Doctoral Thesis

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# Chern Simons effective actions
# &
# 't Hooft anomalies

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# Ringraziamenti[1]

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Durante questi quattro anni ho avuto la fortuna di ricevere il supporto e l'affetto di varie persone. Questa é una lista per forza di cose incompleta, quindi se mi sono dimenticato di voi non sentitevi offesi, ma fatemelo sapere con sottili frecciate nei canali adeguati...

In primo luogo ringrazio la mia famiglia per il supporto durante gli studi e aver accettato la mia bizzarra vocazione per la Fisica.

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Un ringraziamento va anche ai vari (ex) studenti dell'IFT, in particolare Diego (instancabile compagno di Ping Pong e creature delle ore notturne del centro) e David (per il suo "particolare" approccio alla vita e per la sua guida alle vie del risparmio estremo) e ai miei compagni di pranzo preferiti: Javi Laguna, German Sierra e la banda dei matematici centauri (Alberto[2], Victor, Alex, ...) per le lunghe conversazioni sui piú svariati argomenti e la sopportazione del mio difficile carattere.

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[1]This part is written in Italian. I did so because there is no language that encompasses all of the people and I do not like English too much, as many know. Luckily, I can assure you that great translation applications are present on the Web. I have tested them.

[2]Per questo gli sono ancora debitore di una cena.

Questa tesi é frutto di tutte le vostre influenze e delle mie imperfezioni, e direi, per concludere con una nota sardonica[3]:



It ain't much but it's honest work

Buona Lettura,
Christian

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[3]Al momento della scrittura sono due mesi e mezzo che vivo in campagna per sfuggire al coronavirus, da qui il contesto "bucolico" dell'immagine.

# Author's declaration

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I declare that this thesis is the result of my own original work and that it has not been submitted, in whole or in part, for any other degree or academic award. In cases where the work of others is presented, appropriate citations are used. The thesis is based on the following publications:

1. **Anomalous hall viscosity at the weyl semimetal/insulator transition**, C.C. and Karl Landsteiner, *Phys.Rev.B 99 (2019) 19, 195146.* (Chapter 4)

2. **Torsion and anomalies in the warped limit of Lifshitz theories**, C.C. , *JHEP 01 (2020) 190.* (Chapter 4)

3. **Higher spin vortical Zilches from Kubo formulae**, C.C. and Jorge Fernández-Pendás, *Phys. Rev. D, 98(10):105008, 2018.* (Chapter 5)

4. **Thermal Chern-Simons terms from bulk extensions**, C.C. and K. Landsteiner, *To appear.* (Chapter 2)

Furthermore, during my PhD I have also co-authored the following papers:

1. **Axial Hall effect and universality of holographic Weyl semi-metals**, C.C., Jorge Fernández-Pendás, and Karl Landsteiner, *JHEP, 02:138, 2017.*

2. **Anomalous transport and holographic momentum relaxation**, C.C., Jorge Fernández-Pendás, Karl Landsteiner and Eugenio Megías, *JHEP 09 (2017) 004*

3. **Membrane paradigm and RG flows for anomalous holographic theories**, C.C. and Jorge Fernández-Pendás, *JHEP, 04:134, 2018.*

# Summary

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This thesis is devoted to several ramifications of the study of the effects of 't Hooft anomalies on the low energy physics of quantum systems. While primarily used through their matching condition to restrict the possible nature of the infrared degrees of freedom of a quantum theory, in recent years an increasing amount of literature has been developed linking such anomalies with (measurable) low energy transport properties. In turn, most of these can be characterized by time independent (i.e. Euclidean) response theory, which has a natural interpretation in terms of an Euclidean path integral on manifolds of the type $\mathcal{M}_{d-1} \times S^1$ in the presence of sources for the (anomalous) global symmetries. Since the $S^1$ reduction is expected to give rise to a local effective action, it is possible, at first order in derivatives, that Chern-Simons terms on $\mathcal{M}_{d-1}$ may arise. This reformulates the problem of anomalous transport in terms of effective field theory.

These types of problems have attracted quite a lot of attention by diverse communities, notably:

- The theoretical condensed matter community, which is interested in such phenomena as universal signature of possible topological phases of real materials (e.g. Weyl semi-metals)

- The QCD/Holography communities, which are interested in the resilience of these phenomena at strong coupling and their possible relevance for hadronic physics.

- The theoretical high-energy community, which is interested in the way the universal Chern-Simons terms may impact the evaluation of partition functions of anomalous theories.

The report will be centered around various applications and extensions of said paradigms which I have studied during these years of research, in particular:

- The extension of universality arguments for thermal Chern-Simons terms in the presence of gravitational anomalies. This topic has been most elusive due to the high order at which gravitational anomalies enter the derivative expansion. However it has been suggested that examining the theories on curved backgrounds avoids this problem allowing for an effective field theory formulation. We circumvent the need for a curved background by carefully analyzing the Callan-Harvey type inflow mechanism for Lorentz anomalies. This gives a neat derivation of how gravitational Chern-Simons terms in the gravito-magnetic potential field may arise and it is explained in Chapter 2.

- The issue of anomaly-induced transport in non-relativistic systems. This is studied in Chapter 4 and comprises both the direct study of a particular class of fermionic critical points, which have been pointed out to possibly have interesting transport signatures, as well as a more general study of a class of (warped) non-relativistic theories through standard tools of anomalous physics. The two studies match in their region of overlap.

- The extension of this paradigm to higher spin towers of conserved currents, which is discussed in Chapter 5. We point out some interesting examples in two and four dimensions and briefly discuss how these may be interpreted as mixing between current operators in non-trivial backgrounds.

Apart from these topics, we also have two introductory/review Chapters (1 and 3) to lay down some foundational tools that will be needed in the rest of the exposition. We have decided also to render each Chapter self-complete, with Appendices appearing at the end of individual Chapters and containing burdensome computational details.

We conclude each Chapter with a small list of possible future studies, according to the author's preferences.

It has to be said, that this thesis will reflect the style of the author, that is it will contain many excursions about the intuition and the ideas behind the construction, while we will avoid getting into gory details if not needed. Some formulas may thus be slightly imprecise, but in a way that does not affect the final results.

# Resumen

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Esta thesis se centra en varias ramificaciones del estudio de los efectos de las anomalías de 't Hooft sobre la fisica de baja energía de los sistemas cuanticos. Utilizadas en pasado a través de su conservación con el fin de restringir el espectro de las posibles particulas en el infrarrojo, durante los ultimos años varios articulos han relacionado la presencia de estas anomalias con propriedades de transporte medibles a bajas energias. La mayor parte de ellas puede ser caracterizada a traves de teoria de respuesta independiente del tiempo (i.e. Euclidea), que se puede interpretar naturalmente en terminos de integrales de camino sobre variedades del tipo $\mathcal{M}_{d-1} \times S^1$ con fuentes encendidas para las simetrias anomalas. Al ser esperable que la reduccion sobre $S^1$ de lugar a una acción efectiva local, es posible que terminos de Chern-Simons sobre $\mathcal{M}_{d-1}$ aparezcan a primer orden en derivadas. Esto permite reformular el problema del transporte anomalo en terminos de teoria efectiva.

Estos tipos de problemas han atraido mucho interest por varias comunidades, por ejemplo:

- La comunidad de materia condensada teorica, que está interesada en tales fenomenos como signales universales de posibles fases topologicas en materiales reales (por ejemplo, semimetales de Weyl).

- La comunidad de QCD/Holografía, que está interesada en la resiliencia de estos fenomenos en regimenes de fuerte acoplo y su posible relevancia para la fisica de los hadrones.

- La comunidad de Fisica Teorica de altas energias, que quiere estudiar la manera en que estos terminos de Chern-Simons pueden impactar a la evaluación de, por ejemplo, las funciones de partición en teorias anomalas.

Este texto se centrará en vairas aplicaciones y extensiones de dichos paradigmas, que he estudiado durante estos años de investigación, en particular:

- La extensión de los argumentos para la universalidad de los terminos "termicos" de Chern-Simons derivados por anomalias gravitatorias. Este topico ha sido bastante problematico por el alto contenido de derivadas de dichas anomalias. Sin embargo, ha sido sugerido que poner estas teorias sobre fondos curvos pueda resolver estos problemas y permitir la formulación de una teoria efectiva. Evitaremos la necesidad de un fondo curvo analizando con cuidado el mecanismo de influjo de Callan y Harvey en el caso de las anomalias de Lorentz. Esto nos permitira dar una interesante derivación de como estos terminos de Chern-Simons puedan ser derivados y se explicará en el Capitulo 2.

- El problema de transporte anomalos en sistemas no-relativistas. Esto se estudiará en el Capitulo 4 y comprenderá el estudio directo de una clase particular de punto criticos fermionicos, que han sido ipotizados tener interesantes fenomenos de transporte, así como un estudio más general de una clase de dichas teorias no-relativistas con instrumentos de anomalias cuanticas. Los dos estudios estarán de acuerdo en las regiones de coincidencia.

- La extensión de dichos paradigmas a torres de más alto espín de corrientes conservadas, que se discute en el Capitulo 5. Demonstraremos algunos ejemplos interesantes en dos y cuatro dimensiones y discutiremos su interpretación en terminos de mezcla entre las corrientes en fondos no triviales.

Además de estos topicos, también tendremos dos Capitulos introductivos (1 y 3) para introducir algunos conceptos y intrumentos que serán utiles en el resto de la esposición. Hemos decidido hacer cada Capitulo autocontenido a través de Apendices en su final que recogen los detalles más tediosos de las computaciones.

Cada Capitulo se concluye con un pequeño listado de posibles estudios futuros, según las preferncias del autór.

Hay que decir que esta tesis reflejará el estilo del autor, es decir, contendrá muchas excursiones sobre la intuición y las ideas detrás de la construcción, mientras que evitaremos entrar en detalles sangrientos si no es necesario. Por lo tanto, algunas fórmulas pueden ser ligeramente imprecisas, pero de una manera que no afecta los resultados finales.

# Contents

# Chapter 1

# A Review of 't Hooft anomalies

## 1.1    Introduction

Most of this thesis will be concerned with consequences of 't Hooft anomalies for the low energy behavior of Quantum Field Theories. This Chapter serves first as a soft introduction on the subject and as a review of generic results about the form of the effective action for anomalous theories. This first Chapter will be a quick review of well known methods and properties of anomalies, which will be put to use in the second Chapter and in various parts of the rest of this thesis. The focus is onto introducing methods and ideas which are drawn upon in this work, rather than giving a complete overview of the subject, which would require a thesis by itself.

Let us begin by stating what an anomaly is. This term is often origin of confusion in the Quantum Field Theory literature. We will be interested in theories which have either a global symmetry group $G$, or a gauge group $\mathcal{G}$. We will denote *dynamical* gauge fields by lower-case letter $a, b, c$ and external ones by upper case $A, B, C$. Since we mostly work with continuous symmetries, we assume the existence of a current operator $j^\mu$ which acts as the generator of said symmetry. The case of discrete symmetries can be phrased along the lines of the discussion for 't Hooft anomalies towards the end of the list.

We will distinguish these three types of anomalies:

**ABJ Anomalies**, introduced first in the seminal papers [1, 2]. They refer to the possibility that quantum fluctuations of *dynamical* gauge fields may break a classical *global* symmetry. This is translated into a statement about the Noether current $j^\mu$ which no longer fulfills the usual Ward identities; in particular

$$D \star j = c_{ABJ} P(a) \,, \tag{1.1}$$

as an operator statement. It is very important that both sides are quantum operators. Using the equation above in correlators leads to expressions for the anomaly involving simple Feynman graphs. ABJ anomalies lead to the impossibility of quantizing certain theories while maintaining all of their (classical) symmetries.

**Gauge Anomalies** refer to the case in which a classical *gauge* symmetry is broken by quantum fluctuations, that is:

$$D \star j_{\text{gauge}} = c_{\text{gauge}} P_{\text{gauge}}(a) \,, \tag{1.2}$$

again, as an operator statement. Since the conservation of a gauge current is a consistency condition which follows from quantizing the classical equations of motion for the gauge field, the presence of such anomalies spoils the consistency of the quantum theory. In the perturbative (Feynman) expansion, this is reflected in a lack of renormalizability due to the impossibility of applying the Ward identities. The cancellation of gauge anomalies is thus a consistency condition for any physical theory. Another way of saying it, is that the gauge current implements a constraint on the Hilbert space of the quantum theory, conservation renders this constraint consistent and the gauge anomaly spoils such consistency. A more conservative statement is that the gauge field acquires a further propagating degree of freedom, can become massive, and it is not clear whether the theory may still be renormalizable in such a regime. For our purposes cancellation of gauge anomalies can be thought as a set of consistency requirements on quantum field theories.

**'t Hooft Anomalies**. These are, in a certain sense, a more general construction than the previous two, since both ABJ and gauge anomalies may be obtained by gauging parts of the global symmetry group in question. They concern the fate of the *global* symmetries of the theory upon the introduction of *external* gauge fields $A$ for them. The most general definition, which applies to both continuous and discrete symmetries, is to look at the properties of the partition function $Z[A]$ under the gauge transformation $A \to A^\alpha$, where $A^\alpha = A + D\alpha$ for continuous symmetries.

Generally the partition function ought to be invariant under such redefinition due to current conservation, however, in the presence of 't Hooft anomalies it is allowed to vary by a phase

$$Z\left[A^\alpha\right] = \exp\left(i \int c_{\text{t Hooft}} \alpha \, P(A)\right) Z\left[A\right],\tag{1.3}$$

which makes the modulus of the partition function a well defined object, while the phase is not[1].

For continuous symmetries the above condition is most easily stated in terms of the generating functional $W[A] = \log(Z[A])$ as

$$\delta_\alpha W[A] = i \int c_{\text{t Hooft}} \alpha \, P(A) \equiv i \mathcal{A}_\alpha(A).\tag{1.4}$$

For a continuous symmetry (1.3) can be translated as a statement about the one-point function of the divergence of the current operator by expanding to linear order in $\alpha$

$$\langle D \star j \rangle = c_{\text{t Hooft}} \, P(A).\tag{1.5}$$

Notice that this is not an operator equation, however one may turn it into such by summing over the $A$s, thus gauging the symmetry. In this sense 't Hooft anomalies are an *obstruction* to the gauging of a global symmetry.

Another equivalent statement comes from taking functional derivatives with respect to $A$ of (1.3) and then setting $A$ to zero. This gives rise to the well known fact that current correlators of the form $\langle D_\mu j^\mu j^\nu j^\rho ... \rangle$ are non-vanishing at coincident points. Thus, in a theory with

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[1]The reason for this is that the modulus can usually be defined by Pauli-Villars regularization. Such statement ceases to be true if the theory is no longer CPT invariant, as is the case when one introduces complexified gauge backgrounds. This seems to be useful in some recent applications [3]

t'Hooft anomalies, only contact terms of the current operators are affected, this constitutes a characterization of anomalies as a UV phenomenon.

It is important to stress that the symmetry in a theory with 't Hooft anomalies is not broken as in the ABJ case: local operators[2] continue to transform in irreducible representations of the symmetry group and the Hilbert space carries the quantum numbers of the global symmetry. In this thesis we will be always working with 't Hooft anomalies. From now on we will omit such prefix and refer to them only as "anomalies".

Let us now state various pieces of lore about these anomalies, the ones most relevant for this Chapter will be further discussed in the next Section.

- Anomalies for continuous symmetries can be computed perturbatively using standard Feynman diagram techniques. They turn out to exist only in even space-time dimension $d = 2m$, with $P(A)$ a polynomial starting at order $m$ in $A$ and its derivatives (in the abelian case $P(A) = (dA)^m$). Taking functional derivatives one finds that the coefficient $c_{\text{'t Hooft}}$ is given at one loop by an (m+1)-gon diagram of currents (in four dimensions it is the famous triangle diagram). This turns out to be linearly divergent and must be regularized. The finite remainder is scheme independent and gives the anomaly. Furthermore, it can be shown that this computation at one-loop is actually exact, this can be explained by noticing that $c_{\text{'t Hooft}}$ is a quantized function of the charges of the elementary particles [4, 5] (in a weakly coupled description), it thus can only change in a discontinuous way. The one loop term is the only one independent of the continuous parameter $\hbar$ and thus it is the only term that may contribute. Of course a real proof of this statement involves careful analysis of the possible higher loop diagrams.

- In a quantum theory the generating functional is only defined up to the addition of local counter-terms. Thus it is plausible that particular counter terms exist (which are not gauge invariant) that make the anomaly to vanish. Only when such counter-terms do not exist we may really talk about a 't Hooft anomaly. In other words, the equations (1.4) above is defined *modulo* gauge variations of local functionals of $A$. This observation prompts examining the problem of classifying possible anomalies as a cohomological one, whose solution is discussed in the next Section. This analysis has a further implication however, that, even for anomalous theories, a local counter-term living in one dimension higher that cancels the anomaly may always be found. This procedure is called the inflow mechanism [6]. In modern language, one may think as $d + 1$ dimensional space-time as hosting a very massive quantum theory, which at low energy generates the required functional as a topological field theory upon integrating out the massive degrees of freedom.

- While ABJ and gauge anomalies essentially give "destructive" results about the fate of the symmetries of a quantum theory, 't Hooft anomalies are very useful in that they give constructive input about the structure of interacting theories. Indeed, following the original idea by 't Hooft, suppose we are given a UV free Quantum Field Theory. At

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[2]To be precise, care is needed in dealing with current operators, indeed we will see later that defining currents as functional derivative of the effective action leads to objects which do not transform in the correct way under the anomalous symmetry. This can however be remedied by fixing the contact terms in the current-current correlators to define covariant currents.

high energy it can be described as a set of weakly interacting particles, say $\Psi_i$, which however become strongly interacting at lower energies. In the UV we may compute 't Hooft anomalies for its global symmetry $G$ at one-loop. Then, we may add a set of massless free fields $\tilde{\Psi}_a$ which are also charged under $G$, but precisely cancel the 't Hooft anomaly of the rest of the matter content[3]. We can then gauge the $G$ symmetry, since it is anomaly-free and flow into the strongly interacting, low energy, regime. Here we find the same free particles $\tilde{\Psi}_a$ but a different effective description for the $\Psi_i$. However we still can assure that the $G$ current is anomaly-free, since it is gauged, and that the contribution to its anomaly from the $\tilde{\Psi}_a$ particles is the same as in the UV. Thus we conclude that, at low energy, the theory for the $\Psi_i$ still has the same 't Hooft anomaly, that is, the coefficient $c_{\text{'t Hooft}}$ is an RG invariant. This is usually called "'t Hooft anomaly matching". It is a nice and interesting question, object of much recent studies, to classify the possible ways in which a certain anomaly may be matched. This gives insights on the possible G-preserving RG flows that the theory may be subject to. In particular, one may try and ask whether a topological theory in the infrared may saturate the anomalies [7].

- While we have only talked about anomalies involving only one symmetry group $G$, it is often the case in theories with $G = G_1 \times G_2$ global symmetry that mixed anomalies between $G_1$ and $G_2$ may arise. What this means is that the current (non)-conservation equations take the form

$$D \star j_{1,2} = c_{1,2} P_{1,2}(A_1, A_2) \,. \tag{1.6}$$

  In these cases it is always possible to find local counter-terms so that either $c_1$ or $c_2$ vanishes. What these equations imply is that it is not possible to gauge *both* $G_1$ and $G_2$ at the same time in a consistent way. Upon gauging one symmetry the other is broken by an ABJ anomaly. A prime example is the usual massless Dirac fermion, which classically has a $U(1)_V \times U(1)_A$ global symmetry. There is however a well known mixed anomaly between the two $U(1)$s. Preserving the Maxwell vector symmetry $U(1)_V$ and gauging it leads to an ABJ anomaly for the axial current, which breaks the axial symmetry down to a discrete subgroup $\mathbb{Z}_{2q}$, with $q$ the charge of the fundamental fermion.

- It is also common (but ultimately inexact) lore that anomalies are only tied to theories with fermionic degrees of freedom. This will be largely true for the theories we will examine, and it is the historical way in which perturbative anomalies were computed using the Fujikawa method (which we review in the next Section). Two important exceptions are anomalies for discrete symmetries and anomaly matching in the case of bosonization-type of duality. In this last case the bosonic theory must also reproduce the anomalies on the fermionic side. A simple and notable example is the chiral anomaly in $d = 2$ which is matched by the free boson CFT through a mixed anomaly between momentum and winding $U(1)$ symmetries.

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[3]As long as the UV description is weakly coupled these are guaranteed to exist, since they are basically the complex conjugate of the $\Psi_i$ with interactions turned to zero. If the UV theory is also strongly interacting it is less clear that such a system always exists.

The classification given above is not precise nor complete. For example conformal anomalies do not seem to come by inflow from one dimension higher nor satisfy the 't Hooft matching conditions, instead they can be shown to be monotonous functions under Lorentz preserving RG flows [8, 9]. Another interesting aspect not included in our discussion are higher form symmetries [10], 2-Group symmetries [11, 12], in which the "anomaly" is given by a combination of external and dynamical gauge field $D \star j = c_2 P(A) Q(b)$, in this case the symmetry is maintained in the absence of external sources, however the OPE of the global current operators is modified so that they may fuse in the $Q(b)$ operator, which is usually a topological current for extended excitations.[4]

The rest of the Chapter, which is organized as follows: in Section 1.2 we give a systematic description of 't Hooft anomalies for continuous symmetries and generalities for their computation. This still does not give not a complete list of methods, since we leave of some historically important points of view such as the triangle diagrams and Schwinger's point-splitting regularization. The main concern has been to introduce methods which will be relevant for the rest of the thesis. Very good review books are available, for example [13, 14, 15] and lecture notes [16, 17, 18]. The formalism developed will be used in the next chapter 2 where we explain how anomalies constrain the low energy effective action and give general methods to compute such dependence. This gives us the opportunity to review the corresponding literature and to state some interesting results about gravitational anomalies. We conclude with remarks about possible open directions.

## 1.2 How to compute Anomalies

The primary objective of this section is to explain the mathematical structure behind anomalies, that is the possible form of $P(A)$ and to give practical methods to determine the "anomaly coefficient" $c_{\text{'t Hooft}}$.

While the first task is beautifully encoded in the solution of a series of consistency requirements for the effective action $W[A]$ which can in turn be converted in a cohomological problem following Wess and Zumino [19], the second task has no universal solution. For theories with a perturbative description Feynman diagrams are always available, but for the theories we are interested in, which will all involve fermionic excitation, a more elegant method has been proposed by Fujikawa [20].

While the Wess-Zumino conditions are purely geometric-algebraic equations, the Fujikawa method requires regularizing divergent expressions. This may be done in various ways, but it turns out that the computations simplify drastically if one uses a gauge invariant regulator.

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[4]Notice that above we have used "anomaly", indeed the conservation law in the two group case should *not* be viewed as an anomaly, but rather as a different group symmetry. Indeed the partition function may be made gauge invariance by introducing a gauge field $B_Q$ for the topological current and suitably changing its transformation law to cancel the "anomalous" variation. This new transformation law goes under the name of "two group symmetry". This is somewhat similar to what happens with axions in the case of ABJ anomalies. Another way of saying this is that the anomaly is the solution to a cohomological problem, which may be trivialized by enlarging the space of fields. These two group structures may still be anomalous in the usual sense, by having the partition function not be gauge invariant under the full two-group by a complex phase.

This however has an unwanted consequence in that the partition function thus defined will not satisfy the Wess-Zumino equations.

Another way to phrase the problem is that current operators, in the presence of an external gauge field $A$, are susceptible to redefinitions by polynomials in the external fields:

$$j^\mu \to j^\mu + P_{BZ}^\mu(A) \,, \tag{1.7}$$

in the theory without external gauge fields, such redefinitions correspond to different choices of contact terms in current correlators. Let us stress that these redefinitions may or may not be expressible as the addition of local counter-terms to the effective action.

Now, in the presence of 't Hooft anomalies, there is no reason for these polynomials to be gauge invariant and indeed one can define two interesting sets of current operators:

**Consistent currents** $\mathcal{J}^\mu$, which are defined as being functional derivatives of the effective action $W[A]$ satisfying the Wess-Zumino equations:

$$\langle \mathcal{J}^{\mu_1}..\mathcal{J}^{\mu_n} \rangle = -i\frac{\delta}{\delta A_{\mu_1}}... - i\frac{\delta}{\delta A_{\mu_n}}W[A] \,, \tag{1.8}$$

which are however *not* gauge invariant operators as we will shortly see.

**co-variant currents** $J^\mu$ which are gauge invariant operators, defined through a co-variant regularization of the Quantum Field Theory of interest, but cannot be written as functional derivatives of the effective action $W[A]$.

As we have already mentioned, the two choices may be related by a local polynomial in the gauge fields $P_{BZ}^\mu$, called a Bardeen-Zumino polynomial. From the definition of co-variant currents, it follows that $P_{BZ}^\mu$ cannot arise as a local counter-term in $d$ dimensions.

With these definitions in mind, we can now enter the core of this Section. To fix some ideas, in the next Subsection we will always refer to consistent anomalies, while in the following one we will compute co-variant anomalies. We will then conclude with some well known examples of anomalies which will be useful for the second part of this chapter.

### 1.2.1 Consistent anomalies and WZ descent equations

As we have anticipated at the beginning of this Section, there exists a formal way to categorize possible candidates for quantum anomalies. This starts recalling that gauge transformations form a group. Specializing to infinitesimal transformations generated by parameters $\alpha, \beta$ the group law reads

$$\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha = \delta_{[\alpha,\beta]} \,. \tag{1.9}$$

It is natural to ask the effective action of a quantum theory to follow such group law, that is:

$$\left(\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha\right) W[A] = \delta_{[\alpha,\beta]}W[A] \,. \tag{1.10}$$

Equation (1.10) is called the "Wess-Zumino consistency condition". By using (1.4) one may phrase this as a consistency condition for quantum anomalies:

$$\delta_\alpha \mathcal{A}_\beta(A) - \delta_\beta \mathcal{A}_\alpha(A) = \mathcal{A}_{[\alpha,\beta]}(A) \,, \tag{1.11}$$

These equations have a wealth of solutions:

1. One notable class is made up of gauge-invariant effective actions $\delta_\alpha W_{\text{inv}}(A) = 0$, which just give $\mathcal{A}_\alpha(A) = 0$.

2. Another set of solutions is made-up of integrals of arbitrary local (albeit not gauge-invariant) counter-terms, which are polynomials in $A$. In this case one may write, in an appropriate basis $\delta_\alpha = \int D_\mu \alpha^a \frac{\delta}{\delta A_\mu^a}$ and the consistency condition follows swiftly.

Such local counter-terms should not be responsible for defining anomalies, since they may be added or subtracted at will to the effective action. Thus it is natural to define the possible candidates for anomalies as:

$$Anomalies = \{solutions\ to\ (1.10), (1.11)\}/\{local\ counter-terms\} \tag{1.12}$$

which already starts looking like a cohomological problem. Notice that, while this procedure does fix the form of the possible (perturbative) anomalies of a theory given its continuous symmetry group $G$, it does not fix the coefficient with which they might appear (which may as well be zero). That needs to be computed directly by different methods once the theory is fixed.

**BRST cohomology and descent equations**

The main obstacle to having a well defined cohomological problem is that gauge transformations with a fixed parameter $\alpha$ are not naturally nihilpotent. The simple way out of this is to promote gauge parameters to anti-commuting ghosts $c$ valued in the Lie algebra $\mathfrak{g}$ of $G$. The BRST transformation thus defined is usually called $s$ and acts as

$$sA = Dc\,, \tag{1.13}$$

$$sc = \frac{1}{2}[c, c]\,, \tag{1.14}$$

which satisfies $s^2 = 0$. The Wess-Zumino condition for the anomaly then becomes

$$s\mathcal{A}_c(A) = 0\,, \tag{1.15}$$

and $\mathcal{A}$ is defined only modulo $s$-exact terms.

