mathscr{C} \to \mathscr{D}$ a 2-functor. Suppose that Φ is surjective up to invertible 1-morphisms; that is, for any $d \in \mathscr{D}$ there is some $c \in \mathscr{C}$, $f \in \mathrm{Hom}_{\mathscr{D}}(\Phi(c), d)$, $g \in \mathrm{Hom}_D(d, \Phi(c))$ and invertible 2-morphisms $fg \Rightarrow \mathrm{Id}_d$*

and $\mathrm{Id}_{\Phi(c)} \Rightarrow gf$*. Suppose further that*

$$\Phi_{AB} \colon \mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{D}}(\Phi(A), \Phi(B))$$

is an equivalence of categories for all $A, B \in \mathscr{C}$*. Then* $\mathscr{C}, \mathscr{D}$ *are equivalent* 2-*categories.*

This is a adaptation of the statement that a functor between 1-categories is an equivalence if and only if it is fully faithful and essentially surjective.

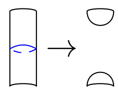
4.3 Extended TQFTs valued in $\mathscr{Var}$

This section analyses (1 + 1 + 1)-TQFTs valued in the 2-category $\mathscr{Var}$. First, the case of TQFTs valued in the affine subcategory $\mathscr{AffVar} \subset \mathscr{Var}$ is considered; since quasi-coherent sheaves of modules over affine schemes are determined by their global sections, which are modules over a ring, this provides a signification simplification. Once this case is considered, the general case of TQFTs valued in $\mathscr{Var}$ is examined.

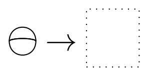
4.3.1 Affine extended TQFTs

In this section, Theorem 1.3 will be proved under the additional assumption that $X = Z(S^1)$ is affine and reduced. This result will be used to prove the general case (without the assumption that X is affine) in Subsection 4.3.2.

In $\mathscr{Bord}_{1+1+1}$, the cap and cup bordisms are an adjoint pair. Following the notation of [6], the unit and counit are given by the 2-morphisms $\mu^\dagger$ and $\nu^\dagger$, defined as follows. The bordism

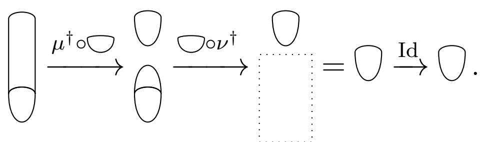
$\mu^\dagger :$ 

is the trace of surgery around the blue circle. The bordism

$\nu^\dagger :$ 

is given by the trace of surgery on the 2-sphere.

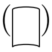
Since these are a unit-counit pair, there is an equality



(4.7)

where whiskering is used to compose the 1-morphism with the 2-morphisms $\mu^\dagger$ and $\nu^\dagger$ (recall in the bordism category, this whiskering is performed by taking the product of the 1-morphism with the unit interval I to give the identity 2-morphism over it, and then gluing the 2-morphisms together along their boundary). In particular, $Z(\mu^\dagger) \circ Z($$)$ is a monomorphism.

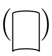
Suppose Z is a (1+1+1)-TQFT valued in $\mathscr{Var}$ such that the induced (1+1)-TQFT is Z_X for some X[1]. Then there are isomorphisms

$Z($$) \cong \mathscr{O}_\Delta$

and

$Z($$) \cong \mathscr{O}_{X \times X}\,,$

so

$Z(\mu^\dagger) \in \mathrm{Hom}(Z($$), Z($$)) \cong \mathrm{Hom}(\mathscr{O}_\Delta, \mathscr{O}_{X\times X})\,.$

In the rest of this section, it will be shown that $\mathrm{Hom}(\mathscr{O}_\Delta, \mathscr{O}_{X\times X}) = \{0\}$ unless X is a field.

