

Topical Review

Chapter 6: Ambitwistor Strings and Amplitudes from the Worldsheet

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Abstract

Starting with Witten's twistor string, chiral string theories have emerged that describe *field theory* amplitudes without the towers of massive states of conventional strings. These models are known as ambitwistor strings due to their target space; the space of complexified null geodesics, also called ambitwistor space. Correlators in these string theories directly yield compact formulae for tree-level amplitudes and loop integrands, in the form of worldsheet integrals fully localized on solutions to constraints known as the *scattering equations*. In this chapter, we discuss two incarnations of the ambitwistor string: a 'vector representation' starting in space–time and structurally resembling the Ramond Neveu–Schwarz (RNS) superstring, and a four-dimensional twistorial version closely related to, but distinct from Witten's original model. The RNS-like models exist for several theories, with 'heterotic' and type II models describing super-Yang–Mills and 10d supergravities respectively, and they manifest the double copy relations directly at the level of the worldsheet models. In the second half of the chapter, we explain how the underlying models lead to diverse applications, ranging from extensions to new sectors of theories, loop amplitudes and to scattering on curved backgrounds. We conclude with a brief discussion of connections to conventional strings and celestial holography.

Keywords: ambitwistor, strings, amplitudes, loop

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1. Introduction

The twistor-string of Witten and Berkovits [1–3] marked a turning point in the study of scattering amplitudes, exposing systematic structures that are not apparent from either standard space–time Lagrangians or from conventional strings. In conventional string theory, the target space is space–time, whereas for the twistor-string, it is the cotangent bundle of twistor

space, where twistor space \mathbb{PT} is a three-complex-dimensional manifold. The string path integral localizes onto holomorphic maps from a Riemann surface Σ , the string worldsheet, into \mathbb{PT} . The theory led to formulae for the complete tree-level S -matrix for four-dimensional super Yang–Mills of unprecedented simplicity. These have by now been generalized to a great variety of theories and to loop integrands and beyond. This review will cover many of these developments. By way of introduction we give a brief history.

1.1. A brief history

The story starts with Nair’s reformulation [4] of the famous Parke–Taylor formula for the colour-ordered MHV Yang–Mills amplitude

$$A^{\text{MHV}}(1^+, \dots, i^-, \dots, j^-, \dots, n^+) = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle (n-1)n \rangle \langle n1 \rangle}, \quad (1.1)$$

where the n particles have massless momenta $k_i = |i\rangle\langle i|$ in spinor helicity notation, and particles i, j have negative helicity with the rest positive. Nair, in an elegant $\mathcal{N} = 4$ supersymmetric formulation including the supermomentum conserving delta-function, expressed this as an integral of a current-algebra correlator over the moduli space of Riemann spheres, \mathbb{CP}^1 s, holomorphically embedded in supertwistor space of degree one (i.e., lines in \mathbb{CP}^3). In the twistor string [1–3], N^{k-2} MHV amplitudes with k negative helicity gluons arise as integrals over the moduli space of degree $k-1$ curves in \mathbb{PT} .

A striking output was the formula of Roiban, Spradlin and Volovich (RSV) [5]. They were able to perform some of the moduli integrals so as to express the full tree-level S -matrix for $\mathcal{N} = 4$ super Yang–Mills tree-amplitudes as a sum over residues. It was soon observed [6] that the equations that determine these residues are underpinned by the *scattering equations*. These equations determine $(n-3)!$ sets of n points $\{\sigma_i\} \subset \mathbb{CP}^1$ up to Möbius transformations, i.e. points in the moduli space $\mathfrak{M}_{0,n}$, in terms of the n null momenta k_i of the scattering particles:

$$\mathcal{E}_i := \sum_{j=1}^n \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} = 0. \quad (1.2)$$

These equations play a pivotal role in the subsequent story.

It emerged that the twistor-string is equivalent to $\mathcal{N} = 4$ super Yang–Mills coupled to a certain conformal supergravity [3]. This does imply that attempts to compute Yang–Mills loop amplitudes via this method would be corrupted by conformal supergravity modes running in the loops; these are not in any case easy to compute [7].

There were two lines of attack to find analogous formulae and theories for Einstein (E) gravity, one by improving our understanding of its MHV amplitude and the other via the double copy [8, 9]. The latter expresses E gravity amplitudes as a ‘square’ of the different colour-ordered Yang–Mills amplitudes using an inner product, the ‘KLT’ momentum kernel, introduced by Kawai, Lewellen and Tye [10]. Following [11] it was conjectured that the residues on which the RSV formulae are supported in fact diagonalize the KLT kernel, so that the double copy can be implemented on the RSV formula to produce amplitudes for E gravity in four dimensions [12]. On the other hand, Hodges had found an optimal version of the MHV formula for gravity amplitudes [13] in terms of reduced determinants that bore little relation to these formulae. This led Cachazo and Skinner to introduce a compact worldsheet formula for $\mathcal{N} = 8$ supergravity amplitudes [14, 15] that was soon seen to arise from a twistor-string for $\mathcal{N} = 8$ supergravity [16]. In this theory, Hodges’ reduced determinants and their generalizations are obtained from fermion correlators on the worldsheet.

Cachazo, He and Yuan (CHY) pursued the relationship between the double copy and the scattering equations, proving that the solutions to the scattering equations do indeed diagonalize the KLT kernel and giving an elegant formulation for its expression as a reduced determinant on $\mathfrak{M}_{0,n}$ [17]. Both the double copy and the scattering equations make sense in arbitrary dimension and the CHY collaboration were soon able to produce expressions for Yang–Mills and gravity amplitudes in all dimensions [18] that perfectly express the double copy within formulæ of the form

$$\mathcal{A} = \int \mathcal{I}^L \mathcal{I}^R \frac{\prod_{i=1}^n \bar{\delta}(\mathcal{E}_i) d\sigma_i}{\text{vol PSL}(2, \mathbb{C}) \times \mathbb{C}^3}. \quad (1.3)$$

Here the $\bar{\delta}(\mathcal{E}_i)$ are delta-functions that impose the scattering equation (1.2) and the PSL(2) quotient by Möbius transformations on the σ_i is intended in the Faddeev–Popov sense, reducing the formula to an $(n-3)$ -dimensional integral over $\mathfrak{M}_{0,n}$, the moduli space of n marked points on the Riemann sphere. This then localizes on $(n-3)!$ residues at the solutions to the scattering equation (1.2).

The factors $\mathcal{I}^{L/R}$ are theory dependent, and can depend on polarization or colour data associated to particles in the theory; this factorization then gives diagonalized expression of the double copy. The zero'th copy is when both are associated to colours. For two different colour orderings, they are given by Parke–Taylor expressions each with denominators like that of (1.1) but with $\langle ij \rangle$ replaced by $\sigma_i - \sigma_j$. The amplitudes are then those of a theory of biadjoint scalars [19], i.e., $\phi^{a\tilde{a}}(x)$ with a being an index associate to a Lie algebra \mathfrak{g} and \tilde{a} associated to another $\tilde{\mathfrak{g}}$; the theory has cubic interactions $f_{abc}\tilde{f}_{\tilde{a}\tilde{b}\tilde{c}}\phi^{a\tilde{a}}\phi^{b\tilde{b}}\phi^{c\tilde{c}}$ determined by the two sets of structure constants f_{abc} and $\tilde{f}_{\tilde{a}\tilde{b}\tilde{c}}$. If one Parke–Taylor is replaced by a certain reduced Pfaffian, see (2.38), then the formula yields Yang–Mills amplitudes, and if both are such Pfaffians, one obtains gravity formulæ. CHY soon extended their framework to include remarkable new expressions for amplitudes of many more theories of interest such as effective theories, nonlinear-sigma models (NLSM), Born–Infeld (BI) and so on [20, 21].

These formulæ have undoubtedly theoretical importance in their own right, but the question remained as to what physical principles generate them; they look unlike anything that arises from a space–time action formulation and the conventional string does not localize on residues in the field theory limit. Such underlying principles should for example give insights into extensions to loop amplitudes or non-perturbative phenomena. Both the CHY formulæ and twistor-strings are now understood under the umbrella of ambitwistor-string theories [22]. These are quantum field theories of holomorphic maps from a Riemann surface to projective ambitwistor space, \mathbb{PA} ; this is defined to be the complexification of the phase space of a massless particle. In four dimensions, ambitwistor space can be realized as the cotangent bundle of twistor space, $\mathbb{A} = T^*\mathbb{PT}$ and of its dual $\mathbb{A} = T^*\mathbb{PT}^*$; twistor space is chiral, and dual twistor space antichiral, so \mathbb{A} is ambidextrous, hence its name³. The original twistor-strings can be of either chirality but can both be understood in this way, and indeed there is an ambidextrous version [26] in the same twistor coordinates that generates formulæ that are distinct from those of RSV and Cachazo–Skinner and we introduce these in section 3. However, the simplest presentation that connects most directly to the CHY formulæ is a presentation of ambitwistor space analogous to the original Ramond–Neveu–Schwarz (RNS) model for the conventional string

³ It was introduced by Witten [23], Isenberg *et al* [24], as a space on which one can encode general (super) Yang–Mills fields generalizing Ward’s twistor construction [25] for self-dual Yang–Mills fields.

and so we start with this in the next section section 2. In particular in section 2.4 we give a more complete proof of the CHY formulæ by means of BCFW recursion, which is not easily found in the current literature.

The second half of the review focuses on one of the key applications of the ambitwistor-string framework, the extension of the tree-level formulæ to those that provide loop integrands. In section 4 we see that following the usual string paradigm, loop integrands can be constructed via higher genus worldsheets. However, this yields formulæ that are at least superficially highly transcendental for loop integrands that should be rational functions. In section 5 we explain how, by means of a residue theorem, such formulæ can be reduced to ones based on nodal Riemann spheres. We go on to explain various new representations of loop integrands at one and two loops and further applications to the double copy. In the final section 6 we briefly discuss further frontiers, extensions to curved backgrounds, and connections with the conventional string and with celestial holography, providing pointers to the literature.

2. Ambitwistor geometry and models

Ambitwistor-string theories are chiral strings, i.e., quantum theories of holomorphic maps from a Riemann surface Σ into a complex manifold. The target space, ambitwistor space \mathbb{A} , is the complexification of the real phase space of null geodesics. In d dimensions, points of \mathbb{A} correspond to complex null geodesics in a complexified space–time (M, g) in which the metric depends holomorphically on the d complex coordinates on M : this can be obtained by complexification of a real space–time with analytic metric.

There are many real worldline actions for massless particles with different couplings to background fields and supersymmetry and there is a simple recipe to go from such a real worldline action to a complex ambitwistor-string action. Here we start with the simplest first-order version in d -dimensional Minkowski space–time (M, η) :

$$S[X, P] = \frac{1}{2\pi} \int P_\mu \, dX^\mu - \frac{\tilde{e}}{2} P^2. \quad (2.1)$$

In this action, the einbein \tilde{e} is a Lagrange multiplier enforcing the constraint $P^2 := \eta^{\mu\nu} P_\mu P_\nu = 0$, and is also the worldline gauge field for the gauge transformations

$$\delta X^\mu = \alpha \eta^{\mu\nu} P_\nu \quad \delta P_\mu = 0 \quad \delta \tilde{e} = d\alpha, \quad (2.2)$$

conjugate to this constraint. Thus P must be null and gauge transformations send fields X to X' along the null translation generated by P . The solutions to the field equations modulo gauge are null geodesics in space–time, parametrized by the scaling of P . The quantization of this action leads to the massless Klein–Gordon equation.

The ambitwistor string replaces the worldline with a Riemann surface Σ and complexifies the target space so that the (P_μ, X^μ) are holomorphic coordinates on the cotangent bundle T^*M of complexified Minkowski space (M, η) ⁴. The algorithm to obtain an ambitwistor-string starts by replacing dX in (2.1) by

$$\bar{\partial}_e X = d\bar{\sigma} \partial_{\bar{\sigma}} X - e_{\bar{\sigma}}^\sigma d\bar{\sigma} \partial_\sigma X, \quad (2.3)$$

⁴For simplicity, we introduce the ambitwistor string for flat Minkowski space; this definition can be extended to a general curved space–time (M, g) at the cost of some complications. References, as well as a brief discussion, can be found in section 6.1.

where σ is a local holomorphic coordinate on Σ . This leads to the bosonic action

$$S_{\text{bos}}[X, P] = \frac{1}{2\pi} \int_{\Sigma} P_{\mu} \bar{\partial}_e X^{\mu} - \frac{\tilde{e}}{2} P^2. \quad (2.4)$$

For the kinetic term of (2.4) to be invariant, we must take P_{μ} to be a complex $(1, 0)$ -form, i.e., with values in the *canonical bundle* $K_{\Sigma} := \Omega_{\Sigma}^{1,0}$ on the worldsheet. Thus, when $e_{\bar{\sigma}}^{\sigma} = 0$, $P_{\mu} = P_{\sigma}^{\mu} d\sigma$ where σ is a local holomorphic worldsheet coordinate. We must take both $e := e_{\bar{\sigma}}^{\sigma} d\bar{\sigma} \partial_{\sigma}$ and \tilde{e} to be $(0, 1)$ -forms on Σ with values in holomorphic vector fields T_{Σ} , i.e., Beltrami differentials. The worldsheet field e plays the same role as in the conventional string, parametrizing complex structures on Σ and gauging worldsheet diffeomorphisms, but now in a chiral model. On the other hand, the geometric interpretation of \tilde{e} is quite different from that of the ordinary string. It imposes the constraint $P^2 = 0$ and gauges the transformations⁵

$$\delta X^{\mu} = \alpha \eta^{\mu\nu} P_{\nu}, \quad \delta P_{\mu} = 0, \quad \delta \tilde{e} = \bar{\partial}_e \alpha := \bar{\partial}\alpha - [\alpha, e]. \quad (2.5)$$

Here we must take α to transform as a holomorphic vector on the worldsheet. Thus (X, P) describe a map into complexified cotangent bundle T^*M of complexified space-time, but imposing the constraint $P^2 = 0$ and quotienting by the gauge symmetry generated by the geodesic spray $P \cdot \partial_X$ reduces the target space of (2.4) to the space of complex null geodesics \mathbb{A} , ambitwistor space, via

$$p : T^*M|_{P^2=0} \rightarrow \mathbb{A} := T^*M|_{P^2=0} / P \cdot \partial_X. \quad (2.6)$$

A subtlety in this description is that the scaling of P_{σ} is tied to that of one-forms on Σ ; P takes values in the canonical bundle of K_{Σ} . Thus there is only a preferred scaling of the geodesic once a worldsheet coordinate σ has been chosen. This reduces the target space further to projective ambitwistor space \mathbb{PA} , quotienting by the rescalings of P .

Following the double copy, all our models will take the form

$$S = S_{\text{bos}} + S^L + S^R, \quad (2.7)$$

where S^L and S^R are two independent choices of worldsheet matter. There are a number of interesting choices outlined in [27], but in order to establish the basic models, we will focus on the following two worldsheet systems:

Current algebras. The first will be a current algebra with action denoted by S_C . It provides ‘currents’ $j^a \in K_{\Sigma} \otimes \mathfrak{g}$, where a is a Lie algebra index associated to the Lie algebra \mathfrak{g} , that satisfy the OPE

$$j^a(\sigma) j^b(\sigma') \sim \frac{k \delta^{ab}}{(\sigma - \sigma')^2} + \frac{f_c^{ab} j^c}{\sigma - \sigma'}. \quad (2.8)$$

Here δ^{ab} is the Killing form and f_c^{ab} the structure coefficients of \mathfrak{g} , and k the level of the current algebra. The simplest S_C arises from free fermions $\rho^{\alpha} \in K_{\Sigma}^{1/2} \otimes \mathbb{C}^N$ with action

$$S_{\rho} := \int \rho^{\alpha} \bar{\partial} \rho^{\beta} \delta_{\alpha\beta}, \quad \rightsquigarrow \quad j^{\alpha\beta} := \rho^{\alpha} \rho^{\beta} \in K \otimes \mathfrak{so}(N), \quad k = 1. \quad (2.9)$$

⁵ In local coordinates, these gauge transformations can be explicitly expressed as follows. Take $e = e^{\sigma} \partial_{\sigma} = e_{\bar{\sigma}}^{\sigma} d\bar{\sigma} \partial_{\sigma}$, $\tilde{e} = \tilde{e}^{\sigma} \partial_{\sigma} = \tilde{e}_{\bar{\sigma}}^{\sigma} d\bar{\sigma} \partial_{\sigma}$ and $\alpha = \alpha^{\sigma} \partial_{\sigma}$, and similarly $\chi = \chi^{\sqrt{\sigma}} \sqrt{\partial_{\sigma}} = \chi_{\bar{\sigma}}^{\sqrt{\sigma}} d\bar{\sigma} \sqrt{\partial_{\sigma}}$ and $\epsilon = \epsilon^{\sqrt{\sigma}} \sqrt{\partial_{\sigma}} = \epsilon_{\bar{\sigma}}^{\sqrt{\sigma}} d\bar{\sigma} \sqrt{\partial_{\sigma}}$, where $(\sqrt{\partial_{\sigma}}, \sqrt{\partial_{\sigma}})$ is a basis for $T_{\Sigma}^{1/2} \oplus \overline{T_{\Sigma}^{1/2}}$. Then $\delta \tilde{e}$ and $\delta \chi$ are given by $\delta \tilde{e}^{\sigma} = \bar{\partial} \alpha^{\sigma} - \alpha^{\sigma} \bar{\partial} e^{\sigma} + e^{\sigma} \partial \alpha^{\sigma}$, $\delta \chi^{\sqrt{\sigma}} = \bar{\partial} \epsilon^{\sqrt{\sigma}} + e^{\sigma} \partial \epsilon^{\sqrt{\sigma}} - \frac{1}{2} \epsilon^{\sqrt{\sigma}} \bar{\partial} e^{\sigma}$.

We will not in general specify the action in detail, and merely denote it by S_C .

Worldsheet superalgebra. This system consists of fermionic spinor $\Psi^\mu \in K_\Sigma^{1/2} \otimes \mathbb{C}^d$ as a fermionic counterpart to X^μ , and a fermionic gauge field $\chi \in \Omega^{0,1} \otimes T_\Sigma^{1/2}$ with action

$$S_\Psi = \int \eta_{\mu\nu} \bar{\Psi}^\mu \bar{\partial}_e \Psi^\nu - \chi P \cdot \Psi. \quad (2.10)$$

The field χ is a gauge field generating degenerate fermionic supersymmetries⁵

$$\delta X^\mu = \epsilon \Psi^\mu, \quad \delta \Psi^\mu = \epsilon \eta^{\mu\nu} P_\nu, \quad \delta P_\mu = 0, \quad \delta \tilde{e} = 2\epsilon \chi, \quad \delta \chi = \bar{\partial}_e \epsilon. \quad (2.11)$$

With these ingredients there are three main consistent models mirroring the closed string models in conventional string theory:

- The biadjoint scalar model is the bosonic model with two current algebras

$$S_{\text{BAS}} := S_{\text{bos}} + S_C + S_{\tilde{C}}, \quad (2.12)$$

The two current algebras, $S_C + S_{\tilde{C}}$ provide currents $j^a, \tilde{j}^{\tilde{a}}$ respectively and generate bi-adjoint scalar amplitudes correctly.