In many applications, however, it is most convenient to work with an "un-integrated" anomaly, that is $\mathcal{A}_\alpha = \int P_\alpha(A)$. Then $P_\alpha(A)$ is also determined by cohomological equations, however only up to total derivatives, that is

$$sP_c(A) = dQ_c(A)\,. \tag{1.16}$$

Luckily, since the operators $d$ and $s$ can be defined in such a way that $\{d, s\} = 0$, this problem can be embedded in a bigger cohomological machinery to which solutions can be generated. Indeed let us denote by $P^{p,g}$ a p-form with $g$ insertions of the BRST ghost $c$ (that is, ghost number $g$). Naturally we are looking for $P^{d,1}$ forms defined in the joint cohomology of $d$ and $s$, which satisfy

$$sP^{d,1}(A) = dP^{d-1,2}(A)\,, \tag{1.17}$$

now suppose we have at our disposal a $d + 2$ form $P^{d+2,0}(A)$ which satisfies:

$$dP^{d+2,0} = 0 \,, s P^{d+2,0} = 0 \,, \tag{1.18}$$

that is, it is both closed and gauge invariant. Then we may (locally) write $P^{d+2,0}(A) = dP^{d+1,0}(A)$. What is now $sP^{d+1,0}(A)$? This is a $d + 1$ form with ghost number 1. We can compute $ds P^{d+1,0}(A) = -ds P^{d+1,0}(A) = s P^{d+2,0}(A) = 0$ to conclude that $s P^{d+1,0}(A)$ is locally closed, thus

$$sP^{d+1,0}(A) = dP^{d,1}(A) \,. \tag{1.19}$$

The same reasoning using now nihilpotency of $s$ shows that

$$sP^{d,1}(A) = dP^{d-1,2}(A) \,, \tag{1.20}$$

making $P^{d,1}(A)$ a candidate anomaly. The chain of equations

$$P^{d+2,0}(A) = dP^{d+1,0}(A) \,, \tag{1.21}$$

$$sP^{d+1,0}(A) = dP^{d,1}(A) \,, \tag{1.22}$$

$$sP^{d,1}(A) = dP^{d-1,2}(A) \,, \tag{1.23}$$

$$\dots \,, \tag{1.24}$$

$$sP^{0,d+1} = 0 \,, \tag{1.25}$$

are known as the Wess-Zumino descent equations. This is an extremely powerful result, since it ultimately allows to encode the information about the anomalous structure of a theory simply by giving the possible top forms $P_i^{d+2,0}(A)$ and the 't Hooft coefficients $c_i$. For example, it simplifies a lot the treatment of mixed anomalies, since in this case there are multiple representatives for $P^{d,1}(A_1, A_2)$ which "move" the anomaly between different sectors, but only a single $P^{d+2,0}(A_1, A_2)$ to start with. Our task at this point is to find a way to determine the possible top forms $P^{d+2,0}(A)$.

**Invariant polynomials,Chern-Simons forms and transgression**

In this section we give a simple class of top forms $P^{d+2,0}(A)$ in the case of a single gauge group, these are called invariant polynomials. The extension to more than one group (that is mixed anomalies) is straightforward as one can take the exterior product of invariant polynomials of lower degree for the two groups.

We start by considering a complex valued $n \times n$ matrix $X$ and the space of polynomials from $\mathbb{C}^{n \times n} \to \mathbb{C}$ which are invariant under $GL(n, \mathbb{C})$ transformations $X \to MXM^{-1}$ and are of fixed degree $m$. Next we substitute $X$ with the curvature two-form for a G-valued connection $A$, $F = dA + A \wedge A$. We take a basis of this vector space which we call $P_i^{d+2}(A)$[5]. The Chern-Weyl theorem (see e.g. [21] for a review) states that these invariant polynomials define closed forms $dP_i^{d+2} = 0$. Also, gauge invariance follows from the $GL(n, \mathbb{C})$ invariance we have required and the transformations property of the curvature $F \to g^{-1}Fg$. The proof of the Chern-Weyl

---

[5]The notation is as follows: $d + 2$ stands for degree of the differential form once we have substituted $X$ for $F$, since $F$ is a two-form this is a polynomial of degree $m = (d + 2)/2$ and is defined only for even $d$.

theorem essentially follows by showing that invariant polynomials may be written as sums and products of $Tr\,(F^n)^6$, which is a closed form by the Bianchi identity $DF = 0$ since

$$dTr\,(F^n) = nTr\,\left(dF \wedge F^{n-1}\right) = -nTr\,\left([A, F]F^{n-1}\right) = 0\,. \tag{1.26}$$

A nice class of these invariant polynomials is given by the Chern characters

$$Ch^n(F) = Tr\,(F^m)\,, \tag{1.27}$$

which are the starting point in deriving anomalies for e.g. $SU(N)$ symmetries. These can be extracted from the generating functional

$$Ch(F) = Tr\,\left(e^{F/2\pi}\right) \tag{1.28}$$

Another important object which will often appear are the Chern characters $c^n(F)$, which are extracted from:

$$c(F) = \det\left(1 + \frac{F}{2\pi}\right)\,, \tag{1.29}$$

in particular, for e.g. $SO(d)$ bundles, $F^T = -F$ and odd terms in the expression above vanish, in this case (which is of relevance for diffeomorphisms and Lorentz transformations $F = R$), one has the Pontryagin classes $p^n(R)$ by

$$c(R) = 1 + p^1(R) + p^2(R) + ...\,, \tag{1.30}$$

with $p^n$ a polynomial of degree $2n$ in the curvature $R$.

Before closing this part, it will be also useful to give a constructive approach, known as transgression formulas, which gives the form of lower lying terms in the Wess-Zumino descent equations. They are also very useful in determining the change in Chern-Simons form after a change in the connection. We will use such formulas in the next Chapter to determine the effects of 't Hooft anomalies on thermal effective actions on $\mathcal{M}_{d-1} \times S^1$.

The idea is to fix an invariant polynomial $P^{d+2,0}(A)$ and consider two connections $A_1$ and $A_2$ which are homotopic to each other. This means that $A_t = A_1 + t(A_2 - A_1)$ is a well defined connection for each $t \in [0, 1]$. We can then define a curvature $F_t = dA_t + A_t \wedge A_t$ and a co-variant derivative $D_t$. We are interested in computing the difference between invariant polynomials built out of the two connections $P^{d+2,0}(A_2) - P^{d+2,0}(A_1)$. The starting point is to write

$$P^{d+2,0}(A_2) - P^{d+2,0}(A_1) = \int_0^1 dt\,\frac{d}{dt}P^{d+2,0}(A_t)\,, \tag{1.31}$$

since $P^{d+2,0}$ depends only on $F_t$ we may use the chain rule. Furthermore we also notice that $\frac{d}{dt}F_t = D_t(A_2 - A_1) \equiv D_t\Delta A$, so that:

$$P^{d+2,0}(A_2) - P^{d+2,0}(A_1) = \int_0^1 dt\,Tr\left(D_tA\frac{\partial P^{d+2,0}(A_t)}{\partial F_t}\right)\,, \tag{1.32}$$

---

[6]Throughout the text we use $Tr$ to denote the trace in the adjoint representation, while we use $tr_R$ or just $tr$ to denote traces in a fixed representation of just the trace of an $n \times n$ matrix respectively.

now one integrates the $D_t$ by parts and uses the Bianchi identity $D_t F_t = 0$ to find our first transgression formula:

$$P^{d+2,0}(A_2) - P^{d+2,0}(A_1) = d \int_0^1 dt \, Tr \left( \Delta A \frac{\partial P^{d+2,0}(A_t)}{\partial F_t} \right), \tag{1.33}$$

this implies that the integral of $P^{d+2,0}(A)$ over a closed manifold is a topological invariant of the gauge bundle.

By our previous discussion the formula above also gives

$$d \left( P^{d+1,0}(A_2) - P^{d+1,0}(A_1) \right) = d \int_0^1 dt \, tr \left( \Delta A \frac{\partial P^{d+2,0}(A_t)}{\partial F_t} \right), \tag{1.34}$$

from which a representative of the Chern-Simons form $P^{d+1,0}(A)$ may be extracted by setting $A_1 = 0$. One can go one step further and derive also a transgression formula for Chern-Simons terms. While part of it is already present in the line above, it does not satisfy the descent equation by itself.

We can start just as before by writing the difference between the two Chern-Simons forms as a time derivative:

$$\begin{aligned} P^{d+1,0}(A_2) - P^{d+1,0}(A_1) &= \int_0^1 dt \, \frac{d}{dt} P^{d+1,0}(A) \\ &= \int_0^1 dt \, Tr \left( \Delta A \frac{\partial P^{d+1,0}(A_t)}{\partial A_t} + D_t \Delta A \frac{\partial P^{d+1,0}(A_t)}{\partial F_t} \right), \end{aligned} \tag{1.35}$$

notice now a further term, since the Chern-Simons form may also depend on $A$. We can go a bit further, taking variations of $P^{d+2,0}(A) = dP^{d+1,0}(A)$ and using $\delta F = D \delta A$ gives

$$Tr \left( \frac{\partial P^{d+2,0}(A)}{\partial F} \delta A \right) = Tr \left( \left[ \frac{\partial P^{d+1,0}(A)}{\partial A} + D \frac{\partial P^{d+1,0}(A)}{\partial F} \right] \delta A \right), \tag{1.36}$$

or

$$\frac{\partial P^{d+1,0}(A)}{\partial A} = \frac{\partial P^{d+2,0}(A)}{\partial F} - D \frac{\partial P^{d+1,0}(A)}{\partial F}. \tag{1.37}$$

Substituting this into (1.35) and integrating by parts the $D_t$ gives

$$\begin{aligned} P^{d+1,0}(A_2) - P^{d+1,0}(A_1) &= \int_0^1 dt \, Tr \left( \Delta A \frac{\partial P^{d+2,0}(A_t)}{\partial F_t} \right) + d \tilde{P}^{d,0}(A) \\ \tilde{P}^{d,0}(A) &= \int_0^1 dt \, Tr \left( \Delta A \frac{\partial P^{d+1,0(A)}}{\partial F} \right). \end{aligned} \tag{1.38}$$

Since $P^{d+2}(F, ..., F)$ is a symmetric polynomial by definition, we can simplify the formulas above by recognizing that

$$Tr \left( G(A) \frac{\partial P^{d+2,0}(F_t)}{\partial F_t} \right) = \frac{d+2}{2} P^{d+2}(G(A), F_t, ..., F_t), \tag{1.39}$$

which might be useful in practical computations.

The transgression formulas deserve further comments in view of the discussion about anomaly inflow. The variation $sP^{d+1,0}$ can be computed by setting $A_2 = A_1 + sA_1$ and restricting to

ghost number one. Perhaps more simply, we can use the transgression formula for the trivial connection $A_t = tA$ and take the $s$ transformation of that. The general formula is quite cumbersome, it has however a very important property: it comes in two terms: one, stemming from $\tilde{P}^{d,0}$ looks like a local contribution in $d$ dimensions, the other, coming from the remainder of the formula, is a purely $d+1$ dimensional piece.

This piece will be given by the ghost parameter times a function of the curvature $F$[7], while the piece coming from $\tilde{P}^{d,0}$ will be a total derivative of a function of $(A, F)$. We can then improve the current operator of our quantum field theory by including this contribution into the current. The formula for the anomaly will then look perfectly gauge invariant, i.e. we will have defined a co-variant anomaly.

Such contribution is usually called a Bardeen polynomial and allows us to define *co-variant* currents. A more straightforward way to define this is to recall that consistent currents are local variations of the effective action, then:

$$s \star j = s\frac{\delta}{\delta A}W[A] = \frac{\delta}{\delta A}sW[A] = \frac{\delta}{\delta A}\mathcal{A}(A) \tag{1.40}$$

which gives in a straightforward (albeit sometimes tedious) way the Bardeen Polynomials. We can also use this formalism to introduce in a concise way *co-variant* anomalies.

As we have already mentioned, the descent procedure gives a candidate *consistent* anomaly $P^{d,1}(\alpha, A)$[8] which satisfies the consistency conditions

$$\delta_\beta P^{d,1}(\alpha, A) - \delta_\alpha P^{d,1}(\beta, A) = P^{d,1}([\alpha, \beta], A)\,. \tag{1.41}$$

Recall that $P^{d,1}$ appears on the right-hand-side of the conservation law for the consistent current. By using the Bardeen Polynomials we can thus define a new quantity $P^{d,1}_{\text{cov}}(\alpha, A)$, namely the co-variant anomaly, that appears on the right-hand side of the co-variant conservation law. Since the co-variant current transforms co-variantly under gauge transformations, $\delta_\alpha J = [\alpha, J]$ we conclude

$$\delta_\alpha P^{d,1}_{\text{cov}}(\beta, A) = P^{d,1}_{\text{cov}}([\alpha, \beta], A)\,, \tag{1.42}$$

we can derive the same formula for $\delta_\beta P^{d,1}_{\text{cov}}(\alpha, A)$ and, since the commutator is anti-symmetric we conclude:

$$\delta_\alpha P^{d,1}_{\text{cov}}(\beta, A) - \delta_\beta P^{d,1}_{\text{cov}}(\alpha, A) = 2P^{d,1}_{\text{cov}}([\alpha, beta], A)\,, \tag{1.43}$$

which is the "consistency" condition for the co-variant anomaly. Notice the important factor of two on the right hand side.

We can also give a more compact albeit formal way of computing the anomaly by packing all of the Wess-Zumino chain of equation is a single ghost-valued polynomial. This is known as "Stora's" approach. The main idea is to define a new exterior derivative $\hat{d} = d + s$ and a shifted connection $\hat{A} = A + c$[9]. With a bit of algebra one can prove the following "horizontality condition"[10]

$$\hat{F}(\hat{A}) = \hat{d}\hat{A} + \hat{A} \wedge \hat{A} = F(A)\,. \tag{1.44}$$

---

[7]Thus transforming co-variantly under gauge transformations.

[8]For the moment let us treat $\alpha$ as an ordinary gauge parameter.

[9]Notice this is only a formal definition, since $\hat{A}$ is not a differential one-form, one should always thinking of expanding such formal expressions in pieces with fixed form degree and ghost number.

[10]This is maybe better known as the "Russian formula".

Since $\hat{F}$ satisfies a Bianchi identity with $\hat{D} = \hat{d} + [\hat{A}, \cdot]$ we can write the same transgression formula for $P^{d+2,0}(\hat{F})$, but now with hatted fields. this leads, after applying the Russian formula, to the following identity:

$$\hat{d}P^{d+1,0}(A + c, F) = dP^{d+1,0}(A, F) , \tag{1.45}$$

which, upon expansion in $c$, gives back the descent equations. Now one can simply input the shifted Chern-Simons term above and algebraically expand to in the ghost fields to write down the full consistent anomaly chain.

As a final comment, it is important to stress that, with no a priori perturbative computation, there are various candidate $P^{d+2,0}$ in higher dimensions, constructed e.g. from $Tr(F^2)^m$, $Tr(F^3)^l$ etc. A particular system will have its own combination of such characteristic polynomials. For a given type of free particles there are known formulas giving the anomaly polynomial in generic dimensions, which is not fixed by consistency alone.

### 1.2.2 Covariant anomalies and the Fujikawa method

A very general method to compute perturbative anomalies in fermionic theories was developed by Fujikawa in the '70s. Let us rewiew the main idea behind it. Let us start from an (Euclidean) fermionic path integral in an external background:

$$Z[A] = \int D\Psi D\bar{\Psi} e^{-S + i\int \star j \wedge A} , \tag{1.46}$$

under a local $G$-symmetry transformation with parameter $\alpha$, of which $j$ is the current,

$$\Psi \to U_\alpha[\Psi] \equiv \Psi_\alpha , \quad A \to A + D\alpha \tag{1.47}$$

Naively, since the symmetry acts unitarily, the measure changes by

$$D\Psi \to D\Psi_\alpha \det\left(\frac{\delta\Psi}{\delta\Psi_\alpha}\right) = D\Psi_\alpha , \tag{1.48}$$

while the gauge transformation affects the minimal coupling $\star j \wedge A$ by an additional $\star j \wedge D\alpha$ term and gauge invariance implies:

$$Z[A + D\alpha] = Z[A] \iff D \star j = 0 , \tag{1.49}$$

as in the usual Noether theorem.

This is a correct reasoning were we to integrate over a finite number of variables. The key insight of Fujikawa is to understand that, in the QFT case, the determinant $\det\left(\frac{\delta\Psi}{\delta\Psi_\alpha}\right)$ is ill-defined and needs regularization. For the purposes of perturbative anomalies we may just work at linear order in $\alpha$ and linearize $\frac{\delta\Psi}{\delta\Psi_\alpha} = 1 + \delta_\alpha$. Then

$$\det\left(\frac{\delta\Psi}{\delta\Psi_\alpha}\right) \approx \det(1 + \delta_\alpha) = \exp\left(tr\log(1 + \delta_\alpha)\right) \approx \exp\left(tr[\delta_\alpha]\right) \tag{1.50}$$

and we are led to consider the regularized trace

$$T_\alpha(\tau) = tr\left[\delta_\alpha e^{-\tau R}\right] , \tag{1.51}$$

with $R$ an $A$ dependent, co-variant regulator. For Dirac fermions the standard choice is $R = \slashed{D}_A^2$. At this point one may perform the heat-kernel expansion for small $\tau$:

$$T_\alpha(\tau) = \sum_{n=-k}^{\infty} \tau^k t_\alpha^{(k)}(A)\,, \tag{1.52}$$

and the co-variant anomaly is identified exactly with $t_\alpha^{(0)}(A)$, so that, one derives the anomaly equation

$$\int (\alpha D \star j) = t_\alpha^{(0)}(A)\,. \tag{1.53}$$

In the standard case of chiral anomalies $\delta_\alpha = i\alpha\gamma_5$ and one can actually show that the regularized trace is independent of $\tau^{11}$. Then it is possible to compute the anomaly in the limit $\tau \to \infty$ in which it only projects onto zero modes of the Dirac operator. Because of chiral symmetry one may decompose

$$\slashed{D}_A = \begin{pmatrix} 0 & \mathcal{D}_A \\ \mathcal{D}_A^\dagger & 0 \end{pmatrix}\,, \tag{1.54}$$

so that say left handed zero modes are in $ker(\mathcal{D}_A)$ while right handed ones are in $ker(\mathcal{D}_A^\dagger)$. Then, at $\tau \to \infty$

$$T_\alpha(\tau) = itr(\gamma_5 e^{-\slashed{D}_A^2}) = \dim ker(\mathcal{D}_A) - \dim ker(\mathcal{D}_A^\dagger) = \operatorname{index}(iD_A)\,, \tag{1.55}$$

which connects the index theorem with 't Hooft anomalies. This can be made into a general statement by proving $\tau$ independence in arbitrary dimension. The upshot is that general formulas for the index of a given differential operator are known, which considerably simplified the computation of anomalies. This has been used to provide general formulas for the anomalies of free field representations [22, 23]

In practical cases, one either uses the results from the mathematical literature to compute the index, or explicitly computes the regularized trace by choosing a convenient basis of one particle states.

### 1.2.3 Explicit formulas and notable examples

We conclude this section by giving some notable applications of the formalism described up until now. We will describe the possible anomalies related to abelian flavor symmetries and gravitational (or Lorentz) symmetries in low dimensionalities and comment on the general structure in higher dimensions. We will also discuss some basic aspects of these anomalies, such as the related conserved currents or the fact that Lorentz and diffeomorphism anomalies are always of mixed type between each other for Riemannian geometries.

---

[11]This follows from expanding the Fermi fields in eigenfunctions of the operator $i\slashed{D}$. Since $\gamma_5$ anticommutes with it, a left-handed eigenvector with eigenvalue $\lambda$ is compensated by a right handed one with eigenvalue $-\lambda$, so only zero modes do not cancel in the regulated sum. Hence the $\tau$ independence.

## Ward identities in the presence of external fields

We start by devoting some lines to the explicit description of the form of abelian and diffeomorphism anomalies in general dimensionality[12] and to setting up some notation. While up until now we have used mostly coordinate-free formulas, here we adopt a more pragmatic approach. We work on a general curved $d$ dimensional manifold $\mathcal{M}_d$ equipped with a metric $g_{\mu\nu}$ and an external gauge field $A_\mu$. We define as usual the Christoffel connection by the compatibility condition

$$\nabla_\mu g_{\alpha\beta} = 0 \,, \tag{1.56}$$

and the absence of torsion [13] to be

$$\Gamma^\nu_{\mu\rho} = \frac{1}{2} g^{\nu\alpha} \left( \partial_\rho g_{\mu\alpha} + \partial_\mu g_{\rho\alpha} - \partial_\alpha g_{\mu\rho} \right) \,, \tag{1.57}$$

being $g^{\mu\nu}$ the inverse metric. The curvature associated with this connection is given by the familiar Riemann tensor

$$R_{\mu\nu}{}^\alpha{}_\beta = \partial_\mu \Gamma^\alpha_{\nu\beta} - \partial_\nu \Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\mu\delta}\Gamma^\delta_{\nu\beta} - \Gamma^\alpha_{\nu\delta}\Gamma^\delta_{\mu\beta} \,. \tag{1.58}$$

We will also use in many occasions a vielbein $e^a_\mu$, with $a$ a fundamental $SO(d)$ index, denoting $E^\mu_a$ its algebraic inverse satisfying

$$e^a_\mu E^\mu_a = \delta^\mu_\nu \quad e^a_\mu E^\mu_b = \delta^a_b \,. \tag{1.59}$$

We will also introduce a spin connection $\omega_\mu{}^{ab}$ for the $SO(d)$ symmetry and demand the compatibility condition

$$\nabla_\mu e^a_\nu = 0 \,, \tag{1.60}$$

which fixes the spin connection in terms of the vielbein variables:

$$\omega_\mu{}^{ab} = \frac{1}{2} E^{\nu a} \left( \partial_\nu e^b_\mu - \partial_\mu e^b_\nu + E^{\sigma b} e^c_\mu \partial_\sigma e_{\nu c} \right) - (a \leftrightarrow b) \,. \tag{1.61}$$

Since the vielbein satisfies $g_{\mu\nu} = e^a_\mu e^b_\nu \delta_{ab}$ the spin connection and the Christoffel symbols are not independent objects once the compatibility conditions are imposed the relation between them is

$$\Gamma^\alpha_{\mu\beta} e^a_\alpha E^{\beta b} = \omega_\mu{}^{ab} + E^{\nu a} \partial_\mu e^b_\nu \,. \tag{1.62}$$

This formula will allow us to relate diffeomorphism and Lorentz anomalies.
In an anomaly-free theory the effective action is a (gauge invariant) functional of $g, \Gamma, A, F$, or alternatively of the frame variables $e^a, \omega^{ab}$ and the $U(1)$ field. Its variation is given by

$$\delta W[g, \Gamma, A] = \int \sqrt{g} \left( t^{\mu\nu} \delta g_{\mu\nu} + G^\alpha_{\mu\beta} \delta\Gamma_\mu{}^\alpha{}_\beta + \mathcal{J}^\mu \delta A_\mu \right) \,,$$

$$\delta W[e, \omega, A] = \int |e| \left( t^\mu_a \delta e^a_\mu + S^\mu_{ab} \delta\omega_\mu{}^{ab} + \mathcal{J}^\mu \delta A_\mu \right) \,, \tag{1.63}$$

---

[12]The existence of anomalies for the coupling to an external curved geometry was pointed in some early works [24, 25, 26] out and elaborated on in the seminal paper [27].

[13]Such conditions will be relaxed in the next chapter in which we will introduce Non-relativistic geometries

In compatible geometries the "spin currents" coming from the dependence on the spin connection and Christoffel symbols may be re-adsorbed in an improvement for the energy-momentum tensor, which reads

$$
\begin{aligned}
\mathcal{T}^{\mu\nu} &= t^{\mu\nu} + \frac{1}{2}\nabla_\rho \left(G^{\rho\mu\nu} + G^{\mu\nu\rho} - G^{\nu\mu\rho}\right), \\
\mathcal{T}_a^\mu &= t_a^\mu + \frac{1}{2}E^{\mu b}\nabla^c \left(S_{cba} + S_{bac} - S_{abc}\right).
\end{aligned}
\tag{1.64}
$$

Notice that we have used script letters for the improved currents, meaning that they are *consistent* ones. Finally, Ward identities come from considering diffeomorphism, Lorentz and gauge variations with parameters $(\xi^\mu, \Omega^{ab}, \alpha)$ and the nontrivial variations of the external fields:

$$
\delta_\xi g_{\mu\nu} = \nabla_{(\mu}\xi_{\nu)}, \quad \delta_\xi \Gamma^\alpha_{\mu\beta} = (\mathcal{L}_\xi \Gamma_\mu)^\alpha{}_\beta + \nabla_\mu \Lambda^\alpha{}_\beta, \quad \Lambda^\alpha{}_\beta = \partial_\beta \xi^\alpha
\tag{1.65}
$$

$$
\delta_\xi A_\mu = \mathcal{L}_\xi A_\mu, \quad \delta_\xi e_\mu^a = \mathcal{L}_\xi e_\mu^a, \quad \delta_\xi \omega_\mu^{ab} = \mathcal{L}_\xi \omega_\mu^{ab},
\tag{1.66}
$$

$$
\delta_\Omega e_\mu^a = \Omega^a{}_b e_\mu^b, \quad \delta_\Omega \omega_\mu^{ab} = \partial_\mu \Omega^{ab} + [\omega_\mu, \Omega]^{ab}
\tag{1.67}
$$

$$
\delta_\alpha A_\mu = \partial_\mu \alpha,
\tag{1.68}
$$

where $\mathcal{L}_\xi$ denotes the Lie derivative and, for the Christoffel symbols, it acts only on the form index. Notice that, in the Christoffel case, we may split a "tensorial" variation given by the Lie derivative from a "connection-like" piece given by $\Lambda^\alpha{}_\beta$. Putting everything together the Ward identity read

$$
\nabla_\mu \mathcal{T}^{\mu\nu} + \mathcal{J}_\mu F^{\mu\nu} - A^\nu \nabla_\mu \mathcal{J}^\mu = 0
\tag{1.69}
$$

$$
\mathcal{T}_{ab} - \mathcal{T}_{ba} = 0,
\tag{1.70}
$$

$$
\nabla_\mu \mathcal{J}^\mu = 0,
\tag{1.71}
$$

with $\mathcal{T}_{ab} = \mathcal{T}_b^\mu e_{\mu a}$. These represent the diffeomorphism, Lorentz and $U(1)$ Ward identities. In the anomalous case the equations above get modified as follows

$$
\nabla_\mu \mathcal{T}^{\mu\nu} + \mathcal{J}_\mu F^{\mu\nu} - A^\nu \mathcal{A} = \mathcal{A}^\nu
\tag{1.72}
$$

$$
\mathcal{T}_{ab} - \mathcal{T}_{ba} = \mathcal{A}^{ab},
\tag{1.73}
$$

$$
\nabla_\mu \mathcal{J}^\mu = \mathcal{A},
\tag{1.74}
$$

with $\left(\mathcal{A}, \mathcal{A}^{ab}, \mathcal{A}^\mu\right)$ being the gauge, Lorentz and diffeomorphism anomalies respectively. To find the allowed form for the triplet, we need to solve the consistency conditions or, equivalently, the descent equations, for a fixed dimensionality.