Definition 4.24. For a ring R and a maximal ideal $\mathfrak{m} \in \mathrm{Specm}(R)$, let $\mathbb{F}_\mathfrak{m} = R/\mathfrak{m}$ be the residue field. Let $\mathrm{Ev}_\mathfrak{m} \colon R \to \mathbb{F}_\mathfrak{m}$ be the map formed by taking the quotient of R by $\mathfrak{m}$. Define $\mathrm{Ev}_{\mathfrak{m},\mathfrak{n}} \colon R \otimes_k R \to \mathbb{F}_\mathfrak{m} \otimes \mathbb{F}_\mathfrak{n}$ by

$$\mathrm{Ev}_{\mathfrak{m},\mathfrak{n}} = (\mathrm{Id}_{\mathbb{F}_\mathfrak{m}} \otimes \mathrm{Ev}_\mathfrak{n}) \circ (\mathrm{Ev}_m \otimes \mathrm{Id}_R)\,.$$

Lemma 4.25. *Let R be a ring with $\mathrm{Jac}(R) = 0$ and $\mathfrak{m} \lhd R$ be a fixed maximal ideal. If*

$$J_\mathfrak{m} = \bigcap_{\mathfrak{n} \in \mathrm{Specm}(R), \mathfrak{n} \neq \mathfrak{m}} \mathfrak{n} \neq \{0\}\,,$$

then $R = \mathfrak{m} \oplus J_\mathfrak{m}$.

[1] In fact, the following analysis holds for a $(1+1+1)$-TQFT valued in any 2-category $\mathscr{C}$ such that $\mathbf{H}\mathscr{C} \subset \mathbf{H}\mathscr{Var}$.

Proof. Pick some $r \in J_{\mathfrak{m}} \setminus \{0\}$. Then

$$J_{\mathfrak{m}} \cap \mathfrak{m} = \mathrm{Jac}(R) = \{0\} \,,$$

so $r \notin \mathfrak{m}$. Hence $\mathfrak{m} \subsetneq \mathfrak{m} + J_{\mathfrak{m}}$, but since $\mathfrak{m}$ is a maximal ideal it must be the case that $\mathfrak{m} + J_{\mathfrak{m}} = R$. Since $J_{\mathfrak{m}} \cap \mathfrak{m} = \mathrm{Jac}(R) = \{0\}$, it follows that $R = \mathfrak{m} \oplus J_{\mathfrak{m}}$. □

Lemma 4.26. *Let R be an algebra with $\mathrm{Jac}(R) = \{0\}$. Suppose R cannot be written as a non-trivial direct sum. Let $S = R \otimes_k R$ and let D be the S-module with underlying abelian group R and with $r \otimes r' \in S$ acting by multiplication by rr'. Then*

$$\mathrm{Hom}_S(D, S) = \begin{cases} R & R \text{ is a field} \\ \{0\} & \text{otherwise} \end{cases}$$

Proof. Say $g \in \mathrm{Hom}_S(D, S)$ and let $g_1 = g(1 \otimes 1)$. Fix maximal ideals $\mathfrak{m}, \mathfrak{n} \in \mathrm{Specm}(R)$, $\mathfrak{m} \neq \mathfrak{n}$. Then $\mathfrak{m} \setminus \mathfrak{n} \neq 0$ (else $\mathfrak{m} \subset \mathfrak{n}$ and since $\mathfrak{m}$ is maximal this would give $\mathfrak{m} = \mathfrak{n}$) so pick some $r_0 \in \mathfrak{m} \setminus \mathfrak{n}$. Then $\mathrm{Ev}_{\mathfrak{m}}(r_0) = 0$, $\mathrm{Ev}_{\mathfrak{n}}(r_0) \neq 0$. Now

$$0 = (r_0 \otimes 1 - 1 \otimes r_0) \cdot (1 \otimes 1) \in D \,,$$

so

$$0 = g(0) = (r_0 \otimes 1 - 1 \otimes r_0) g_1 \in S \,.$$

Applying $\mathrm{Ev}_{\mathfrak{m},\mathfrak{n}}$ gives

$$0 = \mathrm{Ev}_{\mathfrak{m},\mathfrak{n}} \left((r_0 \otimes 1 - 1 \otimes r_0) g_1 \right) = \left(-1 \otimes \mathrm{Ev}_{\mathfrak{n}}(r_0) \right) \mathrm{Ev}_{\mathfrak{m},\mathfrak{n}}(g_1) \,,$$