- The heterotic model has one fermionic matter system S_Ψ and one current algebra S_C , and generates Yang–Mills amplitudes correctly;

$$S_{\text{het}} := S_{\text{bos}} + S_\Psi + S_C. \quad (2.13)$$

- The type II models with two fermionic worldsheet matter systems $S_\Psi, S_{\tilde{\Psi}}$ generate supergravity amplitudes

$$S_{\text{II}} := S_{\text{bos}} + S_\Psi + S_{\tilde{\Psi}}. \quad (2.14)$$

These models already manifest the double copy, and a naive⁶ version of the colour-kinematics duality via the interchangeability of the current algebras with the S_Ψ systems. In the double copy, the biadjoint scalar is the zeroth copy, Yang–Mills the single copy and gravity the double copy.

One can construct many further models with more elaborate choices of worldsheet matter and all models seem to give rise to amplitudes of some field theory, at least at tree level. These amplitude formulae include BI, Dirac–Born–Infeld (DBI), Einstein–Yang–Mills (EYM), harmonic maps and more, manifesting a more extended double copy, see table 1 and [20, 21, 27]. Only the last models S_{II} correspond to conventional supergravity, yielding IIA or IIB supergravity in ten dimensions according to the choice of GSO projection as in conventional string theory.

2.1. BRST gauge fixing and quantization

On the Riemann sphere in the absence of vertex operators, we can gauge fix by setting each of the gauge fields to zero (more generally we can fix the gauge fields to lie within the cohomology class $H^{0,1}(\Sigma, T_\Sigma(-\sigma_1 - \dots - \sigma_n))$.

⁶Naive because of the absence of Jacobi relations, or a suitable analogue of identical relations between S_C and S_Ψ , see also the discussion in section 2.5.

Table 1. Theories arising from the different choices of matter models.

S^L		S^R			
		S_Ψ	S_{Ψ_1, Ψ_2}	$S_{\rho, \Psi}^{(\tilde{m})}$	$S_{\text{YM}, \Psi}^{(\tilde{N})}$
S_Ψ	E				
S_{Ψ_1, Ψ_2}	BI	Galileon			
$S_{\rho, \Psi}^{(m)}$	$\text{EM} _{U(1)^m}$	DBI	$\text{EMS} _{U(1)^m \otimes U(1)^{\tilde{m}}}$		
$S_{\text{YM}, \Psi}^{(N)}$	EYM	extended DBI	$\text{EYMS} _{SU(N) \otimes U(1)^{\tilde{m}}}$	$\text{EYMS} _{SU(N) \otimes SU(\tilde{N})}$	
$S_{\text{YM}}^{(N)}$	YM	NLSM	$\text{YMS} _{SU(N) \otimes U(1)^{\tilde{m}}}$	$\text{gen. YMS} _{SU(N) \otimes SU(\tilde{N})}$	BS

After gauge fixing a ghost system is introduced for each gauge field, the fermionic $(b, c) \in K_\Sigma^2 \oplus T_\Sigma$ for e , and $(\tilde{b}, \tilde{c}) \in K_\Sigma^2 \oplus T_\Sigma$ for \tilde{e} , and the bosonic $(\beta, \gamma) \in K_\Sigma^{3/2} \oplus T_\Sigma^{1/2}$ for χ with free ghost actions

$$S_{(b,c)} = \int b \bar{\partial} c, \quad S_{(\tilde{b},\tilde{c})} = \int \tilde{b} \bar{\partial} \tilde{c}, \quad S_{(\beta,\gamma)} = \int \beta \bar{\partial} \gamma. \quad (2.15)$$

Invariance under the gauge symmetries is then imposed by considering the cohomology associated with the BRST operator Q which takes the form (here for the type II models)⁷

$$Q = \oint c \left(T^m + \frac{1}{2} T^{bc} \right) + \frac{\tilde{c}}{2} P^2 + \gamma P \cdot \Psi + \tilde{\gamma} P \cdot \tilde{\Psi} + \frac{1}{2} \tilde{b} (\gamma^2 + \tilde{\gamma}^2), \quad (2.16)$$

where T^m is the holomorphic part of the stress–energy tensor, and $T^{bc} = (\partial b)c + 2b(\partial c)$. Classically Q^2 vanishes by construction, but, as in standard string theory, the quantum models can be inconsistent as double contractions can give $Q^2 \neq 0$. We have:

- The pure bosonic model S_{bos} is critical in 26 dimensions. However, for $S_{\text{BAS}} = S_{\text{bos}} + S_C + S_{\tilde{C}}$, the critical dimension will depend on the central charges of $S_C + S_{\tilde{C}}$.
- The model S_{het} is critical in 10d with current algebras for $E_8 \times E_8$ or $\text{SO}(32)$.
- The type II models are critical in 10d.

The central charge calculations are analogous to those in conventional string theory. Although generic anomalies obstruct amplitude interpretations of the correlators, models with $Q^2 \neq 0$ due to a central charge anomaly or even gauge anomalies, can still give well-defined tree-level amplitudes even though the underlying ambitwistor-string is not consistent⁸.

2.2. Vertex operators

In string theory, amplitudes are constructed as correlation functions of vertex operators, with each vertex operator corresponding to an external particle. Vertex operators come in various

⁷ For a general gauge system generated by currents j^a of perhaps different spins or statistics, we have ghosts c^a of opposite statistics and $Q = \oint c_a j^a + \frac{1}{2} b^c f_c^{ab} c_a c_b$, where f_c^{ab} are the structure constants.

⁸ This in particular follows when it is possible to add an additional sector S_{ext} to the ambitwistor string that does not interact with the other fields, and hence does not interfere with the amplitudes for the original fields, but cancels their anomalies. For example, when the central charge is too low, S_{ext} could be interpreted as additional (compactified) dimensions to take the dimension up to the critical dimension. However, the extra degrees of freedom would run in the loops at higher genus.

pictures that depend on how residual gauge freedom is fixed after initial gauge fixing [28–30]. For worldsheet diffeomorphisms, the generic case for multiparticle amplitudes are *integrated vertex operators* that require integration over Σ ; these can be understood as the perturbations of the action corresponding to infinitesimal background plane-wave fields. One also needs a small number of *fixed vertex operators* that correspond to the same type of particles, but fix the residual gauge symmetries. These two types of vertex operators are related by pairing the fixed vertex operator with moduli insertions from the gauge-fixing procedure⁹. We here give the basic recipes required for the amplitude formulæ together with some brief intuitions on the geometry following [22, 27]. More sophisticated derivations are given in [31, 32].

Integrated vertex operators. Space–time fields can be represented on ambitwistor space via the *Penrose transform*. This realizes spin s fields on space–time as cohomology classes $H^1(\mathbb{PA}, \mathcal{O}(s-1))$ on projective ambitwistor space \mathbb{PA} : these classes can be represented as $\bar{\partial}$ -closed $(0, 1)$ -forms on \mathbb{PA} of homogeneity degree $s-1$ in P . For spin s plane-waves of the form $\epsilon_{\mu_1} \dots \epsilon_{\mu_s} e^{ik \cdot X}$, such cohomology classes can be written explicitly as

$$(\epsilon \cdot P)^s \bar{\delta}(k \cdot P) e^{ik \cdot x} \in H^1(\mathbb{PA}, \mathcal{O}(s-1)). \quad (2.17)$$

Here, we define the complex delta function $\bar{\delta}(z)$ for a complex variable z by

$$\bar{\delta}(z) := \bar{\partial} \frac{1}{z} = 2\pi i \delta(\Re z) \delta(\Im z) d\bar{z}. \quad (2.18)$$

Although expressed on T^*M , the plane wave representative descends to \mathbb{PA} as $k \cdot P = 0$ on the support of the delta function, so that under $X \rightarrow X + \alpha P$, $k \cdot X$ does not change. For $s = 1$ this provides the Maxwell version of the ambitwistor Yang–Mills correspondence of Witten and Isenberg *et al* [23, 24] and for $s = 2$ this provides the linear version of the transform for gravity introduced by LeBrun [33], see [22, 34] for general linear fields.

More generally, our integrated vertex operators all take the form

$$\mathcal{V} := \int w \bar{\delta}(k \cdot P) e^{ik \cdot x}, \quad (2.19)$$

where w depends on P and the worldsheet matter fields from $S^L + S^R$. With our identification of $\mathcal{O}_P(1) = K_\Sigma$ on the worldsheet, $w \in K_\Sigma^2$, and the integrand defines a $(1, 1)$ -form as $\bar{\delta}(k \cdot P)$ has weight -1 . In general, to manifest the double copy, we take

$$w = v^L v^R, \quad v^L, v^R \in K_\Sigma, \quad (2.20)$$

and v^L, v^R are either of the form $t \cdot j$ where j is a current associated to S_C and $t^a \in \mathfrak{g}$, or $v^{L,R} = \epsilon \cdot P + k \cdot \Psi \epsilon \cdot \Psi$ when associated with S_Ψ . For the models that give the original CHY formulæ for biadjoint scalar amplitudes, Yang–Mills and gravity, this yields

$$\begin{aligned} w_{\text{BAS}} &:= t \cdot j \tilde{t} \cdot \tilde{j}, \quad t^a \in \mathfrak{g}, \tilde{t}^{\dot{a}} \in \tilde{\mathfrak{g}}, \\ w_{\text{YM}} &:= (\epsilon \cdot P + k \cdot \Psi \epsilon \cdot \Psi) t \cdot j, \\ w_{\text{grav}} &:= (\epsilon \cdot P + k \cdot \Psi \epsilon \cdot \Psi) (\tilde{\epsilon} \cdot P + k \cdot \tilde{\Psi} \tilde{\epsilon} \cdot \tilde{\Psi}). \end{aligned} \quad (2.21)$$

⁹From the CFT perspective, the fixed vertex operators are more fundamental, and correlators can be equivalently expressed using fixed vertex operators only, with additional moduli insertions often referred to as picture changing operators (especially for fermionic symmetries such as (2.11)).

The corresponding vertex operators are required to be Q -closed as part of the BRST quantization. Classically, this is automatic, but quantum mechanically, double contractions with the P^2 term imply that $k^2 = 0$. In those models containing S_Ψ , double contraction with the $P \cdot \Psi$ term further impose that $k \cdot \epsilon = 0$. Thus quantum consistency implies that our vertex operators correspond to on-shell fields in Lorentz gauge. (The on-shell condition is *not* a consequence of the classical ambitwistor Penrose transform.) This gives the correct linear theory for bi-adjoint scalar and Yang–Mills theory. In the case of the (Q -invariant) gravity vertex operators, the polarization vectors give the on-shell polarization data for a linearized metric, $g_{\mu\nu} = \epsilon_{(\mu} \tilde{\epsilon}_{\nu)} e^{ik \cdot X}$, B -field $B_{\mu\nu} = \epsilon_{[\mu} \tilde{\epsilon}_{\nu]} e^{ik \cdot X}$ and dilaton $\phi = \epsilon \cdot \tilde{\epsilon} e^{ik \cdot X}$. These fields form the NS sector of 10d supergravity but make sense in all dimensions and have become known as *fat gravity*.

The array of integrated vertex operators given by (2.21) are all of the form $w = v^L v^R$ where v^L and v^R are one-forms on the worldsheet, either of the form $t \cdot j$ or $\epsilon \cdot P + k \cdot \Psi \epsilon \cdot \Psi$. This decomposition gives an elegant microscopic formulation of the double copy with the interchangeability of the two types of operator building up from biadjoint scalars as the zero'th copy to Yang–Mills and then gravity. We note that the biadjoint scalar model also contains gravitational and gauge sectors and there is also a gravitational sector in the heterotic model which we briefly discuss in section 2.5.

Fixed vertex operators. For the remainder of this chapter, we focus on the genus-zero case, higher-genus correlators are discussed in section 4. In all ambitwistor string models, non-trivial tree-level correlators require three fixed vertex operators, related to the integrated vertex operators above via moduli insertions. These operators are inserted at three arbitrarily chosen fixed points on Σ without integration, and saturate the ghost zero modes in the path-integral. Much of this arises in the same way as conventional string theory. For the three models above these come in the form

$$V = c \tilde{c} w e^{ik \cdot X}. \quad (2.22)$$

The lack of integration is associated to fixing residual worldsheet diffeomorphism freedom. Setting the worldsheet gravity $e = 0$ fixes the coordinates up to the three-dimensional group of Möbius transformations $\text{PSL}(2, \mathbb{C})$. This is a non-compact integration in the path integral, handled via the Faddeev–Popov procedure by fixing the insertion points of three vertex operators, $(\sigma_1, \sigma_2, \sigma_3)$ to e.g. $(0, 1, \infty)$. The vertex operators remain Q -invariant despite the lack of integration due to the ghosts c ; in the path integral this amounts to only quotienting by gauge transformations that vanish at these fixed insertions. Since the fermionic ghosts $c \in \Omega^0(T_\Sigma)$ have three zero modes on the sphere (see appendix B), tree-level correlators must include three fixed vertex operators to give non-trivial Berezinian integrals¹⁰. The ghost correlation function provides the needed Faddeev–Popov determinant associated to the gauge fixing.

The most novel part in these models, compared to conventional strings, is the descent associated to the gauge field \tilde{e} that imposes the $P^2 = 0$ constraint. In this case, the residual gauge freedom amounts to adding αP to X where α is a holomorphic section of T_Σ . When $\Sigma = \mathbb{CP}^1$, this is three-dimensional, and is fixed by fixing the values of $X(\sigma_i)$ at three values of σ on the corresponding point of the geodesic. Descent is given by the connecting homomorphism δ described in (A.4) that implements the Penrose transform from space–time fields to their corresponding cohomology classes on ambitwistor space.

¹⁰More precisely $n_c - n_b = 3$ must match the zero-mode count, where $n_{b,c}$ are the numbers of respective ghost insertions. The b -ghost insertions can arise from moduli insertions.

Pictures. The same procedure for the fermionic symmetries associated constraint $P \cdot \Psi$ or $P \cdot \tilde{\Psi}$ leads to vertex operators (both fixed and integrated) in different pictures. We denote this via a superscript (p) , with $p = -1$ or $p = 0$,

$$V^{(p)} = c\tilde{c} w^{(p)} e^{ik \cdot X}, \quad \mathcal{V}^{(p)} = \int w^{(p)} \bar{\delta}(k \cdot P) e^{ik \cdot x}, \quad (2.23)$$

and take again $w^{(p)} = v^{(p)L} v^{(p)R}$ in line with the double copy, where

$$v^{(-1)} = \delta(\gamma) \epsilon \cdot \Psi v^{(0)} = \epsilon \cdot P + k \cdot \Psi \epsilon \cdot \Psi. \quad (2.24)$$

Vertex operators in different pictures are related via so-called picture changing operators $\Upsilon = \delta(\beta)\{Q, \beta\} \cong \delta(\beta)P \cdot \Psi$ (and similarly $\tilde{\Upsilon} \cong \delta(\tilde{\beta})P \cdot \tilde{\Psi}$);

$$V^{(0)}(\sigma) = \lim_{z \rightarrow \sigma} \Upsilon(z) \tilde{\Upsilon}(z) V^{(-1)}(\sigma). \quad (2.25)$$

Here we have used the notation ‘ \cong ’ to indicate equality up to ghost terms that do not contribute to the correlator. Note that the vertex operators of (2.21) are thus given in the picture $p = 0$; with $v = v^{(0)}$. At genus zero each bosonic ghost γ has two zero modes, so tree-level correlators must contain exactly two vertex operators with picture number $p = -1$ ⁹.

2.3. Amplitudes

In general amplitudes are obtained as correlation functions of vertex operators with sufficient fixed vertex operators to precisely saturate the zero-modes of the ghost fields;

$$\mathcal{A}(1, \dots, n) = \langle V_1 V_2 V_3 \mathcal{V}_4 \dots \mathcal{V}_n \rangle. \quad (2.26)$$

In the supersymmetric cases, two of the fixed vertex operators of the form $V^{(-1)}$ and one of type $V^{(0)}$. Our first task is to see that the path-integral that defines this correlation function localizes onto the solutions to the scattering equations.

In all cases, the computation of the correlator involves a worldsheet correlation function of the w 's of (2.21) which we denote by $\langle W \rangle_\Sigma$, as well as the correlators of the c and \tilde{c} ghosts denoted by $\langle C\tilde{C} \rangle_\Sigma$. We will return shortly to the task of evaluating these correlators. The PX -correlator is universal to all models, and we consider it first. Fortunately, we can sidestep the problem of evaluating the contractions between the X 's in the $e^{ik \cdot X}$ factors and P 's in the vertex operators by taking the $e^{ik \cdot X}$ factors into the action in the path-integral, and treating them as sources in the PX -action. After gauge fixing this action, the full correlation function is then given by the path integral

$$\langle V_1 \dots \mathcal{V}_n \rangle = \int D[X, P] \int \prod_{i=4}^n \bar{\delta}(k_i \cdot P(\sigma_i)) d\sigma_i \langle C\tilde{C} \rangle_\Sigma \langle W \rangle_\Sigma e^{S_{\text{eff}}}, \quad (2.27)$$

and we can write the (effective) action with the vertex operator sources as

$$\begin{aligned} S_{\text{eff}} &= \frac{1}{2\pi} \int_\Sigma P \cdot \bar{\partial} X + \sum_{i=1}^n ik_i \cdot \\ X(\sigma_i) &= \frac{1}{2\pi} \int_\Sigma \left(P \cdot \bar{\partial} X + \sum_{i=1}^n k_i \cdot X(\sigma) \bar{\delta}(\sigma - \sigma_i) d\sigma \right). \end{aligned}$$

Since the action is now linear in X , and there is no further X -dependence in the path-integral, we can integrate out the X -field. Its zero-modes provide d momentum-conserving delta-functions, while the non-zero-modes localize the P path-integral onto the solution to the equations of motion of this action:

$$\bar{\partial}P_\mu = \sum_{i=1}^n ik_{i\mu} \bar{\delta}(\sigma - \sigma_i). \quad (2.28)$$

On the sphere, these have a unique solution given by

$$P_\mu(\sigma) = \sum_i \frac{k_{i\mu}}{\sigma - \sigma_i}. \quad (2.29)$$

This solution can be substituted into the delta-functions (and into the w 's) yielding

$$\bar{\delta}(k_i \cdot P(\sigma_i)) = \bar{\delta}(\mathcal{E}_i), \quad \mathcal{E}_i := \sum_{j \neq i} \frac{k_i \cdot k_j}{\sigma_{ij}}, \quad \sigma_{ij} = \sigma_i - \sigma_j. \quad (2.30)$$

We now see that the delta-functions impose the *scattering equations* $\mathcal{E}_i = 0$. Thus the path integral localizes to

$$\langle V_1 \dots V_n \rangle = \delta^d \left(\sum_{i=1}^n k_i \right) \int \prod_{i=4}^n \bar{\delta}(k_i \cdot P(\sigma_i)) d\sigma_i \langle C\tilde{C} \rangle_\Sigma \langle W \rangle_\Sigma. \quad (2.31)$$

The correlator of the three c 's and \tilde{c} 's is elementary and gives a numerator factor of $(\sigma_{12}\sigma_{23}\sigma_{31})^2$. With this we define the CHY measure

$$d\mu_n^{\text{CHY}} := (\sigma_{12}\sigma_{23}\sigma_{31})^2 \prod_{i=4}^n \bar{\delta}(k_i \cdot P(\sigma_i)) d\sigma_i = \frac{\prod_{i=1}^n \bar{\delta}(\mathcal{E}_i) d\sigma_i}{\text{vol PSL}(2, \mathbb{C}) \times \mathbb{C}^3}. \quad (2.32)$$

In the second equality here we have identified one factor of $\sigma_{12}\sigma_{23}\sigma_{31}$ as the Faddeev–Popov determinant for the action of Möbius transformations on the sphere with n points, and the second factor as that for the action of the residual gauge symmetries (2.5) associated to translations along the lightray. This is now a measure on the moduli space $\mathcal{M}_{0,n}$, the moduli space of n points in \mathbb{CP}^1 modulo Möbius transformations.