**d=2**

We start by writing the possible top forms $P^{4,0}$ relevant for the anomalies in two dimensions. There are basically two possibilities:

$$
P^{4,0}(A) = F \wedge F \quad P^{4,0}(\omega) = tr\left(R_\omega \wedge R_\omega\right),
\tag{1.75}
$$

that is the second Chern character and the first Pontryagin class. For mixed anomalies there is also the wedge product of the first Chern characters

$$
P^{4,0}_{\text{mixed}}(A, B) = F_A \wedge F_B,
\tag{1.76}
$$

notice that there is no mixed anomaly between $A$ and $\omega$. using the transgression formula we find the Chern-Simons forms

$$P^{3,0}(A) = A \wedge dA \quad P^{3,0}(\omega) = tr\left(\omega d\omega + \frac{2}{3}\omega^3\right), \tag{1.77}$$

and the mixed Chern-Simons

$$P^{3,0}_{\text{mixed}} = AdF_B + xd(A \wedge B), \tag{1.78}$$

where the second term acts as the Bardeen counterterm in the local QFT and shifts the anomaly to the desired sector, $x = 0$ will have the anomaly in the $A$ current, while $x = 1$ in the $B$ current. We can now act with the BRST operator $s$ to find the consistent anomalies to be

$$P^{2,1}(A) = sF, \quad P^{2,1}(\omega) = sd\omega, \tag{1.79}$$

while for the mixed case we have two anomalies

$$P^{2,1}_A(A, B) = (1 - x)F_B \quad P^{2,1}_B(A, B) = xF_A. \tag{1.80}$$

In the quantum theory these will be multiplied by theory dependent coefficients $c_A$, $c_g$, $c_{AB}$. This fixes the previous section's quantities to be

$$\mathcal{A}_{d=2} = c_A\sqrt{g}\epsilon^{\mu\nu}F_{\mu\nu}, \quad \mathcal{A}^{ab}_{d=2} = c_g\sqrt{g}\epsilon^{\mu\nu}\partial_\mu\omega^{ab}_\nu. \tag{1.81}$$

We now come to the diffeomorphism anomaly. There are two ways to derive its form. One is to think of $\Gamma$ as transforming as a connection with parameter $\Lambda$ as in (1.68) and then integrating by parts a further time to get the anomaly:

$$\mathcal{A}^\mu = \bar{c}_g\sqrt{g}g^{\beta\mu}\epsilon^{\nu\rho}\partial_\nu\partial_\alpha\Gamma^\alpha_{\rho\beta}. \tag{1.82}$$

The other one is to use equation (1.62) to link the two Chern-Simons terms:

$$P^{3,0}(\omega) = P^{3,0}(\Gamma) - dtr\left(\Gamma e^{-1}de\right) + \frac{1}{3}tr\left(e^{-1}de\right)^3 \tag{1.83}$$

where the connection indices in $\Gamma$ are contracted with vielbeins and $(e^{-1}de)^{ab}_\mu = E^{\nu a}\partial_\mu e^b_\nu$. The diffeomorphism anomaly then follows from (minus) the variation of the last two terms on the right hand side. This presentation is also useful since we may think as those terms as Bardeen counterterms that move the anomaly between the Lorentz and the diffeomorphism sector[14], thus $c_g = \bar{c}_g$ upon matching of the two anomalies.

Here we have been quite cavalier about the diffeomorphism group and the descent equations. In a physical situation the diffeomorphism and/or Lorentz group of the Chern-Simons terms should be extended to the one of the $d + 1$ dimensional manifold where the Chern-Simons term lives in order to guarantee (up to boundary terms) covariance. For the purpose of the anomalous Ward identities this is not important, however we will see when dealing with anomalous effective actions that a proper account of this extension is essential for deriving a consistent effective action at finite temperature.

---

[14]While for the first term this is obvious, the second term is a WZW term and is not a total derivative, its variation however is, so that it contributes as a local action to the generating functional

Until now we have discussed consistent anomalies. To find the co-variant ones one needs to fix the Bardeen polynomials. Let us see one example. For the pure $U(1)$ case one may write:

$$\mathcal{A} = 2c_A \partial_\mu \left( \sqrt{g} \epsilon^{\mu\nu} A_\nu \right), \tag{1.84}$$

taking variation with respect to $A$ gives

$$s\mathcal{J}^\mu = -2c_A \epsilon^{\mu\nu} \partial_\nu c, \tag{1.85}$$

so that we may define the co-variant current

$$J^\mu = \mathcal{J}^\mu + 2c_A \epsilon^{\mu\nu} A_\nu \tag{1.86}$$

with the co-variant anomaly

$$\nabla_\mu J^\mu = 2c_A \epsilon^{\mu\nu} F_{\mu\nu}. \tag{1.87}$$

Also instructive is to see what happens for the Lorentz anomaly. Here, as one may guess, we will need to replace the $d\omega$ term with a curvature[15]. There the Bardeen counterterm may be computed from the transgression formula to be $dP_{BZ} = d\omega + 2\omega \wedge \omega$ so that the co-variant anomaly reads

$$T_{ab} - T_{ba} = 2c_g \epsilon^{\mu\nu} R^\omega_{\mu\nu ab}. \tag{1.88}$$

In order to get some familiarity with the numerical values of the coefficients $c_A$ and $c_g$ let us give the values for a single Weyl fermion of charge $q$ under the (chiral) $U(1)$:

$$c_A = \frac{q^2}{4\pi}, \quad c_g = \frac{1}{192\pi}, \tag{1.89}$$

more in general, for a 2d CFT with Kac-Moody levels $k_L$ and $k_R$ and central charges $c_L$ and $c_R$ we have

$$c_A = \frac{k_L - k_R}{2\pi}, \quad c_g = \frac{c_L - c_R}{96\pi}, \tag{1.90}$$

**d=4**

In four dimensions the story is similar, but now there is no Pontryagin class for the purely gravitational terms. We have

$$P^{6,0}(A) = F \wedge F \wedge F, \quad P^{6,0}(A,\omega) = F \wedge p^2(R_\omega), \tag{1.91}$$

and two mixed classes purely for $U(1)$:

$$P^{6,0}_{1,2}(A,B) = F_A \wedge F_B \wedge F_B, \quad P^{6,0}_{2,1}(A,B) = F_A \wedge F_A \wedge F_B. \tag{1.92}$$

The Chern-Simons term follow from the previous discussion:

$$P^{5,0}(A) = A \wedge F \wedge F, \tag{1.93}$$

$$P^{5,0}(A,\omega) = F \wedge P^{3,0}(\omega) + xd\left(A \wedge P^{3,0}(\omega)\right), \tag{1.94}$$

---

[15]Of course, in $d = 2$ the Lorentz group is Abelian and the curvature is just $d\omega$, but let us forget this fact for a moment for the sake of the argument.

and for the mixed $U(1)$s

$$P^{5,0}_{1,2}(A, B) = A \wedge F_B \wedge F_B + yd\left(A \wedge B \wedge F_B\right) \tag{1.95}$$

$$P^{5,0}_{1,2}(A, B) = B \wedge F_A \wedge F_A + zd\left(A \wedge B \wedge F_A\right) . \tag{1.96}$$

Making contact with our previous discussion, in $d = 4$:[16]

$$\mathcal{A} = \sqrt{g}\epsilon^{\mu\nu\rho\sigma}\left(c_A F_{\mu\nu}F_{\rho\sigma} + c_g tr\left[R^\omega_{\mu\nu}R^\omega_{\rho\sigma}\right]\right) \tag{1.97}$$

with the anomaly only in the $U(1)$ sector. The Lorentz anomaly would look like

$$\mathcal{A}^{ab} = c_g \epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}\partial_\rho\omega^{ab}_\sigma , \tag{1.98}$$

and the diffeomorphism one

$$\mathcal{A}^\mu = c_g \sqrt{g}g^{\beta\mu}\epsilon^{\tau\delta\nu\rho}\partial_\alpha\left(F_{\tau\delta}\partial_\nu\Gamma^\alpha_{\rho\beta}\right) . \tag{1.99}$$

Mixed covariant anomalies admit no Bardeen counterterms, so they are uniquely defined. One gets a Bardeen polynomial for the pure $U(1)$ anomaly for the current and a second one for the $U(1$ variation of the stress tensor. The covariant anomalies then read

$$\nabla_\mu J^\mu = \epsilon^{\mu\nu\rho\sigma}\left(3c_A F_{\mu\nu}F_{\rho\sigma} + c_g tr\left[R^\omega_{\mu\nu}R^\omega_{\rho\sigma}\right]\right) \tag{1.100}$$

$$T_{ab} - T_{ba} = -2c_g \epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}R^\omega_{\rho\sigma ab} . \tag{1.101}$$

Here too, it is useful to give the values for the anomalies of a single Weyl fermion:

$$c_A = \frac{q^3}{96\pi^2} , \quad c_g = \frac{q}{768\pi^2} , \tag{1.102}$$

for multiple species one sums over the charges. In the case of a chiral fermion the anomaly coefficients (after preserving vector-like symmetries) are given by the formula above with the understanding that left handed fermions contribute $q_L$ and right handed ones $-q_R$.

**Higher dimensions**

The higher dimensional story tends to be more complicated, as there can be several characteristic classes to start from and the form of the anomaly polynomial depends on the choice of free-field representation we work with. Of course the machinery for determining the possible Chern-Simons terms is still in place and allows us to conclude the following:

- In every even dimensionality there may be (chiral) flavor anomalies for $SU(N)$ gauge fields, coming from the descent procedure applied to the Chern-character $tr\left(F^n\right)$, $d+2 = 2n$.

- Purely gravitational anomalies only exist in $d = 2+4k$ since Pontryagin classes only come with even powers of the curvature due to the rotation group being $SO(d)$, thus $d+2 = 4k$ hence the formula before.

---

[16]Below $R^\omega$ may be substituted also with the Riemann tensor.

- In dimensions $d = 4k$ we can wedge the Pontryagin class of two lower dimensions with a Chern-class. This gives upon descent mixed gauge-gravitational anomalies in all $d = 4k$. In high enough dimension one could in principle also have mixed anomalies in $d = 4k + 2$ with $k \geq 1$, but we will not study systems of such high dimensionality here.

It is worth pointing out that, in the case of a given particle species, there exist closed formulas for the generating function of (co-variant) anomaly polynomials in general dimension. For, example, a well known formula for the anomaly polynomial of a spin $1/2$ particle is:

$$P_{1/2}(A, \omega) = \text{tr}_\Psi \left( e^{i \frac{F}{2\pi}} \right) \wedge \hat{\mathbf{A}}(R), \tag{1.103}$$

with the A-roof genus

$$\hat{\mathbf{A}}(R) = \text{Pf} \left( \frac{R/4\pi i}{\sinh(R/4\pi i)} \right), \tag{1.104}$$

with the Pfaffian Pf. The trace in the first equation is in the relevant flavor representation for the Fermi field. One is instructed to expand the above as a differential form and pick the term(s) of the right dimensionality.

## 1.3  Synopsis

In this first Chapter we have reviewed some well known facts about perturbative anomalies. We have clarified the difference between the *consistent* and *co-variant* formulations, which, albeit somewhat technical, will be very important in the following presentation. Also, we have given some practical tools to select the possible representative of an anomaly polynomial and to fix the coefficient with which it could appear in a given perturbative description.

Let us conclude with some important points that we may have skipped during the presentation. Although not explicitly stated, perturbative 't Hooft anomalies are always related to the presence of "chirality" in the underlying system. That is one has an idempotent operator which commutes with the Lorentz generators and does not act on spacetime, which allows projection of the degrees of freedom. Such operator may be e.g. the $\gamma_{d+1}$ Dirac matrix, the $\star$ Hodge dual. One should think of the results presented as implying that such a chiral projection has been made.

From our presentation it would also seem that anomalies are only present in even dimensions due to the presence of characteristic classes only for $d + 2$ even. This is however not true, since discrete anomalies (such as the parity anomaly) are present in odd dimensions and can be related with Chern-Simons forms [28]. Another exception might be given by perturbative anomalies for higher-form symmetries, which can give rise to odd dimensional characteristic classes as the gauge field is a general $p$-form, indeed for $p$ even $\int dA_p$ is an odd dimensional topological invariant.

Finall, let us conclude with a more broad point of view, which is meant to justify our interest in 't Hooft anomalies throughout this thesis. As we have seen, 't Hooft anomalies are both RG-invariants and strongly constrained by algebraic geometry. This makes them great indicators for universality. Until about ten years ago such universality was in some sense under-appreciated, in that it was not believed that the specific form of the low energy effective action could be strongly

constrained by anomalies[17], Contrary to expectations, a great deals of progress has been made in understanding how and when 't Hooft anomalies can give rise to local contributions upon integrating out heavy modes. At a more simplisitc level, such contributions show themselves in the presence of protected, non-dissipative transport phenomena at very long wavelength, which can be even measured in the laboratory[29, 30].

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[17]The sense in which we mean this is the following: anomalies are not expected to emerge as local contributions to the effective action, which are those which are well defined by a Wilsonian-type of RG flow.

# Chapter 2

# Effective actions and 't Hooft anomalies

## 2.1 History of the problem

Having now finished our small review of 't Hooft anomalies we come to the main part of this chapter. While we have already mentioned the role of 't Hooft anomalies in constraining possible RG flows to lower energies through their matching, this Chapter will be dedicated to the dependence of the effective action on $S^1 \times \mathcal{M}_{d-1}$ on anomaly coefficients such as $c_A, c_g$. More precisely, we will consider fibrations of $S^1$ over $\mathcal{M}_{d-1}$ which have the local form:

$$ds^2 = \left(d\tau + a_i(x)dx^i\right)^2 + h_{ij}(x)dx^i dx^j \,, \tag{2.1}$$

also endowed with a gauge field:

$$A = \mu(d\tau + a_i(x)dx^i) + b_i dx^i \,, \tag{2.2}$$

so that we allow for a nontrivial holonomy $\beta^{-1}\mu$ on the $S^1$.[1] We also have a thermal Killing vector $u^\mu = \partial_\tau$ and the corresponding one-form $u = d\tau + a_i(x)dx^i$.

While at first sight this may seem like a trivial question, since we have already stated in talking about the descent equations that anomalies precisely *do not* appear in a local effective action. It turns out that, upon dimensionally reducing over $S^1$, they can indeed appear as local Chern-Simons terms for the effective action $\tilde{W}[a, b]$ on $\mathcal{M}_{d-1}$ [31, 32]. The Chern-Simons terms are a function of the local $U(1)$ gauge field $b_i$ and the gravito-magnetic potential field $a_i$. For example in $d = 4$ can be essentially of three types:

$$\tilde{W}[a, b] = \int_{\mathcal{M}_3} \left(c_{bb}\ b \wedge db + c_{aa}\ a \wedge da + c_{ab}\ a \wedge db\right) \,, \tag{2.3}$$

while in $d = 2$ there are only two terms

$$\tilde{W}[a, b] = \int_{\mathcal{M}_1} \left(c_a\ a + c_b\ b\right) \tag{2.4}$$

---

[1]One may also add a dilaton field in front (2.1), this makes the local temperature position dependent but does not change the conclusions, in fact, it is tempting to think of the inverse temperature as playing a role analogous to that of the gauge holonomy $\mu$.

with $c_{bb}$, $c_{aa}$, $c_{ab}$, $c_a$, $c_b$ functions of the holonomy $\beta\mu$ and of the inverse temperature $\beta$. In our conventions $b_i$ has dimensions of energy, while $a_i$ has no dimensions. This, together with assuming a smooth vacuum limit, fixes the possible terms which may appear:

$$c_{aa} = \beta^{-2} c_{aa}^{(2,0)} + \beta^{-1}\mu c_{aa}^{(1,1)} + \mu^2 c_{aa}^{(0,2)} \,, \tag{2.5}$$

and so on. We will be interested in determining the relationship of these dimensionless numbers with 't Hooft anomalies.

To answer the previous comment, the presence of Chern-Simons terms on $\mathcal{M}_{d-1}$ can be justified for two reasons.

1. If the spectrum of KK particles on $S^1$ has no zero modes, one may integrate out the whole tower of massive states and expect them to give (term by term) rise to a local effective action. This action is first order in derivatives and should thus satisfy a number of constraints coming from the mother theory. This will translate into the fact that holonomy-dependent terms will match the flavor anomalies in $d$ dimensions, and thus they need not be properly quantized CS terms. The holonomy-independent terms, however, form a different family, since the temperature tempendence may be re-adsorbed in the definition of the gravito-magnetic potential field and the diffeomorphism anomalies enter at higher order in the derivative expansion.

   Their usefulness was explained in [33]. The theories may have global anomalies, which are encoded in the transformation properties of the partition function with respect to *large* gauge transformations of the background fields (that is, transformations that cannot be deformed to the identity and change non-local oservables such as holonomies). These persist upon dimensional reduction and can be matched by Chern-Simons terms. Since global anomalies sit in finite discrete groups, this reasoning can only determine the coffiecients of the effective action modulo certain integers. This is to be expected, as a properly quantized Chern-Simons term is invariant under both small and large gauge transformation and is thus invisible to this line of reasoning.[2] Local correlation functions on $\mathcal{M}_{d-1}$ are sensitive to the full coefficient. This is usually not enough to say that the integer part of the Chern-Simons term should matter, since such correlators in positions space are purely contact terms and may be generated by adding a properly quantized Chern-Simons term to the action, which we are free to do. However here we may raise a further important point: since our theory was $d$ dimensional to begin with, the Chern-Simons term should come from a local counterterm in $d$ dimensions! A moment of thinking shows that there are no such terms and thus the integer part cannot be swept away as usual.

   These facts make the latter type of Chern-Simons terms much more problematic to derive within effective field theory. Our aim in this Chapter is to give a clear physical picture of how they arise.

2. One may think of anomalous theories as boundary modes of gapped systems living on a $d+1$ dimensional manifold $\mathcal{N}_{d+1}$ with boundary $\mathcal{M}_{d+1}$ via anomaly inflow. Then we may

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[2]To fix some conventions, one may rescale $a \to \beta a'$ so that $a'$ has the right dimensionality, however the large gauge transformation shift the holonomy by integers instead of integers times $2\pi$, the convenient normalization to confront with the Chern-Simons quantization condition is $a \to \frac{\beta}{2\pi} a''$.

couple our anomalous theory to such a bulk that cancels the perturbative anomalies. In this way correlators of the full theory should be independent of the anomaly coefficients. One may then try to estimate the effective action by reducing the effective action on $\mathcal{N}_{d+1}$ (which is local) to $\mathcal{M}_{d-1}$. This is in general not a well defined procedure, but it turns out that specifying the boundary holonomies completely determines the action for the extended connections. The inflow procedure was pioneered in [34, 35] in the case of flavor anomalies. Here we extend it to the gravitational case by considering the embedding of the $SO(d)$ frame rotations into the bulk $SO(d+1)$. This turns out to fix completely the anomalous contribution to the effective action. Our strategy then summarizes in the diagram below:

$$S^1 \times \mathcal{M}_{d-1} \xleftarrow[\text{inflow}]{} \mathcal{N}_{d-1}$$

KK reduction $\Big\downarrow$ $\swarrow$ KK, fixed holonomies

$$\mathcal{M}_{d-1}$$

The history of this problem is quite long, and the formulation in terms of thermal effective actions may not be the most familiar, so it is useful here to present some of the historical developments. First, one may think of the effective actions of the type (2.3),(2.4) as computing $(d-1)$-point thermal correlators of electric currents $J^i$ and energy currents $J_\epsilon^i = T^{i\tau}$ on $\mathcal{M}_{d-1}$. Indeed one may consider the following ansatz for the one-point functions of said current operators in the geometries above:

$$\langle J \rangle = c_a \,, \tag{2.6}$$

$$\langle J_\epsilon \rangle = c_b \,, \tag{2.7}$$

in $d=2$ and

$$\langle J^i \rangle = c_{ab}\Omega^i + 2c_{bb}B^i \,, \tag{2.8}$$

$$\langle J_\epsilon^i \rangle = c_{ab}B^i + 2c_{aa}\Omega^i \,, \tag{2.9}$$

with the following conventions: we define electric/magnetic decomposition in the ambient four dimensional geometry through the one-form $u = d\tau + a_i dx^i$ by

$$dA = E \wedge u + (B + 2\mu\Omega) \,, \quad du = E_g \wedge u + \Omega \,. \tag{2.10}$$

The magnetic parts $B$, $\Omega$ are time-independent quantities which descend to two-forms in $d-1$ dimensions. There we define

$$B^i = \frac{1}{2}\epsilon^{ijk}B_{jk} \,, \quad \Omega^i = \epsilon^{ijk}\partial_j a_k \,. \tag{2.11}$$

One recognizes that these are exactly the possible time-independent responses coming from the Chern-Simons actions on $\mathcal{M}_{d-1}$. In higher dimensions one would have more insertions of the magnetic fields, since currents are formally $d-2$ forms.

These in turn can be related to hydrodynamic response functions in the full $d$-dimensional space-time to magnetic fields and vorticity[3] by interpreting the one-form $u$ as a velocity field. Of course the constitutive relations above need to be corrected by introducing dissipative terms, and should allow for a weak time dependence.

Finally there is a standard way to relate the constitutive relations to Euclidean correlators via the Kubo formulas. In this simplified case in which time dependence is absent and the Eulidean formalism is thus expected to capture the relevant features, they amount to taking further functional derivatives to eliminate the external fields and going to momentum space on $\mathcal{M}_{d-1}$ then, e.g. in $d-4$

$$c_{bb} = \lim_{k \to 0} \epsilon_{ijl} \frac{\langle J^i(k) J^j(-k) \rangle}{ik^l} \,, \tag{2.12}$$

$$c_{ab} = \lim_{k \to 0} \epsilon_{ijl} \frac{\langle J^i(k) J^j_\epsilon(-k) \rangle}{ik^l} \,, \tag{2.13}$$

$$c_{aa} = \lim_{k \to 0} \epsilon_{ijl} \frac{\langle J^i_\epsilon(k) J^j_\epsilon(-k) \rangle}{ik^l} \,, \tag{2.14}$$

up to possible contact terms. In Lorentzian signature, these should be analytically continued to retarded two-point functions, but in this time-independent case the difference is negligible. Such formulas can be evaluated in weakly coupled models, such as free fermions [36, 37] and result to be dependent on the anomaly content of the theory.

For flavor anomalies a justification of the universality of this results comes from consistency of the hydrodynamic expansion with the Ward-Takahashi identities, since both the response and the anomaly enter at the same order in the derivative expansion, as it was shown by Son and Surowka [38].

For gravitational anomalies, while explicit computations both at strong and weak coupling [39, 37, 40] gave indications of response induced by the gravitational anomaly. The KK analysis of [32] shows that these Chern-Simons terms arise as regularized sums of the effective actions of the single Matsubara modes. The hydrodynamics arguments fail to give a proper account of such effects, since the anomaly only enters the constitutive relations at higher order.

An interesting idea, first proposed in [41] and then expanded upon in [42], is to consider the theory on a nontrivial curved geometry which asymptotically looks like the thermal cyclinder. This may either be a cone or an Eulidean cigar:

$$ds^2 = dr^2 + f(r)ds^2_{M_{d-1}} \,, \tag{2.15}$$

In this case derivatives in the "radial" direction violate the naive derivative expansion and allow to use arguments similar to those of Son to fix the response functions. This argument however fails if the geometry is just an Euclidean cylinder.

Another idea comes from holographic computations [39, 43, 44]. In holography these effects arise due to the extension of the gravitational Chern-Simons terms to the full group off bulk diffeomorphisms. Such extension cannot be trivial since the holographic background for a thermal state is an Euclidean black-hole, which has a non-vanishing extrinsic curvature. The appearance of the extrinsic curvature also breaks the derivative expansion. In [43] these effects were shown to be universal and independent on the fine details of the dynamical bulk geometry.

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[3]In that setup that is the name for $\Omega^i$.

The idea here is to further relax these assumptions, dropping the need for dynamical gravity in the bulk, but basic the universality of the result on metric compatibility and the structure of the Chern-Simons action. Indeed it is clear, as will be expanded upon later, that the inflow argument for gravitational terms needs an extension of the gauge group in the bulk. Such extension cannot be trivial, since the requirement of a single boundary does not allow for a nowhere-vanishing extrinsic curvature.

In the rest of the Chapter we develop the details of this intuitive picture, setting up the precise formulation for the inflow and connecting with the previous results, high-lighting the physical reasons why such line of reasoning indeed works and making contact with a method for computing partition functions in the presence of nontrivial holonomies.

As a final precisation, let us add that the Chern-Simons coefficients introduced above are universal objects as far as the anomalous symmetry is unbroken. By either explicitly breaking it in the UV by a relevant deformation (as studied e.g. in [45]) or by making background gauge field dynamical, to have an ABJ anomaly [46] the coefficients may run. In some cases this can be simply traced back to wavefunction renormalization of the gauge fields, however the problem is far from understood. In this aspect, the results presented from now on assume that no explicit breaking can be present at any energy scale and only 't Hooft anomalies are allowed.

## 2.2 Geometric argument and Abelian Inflow

We use this Section as a warm-up to concretely present the machinery at work in a simpler setup, review the arguments of [34] and put them under a slightly different light which will reveal useful in the next Section.

The geometric setup is given by equations (2.1),(2.2). We consider a theory with a $U(1)$ flavor anomaly $c_A$ and determine the dependence of the effective action on $\mathcal{M}_{d-1}$ on $c_A$. A way to evaluate this was given in [34] by considering an auxiliary connection $\hat{A}$ with no holonomy around the $S^1$. In our gauge choice this just means

$$\hat{A} = b_i dx^i \, , \tag{2.16}$$

it is also important that the electric field[4] $\hat{E} = i_u d\hat{A} = 0$ is vanishing, while for the full connection $\hat{E} = d\mu$. In the same way the magnetic fields are slightly different $B = db + \mu da$ while $\hat{B} = db$. This is the key property of the hatted connection which helps in practical computations, since top-degree forms of the hatted connection will automatically vanish.

Now let us give a practical flavor of how this connection enters in the computation of the partition function by a small geometrical detour. Our objective here is to compute the partition function on a fibration of $S^1$ over $\mathcal{M}_{d-1}$ for an anomalous quantum field theory, in the presence of nontrivial holonomies. Let us call it $Z[A]$ for short. This computation is of course extremely complicated, however we might be intersted in the phase of $Z[A]$, which should be determined by 't Hooft anomalies (we assume this to be the case). Since we are interested in the contribution

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[4]Since we have a Killing vector $u^\mu$, electric and magnetic fields are defined by decomposing the curvature $F = E \wedge u + B$.

coming from non-trivial holonomies we could as well compute

$$Z[A]Z^*[\hat{A}] \, . \tag{2.17}$$

This setup is not gauge invariant, since both partition functions suffer from 't Hooft anomalies but only the first one will transform non-trivially.

We can make the setup gauge invariant by imagining a fictitious bulk connecting the two copies of our spacetime with opposite orientation, the bulk connection will need to interpolate between $A$ and $\hat{A}$ at the two ends, and as such is it somewhat constrained, thus we examine instead

$$Z[A, \hat{A}]_{\text{inv}} = Z[A]e^{-ic_A P^{d+1}(A_{\text{bulk}})}Z^*[\hat{A}] \, , \tag{2.18}$$

where, for perturbative anomalies, $P^{d+1}$ is just a Chern-Simons form, as introduced previously, while $c_A$ is the anomaly coefficient of the theory. In this setup the boundary theories are anomaly-free, this means that the contribution of the anomaly coefficient to the effective action should vanish. Thus $Z[A, \hat{A}]_{\text{inv}}$ is a real number and we may evaluate

$$Z[A]Z^*[\hat{A}] = e^{ic_A P^{d+1}(A_{\text{bulk}})} \, . \tag{2.19}$$

In writing these expressions, we should make sure that the right-hand-side is *not* a functional of the bulk interpolation between the connections $A$, $\hat{A}$. If this can be shown, then the right hand side is secretly a functional of $A$ , $\hat{A}$ and we may in particular take well defined "boundary" variations, which we denote as a $\hat{\delta}$, that is, variations of the boundary conditions for the bulk functional.

