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Relationship between Φ^4-matrix model and N-body harmonic oscillator or Calogero-Moser model

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Abstract. We study some Hermitian Φ^4-matrix model and some real symmetric Φ^4-matrix model whose kinetic terms are given by $\mathrm{Tr}(E\Phi^2)$, where E is a positive diagonal matrix without degenerate eigenvalues. We show that the partition functions of these matrix models correspond to zero-energy solutions of a Schödinger type equation with N-body harmonic oscillator Hamiltonian and Calogero-Moser Hamiltonian, respectively. The first half of this paper is primarily a review of previous work of us. The discussion of the properties of zero-energy solutions and the discussion of systems of differential equations satisfied by partition functions derived from the Virasoro algebra in the latter half of this paper contain novel material.

1 Brief Background

First, let us take a moment to review some historical background on matrix models.

In the 1990s, deep connections between matrix models and two-dimensional quantum gravity were discovered, and many important developments have been made. We refer to [2] for an early review that covers most of these achievements. Among them, the matrix model that is closest to the content of this paper is the Kontsevich model [9]. Its action is given by $S_K = N\ Tr\{E\Phi^2 + \frac{\lambda}{3}\Phi^3\}$, where Φ is an $N \times N$ Hermitian matrix, E is a positive diagonal $N \times N$ matrix $E := diag(E_1, E_2, \cdots, E_N)$ without degenerate eigenvalues, and λ is a complex number as a coupling constant. This model was proposed to prove the Witten conjecture [13]. The model we will study is given by replacing the interaction term $\frac{1}{3}\Phi^3$ by $\frac{1}{4}\Phi^4$.

In the 2000s, a different motivation led to the study of matrix models. There has been a lot of interest in analyzing matrix models as quantum field theories on noncommutative spaces. A noncommutative space is realized, for example, as a noncommutative algebra obtained by deforming a function algebra on that space into a noncommutative product. Therefore, by attaching an appropriate matrix representation to the noncommutative algebra, the field theory can be described as a matrix model. However, those quantum theories generally have no chance to be renormalizable because of the UV/IR problem. The scalar Φ^4 field theory on Moyal space by Grosse and Wulkenhaar [5] and the scalar Φ^3 theory by Grosse and Steinacker [6] [7] appeared as theories to avoid the UV/IR problem. In other words, they showed that those field theories to which certain counter-Lagrangian terms are added are renormalizable.

This scalar Φ^4 quantum field theory on Moyal space (Grosse-Wulkenhaar model) is the model we discuss in this paper.

2 N-body harmonic oscillator system and Calogero-Moser model from Φ^4-matrix model

We study a Hermitian Φ^4-matrix model and a real symmetric Φ^4-matrix model whose kinetic terms are given by $\mathrm{Tr}(E\Phi^2)$, where E is a positive diagonal matrix $E := diag(E_1, E_2, \cdots, E_N)$ without degenerate eigenvalues. In this paper, we treat both cases of Hermitian and real symmetric matrices as Φ, and write them with subscript β when equations can be described in a unified manner. When β is 2, it is the case of a Hermitian matrix, and when β is 1, it is the case of a real symmetric matrix. We show that their partition functions of these matrix models correspond to zero-energy solutions of a Schödinger type equation with N-body harmonic oscillator Hamiltonian and Calogero-Moser Hamiltonian [1], respectively. (See also related works [10, 11, 12].) In more detail, they are as follows.

Let $Z(E, \eta)$ be the partition function defined by

$$Z(E, \eta) = \int_{M_\beta} d\Phi \; e^{-S_E[\Phi]}, \tag{1}$$

where

$$S_E = N \; \mathrm{Tr}\{E\Phi^2 + \frac{\eta}{4}\Phi^4\} \tag{2}$$

η is a positive real number as a coupling constant, M_2 is the space of $N \times N$ Hermitian matrices, and M_1 is the space of $N \times N$ real symmetric matrices.

Here the integral measure is the ordinary Haar measure. Let $\Delta(E)$ be the Vandermonde determinant $\Delta(E) := \prod_{k<l}(E_l - E_k)$. Then the function

$$\Psi(E, \eta) := e^{-\frac{N}{\beta\eta}\sum_{i=1}^N E_i^2} \Delta(E)^{\frac{\beta}{2}} Z(E, \eta)$$

is a zero-energy solution of the Schrödinger type equation for the N-body harmonic oscillator system when we consider the Hermitian matrix model with $\beta = 2$, and for Calogero-Moser model when we consider the real symmetric matrix model with $\beta = 1$.