To finish the correlation function computation we need to evaluate $\langle W \rangle_\Sigma$. Because of the construction of the w 's as $w = v^L v^R$, where v^L and v^R are constructed from independent worldsheet matter systems, this correlation function naturally factorizes as

$$\langle W \rangle_\Sigma = \mathcal{I}^L \mathcal{I}^R, \quad (2.33)$$

where $\mathcal{I}^L, \mathcal{I}^R$ are respectively the correlators of the v^L 's and v^R 's. This gives the final amplitude formula as

$$\mathcal{A}_n^{\text{CHY}} = \int d\mu_n^{\text{CHY}} \mathcal{I}_n, \quad \mathcal{I}_n := \mathcal{I}_n^L \mathcal{I}_n^R. \quad (2.34)$$

All ambitwistor strings give formulae of this CHY form with $\mathcal{I}^L, \mathcal{I}^R$ given as follows.

For the current algebra S_C this correlation function is standard, breaking up into single-trace and multi-trace terms. The single-trace terms are a sum over permutations $\alpha \in S_n$ of

$\text{tr}(t_{\alpha(1)} \dots t_{\alpha(n)}) \text{PT}(\alpha)$ where the $\text{PT}(\alpha)$ are *Parke–Taylor* factors defined by

$$\text{PT}(\alpha) = \prod_{i=1}^n \frac{1}{\sigma_{\alpha(i)\alpha(j)}}. \quad (2.35)$$

The multi-trace terms are also part of the field theory defined by the ambitwistor string, and can be interpreted as tree amplitudes with one of the scalars of the corresponding gravity theory running along an internal propagator [3, 35, 36].

The most interesting ingredient is the correlator arising from the v^L 's when $S^L = S_\Psi$ leading to the CHY Pfaffian, defined via a skew symmetric the $2n \times 2n$ matrix

$$M := \begin{pmatrix} A & C \\ -C^t & B \end{pmatrix}, \quad (2.36)$$

with components

$$A_{ij} := \frac{k_i \cdot k_j}{\sigma_{ij}}, \quad B_{ij} := \frac{\epsilon_i \cdot \epsilon_j}{\sigma_{ij}}, \quad C_{ij} := \begin{cases} \frac{\epsilon_j \cdot k_i}{\sigma_{ij}}, & i \neq j, \\ -\sum_k \frac{\epsilon_i \cdot k_k}{\sigma_{ik}}, & i = j. \end{cases} \quad (2.37)$$

The matrix has a two-dimensional kernel on the support of the scattering equations given by the row vectors $(1, \dots, 1 | 0, \dots, 0)$ and $(\sigma_1, \dots, \sigma_n | 0, \dots, 0)$. This allows us to define a reduced Pfaffian by

$$\text{Pf}'(M) = \frac{1}{\sigma_{12}} \text{Pf}(M^{12}), \quad (2.38)$$

where M^{12} is M with the first two rows and columns deleted. Importantly, this reduced Pfaffian is permutation invariant. We now have the main statement that the correlation function of the v^L 's from S_Ψ is given as

$$\langle v_1^0 v_2^0 v_3 \dots v_n \rangle = \text{Pf}'(M). \quad (2.39)$$

With these ingredients, we have now arrived at the original three main CHY formulæ,

$$\mathcal{I}_n = \mathcal{I}^L \mathcal{I}^R = \begin{cases} \text{PT}(\alpha) \text{PT}(\beta), & \text{Biadjoint scalar} \\ \text{PT}(\alpha) \text{Pf}'(M), & \text{Yang–Mills theory} \\ \text{Pf}'(M) \text{Pf}'(\tilde{M}), & \text{NS gravity.} \end{cases} \quad (2.40)$$

2.4. Proof of the CHY formulæ

The CHY formulæ (2.34) and (2.40) are strikingly compact, valid for all multiplicity and all dimensions, with a tantalizing worldsheet origin. They are quite remote from standard formulations of field theory scattering amplitudes; so how do we know they correctly describe amplitudes? A straightforward sanity check verifies that they give the correct three- and four-particle amplitudes, which is already nontrivial at four points. In this section, we give a full proof for the ambitwistor string correlators $\mathcal{A}_n^{\text{CHY}}$ for any number of external particles. Along the way, we will gain a better understanding of the role the scattering equations play for

massless amplitudes, and explore how they relate geometric factorization in the moduli space $\mathfrak{M}_{0,n}$ to kinematic factorization of the scattered particles which occurs when partial sums of the momenta become null.

The proof is based on the Britto–Cachazo–Feng–Witten (BCFW) recursion relation for scattering amplitudes [37–40], reviewed in chapter 1 [41]. Employed constructively, the BCFW recursion allows us to build the full tree-level S -matrix recursively from lower point amplitudes, using the three-particle amplitudes as seeds. Thus, BCFW also guarantees that any proposed expression satisfying the recursion relation, with the correct three-point seed amplitudes, is a representation of the S -matrix. This means that we can prove the CHY formulæ for Yang–Mills and gravity by verifying that they obey the assumptions that lead to the BCFW recursion relations.

BCFW recursion. The on-shell recursion relations exploit elementary complex analysis and knowledge of singularities of the amplitudes. The poles occur precisely at *factorization channels* of the amplitude when a partial sum of the external momenta becomes null so that some intermediate propagator becomes singular. If the theory is both *local* and *unitary*, then these poles are all simple with residues given by the sum of products of two tree amplitudes

$$\lim_{K_I^2 \rightarrow 0} K_I^2 \mathcal{A}_n = \sum_r \mathcal{A}_{n_I+1}(K_I, r) \mathcal{A}_{n_{\bar{I}}+1}(-K_I, r), \quad (2.41)$$

where $K_I = \sum_{i \in I} k_i$ is the partial sum of momenta of the particles $i \in I$, and the sum is over polarization states r that can run in the propagator; here we have denoted the sets of external particles in each of the subamplitudes by I and \bar{I} , with multiplicities $n_I = |I|$ and $n_I + n_{\bar{I}} = n$. This is the content of the *optical theorem*, here restricted to tree-level amplitudes (stripped of their momentum-conserving delta-functions). It ensures that tree-level amplitudes are meromorphic functions of the external momenta, with only simple poles.

This allows us to harness the power of complex analysis in which one can reconstruct a holomorphic function on the Riemann sphere from its residues. To exploit this idea, we choose a complex one-parameter deformation of the external momenta,

$$k_1 \rightarrow \hat{k}_1(z) = k_1 + zq, \quad k_n \rightarrow \hat{k}_n(z) = k_n - zq, \quad (2.42)$$

where z is a complex variable and q_μ is a reference vector satisfying $q^2 = q \cdot k_1 = q \cdot k_n = 0$ such that all external particles remain on-shell¹¹. Then Cauchy's residue theorem allows us to express the original undeformed amplitude as the sum over all other residues, including a boundary term \mathcal{B}_∞ from a potential residue as $z \rightarrow \infty$,

$$\mathcal{A}_n = \frac{1}{2i\pi} \oint_{z=0} \frac{\mathcal{A}_n(z)}{z} = \sum_{I, r_I} \frac{1}{K_I^2} \mathcal{A}_{n_I+1}(z_I, r_I) \mathcal{A}_{n_{\bar{I}}+1}(z_{\bar{I}}, r_{\bar{I}}) + \mathcal{B}_\infty. \quad (2.43)$$

If for some good choices of q , the boundary term vanishes, $\mathcal{B}_\infty = 0$, a theory is *on-shell constructible*. The remaining residues away from infinity correspond to singular kinematic configurations, where the optical theorem guarantees that the amplitude factorises into a product of on-shell lower-particle amplitudes, giving the BCFW recursion for scattering amplitudes. Thus the tree S -matrix can be built from three-point amplitudes. This is the case for theories such as Yang–Mills, gravity [42] or indeed any 4d renormalizable QFT [43]; see the review chapter 1 [41], or [44–47].

¹¹ For particles transforming in non-trivial representations of the little group, the polarization vectors have to be shifted as well, which is best seen in covariant gauge, see e.g. [42] for details.

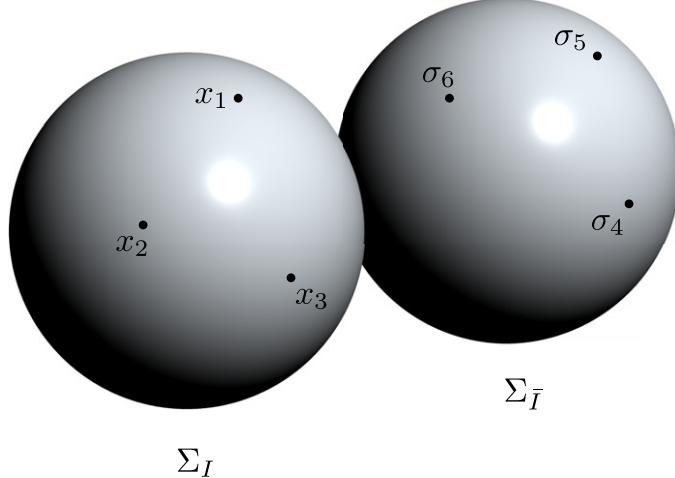


Figure 1. Boundary divisor $\mathfrak{M}_{0,n_I+1} \times \mathfrak{M}_{0,n_{\bar{I}}+1} \subset \partial \widehat{\mathfrak{M}}_{0,n}$ of the moduli space, corresponding to a pair of marked spheres Σ_I and $\Sigma_{\bar{I}}$. We parametrize the spheres by $x \in \Sigma_I$, and $\sigma \in \Sigma_{\bar{I}}$, with a nodal point $\sigma_I \in \Sigma_{\bar{I}}$ and $x_I = \infty \in \Sigma_I$, and subject to $\sigma = \sigma_I + \varepsilon x$. Depicted is the case for $n = 6$, with $I = \{1, 2, 3\}$ and $\bar{I} = \{4, 5, 6\}$.

Factorization proof of the CHY representation. The CHY formulæ will satisfy the BCFW recursion (2.43) if we can prove that it satisfies factorization (2.41) and $\mathcal{B}_\infty = 0$. The solution to the recursion is unique, given appropriate three-point seeds, and so we can deduce that the CHY formulæ give valid representations of the amplitude. We will use this strategy following reference [48].

Recalling the general structure of $\mathcal{A}_n^{\text{CHY}}$, we can see that the formulæ only have poles when there are residues where a subset I of the marked points collide,

$$\sigma_i = \sigma_I + \varepsilon x_i \quad \text{for } i \in I \quad \text{and} \quad \varepsilon \rightarrow 0. \quad (2.44)$$

This parametrizes a boundary $\partial \widehat{\mathfrak{M}}_{0,n}$ of the (Deligne–Mumford compactified [30, 49]) moduli space, corresponding to a separating degeneration of the worldsheet into a pair of spheres connected by a node σ_I , see figure 1. Thus, the CHY formula only generates poles when solutions to the scattering equations approach the boundary of the moduli space. We now show that this can only happen if the corresponding K_I is null.

The scattering equations relate these boundaries $\partial \widehat{\mathfrak{M}}_{0,n}$ to the factorization channels of the amplitude as follows. The key observation is that the scattering equations descend naturally onto the two component spheres Σ_I and $\Sigma_{\bar{I}}$,

$$\mathcal{E}_i = \begin{cases} \frac{1}{\varepsilon} \mathcal{E}_i^{(I)} & i \in I \\ \mathcal{E}_i^{(\bar{I})} & i \in \bar{I}, \end{cases} \quad (2.45)$$

where the equality holds to leading order in the degeneration parameter ε , and where $\mathcal{E}^{(I)}$ and $\mathcal{E}^{(\bar{I})}$ are the scattering equations on Σ_I and $\Sigma_{\bar{I}}$ respectively,

$$\mathcal{E}_i^{(I)} := \sum_{j \in I} \frac{k_i \cdot k_j}{x_{ij}}, \quad \mathcal{E}_i^{(\bar{I})} := \sum_{p \in \bar{I}} \frac{k_i \cdot k_p}{\sigma_{ip}} + \frac{k_i \cdot K_I}{\sigma_{iI}}, \quad (2.46)$$

with $K_I = \sum_{i \in I} k_i$ the momentum of the internal propagator¹². Under the degeneration (2.44) and on the support of the full scattering equations, both of these sets of scattering equations are of $\mathcal{O}(\varepsilon)$. This guarantees that also the momentum $K_I^2 = \mathcal{O}(\varepsilon)$ of the propagator connecting the two sub-amplitudes is on-shell because

$$K_I^2 = \frac{1}{2} \sum_{i,j \in I} k_i \cdot k_j = \sum_{\substack{i,j \in I \\ i \neq j}} \frac{x_i k_i \cdot k_j}{x_i - x_j} = \sum_{i \in I} x_i \mathcal{E}_i^{(I)} = \mathcal{O}(\varepsilon). \quad (2.47)$$

The scattering equations thus map the boundary of the moduli space to factorization channels, with singular kinematic configurations. This property lies at the heart of the CHY formalism, and it is the reason the scattering equations are universal to massless theories, elegantly encoding massless propagators. The role of the integrand on the other hand is to determine which poles occur in the amplitude, and to provide the correct numerator structure—all information specific to a given theory.

The essence of factorization in this framework now reduces to a straightforward scaling argument in the degeneration parameter ε to determine the degrees of and residues at the poles arising for the factorization channels. Consider first the measure $d\mu_n^{\text{CHY}}$: the scattering equations and factors $d\sigma_i$ descend naturally to $\Sigma_{I,\bar{I}}$, each providing one power of ε for particles $i \in I$. The only subtlety stems from Möbius invariance on Σ_I , which allows us to fix $x_m = 1$ for some $m \in I$, such that $d\sigma_m = d\varepsilon$. Moreover, on the support of the remaining scattering equations, one equation enforces the constraint $K_I^2 = \mathcal{O}(\varepsilon)$. Combining all these factors, we see that the measure factorizes neatly into

$$d\mu_n = \varepsilon^{2(n_I-1)} \frac{d\varepsilon}{\varepsilon} \bar{\delta}(K_I^2 + \varepsilon \mathcal{F}) d\mu_{n_I+1} d\mu_{n_{\bar{I}}+1}. \quad (2.48)$$

Here, the measure $d\mu_{n_I+1}$ is the natural CHY measure for the sphere Σ_I ; with n_I marked points corresponding to the external particles and the additional nodal point associated with the internal propagator. As we saw above, the delta-function restricts K_I^2 to vanish to order ε , thereby restricting to singular kinematic configurations. The factor \mathcal{F} denotes a function of the marked points and kinematics, and will drop out of the final formula for a unitary theory with simple poles.

At this point, we can see that the integrand \mathcal{I}_n for a theory respecting the factorization property (2.41) must split into respective integrands on each sphere;

$$\mathcal{I}_n = \varepsilon^{-2(n_I-1)} \sum_{\text{states}} \mathcal{I}_{n_I+1} \mathcal{I}_{n_{\bar{I}}+1}. \quad (2.49)$$

We can phrase this equivalently as a condition on the ‘half-integrands’ $\mathcal{I}^{L/R}$, now with a scaling factor of $\varepsilon^{-(n_I-1)}$. This property can easily be verified for the Parke–Taylor factors, and was proven for the reduced Pfaffian $\text{Pf}'M$ in [48]. For the bi-adjoint scalar, Yang–Mills theory and gravity we can then combine the behaviour of the measure (2.48) and the integrands (2.49),

$$\mathcal{A}_n = \sum_{\text{states}} \int \frac{d\varepsilon}{\varepsilon} \bar{\delta}(K_I^2 + \varepsilon \mathcal{F}) (d\mu_{n_I+1} \mathcal{I}_{n_I+1}) (d\mu_{n_{\bar{I}}+1} \mathcal{I}_{n_{\bar{I}}+1}), \quad (2.50)$$

¹²In these expressions, we implicitly gauge-fixed the nodal point (from the perspective of Σ_I) to $x_I = \infty$.

to see that the amplitude exhibits factorization properties in line with the optical theorem and the BCFW recursion,

$$\mathcal{A}_n = \frac{1}{K_I^2} \sum_{\text{states}} \mathcal{A}_{n_I+1} \mathcal{A}_{n_{\bar{I}}+1}. \quad (2.51)$$

The pole structure of the amplitude is thus determined by an interplay between the integrand and the measure including the scattering equations: while the measure guarantees the correct form for *all* possible poles, the integrand selects a subset of the poles suitable for the theory, and determines the residue on the pole via the factorized integrands \mathcal{I}_{n_I+1} and $\mathcal{I}_{n_{\bar{I}}+1}$. For example, for a Parke–Taylor factor, we only find a pole when the subset I is connected in the cyclic ordering of the Parke–Taylor.

The boundary term: for both gravity and Yang–Mills theory, the absence of the boundary term can be verified in covariant gauge [42] with a careful choice of shift vector $q = \epsilon_1$. Considering the shift (2.42) in the limit $z \rightarrow \infty$, the two shifted momenta become to leading order $k_{1,n} = \pm zq$, so that the amplitude has an interpretation as a hard light-like particle propagating in a soft background. On the support of the scattering equations, the dominant contribution to the amplitude then stems from the boundary of moduli space where $\sigma_n - \sigma_1 = \varepsilon$, with the degeneration parameter ε scaling as z^{-1} . The calculation for the overall scaling behaviour in z is then lengthy but straightforward: we find that the measure scales as $d\mu^{\text{CHY}} \sim z^{-2}$, whereas the integrands¹³ behave as $\text{PT} \sim z$ and $\text{Pf}'M \sim z^0$. For both gauge theory and gravity, this ensures that the boundary term vanishes,

$$\mathcal{B}_\infty = \lim_{z \rightarrow \infty} \mathcal{A}_n^{\text{CHY}} = 0. \quad (2.52)$$

This concludes our proof of the CHY representation for Yang–Mills theory and gravity.

2.5. Discussion

The ambitwistor string theories of this section are not simply vehicles to arrive at CHY formulæ, but contain much more information. In section 4 we will see that they provide a stepping stone to loop integrands. Here we mention other features. Theories often contain extra vertex operators beyond those originally envisaged. These extend the tree formulæ to amplitudes of more elaborate theories combining gravity, gauge theory and scalars. Furthermore, different forms of worldsheet matter lead to different theories, and different representations of ambitwistor space give different amplitude representations. We also make some brief remarks on connections with the colour-kinematic duality.

Although the model S_{BAS} leads directly to the CHY biadjoint scalar amplitude formula above, it also contains gauge theory and gravity vertex operators that lead to formulæ for gauge and gravity amplitudes. These were initially hard to interpret with the simplest having $\mathcal{I}^R = \prod_i \epsilon_i \cdot P(\sigma_i)$ for the gauge theory, and doubled for the gravity theory. They are now understood to be parts of a 4th order gauge theory that is conformally invariant in 6d as described in [50] and a 6th order gravity theory whose linearization is given in [35], see [51] for discussion of these theories in 6d.

Similarly S_{het} contains vertex operators for a 4th order gravity that is conformally invariant in 4d and is thought to agree [50] in 4d with that found by Berkovits and Witten [3], but extends

¹³If the shifted particles are non-adjacent in the planar ordering α , this improves to $\text{PT}(\alpha) \sim z^0$

to all dimensions [36]. What is remarkable in both these examples is that the ambitwistor-string models are able to generate amplitudes for complete theories including gauge and gravity sectors, albeit ones that are pathological with higher-order equations of motion.