A shortcoming of this formulation is that the interpolating connections may be not simple to define. However we can alternatively think of evaluating the same quantity by gluing a one-sided bulk to both $Z[A]$ and $Z[\hat{A}]$. This is consistent provided the integral on a closed $d+1$ surface with boundary conditions coming from each piece is trivial. In a one-sided bulk the gauge field should interpolate between the boundary connection and a trivial one where the local $S^1$ shrinks to zero. Then we may also write

$$Z[A]Z^*[\hat{A}] = e^{ic_A\left[P^{d+1}(A)-P^{d+1}(\hat{A})\right]} \, , \tag{2.20}$$

where on the right-hand side we mean bulk extensions. The advantage of this formula is that the right hand side may be evaluated with the help of transgression formulas. An artistic interpretation of this whole construction can be found in Figure 2.1.

Until now we have been somewhat careless about boundary terms. Recall that the transgression formula leads a two-term expression in which the second one is a total derivative. There will be then two kinds of contributions to (2.20) depending on whether they enter as bulk integrals or through boundary values.

We have already discussed that the Chern-Simons action in the anomaly inflow mechanism does two things: it cancels the consistent anomalies, but also provides Bardeen polynomials to render the current operators co-variant. The last of these came from the total derivative part in the transgression formula, so that we may identify the generator of the effective action for co-variant currents as the purely bulk piece of the transgression formula. Our goal is then to find a compact expression for such co-variant effective action. This also clarifies a point that
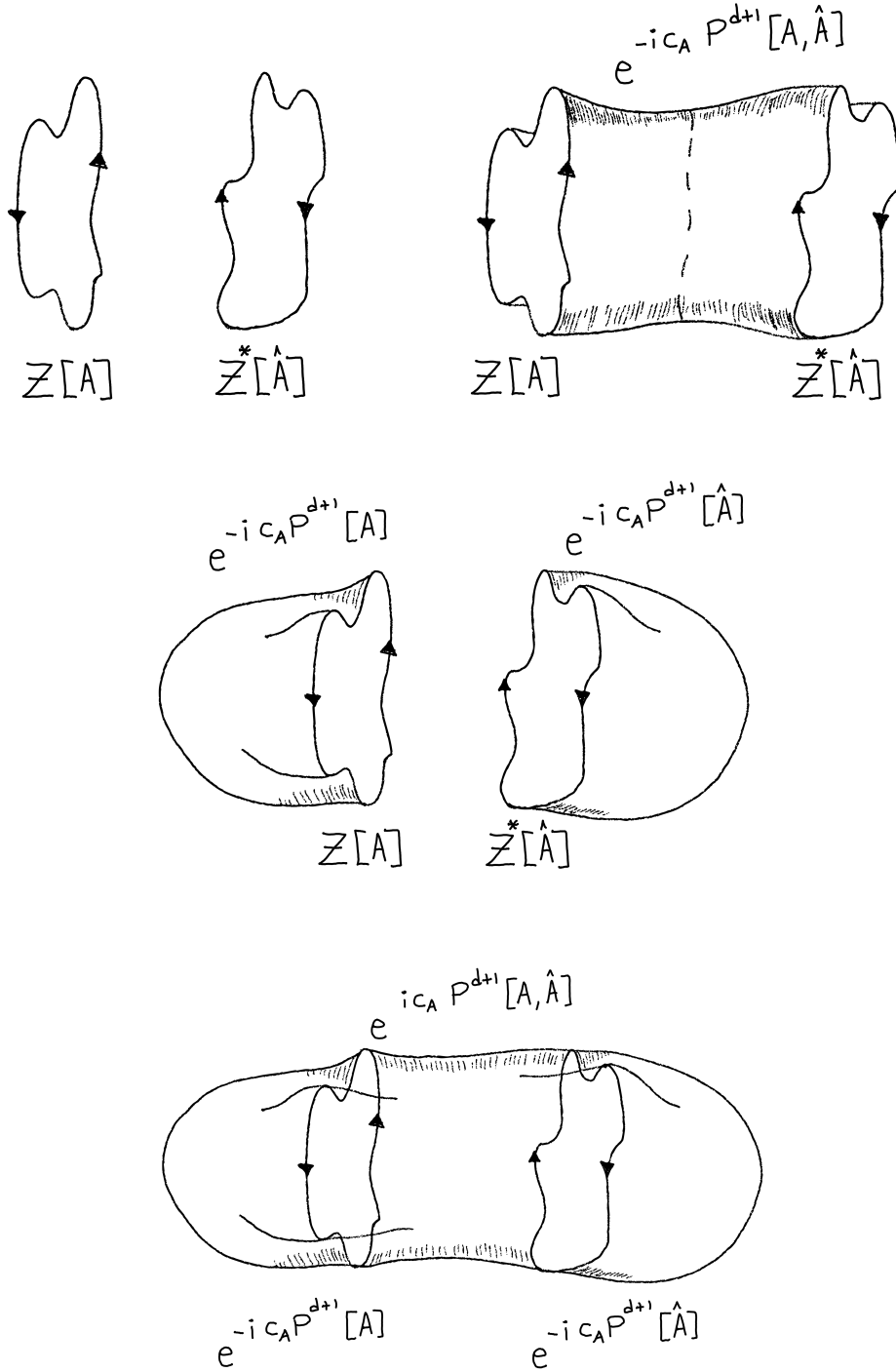
Figure 2.1: Steps of the inflow construction. From top-left to bottom right: holonomy-dependence of the phase of $Z[A]$, anomaly cancellation through inflow, inflow "piece by piece", full trivialized bulk description.

the careful reader may have already noticed was missing in our discussion, that is what current operators is the effective action (2.4), (2.3) describing. The answer is that we will take it to describe co-variant current correlators.

A second point of great importance is to understand why the seemingly $d + 1$ dimensional expression for the co-variant effective action obtained from the inflow argument should be describing a $d - 1$ dimensional effective action instead. The intuitive picture is that the $d + 1$ dimensional effective action is not computed on arbitrary bulk gauge field configurations. All admissible gauge fields have fixed components on $\mathcal{M}_{d-1}$ and fixed holonomies around the $S^1$ at the boundary. This second condition, since the $S^1$ will be contractible in the bulk, will fix certain two dimensional integrals of the curvature over the bulk "cigar" obtained by extending the $S^1$ in the bulk. Doing these integrals effectively lands on a $d - 1$ dimensional contribution. Let us now come to the formalization of the above arguments in the simplest case of pure $U(1)$ anomalies in $d = 2$ and $d = 4$. Most expressions readily generalize, although they become more cumbersome.

In the $U(1)$ case the formulas can be simplified quite a bit. This is due the following property (which is proved in [34]):in our variables, the transgression $P^{d+1}(A) - P^{d+1}(\hat{A})$ is a polynomial of strictly positive degree in $du = da$. This allows to use the formal identity $1 = d\left(u/da\right)$ to simplify:

$$P^{d+1}(A) - P^{d+1}(\hat{A}) = d\left(\frac{u}{da}\right)\left[P^{d+1}(A) - P^{d+1}(\hat{A})\right]$$
$$= d\left(u \wedge \frac{P^{d+1}(A) - P^{d+1}(\hat{A})}{da}\right) - u \wedge \frac{P^{d+2}(A) - P^{d+2}(\hat{A})}{da}, \qquad (2.21)$$
$$= V(A) + dW(A).$$

In the notation above, $V(A)$ generates the effective action for the co-variant currents, which will be our main interest in this Section. We write more clearly these important formulas once again

$$V(A) = c_A u \wedge \frac{P^{d+2}(A) - P^{d+2}(\hat{A})}{da}, \qquad (2.22)$$

$$W(A) = -c_A u \wedge \frac{P^{d+1}(A) - P^{d+1}(\hat{A})}{da}, \qquad (2.23)$$

where we have restored (minus) the anomaly coefficient for the anomaly cancellation on the boundary.

The evaluation is further simplified since both terms are wedged with $u$. Then only the magnetic part of the forms inside $P^{d+1}$ and $P^{d+2}$ may give a nonzero contribution. Let us see a couple of examples.

In $d = 2$ we have $P^4(A) = dA \wedge dA$ so

$$u \wedge \frac{P^4(A) - P^4(\hat{A})}{da} = u \wedge \frac{\mu^2 da \wedge da + 2\mu da \wedge db}{da} = u \wedge \left(\mu^2 da + 2\mu db\right), \qquad (2.24)$$

while

$$u \wedge \frac{P^3(A) - P^3(\hat{A})}{da} = u \wedge \mu b, \qquad (2.25)$$

on the other hand, in $d = 4$ $P^6 = dA^3$ so that:

$$u \wedge \frac{P^6(A) - P^6(\hat{A})}{da} = u \wedge \left(3\mu db^2 + 3\mu^2 da \wedge db + \mu^3 da^2\right) , \tag{2.26}$$

while

$$u \wedge \frac{P^5(A) - P^5(\hat{A})}{da} = u \wedge b \wedge \left(2\mu db + \mu^2 da\right) . \tag{2.27}$$

How does one derive the Chern-Simons action from this formulation? There are basically two ways.

One is to take functional derivative with respect to the magnetic sources. For $W$ this is trivial, since it is already a suface term, while for $V$ one has to use the fact that the magnetic sources only can enter through their curvature and may integrate by parts. This is the method described in [34], which is extremely elegant and allows for rather compact formulas in arbitrary dimension. One then ends up with the expression for the currents on the physical boundary, and may further notice that, since they are proportional to $u$, they can be reduced onto the base manifold. Confronting this with the effective Chern-Simons description fixes the effective action. Let us see this in the examples above:

In $d = 2$ we have

$$V(A)^{d=2} = c_A u \wedge \left(\mu^2 da + 2\mu db\right) , \tag{2.28}$$

upon variation this gives rise to a boundary term

$$\star J \wedge \delta b + \star J_\epsilon \wedge \delta a = 2c_A u\mu \wedge \delta b + c_A u\mu^2 \wedge \delta a \tag{2.29}$$

which is compatible with a co-variant Chern-Simons action

$$W_{cov}[a, b]^{d=2} = c_A \beta \int \left(2\mu b + \mu^2 a\right) . \tag{2.30}$$

Notice that the current $J$ satisfies the co-variant anomaly equation (in our choice of background)[5]

$$d \star J = 2c_A dA = 2c_A d(\mu u) . \tag{2.31}$$

The consistent current $\star\mathcal{J} = \star J - c_A u\mu$ comes from the full action and satisfies the consistent anomaly

$$d \star \mathcal{J} = c_A dA . \tag{2.32}$$

In four dimensions the procedure is similar, although slightly longer, starting from

$$V(A)^{d=4} = c_A u \wedge \left(3\mu db^2 + 3\mu^2 da \wedge db + \mu^3 da^2\right) \tag{2.33}$$

which gives

$$\star J \wedge \delta b + \star J_\epsilon \wedge \delta a = c_A u \left(6\mu db + 3\mu^2 da\right) \wedge \delta b + c_A u \left(2\mu^3 da + 3\mu^2 db\right) \wedge \delta a \tag{2.34}$$

---

[5]Is is useful here to comment a bit on how the equation below should be interpreted. Usually one is not allowed to identify chemical potential gradients and electric fields in anomalous quantum field theories (we expand on this point further through the Chapter). This always generates a great deal of confusion. However, as long as we are in thermal equilibrium, such identification can be justified [34]. The divergence equation below should be interpreted in this way.

from which

$$W_{cov}[a,b]^{d=4} = c_A \, \beta \int \left( 3\mu b \, db + 3\mu^2 a \, db + \mu^3 a \, da \right) . \tag{2.35}$$

Which indeed gives a currents satisfying the co-variant anomaly

$$d \star J = 3c_A dA \wedge dA . \tag{2.36}$$

the consistent current, on the other hand, is $\star \mathcal{J} = \star J - c_A \, u \wedge (4db + 2da)$ satisfies the consistent anomaly

$$d \star \mathcal{J} = c_A \, dA \wedge dA . \tag{2.37}$$

However, we might also want to derive *directly* the dimensional reduction by starting with the full Chern-Simons action. While this maybe somewhat less elegant in general, it give s a clear physical intuition about the problem and will be of great use in extending the treatment to gravitational anomalies, in which the simple functional differentiation is not sufficient. The tactic here can be resumed in isolating the bulk derivatives of the holonomy and use the regularity conditions to fix their value and dimensionally reduce the system to $d-1$ dimensions. For flavor symmetries this is not very convenient, since Bardeen-Zumino polynomials have to be added essentially by hand to find the co-variant currents, however there are no such polynomials for the gravitational case.

To continue the discussion, however, it is useful to properly set up the problem of bulk extensions so as to make the computation of the effective action as clear as possible. We do this, as well as discuss the incorporation of gravitational anomalies, in the next Section.

## 2.3 Thermal & abelian inflow from bulk extensions

In this section we introduce the generalization of the inflow arguments for gravitational and mixed-gravitational anomalies. We will work always in terms of Lorentz anomalies, however the conclusions are the same for the diffeomorphism case, since the two are related by a Bardeen counterterm. We will consider mainly embedding in a bulk with only one boundary, since those are the relevant ones to capture the purely gravitational contributions.

The main idea is that, to define the inflow, we need to continue the boundary Riemannian geometry into the bulk. This can be done in several ways, but we will stress that all allowed ways should be co-variant under diffeomorphisms. The simplest such procedure is to start from a boundary geometry equipped with a compatible vielbein $e^a$, $\nabla_\mu e^a_\nu = 0$ and a spin connection determined by the previous equation. In embedding this into the bulk geometry we introduce an *extended* vielbein $e^A$, with $A$ an $SO(d+1)$ index. Notice that this is a double extension in both the "form" components, which now are $e^A_M$, $M = 0, .., d$ and the tangent space components.

We will also treat in more detail the extension of Abelian gauge fields, which will be useful in giving an alternative derivation of the Chern-Simons action in $d-1$ dimensions. In this case there is no group extension, but regularity conditions in the bulk should be carefully imposed. To make this compatible with the boundary geometry we ask the following

1. The restriction of $e^A$ on $\mathcal{M}_{d-1} \times S^1$ coincides (up to gauge transformations) with $e^a$.

2. In the bulk the compatibility condition is automatically updated to $\nabla_M e_N^A = 0$. This fixes univocally the form of the bulk spin connection $\omega^{AB}$.

3. We will consider extensions with vanishing extrinsic curvature $K_{MN}$ at the boundary. While a nontrivial extrinsic curvature in the bulk is needed to "close off" the manifold, boundary extrinsic curvature only results in the necessity of adding $K$-dependent counterterms to the effective action. This is well known from standard anomaly inflow, but in our cases it will only produce unnecessary clutter.

4. As far as the gauge field is concerned, it needs to have a fixed boundary along the $S^1$ given by $\mu\beta$, as well as a regular bulk extension. Since the $S^1$ is contractible inside the bulk, this will give us also some local regularity conditions in a given coordinate system for the field strength at the point where the $S^1$ shrinks to zero size.

This is by no means the most general extension, it is however the most familiar one. Another interesting idea may be to extend the boundary Riemannian geometry to a Newton-Cartan geometry in the bulk. We will introduce such geometries in Chapter 3 with a different end, however it would be an interesting case to study on its own.

Before going forward it will be useful to present an intuitive picture of the role of extrinsic curvature and of the regularity conditions for the gauge field. For that we examine the simplest extension of $S^1$ into an Euclidean cigar/ disk. If $\tau \sim \tau + \beta$ is the coordinate on the $S^1$, we consider metrics of the type

$$ds^2 = \frac{dR^2}{f(R)} + f(R)d\tau^2 \,, \tag{2.38}$$

we furthermore choose the boundary to lay at, say $R = R_0$ and $f(R \to R_0) = 1 + O(R - R_0)^2$. The normal vector to the $S^1$ is just $n = f(R)^{1/2}\partial_R$, which is normalized to unity. Defining $n_M$ in the obvious way, the metric reads $g_{MN} = n_M n_N + h_{MN}$. The extrinsic curvature is defined as

$$K_{MN} = \frac{1}{2}\mathcal{L}_n h_{MN} \,, \tag{2.39}$$

in our coordinates this just reads

$$K_{\tau\tau} = \frac{1}{2}f(R)^{1/2}\partial_R f(R) \,. \tag{2.40}$$

As one can check this indeed vanishes at the boundary[6]. Now we would like to deal with a geometry with only one boundary. In these coordinates the only possibility is to shrink the $\tau$ cycle to zero size for some $R = R_H$. Assuming this is a simple zero we have $f(R \to R_H) = f'(R_H)(R - R_H) + O(R - R_H)^2$. Then we may notice that the scalar

$$\int_{R_H} d\tau \sqrt{h}K = \frac{1}{2}\beta f'(R_H) \,, \tag{2.41}$$

with $K = K_{MN}h^{MN}$, is non-vanishing. Furthermore, the geometry is even more constrained, since we may introduce a coordinate $\rho^2 = (R - R_H)$ to write, near $R_H$

$$ds^2 = \frac{4\rho^2}{f'(R_H)} + f'(R_H)\rho^2 d\tau^2 \,, \tag{2.42}$$

---

[6]More precisely, the scalar $\int \sqrt{h}K$ vanishes, when one deals with the other end of the geometry it is always possible to raise or lower indices to make the resulting expressions vanishing or divergent. The scalar we have just introduced will however be a constant.

which is just the metric on a cone with angular coordinate $\theta = \frac{f'(R_H)}{2}\tau$. Imposing the absence of conical defects, that is $\theta \sim \theta + 2\pi$ and taking in account the periodicity of $\tau$ we get

$$f'(R_H) = 4\pi\beta^{-1}\,, \tag{2.43}$$

and the integral (2.41) becomes just $2\pi$. From this it should be clear that, in *regular*, one sided geometries bulk extrinsic curvature is strongly constrained by the boundary holonomy $\int d\tau = \beta$. In our story something very similar will be used to fix the thermal part of the effective Chern-Simons action.

We can also add a gauge field to this story, at the boundary this is given by just $A = \mu d\tau$, with $\mu$ a constant. In the bulk this must be extended. Particularly important is the presence of a non-vanishing curvature $F_{\tau R}$. Regularity would amount to ask the holonomy to vanish at the center $R_H$ of the cigar, while assuming that no electric charge seats there also fixes $F_{\tau R})(R_H) = 0$.

Computing the effective action with this extension, however, raises various subtleties, due to the presence of boundary Bardeen-Zumino currents. This discussion has already been widely addressed in the context of holography [47] and Quantum Field Theory [48]. In the holographic context, solutions to the dynamical equations allow for an unspecified boundary holonomy, which leads to different consistent currents after computation. In quantum field theory the explanation is the following. Let $Z_{\text{twisted}}[A]$ be the partition function obtained in a background with trivial holonomy for the external gauge fields but with twisted boundary conditions around the thermal circle for the dynamical fields

$$\Psi(\tau - \beta) = e^{-\beta\mu}(-)^F \Psi(\tau)\,, \tag{2.44}$$

for non anomalous theories, one can alternatively give a non-vanishing holonomy to the gauge field which twists the Hamiltonian of the system, but leaves boundary conditions untouched. In the anomalous case such large gauge transformation gives rise to a non-trivial $\theta$-term which is finite under partial integration, schematically

$$Z_{\text{twisted}}[A] = Z_{\text{untwisted}}(A') \exp\left(-c_A \beta \tilde{P}^{d-1}(A', R)\right)\,. \tag{2.45}$$

Once we attach a bulk for the inflow mechanism, the system is invariant under such change, which may however be achieved through a bulk gauge transformation in the extra direction. The right-hand side above will be represented by the bulk extension with nontrivial boundary holonomy, while the left-hand side will have a trivial holonomy at the boundary, but a nontrivial one (equal to $-\beta\mu$) at $R_H$. The invariant quantity, which is given by the bulk integral of the field strength, is of course fixed and gauge invariant. We will choose this second option in the calculations to follow.

After the extension is defined, one may ask in a meaningful way how this impacts the Chern-Simons term in the inflow mechanism. To answer such question it is useful to decompose the bulk spin connection in the following way:

$$\omega = \frac{1}{2}\omega^{ab}J_{ab} + B^a H_a\,, \tag{2.46}$$

with $J^{ab}$ the $SO(d)$ generators for the boundary rotations and $H^a$ the generators of the coset $SO(d+1)/SO(d)$. The way to define the splitting will be natural once an approriate gauge for the metric is fixed.

It is then meaningful to define the following transgression form from $SO(d)$ to $SO(d+1)$:

$$\mathcal{I}^{d+1}(\omega^{ab}, B^a) = P^{d+1}(SO(d+1)) - P^{d+1}(SO(d)), \tag{2.47}$$

with the right hand side defined by the transgression formula of the second kind (1.35) applied to $A_t = \omega^{ab}J_{ab} + tB^aH_a$. Since $B^a$ is an $SO(d)$ vector, it follows from the transgression formula that $\mathcal{I}^{d+1}$ is gauge invariant under $SO(d)$. Furthermore, if we keep the extrinsic curvature vanishing, also the transformation generated by $H_a$ cannot reach the boundary. We have thus splitted the contribution from the Lorentz anomaly from a *co-variant* contribution coming from the bulk extension. The main goal of this Section is to study the properties of (2.47) to argue how it fixes the purely temperature dependent Chern-Simons terms in the low energy effective action.

The discussion above has been for purely gravitational anomalies. The extension to mixed anomalies is simply given by wedging (2.47) with the appropriate Chern-class for the $U(1)$ connection.

### 2.3.1 ADM gauge and relevant quantities

Let us start by fixing a convenient gauge for our study. This will be instrumental to define the splitting of the rotation group as well as in deriving explicit formulas for evaluation. There are two such choices from the author's perspective. Let $n^M$ be the normal vector to the foliation we want to introduce, one such choice is to set $i_N\omega^{AB} = 0$. Another, more convenient choice is to use the Feffermann-Graham gauge in ADM coordinates

$$ds^2 = dr^2 + h_{\mu\nu}dx^\mu dx^\nu, \tag{2.48}$$

with $h_{\mu\nu}$ reducing to the boundary metric at $r \to 0$[7]. In this coordinates the extrinsic curvatures is $K_{\mu\nu} = \frac{1}{2}\dot{h}_{\mu\nu}$. In this coordinates one will have a fictitious singularity near the point where the $S^1$ shrinks to zero size (the choice of such bulk point is also a coordinate artifact), however physical quantities will be singularity-free. We also have the usual ADM decomposition for the Christoffel symbols

$$\Gamma^r{}_{\mu\nu} = K_{\mu\nu}, \quad \Gamma^\mu{}_{\nu r} = -h^{\mu\rho}K_{\rho\nu}, \tag{2.49}$$

given the metric (2.48) we may define the vielbein $e^A = (dr, e^a)$ with $e^a_\mu e^b_\nu \delta_{ab} = h_{\mu\nu}$. This also defines the splitting of the generators of $SO(d+1)$, since we take the $a$ indeces to transform under the $SO(d)$ subgroup. It remains to impose the compatibility conditions to determine the spin connection components $\omega^{ab}$ and $B^a = \omega^{ar}$. We will do this with care since their form will be needed.

Explicitly the compatibility condition reads

$$0 = \nabla_M e^A_N = \partial_M e^A_N - \Gamma^P{}_{MN}e^A_P - \omega_M{}^A{}_B e^B_N. \tag{2.50}$$

We solve these equations component by component starting with $A = r$ which gives

$$K_{\mu\nu} = B^a_\mu e_{\nu a}, \quad B^a_r = 0, \tag{2.51}$$

---

[7]We choose the radial coordinate so that the normal vector points towards the tip of the cigar.

or

$$B^a_\mu = K_{\mu\nu} E^{\nu a} \,. \tag{2.52}$$

This is a very important formula for us, since it allows to conclude that one needs to extend the spin connection to $SO(d+1)$ in the presence of extrinsic curvature. Recalling the definition of $K_{\mu\nu}$ one finds explicitly

$$B^a_\mu = \dot{e}^a_\mu - e_{\nu b} \left( e^b_\mu \dot{E}^{\nu a} \right) , \tag{2.53}$$

where the second term arises to assure that $B^a$ transforms as a vector under $SO(d)$. The rest of the equations give constraints involving $e^a_\mu$. Setting $N = r$ gives the same constraint that we have already solved, the remaining two equations read

$$\dot{e}^a_\mu - K_{\mu\nu} E^{\nu a} - \omega_r{}^a{}_b e^b_\nu = 0 \,, \tag{2.54}$$

$$\partial_\mu e^a_\nu - \Gamma^\rho_{\mu\nu} e^a_\rho - \omega_\mu{}^a{}_b e^b_\nu = 0 \,, \tag{2.55}$$

so that $\omega_\mu{}^{ab}$ is given by the usual $SO(d)$ formula, while

$$\omega_r{}^{ab} = E^{\mu b} e_{\nu c} \left( e^c_\mu \dot{E}^{\nu a} \right) , \tag{2.56}$$

this allows to write in a coordinate-free manner

$$B^a = \mathcal{L}^\omega_n \, e^a \,, \tag{2.57}$$

with $\mathcal{L}^\omega_n = \mathcal{L}_n - [\omega, \cdot]$ is a lie derivative co-variantized with respect to $SO(d)$. With the boundary conditions of vanishing extrinsic curvature it is simple to see that, as we approach the boundary, the field $B^a$ dies off, while $\omega^{ab}$ naturally coincides with the boundary spin connection. This last observation is important, since it implies that total derivative terms in the transgression formula (2.47) do not give any contribution.

Let us also consider a simplified setup in which we extend as before $S^1$ to an Euclidean cigar in these coordinates. The metric reads

$$ds^2 = dr^2 + h(r)d\tau^2 \,, \tag{2.58}$$

in confronting with the previous coordinate $R$, we have

$$\frac{dR}{dr} = \sqrt{h(r)} \,, \quad h(r) = f(R(r)) \,, \tag{2.59}$$

so that the regularity condition reads, if $r(R_H) = 1$

$$f'(R_H) = \lim_{r \to 1} \frac{\dot{h}(r)}{\sqrt{h(r)}} = 2\sqrt{\dot{h}(1)} = 4\pi\beta^{-1} \,, \tag{2.60}$$

Using this we can evaluate the gauge field $B = K_{\tau\tau} E^\tau d\tau$ , which is

$$B_\tau = \frac{1}{2} f'(R(r)) \,, \tag{2.61}$$

where the prime denotes derivative with respect to $R$. Notice that the scalar invariant we had used before $\int \sqrt{h} K = \int B$ is just the line integral of the gauge field $B$. One can generalize this situation to the case at hand. First let us suppose that the bulk geometry has also a global

Killing vector $\beta^M$ extending the boundary thermal isometry. In this case we may define a frame field $\beta^A = e_M^A \beta^M$ which is co-variantly constant

$$\nabla_M \beta^A = 0\,, \tag{2.62}$$

this equation follows from asking the vielbein $e^A$ to be invariant under the isometry generated by $\beta^M$ up to a frame rotation. One we have this vector at our disposal, we may use it to contract $A$ indices, and define the holonomies

$$w_\beta[e] = \oint \sqrt{\gamma} e_M^A \beta_A dx^M\,, \tag{2.63}$$

$$w_\beta[B] = \oint \sqrt{\gamma} B_M^A \beta_A dx^M\,, \tag{2.64}$$

where the integral can be thought as being taken around the $S^1$ at some fixed bulk coordinate $r$ and $\gamma$ is the induced metric. The first line computes the length of the local thermal circle, while the second is the extrinsic curvature invariant we had introduced before, as one can check by direct computation.

In treating flavor and mixed anomalies, we will also need to extend the gauge field into the bulk. Following our previous discussion we will choose a gauge such that:

$$A_r = 0 \tag{2.65}$$

and

$$\lim_{r \to 0} A_\tau = 0\,, \quad \lim_{r \to 1} A_\tau = -\mu\,, \tag{2.66}$$

through

$$A = g(r)(d\tau + a_i(x)dx^i) + b_i dx^i\,, \tag{2.67}$$

with $\lim_{r \to 0} g(r) = 0$, $\lim_{r \to 1} g(r) = -\mu$.