and hence $\mathrm{Ev}_{\mathfrak{m},\mathfrak{n}}(g_1) = 0$. Let $g_{\mathfrak{m}} = (\mathrm{Ev}_{\mathfrak{m}} \otimes \mathrm{Id}_R)(g_1)$, so $(\mathrm{Id}_{\mathbb{F}_{\mathfrak{m}}} \otimes \mathrm{Ev}_{\mathfrak{n}})(g_{\mathfrak{m}}) = 0$ and hence $g_{\mathfrak{m}} \in \mathbb{F}_{\mathfrak{m}} \otimes \mathfrak{n}$. As this holds for all $\mathfrak{n} \neq \mathfrak{m}$, it must be the case that $g_{\mathfrak{m}} \in \mathbb{F}_{\mathfrak{m}} \otimes J_{\mathfrak{m}}$.

Suppose first $g_{\mathfrak{m}} \notin \mathbb{F}_{\mathfrak{m}} \otimes \mathfrak{m}$. Then there is some $r_{\mathfrak{m}} \in J_{\mathfrak{m}} \setminus \mathfrak{m}$ (and in particular

$J_{\mathfrak{m}} \neq \{0\}$). By Lemma 4.25, R can be written as a direct sum $R = \mathfrak{m} \oplus J_{\mathfrak{m}}$. By assumption this direct sum must be trivial, so $\mathfrak{m} = 0$ and R is a field.

Suppose instead $g_{\mathfrak{m}} \in \mathbb{F}_{\mathfrak{m}} \otimes \mathfrak{m}$ for all $\mathfrak{m} \in \mathrm{Specm}(R)$. Then $g_{\mathfrak{m}} \in \mathbb{F}_{\mathfrak{m}} \otimes \mathrm{Jac}(R) = 0$, and hence

$$g_1 \in \bigcap_{\mathfrak{m} \in \mathrm{Spec}(R)} \ker(\mathrm{Ev}_{\mathfrak{m}} \otimes \mathrm{Id}_R) = \bigcap_{\mathfrak{m} \in \mathrm{Spec}(R)} \mathfrak{m} \otimes R = \mathrm{Jac}(R) \otimes R = \{0\}$$

and so $g = 0$. □

Corollary 4.27. *Let X be an affine reduced smooth scheme of finite type. Then*

$$\mathrm{Hom}_{\mathscr{AffVar}}(\mathscr{O}_\Delta, \mathscr{O}_{X\times X}) = \begin{cases} \mathbb{F} & X \cong \mathrm{Spec}(\mathbb{F}) \textit{ for some field } \mathbb{F} \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Since X is an affine reduced smooth scheme of finite type, $X = \mathrm{Spec}(R)$ for some finitely generated k-algebra R with no nilpotents. Since R is a finitely generated k-algerab, its Jacobson radical equals its nilradical, so $\mathrm{Jac}(R) = \{0\}$ and we can apply Lemma 4.26. □

It is now possible to prove Theorem 1.3 under the assumption that $Z(S^1)$ is affine and reduced.

Partial proof of Theorem 1.3. Let R be such that $X = \mathrm{Spec}(R)$. Then

$$Z(\mu^\dagger) \in \mathrm{Hom}_{\mathscr{AffVar}}(\mathscr{O}_\Delta, \mathscr{O}_{X\times X}).$$

Since $Z(\mu^\dagger) \circ Z(\smile)$ is the unit of an adjunction, it must be non-zero, so by Corollary 4.27 R is a field. □

4.3.2 Extended TQFTs

Let Z be a $(1+1+1)$-TQFT valued in $\mathscr{Var}$. Recall that the TQFT Z is said to be *based* on the scheme $X = Z(S^1)$. Given such a TQFT, it is natural to ask if it is possible to use this to construct a TQFT based on an open affine subscheme $U \subset X$.