Like the conventional string, the type II gravity models S_{II} require a GSO projection to project out unwanted states; we only gave vertex operators consistent with the GSO projection that are even under $(\gamma, \Psi) \rightarrow (-\gamma, -\Psi)$ and similarly for $(\tilde{\gamma}, \tilde{\Psi})$. These models also admit a Ramond sector constructed in the usual way from the Ψ spin-field, with vertex operators for NS–R and R–R sectors as in the conventional string in addition to the NS–NS vertex operators introduced above. They require the GSO projection to be applied independently to the ‘tilded’ and ‘untilded’ states to yield the 10d type IIA and IIB supergravity theories [31]. In principle, correlators give amplitude formulae for all sectors and any multiplicity, but Ramond-sector correlators are hard to compute explicitly beyond three and four points. Again we see that the ambitwistor-string model naturally completes the NS–NS–sector CHY-formulæ to the well-known type-II supergravity theories.

Soon after their original formulæ, CHY introduced expressions for amplitudes in many more theories [20, 21], going beyond E, to include BI, DBI, NLSM, Einstein–Maxwell (EM), EYM, Yang–Mills scalar (YMS) and even galileons. Further forms of worldsheet matter can be introduced to yield models that generate these formulæ as in table 1. Here for example S_{Ψ_1, Ψ_2} is an $N = 2$ version of the worldsheet superalgebra [32] discussed above, and we refer to [27] for full details of all the models.

Potentially the most remarkable of these models would be that for EYM, critical in 10d. However, it has twice as many gluon vertex operators as appropriate for conventional Yang–Mills theory and describes amplitudes for a theory with action $\int \text{tr}(B \wedge D_A^* F_A)$ where A is a standard gauge field, and B a Lie-algebra valued one-form serving as a Lagrange multiplier for the Yang–Mills equations on A .

The proof of the novel CHY formulæ arising from the massless theories listed in table 1 is not immediate: the factorization arguments extend to these models, but the BCFW shift is not directly applicable because of higher powers of momentum dependence in the vertices of many of these theories. However, different on-shell recursion relations generalizing the BCFW construction [52, 53], can be used instead. The arguments given above then extend to the whole zoo of theories with CHY-representations with only minor adjustments.

Shortly after the original ambitwistor string [22], a pure spinor analogue was introduced in [54], and its corresponding amplitude formulæ verified in [55]. Models based on the Green–Schwarz worldline model appears in [56], and progress towards 11d models from their worldline counterparts are treated in [57, 58]. Conformally invariant models, describing 6d biadjoint scalar theory, 4d gauge theory and 2d gravity, were introduced in [59].

Color-kinematics duality. The double copy [8] is built into the structure of the CHY formulæ and the ambitwistor string vertex operators. This is much as in the conventional string, where vertex operators are a product of left and right moving parts; however, here in the ambitwistor string, both are holomorphic on the worldsheet. In a field theory framework, the double copy is build on colour-kinematics duality [60]: it asserts that Yang–Mills amplitudes can be expressed as a sum over trivalent graphs Γ of the form

$$\mathcal{A} = \sum_{\Gamma} \frac{N_{\Gamma} c_{\Gamma}}{D_{\Gamma}}, \quad (2.53)$$

where c_{Γ} are the colour factors associated with the graph with vertices determined by the Lie algebra structure constants, D_{Γ} the propagator denominators associated to Γ and N_{Γ} the *kinematic numerators* depending on the polarizations and momenta. If polynomial in the momenta

they are said to be local. Colour-kinematic duality is the assertion that they can be constructed so as to satisfy the same identities as the colour factors c_Γ that arise as a consequence of Jacobi identities. This implies a double copy in the following form: that by replacing the c_Γ by another set of N_Γ , we obtain E gravity amplitudes or loop integrands [9]. Although such local kinematic numerators are known to exist at low loop order, their general theory and underlying kinematic algebra structure remains obscure. The CHY formulae have provided a powerful tool for construction of such numerators starting with [19] followed by a construction for local numerators in [61] based on the EYM formulae, see also [9, 62–64] for more recent works with many more references to progress in this very active area.

The structure of the colour factors is best understood in the language of free Lie algebras or Lie polynomials [65–67]. These are embedded in the geometry of the boundary structure of $\mathfrak{M}_{0,n}$ and play a key role [65] in the CHY formula and in the polytope constructions of [68]. See [69, 70] for more on numerators in this framework.

3. The twistor-string and the 4d ambitwistor string

For the models discussed in the previous section, the $P^2 = 0$ constraint is gauged in the quantum theory. Nevertheless, the fact that P^2 has vanishing OPE with itself¹⁴ allows this constraint to be solved essentially classically so that the integrated vertex operators are Penrose transforms of the space–time fields on \mathbb{PA} and hence localize on the scattering equations. In this section we discuss models in which the $P^2 = 0$ constraint is solved explicitly rather than gauged, using spinors classically before quantization. We achieve this by setting

$$P_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}, \quad \alpha = 0, 1, \quad \dot{\alpha} = \dot{0}, \dot{1}, \quad (3.1)$$

where $\lambda_\alpha, \tilde{\lambda}_{\dot{\alpha}}$ are two-component spinors defined up to $(\lambda_\alpha, \tilde{\lambda}_{\dot{\alpha}}) \rightarrow (s\lambda_\alpha, s^{-1}\tilde{\lambda}_{\dot{\alpha}})$ for $s \neq 0$.

Both the original twistor-string and the 4d ambitwistor-string have the same classical target, \mathbb{PA} , now specifically in four dimensions and so coincides with the original 4d ambitwistor space. However there is now the freedom to make different choices of twists for the line bundles associated with the scalings of the spinors λ_α and $\tilde{\lambda}_{\dot{\alpha}}$. This leads to two distinct models (perhaps three models if one includes a ‘dual-twistor’-string). They even share some of their vertex operators, and so we present them alongside each other. The 4d ambitwistor-string is most analogous to the RNS type models introduced above and so we present its amplitude formulae first.

The term ambitwistor space arose from the fact that in four dimensions, \mathbb{A} is both the cotangent bundle of chiral projective twistor space \mathbb{PT} and its antichiral dual, \mathbb{PT}^* . We can supersymmetrize so that twistor space $\mathbb{T} = \mathbb{C}^{4|\mathcal{N}}$ has coordinates $Z = (\lambda_\alpha, \mu^{\dot{\alpha}}, \chi^I)$, where χ^I are fermionic coordinates with I an \mathcal{N} -component R -symmetry index. Similarly we denote a dual twistor by $\tilde{Z} = (\tilde{\lambda}_{\dot{\alpha}}, \tilde{\mu}^\alpha, \tilde{\chi}_I) \in \mathbb{T}^*$ with the duality defined by

$$Z \cdot \tilde{Z} := \lambda_\alpha \tilde{\mu}^\alpha + \mu^{\dot{\alpha}} \tilde{\lambda}_{\dot{\alpha}} + \chi^I \tilde{\chi}_I. \quad (3.2)$$

The original supersymmetric ambitwistor space of [23, 24] is a supersymmetric extension of the space of scaled complex null geodesics which we shall again denote by \mathbb{A} and can be expressed as

$$\mathbb{A} := \{(Z, \tilde{Z}) \in \mathbb{T} \times \mathbb{T}^* \mid Z \cdot \tilde{Z} = 0\} / \{Z \cdot \partial_Z - \tilde{Z} \cdot \partial_{\tilde{Z}}\}. \quad (3.3)$$

¹⁴ Chiral strings for which this is not the case are discussed in section 6.2.

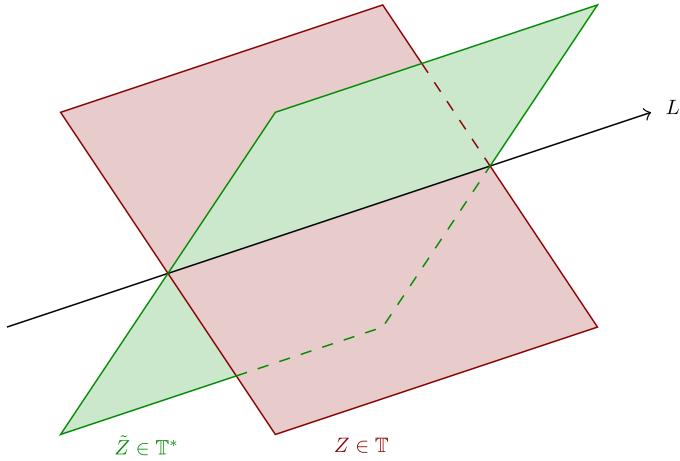


Figure 2. Illustration of ambitwistor space. A fixed twistor $Z \in \mathbb{PT}$ corresponds to a totally null two-plane on space–time, known as an α -plane; similarly for $\tilde{Z} \in \mathbb{PT}^*$ and β -planes. When $Z \cdot \tilde{Z} = 0$, these planes intersect in a light-ray L .

To see the connection with null geodesics, we first introduce the supertwistor incidence relations

$$\mu^{\dot{\alpha}} = (ix^{\alpha\dot{\alpha}} - \theta^{I\alpha}\tilde{\theta}_I^{\dot{\alpha}})\lambda_{\alpha}, \quad \chi^I = \theta^{I\alpha}\lambda_{\alpha}, \quad (3.4)$$

which defines an α -plane, a totally null self-dual $2|3\mathcal{N}$ -plane in complex Minkowski space $\mathbb{M}^{4|4\mathcal{N}}$ with coordinates $(x^{\alpha\dot{\alpha}}, \theta^{I\alpha}, \tilde{\theta}_I^{\dot{\alpha}})$. A β -plane, again a totally null $2|3\mathcal{N}$ -plane, but now anti-self-dual, is given by the dual-twistor incidence relations

$$\tilde{\mu}^{\alpha} = (-ix^{\alpha\dot{\alpha}} - \theta^{I\alpha}\tilde{\theta}_I^{\dot{\alpha}})\tilde{\lambda}_{\dot{\alpha}}, \quad \tilde{\chi}_I = \tilde{\theta}_I^{\dot{\alpha}}\tilde{\lambda}_{\dot{\alpha}}. \quad (3.5)$$

An α -plane and a β -plane intersect in a super-null geodesic when $Z \cdot \tilde{Z} = 0$. Fixing $(Z, \tilde{Z}) \in \mathbb{A}$, the coordinates $(x, \theta, \tilde{\theta})$ in super Minkowski space then vary over a $1|2\mathcal{N}$ -dimensional *super light ray* as illustrated in figure 2. Note that ambitwistor space \mathbb{A} is a phase space with symplectic potential

$$\Theta = \frac{i}{2}(Z \cdot d\tilde{Z} - \tilde{Z} \cdot dZ) = P_{\alpha\dot{\alpha}} dx^{\alpha\dot{\alpha}} + \text{fermionic coordinates}. \quad (3.6)$$

For more details on twistor- and ambitwistor space, as well as the twistor correspondence, we refer the reader to the excellent reviews and textbooks [71–75].

Both the twistor-string and the 4d ambitwistor string consist of worldsheet fields (Z, \tilde{Z}) on the worldsheet Riemann surface Σ and taking values in $\mathbb{T} \times \mathbb{T}^*$ tensored with line bundles on Σ . For the 4d ambitwistor string, (Z, \tilde{Z}) are valued also in the spin bundle $K_{\Sigma}^{1/2}$, whereas for the twistor-string, Z takes values in a line bundle $L \rightarrow \Sigma$ of degree $d \geq 0$ and \tilde{Z} in $K_{\Sigma} \otimes L^*$. In both cases the action is based on the symplectic potential which together with the constraint $Z \cdot \tilde{Z} = 0$ imposed by a Lagrange multiplier $a \in \Omega_{\Sigma}^{(0,1)}$ gives

$$S = \frac{1}{2\pi} \int_{\Sigma} \tilde{Z} \cdot \bar{\partial}_e Z - Z \cdot \bar{\partial} \tilde{Z} + a Z \cdot \tilde{Z}. \quad (3.7)$$

With either of the above choices for the line bundles for (Z, \tilde{Z}) , Θ takes values in the canonical bundle K_Σ so that the integral is well-defined. The gauge symmetry associated with a implements the quotient by $Z \cdot \partial_Z - \tilde{Z} \cdot \partial_{\tilde{Z}}$. The key distinction¹⁵ between the two theories at this stage is the assignment of degrees of the line bundles for (Z, \tilde{Z}) on the worldsheet.

As before we should include worldsheet gravity by starting with the operator $\bar{\partial}_e = d\bar{\sigma}\partial_{\bar{\sigma}} - e_{\bar{\sigma}}^\sigma d\bar{\sigma}\partial_\sigma$. We gauge fix $e_{\bar{\sigma}}^\sigma = 0 = a$ leading to the usual ghost (b, c) -system and ghosts (u, v) for a , leading to the BRST operator

$$Q = \int cT + uZ \cdot \tilde{Z}, \quad (3.8)$$

where T is the worldsheet stress tensor. The potential gauge anomaly vanishes precisely for maximal supersymmetry with $\mathcal{N} = 4$. To have vanishing central charge, we must include additional worldsheet matter, which for super Yang–Mills theory we take to be a current algebra S_C with central charge $c = 12$.

Amplitudes are again obtained as correlation functions of vertex operators. In the following we just give integrated vertex operators (they simply differ by factors of c and u from their fixed counterparts), and divide by the volume of $GL(2, \mathbb{C})$ in the final formula, understood in the usual Faddeev–Popov sense.

In order to construct vertex operators, we need to encode plane waves on twistor space. This is achieved by the *Penrose transform*, which realizes massless fields of helicity h as cohomology classes $H^1(\mathbb{PT}, \mathcal{O}(2h-2))$ and $H^1(\mathbb{PT}^*, \mathcal{O}(-2h-2))$ on bosonic twistor space or its dual, and these can be represented as $\bar{\partial}$ -closed as $(0, 1)$ forms of homogeneity $\pm 2h-2$ on \mathbb{T} or \mathbb{T}^* respectively. For Maxwell super-momentum eigenstates with bosonic four-momentum $k_{\alpha\dot{\alpha}} = \kappa_\alpha \tilde{\kappa}_{\dot{\alpha}}$ and fermionic supermomenta q^I or \tilde{q}_I , we have on twistor and dual twistor space respectively

$$a = \int \frac{ds}{s} \bar{\delta}^2(\kappa_\alpha - s\lambda_\alpha) e^{is([\mu\tilde{\kappa}] + \chi^I \tilde{q}_I)} \in H^1(\mathbb{PT}, \mathcal{O}), \quad (3.9)$$

$$\tilde{a} = \int \frac{ds}{s} \bar{\delta}^2(\tilde{\kappa}_{\dot{\alpha}} - s\tilde{\lambda}_{\dot{\alpha}}) e^{is([\tilde{\mu}\kappa] + \tilde{\chi}_I q^I)} \in H^1(\mathbb{PT}^*, \mathcal{O}). \quad (3.10)$$

As before, for a complex variable z , $\bar{\delta}(z) = \bar{\partial}(1/z)$ is a complex double delta-function $(0, 1)$ -form. These representatives encode the full supermultiplet, and the individual fields can be identified as the coefficients in the expansion in χ^I , each an element of the cohomology class $H^1(\mathbb{PT}, \mathcal{O}(2h-2))$ with the appropriate helicity h . For each field, the (dual) twistor representatives are related to space–time fields by explicit integral formulae [73]. Using these it is straightforward to check that at $\mathcal{N} = 0$, a and \tilde{a} are representatives for classes in $H_{\bar{\partial}}^1(\mathbb{PT})$ and $H_{\bar{\partial}}^1(\mathbb{PT}^*)$ respectively that generate Maxwell field momentum eigenstates $\tilde{\kappa}_{\dot{\alpha}} \tilde{\kappa}_{\dot{\beta}} e^{ik \cdot x}$ and conjugate. These degree-zero cohomology classes pull back to ambitwistor space \mathbb{A} and combine to give the spin-1 plane wave representative (2.17). For gravity the relationship is more subtle.

Yang–Mills vertex operators. For both the twistor-string and the 4d ambitwistor-string, the Yang–Mills vertex operators are constructed from the representatives above, multiplied by current-algebra generators $t \cdot j$ as for the ‘vector’ models discussed in the last section. In the twistor string, vertex operators are built from the supertwistor representatives $a(Z) \in H^1(\mathbb{PT}, \mathcal{O})$ with full $\mathcal{N} = 4$ supersymmetry; for these, the expansion in χ^I gives the full

¹⁵ Strictly speaking, for the realization $\mathbb{A} = T^*\mathbb{PT}$ we should base the action on the symplectic potential $\Theta = i\tilde{Z} \cdot dZ$; this differs from that above by the exact form $d(Z \cdot \tilde{Z})$ adding an exact term to the stress–energy tensor.

spin-1 supermultiplet represented on twistor space. In the ambitwistor string, vertex operators can also be constructed from the dual $\tilde{a}(\tilde{Z}) \in H^1(\mathbb{PT}^*, \mathcal{O})$, again with maximal supersymmetry. Explicitly,

$$\mathcal{V}'_a = \int_{\Sigma} a_a j \cdot t_a, \quad \tilde{\mathcal{V}}_a = \int_{\Sigma} \tilde{a}_a j \cdot t_a, \quad (3.11)$$

where a is a particle label, $a = 1, \dots, n$. These vertex operators are straightforwardly consistent and Q -invariant. However, the supermomenta q^I and \tilde{q}_I are not independent, but are Fourier transforms of each other; a uniform representation is obtained by a fermionic Fourier transform on \mathcal{V}'_a , giving

$$\mathcal{V}_a = \int \frac{ds_a}{s_a} \bar{\delta}^{2|\mathcal{N}}(\kappa_a - s_a \lambda | q_i - s_i \chi) j \cdot t_a, e^{is_a [\mu \tilde{\kappa}_a]}, \quad (3.12)$$

where for a fermionic variable χ , $\delta(\chi) = \chi$.