### 2.3.2 Extension independence and the eta-invariant

A key fact which we need in order to justify our computation is the ability to evaluate the bulk Chern-Simons action through a simple choice of background.

In the Abelian case this is not very hard to show. The idea is to take two geometries with different bulk extensions, say $A_1$ and $A_2$ with bulks $\mathcal{N}_1$ and $\mathcal{N}_2$. The gauge fields need to coincide (up to small gauge transformations) at the boundary of $\mathcal{N}_{1,2}$. We can then take the Chern-Simons action for $A_2$, reverse its orientation and glue it back to the one for $A_1$. This gives the variation of the Chern-Simons action due to the bulk extension, and it is evaluated as a Chern-Simons term on a closed manifold.

In this evaluation one can always single out an integral of the curvature over the two Euclidean cigars (which together are topologically a sphere). Since we have chosen the holonomies to coincide on the common "boundary" $S^1$ this makes the first Chern-class to vanish and so does the full integral. This is a particular application of a general statement regarding extension independence (mathematically, cobordism invariance) of fermionic partition functions.

For the general, gravitational case, there is no such simple proof of background independence, but there are explicit arguments for a given system which use the APS $\eta$ invariant. It is

well known, see e.g. [49, 50, 51], that the path integral for a Dirac fermion in the presence of background gauge field and spin connection can be made a well defined object (in a geometrical sense) by defining it though a massive bulk fermion subject to particular boundary conditions. This path integral can be shown to give

$$Z(bulk) = |Z_{Dirac}|e^{-i\pi\eta_{\mathcal{N}}/2} \tag{2.68}$$

with the APS $\eta$ invariant:

$$\eta_{\mathcal{N}} = \sum_{\lambda} sign(\lambda) \,, \tag{2.69}$$

with $\lambda$ the eigenvalues of the Dirac operator in the bulk and the convention $sign(0) = 1$ and an appropriate regularization. This is a generalization of the perturbative result of the anomaly inflow, in which the phase of $|Z_{Dirac}|$ is cancelled by a Chern-Simons term. Here the presence of the $\eta$ invariant signals possible non-perturbative anomalies.

From this it is possible to define a partition function for the Dirac fermion which is independent of the bulk extensions of the fields on $\mathcal{N}$:

$$Z_{\Psi}[A] = |Z_{\Psi}| \exp\left( i \int_{\mathcal{N}} P^{d+1}(A) \right) \exp\left( \pi i \eta_{\mathcal{N}}/2 \right) \,, \tag{2.70}$$

While the Chern-Simons term gives the perturbative anomalies, the $\eta$ invariant carries global information only. The $\eta$ invariant has nice properties under gluing and cutting of manifold. In particular, if $\mathcal{N}_1$ and $\mathcal{N}_2$ are glued together through a common boundary $\mathcal{M}$, then

$$\exp\left( \pi i \eta_{\mathcal{N}_1} \right) \exp\left( -\pi i \eta_{\mathcal{N}_2} \right) = \exp\left( \pi i \eta_{\mathcal{N}_1 \cup -\mathcal{N}_2} \right) \,, \tag{2.71}$$

furthermore, due to the APS index theorem, on a closed manifold $\bar{\mathcal{N}}$ the two terms are related by the index theorem

$$\mathcal{I} = \int_{\bar{\mathcal{N}}} P^{d+1}(A) - \eta/2 \,, \tag{2.72}$$

thus if the index is even the difference in the phase of $Z_{\Psi}[A]$ between different extensions cancels. This property makes the expression (2.70) independent on the chosen bulk extension of the fields.

Thus, if our proposal for the phase of the partition function is sensible, it *needs* to be extension-independent too. It is important to notice that, in equation (2.70), the putative Chern-Simons term for an $SO(d)$ spin connection contains the $SO(d)$ Riemannian curvature extended in the bulk. Our idea then can be resumed by saying that the extension of the Riemann tensor from $SO(d)$ to $SO(d+1)$ gives a way to compute the $\eta$ invariant. Let us give some intuition behind this, although a completely general argument is still lacking.

One clue comes by using Feffermann-Graham coordinates to evaluate the gravitational Chern-Simons terms for $SO(d+1)$ on a closed manifold $\mathcal{N}$. This can be done by evaluating the respective Pontryagin class on a new manifold $\mathcal{P}$ with boundary $\mathcal{N}$. This gives a relation between the $SO(d)$ Chern-Simons term and the $SO(d)$ invariant part given by the transgression. For example, in the case of $p_1(R)$

$$p_1(SO(d+1)) = p_1(SO(d)) + \frac{\nabla_J B^a}{2\pi} \wedge \frac{\nabla_J B^a}{2\pi} \,, \tag{2.73}$$

on a closed three manifold for which the four dimensional $p_1(SO(d+1))$ is trivial, this gives the desired relation by looking at (2.72).

Another interesting idea may come from something that often works for flavor anomalies. When one wants to study global anomalies [52, 53], one looks for elements of the global symmetry group which cannot be deformed to the identity. However, it is often the case that such elements can be seen as continuously connected to the identity if one extends the symmetry group. This indeed happens for rotations, for example a parity transformation in two dimensions cannot be brought to the identity by an $SO(2)$ rotation, but it can be brought to the identity by a continuous $SO(3)$ transformation (of course, disregarding the action on the added dimension). In such a framework, one may try and see non-perturbative anomalies as perturbative anomalies of extended groups. In would be nice to understand our construction in these terms, however one cannot embed the $T$ transformation which gives rise to the global gravitational anomalies in $SO(3)$.

A way to phrase our construction then is the following: starting from the extension-invariant partition function we may want to compute the phase when the boundary fermions live on a cylinder. The total phase will be independent on the extension, however the single terms $\eta/2$ and $\int P^{d+1}(A)$ are not. If one chooses to keep the bulk fields $SO(d)$ connections, one needs to go through a complicated analysis to determine the $\eta$ invariant for the bulk Dirac operator (This is not simply given by the index theorem, since the bulk manifold is not closed). Otherwise, one may extend the Chern-Simons term in a co-variant way, incorporating the result of a difficult computation of the $\eta$ invariant in a geometric substitution.

This is not guaranteed to always work, indeed there is no a-priori way to argue that no further $d-1$ Chern-Simons contributions appear in the $\eta$ invariant. In some systems this does not happen, in which case the substitution rule of [54, 55, 56] holds. For more complicated systems, however (e.g. gravitons and gravitinos) the may be further massless modes in the bulk near the coordinate singularity which should be treated with care. These may contribute to the formula for the partition function and give further Chern-Simons terms in $d-1$ dimensions. These will be however properly quantized, as the global anomalies for such systems have been determined by using the index theorems such as (2.72).

The explanation of such contributions are an interesting object for future studies and, if computed, would finally seal the deal on this by now long story of thermal Chern-Simons terms. This also allows to give a tentative explanation for the coefficient of such Chern-Simons terms

1. They are *not* completely fixed by analyzing the global anomalies. Indeed, such analysis is blind to properly quantized Chern-Simons terms in $d-1$ dimensions, which however cannot descend from any local counterterm in $d$ dimensions. One explanation in view of what we have said before is that the contributions of further massless modes at coordinate singularities cancel on closed manifolds (since they have two such singularities with opposite orientation).

2. They are *not* completely fixed by perturbative anomalies either, for the (speculative) reasons above.

Rather they are fixed by a more complicated procedure which should insist on defining a bordism invariant partition function for the anomalous theory. This needs to contain both

kinds of anomalies which interact in non-obvious ways. In what follows we will assume to work with a system to which the substitution rule applies and show explicitly how it arises by the bulk extension $SO(d) \to SO(d+1)$.

### 2.3.3 A different perspective on Abelian inflow

Let us go through the two dimensional case for simplicity. The Chern-Simons term is just

$$P^{3,0}(A) = -c_A \int A \wedge dA \,, \tag{2.74}$$

and we decompose $A = \mu \wedge u + b$, thus isolating the holonomy. Now we evaluate the Chern-Simons action with fixed boundary holonomy explicitly, and subtract the cuadratic part in the $b$-field. The linear part in $b$ reads

$$- c_A \int 2d\,(\mu u) \wedge b + c_A \int_{\mathcal{M}_2} \mu u \wedge b \,, \tag{2.75}$$

using the fact that the holonomy integrated over the Euclidean cylinder is fixed and dimensionally reducing the second term we get (and taking into account the minus sign from the inflow)

$$W(b) = c_A \beta \int \mu \, b \,, \tag{2.76}$$

the co-variant Chern-Simons action is found by adding back the Bardeen-Zumino polynomial to be

$$W_{cov}(b) = 2c_A \beta \int \mu \, b \,, \tag{2.77}$$

as it should. Notice that in this case we did not need any variation but only to properly taking into account the analytic continuation into the bulk with fixed holonomies. Alternatively, we may work in our gauge choice and extract directly the co-variant Chern-Simons action.

Now we deal with the effective action for the energy current. In our background choice the Chern-Simons term simply reads

$$- c_A \int \frac{d}{dr}(g(r)^2)drd\tau \wedge a \,, \tag{2.78}$$

which can be evaluated using the boundary conditions on $g(r)$ to give

$$W_{cov}(a,b) = 2c_A \beta \int \mu \, b + c_A \beta \int \mu^2 a \,. \tag{2.79}$$

The proof of the same statements in four dimension is completely analogous. In that case to it is apparent that the boundary conditions on the holonomy uniquely fix the answer.

### 2.3.4 Computing the transgression $\mathcal{I}^{d+1}$

Now we may come to the expression for $\mathcal{I}^{d+1}$, which we have introduced in (2.47). We first derive a formal representation, then we evaluate its boundary variation in a simple bulk extension to give the effective action. It pays to start with a formal computation at the level of Lie algebra.

We consider a subalgebra $\mathfrak{h}$ of the full Lie algebra $\mathfrak{g}$ and the splitting $\mathfrak{g} = \mathfrak{j} \oplus \mathfrak{h}$ such that $[\mathfrak{j}, \mathfrak{j}] \in \mathfrak{j}$ , $[\mathfrak{j}, \mathfrak{h}] \in \mathfrak{h}$ and $[\mathfrak{h}, \mathfrak{h}] \in \mathfrak{j}$. Using components $J_i$, $H_a$ this reads

$$[J_i, J_j] = f_{ij}{}^k J_k \,, \tag{2.80}$$

$$[H_a, H_b] = h_{ab}{}^i J_i \,, \tag{2.81}$$

$$[J_i, H_a] = l_{ia}{}^b H_b \,, \tag{2.82}$$

the $J_i$ generate a subgroup $J$ of $G$ and the $H_a$ generate the coset $G/J$. We will work with e a $\mathfrak{g}$-valued connection $A_G$ and split it into

$$A_G = A_J + B_H \,, \quad A_J = A_J^i J_i \,, \quad B_H = B_H^a H_a \,. \tag{2.83}$$

Let us give also the formulas for our specific case. Recall the splitting (2.46). The commutation relations between the generators $J^{ab}$, $H_a$ take the form

$$[J_{ab}, J_{cd}] = f_{ab,cd}{}^{ef} J_{ef} \,, \tag{2.84}$$

$$[H_a, H_b] = h_{ab}{}^{ef} J_{ef} \,, \tag{2.85}$$

$$[J_{ab}, H_c] = l_{ab,c}{}^{d} H_d \,, \tag{2.86}$$

with $f_{ab,cd}{}^{ef}$ the $SO(d)$ structure constants, $h_{ab}{}^{ef} = \delta_a^e \delta_b^f - \delta_a^f \delta_b^e$ and $l_{ab,c}{}^{d} = \delta_{ac}\delta_b^d - \delta_a^d \delta_{bc}$. Recalling that we are interested in cases in which $B_H$ vanishes at the boundary, the transgression formula becomes

$$\mathcal{I}[A_J, \ B_H] = \int_0^1 dt \, \mathrm{tr}\left[\partial_t A_t \wedge \frac{\partial P^{d+2}(A_t)}{\partial F_t}\right] \,, \tag{2.87}$$

with the interpolating connection $A_t = A_J + t B_H$. As for the connection, the curvature $F_t$ also admits a split

$$F_t = F_{t,J} + F_{t,H} \,, \tag{2.88}$$

with

$$F_{t,J} = (dA_J + A_J \wedge A_J) + t^2 B_H \wedge B_H = F_J + t^2 B_H \wedge B_H \,, \tag{2.89}$$

and

$$F_{t,H} = t\,(dB_H + [A_J, B_H]) \equiv t\nabla_J B_H \,. \tag{2.90}$$

The strategy is then to expand the partial derivative in (2.87) and compute the non-vanishing traces. This can become increasingly hard as the dimensionality grows since

1. The number of possible characteristic polynomials which enters $P^{d+2}$ gets significantly bigger in more than four dimensions. For specific theories the way in which they appear is known and fixed, but the expressions can become far from trivial.

2. One needs to know how to decompose the trace operation over the $\mathfrak{so}(d+1)$ Lie algebra in traces over the chosen $\mathfrak{so}(d)$ subalgebra. Which is a cumbersome exercise in representation theory.

It is instructive to guess first what non-trivial terms may arise. In the end we are mostly interested in contributions that are of low order in derivatives over $\mathcal{M}_{d-1}$, indeed the Chern-Simons terms we wish to fix are the lowest order of them all compatible with gauge invariance.

Recall that $\mathcal{I}$ is, by definition, an j-invariant expression. j-invariant building blocks are polynomials in $B_H$, $\nabla_J B_H$ and the curvature $F_J$. The latter can either combine between themselves as characteristic polynomials of the j algebra, of contract with the $B_H$s. The way in this such contractions appear is however fixed by the form of $F_{t,J}$. The precise form of the final answer heavily depends on the choice of j. What we may point out is the existence of a universal term, which exists even for vanishing $F_J$. This reads

$$\mathcal{I}[B_H, A_J^{\text{flat}}] = \mathcal{P}_{a_1,\dots,a_m}^{(m)} \left[ B_H^{a_1} \wedge (\nabla_J B_H)^{a_2} \wedge \dots \wedge (\nabla_J B_H)^{a_m} \right], \quad 2m - 2 = d, \tag{2.91}$$

where $\mathcal{P}^{(m)}$ contains the different traces coming from the characteristic polynomial $P^{d+2}$. Notice that a nontrivial contribution to $\mathcal{P}^m$ should always be present in $d = 4k + 2$ by the k+1-th power of the first Pontryagin class, which would give

$$\mathcal{P}^{m=2k+2} \sim \delta_{a_1 a_2} \dots \delta_{a_{m-1} a_m}, \tag{2.92}$$

also, knowing that pure gravitational anomalies only exist in $d = 2 + 4k$ already tells us that this is where such term will appear. In other even dimensions one can still get a mixed anomaly with a lower dimensional gravitational contribution. In this case one will get a characteristic class of the $U(1)$ bundle times a $\mathcal{P}^r$ with $r < m$.

Let us treat an important example, for $\mathfrak{g} = \mathfrak{so}(d+1)$, $\mathfrak{j} = \mathfrak{so}(d)$ and $d = 2$, one starts with $P^4 = p_1$ the first Pontryagin class and:

$$\mathcal{I}[B_H, A_J^{\text{flat}}]^{d=2} = 2 \left[ B^a \wedge \nabla_\omega B_a \right], \tag{2.93}$$

while in $d = 4$ there is no such term since the $P^6(SO(5))$ vanishes identically. Also, for $d = 2$ one may compute the full trace and show that the universal piece is the complete answer:

$$\mathcal{I}[B_H, A_J]^{d=2} = 2 \left[ B^a \wedge \nabla_J B_a \right], \tag{2.94}$$

this will be the main ingredient in our examples in two and four dimensions. The four dimensional case comes by wedging the $U(1)$ curvature $F$ with the previous equation.

For further terms, it will be useful to make contact with the substitution rule of [35]. The authors of the paper introduce an auxiliary gauge field $A_T = 2\pi T u$ and argue that the correct Chern-Simons terms may be derive by using the normal inflow methods, as in the abelian case, but substituting $P^{d+2}$ with $P_T^{d+2}$ given by

$$P_T^{d+2} = P^{d+2} \left\{ p_k(R) \to p_k(R) - \left( \frac{dA_T}{2\pi} \right)^2 \wedge p_{k-1}(R) \right\}, \tag{2.95}$$

with $p_k$ the k-th Pontryagin class and $p_0 = -1$, while $R$ is the $SO(d)$ curvature. In their case the auxiliary gauge field had to introduced somewhat "ad-hoc". In our case, however, as the reader might have guessed, it will come naturally from the embedding into $SO(d+1)$ and is roughly the value of $B^a$ at the tip of the geometry (given a particular choice of coordinates). The formulas above should serve as a first step in proving the substitution formula, however evaluating higher traces appears to be necessary test our hypotesis. This is a daunting task. We can however simplify it considerably by looking at the generating function of the Pontryagin classes

$$\det\left( \mathbb{I} + \frac{tR}{2\pi} \right) = \sum_k t^{2k} p_k(R), \tag{2.96}$$

and evaluate it for the $SO(d+1)$ extension. In this case the matrix $R$ reads:

$$R = \begin{pmatrix} R^{ab} + B^a \wedge B^b & \nabla_J B^a \\ \nabla_J B^a & 0 \end{pmatrix}, \tag{2.97}$$

using an $SO(d)$ transformation (recall that $\mathcal{I}$ is invariant, so it is allowed) we can set $\nabla_J B^a$ to have only one non-vanishing component. Then the determinant can be computed by cancelling rows and columns to be

$$\det\left(\mathbb{I} + \frac{tR}{2\pi}\right) = \det\left(\mathbb{I} + \frac{t(R^{ab} + B^a \wedge B^b)}{2\pi}\right) - t^2 \frac{\nabla_J B^a \nabla_J B^a}{(2\pi)^2} \det\left(\mathbb{I} + \frac{tR'}{2\pi}\right), \tag{2.98}$$

The first term is just the generator of Pontryagin classes for the $J$-connections, which is going to be subtracted by the the transgression. In the second term $R'$ is the $SO(d-1)$ curvature, since we have subtracted terms in the direction of $B^a$. Expanding this in $t$ we find that, upon transgression:

$$p_k(R) \to p_k(R_J) - \frac{\nabla_J B^a \nabla_J B^a}{(2\pi)^2} p_{k-1}(R'), \tag{2.99}$$

which is our "adapted" substitution rule. Notice that it looks very similar to the one proposed in [54, 35], but it is more apt to the dimensional reduction we are about to perform. Indeed, the $\frac{\nabla_J B^a \nabla_J B^a}{(2\pi)^2}$ terms will be reduced over an Euclidean cigar and give rise to Chern-Simons terms in the gravito-magnetic potential field, while the contributions from higher Pontryagin classes make up further topological terms allowed from nontrivial $SO(d-1)$ bundles on $\mathcal{M}_{d-1}$. It thus remains to prove the connection between our $B^a$ field and the gravito-magnetic potential Chern-Simons terms.

### 2.3.5 Thermal inflow in $d = 2, 4$

To study the bulk integral of the Chern-Simons term it is convenient to fix a nice extension into the bulk and use the independence on such extension to generalize the computation. We will thus study metrics of the form (in the Fefferman-Graham gauge)

$$ds^2 = dr^2 + h(r)(d\tau + a_i(x)dx^i)^2 + \gamma_{ij}(x)dx^i dx^j, \tag{2.100}$$

with $h(r)$ chosen in such a way that:

1. The boundary extrinsic curvature vanishes $\lim_{r\to 0} K_{\mu\nu} = 0$. For us this will translate in

$$\lim_{r\to 0} \sqrt{\dot{h(r)}} = 0. \tag{2.101}$$

2. At the coordinate singularity, say $r = 1$, it fulfills

$$\lim_{r\to 1} \sqrt{\dot{h(r)}} = 2\pi\beta^{-1} \tag{2.102}$$

so that there is no conical singularity there.

Notice that we take the metric on $\mathcal{M}_{d-1}$ to be constant in the $r$ direction. This can be done without problems since we only need one cycle to shrink to zero size. We also do not introduce any r-dependence in the gravito-magnetic potential field, which greatly simplifies our task. The vielbein for this metric is very simple (we split the index $a$ into $0, i$ for simplicity)

$$e^r = dr \,, \tag{2.103}$$

$$e^0 = \sqrt{h(r)}(d\tau + a_i dx^i) \,, \tag{2.104}$$

$$e^i = \hat{e}^i_j dx^j \,, \tag{2.105}$$

with $\hat{e}^i$ the boundary vielbein on $\mathcal{M}_{d-1}$. The inverse vielbein is, instead

$$E_r = \partial_r \tag{2.106}$$

$$E_0 = h(r)^{-1/2}\partial_\tau \,, \tag{2.107}$$

$$E_i = \hat{E}^j_i \partial_j - h(r)^{-1/2} \hat{E}^j_i a_j \partial_\tau \,, \tag{2.108}$$

now a straightforward but lengthy computation gives the compatible spin connection:

$$B^r = B^i = 0 \,, \tag{2.109}$$

$$B^0 = \sqrt{\dot{h}(r)}(d\tau + a_i dx^i) \,, \tag{2.110}$$

$$\omega_r{}^{ab} = 0 \,, \tag{2.111}$$

$$\omega_\mu{}^{i0} = \frac{1}{2}\delta^k_\mu \hat{E}^{ji}\left(\partial_j e^0_k - \partial_k e^0_j\right) \,, \tag{2.112}$$

$$\omega_\mu{}^{ij} = \frac{1}{2}\hat{\omega}_\mu{}^{ij} + e^0_\mu \frac{1}{2}\hat{E}^{ki}\hat{E}^{lj}\left(\partial_l e^0_k - \partial_k e^0_l\right) \,, \tag{2.113}$$

with this in hand we may compute the boundary variation $\hat{\delta}$ of $\mathcal{I}$. A quick computation shows, in $d=2$

$$\delta\mathcal{I}^{d=2}(B, \omega) = -4c_g \int B^a \wedge \nabla_J \delta B_a \,, \tag{2.114}$$

where we have restored (minus) the gravitational anomaly $c_g$ and dropped boundary terms which vanish due to the absence of extrinsic curvature. If we fix the boundary holonomy of $B^a$ the second term must vanish one integrated over two dimensions and thus the action should not depend on the chosen bulk extension.
We thus evaluate it in our background above and find

$$\mathcal{I}^{d=2}(B, \omega) = 2c_g \epsilon^{01} \int_{\mathcal{C}} \frac{d}{dr}\left(\sqrt{\dot{h}(r)}\right)^2 dr d\tau \times \int a dx = 2c_g \,\beta \left(\sqrt{\dot{h}(1)}\right)^2 \epsilon^{01} \int a dx \tag{2.115}$$

where $\mathcal{C}$ is the Euclidean cigar. Since only the radial derivative of $B^0$ can contribute, this gives rise to a total derivative contribution which is then evaluated at the coordinate singularity $r=1$. Now we use the regularity condition to compute

$$\mathcal{I}^{d=2}(B, \omega) = 8\pi^2 c_g \beta^{-1} \int a(x) dx \,, \tag{2.116}$$

which coincides with the classical result [41].

We have thus fixed the thermal Chern-Simons action to be

$$W^{d=2}(a) = 8\pi^2 c_g \; \beta^{-1} \int a(x) dx \,, \tag{2.117}$$

as a consequence of properly implementing the inflow mechanism in a diffeomorphism invariant bulk.

A very similar reasoning can be done in four dimensions, here one furthermore needs a gauge background for the $U(1)$ field. Let us first start with a system with no holonomy, then

$$A = b_i(x) dx^i \,, \tag{2.118}$$

doesn't need any bulk extension, the exact same computation as before, now starting with

$$\mathcal{I}^{d=4}(B, \omega) = -2c_m \int F \wedge B^a \wedge \nabla_J B_a \,, \tag{2.119}$$

This gives for the Chern-Simons action

$$W^{d=4}(a, b) = 8\pi^2 c_m \beta^{-1} \int db \wedge a \,, \tag{2.120}$$

as expected.

In the presence of holonomy for the $A$ field, we may also have further terms in which the chemical potential enters. The ones coming from the gauge anomaly have already been discussed. For the gravitational case one notices that, yet another time, the relevant terms group together in a total radial derivative, now of the form:

$$- c_m \int \partial_r \left( g(r)(\sqrt{\dot{h(r)}})^2 \right) d\tau dr \int a \wedge da \tag{2.121}$$

which gives the further term to complete

$$W^{d=4}(a, b) = 8\pi^2 c_m \beta^{-1} \int a \wedge db + 8\pi^2 c_m \beta^{-1} \int \mu a \wedge da \,, \tag{2.122}$$

consistent with the previous literature.

### 2.3.6 Higher dimensional generalizations

The main identity that bring to generalization in higher dimension is the fact that, in the background above, higher dimensional terms of the form

$$\int B^a \wedge \nabla_J B_a \wedge \left( \nabla_J B^b \nabla_J B_b \right)^m = \int_{\mathcal{C}} d\tau dr \frac{d}{dr} \left( h^{1/2} \right)^{2m+2} \int a \wedge (da)^{2m} \,. \tag{2.123}$$

Notice that the spin connection drops out from the equations since only $B^0$ is non-vanishing. Integrating over the Euclidean cigar then gives a factor of $\beta(2\pi\beta^{-1})^{2m+2}$, so that such term will descend to the $d-1$ dimensional Chern-Simons action:

$$\int B^a \wedge \nabla_J B_a \wedge \left( \nabla_J B^b \nabla_J B_b \right)^m = (2\pi)^{2m+2} \beta^{-(2m+1)} \int_{\mathcal{M}_{d-1}} a \wedge (da)^{2m} \,. \tag{2.124}$$

Such a term, as we have already anticipated, will be present in any dimension $d = 2 + 4k$, and, together with flavor gauge fields, also in $d = 4k$ due to the replacement of the first Pontryagin class. It gives a universal representative for coming from the bulk extension of $SO(d)$ to $SO(d+1)$. It can be checked that this conclusion coincides with the replacement rule for $p_1(R)$ given that we remember that each $p_1$ will give a factor of $2\nabla_J B^a \wedge \nabla_J B_a$ taking into account the trace.

We can now also discuss in more detail the substitution rule we have derived previously, and in particular the role of the tensor $R'$ which has not been completely specified. As anticipated, $R'$ is computed from $R_J$ by projecting out the tangent space directions of the vector $\nabla_J B^a$. Computing such gradient in our gauge choice:

$$\nabla_J B^0 = (dB^0, \omega^i{}_0 \wedge B^0), \tag{2.125}$$

shows that such procedure serves to eliminate the a-dependent parts from the curvature tensor, so that $R'$ indeed is the curvature associated to the spin connection on $\mathcal{M}_{d-1}$.

Applying the substitution rule to higher characteristic classes or, in our notation, computing the transgression for higher characteristic classes, gives higher derivative Chern-Simons terms of the form:

$$(2\pi)^{2r+2} \beta^{-(2r+1)} \int_{\mathcal{M}_{d-1}} a \wedge (da)^r \wedge P^{d-2-2r}(SO(d-1)), \tag{2.126}$$

which are, however, only relevant in dimensions $d = 6$ and higher and have not been studied extensively. They gives rise to higher derivative contributions to the Chern-Simons action on $\mathcal{M}_{d-1}$.x'

## 2.4 Conclusions and future directions

In this Chapter we have reviewed a general construction to determine certain Chern-Simons terms in the thermal effective action in $d-1$ dimensions and how this allows to relate them to 't Hooft anomalies. We have further given a new construction, which takes inspiration from the holographic treatment, which explains the presence of fractionally quantized, but otherwise appropriate Chern-Simons terms through careful analysis of the anomaly inflow mechanism and the necessity for a non-trivial embedding of the Lorentz group $SO(d)$ in $SO(d+1)$ in order to have a co-variant topological theory in the bulk. The properties of this extension turn out to be rather universal, and are essentially fixed by regularity conditions and the boundary values of certain holonomies. This perspective further allows for a different justification of the replacement rule [55, 35].