Theorem 2.1. *Let $\Psi(E, \eta)$ be a function defined by*

$$\Psi(E, \eta) := e^{-\frac{N}{\beta\eta}\sum_{i=1}^N E_i^2} \Delta(E)^{\frac{\beta}{2}} Z(E, \eta).$$

Then $\Psi(E, \eta)$ is a zero-energy solution of the Schrödinger type equation

$$\mathcal{H}\Psi(E, \eta) = 0.$$

Here $\mathcal{H}$ is the Hamiltonian $\mathcal{H}_{HO}$ for the N-body harmonic oscillator system when we consider the Hermitian matrix model with $\beta = 2$:

$$\mathcal{H}_{HO} := -\frac{\eta}{N}\sum_{i=1}^{N}\left(\frac{\partial}{\partial E_i}\right)^2 + \frac{N}{\eta}\sum_{i=1}^{N}(E_i)^2. \tag{3}$$

When we consider the real symmetric matrix model with $\beta = 1$, then $\mathcal{H}$ is the Hamiltonian $\mathcal{H}_{CM}$ for Calogero-Moser model:

$$\mathcal{H}_{CM} := \frac{-\eta}{2N}\left(\sum_{i=1}^{N}\frac{\partial^2}{\partial E_i^2} + \frac{1}{4}\sum_{i\neq j}\frac{1}{(E_i - E_j)^2}\right) + 2\frac{N}{\eta}\sum_{i=1}^{N}E_i^2. \tag{4}$$

Let us look at a rough outline of the proof of this theorem, in this section. The details are given in [3] [4]. The flow of the proof is the same in the case of the Hermitian matrix model as in the case of the real symmetric matrix. Therefore, the Hermitian matrix case ($\beta = 2$), which is computationally simpler, will be discussed below.

To derive the above differential equation, E is not enough, so let us introduce H as a positive Hermitian $N \times N$ matrix with nondegenerate eigenvalues $\{E_1, E_2, \cdots, E_N \mid E_i \neq E_j \text{ for } i \neq j\}$. Using this H, we consider the new action

$$\begin{aligned} S &= N\ Tr\{H\Phi^2 + \frac{\eta}{4}\Phi^4\} \\ &= N\left(\sum_{i,j,k}^{N} H_{ij}\Phi_{jk}\Phi_{ki} + \frac{\eta}{4}\sum_{i,j,k,l}^{N}\Phi_{ij}\Phi_{jk}\Phi_{kl}\Phi_{li}\right). \end{aligned} \tag{5}$$

The partition function defined by this S

$$Z(E,\eta) := \int_{M_2} d\Phi\ e^{-S}, \tag{6}$$

is the same one defined by (1), because the integral measure is $U(N)$ invariant. We use the symbol $\langle O\rangle$ as a non-normalized vacuum expectation value defined by $\langle O\rangle := \int_{M_2} d\Phi\ Oe^{-S}$.

The Schwinger-Dyson equation is derived from

$$\int_{M_2}\frac{\partial}{\partial\Phi_{ij}}\left(\Phi_{ij}e^{-S}\right) = 0, \tag{7}$$

which is expressed as

$$Z(E,\eta) - N\sum_k(\langle H_{ki}\Phi_{ij}\Phi_{jk}\rangle + \langle H_{jk}\Phi_{ki}\Phi_{ij}\rangle) - N\eta\sum_{k,l}\langle\Phi_{jk}\Phi_{kl}\Phi_{li}\Phi_{ij}\rangle = 0. \tag{8}$$

The key to obtaining the desired partial differential equation is that the following expectation values are expressed in terms of partial derivatives:

$$\frac{\partial Z(E,\eta)}{\partial H_{ij}} = -N\sum_k\langle\Phi_{jk}\Phi_{ki}\rangle, \quad \frac{\partial^2 Z(E,\eta)}{\partial H_{ij}\partial H_{mn}} = N^2\sum_{k,l}\langle\Phi_{jk}\Phi_{ki}\Phi_{nl}\Phi_{lm}\rangle. \tag{9}$$

After summing (8) overindices i and j and substituting (9) for it, we get

$$\mathcal{L}_{SD}^{H}Z(E,\eta) = 0. \tag{10}$$

Here $\mathcal{L}_{SD}^H$ is a second order differential operator defined by

$$\mathcal{L}_{SD}^H := N^2 + 2\sum_{i,k} H_{ki}\frac{\partial}{\partial H_{ki}} - \frac{\eta}{N}\sum_{i,k}\left(\frac{\partial}{\partial H_{ki}}\frac{\partial}{\partial H_{ik}}\right). \tag{11}$$