3.1. Yang–Mills amplitudes in the 4d ambitwistor string

N^{k-2} MHV Yang–Mills amplitudes can be obtained as correlation functions of the above vertex operators, taking k from dual twistor space and $n - k$ from twistor space:

$$\mathcal{A} = \langle \tilde{\mathcal{V}}_1 \dots \tilde{\mathcal{V}}_k \mathcal{V}_{k+1} \dots \mathcal{V}_n \rangle. \quad (3.13)$$

The current algebra correlator gives the usual Parke–Taylor factor (together with some multi-trace terms to be discussed later). As before, we take the exponentials from the vertex operators into the action to provide source terms

$$\int_{\Sigma} \sum_{i=1}^k is_i, ((\tilde{\mu} \kappa_i) + \tilde{\chi} \cdot q_i) \bar{\delta}(\sigma - \sigma_i) + \sum_{p=k+1}^n is_p [\mu \tilde{\kappa}_p] \bar{\delta}(\sigma - \sigma_p). \quad (3.14)$$

With these sources, the equations of motion for Z and W become

$$\bar{\partial}Z = \sum_{i=1}^k s_i(\kappa_i, 0, q_i) \bar{\delta}(\sigma - \sigma_i), \quad \bar{\partial}W = \sum_{p=k+1}^n s_p(\tilde{\kappa}_p, 0, 0) \bar{\delta}(\sigma - \sigma_p). \quad (3.15)$$

Since (Z, \tilde{Z}) are worldsheet spinors, the solutions exist and are unique, given by

$$Z(\sigma) = \sum_{i=1}^k \frac{s_i(\kappa_i, 0, q_i)}{\sigma - \sigma_i}, \quad \tilde{Z}(\sigma) = \sum_{p=k+1}^n \frac{s_p(\tilde{\kappa}_p, 0, 0)}{\sigma - \sigma_p}. \quad (3.16)$$

Thus the correlator localizes on the integrals

$$\mathcal{A} = \int \text{PT}(\alpha) d\mu_n^{(4d)}, \quad (3.17)$$

where $d\mu_n^{(4d)}$ is the 4d *polarized scattering equations* (explained below) measure:

$$d\mu_n^{(4d)} := \frac{\prod_{a=1}^n d\sigma_a ds_a/s_a}{\text{vol GL}(2, \mathbb{C})} \prod_{i=1}^k \bar{\delta}^2 \left(\tilde{\kappa}_i - s_i \tilde{\lambda}(\sigma_i) \right) \times \prod_{p=k+1}^n \bar{\delta}^{2|\mathcal{N}|} \left(\kappa_p - s_p \lambda(\sigma_p) \mid q_p - s_p \chi(\sigma_p) \right).$$

This can be expressed in homogeneous coordinates on the Riemann sphere $\sigma_{\tilde{\alpha}} = \frac{1}{s}(1, \sigma)$ using the notation $(i j) = \sigma_{i\tilde{\alpha}} \sigma_j^{\tilde{\alpha}}$ (with indices raised and lowered by the usual skew symmetric $\epsilon_{\tilde{\alpha}\tilde{\beta}}$, but note that these here are not Lorentz spinor indices) as follows;

$$Z(\sigma) = \sum_{i=1}^k \frac{(\kappa_i, 0, q_i)}{(\sigma \sigma_i)}, \quad \tilde{Z}(\sigma) = \sum_{p=k+1}^n \frac{(\tilde{\kappa}_p, 0, 0)}{(\sigma \sigma_p)}, \quad (3.18)$$

where we have rescaled W and Z by a factor of $1/s$. Then

$$d\mu_n^{(4d)} = \frac{\prod_{a=1}^n d^2\sigma_a}{\text{vol GL}(2, \mathbb{C})} \prod_{i=1}^k \bar{\delta}^2 \left(\tilde{\kappa}_i - \tilde{\lambda}(\sigma_i) \right) \times \prod_{p=k+1}^n \bar{\delta}^{2|\mathcal{N}|} \left(\kappa_p - \lambda(\sigma_p) \mid q_p - \chi(\sigma_p) \right). \quad (3.19)$$

This defines a residue formula on $G(2, n)$, the $2n - 4$ -dimensional Grassmannian of two-planes in n -dimensions parametrized by the $\sigma_{a\tilde{\alpha}}$ modulo $\text{GL}(2, \mathbb{C})$. This measure has several notable features:

- The $2n$ variables $\sigma_{a\tilde{\alpha}}$ are supported on the *polarized scattering equations*

$$\tilde{\lambda}_{\tilde{\alpha}}(\sigma_i) = \tilde{\kappa}_{i\tilde{\alpha}}, \quad \lambda_{\alpha}(\sigma_p) = \kappa_{p\alpha}, \quad (3.20)$$

for $i = 1 \dots k$ and $p = k + 1 \dots n$. These imply the ordinary scattering equations for σ_a ; defining $P(\sigma)_{\alpha\tilde{\alpha}} = \lambda(\sigma)_{\alpha} \tilde{\lambda}(\sigma)_{\tilde{\alpha}}$ it is straightforward to show that P has simple poles at σ_a with residue k_a and that $k_a \cdot P(\sigma_a) = 0$ follows on the support of (3.20). However, the polarized scattering equations also incorporate the choice of polarization data via the s_a , and are refined to give just those $A(n - 3, k - 2)$ solutions appropriate to N^k MHV degree, where $A(p, q)$ are the Eulerian numbers¹⁶ [12].

- For maximal supersymmetry, the measure $d\mu_n^{(4d)}$ only depends on the MHV sector, not the specific helicity assignment of the n gluons. However, this ‘dihedral symmetry’ of the N^k MHV amplitude is not manifest, and the above formulæ require a choice of helicity assignment, here with the first k particles of negative helicity.
- The measure contains $2n$ bosonic delta functions but only $2n - 4$ integrals; with the difference due to the $\text{vol}(\text{GL}(2, \mathbb{C}))$ quotient. The remaining delta-functions encode momentum-conserving delta functions, as can be seen from

$$\sum_{p=k+1}^n \kappa_p \tilde{\kappa}_p = \sum_{p=k+1}^n \tilde{\kappa}_p \sum_{j=1}^k \frac{\kappa_j}{(p j)} = - \sum_{j=1}^k \kappa_j \tilde{\kappa}_j, \quad (3.21)$$

where we used the first (second) set of delta functions in (3.19) to get the first (second) equality; similarly we obtain supermomentum conservation $\sum_{a=1}^n \tilde{\kappa}_a q_a = 0$.

- The amplitude formula (3.17) can be verified at $\mathcal{N} = 0$ by integrating out the moduli in equation (3.22) in [6], see [75] for details.
- While maximal supersymmetry with $\mathcal{N} = 4$ naively seems to double the spectrum of the model, both vertex operators (3.11) are representatives of the *same* space-time field, as

¹⁶ Satisfying the recursion $A(p, q) = (p - q)A(p - 1, q - 1) + (q + 1)A(p - 1, q)$.

established by the Penrose transform that maps, for example, the same ASD Maxwell field to an element of $H^1(\mathbb{PT}, \mathcal{O}(-4))$ or to one in $H^1(\mathbb{PT}^*, \mathcal{O})$. Thus vertex operators \mathcal{V}' on \mathbb{PT} and $\tilde{\mathcal{V}}$ on \mathbb{PT}^* represent the same space–time multiplet for $\mathcal{N} = 4$.

- The model also contains vertex operators for non-minimal conformal gravity states, believed to coincide with the analogous states in the original twistor string [2]. The full model is understood to give amplitudes for this combined $\mathcal{N} = 4$ super-conformal combination of conformal gravity and Yang–Mills theory, with the multi-trace terms in the current correlator corresponding to interactions mediated by scalars of conformal supergravity.

3.2. The twistor-string formulæ

Soon after the original twistor-string, RSV [5] simplified its correlation function to give the (historically) first fully localized worldsheet formula for a field theory amplitude. We present it out of historical order because it has additional moduli integrals compared to the formulæ discussed so far. This formula also requires $\mathcal{N} = 4$ supersymmetry for anomaly cancellation and for the expression to be well-defined. It does then have the benefit of manifesting the full dihedral symmetry of the N^k MHV amplitude irrespective of the specific helicities of individual gluons. In contrast, the ambitwistor string formulæ do not manifest this symmetry.

The key distinction from the 4d ambitwistor-string is that the twistors Z are understood to be sections of line bundles $L \rightarrow \Sigma$ of degree $d > 0$, and the dual twistors \tilde{Z} sections of $L^* \otimes K_\Sigma$. Moreover, only twistorial vertex operators are used so that it is more natural to refer to the model as a twistor-string. At genus zero and degree d , the maps $Z : \Sigma \rightarrow \mathbb{PT}$ have moduli $M_r \in \mathbb{T}$, $r = 0, \dots, d$ given by

$$Z(\sigma) = \sum_{r=0}^d M_r \sigma^r, \quad M_r = (m_{r\alpha}, m_r^{\dot{\alpha}}, m_r^I), \quad (3.22)$$

where σ is an affine coordinate on $\Sigma = \mathbb{CP}^1$ and the m_r^I are fermionic with the remaining components of the M_r bosonic.

At $\mathcal{N} = 4$, the vertex operators (3.11) or (3.12) contain the full multiplet including gluons of both helicities. The full tree-level Yang–Mills super-amplitude is obtained as the correlator of n such vertex operators of one type or other. There are no contractions between the vertex operators beyond the current algebra which give the usual Parke–Taylor factor, with the usual caveats about multi-trace terms. Thus the path-integral immediately localizes onto the zero-modes (3.22), yielding the amplitude formula

$$\mathcal{A}_n = \int \frac{d^{4d+4|4d+4} M}{\text{vol GL}(2)} \prod_{a=1}^n \frac{d\sigma_a ds_a}{s_a} \bar{\delta}^2(\kappa_a - s_a \lambda(\sigma_a)) e^{is_a(\mu \tilde{\kappa}_a + \chi^I \tilde{q}_{aI})} \text{PT}(\alpha). \quad (3.23)$$

In order to simplify this further, we can formally integrate out the moduli $m_r^{\dot{\alpha}}, m_r^I$. Since they only appear in the exponentials, this leads to additional delta-functions,

$$\mathcal{A}_n = \int \frac{d^{2d+2} m_{r\alpha}}{\text{vol GL}(2)} \prod_{a=1}^n \frac{d\sigma_a ds_a}{s_a} \bar{\delta}^2(\kappa_a - s_a \lambda(\sigma_a)) \times \prod_{r=0}^d \delta^{2|4} \left(\sum_{a=1}^n \sigma_a^r s_a \tilde{\kappa}_{a\alpha} \right) \text{PT}(\alpha). \quad (3.24)$$

In this formula, (3.22) gives $\lambda(\sigma_a) = \sum_r m_{r\alpha} \sigma_a^r$. Once the $\text{GL}(2)$ quotient is taken into account, there are four more delta-functions than integrals encoding supermomentum conservation.

This can be made explicit by summing the arguments of the $(d+1)$ $\bar{\delta}^{2|4}$ -functions, multiplied by m_{ra} , and using the support of the first n delta-functions. Thus the moduli integrals over (s_a, σ_a, m_{ra}) modulo $\text{GL}(2, \mathbb{C})$ can be performed against the remaining delta functions to give a sum of residues multiplied by the momentum conservation delta-functions. This is analogous to the 4d-ambitwistor string, but with an additional $2d+2$ moduli integrals and delta-functions.

- The fermionic delta functions relate the MHV degree k of the amplitude to the degree of the line bundle by $k = d-1$
- In [5, 6], the formula was shown to be parity invariant, which is far from manifest, with a number of further checks. A BCFW recursion proof was given in [76].
- In [3, 77] it was shown that one can further introduce vertex operators for $\mathcal{N} = 4$ superconformal gravity. There are two multiplets, one containing the ASD Weyl tensor, determined by a divergence-free vector field $f(Z) \cdot \partial_Z$ and realized in the twistor-string by the vertex operator $f \cdot \tilde{Z}$, and one containing the SD Weyl tensor, given by a one-form $g(Z) \cdot \partial_Z$. Correlators involving these states are more complicated because the \tilde{Z} factors must be contracted completely before one can reduce to a moduli integral¹⁷.
- The equivalence with the 4d ambitwistor formula was proved in [75]; it essentially uses the moduli integrals in the twistor string formula (3.23) to perform a *twistor-transform* of $d+1$ of the cohomology classes in the vertex operators. This for example maps an element of $H^1(\mathbb{PT}, \mathcal{O}(-4))$ describing an ASD Maxwell field to that in $H^1(\mathbb{PT}^*, \mathcal{O})$ for the same field.
- Both in the twistor string and in the ambitwistor string, a number of open questions remain. In the twistor string, we do not include vertex operators $\tilde{\mathcal{V}}_a$ constructed from dual twistor cohomology classes, but there does not seem to be a good reason not to. Related to this, gauging the current $Z \cdot \tilde{Z}$ in the 4d ambitwistor string should also result in a sum over the degree of line bundle associated to the gauge field a , but this does not seem to play a role in the amplitude formulæ; see also [75].

3.3. Einstein supergravity models and amplitudes

The twistor string for $\mathcal{N} = 8$ Einstein supergravity was introduced by Skinner [16] as the model underpinning the earlier Cachazo–Skinner formula [14] (proved in [15]). A 4d ambitwistor version was then introduced in [26]. Again, both have essentially the same underlying worldsheet model but with different worldsheet assignments of twists for the twistor and dual twistor target fields and ghosts. We focus here on the 4d ambitwistor version for brevity; see [14, 16] for the twistor-string version. These have the advantage of full permutation invariance, at the price of additional moduli integrals.

In order to break conformal invariance we introduce skew bilinear forms on twistor space and its dual, the infinity twistor and its dual. When non-degenerate, these encode a cosmological constant and a gauging of R -symmetry. Although the model was originally introduced incorporating a cosmological constant, here we restrict to the Minkowski space model and its amplitudes. In this case the infinity twistors have rank 2 and we will denote contractions with a pair of twistors by $\langle Z_1, Z_2 \rangle := \langle \lambda_1 \lambda_2 \rangle$ and with a pair of dual twistors by $[\tilde{Z}_1, \tilde{Z}_2] := [\tilde{\lambda}_1 \tilde{\lambda}_2]$.

We introduce a worldsheet superalgebra by extending the target to include $(\rho, \tilde{\rho}) \in \mathbb{T} \times \mathbb{T}^*$ that are parity reversed, taking values in $\mathbb{C}^{\mathcal{N}|4} \otimes K_{\Sigma}^{1/2}$ (rather than $\mathbb{C}^{4|\mathcal{N}}$); they are taken to

¹⁷ Only $f \cdot \tilde{Z}$ and its conjugate need to be used in the 4d ambitwistor model.

be worldsheet spinors in both the twistor-string and the 4d ambitwistor string versions. An additional set of constraints are gauged, and including the original $Z \cdot \tilde{Z}$, the total set of currents that are gauged in this model becomes

$$K_a = (Z \cdot \tilde{Z}, \rho \cdot \tilde{\rho}, \tilde{Z} \cdot \rho, [\tilde{Z}, \tilde{\rho}], Z \cdot \tilde{\rho}, \langle Z, \rho \rangle, \langle \rho, \rho \rangle, [\tilde{\rho}, \tilde{\rho}]). \quad (3.25)$$

In the BRST quantization, this introduces corresponding ghosts (β_a, γ^a) , together with the fermionic (b, c) ghosts as before [78], and leads to a BRST Q -operator

$$Q = \int cT + \gamma^a K_a - \frac{i}{2} \beta_a \gamma^b \gamma^c C_{bc}^a, \quad (3.26)$$

where C_{bc}^a are the structure constants of the current algebra K_a . The model again has a potential gauge anomaly, whose coefficient vanishes for $\mathcal{N} = 8$ maximal supersymmetry.

In these E gravity models, Q -invariance implies that vertex operators are built from $\bar{\partial}$ -closed $(0, 1)$ -forms h of weight two on twistor space, as well as \tilde{h} on dual twistor space for the ambitwistor string. For momentum eigenstates, h and \tilde{h} are

$$h_a = \int \frac{ds_a}{s_a^3} \bar{\partial}^2 |\mathcal{N}| (\kappa_a - s_a \lambda |q_a - s_a \chi) e^{is_a [\mu \tilde{\kappa}_a]},$$

$$\tilde{h}_a = \int \frac{ds_a}{s_a^3} \bar{\partial}^2 (\tilde{\kappa}_a - s_a \tilde{\lambda}) e^{is_a (\langle \tilde{\mu} \kappa_a \rangle + \tilde{\chi}_r q_a^r)}.$$

These yield two types of vertex operators, appearing in integrated or fixed form, here integration being with respect to ghost zero modes. The ghosts $\nu = (\gamma^3, \gamma^4)$, $\tilde{\nu} = (\gamma^5, \gamma^6)$ each have one zero mode that are fixed by the insertion of one each of

$$V_h = \int_{\Sigma} \delta^2(\nu) h, \quad \tilde{V}_{\tilde{h}} = \int_{\Sigma} \delta^2(\tilde{\nu}) \tilde{h}. \quad (3.27)$$

The remaining particles are represented by integrated vertex operators

$$\mathcal{V}_h = \int \left[\tilde{Z}, \frac{\partial h}{\partial Z} \right] + \left[\tilde{\rho}, \frac{\partial}{\partial Z} \right] \rho \cdot \frac{\partial h}{\partial Z}, \quad \tilde{\mathcal{V}}_{\tilde{h}} = \int \left\langle Z, \frac{\partial \tilde{h}}{\partial \tilde{Z}} \right\rangle + \left\langle \rho, \frac{\partial}{\partial \tilde{Z}} \right\rangle \tilde{\rho} \cdot \frac{\partial \tilde{h}}{\partial \tilde{Z}}. \quad (3.28)$$

Amplitudes are now given by the worldsheet correlation function

$$\mathcal{M} = \left\langle \tilde{\mathcal{V}}_{\tilde{h}_1} \prod_{i=2}^k \tilde{\mathcal{V}}_{\tilde{h}_i} \prod_{p=k+1}^{n-1} \mathcal{V}_{h_p} V_{h_n} \right\rangle. \quad (3.29)$$

The correlator of the $(\rho, \tilde{\rho})$ fermion system is given by the determinant of the following $n \times n$ matrix:

$$\mathcal{H} = \begin{pmatrix} \mathbb{H} & 0 \\ 0 & \tilde{\mathbb{H}} \end{pmatrix}, \quad (3.30)$$

where, for $i, j \in \{1, \dots, k\}$ and $p, q \in \{k+1, \dots, n\}$

$$\mathbb{H}_{ij} = \begin{cases} \frac{\langle i j \rangle}{(i j)}, & i \neq j, \\ - \sum_{l=1, l \neq i}^k \mathbb{H}_{il}, & i = j, \end{cases} \quad \tilde{\mathbb{H}}_{pq} = \begin{cases} \frac{[p q]}{(p q)}, & p \neq q, \\ - \sum_{r=k+1, r \neq p}^n \tilde{\mathbb{H}}_{pr}, & p = q. \end{cases} \quad (3.31)$$

The off-diagonal element \mathcal{H}_{ij} comes from the contraction of the ρ -term in the i th vertex operator with the $\tilde{\rho}$ -term in the j th, and the diagonal elements of \mathcal{H} stem from the first term in the integrated vertex operator (3.28). By an analogous calculation as for Yang–Mills theory, we then obtain the gravity amplitudes

$$\mathcal{M}_n = \int d\mu_n^{(4d)} \det'(\mathcal{H}), \quad (3.32)$$

where $\det' \mathcal{H}$ is the determinant omitting a row and column from each of $\tilde{\mathbb{H}}$ and \mathbb{H} corresponding to the fixed vertex operators; the answer is independent of this choice because each has kernel $(1, \dots, 1)$.

- Through lack of space we do not include the Cachazo–Skinner twistor-string gravity formula, but refer the reader to [14–16]. The equivalence between these formulæ above follows the corresponding story for Yang–Mills, see [75].
- It is possible to consider higher rank versions of the infinity twistor with bosonic rank four corresponding to the inclusion of a cosmological constant [79] and higher rank in the fermionic directions corresponding to gauged supergravity theories with gauged R -symmetry [80]. The Skinner model [16] was originally formulated in this way. Corresponding amplitude formulæ with non-zero cosmological constant have been explored in [81–83] but remain conjectural.
- Comparison between the Yang–Mills versus gravity formula and the corresponding CHY formula restricted to 4d makes clear that the CHY Pfaffian must be equal to $\det' \mathcal{H}$. This is shown via analyticity and conformal field theory (CFT) arguments in [84] and used to extend the formulæ to include EYM amplitudes improving [85].
- The double copy is *not* manifest in these formulæ. To see it, one must integrate out the s_a coordinates in the polarized scattering equations measure. This yields a second $\det' \mathcal{H}$ multiplied by the CHY measure, providing a geometric origin to this second ‘copy’ in the gravity formulæ [86]. See also [87, 88] which include some interesting extensions.
- The polarized scattering equations have been extended to higher dimensions, including 6d in [86, 89], 10 & 11d in [90], with underlying models in 5 & 6d in [91]. These 6d formulæ are distinct from the earlier ones of [92–94] which give twistor-string like expressions for D5 and M5-branes. The corresponding Yang–Mills and gravity formulæ however become awkward for odd numbers of particles, an issue that does not arise in the brane theories, since brane amplitudes are only nontrivial for even numbers of particles.
- Twistorial models in 10d that solve the $P^2 = 0$ constraint have been introduced using ‘impure’ 10d twistors in [95, 96] and pairs of pure twistors in [97].
- There exists a two-twistor representation of the space of massive particles [98] and its complexification can be used as a target space for a massive twistor string [99]. This yields

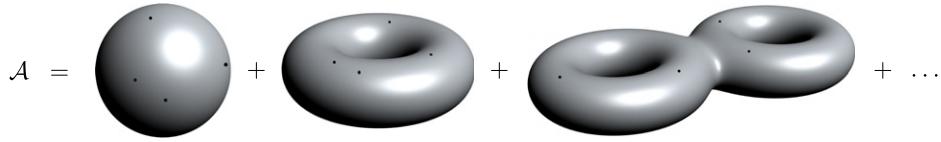


Figure 3. Schematic loop expansion of the amplitude. In worldsheet models such as the ambitwistor string, g -loop amplitudes correspond to correlators on genus- g Riemann surfaces.

massive amplitudes in four dimensions including half-integral spins and manifest supersymmetry in formulae localized again on the polarized scattering equations. These formulae also arise by dimensional reduction from 6d and were first found in [86] (again preceded by related massive amplitude formulae in [94]). Such massive formulae have also been used to construct one-loop integrands [100].