It would be nice to generalize and more clearly formulate the set of constraints given by this "geometric" extension through tools of algebraic geometry, which may also help to generalize this analysis to different systems. Particularly interesting for the author seems to be the question of whether there exist a bulk extension through anomaly inflow of anomalous partition function for arbitrary data (gauge fields, holonomy, discrete data such as spin structure...), or, through geometric reasoning, some of these partition functions may be shown to vanish.

Another nice direction for further study may be the derivation of Cardy formulas for Super-symmetric theories, [32], which are essentially found by embedding the Chern-Simons terms on

$\mathcal{M}_{d-1}$ in a full super-gravity multiplet. The inflow analysis may help to clarify the generality of such arguments.

It is also important to point out the possible restrictions on this construction. Indeed it is well known that the substitution rule's answer does not coincide with the perturbation theory computation for gravitinos and gravitons [57]. The lack of agreement could however be solved by adding a properly quantized Chern-Simons term in $d-1$ dimensions. An important point to explain is why the inflow strategy here must be modified and how. For the first part of the question we have a (partial) answer, since the inflow mechanism actually needs the definition of a massive bulk system in order to be carried out precisely. For fermions this is not a problem, nor it is for $p$ forms in $d = 2p$ dimensions, in in this case one may make the bulk theory "topologically" massive by adding a Chern-Simons interaction. For higher spin particles the situation is more subtle, as there is no obvious way to introduce e.g. massive spin 2 fields. The Dai-Freed-like formula for the partition function (2.70) then is not guaranteed to apply[8]. Understanding the way in which the inflow mechanism should be modified poses an interesting question for the future.

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[8]It is true that, for example in the gravitino case, one can gauge fix in order to end up with lower spin degrees of freedom, however it is not at all clear that such a gauge fix can be imposed globally in the bulk. This is not a problem for the computation of e.g. perturbative anomalies, since in that case only the geometry very close to the boundary is meaningful, but in our computation we have seen that the $d-1$ dimensional locally gauge invariant terms arise from physics near the "apparent" singularity. A similar comment was made in [41]. We think such situation deserves further study.

# Chapter 3

# Non-relativistic geometries

## 3.1   Introduction

In this chapter we switch gears and treat a couple of specific problems related to non-relativistic systems. The topic of classifying the possible low energy effective actions related to (emergent) non-relativistic invariance at low energy is receiving a renewed amount of attention during the last years.

One motivation for this resurgence may be found in the description of fractional quantum Hall systems, which at low energy can often be thought as interacting non-relativistic fermions in a magnetic field[58]. Another very interesting direction has been the treatment of non-relativistic gravitational theory, that is, dynamical theories of gravity whose first order formulation describes a non-relativistic gauge theory (and thus are seen as particular, non-Riemannian, geometries). From our point of view this is indeed an interesting direction, since this also classifies the possible contributions to the effective action $W[A]$ of a non-relativistic system.

Our studies here thread between the two fields, in that we study the linear response (equivalently, the quadratic part of the effective action) of a particular class of non-relativistic fermionic systems and argue for the emergence of some non-universal features in a particular "warped" limit. To do this one must first specify which kind of non-relativistic theory we are talking about and then introduce an appropriate way to couple it to a non-relativistic geometry. This is done in the next Section 3.2. Once this is taken care of, it is possible to study in full generality the (linear) dependence of the theory's one-point functions on the external Non-relativistic data, that is setting up an appropriate linear response theory. This is done in Section 4.1, focusing on a particular class of systems. At this point, if universal features appear in the computations, one may try to explain them via an effective field theory reasoning akin to what we have developed in the first two Chapters of this thesis. This is done is Section 4.2, arguing that non-trivial anomalies may emerge in a particular limit, hence generating universal features in the linear response.

We will now introduce some preliminary notions about non-relativistic groups. A simple way to think about them may be as the set of Inonu-Wigner contractions of the relativistic Poincaré algebra $ISO(D-1,1)$. As a matter of convention, the discussion for the most part can be carried out in either Euclidean or Lorentzian signatures. We denote the space-time dimension by $D = d+1$, with $d$ being the dimension of space(-time) where the residual $SO(d)$ $(SO(d-1,1))$

rotations acts. Usually this is a spatial slice, in our case it will not be. Then we describe Non-relativistic groups, such as the Galilei group $Gal_d$ as symmetry groups of $D = d+1$ dimensional systems.

Inonu-Wigner contractions of a Lie algebra $\mathfrak{g}$ are performed by introducing an auxiliary parameter $c$ in the definition of the generators in such a way that the $c \to \infty$ limit is well defined. A generic way to implement this construction is to divide the Lie algebra in two sub-spaces $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{j}$ in such a way that:

$$[\mathfrak{j}, \mathfrak{j}] \in \mathfrak{j}, \quad [\mathfrak{h}, \mathfrak{j}] \in \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{h}] \in \mathfrak{j}, \tag{3.1}$$

or, in components:

$$[J_a, J_b] = c_{ab}{}^c J_c, \quad [H_i, J_a] = d_{ia}{}^j H_j, \quad [H_i, H_j] = f_{ij}{}^a J_a. \tag{3.2}$$

To each such decomposition of $\mathfrak{g}$ we may associate a contraction by the redefinitions[1]:

$$H_i \to c\tilde{H}_i, \tag{3.3}$$

taking the limit $c \to \infty$ and eliminating $c$ from the equations the algebra now results in

$$[J_a, J_b] = c_{ab}{}^c J_c, \quad [\tilde{H}_i, J_a] = d_{ia}{}^j \tilde{H}_j, \quad [\tilde{H}_i, \tilde{H}_j] = 0. \tag{3.4}$$

The most famous example is the Galilei group. This comes by taking the Poincaré generators $(J^{ab}, P^a)$ and fix a direction (say 0) to separate them in $\mathfrak{j} = (J^{AB}, P^0)$ and $\mathfrak{h} = (J^{0A}, P^A)$ with $a = (0, A)$. We re-scale $J^{0A} = cG^a$, $P^A = cP^A$ and call $P^0 = H$ to get the non-trivial commutators

$$[J^{AB}, J^{CD}] = \delta^{AC} J^{BD} + ... \quad [J^{AB}, P^C] = \delta^{AC} P^B - \delta^{BC} P^A, \tag{3.5}$$

$$[J^{AB}, G^C] = \delta^{AC} G^B - \delta^{BC} G^A, \quad [G^A, H] = P^A, \tag{3.6}$$

where the dots stand for the usual permutations of the $SO(d)$ Lie algebra. One can be more general and also notice that the last term in (3.4) may admit a central extension $N$, for example in the Galilei case this is the particle number symmetry

$$[G^A, P^B] = \delta^{AB} N. \tag{3.7}$$

This construction allows to generate various other new Non-relativistic algebras. Another example which will be important for us is the Carroll algebra[59, 60, 61]. Physically, this corresponds to a high energy limit in which one moves to an extremely boosted frame.

This is obtained by taking $\mathfrak{j} = (J^{AB}, P^A)$ and $\mathfrak{h} = (J^{0A}, P^0)$. Re-scaling $J^{0A} = c^{-1} C^A$ and $P^0 = c^{-1} \Pi$ and taking $c \to 0$[2] we get the algebra:

$$[J^{AB}, J^{CD}] = \delta^{AC} J^{BD} + ... \quad [J^{AB}, P^C] = \delta^{AC} P^B - \delta^{BC} P^A, \tag{3.8}$$

$$[J^{AB}, C^C] = \delta^{AC} C^B - \delta^{BC} C^A, \quad [C^A, P^B] = \delta^{AB} \Pi. \tag{3.9}$$

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[1]This is a particular case of a more generic procedure, called Lie algebra expansions, which will however not need here.

[2]For this reason, the Carroll algebra is usually called an "ultra-relativistic" limit.

This algebra turns out to have no interesting central extension in $D > 3$ (in $D = 3$ one may use the $SO(2)$ epsilon tensor $\epsilon^{AB}$). One can also have more general "brane"-like algebras, by selecting different subgroups $SO(p)$ of $SO(D)$. These are object of much recent study, however we will not have anything to say about them.

One thing that we still may add from the purely algebraic perspective is that at this point one may also think of adding scaling generators to this algebra. One usually does this by first introducing the dilatation generator $D$ asking that all of the generators have prescribed scaling dimensions. In the Galilei case, since the central extension must be dimensionless and normalizing $D$ such that $[D, P^A] = P^A$, $[D, G^A] = -G^A$ then $[D, H] = 2H$ and there is only one such extension. For Carroll there is a whole family of them with

$$[D, P^A] = P^A \,, \quad [D, C^A] = (z - 1)C^A \,, \quad [D, \Pi] = z\Pi \,. \tag{3.10}$$

These algebras should describe Carrollian-Lifshitz theories. We will be particularly interested in the special case $z = 0$, which we will call the warped limit.

Having introduced the Non-relativistic algebras which will be relevant for our presentation, we now move to describe the geometry in which such algebras may be realized.

## 3.2 Newton-Cartan and Carrollian geometries

In this Section we describe the kind of geometric data that will be needed in describing non-relativistic theory on arbitrary manifold. From the Quantum Field Theory perspective, it is clear that coupling to a Riemannian geometry is problematic, since the stress tensor would have to satisfy the full $SO(D)$ Ward identities

$$T_{\mu\nu} - T_{\nu\mu} = 0 \,, \tag{3.11}$$

while such symmetry is clearly absent from our system. One may want to solve this problem by breaking general co-variance, which however we would like to keep since it assures energy and momentum conservation, which will be present in the systems of interest. One thus needs a diffeomorphism-co-variant geometric formulation which also only allows for a "reduced" tangent space symmetry.

There are various possible such formulations, which roughly speaking boil down to the kind of non-relativist algebra that the underlying Quantum Theory has as its symmetries. We will start with the most well known such geometry, Newton-Cartan geometries, following the presentation of [62], which are the most well known and are apt to describe Galileian theories and we will also introduce Carrollian geometries [63], which are the natural setup for Carrollian theories. This introduces a set of omnipresent ingredients, such as the non-vanishing torsion, which will speed up our later presentation. There is, of course, also a first order formulation of these, which we present for Carrollian theories in 4.2.

### 3.2.1 Newton-Cartan Geometry

Newton-Cartan Geometry starts with assuming the existence of a nowhere-vanishing one form $n$, which can be loosely thought as describing the non-relativistic direction. This is sometimes

taken to be closed $dn = 0$, but in our case the non-vanishing of $dn$ will be crucial. Together with this one-form, one can define a symmetric tensor $h^{\mu\nu}$ whose kernel is spanned by $n$:

$$n_\mu h^{\mu\nu} = 0 \,. \tag{3.12}$$

In the intuitive picture of $n$ defining a foliation of the ambient space-time, $h^{\mu\nu}$ can be thought as defining an (inverse) metric on the $D-1$ dimensional slices. For our purposes it will also be useful to introduce (co)-frame variables $E_A^\mu$ satisfying[3]:

$$E_A^\mu E_B^\nu \, \delta^{AB} = h^{\mu\nu} \,, \quad E_A^\mu n_\mu = 0 \tag{3.13}$$

At this point one would like to introduce algebraic inverses of our data to be able to define e.g. a metric on the ambient space-time. These are a vector field $v^\mu$ and a symmetric tensor $h_{\mu\nu}$ (or, equivalently a frame $e_\mu^A$) satisfying the algebraic relations

$$v^\mu n_\mu = 1 \,, \quad h_{\mu\nu} v^\mu = 0 \,, \quad h^{\mu\rho} h_{\rho\nu} = \delta_\nu^\mu - v^\mu n_\nu \equiv P^\mu{}_\nu \,, \tag{3.14}$$

for the vielbein fields this means

$$e_\mu^A v^\mu = 0 \,, \quad e_\mu^A E_A^\nu = \delta_\nu^\mu - v^\mu n_\nu \,, \quad e_\mu^A E_B^\mu = \delta_B^A \,. \tag{3.15}$$

This also allows us to define a metric on our $D$ dimensional manifold

$$g_{\mu\nu} = n_\mu n_\nu + h_{\mu\nu} \,. \tag{3.16}$$

In contrast to the Riemannian case, where the analog of $h^{\mu\nu}$, the metric $g^{\mu\nu}$ is an invertible $D \times D$ matrix with a unique inverse, in the Newton-Cartan case we deal with non-invertible algebraic objects. This is reflected in the algebraic relations above, which are insufficient to fully determine $\left( v^\mu, \, h_{\mu\nu}, \, e_\mu^A \right)$. This gives rise to the Milne-boost redundancy:

$$v^\mu \to v^\mu + h^{\mu\nu}\gamma_\mu \,, \quad h_{\mu\nu} \to h_{\mu\nu} - \left( n_\mu P^\rho{}_\nu + n_\nu P^\rho{}_\mu \right)\gamma_\rho + n_\mu n_\nu h^{\alpha\beta}\gamma_\alpha\gamma_\beta \,, \tag{3.17}$$

$$e_\mu^A \to e_\mu^A - n_\mu E^{\rho A}\gamma_\rho \,, \tag{3.18}$$

which has the natural interpretation of a geometric implementation of boost symmetry.

Now that we have some pieces in place we still need to define parallel transport in this geometry, to do that we need to fix a connection $\Gamma^\rho{}_{\mu\nu}$. While in the Riemannian setup one usually requires metric compatibility, in the Newton-Cartan case it is natural to require the boost-invariant quantities to be co-variantly conserved:

$$\nabla_\mu n_\nu = \nabla_\mu h^{\alpha\beta} = 0 \,. \tag{3.19}$$

the first equation relates the exterior derivative of the foliation vector to the torsion $T_{\mu\nu} = \left( \Gamma^\rho{}_{\mu\nu} - \Gamma^\rho{}_{\nu\mu} \right) n_\rho$ as

$$- dn = T \,, \tag{3.20}$$

on the other hand, taking into account also the second equation one obtains a more complete formula

$$\Gamma^\rho{}_{\mu\nu} = v^\rho \partial_\mu n_\nu + \hat{\Gamma}^\rho{}_{\mu\nu}[h] + h^{\rho\sigma} n_{(\nu} F_{\mu)\sigma} \,, \tag{3.21}$$

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[3]Although here we use the Kronecker delta $\delta^{AB}$, in our applications this will be substituted by the Minkowsky metric $\eta^{AB}$

Where $\hat{\Gamma}^\rho_{\mu\nu}[h]$ is the usual expression for the Christoffels with $h$ instead of the metric $g$ and $F_{\mu\nu}$ is an undetermined two form. In the absence of torsion this form il closed, and so it may be thought of as defining a $U(1)$ connection, which is precisely the central extension of the Galilei algebra. Always in this case, one can check that imposing $A$ to transform as

$$A_\mu \to A_\mu + P^\nu_\mu \gamma_\nu - \frac{1}{2} n_\mu h^{\alpha\beta} \gamma_\alpha \gamma_\beta \,, \tag{3.22}$$

renders the connection boost invariant. Such a redefinition is not possible in the presence of torsion. One can also run a similar story for the spin connection by solving the equation

$$\nabla_\mu E^\nu_A = 0 \,, \tag{3.23}$$

which gives

$$\omega^{AB}_\mu = \hat{\omega}^{AB}_\mu[e] + n_\mu \, E^{\nu A} E^{\rho B} F_{\nu\rho} \,. \tag{3.24}$$

Linearizing this set of transformations and computing their commutators indeed confirms that this geometry gives a representation of the Galilei algebra with central extension.

One might want to simplify the geometry in two ways:

1. Eliminate the central extension.

2. Defined a preferred frame where Milne boost are not present (after all, many theories have reduced rotation symmetry but still no boosts).

The first item essentially amounts to not assume that $F$ is a closed two form, however it should still transform under Milne boosts to have an invariant connection. The lack of an invariant connection doesn't have to be a problem, however it raises the interesting question of how to properly define geometric and topological invariant in the non-relativistic setting. The second item can also be taken care of as shown in [64]. The idea is to impose a further, non-boost-invariant condition which specifies a preferred frame. This condition can be the co-variant constancy of $v^\mu$ (equivalently of $h_{\mu\nu}$). This gives the equations:

$$\nabla_\mu v^\nu = \frac{1}{2} h^{\alpha\nu} \mathcal{L}_v h_{\alpha\mu} \tag{3.25}$$

$$\nabla_\mu h_{\alpha\beta} = n_{(\alpha} \mathcal{L}_v h_{\beta)\mu} \,, \tag{3.26}$$

since the Lie derivative is not boost invariant, one may choose a boosted frame which makes the right-hand-side to vanish[4]. When working in this setup we will simplify the notation by renaming $(v^\mu \,, \, n_\mu) = (l^\mu \,, \, l_\mu)$. We will also set the two-form $F$ to zero.

### 3.2.2 Carrollian geometries

A close relative of this construction allows also the define Carrollian geometries [63]. As we have discussed in the Introduction, the Carroll group arises as an Inonu-Wigner contraction in which the roles of momenta $P^A$ and $P^0$ are reversed in their belonging to the sub-algebras $\mathfrak{H}$

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[4]Since this is essentially given by the extrinsic curvature of the foliation, there may be issues with imposing such equation globally, for our purposes we will be content with restricting to geometries in which this is possible.

or $\mathfrak{J}$. This gives a hint that Carrollian geometries may be defined by inverting the roles of $n_\mu$ and $e^A_\mu$ in the Galilei case. This prompts one to start with a vector field $v^\mu$ and a metric on spatial slices $h_{\mu\nu}$ with

$$v^\mu h_{\mu\nu} = 0 \,, \tag{3.27}$$

alternatively, we introduce a frame $e^A_\mu$. The inverse variables are introduced algebraically as before, but this time we will have a Carrollian boost symmetry instead:

$$n_\mu \to n_\mu + h_{\mu\nu}\kappa^\mu \,, \quad h^{\mu\nu} \to h^{\mu\nu} - (v^\mu P^\nu{}_\rho + v^\nu P^\mu{}_\rho)\,\kappa^\rho + v^\mu v^\nu h_{\alpha\beta}\kappa^\alpha\kappa^\beta \,, \tag{3.28}$$

while

$$E^\mu_A \to E^\mu_A - v^\mu e_{\rho A}\kappa^\rho \,. \tag{3.29}$$

In the same spirit as before, the connection is determined by demanding co-variant constancy of the boost invariant data

$$\nabla_\mu v^\mu = \nabla_\mu h_{\alpha\beta} = 0 \,. \tag{3.30}$$

The determination of the connection through this procedure turns out to be more complicated than in the Galileian case. Let us give some plausibility for this conclusion before giving the results. First one may consider the torsion $T_{\mu\nu}$ defined as in the Galilei case; now this quantity is not boost invariant (we had to contract a $n_\rho$ to define it), so the extra data will be expressed in a more convoluted way. Secondly, now the extrinsic curvature of the slices $K_{\mu\nu} = \frac{1}{2}\mathcal{L}_v h_{\mu\nu}$ is not a boost invariant object and thus may appear at various stages of the construction.

After a tedious but straightforward computation one finds [63]:

$$\Gamma^\rho{}_{\mu\nu} = -v^\rho\partial_\mu n_\nu + \hat{\Gamma}^\rho{}_{\mu\nu}[h] - n_\nu h^{\rho\sigma}K_{\mu\sigma} - v^\rho X_{\mu\nu} + \frac{1}{2}h^{\rho\sigma}Y_{\sigma\mu\nu} \,, \tag{3.31}$$

with $X_{\mu\nu} = \nabla_\mu n_\nu$ and $Y_{\sigma\mu\nu}$ undetermined apart from the properties $v^\sigma Y_{\sigma\mu\nu} = v^\nu Y_{\sigma\mu\nu} = 0$. Notice that indeed the torsion is not as simple as in the Galilei case and given by

$$\Gamma^\rho{}_{\mu\nu} - \Gamma^\rho{}_{\nu\mu} = -v^\rho\left(\partial_\mu n_\nu - \partial_\nu n_\mu + 2X_{[\mu\nu]}\right) + h^{\rho\sigma}\left(Y_{\sigma[\mu\nu]} - 2n_{[\nu}K_{\mu]\sigma}\right) \,. \tag{3.32}$$

A nice solution proposed in [63] which simplifies the equations, is to introduce a vector field $M^\mu$ and to use $h_{\mu\nu}M^\nu$ as a Stueckelberg field to cancel parts of the boost variation of the geometry. This allows to define the boost invariant quantities[5]

$$\hat{n}_\mu = n_\mu - h_{\mu\nu}M^\nu \quad \hat{E}^\mu_A = E^\mu_A + v^\mu e^A_\nu M^\nu \,, \tag{3.33}$$

and find solutions for $X_{\mu\nu}$ and $Y_{\rho\mu\nu}$ such that the connection and torsion now simply read

$$\Gamma^\rho{}_{\mu\nu} = -v^\rho\partial_\mu \hat{n}_\nu + \hat{\Gamma}^\rho{}_{\mu\nu}[\hat{h}] - \hat{n}_\nu\hat{h}^{\rho\sigma}K_{\mu\sigma} + \hat{n}_\mu\hat{h}^{\rho\sigma}K_{\nu\sigma} \,, \tag{3.34}$$

$$\Gamma^\rho{}_{\mu\nu} - \Gamma^\rho{}_{\nu\mu} = -v^\rho\left(\partial_\mu\hat{n}_\nu - \partial_\nu\hat{n}_\mu\right) - 2\hat{h}^{\rho\sigma}\hat{n}_{[\nu}K_{\mu]\sigma} \,. \tag{3.35}$$

Notice that imposing the extrinsic curvature to vanish we go back to a situation similar to Newton-Cartan without boosts.

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[5]Notice that, since $M^\mu$ is a Stueckelberg field, the boost co-variant algebraic equations for orthogonality still hold, albeit "hatting" the redefined variables everywhere.

### 3.2.3 The view from $D+1$

A nice interpretation of these construction comes from embedding these non-relativistic geometries in a higher dimensional space-time. While giving an intuitive picture for some of the properties of these geometries, this point of view will also be helpful to tie the discussion with the anomaly-inflow type of argument that we have already put forward in the previous Chapter. The higher dimensional interpretation for Galileian geometries is quite well known that one may start from a $D+1$ dimensional manifold with a null isometry $n^M \partial_M$. Then the metric can be taken in local coordinates to be of the form

$$ds^2_{D+1} = 2n_\mu dx^\mu \left( du + A_\mu dx^\mu \right) + h_{\mu\nu} dx^\mu dx^\nu \,, \tag{3.36}$$

with all fields $u$-independent functions and $h_{\mu\nu}$ of rank $D+1-2 = d-1$. There are actually infinite ways to choose the fields $A_\mu$ and $h_{\mu\nu}$ for the metric to have this form, indeed:

$$A_\mu \to A_\mu + \chi_\mu \quad h_{\mu\nu} \to h_{\mu\nu} + n_\mu \chi_\nu + n_\nu \chi_\mu \,, \tag{3.37}$$

all do the trick. Requiring the rank of $h_{\mu\nu}$ to be conserved by this transformation (which are just the orthogonality conditions in the Galileian setup) fixes the one-form $\chi_\mu$ to give back the Milne boosts $\chi_\mu = P^\mu_\nu \gamma_\nu - \frac{1}{2} n_\mu h^{\alpha\beta} \gamma_\alpha \gamma_\beta$. Finally, one can check that the symmetric part of the connection induced on the $d$ dimensional manifold coincides with the one in (3.21) provided $F = dA$. The torsion, which is absent since the higher dimensional geometry, can be found sitting in the co-variant derivative of the null generator $\nabla_M n_N = -T_{MN}$.

The physical idea behind this construction is that fields in $D+1$ dimensions will decompose according to their null momentum in the $u$ direction. This is simply the charge under the central $U(1)$[6] of the Galilei group. One may then perform a null Kaluza Klein reduction to end up with a tower of Galileian invariant theories. While this process readily generates free theory examples, since in this case the KK modes decouple, it does not generate interacting example with a finite number of particle species. It can be used, however, to derive some properties of effective actions.

One example regards the presence of perturbative (non-conformal) anomalies in Galileian theories. Since even dimensional Galileian theories can be loosely thought of null reductions of odd dimensional standard QFTs and vice-versa, one can exclude them automatically to have such perturbative anomalies[7] (for $D+1$ odd it is obvious, since perturbative anomalies are only even dimensional, for $D+1$ even it follows from the fact that the dimensional reduction only gives rise to Chern-Simons terms, which would require holonomy around the null direction.). Conformal anomalies however can still be present, and have indeed been studied in the literature [65, 66].

The Carrollian case has a similar but less known story. In this case the Carrollian geometry does not come from null reduction, but from null *embeddings*.

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[6]In order to have $U(1)$ rather than $\mathbb{R}$ we need to take the null direction as a compact circle.

[7]One small counterargument could exist for theories which cannot be realized by integrating out a tower of KK modes, which then would evade this kind of argument. However anomalies are essentially independent from interactions and free Galileian theories admit a bulk interpretation.

To define a null embedding we can choose a light-cone coordinate, say $u$, and define our surface through the equation:

$$u = u_0 = \text{constant} \,, \tag{3.38}$$

the fact that the normal vector to the foliation $\partial_M u$ is null gives the condition $g^{uu} = 0$ for the ambient metric. Diffeomorphisms that preserve the null surface are given by:

$$x'^\mu = f^\mu(x^\nu, u) \,. \tag{3.39}$$

Let us parametrize the inverse metric as:

$$g^{-1} = 2v^\mu \partial_\mu \otimes \partial_u + h^{\mu\nu} \partial_\mu \otimes \partial_\nu \,, \tag{3.40}$$

then the diffeomorphisms preserving the surface act on $g^{-1}$ as:

$$\left(g^{-1}\right)' = 2v^\mu \partial_\mu \otimes \partial_u + \left(h^{\mu\nu} + v^\mu \partial_u f^\nu + v^\nu \partial_u f^\mu\right) \partial_\mu \otimes \partial_\nu \,, \tag{3.41}$$

displaying the transformation properties of $h^{\mu\nu}$. This are still not the Carroll transformations. However one can play a similar game as in the Galileian case, requiring that the rank of $h$ be $D - 1$, this incidentally also defines $n_\mu$ as the kernel of $h^{\mu\nu}$. Then transformations of the form above preserving $n_\mu h^{\mu\nu} = 0$ and $n_\mu v^\mu = 1$ are precisely the Carrollian diffeomorphisms.

The appearance of the Stueckelberg field $M^\mu$ can also be justified in a similar way. If one decomposes $h^{\mu\nu} = \hat{h}^{\mu\nu} + v^\mu M^\nu + v^\nu M^\mu$, the transformations properties of $h^{\mu\nu}$ under boosts may be adsorbed entirely by $M^\mu$. If one writes the metric for this system the answer is:

$$ds^2 = 2du\left(\Phi du - \hat{n}_\mu dx^\mu\right) + h_{\mu\nu} dx^\mu dx^\nu \,, \tag{3.42}$$

with $\Phi = -n_\mu M^\mu + \frac{1}{2} h_{\rho\sigma} M^\rho M^\sigma$. Notice here there is no $U(1)$ redefinition of $n_\mu$ that makes a further central term appear, since it is adsorbed by the presence of $M$.

We can now try to make a similar reasoning regarding perturbative anomalies in the Carrollian case. Now an even dimensional Carrollian system is seen as a null embedding in a $D+1 = d+2$ odd dimensional bulk. In the bulk we may have a topological theory which creates a consistent anomaly on the Carrollian manifold upon inflow without any contradiction. This in principle allow Carrollian theory to have a richer structure in this respect than Galileian ones, which we will study in Section 4.2.