Next we rewrite this Schwinger-Dyson equation in terms of $E_n (n = 1, 2, \cdots, N)$. After small calculations, we find that the second term is expressed as

$$\sum_{i,j} H_{ij}\frac{\partial Z(E,\eta)}{\partial H_{ij}} = \sum_k E_k \frac{\partial Z(E,\eta)}{\partial E_k}, \tag{12}$$

and the Laplacian is rewritten as

$$\sum_{i,k}\left(\frac{\partial}{\partial H_{ki}}\frac{\partial}{\partial H_{ik}}\right) Z(E,\eta) = \left\{\sum_{i=1}^N \left(\frac{\partial}{\partial E_i}\right)^2 + \sum_{i\neq j}\frac{1}{E_i - E_j}\left(\frac{\partial}{\partial E_i} - \frac{\partial}{\partial E_j}\right)\right\} Z(E,\eta). \tag{13}$$

Here $\sum_{i\neq j}$ means $\sum_{i,j=1,i\neq j}^N$. From (10) , (12) , and (13), we find that the partition function defined by (1) satisfies

$$\mathcal{L}_{SD} Z(E,\eta) = 0, \tag{14}$$

where

$$\mathcal{L}_{SD} := \left\{\frac{\eta}{N}\sum_{i=1}^N \left(\frac{\partial}{\partial E_i}\right)^2 + \frac{\eta}{N}\sum_{i\neq j}\frac{1}{E_i - E_j}\left(\frac{\partial}{\partial E_i} - \frac{\partial}{\partial E_j}\right) - 2\sum_k E_k\frac{\partial}{\partial E_k} - N^2\right\}. \tag{15}$$

The next step is to diagonalize $\mathcal{L}_{SD}$. By direct calculations, we can prove the following proposition.

Proposition 2.2. *The differential operator $\mathcal{L}_{SD}$ defined in (15) is transformed into the Hamiltonian of the N-body harmonic oscillator as*

$$-e^{-\frac{N}{2\eta}\sum_i E_i^2}\Delta(E)\mathcal{L}_{SD}\Delta^{-1}(E)e^{\frac{N}{2\eta}\sum_i E_i^2} = -\frac{\eta}{N}\sum_{i=1}^N\left(\frac{\partial}{\partial E_i}\right)^2 + \frac{N}{\eta}\sum_{i=1}^N (E_i)^2. \tag{16}$$

Note that this transformation is invertible. We can regard this as some kind of gauge transformation. We denote the right-hand side by $\mathcal{H}_{HO}$ as (3). To cancel the gauge transformation $\Delta^{-1}(E)e^{\frac{N}{2\eta}\sum_i E_i^2}$, we introduce a transformed partition function $\Psi(E,\eta)$ by

$$\Psi(E,\eta) := e^{-\frac{N}{2\eta}\sum_i E_i^2}\Delta(E)Z(E,\eta). \tag{17}$$

From Proposition 2.2, we find that the transformed partition function $\Psi(E,\eta)$ is a zero-energy solution of the Schrödinger-type differential equation:

$$\mathcal{H}_{HO}\Psi(E,\eta) = 0. \tag{18}$$

Theorem 2.1 for the case of the Hermitian matrix model is thus proved. $\mathcal{H}_{HO}$ is the N-body harmonic oscillator Hamiltonian (3). This N-body harmonic oscillator system has no interaction terms between the oscillators, so it is a trivial quantum integrable system.

We can perform parallel discussions as above for the case of real symmetric matrix model. We refer to [4] for details.

3 Zero-energy solution of N-body harmonic oscillator system

In this section, we will investigate the partition function, or Ψ in the above section, which is the zero-energy solution of the N-body harmonic oscillator system, in more detail. In particular, it is a well-known fact that $L^2(\mathbb{R}^N)$ does not contain a solution of the N-body harmonic oscillator system with zero energy. What kind of function is this Ψ? Let us consider this question.