- Both the twistor-string and the ambitwistor-string formulae naturally embed [101, 102] into the Grassmannians of [103], where they can be related to twistor-string formulae for leading singularities and BCFW terms. In 6d a Lagrangian Grassmannian approach $LG(n, 2n)$ was proposed in [94] that facilitates the comparison [104] between the different 6d amplitude formulae [89, 94].

4. Loop amplitudes at higher genus

The ambitwistor string not only provides a beautiful geometric explanation of the CHY formulae but it allows us to extend these tree-level amplitude formulae in a variety of directions. One important such avenue are extensions of the worldsheet CHY formulae to loop amplitudes. On a practical level, it is clear that the only way to approach this problem is via a *model*—guessing loop-level formulae is simply not generally feasible.

In the first instance, as a closed-string worldsheet model, we will see that g -loop amplitudes in the ambitwistor string should be given by correlators over genus- g Riemann surfaces; this is already a significant simplification from the *super* Riemann surfaces usually encountered in the superstring [30]. The full amplitude then has a perturbative expansion as a sum over topologies, schematically expressed in figure 3.

In this section, we aim to give a sketch of the correspondence between higher-genus correlators and loop amplitudes, many additional details can be found in the original papers carrying out the calculations at one [31, 105] and two loops [106, 107]. The focus, both here and in the original papers, lies on the RNS model S_{II} describing supergravity, because the others models are ‘contaminated’ by their unphysical gravitational states propagating in the loop¹⁸. Rather than providing a fully self-contained derivation of the loop amplitudes, we emphasize here the key features distinguishing the ambitwistor string from the superstring, and reflecting its *field theory* nature. We will encounter this repeatedly via calculations that bear, superficially, a close similarity to the superstring [110–116], but with considerable simplifications and important differences, all originating in the chiral nature of the ambitwistor string and its supersymmetry structure—the worldsheet super gauge algebra discussed in section 2, as opposed to the worldsheet super-diffeomorphisms familiar from the superstring. In the superstring, the fermionic constraint squares to give the stress–energy tensor generating worldsheet diffeomorphisms, so

¹⁸ Progress for the twistorial models has also been rather limited, see [7, 100, 108, 109].

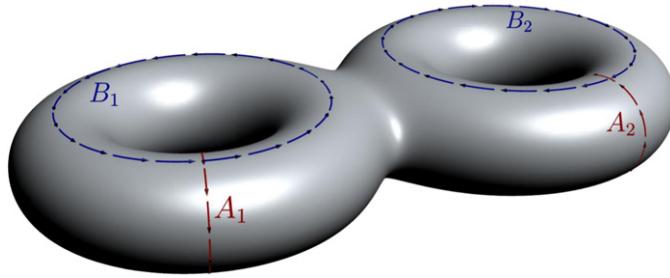


Figure 4. The homology basis at genus two.

the correlator is an integral over the moduli space of a supermanifold. In the ambitwistor string on the other hand, the worldsheet fermionic symmetry squares to give P^2 which, in generating translations along the geodesics, does not complicate the moduli space of the Riemann surface.

4.1. From higher-genus correlators to loop amplitudes

The ambitwistor string provides a clear prescription for calculating loop amplitudes as a correlator over a higher-genus worldsheet. Our first goal will be to understand this prescription, before we proceed to evaluate the correlator.

The correlator: the genus- g ambitwistor string correlator with n vertex operators involves an integral over the moduli space $\mathfrak{M}_{g,n}$, stemming from the integration over inequivalent metrics under conformal transformations [117, 118]. A convenient description of this moduli space (up to genus three) is given by the *period matrix*, which is defined as follows. For a genus- g Riemann surface, we choose a homology basis of cycles A_I and B_I , with $I = 1, \dots, g$ such that the intersection form is canonical, as illustrated in figure 4 for genus two. Transformations acting on the homology basis (A_I, B_I) while leaving the intersection form invariant are known as modular transformations, and form the *modular group* $\mathrm{Sp}(2g, \mathbb{Z})$. If we normalize the holomorphic abelian differentials ω_I against the A -cycles, then the period matrix $\Omega_{IJ} = \Omega_{(IJ)}$ is given by the B -periods;

$$\oint_{A_I} \omega_J = \delta_{IJ}, \quad \oint_{B_I} \omega_J = \Omega_{IJ}. \quad (4.1)$$

Up to genus three, $\mathfrak{M}_{g,n}$ can then be parametrized by such period matrices Ω up to the modular group as the dimension count for these two spaces agrees up to genus three, with $\dim \mathfrak{M}_{g,n} = 3g - 3$ and $\dim \Omega = \frac{1}{2}g(g + 1)$. This is particularly simple in the case of the torus, where the moduli space is parametrized by the complex parameter $\tau \in \mathcal{F}$, where

$$\mathcal{F} = \left\{ \tau \in \mathbb{C} \mid |\tau| \geq 1, -\frac{1}{2} \leq \Re(\tau) \leq \frac{1}{2} \right\}, \quad (4.2)$$

is the fundamental domain obtained by quotienting the upper half-plane by the Dehn twists generating the modular group $\mathrm{PSL}(2, \mathbb{Z})$, see figure 5.

At loop level, the GSO projection—required to restrict to the correct supergravity degrees of freedom—results in a sum over worldsheet spin structures $\kappa \in (\mathbb{Z}/2\mathbb{Z})^{2g}$. Since the GSO projection is implemented independently for each of the two chiral spinors, this leads to a double sum over spin structures whose relative phases $\eta_\kappa, \eta_{\bar{\kappa}}$ are entirely determined by modular

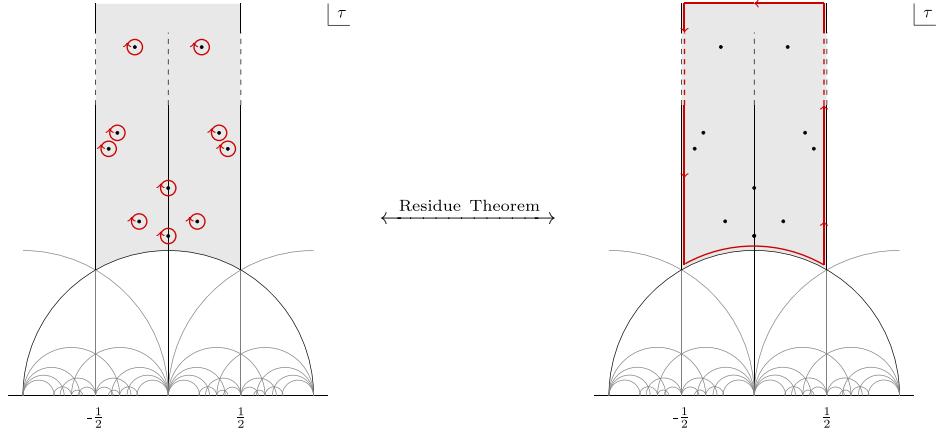


Figure 5. A graphic depiction of the residue theorem on the fundamental domain for $g = 1$. Illustrated on the left is the localisation of the integrand on solutions to $u = 0$ on the support of the remaining scattering equations. This is *equal* to the integrand, now localised on the boundary $\tau = i\infty$ of the moduli space, with the two expressions related by a residue theorem on the fundamental domain.

invariance and unitarity [119]. The n -particle g -loop ambitwistor string correlator can then be schematically expressed for $g > 1$ as follows:

$$\mathcal{A}_n^{(g)} = \int_{\mathfrak{M}_{g,n}} \prod_{I \leq J} d\Omega_{IJ} \sum_{\kappa, \tilde{\kappa}} \eta_{\kappa} \eta_{\tilde{\kappa}} \left\langle \prod_{r=1}^{3g-3} b_r \tilde{b}_r \bar{\delta}(\langle \mu_r P^2 \rangle) \prod_{\alpha=1}^{2g-2} \Upsilon_{\alpha} \tilde{\Upsilon}_{\alpha} \prod_{i=1}^n \mathcal{V}_i \right\rangle_{\kappa \tilde{\kappa}}. \quad (4.3)$$

The torus is exceptional needing one fixed vertex operator due to the remaining symmetry and associated zero mode of the c and \tilde{c} ghosts; the first product are then nontrivial with a single entry. Here $\mu_r \in \Omega^{(0,1)} \otimes T_{\Sigma}^{(1,0)}$ is a Beltrami differential, and $\Upsilon_{\alpha}, \tilde{\Upsilon}_{\alpha}$ are the *picture changing operators* introduced in section 2.2.

A key feature is that the additional delta-functions, $\bar{\delta}(\langle \mu_r P^2 \rangle)$ enforces that P^2 vanishes. These form an important part of the loop-level scattering equations, as we will see below, and give n_b new constraints, with $n_b = 1$ on the torus and $n_b = 3g - 3$ at higher genus leading again to a sum over residues as at tree-level. Moreover, we note that for $g \geq 1$, not all picture-changing operators $\Upsilon_{\alpha} \tilde{\Upsilon}_{\alpha}$ can be absorbed by the vertex operators; this will affect the structure of the integrand.

Scattering equations: since the correlator only depends on X via the exponentials in the vertex operators, the PX -path integral can be evaluated following the same strategy as at tree-level. As discussed in section 2, the X zero-mode integral then gives $d = 10$ momentum-conserving delta-functions, and the non-zero modes localize P onto the solution to its equations of motion (2.28). Although the genus- g Green's function for the PX -system is complicated¹⁹, the equations of motion fortunately have a simple solution in terms of holomorphic and meromorphic differentials;

$$P_{\mu}(z) = \ell_{\mu}^I \omega_I + \sum_i k_{i\mu} \omega_{i*}(z). \quad (4.4)$$

¹⁹ Appendix A of reference [111] serves as an excellent summary for CFTs on higher-genus worldsheets.

It is traditional at higher genus to use the worldsheet coordinate z rather than σ to avoid confusion when simplifying the resulting amplitude formulae, see section 5. Here, in the second term, the meromorphic differentials ω_{ij} are so-called *Abelian differentials of the third kind*, i.e., one-form in z , with a simple poles at two marked points z_i, z_j with respective residues ± 1 ; z_* is a reference point, but momentum conservation ensures that P has no pole at that point. The first term arises because, at loop level, the equations $\bar{\partial}P = 0$ admits homogeneous solutions $\ell_\mu^I \omega_I$, given by the holomorphic differentials ω_I multiplied by the zero-modes ℓ_μ^I . The path integral now includes an integral over these zero-modes, named to suggest already their role as the loop momentum of the amplitude.

The scattering equations—both geometrically and from the CFT point-of-view—still play the same role as at tree-level, enforcing the constraint $P^2 = 0$, now however on the genus- g worldsheet. As evident from the correlator, there are now two types of constraints: n scattering equations (or $n - 1$ on the torus, due to its translation invariance) are still associated to the moduli of the insertion points of the vertex operators, and resemble the tree-level constraints,

$$\mathcal{E}_i = \text{Res}_i P^2 = 2k_i \cdot P(z_i). \quad (4.5)$$

In contrast to tree-level however, these constraints are not sufficient to ensure $P^2 = 0$ on higher-genus surfaces due to the homogeneous solutions. This is reflected in the additional n_b delta-functions in the correlator, which provide the remaining constraints. They are best parametrized by expanding P , on the support of the remaining scattering equations \mathcal{E}_i , into a basis of quadratic holomorphic differentials,

$$P^2 = u dz^2, \quad P^2 = u^{IJ} \omega_I \omega_J, \quad (4.6)$$

given here at one and two loops respectively. The new, loop-level scattering equations then enforce $u = 0$ and $u_{IJ} = 0$ respectively. For fixed loop momenta ℓ^I , these constraints are solved by localizing the moduli Ω_{IJ} themselves so that, as we will see, the *loop integrand* reduces to a sum of residues in $\mathfrak{M}_{g,n}$.

Loop amplitudes: the ambitwistor string thus leads to the following formulæ,

$$\mathcal{A}_n^{(g)} = \delta^{10} \left(\sum_{i=1}^n k_i \right) \int \prod_{I=1}^g d^{10} \ell_I \mathcal{I}_n^{(g)}, \quad (4.7)$$

where the integral is over the zero-modes ℓ_μ^I of P_μ , and the ‘loop integrand’ $\mathcal{I}_n^{(g)}$ is given by an integral over the moduli space $\mathfrak{M}_{g,n}$ ²⁰,

$$\mathcal{I}_n^{(g)} = \int_{\mathfrak{M}_{g,n}} \prod_{I \leq J} d\Omega_{IJ} \bar{\delta}(u_{IJ}) \prod_i \bar{\delta}(\text{Res}_i P^2) \mathcal{I}_n^{\text{chi}} \tilde{\mathcal{I}}_n^{\text{chi}}. \quad (4.8)$$

The chiral integrands $\mathcal{I}_n^{\text{chi}}$ stem from the remainder of the correlator, including the sum over spin structures, and contain the non-trivial chiral partition functions $\mathcal{Z}_\kappa^{\text{chi}}$ of all fields, as well as Pfaffian factors from the $\Psi, \tilde{\Psi}$ correlators.

For brevity, we only give explicit expressions for even spin structures at one loop; this covers many cases of interest, as odd spin structures do not contribute in $d \leq 9$ dimensions or for four particles at $g \leq 3$. Formulae for odd spin structures and two-loops can be found in

²⁰ Note that for the torus, this only contains $n - 1$ scattering equations $\mathcal{E}_i = \text{Res}_i P^2$ as discussed above.

[31, 106, 107] respectively. For an even spin structure δ ,

$$\mathcal{I}_n^{\text{chi}} = \sum_{\delta} (-1)^{\delta} \mathcal{Z}_{\delta}^{\text{chi}} \text{Pf}(M_{\delta}), \quad \text{with } M_{\delta} = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}, \quad (4.9)$$

where the chiral partition function $\mathcal{Z}_{\delta}^{\text{chi}} = \vartheta_{\delta}(0|\tau)^4 \eta(\tau)^{-12}$ is given in terms of Jacobi theta-functions and the Dedekind eta-function, and where M_{δ} is a $2n \times 2n$ matrix with components $A_{ii} = B_{ii} = 0$, $C_{ii} = -\epsilon_i \cdot P(z_i)$ and

$$A_{ij} = k_i \cdot k_j S_{\delta}(z_{ij}|\tau), \quad C_{ij} = \epsilon_i \cdot k_j S_{\delta}(z_{ij}|\tau), \quad B_{ij} = \epsilon_i \cdot \epsilon_j S_{\delta}(z_{ij}|\tau). \quad (4.10)$$

The structure of the matrix is reminiscent of that at tree-level, but its components now depend on the punctures z_i via the Green's function for the fermion Ψ -system; the Szegö kernel

$$S_{\delta}(z - w|\tau) = \frac{\vartheta_1(0|\tau)}{\vartheta_1(z - w|\tau)} \frac{\vartheta_{\delta}(0|\tau)}{\vartheta_{\delta}(z - w|\tau)} \sqrt{dz} \sqrt{dw}. \quad (4.11)$$

Four particle amplitude: for four external particles, these amplitude expressions simplify drastically, and the sum over spin structures can be performed explicitly. In this case the chiral integrand is commonly written as $\mathcal{I}_4^{\text{chi}} = \mathcal{K} \mathcal{Y}^{(g)}$, where \mathcal{K} is a universal kinematic prefactor defined by the tree amplitudes $\mathcal{A}_4^{(0)} = \mathcal{K} \tilde{\mathcal{K}} / stu$, and

$$\mathcal{Y}^{(1)} = \mathcal{K} \prod_{i=2}^4 dz_i, \quad \mathcal{Y}^{(2)} = \mathcal{K} (s \Delta_{14} \Delta_{23} - t \Delta_{12} \Delta_{34}). \quad (4.12)$$

Here, $\Delta_{ij} = \varepsilon^{IJ} \omega_I(z_i) \omega_J(z_j)$ denotes the natural modular form of weight -1 on a $g = 2$ Riemann surface, and $s = 2k_1 \cdot k_2$ and $t = 2k_1 \cdot k_4$ are Mandelstam variables.

4.2. Properties and further developments

Since supergravity amplitudes are—in contrast to the superstring—not UV finite, the ambitwistor string correlators (4.7) are divergent and so should be understood as formal expressions only before regularization. The natural object of interest is the *loop integrand* $\mathcal{I}_n^{(g)}$ and is well-defined. This loop integrand has indeed many interesting properties:

- First and foremost, $\mathcal{I}_n^{(g)}$ is an integral over the moduli space $\mathfrak{M}_{g,n}$ of marked Riemann surfaces. Aspects of modular invariance, while far from manifest, follow as a consequence of the GSO projection and the resulting sum over spin structures [31, 107]. However, modular transformations affect the loop momentum ℓ^I , so (4.8) should be interpreted in the first instance as an integral over a specific fundamental domain and the integrand is not in itself expected to be modular invariant; full modular invariance is only restored after accounting for the transformation properties of the loop momenta. Unlike in the superstring case, modular invariance does *not* guarantee finiteness of the amplitude, and, as we will see in the next section, we are dealing with field theory loop integrands, so that the moduli integral over the zero modes ℓ^I that remain to be performed are infinite.
- In analogy with the tree-level ambitwistor string correlator, and in contrast with the superstring, the moduli space integrals in $\mathcal{I}_n^{(g)}$ are *fully localized* on solutions to loop-level scattering equations. This is evident in the explicit formulae (4.8), but also has a clear CFT origin since both the integration measure and the delta-function insertions are tied to the dimension of the moduli space $\dim \mathfrak{M}_{g,n} = n + 3g - 3$.

- The correlator formulæ (4.8) also exhibit a tantalizing resemblance to the superstring amplitude in the chiral splitting formalism [117, 120]. This manifests itself both via closely analogous calculations (compare, for example, the two-loop calculations in [107, 110–115]) and in the final answer; in fact the *same* chiral half-integrands $\mathcal{Y}^{(g)}$ appear in the superstring

$$\mathcal{A}_{\mathbb{S}}^{(g)} = \mathcal{K}\tilde{\mathcal{K}} \int_{\mathfrak{M}_{g,4}} \left| \prod_{I \leq J} d\Omega_{IJ} \right|^2 \int d\ell \, |\mathcal{Y}^{(g)}|^2 \prod_{i < j} |E(z_i, z_j)|^{\frac{\alpha_{ij}}{2}} |C(\ell^I)|^2, \quad (4.13)$$

and the ambitwistor string for $n = 4$ particles. Here the product over the prime form $E(z_i, z_j)$ and the chiral splitting factor $C(\ell^I)$ form the loop-level Koba–Nielsen factor characteristic of the string. Gaining a better understanding of the close similarity between such different theories—the superstring and supergravity—still remains an open problem. Various recent developments have aimed to clarify various aspects, such as the role of loop momentum in string theory [121], the ambitwistor string moduli integrals [32], detail of the chiral splitting in the ambitwistor string [122], and the relation to other chiral strings [123–126].