# Chapter 4

# Lifshitz fermions and universality

## 4.1 Torsional response for Lifshitz fermions

In this Chapter we begin the study of the response properties of a particular class of $d = 4$ fermionic systems. A representative of this class is given by a massive Dirac fermion in an axial background field $b^\mu$, with action

$$S = \int d^4x \bar{\Psi} \left( i\slashed{D} - m + \gamma_5 \slashed{b} \right) \Psi \, . \tag{4.1}$$

This action has been widely used in the literature as a continuous toy model describing a quantum (i.e. zero temperature) phase transition between a trivial gapped system (insulator) and a system of two Weyl fermions (Weyl-semimetal). The analysis of the RG flow is extremely simple since the model is free. One first finds the energy bands for the system:

$$\epsilon(k)^2_\pm = k^2 + m^2 + b^2 \pm 2|b|\sqrt{m^2 + (\hat{b} \cdot k)^2} \, , \tag{4.2}$$

where we have taken $b^\mu$ to be a spatial vector. The bands responsible for the low energy behavior are those for which the minus sign is chosen above. The low energy phase is determined by the respective magnitude of $b$ ,$m$. For $|b| > |m|$ the lowest bands touch at $\vec{k}_\pm = \pm\alpha\vec{b}$, where $\alpha = \sqrt{1 - m^2/b^2}$ may be interpreted as a screening factor for the chiral charge. In the opposite case the system is gapped, with the gap given by $\Delta_{gap} = 2\sqrt{m^2 - b^2}$. The theory at low energy is then either described by a massive Dirac fermion, which can be integrated out, or a couple of two Weyl fermions with a nontrivial axial field $b^\mu_{\text{eff}} = \alpha b^\mu$ and action

$$S_{eff} = \int d^4x \bar{\Psi}_{eff} \left( i\slashed{D} + \gamma_5 \slashed{b}_{eff} \right) \Psi_{eff} \, . \tag{4.3}$$

In this phase one may extract a nontrivial chiral effective action by rotating away the axial vector though a chiral transformation, the chiral anomaly then gives an effective action:

$$W[A, b_{eff}] - W[A, 0] = \frac{1}{24\pi^2} \int d^4x b_{eff} \wedge A dA \, , \tag{4.4}$$

which gives rise to a non-trivial electric current in a background magnetic field, which is a signature of the low energy phase.

Less studied is the quantum critical point of this model, which appears as the difference $|m| - |b|$ approaches zero. It is simple to expand the energy bands in this regime to give:

$$\epsilon^2/m^2 = \frac{k_\perp^2}{m^2} + \frac{(k \cdot \hat{b})^4}{4m^4} + O\left((k \cdot b/m)^6\right) , \qquad (4.5)$$

so that the model enjoys an emergent Lifshitz scaling symmetry anisotropic scaling $z = 1/2$ (see Figure below).[1] It would be interesting to determine whether this critical point also possesses
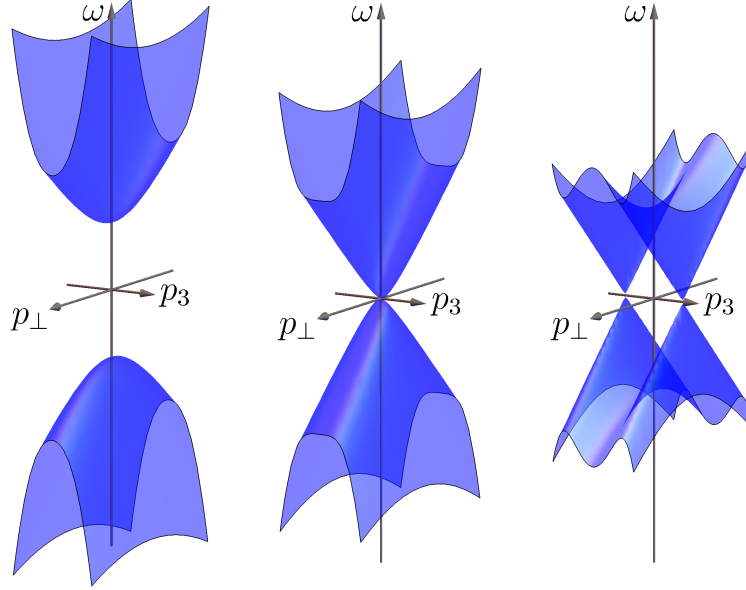


Figure 4.1: Lowest energy bands of the fermionic system (4.1). The figure shows the dispersion relation as function of $p_\perp$ and the anisotropic momentum $p_3$. The left figure is the insulating phase. The right figure the is deep in the Weyl semi-metal phase and the middle figure shows the critical point in between the two.

some distinctive feature. In this case it cannot directly come from chiral physics, since the vanishing of $b_{eff}$ may be interpreted as a decoupling of the $U(1)_A$ symmetry. It was however shown in [67] using an holographic model [68, 69] that nontrivial correlators of the stress tensor of the theory are present in the vicinity of the critical point, giving rise to an anisotropic, non-dissipative Hall viscosity once the system is put at finite temperature.[2] It has been known for a long time [70] that such terms may only arise if both rotational invariance and either parity or time reversal are broken. Both breakings come in this model directly from the introduction of the external axial field. However in both low energy phases, rotational invariance is effectively restored up to the contribution from the chiral anomaly. Thus such an observable, if present, must be a property of the critical point.

To properly answer this question from a field theoretical perspective, we employ a simplified

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[1]Notice we use a convention where the "isotropic" directions scale uniformly, while the $\hat{b}$ direction scales by a factor $\lambda^z$.

[2]Since the precise definition of the operators entering in the Kubo relations requires some formalism in this case, the precise discussion of the observables we will be interested in is developed in 4.1.1

two-band model for the critical Lifshitz fermion:

$$S_{Lif} = \int d^4x \left\{ \chi^\dagger \left[ i\sigma^A \partial_A + \frac{\sigma \cdot \hat{b}}{b} \left( i\partial \cdot \hat{b} \right)^2 \right] \chi \right\},$$ (4.6)

where we denote the Lorentz group indices in the remaining $SO(1,2)$ directions using capital Latin letters $A, B, C$. This model indeed reproduces (4.5) up to the desired order. It can also be interpreted in a more transparent way by using representations of the $SO(1,2)$ Clifford algebra instead, which amounts to introducing $\bar{\chi} = \chi^\dagger i\sigma \cdot \hat{b}$ so that the Lagrangian takes the form of the one for a $2 + 1$ dimensional Dirac fermion with a momentum-dependent mass $\mu(p) = i \left( p \cdot \hat{b} \right)^2 / b$. A nice property of this system is that is still charge conjugation invariant, so that one may also define it for a Majorana representation $\varphi$ of $SO(1,2)$. We can take for example $\gamma^A = (i\sigma_3, \sigma_2, -\sigma_1)$ and the charge conjugation matrix $\mathcal{C} = i\sigma_2$. The Majorana condition reads $C\bar{\chi}^T = \chi$.

Furthermore, we can also generalize the scaling exponent to be an arbitrary $z = 1/2n$, $n \in \mathbb{N}$ to find the model:

$$S_z = \int d^4x \left\{ \bar{\varphi} \left( i\gamma^A \partial_A \right) \varphi + s\, \varphi^T M (\partial \cdot \hat{b})^{1/2z} C^{-1} \varphi \right\},$$ (4.7)

where $M(\partial \cdot \hat{b}) = \overleftarrow{\partial} \cdot \hat{b} \hat{b} \cdot \overrightarrow{\partial}$ and $s = \pm 1$ sets the sign of the mass. We are also working in units of $b = 1$ for simplicity, this is not a problem since it cam be reinstated by dimensional analysis and is a dimensionless parameter according to the Lifshitz scaling. Taking $z = 1/2n$ as anticipated allows the model to remain local. Symmetry-wise, our model respects charge conjugation, however breaks time reversal (due to the 2+1 dimensional mass term) and rotational invariance. The Lifshitz scaling symmetry can be seen as acting as follows:

$$x^A \to \lambda x^A \quad x \cdot b \to \lambda^z x \cdot b \quad \varphi \to \lambda^{-(1+z/2)} \varphi.$$ (4.8)

One may ask whether the model (4.7) may be obtained from the fermionic Lagrangian via a specific deformation too:

$$S_{eff} \to S_{eff} + \sum_{s=2}^{n} \int d^4x \lambda^s \mathcal{O}_s,$$ (4.9)

with specific (fine tuned) $\lambda^s$.

The answer is positive, at least at intermediate energy scales. Indeed one may try to add irrelevant operators to cancel all derivatives up to the $1/z$-th at the band touching point $k = 0$. These deformations are simply given by higher spin chiral currents, schematically:

$$\mathcal{O}_s = b^{\mu_1}...b^{\mu_s} \bar{\Psi} \gamma_5 \gamma^{\mu_1} \partial^{\mu_2}...\partial^{\mu_s} \Psi.$$ (4.10)

Imposing the dispersion relation to have Lifshitz scaling $1/z$ at small momenta and $|b| = m$ fixes

$$\lambda^{2s} = \frac{(1/2)_s}{s!} m^{1-2s} \, 2s \le 1/z.$$ (4.11)

So that one can indeed reach the higher $z$ critical point by fine tuning a finite number of couplings[3].

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[3]This procedure can only be seen as working for intermediate energy scales, indeed in the far UV the theory should be UV completed.

The next step that will be crucial for our computations is the coupling of this model to curved space-time. Now is where our introduction to non-relativistic geometries starts to pay off. Notice that the model 4.7 still has $SO(2,1)$ symmetry. This leads us to a version of Newton-Cartan geometry where the boosts have been fixed by canceling the extrinsic curvature. In the notation of Section 3.2 the model should couple through:

$$S_z[e^A, l] = \int \sqrt{-g} \left\{ \bar{\varphi} \left( i\gamma^A E^\mu_A \nabla_\mu \right) \varphi + s \; \varphi^T M (\nabla_l)^{1/2z} C^{-1} \varphi \right\} , \tag{4.12}$$

where we introduce the shorthand notation $\nabla_l = l^\mu \nabla_\mu$, with the understanding that, in the flat space-time limit, $l^\mu = b^\mu$. The co-variant derivative $\nabla_\mu$ acts through the Newton-Cartan connection and through the spin connection $\omega^{AB}_\mu$, which in this case we may take simply as given by the usual formula in terms of the triad $e^A_\mu$.

Using this background we can write down the Ward identities obeyed by the theory and introduce its relevant currents. The general variation of the action reads:

$$\delta S_z[e^a, l] = \int \sqrt{g} \left( t^\mu_a \delta e^A_\mu + p^\mu \delta l_\mu + S^\mu{}_{AB} \delta \omega^{AB}_\mu + \Omega^{\mu\nu} \delta T_{\mu\nu} \right) , \tag{4.13}$$

being $T_{\mu\nu}$ the torsion. We will call $t^\mu_A$ the "isotropic stress tensor" and $p^\mu$ the "anisotropic momentum current". The equation above is meant to signify that variations of the inverse triad and $l^\mu$ are to be re-expressed as variations of $e^A_\mu$, $l_\mu$ through the equations:

$$\delta E^\nu_B = -E^\mu_B E^\nu_A \delta e^A_\mu - l^\nu E^\mu_B \delta l_\mu \tag{4.14}$$

$$\delta l^\nu = -l^\mu E^\nu_A \delta e^A_\mu - l^\nu l^\mu \delta l_\mu. \tag{4.15}$$

The Ward identities then come from the diffeomorphism and Lorentz variations of the $S_z$, on our fields they act as:

$$\delta_\xi l_\mu = \nabla_\mu (\xi^\nu l_\nu) - T_{\nu\mu} \xi^\nu , \tag{4.16}$$

$$\delta_\xi e^A_\mu = \nabla_\mu (\xi^\nu e^A_\nu) - \xi^\lambda \omega_\lambda{}^A{}_B e^B_\mu , \tag{4.17}$$

for the diffeomorphism generated by $\xi^\mu$ and

$$\delta_\Omega l_\mu = 0 \tag{4.18}$$

$$\delta_\Omega e^A_\mu = \Omega^A{}_B e^B_\mu . \tag{4.19}$$

for tangent space rotations generated by $\Omega_{AB} = -\Omega_{BA}$. The last term in (4.17) is not co-variant under tangent space transformations, as is the case for connections. However we may combine it together with a Lorentz variation with $\Omega^{AB}_\xi = \xi^\lambda \omega_\lambda{}^{AB}$ to cancel it. We will use such "co-variantized" variation in what follows.

Another useful organizing principle is to make the Lorentz symmetry manifest, which allows us to split the equations in irreps of $SO(1,2)$ and will be very useful for the Kubo formulation. This means that we split any vector field $V^\mu = V l^\mu + V^A E^\mu_A$. This splitting makes sense in the absence of boosts, otherwise the parameters $(V, V^A)$ will mix among themselves. This allows to split e.g. the diffeomorphism generator:

$$\xi^\mu = \theta l^\mu + \xi^A E^\mu_A , \tag{4.20}$$

and rewrite (4.16)(4.17) as

$$\delta_\theta l_\mu = \partial_\mu \theta - \theta G_\mu \,, \tag{4.21}$$

$$\delta_\theta e_\mu^A = 0 \,, \tag{4.22}$$

$$\delta_\xi l_\mu = -T_{A\mu}\xi^A \,, \tag{4.23}$$

$$\delta_\xi e_\mu^A = \nabla_\mu \xi^A \,. \tag{4.24}$$

Finally, Ward identities have a more compact form if we use the explicit dependence of $\omega_\mu^{AB}$ and $T_{\mu\nu}$ on the data $\left(e_\mu^A, l_\mu\right)$. This allows us to define

$$\begin{aligned}
\tau^\mu{}_A &= t^\mu{}_A + \frac{1}{2}l^\mu(\nabla^B - G^B)\sigma_{BA} \\
&+ \frac{1}{2}\left[E^{\mu B}(\nabla^C - G^C)\left(s_{CBA} + s_{BAC} - s_{ABC}\right) + \nabla_l\sigma_{BA}\right] \,,
\end{aligned} \tag{4.25}$$

and

$$\pi^\mu = p^\mu - (\nabla_\nu - G_\nu)\Lambda^{\nu\mu} \,, \tag{4.26}$$

where $\nabla_l \equiv l^\mu \nabla_\mu$, $G_\nu = l^\mu T_{\mu\nu}$ whereas $s_{ABC}$ and $\sigma_{AB}$ are defined through the splitting of the spin connection by

$$S^\mu{}_{AB} = E^{\mu C}s_{CAB} + l^\mu \sigma_{AB} \,. \tag{4.27}$$

Finally in integrating by parts in the presence of torsion it is useful to keep in mind the following equality

$$\frac{1}{\sqrt{g}}\partial_\mu \sqrt{g} = \Gamma^\nu_{\mu\nu} = \Gamma^\nu_{\nu\mu} + G_\mu \,. \tag{4.28}$$

After all this preparation work we write down the Ward identities:

$$\left(\nabla_\mu - G_\mu\right)\tau^\mu{}_A = T_{A\mu}\pi^\mu \,, \tag{4.29}$$

$$\left(\nabla_\mu - 2G_\mu\right)\pi^\mu = 0 \,, \tag{4.30}$$

$$e_{\mu[A}\tau^\mu{}_{B]} = 0 \,. \tag{4.31}$$

which can be recast by further saturating the contracted space-time indices through

$$\tau^\mu{}_A = E^{\mu B}\tau_{BA} + l^\mu \Sigma_A \,, \tag{4.32}$$

$$\pi^\mu = E^{\mu A}\pi_A + l^\mu \pi \,, \tag{4.33}$$

into the form

$$\left(\nabla_A - G_A\right)\tau^{AB} + \nabla_l \Sigma^B = T^B{}_A \pi^A + G^B \pi \,, \tag{4.34}$$

$$\left(\nabla_A - 2G_A\right)\pi^A + \nabla_l \pi = 0 \,, \tag{4.35}$$

$$\tau_{[AB]} = 0 \,, \tag{4.36}$$

while Lifshitz invariant implies:

$$\tau^A{}_A + z\pi = 0 \,, \tag{4.37}$$

which may be derive via the following Weyl rescalings:

$$\delta_\sigma l_\mu = -z\sigma l_\mu \,, \tag{4.38}$$

$$\delta_\sigma e_\mu^A = -\sigma e_\mu^A \,. \tag{4.39}$$

Equations (4.34),(4.35),(4.36) reflect conservation of energy and momentum in both the isotropic and anisotropic directions and the Lorentz symmetry for $\tau_{AB}$, which may be thought of an effective stress tensor living in $d = 3$ dimensions. It is important to notice that, in contrast with the fully isotropic case, the current components $(\tau_{AB}, \pi_A, \Sigma_A, \pi)$ sit in different multiplets. This allows for a wider variety of nontrivial correlations which we will explore in the next Section.

### 4.1.1 Lifshitz hydrodynamics and Kubo formulas

The aim of this section it to develop an "hydrodynamic" theory of response for systems coupled to the geometry discussed above. This will allow us to link Quantum Field Theory correlators to low energy observables and give a clear definition of "anisotropic Hall viscosity". The theory of non-relativistic hydrodynamics has a long story, indeed it precedes relativistic hydrodynamics. However the usual discussion rests essentially on the natural identification of the fluid velocity vector $u^\mu$ with the Newton-Cartan vector field $v^\mu$. In our case this is not possible, since $l^\mu$ is space-like rather than time-like. One must then introduce the fluid velocity in an independent manner and explain the differences with respect to the "usual" setup.

As mentioned before, it is expedient to decompose:

$$u^\mu = \theta l^\mu + v^A E_A^\mu \,, \tag{4.40}$$

in the absence of boosts $\theta$ is an invariant of the flow and thus may be taken as a separate piece of data. One can then simply discriminate flow with $\theta = 0$ and $\theta \neq 0$. For $\theta = 0$ (which is the case we will be mostly interested in) it makes sense to normalize $v^A v_A = -1$ so that we might bring $v^A = (1, 0, 0)$ to rest by a local Lorentz transformation. The interpretation for the curious role of $\theta$ comes from the fact that the usual stress tensor multiplet, to which the fluid velocity is expected to couple as a chemical potential through[4]:

$$W[u, T] = \int u^\mu T_{\mu 0} \,, \tag{4.41}$$

splits in the $\tau_{AB}$ and $\pi_A$ currents. The $\theta$ component then just acts as a chemical potential for the abelian current $\pi^A$, and will exactly behave as one through the whole formulation.

### Lifshitz Hydrodynamics

Hydrodynamics is then constructed by considering the most generic one-point functions for conserved currents in an expansion through gradients of the velocity. In our case this is achieved by splitting[5]

$$\nabla_\mu v_A = l_\mu \nabla_l v_A + e_\mu^B \left( \hat{\sigma}_{AB} + \eta_{AB}\Theta + \epsilon_{AB}\omega \right) \,, \tag{4.42}$$

in terms of the shear $\hat{\sigma}_{AB} = \nabla_{(A} v_{B)} - \frac{1}{2}\eta_{AB}\nabla_C v^C$, the expansion $\Theta = \nabla_C v^C$ and the vorticity $\omega = \epsilon^{ABC} v_A \nabla_B v_C$, with $\epsilon_{AB} = \epsilon_{ABC} v^C$. In the Newton-Cartan setup, however, we have

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[4]Here we use $T_{\mu\nu}$ to denote the stress tensor in anisotropic theories, this is the same notation as the one for the torsion, but it only appears in this formula. Henceforth the letter $T$ will always denote torsion.

[5]In hydrodynamics one usually considers quantities orthogonal to the flow vector $v^A$. Gradients automatically are so due to the normalization condition. It is however convenient to consider all Lorentz indices form now on to be orthogonalized with respect to $v^A$, we dispense the required projectors to avoid cluttering.

another piece of independent data given by the torsion, which does not appear in any co-variant derivatives of the above type. It is then necessary to include this piece of data into the formalism by also expanding the torsion tensor as:

$$T_{\mu\nu} = 2(l_{[\mu}e^A_{\nu]}G_A + e^A_{[\mu}e^B_{\nu]}\left(\zeta_{[B}v_{A]} + \epsilon_{AB}m)\right), \tag{4.43}$$

this essentially amounts to an electric-magnetic decomposition in the 2+1 anisotropic directions, with "electric" field $\zeta_A$ and "magnetic field $m\epsilon^{AB}$, plus a vector field $G^A$. It is then clear that $D$ dimensional hydrodynamics in our geometry will be similar to $d$ dimensional relativistic hydro with an additional electromagnetic field, which is a very well-known subject. The "data" that enter in the hydrodynamic expansion are just functionals of

$$(\hat{\sigma}_{AB}, \ \Theta, \ \omega, \ \zeta_A, \ m, \ G_A) \ . \tag{4.44}$$

The link with quantum field theory correlators is found by expanding this set of data in the fluid rest frame and extracting their dependence on geometry. Taking functional derivatives then gives the desired relations. In our case we will need the formulas for a time-dependent, but otherwise constant, background. The formulas in the other cases are very simple to derive once the analogy between torsion and electromagnetism has been understood.

The main new feature of this expansion is in the fact that $\nabla_\mu u^\nu$ only contains the geometric response to triad perturbations. This follows from co-variant constancy of $l^\mu$ so that

$$\nabla_\mu u^\nu = l^\nu \partial_\mu \theta + E^\nu_a \nabla_\mu v^A \,, \tag{4.45}$$

and

$$\nabla_\mu v^A = \omega^A_{\mu\,B} v^B \,, \tag{4.46}$$

in the rest frame. As one can imagine, the missing response in the gradients of $l^\mu$ can be found in the torsion. Indeed, to first order in the backgrounds of interest for us

$$\nabla_\mu v^A \sim \partial_t e^A_\mu \quad \zeta_A \sim E^\mu_A \partial_t l_\mu \,. \tag{4.47}$$

**Anisotropic Hall viscosity**

Now the time is ripe to properly define the observables we are going to compute. To do so it is useful to have a short excursion to the isotropic case to introduce the concept of Hall viscosity. According to the standard definition the viscosity tensor encodes the response in the spatial components of the stress tensor (called strain tensor) to external gradients of the velocity fields:

$$\langle \tau^{\mu\nu} \rangle = \eta^{\mu\nu\rho\sigma} \nabla_\rho u_\sigma + O(\nabla^2) \,, \tag{4.48}$$

with $\eta^{\mu\nu\rho\sigma} = \eta^{\nu\mu\rho\sigma} = \eta^{\mu\nu\sigma\rho}$ by rotation invariance and exchange symmetry of the two-point functions. There is a further decomposition according to the last exchange symmetry in a "Dissipative" and "Hall" parts:

$$\eta^{\mu\nu\rho\sigma}_D = \eta^{\rho\sigma\mu\nu}_D \,, \quad \eta^{\mu\nu\rho\sigma}_H = -\eta^{\rho\sigma\mu\nu}_H \,. \tag{4.49}$$

The dissipative part gives rise to the standard shear and bulk viscosities, upon decomposing $\eta^D$ in symmetric irreps of the rotation group. This divides the response to the traceless symmetric

part of velocity gradients from that to the divergence of the velocity field. Such quantities are usually not universal and strongly depend on the presence of interactions, see for example [71] for a review of their definition through the Kubo formalism.

The non-dissipative part may only exist in $2+1$ dimensions for isotropic systems, due to the absence of appropriate tensor structures in higher dimensionality. In $D = 2+1$ one writes

$$P^H_{\mu\nu\rho\sigma} = \frac{1}{4} \left( h_{\mu\rho}\epsilon_{\nu\sigma} + h_{\nu\rho}\epsilon_{\mu\sigma} + h_{\mu\sigma}\epsilon_{\nu\rho} + h_{\nu\sigma}\epsilon_{\mu\rho} \right) , \qquad (4.50)$$

with $\epsilon_{\mu\nu} = \epsilon_{\mu\nu\rho}u^\rho$ and $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$. A nice feature of this coefficients is that, in various systems, it has a *universal* form. Thus we can be hopeful that our free field construction will compute a meaningful quantity.

To construct odd tensor structures in higher dimensionalities one needs the presence of another vector field $V^\mu$ to contract into the epsilon tensor, that is, one needs an-isotropy[6]. Now using $\tilde{\epsilon}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}V_\rho u_\sigma$ to construct the projector

$$\tilde{P}^H_{\mu\nu\rho\sigma} = \frac{1}{4} \left( h_{\mu\rho}\tilde{\epsilon}_{\nu\sigma} + h_{\nu\rho}\tilde{\epsilon}_{\mu\sigma} + h_{\mu\sigma}\tilde{\epsilon}_{\nu\rho} + h_{\nu\sigma}\tilde{\epsilon}_{\mu\rho} \right) . \qquad (4.51)$$

This is not however the only tensor structure with the required properties, in fact

$$\Pi^{(1)}_{\mu\nu\rho\sigma} = V_\mu V_\rho \tilde{\epsilon}_{\nu\sigma} , \quad \Pi^{(2)}_{\mu\nu\rho\sigma} = \Pi^{(1)}_{\nu\mu\sigma\rho} , \quad \Pi^{(3)}_{\mu\nu\rho\sigma} = \Pi^{(1)}_{\mu\nu\sigma\rho} + \Pi^{(1)}_{\nu\mu\rho\sigma} , \qquad (4.52)$$

also satisfy the required conditions. Thus one expects four independent Hall viscosity components to be present. In our setting $V^\mu = b^\mu = l^\mu$ and saturating such indices we find two "Hall" projectors:

$$P_{ABCD} = \epsilon^{(A(C}\eta^{B)D)} , \qquad (4.53)$$

$$\epsilon_{AB} = \epsilon_{ABC}v^C , \qquad (4.54)$$

together with three relevant operators coming from the stress tensor multiplet $(\tau_{AB} , \pi_A , \Sigma_A)$. The then write down the most general expansion for the one point functions of these operators in the projectors above and the hydrodynamical data (4.44):

$$\langle \tau^{AB} \rangle = \eta^{ABCD}_\tau \hat{\sigma}_{CD} \qquad (4.55)$$

$$\langle \pi^A \rangle = \eta^\pi \epsilon^{AB}\zeta_B + \eta^{\pi\Sigma}\epsilon^{AB}\nabla_l v_B \qquad (4.56)$$

$$\langle \Sigma^A \rangle = \eta^\Sigma \epsilon^{AB}\nabla_l v_B + \eta^{\pi\Sigma}\epsilon^{AB}\zeta_B \qquad (4.57)$$

where $\eta^{ABCD}_\tau = \eta_\tau P^{ABCD}$. To derive Kubo formulae for the above coefficients we expand to first order in the external geometric data, setting $v^A = (1,0,0)$ to its rest frame value:

$$\langle \tau^{AB} \rangle = \eta^{ABCD}_\tau E^\mu_C \partial_t e_{\mu D} \qquad (4.58)$$

$$\langle \pi^A \rangle = \eta^\pi \epsilon^{AB} E^\mu_B \partial_t l_\mu + \eta^{\pi\Sigma}\epsilon^{AB}l^\mu \partial_t e_{\mu B} \qquad (4.59)$$

$$\langle \Sigma^A \rangle = \eta^\Sigma \epsilon^{AB}l^\mu \partial_t e_{\mu B} + \eta^{\pi\Sigma}\epsilon^{AB} E^\mu_b \partial_t l_\mu . \qquad (4.60)$$

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[6]It is also possible to implement an-isotropic with objects not transforming in the fundamental representation, such as higher rank tensors, for example [72]. We will have nothing to say about such constructions here.