After using the Harish-Chandra-Itzykson-Zuber integral [8] for the unitary group $U(N)$ and some calculations, we get

$$Z(E,\eta) = \sum_{\sigma\in S_N} \frac{c_N}{\Delta(E)} \int_{\mathbb{R}^N} (\prod_{i=1}^{N} dx_i\ e^{-NV(x_i)}) \left(\prod_{l<k} \frac{x_k - x_l}{x_k + x_l}\right) \prod_{j=1}^{N} e^{-NE_j x_j^2}$$
$$= \frac{N!c_N}{\Delta(E)} \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} dx_i\ e^{-N(V(x_i)+E_i x_i^2)}\right) \left(\prod_{l<k} \frac{x_k - x_l}{x_k + x_l}\right), \tag{19}$$

or

$$\Psi(E,\eta) = N!c_N \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} dx_i\ e^{-N(\frac{\eta}{4}x_i^4 + E_i x_i^2 + \frac{1}{2\eta}E_i^2)}\right) \left(\prod_{l<k} \frac{x_k - x_l}{x_k + x_l}\right). \tag{20}$$

These integral representations should be regarded as Cauchy principal values.

Let us look at the simplest case $N = 1$. Introducing new variables $u_i := \sqrt{\frac{N}{\eta}} E_i$, the Schrödinger-type equation (18) is deformed into

$$\sum_{i=1}^{N} \left(\frac{\partial}{\partial u_i}\right)^2 y(u) = \sum_{i=1}^{N} u_i^2 y(u). \tag{21}$$

So in the $N = 1$ case, this is a kind of the Weber equation:

$$y''(u) = u^2 y(u). \tag{22}$$

For the $N = 1$ case, the partition function is

$$Z(E,\eta) := \int_{-\infty}^{\infty} dx\ e^{-Ex^2 - \frac{\eta}{4}x^4} = \int_{-\infty}^{\infty} dx\ e^{-\sqrt{\eta}ux^2 - \frac{\eta}{4}x^4} =: Z(u,\eta), \tag{23}$$

and using this $Z(u,\eta)$, (17) implies that the solution of (22) is given as

$$\Psi(u) := e^{-\frac{E^2}{2\eta}} Z(u,\eta) = e^{-\frac{u^2}{2}} \int_{-\infty}^{\infty} dx\ e^{-\sqrt{\eta}ux^2 - \frac{\eta}{4}x^4}. \tag{24}$$

Furthermore, for the $u > 0$, in terms of the modified Bessel function of the second kind $K_{\frac{1}{4}}(u)$, $\Psi(u)$ can also be written as

$$\Psi(u) = \frac{1}{\eta^{\frac{1}{4}}} \sqrt{u} K_{\frac{1}{4}}(\frac{u^2}{2}). \tag{25}$$

We find that the boundary condition for this solution is required as

$$\Psi(0) = Z(0,\eta) = \int_{-\infty}^{\infty} dx e^{-\frac{\eta}{4}x^4} = \frac{\Gamma(\frac{1}{4})}{\sqrt{2}\eta^{\frac{1}{4}}},$$
$$\Psi'(0) = -\sqrt{\eta}\int_{-\infty}^{\infty} dx\; x^2 e^{-\frac{\eta}{4}x^4} = -\frac{\sqrt{2}\Gamma(\frac{3}{4})}{\eta^{\frac{1}{4}}}.$$

Thus, in the case of $N = 1$, the results are expressed in terms of known special functions.

Let us now consider the first question of this section. It is well known that there are no zero-energy solutions which belong to $L^2(\mathbb{R})$.

By (25), the graph of ψ in the range $u > 0$ is shown in Fig. 1. However, the graph of ψ when the definition range of u is extended to the whole real numbers using the original definition (24) is shown in Fig. 2. This figure clearly shows that ψ is not included in $L^2(\mathbb{R})$.

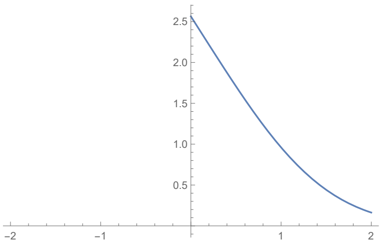

Figure 1: $u > 0$

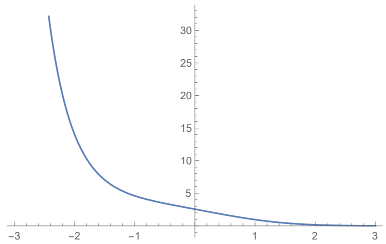

Figure 2: $-\infty < u < \infty$

In the case of $N = 1$, the properties of Ψ can be clearly identified, as we have seen above. For the general case of N, it is not yet known what kind of function Ψ is and what kind of boundary conditions can be imposed on it. On the contrary, even the case $N = 2$ is still an unknown problem.

4 Virasoro (Witt) algebra

These quantum integrable systems often lead to representations of Virasoro (Witt) algebras, reflecting their nature.