The obvious counterpart to the problem of relating the ambitwistor string and the superstring concerns the field theory status of these formulæ. Many important properties of field theory integrands are obscured on the higher genus worldsheet; in particular, it remains mysterious how expressions such as (4.8) could possibly give rise to the simple rational functions of the external kinematics and loop momenta that constitute the supergravity integrand. In the words of the original paper [31], this would surely require ‘miraculous simplifications’. In the next section, we will see investigate these ‘miraculous simplifications’, and explore many of their consequences.

5. From higher genus to the nodal sphere

Despite the compactness of the formulæ, the higher genus correlators are hard to compute and would seem to give highly transcendental functions rather than the rational functions we expect for field theory integrands. Remarkably one can relate the complicated higher-genus correlators of the previous section to the familiar rational functions of field theory integrands. In doing so we obtain a new formulation for loop integrands on the Riemann sphere but with an extra node or double point for each loop order. We also gain control over which fields run in the loop and can therefore generate loop integrand formulæ for the full range of theories in different dimensions that the CHY framework applies to, taking us far beyond the restriction to type II supergravity in 10d of the previous section.

5.1. Residue theorem to the nodal sphere

The key insight [127, 128] is that the full correlator can, via residue theorems, be localised on the boundaries of the moduli space, corresponding geometrically to Riemann spheres with pairs of identified points called a node; these are *non-separating degenerations*. At one loop, this is achieved by a residue theorem on the moduli space $\mathfrak{M}_{1,n}$, trading the localisation on one of the higher-genus scattering equations, $u = 0$, for a localisation on the non-separating boundary divisor $\tau = i\infty$. This residue theorem, illustrated in figure 5, relies on three properties of the integrand: modular invariance, complete localisation on the scattering equations, and

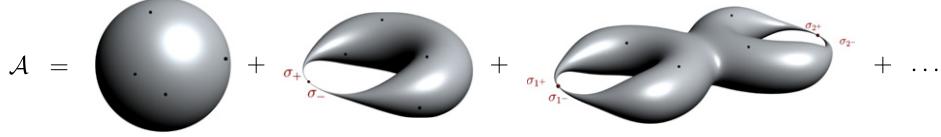


Figure 6. Nodal expansion of the amplitude. The g -loop integrand can be expressed as a fully localised integral over a g -nodal sphere.

the presence of a simple pole at the non-separating boundary divisor. Once localised on the nodal sphere, the integrand simplifies drastically, resembling closely a forward limit of an $(n+2)$ -particle tree amplitude.

To illustrate this further, consider the integrand $\mathcal{I}_n^{(1)}$ with the change of variables $q = e^{2i\pi\tau}$, designed to manifest the pole at the non-separating divisor $q = 0$. Introducing also a short-hand for better readability, the residue theorem implies that

$$\mathcal{I}_n^{(1)} := \int_{\mathfrak{M}_{1,n}} \frac{dq}{q} \bar{\delta}(u) \mathcal{I}(q) \stackrel{\text{RT}}{=} \int_{\mathfrak{M}_{1,n}} \frac{dq}{u} \bar{\delta}(q) \mathcal{I}(q) = \frac{1}{\ell^2} \int_{\mathfrak{M}_{0,n+2}} \mathcal{I}(0). \quad (5.1)$$

The first equality is the definition of the short-hand, all remaining terms in (4.8) have been absorbed into $\mathcal{I}(q)$. The residue theorem, in the second equality, relates this expression to the boundary divisor $q = 0$, and in the final equality we have used that $u = \ell^2$ on the nodal sphere. Thus, contrary to the string, the integrand $\mathcal{I}_n^{(1)}$ is localised on the boundary divisor, reflecting the field-theory status of the ambitwistor string.

While more involved, the argument can be extended to two loops [107, 129], suggesting that the loop expansion in the ambitwistor string has two equivalent representations: an expansion in the worldsheet genus as in figure 3, and a nodal expansion as illustrated in figure 6.

5.2. The integrand on the nodal sphere

On the nodal sphere, the one-loop integrand takes the following simple form:

$$\mathcal{I}_n^{(1)} = \frac{1}{\ell^2} \int_{\mathfrak{M}_{0,n+2}} d\mu_{1,n}^{(\text{nod})} \mathcal{I}^{(1)}, \quad d\mu_{1,n}^{(\text{nod})} \equiv \frac{\prod_A \bar{\delta}(\mathcal{E}_A^{(\text{nod})}) d\sigma_A}{\text{vol } \text{SL}(2, \mathbb{C}) \times \mathbb{C}^3}. \quad (5.2)$$

Here the index A runs over the particle labels $i = 1, \dots, n$, as well as the two marked points σ_+ and σ_- describing the node. The *nodal scattering equations* $\mathcal{E}_A^{(\text{nod})}$, given by the torus scattering equations $\mathcal{E}_i^{(1)}$ on the non-separating boundary, closely resemble the tree-level scattering equations for $n+2$ particles in a forward limit,

$$\mathcal{E}_\pm^{(\text{nod})} = \pm \sum_i \frac{\ell \cdot k_i}{\sigma_{\pm i}}, \quad \mathcal{E}_i^{(\text{nod})} = \frac{k_i \cdot \ell}{\sigma_{i+}} - \frac{k_i \cdot \ell}{\sigma_{i-}} + \sum_{j \neq i} \frac{k_i \cdot k_j}{\sigma_{ij}}. \quad (5.3)$$

Geometrically, they impose that the quadratic differential $\mathfrak{P}^{(1)} = P^2(\sigma) - \ell^2 \omega_{+-}^2$ vanishes globally on the sphere, where P^μ is now given by

$$P^\mu(\sigma) = \ell^\mu \omega_{+-}(\sigma) + \sum_{i=1}^n \frac{k_i^\mu d\sigma}{\sigma - \sigma_i}, \quad \omega_{+-}(\sigma) = \frac{\sigma_{+-} d\sigma}{(\sigma - \sigma_+)(\sigma - \sigma_-)}. \quad (5.4)$$

This is a consequence of the residue theorem, which traded precisely the constraint $u = 0$ for the localisation on the boundary divisor. As expected, the integrand on the nodal sphere is

thus localised on $P^2 = u \omega_{+-}^2 = \ell^2 \omega_{+-}^2$. The nodal scattering equations can then be written compactly as

$$\mathcal{E}_A^{(\text{nod})} = \text{Res}_A \mathfrak{P}^{(1)} = \text{Res}_A (P^2 - \ell^2 \omega_{+-}^2). \quad (5.5)$$

On the nodal sphere, the double copy relations provide an extraordinary ‘free lunch theorem’: although the genus-one representation of the loop integrand only exists for 10d type II supergravity (via the RNS ambitwistor string), nodal sphere expressions exist in any dimension $D \leq 10$, as well as for super Yang–Mills [127]. This is achieved by combining the forward limit interpretation of the nodal sphere formulae with the insights from the *double copy*, which motivates worldsheet integrands of the form

$$\mathcal{I}_{\text{sYM}}^{(1)} = \mathcal{I}_{\text{col}}^{(1)} \mathcal{I}_{\text{susy}}^{(1)}, \quad \mathcal{I}_{\text{sugra}}^{(1)} = \mathcal{I}_{\text{susy}}^{(1)} \tilde{\mathcal{I}}_{\text{susy}}^{(1)}. \quad (5.6)$$

The ‘kinematic’ half-integrand $\mathcal{I}_{\text{susy}}^{(1)}$ is the nodal sphere limit of the chiral loop integrand I_n^{chi} and can be rigorously derived, whereas the forward-limit interpretation of the nodal sphere formalism suggests that its colour counterpart can be obtained by ‘gluing’ the colour indices of the loop punctures in the tree-level expression. This leads to a cyclic sum over Parke–Taylor factors with the colour ‘running in the loop’, analogous to the illustration in figure 7. The half-integrands thus take the form [128]

$$\mathcal{I}_{\text{susy}}^{(1)} = \mathcal{I}_{\text{NS}}^{(1)} + \mathcal{I}_{\text{R}}^{(1)}, \quad \mathcal{I}_{\text{col}}^{(1)} = \sum_{\rho \in S_n} \frac{\text{tr}(T^{\rho(a_1)} \dots T^{\rho(a_n)})}{\sigma_+ \rho(1) \sigma_{\rho(1)} \rho(2) \dots \sigma_{\rho(n)} - \sigma_-}. \quad (5.7)$$

Due to its ambitwistor-string origin, and following the corresponding analysis in the superstring [130], contributions from individual GSO sectors can be identified in $\mathcal{I}_{\text{susy}}^{(1)}$:

$$\mathcal{I}_{\text{NS}}^{(1)} = \sum_r \text{Pf}'(M_{\text{NS}}^r), \quad \mathcal{I}_{\text{R}}^{(1)} = -\frac{c_D}{\sigma_{+-}^2} \text{Pf}(M_2). \quad (5.8)$$

In the Ramond contribution, c_D is a dimension-dependent constant, and the matrix M_2 is defined as on the torus (4.9), but with the nodal sphere Szegő kernel

$$S_2 = \sigma_{ij}^{-1} \left(\sqrt{\frac{\sigma_{i+}\sigma_{j-}}{\sigma_{i-}\sigma_{j+}}} + \sqrt{\frac{\sigma_{i-}\sigma_{j+}}{\sigma_{i+}\sigma_{j-}}} \right).$$

The NS-contribution $\mathcal{I}_{\text{NS}}^{(1)}$ manifests the forward-limit interpretation,

$$M_{\text{NS}}^r = M_{n+2}^{\text{tree}} \Big|_{\ell^2=0, \epsilon_+=\epsilon^r, \epsilon_-=(\epsilon^r)^\dagger}, \quad (5.9)$$

where the additional particles at the nodal points σ_\pm have back-to-back momenta $\pm \ell$, and the sum runs over a basis ϵ^r of polarisation vectors for these two particles.

Similar representations exist for NS–NS-gravity and pure Yang–Mills theory in various dimensions [128], obtained by replacing $\mathcal{I}_{\text{susy}}^{(1)}$ with $\mathcal{I}_{\text{NS}}^{(1)}$. The forward-limit has also been successfully used to construct nodal sphere representation for other theories, including the bi-adjoint scalar [131–134] and EYM theory [135], and played an important role in gaining a better understanding of nodal scattering equations [132] in non-supersymmetric theories.

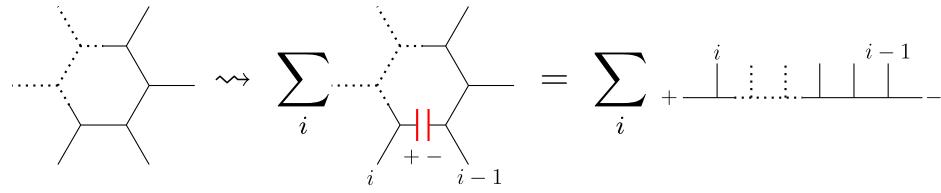


Figure 7. Interpretation of the \mathcal{J}_{lin} representation of loop integrands as $(n+2)$ -particle tree diagrams, summed over different ways of ‘cutting open’ the loop.

It is striking that in all of these formulæ, the integrand has the same simplicity as a tree-level amplitude, and is in particular a rational function of the kinematic data. The residue theorem thus resolves the puzzle of how a field theory integrand can emerge from the complicated higher-genus expressions of the last section²¹.

5.3. Representation of the loop integrand

The remarkable similarity of the loop integrand $\mathcal{J}_n^{(1)}$ to tree-amplitudes in a forward limit has another important consequence: after evaluating the moduli space integrals, the loop integrand appears in a non-standard representation on momentum space. This is already evident from the general form of the (5.2), which involves inverse quadratic powers of the loop momentum only via the overall factor ℓ^{-2} , whereas the nodal scattering equation (5.3) depend linearly on ℓ . This intuition can be made precise by factorisation arguments analogous to section 2.4, showing that the integrand contains poles at $2\ell \cdot K + K^2$ instead of the conventional Feynman loop propagators $(\ell + K)^2$ [128].

Fortunately this novel, ‘linear’ representation can be obtained from the standard loop integrand by a simple prescription, based on a deformation and residue theorem reminiscent of the BCFW recursion [138]. This can be achieved as follows: shift the loop momentum in the standard representation \mathcal{J}_{std} by $\ell \rightarrow \tilde{\ell} = \ell + \eta$, where η points in some auxiliary dimension such that $\ell \cdot \eta = k_i \cdot \eta = \epsilon_i \cdot \eta = 0$, and the Lorentz invariants are unaffected except for $\ell^2 \rightarrow \ell^2 + \eta^2 \equiv \ell^2 + \zeta$. Cauchy’s residue theorem then relates the \mathcal{J}_{std} , expressed now as the residue at $\zeta = 0$, to a sum of terms where all but one of the propagators are linear. A further shift in the loop momentum $\ell \rightarrow \ell - K_a$, where $K_a = \sum_{i \in I_a} k_i$ is the sum of external momenta in a propagator D_a , brings all remaining quadratic propagators into the form ℓ^{-2} , and gives the linear representation of the loop integrand;

$$\mathcal{J}_{\text{std}} = \sum_{\Gamma} \frac{N(\ell, \ell^2)}{\prod_{a \in \Gamma} D_a} \quad \rightsquigarrow \quad \mathcal{J}_{\text{lin}} = \frac{1}{\ell^2} \sum_{\Gamma} \sum_{a \in \Gamma} \frac{N(\ell - K_a, -2\ell \cdot K_a + K_a^2)}{\prod_{b \neq a} (D_b - D_a)|_{\ell \rightarrow \ell - K_a}}. \quad (5.10)$$

Here $D_a = (\ell + K_a)^2$ are the standard Feynman loop propagators, and we have kept the dependence on ℓ^2 explicit in the numerators $N(\ell, \ell^2)$ for better readability. At this stage, it is easily verified that the remaining propagators in \mathcal{J}_{lin} are of the form $2\ell \cdot K + K^2$ expected from the nodal scattering equations²². Schematically, the sum over different propagators a in \mathcal{J}_{lin} can be

²¹ At one loop, this is strongly reminiscent of the Feynman tree theorem [136, 137], but the nodal representations extend to higher loop order, as we will see in section 5.5.

²² In the simplest case when the numerators are independent of ℓ^2 , the residue theorem reduces to repeated partial fraction identities, followed by shifts in the loop momentum $\ell \rightarrow \ell - K_a$ as above;

given an interpretation as different ways of ‘cutting open’ the loop, with each term associated to a tree-diagram involving two back-to-back on-shell momenta $\pm\tilde{\ell} = \pm(\ell + \eta)$; see figure 7.

The procedure (5.10) serves as an algorithm for deriving the linear representation \mathcal{I}_{lin} from a standard integrand \mathcal{I}_{std} . To date, there exists no general algorithm for the reverse direction, impacting our ability to apply established integration techniques as in review chapters 1 and 3 [41, 139] for its evaluation. On the other hand, its non-standard structure also has clear advantages: as we shall see below, the forward-limit structure facilitates the extension of tree-level results to loop level, impacting the double copy at loop level (section 5.4), and allowing for scattering equations-based formulae for the bi-adjoint scalar at one loop [131–134].

$$\frac{1}{\prod_a D_a} = \sum_a \frac{1}{D_a \prod_{b \neq a} (D_b - D_a)}, \quad \text{where } D_a = (\ell + K_a)^2 \text{ and } K_a = \sum_{i \in I_a} k_i. \quad (5.11)$$

Following an alternative direction, there has also been work on obtaining standard loop integrands with Feynman propagators from the scattering equations formalism in [140–143], but at the cost of more complicated expressions, and the origin of these formulae from the ambitwistor string correlator remains unclear.

5.4. Double copy at loop level

At loop level, the double copy is conjectural, but explicit constructions exist for loop integrands in a variety of theories, [8, 144–151], see also the review chapter 2 [9]. In the ambitwistor string, the first incarnation of the double copy at loop level can already be found in the structure of the worldsheet integrands (5.6). Extending the corresponding tree-level result, the BCJ relations also embed straightforwardly [152],

$$\sum_{j=1}^{n-1} \frac{\ell \cdot k_{12\dots j}}{\sigma_{12} \dots \sigma_{j+} \sigma_{+ (j+1)} \dots \sigma_{n-} \sigma_{-1}} = 0, \quad \text{mod } E_a^{(\text{nod})}. \quad (5.12)$$

We can make contact with the double copy on momentum space by expanding both the Yang–Mills integrand \mathcal{I}_{YM} and the gravity integrand $\mathcal{I}_{\text{grav}}$ in a Dixon–Del Duca–Maltoni half-ladder basis,

$$\mathcal{I}_{\text{YM}} = \sum_{\rho \in S_n} c(+, \rho, -) \mathcal{I}_{\text{YM}}(+, \rho, -), \quad \mathcal{I}_{\text{grav}} = \sum_{\rho \in S_n} N(+, \rho, -) \mathcal{I}_{\text{YM}}(+\rho, -), \quad (5.13)$$

where $\mathcal{I}_{\text{YM}}(+, \rho, -)$ are colour-ordered Yang–Mills integrands. Both the colour factors

$$c(+, \rho, -) = f^{a+a_{\rho(1)}b_1} f^{b_1 a_{\rho(2)}b_2} \dots f^{b_{n-1} a_{\rho(n)}a_{-}} \delta^{a+a_{-}}, \quad (5.14)$$

and the kinematic numerators $N(+, \rho, -)$ are associated to cubic diagrams forming a ‘half-ladder’, with legs + and – at opposite endpoints, as on the right of figure 7. If such an expansion can be found, the integrands $\mathcal{I}_{\text{grav}}$ and \mathcal{I}_{YM} are related by replacing the colour factors by the numerators N , and so they satisfy the double copy structure. Such numerators $N(+, \rho, -)$ are also known as BCJ numerators or master numerators, since they generate the numerators for all other diagrams by Jacobi relations. For the ambitwistor string integrands $\mathcal{I}^{(1)}$, this expansion becomes

$$\mathcal{I}_{\text{col}}^{(1)} = \sum_{\rho \in S_n} \frac{c(+, \rho, -)}{\sigma_{+ \rho(1)} \dots \sigma_{\rho(n)-} \sigma_{-+}}, \quad \mathcal{I}_{\text{kin}}^{(1)} \cong \sum_{\rho \in S_n} \frac{N(+, \rho, -)}{\sigma_{+ \rho(1)} \dots \sigma_{\rho(n)-} \sigma_{-+}}, \quad (5.15)$$

and is guaranteed to exist due the one-loop KLT orthogonality [152, 153] on the support of the nodal scattering equations, as expressed by the \cong -symbol. Various strategies have been successfully used to determine the BCJ numerators N for supersymmetric and non-supersymmetric theories [152–155], often generalising tree-level methods by exploiting the forward-limit structure of the loop integrand, and some constructions have been extended to two loops [156]. The relative ease with which BCJ numerators can be constructed in the linear, ambitwistor-string inspired integrand representation stands in stark contrast to the status in the standard representation, where serious obstacles arise already for six external particles at one loop [157, 158], see also [152, 153] for a concise juxtaposition.