Let $\iota \colon U \hookrightarrow X$. Then $(\iota^l)^* \colon \mathscr{D}(X^l) \to \mathscr{D}(U^l)$, and this can be used to construct functors

$$\iota^{m+n} \colon \operatorname{Hom}_{\mathscr{Var}}(X^m, X^n) = \mathscr{D}(X^{m+n}) \to \mathscr{D}(U^{m+n}) = \operatorname{Hom}_{\mathscr{Var}}(U^m, U^n)\,.$$

However, this need not respect composition of morphisms. For example, take $X = \mathbb{P}^1$ to be the projective line and U to be an affine patch. Let $\mathscr{E} = \mathscr{O}_X \in \operatorname{Hom}(\{\mathrm{pt}\}, X)$ and $\mathscr{F} = \mathscr{O}_X \in \operatorname{Hom}(X, \{\mathrm{pt}\})$. Then $(\iota^0)^*(\mathscr{E} \circ \mathscr{F}) = \pi^1_{\emptyset *}(\mathscr{O}_X) = k \oplus k$, but $(\iota^1)^*(\mathscr{O}_X) \otimes (\iota^1)^*(\mathscr{O}_X) = \pi^1_{\emptyset}\mathscr{O}_U = k$.

However, the inverse image functors ι^* do allow us to partially construct a TQFT based on $U \subset X$. This idea allows us to prove the main theorem.

Theorem 4.28. *Let Z be a $(1+1+1)$-TQFT valued in $\mathscr{Var}$ such that the induced $(1+1)$-TQFT corresponds to the Frobenius algebra object in Proposition 4.7, where $X = Z(S^1)$. If X is of finite type and reduced, then it must be discrete. In this case, Z is isomorphic to a direct sum of extended TQFTs, each of which sends S^1 to a single point.*

Proof. Let $U \subset X$ be an reduced open affine subset and $\iota \colon U \to X$ be the inclusion map. There is a fibre square

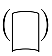
$$\begin{array}{ccc} U & \xrightarrow{i^1_{12}} & U^2 \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota^2} \\ X & \xrightarrow{i^1_{12}} & X^2 \end{array}$$

where in the top line we abuse notation and use the same notation for the map $i^1_{12} : U \to U^2$ as the map $i^1_{12} : X \to X^2$. Using the base-change formula, it follows that

$$\iota^{2*} \circ i_{12}^1 \cong i_{12*}^1 \circ \iota^* ,$$

so $\iota^{2*}(\mathscr{O}_\Delta) = i_{12*}^1(\mathscr{O}_U)$. Thus

$$\iota^{2*} \circ Z(\mu^\dagger) \in \operatorname{Hom}(i_{12*}^1(\mathscr{O}_U), \mathscr{O}_{U^2})$$

$\cong \operatorname{Hom}(Z_U($  $), Z_U($  $))$.

In particular, Corollary 4.27 can be applied: either U is a point, or $\iota^{2*} \circ Z(\mu^\dagger) = 0$.

Let $f = Z(\mu^\dagger)$. Then

$Z(\mu^\dagger \circ$  $) = \pi_{2*}\left(\pi_1^*(\mathscr{O}_X) \otimes f\right)$

$$= \pi_{2*}\left(f\right) .$$

Since $Z(\mu^\dagger \circ$  $)$ is a monomorphism, and ι^* is exact, $\iota^* \circ Z(\mu^\dagger \circ$  $) = \iota^* \circ \pi_{2*}(f)$ is also a monomorphism.

Let $f = Z(\mu^\dagger) \in \operatorname{Hom}(\mathscr{O}_\Delta, \mathscr{O}_{X\times X})$. Suppose that $\iota^{2*}(f) = 0$. Then since $\mathscr{O}_\Delta$ is supported on the diagonal, f is determined by a neighbourhood of the diagonal, so $(\operatorname{Id} \times \iota)^*(f) = 0$. There is a fibre diagram

$$\begin{array}{ccc} X \times U & \xrightarrow{\pi_2} & U \\ \big\downarrow{\scriptstyle \operatorname{Id}\times\iota} & & \big\downarrow{\scriptstyle \iota} \\ X \times X & \xrightarrow{\pi_2} & X . \end{array}$$

This gives a natural isomorphism $\pi_{2*} \circ (\operatorname{Id} \times \iota) \cong \iota^* \circ \pi_{2*}$, and hence $\iota^* \circ \pi_{2*}(f) = 0$, giving a contradiction.