We saw that the partition function of the matrix model of the Hermitian matrix corresponds to the solution of the Schrödinger equation for the N-body harmonic oscillator, and the matrix model of the real orthogonal matrix corresponds to the solution of the Calogero model. Both of these are quantum integrable systems, leading to a representation of the Virasoro (Witt) algebra. We would like to mention this, but the Calogero model requires a complicated setup. We refer to [4] for details. In the following, we restrict our attential to the case of the matrix model of Hermitian matrices corresponding to the solution of harmonic oscillators.

For simplicity, we use variables $u_i := \sqrt{\frac{N}{\eta}}E_i$, then the Hamiltonian is written as

$$\mathcal{H}_{HO} = \sum_{i=1}^{N}\left(-\frac{\partial^2}{\partial u_i^2} + u_i^2\right) = \sum_{i=1}^{N}\{a_i\;,\;a_i^\dagger\},$$

where

$$a_i = \frac{1}{\sqrt{2}}\left(u_i + \frac{\partial}{\partial u_i}\right), \quad a_i^\dagger = \frac{1}{\sqrt{2}}\left(u_i - \frac{\partial}{\partial u_i}\right).$$

Let us introduce the Virasoro generator:

$$L_{-n} = \sum_{i=1}^{N} \left(\alpha \left(a_i^\dagger \right)^{n+1} a_i + (1-\alpha) a_i \left(a_i^\dagger \right)^{n+1} \right), \quad (n \geq -1)$$

that satisfies the following commutation relations:

$$[L_n, L_m] = (n-m)L_{n+m}.$$

Here α is a free parameter. In particular, noting that

$$L_0 = \frac{1}{2}\mathcal{H}_{HO} + \left(\frac{1}{2} - \alpha\right) N,$$

we obtain

$$\left[\frac{1}{2}\mathcal{H}_{HO}, L_{-m}\right] = mL_{-m}.$$

Using $\mathcal{G} = \Delta^{-1}(E)e^{\frac{N}{2\eta}\sum_i E_i^2}$, we define $\tilde{L}_n$ by $\tilde{L}_n := \mathcal{G}L_n\mathcal{G}^{-1}$ satisfying $\left[\tilde{L}_n, \tilde{L}_m\right] = (n-m)\tilde{L}_{n+m}$. Using this $\tilde{L}_n$, we immediately obtain the following result,

$$\left[\mathcal{L}_{SD}, \tilde{L}_{-m}\right] = -2\left[\tilde{L}_0, \tilde{L}_{-m}\right] = -2m\tilde{L}_{-m},$$

which implies the following theorem.

Theorem 4.1. *The partition function defined by (1) satisfies*

$$\mathcal{L}_{SD}(\widetilde{L}_{-m}Z(E,\eta)) = -\,2m(\widetilde{L}_{-m}Z(E,\eta)). \tag{26}$$

Here $\mathcal{L}_{SD}$ is a differential operator such that the Schwinger-Dyson equation for the partition function is given by

$$\mathcal{L}_{SD}Z(E,\eta) = 0. \tag{27}$$

$\mathcal{L}_{SD}$ for the Hermitian matrix model is given in (15).

This means that $\widetilde{L}_{-m}Z(E,\eta)$ is an eigenfunction of $\mathcal{L}_{SD}$ with the eigenvalue $-2m$. This theorem is exactly the same as the theorem obtained in the case of the matrix model of a real symmetric matrix (the Calogero model).

5 Summary

While mainly reviewing [3] and [4], we have seen that the partition function of the matrix model in which the potential of the Kontsevich model is replaced by Φ^4 implies a zero-energy solution of the Schrödinger equation for an integrable system by some gauge transformation. In the case of the Hermitian matrix model, it gives the solution to the Schrödinger equation for a system of N-body harmonic oscillators as an integrable system, and in the case of the real symmetric matrix model, it gives the zero-energy solution of the Calogero model. Since it is known that these integrable systems lead to representations of Virasoro algebras (Witt algebras), a sequence of differential equations that the partition function satisfies was obtained using them.

It is known that there are no such solutions in $L^2(\mathbb{R}^N)$, so the solution obtained here is not such a function. In fact, we discussed the case of $N = 1$ in detail, and by looking at the specific functional form of the solution, we were able to see that it was not in $L^2(\mathbb{R})$. In the case of both the harmonic oscillator and the Calogero model, these zero-energy solutions have not been well studied, and in the general case, as well as in the case of $N = 2$, they are unknown. We look forward to further progress in this area.

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