5.5. Two loops

The nodal sphere formalism has been successfully extended to two loops [107, 129]. In this case, the residue theorem is more subtle, but it remains true that the full integrand localises on the maximal non-separating boundary of the moduli space, corresponding to a bi-nodal sphere, parametrised by four ‘loop marked points’, one pair per node,

$$\mathcal{I}_4^{(2)} = \frac{\mathcal{K}\bar{\mathcal{K}}}{\prod_I (\ell^I)^2} \int_{\mathfrak{M}_{0,4+2g}} c^{(g)} (\mathcal{J}^{(g)} \mathcal{V}^{(g)})^2 \prod_{A=1}^{4+4} {}' \bar{\delta}(\mathcal{E}_A). \quad (5.16)$$

The resulting integrand formula, presented here for $n = 4$ particles for simplicity, takes the form of an integral over the moduli space $\mathfrak{M}_{0,n+2g}$, fully localised on solutions to the nodal scattering equations $\mathcal{E}_A = 0$. While the structure is reminiscent of the one-loop case, new features appear as well; in particular the factors $c^{(g)}$ and $\mathcal{J}^{(g)}$ arise from the degeneration of $\mathfrak{M}_{g,n}$ to $\mathfrak{M}_{0,n+2g}$. We briefly discuss these ingredients below, more detailed expositions, as well as n -point formulae for supergravity and super Yang–Mills theory (constructed again using the double copy) can be found in the original paper [107].

- *Moduli:* the residue theorem localises the integrand on the non-separating boundary, where $q_{II} = e^{i\pi\Omega_{II}} = 0$. In this limit, the remaining moduli, given by the off-diagonal components of the period matrix, become cross-ratios of the nodal marked points σ_{I^\pm} ,

$$q_{IJ} = e^{2i\pi\Omega_{IJ}} = \frac{\sigma_{I+J+}\sigma_{I-J-}}{\sigma_{I+J-}\sigma_{I-J+}}. \quad (5.17)$$

The measure is then naturally expressed in terms of all marked points (including the nodes) modulo Möbius transformation, leading to a Jacobian $\mathcal{J}^{(g)}$ with

$$\prod_{I < J} \frac{dq_{IJ}}{q_{IJ}} = \frac{\mathcal{J}^{(g)}}{\text{vol SL}(2, \mathbb{C})}, \quad \mathcal{J}^{(g)} = \frac{1}{\sigma_{1+2+}\sigma_{1+2-}\sigma_{1-2+}\sigma_{1-2-}} \prod_{I^\pm} d\sigma_{I^\pm}. \quad (5.18)$$

The analogous change of variables in the scattering equations results in another copy of the same Jacobian factor. The cross-ratio factor $c^{(2)} = 1/(1 - q_{12})$ originates in the degeneration of the moduli space $\mathfrak{M}_{2,n}$ to $\mathfrak{M}_{0,n+4}$; to be precise, from mapping the last modular parameter q_{12} to the nodal sphere [107]. It can be given a concrete physical interpretation in projecting out unphysical poles from the integrand $\mathcal{I}_n^{(2)}$.

- *Nodal scattering equations:* on the nodal sphere, P_μ takes the following form;

$$P_\mu(\sigma) = \ell_\mu^I \omega_{I+I-}(\sigma) + \sum_i \frac{k_{i\mu}}{\sigma - \sigma_i} d\sigma, \quad (5.19)$$

where $\omega_{I+I^-}(\sigma)$ are the genus-two holomorphic abelian differentials in the maximal non-separating degeneration. In this limit, these differentials acquire simple poles at the corresponding nodes:

$$\omega_I = \frac{\omega_{I+I^-}}{2\pi i}, \quad \omega_{I+I^-}(\sigma) = \frac{(\sigma_{I+} - \sigma_{I^-}) d\sigma}{(\sigma - \sigma_{I+})(\sigma - \sigma_{I^-})}, \quad (5.20)$$

The nodal scattering equations can then be compactly expressed as the vanishing of a meromorphic quadratic differential $\mathfrak{P}^{(g)}$ with only simple poles,

$$\mathcal{E}_A = \text{Res}_{\mathfrak{P}^{(g)}} \mathfrak{P}^{(2)}, \quad \mathfrak{P}^{(2)} = P^2 - (\ell^I \omega_{I+I^-})^2 + (\ell_1^2 + \ell_2^2) \omega_{1+1^-} \omega_{2+2^-}. \quad (5.21)$$

Note in particular the last term, a novel feature at genus two that plays a crucial role in obtaining the correct loop propagators²³.

- *Integrand:* the chiral integrand is defined straightforwardly by the nodal sphere-limit of the genus-two expression, $\mathcal{Y} = \mathcal{Y}^{(2)}|_{\text{nodal}}$, and can be calculated using (5.20).

5.6. Further topics

The nodal sphere formulation of loop integrands proved to be a starting point for many further exciting avenues of research.

- At the level of the worldsheet model, the simple structure of one-loop correlators, supported on a nodal sphere is reflected by the presence of a so-called ‘gluing operator’ in the ambitwistor string [159]. This gluing operator Δ encodes the propagator of the target-space field theory, and is thus a BRST-invariant but non-local worldsheet operator. Genus zero correlators with an insertion of Δ directly give the one-loop integrand formulae (5.2) localized on the nodal sphere, without need for further simplifications. Extensions to higher loops are currently not known.
- Focussing on the integrand expressions, there has been tremendous progress on extending many of the tree-level evaluation techniques to loop level, [160–163].
- The nodal sphere, and in particular the forward limit structure of the integrand, also inspired loop formulae in the twistorial models [100]. These were obtained from a forward limit of the 6d spinorial tree-level amplitudes mentioned in section 3.
- Very recently, a proposal has also appeared for one-loop correlators in massive ϕ^4 theory on de Sitter spacetime, based on the nodal sphere [164]. This builds on earlier work expressing (tree-level) de Sitter ‘cosmological correlators’ as worldsheet integrals, supported on so-called cosmological scattering equations, which are now differential operators expressed as functions of the conformal generators [165].
- Finally, the ambitwistor string progress at loop level has also inspired calculations and proposals in the full superstring. In reference [166], the authors used an ambitwistor-string-inspired method, based on forward limits of the moduli space integrals, to construct one-loop matrix elements with insertions of operators $D^{2k} F^n$ and $D^{2k} R^n$ in the tree-level effective action. Progress has also been made at higher loop orders, where the double copy and the close relation between the ambitwistor string and the superstring chiral integrands has been used to propose a formula for the three-loop four-particle superstring

²³ And is closely related to the absence of a straightforward Feynman tree-theorem at two loops.

integrand [91]. This is achieved by constructing first an expression on the three-nodal sphere using BCJ numerators, and then lifting this to a fully modular invariant proposal for the superstring chiral integrand on a $g = 3$ Riemann surface.

6. Frontiers

As we have seen, ambitwistor-strings give one of the most direct routes from a physical theory to compact formulæ for tree amplitudes and loop integrands. Despite these successes, many open questions remain and it is questionable as to whether it will one day be possible to understand these worldsheet models as providing secure basic formulations of physical theories. To consolidate them, more work needs to be done to relate them to more standard fully nonlinear formulations of physical theories, either via field theory, string theory or holography. We briefly expand on these connections.

6.1. Curved backgrounds

Amplitudes on curved backgrounds are a relatively new subject with the frontier being, until recently, at three points at tree level. They provide a stepping stone to connect with conventional nonlinear field theory. Spaces of complexified null geodesics make good sense on an analytic curved space with metric $g(x)_{\mu\nu}$ [33] and one can ask whether ambitwistor strings can be defined on such curved ambitwistor spaces. In particular in the RNS models of section 2 we can replace $P^2 \rightarrow H = g^{\mu\nu}(x)P_\mu P_\nu + \dots$. It was shown in [167] that, in the type II case, curved analogues of the constraints H , $G = \Psi \cdot P$ and $\tilde{G} = \tilde{\Psi} \cdot P$ can be constructed so that they satisfy the flat space OPEs iff they are obtained from a solution to the NS–NS-sector of 10d type II supergravity. This can be used to construct vertex operators and amplitudes at three points on a plane wave [168, 169], and one can similarly encode the Yang–Mills equations in the heterotic model [170]. This provides a completely different perspective to that pursued in the 1970s and 1980s when space–time field equations in 4d were shown [24, 34, 171] to correspond to the existence of formal neighbourhoods of \mathbb{A} inside $\mathbb{PT} \times \mathbb{PT}^*$ or supersymmetric extensions [23, 172]. In the ambitwistor string, the field equations are encoded in the quantum consistency of the worldsheet model on a curved ambitwistor space. In all these cases and unlike the conventional string, one can use field redefinitions to make the gauge-fixed action linear, so that correlators are relatively easy to compute. However, the complexity of the curved background constraints makes it problematic to identify the generic integrated vertex operators.

The 4d twistor models and twistor-strings can also be defined on curved backgrounds built from curved twistor spaces. These are equivalent to curved but self-dual Yang–Mills or E backgrounds, which are integrable. Exploiting this integrability, one can construct amplitudes of arbitrary multiplicity [83, 173, 174], with explicit formulæ for certain classes of backgrounds.

6.2. Relationship with conventional, null and chiral strings

All the amplitude formulæ we have discussed localize on the solutions to the scattering equations. These equations were first obtained by Fairlie [175, 176] in a semiclassical study of string solutions, and made famous in the work of Gross and Mende [177, 178], where they were shown to govern the string path integral in the limit of high energy scattering at fixed angle ($s \gg 1/\alpha'$). The scattering equations thus play a prominent role both in the $\alpha' \rightarrow \infty$ limit and in the field theory limit $\alpha' \rightarrow 0$, albeit indirectly via the ambitwistor string. This clash, with the scattering equations appearing in both the low tension and high tension field theory limit, has

so far impeded attempts to connect the ambitwistor string directly with conventional strings. In this context, it is interesting to note that the $\alpha' \rightarrow \infty$ limit can also be understood as a null limit in which the worldsheet becomes null ruled by null geodesics [124].

More generally Siegel proposed that so-called chiral string theories could be obtained by flipping certain worldsheet boundary conditions [123], so that both left-moving and right-moving modes in the conventional string become holomorphic on the worldsheet in the sectorized chiral string. A bosonic such model had already been introduced in [179], and pure spinor versions discussed in [180]. In these models both the left moving and right-moving Virasoro constraints become holomorphic but commute with each other and each satisfies the holomorphic Virasoro algebra. Ambitwistor have a similar character in the sense that X^μ and $\int^\sigma P^\mu$ can be thought of as being independent holomorphic functions on the worldsheet playing roles as different combinations of left and right movers. However, the ambitwistor-string only contains one holomorphic Virasoro generator, $P \cdot \partial X$. The gauged constraint P^2 is analogous to difference between the two Virasoro generators, but has trivial OPE with itself, even on a curved background. This reflects the degeneration of the two copies of the Virasoro algebra to a Galilean conformal algebra [124, 125]. On the other hand, in the model of [179], the gauged constraint P^2 is replaced by a more general quadratic expression in P and ∂X

$$P^2 \rightarrow \mathcal{H} := A_{\mu\nu}(X)P^\mu P^\nu + B_{\mu\nu}P^\mu \partial X^\nu + C(x)_{\mu\nu}\partial X^\mu \partial X^\nu, \quad (6.1)$$

that is constrained to obey a nontrivial OPE. The imposition of these OPEs yields field equations for the background with a finite number of α' corrections. However, since the OPE of \mathcal{H} with itself is nontrivial, there no longer appears to be a reduction to ambitwistor space nor localization on the scattering equations. The connections between sectorized strings, null strings and ambitwistor strings are now well studied in different models [124, 125, 180, 181], see also [182, 183]. For example, versions of T -duality become possible in sectorized strings whereas they are not in the ambitwistor-string [184].

The amplitude formulæ to which such sectorized chiral strings give rise appear to be problematic. The Koba–Nielsen factor of the conventional string, consisting of a product $\prod_{i < j} |\sigma_{ij}|^{\alpha' s_{ij}}$, is replaced by a product $\prod_{i < j} (\sigma_{ij}/\bar{\sigma}_{ij})^{\alpha' s_{ij}}$ so that the branching would seem to make the contour prescription problematic. This is resolved in the work of Mizera who uses twisted cohomology and residues to define the amplitudes of these theories [185]. As $\alpha' \rightarrow \infty$ he shows localization on the scattering equations.

6.3. Celestial holography and soft theorems

Celestial holography [186] seeks to understand the S -matrices of massless theories by re-expressing them as correlation functions for theories on the conformal boundary of asymptotically flat space-times, \mathcal{I} , the *light-cone at infinity*. This approach emerged from the study of connections between soft theorems for amplitudes and asymptotic BMS symmetries of space-time [187, 188], and aims to establish a holographic dictionary for the S -matrix from \mathcal{I} or the celestial sphere [189, 190]. In an asymptotically simple space-time, all light rays reach \mathcal{I} and we can represent ambitwistor space $\mathbb{A} = T^* \mathcal{I}_{\mathbb{C}}$ as the cotangent bundle of the complexification of null infinity. Both the RNS and 4d twistor ambitwistor models can be expressed in this representation [191–193] and from this perspective, the connection between soft theorems and asymptotic symmetries can be understood directly at the level of the vertex operators as generators of supertranslations and super-rotations in their soft limits. The ambitwistor strings then provide an underpinning theory for the generation of so-called *celestial amplitudes*— S -matrix elements based at \mathcal{I} , with asymptotic states in a conformal primary basis on

the $(d - 2)$ -sphere of null generators of \mathcal{I} , [193]. In this formulation the recently discovered gravitational ‘ $w_{1+\infty}$ -symmetry’ [194] can be seen as arising directly from the geometry of the asymptotic twistor space at \mathcal{I} [195]. With this, the worldsheet OPE of the ambitwistor string vertex operators generate the tree-level celestial OPE coefficients of the conformal primaries [195–197].

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Data availability statement

No new data were created or analysed in this study.

Appendix A. Penrose transform

The Penrose transform for a general ambitwistor space \mathbb{PA} maps spin s fields on space–time to elements of $H^1(\mathbb{PA}, \mathcal{O}(s - 1))$, with the classes corresponding to spin s plane-waves of the form $\epsilon_{\mu_1} \dots \epsilon_{\mu_s} e^{ik \cdot X}$ mapping to $(\epsilon \cdot P)^s \bar{\delta}(k \cdot P) e^{ik \cdot x}$.

Briefly, this follows by taking a cohomology class $\phi \in H^1(\mathbb{PA}, \mathcal{O}(s - 1))$ on \mathbb{PA} , pulling it back to $p^* \phi$ on $PT^*M|_{P^2=0}$, the lightcone inside the projective cotangent bundle, where it becomes necessarily trivial as there is no first cohomology on this space for the given homogeneity weights. Thus we can find g of weight $s - 1$ on $T^*M|_{P^2=0}$ such that

$$p^* \phi = \bar{\partial} g, \quad \text{e.g. for } \phi \text{ in (2.17)} \quad g = (\epsilon \cdot P)^s \frac{e^{ik \cdot X}}{k \cdot P}. \quad (\text{A.1})$$

Then we can obtain the field via

$$(\epsilon \cdot P)^s e^{ik \cdot x} = P \cdot \partial_X g. \quad (\text{A.2})$$

Since g is defined up to the gauge freedom, $\delta g = (\epsilon \cdot P)^{s-1} e^{ik \cdot X}$ (or indeed some a general global holomorphic function of P), we have the usual gauge freedom

$$\delta(\epsilon_{\mu_1} \dots \epsilon_{\mu_s} e^{ik \cdot X}) = k_{(\mu_1} \epsilon_{\mu_2} \dots \epsilon_{\mu_s)} e^{ik \cdot X}. \quad (\text{A.3})$$

This can be proven more abstractly as follows. The Penrose transform is the connecting map

$$H^0(PT^*M|_{P^2=0}, \mathcal{O}(s))/P \cdot \partial_X \times (H^0(PT^*M|_{P^2=0}, \mathcal{O}(s - 1))) \xrightarrow{\delta} H^1(\mathbb{PA}, \mathcal{O}(s - 1)), \quad (\text{A.4})$$

from the long exact sequence in cohomology arising from the short exact sequence:

$$0 \rightarrow \mathcal{O}(s-1)_{\mathbb{PA}} \rightarrow \mathcal{O}(s-1)_{T^*M_{P^2=0}} \xrightarrow{P \cdot \partial_X} \mathcal{O}(s)_{T^*M_{P^2=0}} \rightarrow 0 \quad (\text{A.5})$$

that defines the functions on \mathbb{PA} , see [22, 34] for a full discussion.

Appendix B. Conformal field theory basics

In this appendix, we present a lightning review of some aspects of chiral two-dimensional CFTs particularly relevant for the ambitwistor string. For a more extensive introduction, we refer the interested reader to the multitude of excellent textbooks, for example [28, 29, 118, 198–200].

All ambitwistor string models are two-dimensional CFTs with a local action in terms of a set of fields defined over a closed Riemann surface Σ , referred to as the worldsheet. These fields can be characterized by their statistics (bosonic vs fermionic) and their conformal weight, as well as additional quantum numbers such as the spin structure at higher genus. The conformal weight of a field Φ is a pair of half-integers $(h, \bar{h}) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ labeling the transformation properties of Φ under two-dimensional conformal transformations²⁴. Since 2d local conformal transformations are equivalent to holomorphic coordinate transformations, we can identify the conformal weight with the form degree. This means that Φ is section of (h, \bar{h}) powers of the holomorphic and antiholomorphic canonical bundles $K_\Sigma^h \otimes \bar{K}_\Sigma^{\bar{h}}$, i.e.

$$\Phi \in \Omega^0(\Sigma, K_\Sigma^h \otimes \bar{K}_\Sigma^{\bar{h}}). \quad (\text{B.1})$$

Here, negative weights are to be interpreted as sections of the respective tangent bundles, using the isomorphism $K_\Sigma^{-1} \cong T_\Sigma$. Equivalently, Φ may be expressed locally in affine worldsheet coordinates z as

$$\Phi(z, \bar{z}) = \phi_{h, \bar{h}}(dz)^h (d\bar{z})^{\bar{h}}. \quad (\text{B.2})$$

In the ambitwistor string, we will only encounter *chiral* CFTs, known as $\beta\gamma$ -systems (for bosons) or bc -systems (for fermions). We will describe these jointly below, keeping track of the statistics via a variable ϵ , with $\epsilon = -1$ for fermionic statistics, and $\epsilon = 1$ for bosons. A chiral CFT is then defined by the action

$$S = \frac{1}{2\pi} \int b \bar{\partial} c, \quad (\text{B.3})$$

in conformal gauge. The fields b has conformal weight $(h, 0)$, giving conformal weight $(1-h, 0)$ to the conjugate field c . Fields such as these with conformal weight $\bar{h} = 0$ are often referred to as ‘left-moving’. The OPE between these conjugate fields is

$$c(z) b(w) \sim \frac{1}{z-w}, \quad b(z) c(w) \sim -\frac{\epsilon}{z-w}. \quad (\text{B.4})$$

A standard calculation gives the holomorphic stress–energy tensor

$$T_{bc} = -h b \bar{\partial} c + (1-h) (\partial b) c. \quad (\text{B.5})$$

From this expression, we find the central charge anomaly as (twice) the coefficient of the fourth order pole in the $T(z)T(w)$ OPE,

$$\mathfrak{c} = 2\epsilon(6h^2 - 6h + 1). \quad (\text{B.6})$$

²⁴A field is called *primary* if its conformal weights are well-defined.

Finally we note the following useful formula for the number of zero modes n_b and n_c of the two fields on a Riemann surface of genus g , derived via the Riemann–Roch theorem,

$$n_c - n_b = \frac{1}{2}(2h - 1)\chi, \quad \text{with } \chi = 2(1 - g). \quad (\text{B.7})$$

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