Thus $\iota^{2*}(f) \neq 0$. Since U is an reduced affine open, by Corollary 4.27 $U = \operatorname{Spec}(\mathbb{F})$ for some field $\mathbb{F}$. This holds for any reduced affine open U, so X is union of discrete points as required. $\square$

4.3.3 Chain maps of non-zero degree

A natural modification of the category $\mathscr{Var}$ would be to allow the 2-morphisms to be formed from chain maps with possibly non-zero degree (that is, elements of $\mathrm{Ext}(\mathscr{E}, \mathscr{F})$, rather than $\mathrm{Hom}(\mathscr{E}, \mathscr{F})$). However, the map $\mu^\dagger$ can be seen to have degree 0 as follows. The composition in Equation (4.7) is the identity, and hence has degree 0. In particular, the degrees of $\nu^\dagger$ and $\mu^\dagger$ sum to zero. Now

$$\mu^\dagger \in \mathrm{Ext}(\mathscr{O}_\Delta, \mathscr{O}_{X\times X})$$

and since $\mathscr{O}_\Delta$ and $\mathscr{O}_{X\times X}$ are complexes concentrated in degree 0, $\mu^\dagger$ has non-negative degree. Similarly,

$$\nu^\dagger \in \mathrm{Ext}(H^*(\mathscr{O}_X), k)$$

and since k is concentrated in degree 0 and all the terms in $H^*(\mathscr{O}_X)$ are in non-negative degrees it is also the case that the degree of $\nu^\dagger$ is non-negative. Since the degrees of $\mu^\dagger$ and $\nu^\dagger$ sum to 0, they must both be zero and the previous result applies.

Bibliography

[1] Marian Aprodu. *Koszul cohomology and algebraic geometry*. University lecture series. Providence, RI: American Mathematical Society, 2010. URL: https://cds.cern.ch/record/1350778.

[2] Michael F. Atiyah. "Topological quantum field theory". en. In: *Publications Mathématiques de l'IHÉS* 68 (1988), pp. 175–186. URL: http://www.numdam.org/item/PMIHES_1988__68__175_0.

[3] John Baez and Mike Stay. "Physics, Topology, Logic and Computation: A Rosetta Stone". In: *http://www.cs.auckland.ac.nz/staff-cgi-bin/mjd/secondcgi.pl?serial* (Mar. 2009).

[4] Bojko Bakalov and A.A. Kirillov. "Lectures on tensor categories and modular functors". In: *Amer. Math. Soc. Univ. Lect. Ser.* 21 (Jan. 2001).

[5] Djordje Baralic, Z. Petric, and Sonja Telebakovic. "Spheres as Frobenius objects". In: *arXiv: Category Theory* (2016).

[6] Bruce Bartlett et al. *Modular categories as representations of the 3-dimensional bordism 2-category*. 2015. arXiv: 1509.06811 [math.AT].

[7] Jean Bénabou. "Introduction to bicategories". In: *Reports of the Midwest Category Seminar*. Berlin, Heidelberg: Springer Berlin Heidelberg, 1967, pp. 1–77. ISBN: 978-3-540-35545-8.

[8] Alexey Bondal and Michel Bergh. “Generators and Representability of Functors in Commutative and Noncommutative Geometry”. In: *Moscow Mathematical Journal, v.3, 1-36 (2003)* 3 (May 2002). DOI: 10.17323/1609-4514-2003-3-1-1-36.

[9] Alberto Canonaco and Paolo Stellari. “Fourier-Mukai functors: a survey”. In: *EMS Ser. Congr. Rep.* (Sept. 2011). DOI: 10.4171/115-1/2.

[10] Nils Carqueville. “Lecture notes on 2-dimensional defect TQFT”. In: *Banach Center Publications* 114 (July 2016). DOI: 10.4064/bc114-2.

[11] Nils Carqueville and Flavio Montiel Montoya. “Extending Landau-Ginzburg models to the point”. In: *Commun. Math. Phys.* 379.3 (2020), pp. 955–977. DOI: 10.1007/s00220-020-03871-5. arXiv: 1809.10965 [math.QA].

[12] Andrei Căldăraru and Simon Willerton. “The Mukai pairing. I: A categorical approach.” eng. In: *The New York Journal of Mathematics* 16 (2010), pp. 61–98. URL: http://eudml.org/doc/232651.

[13] Alexei Davydov, Liang Kong, and Ingo Runkel. “Field theories with defects and the centre functor”. In: (July 2011). Ed. by Hisham Sati and Urs Schreiber, pp. 71–128. arXiv: 1107.0495 [math.QA].

[14] Charles Ehresmann. “Catégories structurées”. fre. In: *Annales scientifiques de l’École Normale Supérieure* 80.4 (1963), pp. 349–426. URL: http://eudml.org/doc/81794.

[15] N Ganter and M M Kapranov. *Representation and character theory in 2-categories.* Tech. rep. math.KT/0602510. Feb. 2006. URL: http://cds.cern.ch/record/931561.

[16] Nathan Habegger and George Thompson. “The Universal perturbative quantum 3 manifold invariant, Rozansky-Witten invariants, and the generalized Casson invariant”. In: (Nov. 1999). arXiv: math/9911049.

[17] Linde Wester Hansen and Michael Shulman. *Constructing symmetric monoidal bicategories functorially.* 2019. arXiv: 1910.09240 [math.CT].

[18] Robin Hartshorne. *Algebraic geometry.* Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977, pp. xvi+496. ISBN: 0-387-90244-9.

[19] D. Huybrechts. *Fourier–Mukai transforms in algebraic geometry.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2006, pp. viii+307. ISBN: 978-0-19-929686-6. DOI: 10.1093/acprof:oso/9780199296866.001.0001. URL: https://doi.org/10.1093/acprof:oso/9780199296866.001.0001.

[20] Daniel Huybrechts. *Fourier-Mukai Transforms in Algebraic Geometry.* Clarendon Press, 2006. ISBN: 9780199296866.

[21] András Juhász. "Defining and classifying TQFTs via surgery". In: *Quantum Topology* 9 (Feb. 2018). DOI: 10.4171/QT/108.

[22] M. Kapranov. "Rozansky–Witten invariants via Atiyah classes". In: *Compositio Mathematica* 115.1 (1999), 71–113. DOI: 10.1023/A:1000664527238.

[23] Gregory M. Kelly. "Doctrinal adjunction". In: *Category Seminar.* Berlin, Heidelberg: Springer Berlin Heidelberg, 1974, pp. 257–280. ISBN: 978-3-540-37270-7.

[24] Joachim Kock. *Frobenius Algebras and 2-D Topological Quantum Field Theories.* London Mathematical Society Student Texts. Cambridge University Press, 2003. DOI: 10.1017/CBO9780511615443.

[25] Maxim Kontsevich. "Rozansky–Witten Invariants Via Formal Geometry". In: *Compositio Mathematica* 115.1 (1999), 115–127. DOI: 10.1023/A:1000619911308.

[26] A.A. Kosinski. *Differential Manifolds.* Dover Book on Mathematics. Dover Publications, 2007. ISBN: 9780486462448. URL: https://books.google.co.uk/books?id=L1B2dJ2j6h0C.

[27] Saunders Mac Lane. "Natural associativity and commutativity". In: *Rice University Studies* (1963).

[28] F. W. Lawvere. "Metric spaces, generalized logic, and closed categories". In: *Rendiconti del Seminario Matematico e Fisico di Milano* 43 (1973), pp. 135–166.

[29] Tom Leinster. "Basic Bicategories". In: *arXiv Mathematics e-prints*, math/9810017 (Oct. 1998), math/9810017. arXiv: `math/9810017 [math.CT]`.

[30] Joseph Lipman. "Notes on Derived Functors and Grothendieck Duality, Foundations of Grothendieck Duality for Diagrams of Schemes". In: *Lecture Notes in Math. 1960*. Springer, 2009, pp. 1–259.

[31] Jacob Lurie. "On the classification of topological field theories". In: *Current Developments in Mathematics, 2008*. Vol. Volume 2008. Current Developments in Mathematics. Boston, MA: International Press of Boston, Oct. 2009, pp. 129–280. URL: `https://projecteuclid.org/euclid.cdm/1254748657`.

[32] John Milnor, L. Siebenmann, and J. Sondow. *Lectures on the H-Cobordism Theorem*. Princeton University Press, 1965. ISBN: 9780691079967. URL: `http://www.jstor.org/stable/j.ctt183psc9`.

[33] Dmitri Orlov. "Derived categories of coherent sheaves and equivalences between them". In: *Russian Mathematical Surveys* 58 (June 2003), 511–591. DOI: `10.1070/RM2003v058n03ABEH000629`.

[34] Dmitri Orlov. "Equivalences of derived categories and K3 surfaces". In: *Journal of Mathematical Sciences* 84 (June 1997), 1361–1381. DOI: `10.1007/BF02399195`.

[35] Piotr Pragacz, Vasudevan Srinivas, and Vishwambhar Pati. "Diagonal Subschemes and Vector Bundles". In: *Pure and Applied Mathematics Quarterly* 4 (Oct. 2006). DOI: `10.4310/PAMQ.2008.v4.n4.a11`.

[36] Ryan Reich. “Obvious natural morphisms of sheaves are unique”. In: *Theory and Applications of Categories* 29 (July 2013).

[37] Nicolai Reshetikhin and Vladimir Turaev. “Invariants of 3 - Manifolds Via Link Polynomials and Quantum Groups Invent”. In: *Inventiones Mathematicae* 103 (Dec. 1991), pp. 547–597. DOI: 10.1007/BF01239527.

[38] Justin Roberts. “Rozansky-Witten theory”. In: *Topology and geometry of manifolds (Athens, GA, 2001)*. Vol. 71. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 2003, pp. 1–17. DOI: 10.1090/pspum/071/2024627. URL: https://doi.org/10.1090/pspum/071/2024627.

[39] Justin Roberts and Simon Willerton. “On the Rozansky–Witten weight systems”. In: *Algebr. Geom. Topol.* 10.3 (2010), pp. 1455–1519. DOI: 10.2140/agt.2010.10.1455. URL: https://doi.org/10.2140/agt.2010.10.1455.

[40] L. Rozansky and E. Witten. “Hyper-Kähler geometry and invariants of three-manifolds”. In: *Selecta Mathematica, Vol. 3, Iss. 3, p. 401-458* 3 (Sept. 1997), pp. 401–458. DOI: 10.1007/s000290050016. eprint: hep-th/9612216.

[41] Stephen Sawin. “Direct sum decompositions and indecomposable TQFTs”. In: *J. Math. Phys.* 36 (1995), pp. 6673–6680. DOI: 10.1063/1.531180. arXiv: q-alg/9505026 [q-alg].

[42] Christopher Schommer-Pries. “The Classification of Two-Dimensional Extended Topological Field Theories”. In: (Dec. 2011).

[43] Graeme Segal. “Categories and cohomology theories”. In: *Topology* 13.3 (1974), pp. 293 –312. ISSN: 0040-9383. DOI: https://doi.org/10.1016/0040-9383(74)90022-6. URL: http://www.sciencedirect.com/science/article/pii/0040938374900226.

[44] The Stacks project authors. *The Stacks project*. https://stacks.math.columbia.edu. 2019.

[45] Ross Street. "Frobenius monads and pseudomonoids". In: *Journal of Mathematical Physics* 45.10 (2004), pp. 3930–3948. DOI: 10.1063/1.1788852. eprint: https://doi.org/10.1063/1.1788852. URL: https://doi.org/10.1063/1.1788852.

[46] Ravi Vakil. *MATH 216: Foundations of Algebraic Geometry*. URL: https://math.stanford.edu/\textasciitildevakil/216blog/.

[47] K. Walker. "On Witten's 3-manifold Invariants". In: 2003.

[48] Charles A. Weibel. *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1994. DOI: 10.1017/CBO9781139644136.

[49] Edward Witten. "Quantum field theory and the Jones polynomial". In: *Comm. Math. Phys.* 121.3 (1989), pp. 351–399. URL: https://projecteuclid.org:443/euclid.cmp/1104178138.