Upon functional differentiation with respect to $e_\mu^A$, $l_\mu$, we find the Kubo formulae:

$$\eta^\tau = \lim_{\omega \to 0} \frac{-i}{\omega} P_H^{ABCD} \left( G_{ABCD}^{\tau\tau}(\omega, 0) + C_{ABCD}(\omega, 0) \right) \tag{4.61}$$

$$\eta^\pi = \lim_{\omega \to 0} \frac{-i}{\omega} \epsilon^{AB} G_{AB}^{\pi\pi}(\omega, 0) \tag{4.62}$$

$$\eta^\Sigma = \lim_{\omega \to 0} \frac{-i}{\omega} \epsilon^{AB} \left( G_{AB}^{\Sigma\Sigma}(\omega, 0) + C_{AB}(\omega, 0) \right) \tag{4.63}$$

$$\eta^{\pi\Sigma} = \lim_{\omega \to 0} \frac{-i}{\omega} \epsilon^{AB} G_{AB}^{\Sigma\pi}(\omega, 0) \,, \tag{4.64}$$

s where we have defined the retarded Green's function

$$G^{UV}(\omega, \vec{k}) = \int d^4 x \, e^{i(\omega t - \vec{k} \cdot \vec{x})} \mathrm{tr} \left( \rho_\beta [U(\vec{x}, t), V(0, 0)] \right) \theta(t) \,, \tag{4.65}$$

and $C_{ABCD}$, $C_{AB}$ stand for contact terms which arise due to the explicit connection dependence of the relevant operators. For our specific model they are computed in Appendix 4.A. Standard one-loop calculations will lead to the viscosities in the next Section.

## 4.1.2 Evaluation of Hall viscosity and interpretation

Before coming down to the final results it is useful to collect some information coming form time reversal symmetry and Lifshitz invariance. First, it is a straightforward computation to determine the scaling dimension of the four Hall viscosities $\left( \eta^\tau, \eta^\pi, \eta^\Sigma, \eta^{\pi\Sigma} \right)$ by looking at the Kubo formulas and applying (4.38) and (4.39). This gives

$$[\eta^\tau]_L = 2 + z \,, \quad [\eta^\pi]_L = 3z \,, \quad [\eta^\Sigma]_L = 4 - z \,, \quad [\eta^{\pi\Sigma}]_L = 2 + z \,. \tag{4.66}$$

furthermore, all of these coefficients need to be odd under time reversal. Since we work in a thermal state, the temperature is the only dimensionfull parameter according to the Lifshitz scaling, thus we conclude

$$\eta^\tau \sim sT^{2+z} \,, \quad \eta^\pi \sim sT^{3z} \,, \quad \eta^\Sigma \sim sT^{4-z} \,, \quad \eta^{\pi\Sigma} \sim sT^{2+z} \,. \tag{4.67}$$

For the Kubo formulas we will also need the expressions for the improved currents in our model. These are given by

$$\tau_{AB} = i\chi^T \beta_{(A} \overset{\leftrightarrow}{\nabla}_{B)} \chi \,, \quad \pi_A = i\chi^T \beta_A \overset{\leftrightarrow}{\nabla}_l \chi \,, \tag{4.68}$$

$$\Sigma_A = \frac{s}{2z} \chi^T \left[ \overset{\leftarrow}{\nabla}_l M(\nabla_l)^{1/2z-1} \overset{\rightarrow}{\nabla}_A + \overset{\leftarrow}{\nabla}_A M^{1/2z-1}(\nabla_l) \overset{\rightarrow}{\nabla}_l \right] C^{-1} \chi + \frac{1}{2} \nabla^B \sigma_{BA} \,, \tag{4.69}$$

having introduced for commodity the basis of $\beta^A$ matrices defined by $\beta^A = C^{-1} \gamma^A$. They may be represented as $\beta_0 = -1$, $\beta_1 = -\sigma_x$, $\beta_2 = \sigma_z$. For $A$ a spatial index these fulfill $\{\beta_A, C^{-1}\} = 0$, $[\beta_1, \beta_2] = 2C^{-1}$.

To evaluate the two point correlators we first continue these expressions to Euclidean signature following [73]. Then we evaluate the imaginary time Feynman diagrams by summing over internal Matsubara frequencies $\omega_m = 2\pi T(m + 1/2)$ using the integral representation of the fermionic sums

$$\frac{1}{\beta} \sum_n f(\omega_n) = \frac{1}{2} \int_C \frac{dz}{2\pi i} \tanh \left( \beta z/2 \right) f(z) \,. \tag{4.70}$$

where $C$ is a contour encircling the poles of the hyperbolic tangent. The answer thus obtained is continued to Lorentzian frequencies

$$G(\omega, \vec{k}) = -iG_E(\omega + i\epsilon, \vec{k}), \tag{4.71}$$

while momentum sums are evaluated with the help of the integral representations

$$\eta_D(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} n_F(t), \tag{4.72}$$

for the Dirichlet eta function and

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)} = 2 \int_0^{\pi/2} d\phi \sin(\phi)^{2b-1} \cos(\phi)^{2a-1}, \tag{4.73}$$

for the Euler beta function. We give details of the various computations in Appendix 4.B and of the determination of Seagull terms in Appendix 4.A to avoid clutter of the presentation. One interesting thing that comes from these computations is the absence of constant terms in the "Hall" part of the two point functions of our operators, which means that there is no $\delta(\omega)$ singularity in the viscosities, as should be the case for a non-dissipative contribution.

After a long but straightforward computation we are led to the following result, in accordance with the expectations described above:

$$\eta^\pi = \frac{s}{4\pi^2} \frac{z}{3z + 1} T^{3z} \Gamma(3z) \eta_D(3z), \tag{4.74}$$

$$\eta^\tau = \frac{s}{4\pi^2} T^{2+z} \frac{z(z + 4)}{(z + 1)(z + 3)} \Gamma(z + 2) \eta_D(z + 2), \tag{4.75}$$

$$\eta^{\pi\Sigma} = \frac{s}{4\pi^2} T^{2+z} \frac{(z + 4)}{(z + 1)(z + 3)} \Gamma(z + 2) \eta_D(z + 2), \tag{4.76}$$

$$\eta^\Sigma = \frac{s}{4z\pi^2} T^{4-z} \frac{(6 - z)}{(5 - z)(3 - z)} \Gamma(4 - z) \eta_D(4 - z). \tag{4.77}$$

It holds that $\eta^\tau = z\eta^{\pi\Sigma}$. In this way, rescaling $\Sigma \to z\Sigma$ the last three viscosities obey the compact relation

$$\eta^{\text{Hall}}(\xi) = z \frac{s}{4\pi^2} T^\xi \frac{(\xi + 2)}{(\xi + 1)(\xi - 1)} \Gamma(\xi) \eta_D(\xi), \tag{4.78}$$

being $\xi$ their Lifshitz scaling dimension.

It is interesting here to discuss the question of universality of these results. First we may ask ourselves whether interactions might modify such predictions. On the one hand the coefficients of the anisotropic viscosity do not appear to be protected by any quantization condition, on the other hand it is not possible to construct marginal deformations to introduce interactions. Furthermore, it may be that, at very low energy, the introduction of such interactions leads to another free Lifshitz theory, in which case our prediction would apply. This should make clear to which extent the result above should be trustworthy.

While the numerical values for the viscosities thus computed do not seem universal, we may focus on the interesting (warped) limit in which $z \to 0$, where all of the above vanish at zero temperature apart from:

$$\lim_{z \to 0} \eta^\pi = \frac{s}{24\pi^2}. \tag{4.79}$$

Which is somehow reminiscent of a result obtained from Chern-Simons theory. Indeed an hint for dimensional reduction in this limit comes from the density of states of the free system, which scales as $\rho(\epsilon) \sim \epsilon^{1+z}$, interpolating between the four-dimensional $z = 1$ result and the three dimensional $z = 0$ one. Indeed it is plausible that, upon regularization of a tower of KK modes coming from reducing the anisotropic direction, a Chern-Simons term is generated by integrating out the massive $2 + 1$ dimensional fermion in the standard way. This would give rise to an action[7]

$$W[l]_{3d} = -s\frac{\kappa}{4\pi} \int l \wedge dl \,, \tag{4.80}$$

with $\kappa$ the regulated sum over the KK modes. We would need $-\frac{s\kappa}{8\pi} = \frac{s}{48\pi^2}$ to match our previous computation. As usual, what really makes sense is the difference between the Chern-Simons coefficients for $s$ and $-s$. Assuming the anisotropic direction to be compact with anti-periodic boundary conditions (so that the KK modes are all gapped) then $\kappa$ is given by regulating

$$\kappa = \frac{1}{2\pi} \sum_{n=1}^{\infty} (2n-1)^2 = -\frac{1}{12\pi} \,, \tag{4.81}$$

where the factor $1/\pi$ comes from the momentum integrals and $(2n - 1)$ is the momentum in the anisotropic direction of the mode, the final result is obtain by $\zeta$-function regularization. Putting all together one indeed finds[8]:

$$-\frac{s\kappa}{4\pi} = \frac{s}{48\pi^2} \,. \tag{4.82}$$

This computation, of course, should be taken *cum grano salis* as it required various assumptions which are not fully justified. Its purpose is to show that the warped value for the viscosity can indeed be inferred by universal considerations. In the next section we will develop the effective theory for a warped Lifshitz theory with Carrollian boosts, showing how it indeed predicts the appearance of the same type of coefficient and how it can be related to a "torsional" anomaly.

## 4.2 Universality in the warped limit

Prompted by the results presented in the previous Section, we elaborate more about the behavior of the system in the warped limit. While a partial explication of our results was given by dimensionally reducing over the anisotropic direction, here we study the purely four dimensional perspective. First, keeping the same geometric setup we show that a Fujikawa-type computation indeed suggests a new torsional anomaly to be present in the warped limit, we then discuss a brief history of such appearances in the literature pointing out the relevant differences in our case, finally we examine warped theories such as those introduced in [75, 76] and expand on the study of their anomalies initiated in [77]. The last part will comprise most

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[7]This should be thought as the differentce between the actions for positive and negative $s$, as it happens in the case of the $2 + 1$ dimensional fermion which has properly quantized Hall conductivity, even though a single massive fermions only gives half the amount.

[8]A similar conclusion is reached by inspecting the standard computation of Chern-Simons terms at finite temperature [74].

of the Section, as it requires the introduction of a first-order formalism for Carrollian theories to simplify the study of the consistency conditions. The final conclusion is that torsion seems indeed to behave as a "chiral" $U(1)$ field in the warped limit and we give an intuitive picture of why it is so. This results were published in [78].

### 4.2.1 Torsional contributions to the fermionic determinant

We start by fixing the free Majorana action (4.7). We also introduce a dimensionless (according to the Lifshitz counting) parameter $q$ in order to capture the usual scaling dimensions by rescaling $\nabla_l$ derivatives to $\nabla_l/q$ and normalizing the anisotropic kinetic term by an overall factor of $q$, so that now it is given by $sqM(\nabla_l/q)$. This procedure introduces a spurionic Weyl symmetry, which acts by:

$$q \to \lambda^{-1}q \quad (l^\mu, \ E_A^\mu) \to \lambda(l^\mu, \ E_A^\mu), \ \Psi \to \lambda^{3/2}\Psi. \tag{4.83}$$

We will require such transformations to be preserved in the quantum theory. This leads to an identification of $q$ in the warped limit as a coupling with torsion.

We then split a diffeomorphism generator $\xi^\mu = \theta l^\mu + \xi^A E_A^\mu$ and take $\xi^A$ to zero. We consider the regularized trace of the fermionic diffeomorphism variation:

$$\mathcal{T}(\delta_\Psi) = \lim_{\tau \to 0} tr \left[ \delta_\Psi e^{-\mathcal{R}[\tau]} \right], \tag{4.84}$$

with a co-variant regulator $\mathcal{R}$. We fix $\mathcal{R}$ by demanding:

1. $e^\mathcal{R}$ to have finite trace, that is to decay fast enough in all directions in momentum space.

2. $\mathcal{R}$ to be co-variant under Lifshitz transformations and invariant up to a rescaling of the $\tau$s.

3. $\mathcal{R}$ needs to couple consistently to the background geometry.

4. We will also assume the regulator respects the spurionic scaling symmetry. This fixes how $q$ should appear. Notice that this means that the $\tau$s may not transform under such a symmetry.

The simplest candidate which satisfies these requirements is given by

$$\mathcal{R} = A^\dagger A, \tag{4.85}$$

with $A$ related to the Dirac operator as

$$A = i\tau_1 \gamma^a \nabla_a/q + s\tau_2 \left(i\nabla_l/q\right)^{1/z}, \tag{4.86}$$

and we have the freedom of introducing two independent regulators for the isotropic and anisotropic part of the kinetic term. In performing the computation, we will take the $\tau$s small but finite, and take the warped limit *before* the $\tau \to 0$ limit. Furthermore, since we are interested in $T$-odd terms, we focus on those terms which are proportional to the odd parameter $s$. We will perform the computation in both two and four dimensions, the further work needed is very little, but warped theories have been mostly studied in the two dimensional case and having such results will be useful for cross-checks.

$D = 2$

We start by examining the two dimensional case. Here there is only one $\gamma$ matrix which is the identity. Using the standard manipulations presented in Appendix 4.C we find the following expansion for the regulator:

$$\mathcal{R} = \tau_1^2 \nabla_\perp / q^2 + is \sum_k^{1/z} c_k \tau_1 \tau_2 (\nabla_l/q)^k G_\mu E^\mu (\nabla_l/q)^{1/z-k} + \tau_2^2 (i\nabla_l/q)^{2/z} \,. \tag{4.87}$$

This expansion can readily be combined with a basis of plane waves to evaluate the trace order by order. Indeed, since $\delta_\Psi = \theta \left( \nabla_l + ik_l \right)$ in this basis:

$$\mathcal{T}(\delta_\Psi)_D = tr \left[ \theta \nabla_l e^{-\mathcal{R}[\nabla;\tau]} \right] = \int \frac{d^d k_a}{(2\pi)^d} \int \frac{dk_l}{(2\pi)} \theta(\nabla_l + ik_l) e^{\mathcal{R}[\nabla+ik;\tau]} \,, \tag{4.88}$$

expansion around the Gaussian contribution leads to a heat-kernel expansion in $\tau_1$ and $\tau_2$ of the following form:

$$\mathcal{T}(\delta_\Psi)_{D=2} = \sum_{a_1,a_2} \mathcal{T}_{D=2}^{a_1,a_2} \tau_1^{a_1} \tau_2^{a_2} \,, \tag{4.89}$$

with $a_1$ , $a_2$ either integers or multiples of $z$. We are interested in terms in which both $a_1$ and $a_2$ are $O(z)$, which give rise to the finite contributions in the warped limit[9]. We also demand these contributions to be proportional to $s$. This simplifies the analysis a lot. Introducing rescaled variables $k_a = q\, \tau_1^{-1} u_a$, $k_l = q\, \tau_2^{-z} v$ and using that $c_{1/z} = 1/z$ we get only the following contribution:

$$\mathcal{T}_{D=2}^{0,-2z} = \theta \frac{sq^2}{z(2\pi)^2} \int du dv \; v^{1+1/z} \exp(-u^2 - v^{2/z}) E^\mu G_\mu \,, \tag{4.90}$$

going to polar coordinates this reduces to a combination of Euler's Beta and Gamma functions that give:

$$\mathcal{T}_{D=2}^{0,-2z} = \theta \frac{sq^2}{4\pi} \Lambda_2^{2z} \frac{\Gamma(z+1/2)}{\Gamma(1/2)} E^\mu G_\mu \,. \tag{4.91}$$

We now take the warped limit and substitute $\epsilon^{\mu\nu} T_{\mu\nu} = 2E^\mu G_\mu$ which is valid in $D = 2$ to get the warped anomaly:

$$\mathcal{A}_{D=2}(\theta) = \int \sqrt{g} \; \theta \; \frac{sq^2}{8\pi} \epsilon^{\mu\nu} T_{\mu\nu} \,. \tag{4.92}$$

$D = 4$

Very similar methods lead to the four-dimensional result. Here the expansion of the regulator happens to be slightly for complex due to the nontrivial Clifford algebra:

$$\mathcal{R}[\tau] = \tau_1^2 \nabla_\perp^2 / q - \frac{i\tau_1^2}{2} \epsilon^{abc} \gamma_a T_{bc} \nabla_l / q + is\gamma^a \sum_k^{1/z} c_k \tau_1 \tau_2 (\nabla_l/q)^k G_a (\nabla_l/q)^{1/z-k} + \tau_2^2 (i\nabla_l/q)^{2/z} + \mathcal{R}[R^\omega; \tau] \,, \tag{4.93}$$

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[9]By dimensional analysis, we do not expect any finite contributions away from said limit.

with $\mathcal{R}[R^\omega\,;\,\tau]$ a torsion-independent part which will not concern us. Expanding the exponential in (4.88) and looking for the same kind of terms as before now also leads to a single integral that may be evaluated with the same methods as in the $D = 2$ case to give

$$\mathcal{T}_{D=4}^{0,-3z} = \theta \frac{sq^3 \Lambda_2^{3z}}{8\pi^2} \frac{\Gamma(3z/2 + 1/2)}{\Gamma(1/2)} \epsilon^{abc} G_a T_{bc}\,. \tag{4.94}$$

Using now

$$\epsilon^{abc} G_a T_{bc} = \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} T_{\mu\nu} T_{\rho\sigma}\,, \tag{4.95}$$

we find

$$\mathcal{T}_{D=4}^{0,-3z} = \theta \frac{sq^3 \Lambda_2^{3z}}{32\pi^2} \frac{\Gamma(3z/2 + 1/2)}{\Gamma(1/2)} \epsilon^{\mu\nu\rho\sigma} T_{\mu\nu} T_{\rho\sigma}\,, \tag{4.96}$$

whose warped limit gives the anomaly

$$\mathcal{A}_{D=4}(\theta) = \int \sqrt{g}\,\theta\,\frac{sq^3}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} T_{\mu\nu} T_{\rho\sigma}\,. \tag{4.97}$$

Let us notice in both cases the similarity with the standard (co-variant) chiral anomaly in $D = 2, 4$. A tentative dictionary would be that, in the warped limit, the torsion plays a role akin to an electromagnetic field, while the parameter $q$ is the charge. Of course one may use the fact that torsion is dimension-full according to the spurious counting to adsorb $q$ into its normalization, similarly to what happens in standard electrodynamics. It is also worth noticing that the physical significance of $q$, if any, comes from the fact that we have imposed the spurionic symmetry in our regularization scheme, otherwise it could have been seen as a remainder of the regularization of the anomalous trace.

### 4.2.2 Gauging the Carroll algebra

After having studied the co-variant anomalies in our fermionic theory we move onto a consistency check from the Wess-Zumino conditions introduced in Chapter 1. To do so we will introduce a first order formulation of the warped Carroll algebra which we expect to describe the warped limit of our theory. A justification of the emergence of Carrollian boosts will be given in (4.2.4). Let us start by recalling the Carroll algebra with the scaling generator, for the most part it is safe to think that $z = 0$

$$[J^{AB}, J^{CD}] = \delta^{AC} J^{BD} + ... \quad [J^{AB}, P^C] = \delta^{AC} P^B - \delta^{BC} P^A\,, \tag{4.98}$$

$$[J^{AB}, C^C] = \delta^{AC} C^B - \delta^{BC} C^A\,, \quad [C^A, P^B] = \delta^{AB} \Pi \tag{4.99}$$

$$[D, P^A] = P^A\,, \quad [D, C^A] = (z-1) C^A\,, \quad [D, \Pi] = z\Pi\,. \tag{4.100}$$

We not introduce a Lie-algebra-valued connection $\mathcal{A}$ which in components reads:

$$\mathcal{A} = n\Pi + e^A P_A + f^A C_A + \frac{1}{2} \omega^{AB} J_{AB}\,, \tag{4.101}$$

to which one may associate a curvature

$$\mathcal{F} = d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] = F(\Pi)\Pi + F(C)^A C_A + F(P)^A P_A + \frac{1}{2} F(J)^{AB} J_{AB}\,, \tag{4.102}$$

which in components is

$$F(\Pi) = (dn - f^a \wedge e_a) \,, \tag{4.103}$$

$$F(C)^a = Df^a \,, \tag{4.104}$$

$$F(P)^a = De^a \,, \tag{4.105}$$

$$F(J)^{ab} = d\omega^{ab} + \frac{1}{2}[\omega, \omega]^{ab} \,, \tag{4.106}$$

with $D$ shorthand for the co-variant derivative with respect to the $SO(d)$ spin connection. Gauge transformations correspond to $\delta_\alpha \mathcal{A} = \mathcal{D}\alpha$, $\mathcal{D} = d + [\mathcal{A}, ]$ with:

$$\alpha = \theta\Pi + \xi^A P_A + \lambda^A C_A + \Omega^{AB} J_{AB} \,, \tag{4.107}$$

so that the gauge variation reads in components:

$$\begin{aligned} \delta_\alpha \mathcal{A} &= \delta_\alpha n \ \Pi + \delta_\alpha e^A \ P_A + \delta_\alpha f^A \ C_A + \delta_\alpha \omega^{AB} \ J_{AB} \\ &= \left( d\theta + \lambda^A e_A - \xi^A f_A \right) \Pi + \left( D\xi^A + \Omega^A{}_B e^B \right) P_A \\ &+ \left( D\lambda^A + \Omega^A{}_B f^B \right) C_A + D\Omega^{AB} J_{AB} \,, \end{aligned} \tag{4.108}$$

It is notationally useful to also introduce a one form $\Sigma = \lambda^A e_A - \xi^A f_A$ so that under Carrollian boosts $n \to n + \Sigma$. While this fields respect the gauge algebra, the do not, in general, allow the implementation of diffeomorphisms. This is a familiar problem from the gauging of the Poincaré algebra, where imposing diffeomorphisms of $\mathcal{A}$ to be realizable as gauge transformations is usually achieved through the constraint of vanishing torsion $F(P)^a = 0$. Once this equation is imposed one notices two facts:

1. The $SO(D)$ spin connection becomes expressible in terms of the vielbein, seen as the "momentum" components of the gauge field.

2. A diffeomorphism generated by the vector field $\xi^\mu$ may be interpreted as a $P^a$ transformation generated by the parameter $\xi^a = i_\xi e^a$. This is a consequence of the identity

$$\mathcal{L}_\xi \mathcal{A} = i_\xi \mathcal{F} + \delta_{\alpha_\xi} \mathcal{A} \,, \quad \alpha_\xi = i_\xi \mathcal{A} \,. \tag{4.109}$$

   This follows from rewriting the Lie derivative on the left hand side as combination of a gauge transformation plus a spurious term $i + \xi \mathcal{F}$. The curvature constraints in this case make such a term drop out. In general such spurious terms do not cancel, however one may either forget about the transformation laws of some components of the connection by making them not independent, or perform a "compensating" gauge transformation with only certain non-zero components to take care of them. Indeed the Lie derivative on the right hand-side is only defined modulo gauge transformations.

We are tasked to solved a similar problem for the Carroll group. In general there are many solutions. This is not a novelty, since in the Poincaré case one may define a so called Weitzenbock connection which comes by trivializing the $SO(D)$ curvature.
Here we do not attempt to give a complete analysis of what possible constraints can be imposed on the Carrollian geometries, but we point out one which makes the Wess-Zumino problem

particularly simple to solve. Also, prompted by our previous results, we want to find constraints which allow the curvature associated to the $\Pi$ generator to be non-vanishing. This will play the role of the torsion in the Carrollian geometry. Recall also that the "isotropic" part of the torsion in the geometric formulation does not have to vanish, thus we may expect a kind of Weitzenbock-like constraint to exist, indeed in two dimensions the Carrollian connection takes the Weitzenbock form [77].

Based on these ideas we can introduce three sets of constraints:

1. $F(\Pi) = F(P)^A = 0$. This constraint is the natural generalization of the torsion-less condition for the Poincaré algebra. It allows to re-express the spin connection $\omega_\mu^{AB}$ and boost connection $f_\mu^A$ in terms of the fields $n_\mu$ and $e_\mu^A$. The splitting of the diffeomorphisms $\xi^\mu = \theta v^\mu + \xi^A E_A^\mu$ is also recovered once one imposes $i_v e^A = i_{E^A} n = 0$. This is the approach used in [79] to construct a version of Carrollian gravity. After solving the curvature constraints one finds the usual vielbein expression for $\omega^{AB}$, while

$$f_\mu^A = n_\mu v^\nu E^{\rho A} \partial_{[\nu} n_{\rho]} + E^{\nu A} \partial_{[\mu} n_{\nu]} + S^{AB} e_{\mu B} \,, \tag{4.110}$$

with $S^{AB}$ a symmetric tensor. This conditions however to dot allow for torsion field to be introduced in the Carrollian geometry and thus we will not use them.

2. $F(C)^A = 0$ , $F(P)^A = 0$. Here the second constraint will fix the spin connection, while the first defines a Stueckelberg one-form $M$ through the equation

$$dM = f^A \wedge e_A \,, \tag{4.111}$$

whose consistency can be checked by explicitly taking the exterior derivative and using the curvature constraints, which yield

$$d\left(f^A \wedge e_A\right) = Df^A \wedge e_A - f^A \wedge De_A = F(C)^A \wedge e_A - f^A \wedge F(P)_A = 0 \,. \tag{4.112}$$

The transformation properties of $M$ under $C^A$ and $P^A$ follow from the definition to be

$$\delta_{P,C} M = \Sigma \,. \tag{4.113}$$

These exactly cancel the transformation properties of $n$ and allow to define a boost-invariant one-form

$$\hat{n} = n - M \,, \tag{4.114}$$

so that

$$F(\Pi) = d\hat{n} \,, \tag{4.115}$$

becomes an Abelian curvature. This is not totally trivial, since the gauge variation of $F(\Pi)$ is a combination of the constraints. The formulas above should be confronted with (3.33) and (3.35). The shortcoming of these constraints is that only a subset (although not too restricted) of diffeomorphisms is expressible through gauge transformations. To see this it is useful to introduce algebraic inverses for $n$ and $e^A$ and decompose diffeomorphisms as usual $\xi^\mu = \theta v^\mu + \xi^A E_A^\mu$. Now of course this decomposition is not uniquely defined, since $E_A^\mu$ is not boost invariant. However there are two classes of transformations, depending

on which $\xi^A$ can be made to vanish or not. The first class can be realized by a gauge transformation with gauge parameter $\alpha = \theta i_v \mathcal{A}$, supplemented by a pure boost $\Sigma_\theta = \beta_A e^A = \theta i_v dn$. The second class is not completely expressible as gauge transformations, indeed the condition coming from (4.109) requires $i_\xi dn = \tilde{\beta}^a e_a$. Taking the interior product with $v$ and using the standard formulas for Lie derivation gives a condition on $v$

$$0 = i_v \mathcal{L}_\xi n = -i_{\mathcal{L}_\xi v} n \,. \tag{4.116}$$

transformations So that only generators $\xi^\mu$ subject to this may be allowed. The simplest family of such solutions comes from requiring just $\mathcal{L}_\xi v = 0$. This are essentially the so called Carrollian diffeomorphisms [80]. To get a better idea of their form, suppose that $v^\mu = (1, \vec{0})$ in certain coordinates, then the condition above just tells us $\partial_v \xi^\mu = 0$. Alternatively, one might ask directly $i_\xi F(\Pi)$ to be a pure translation. This gives the condition $\mathcal{L}_\xi \hat{n} = 0$.

3. $F(C)^A = 0$ , $F(J)^{AB} = 0$ This is the aforementioned analog of the Weitzenbock constraints, from which it should indeed come as an ultra-relativistic limit. These can be solved as follows: first one notices that, in the absence of $SO(d)$ curvature, the equation $F(C)^A = Df^A = 0$ has the solution

$$f^A = DM^A \,, \tag{4.117}$$

for a zero form $M^A$. This is consistent with the constraints since $F(C)^A = D^2 M^A = F(J)^A{}_B M^B = 0$. As one might expect $M^A$ will be the analog of the Stueckelberg field $M$ above. As before we can construct an invariant curvature

$$\bar{F} = d(n - M^A e_A) = F(\Pi) - M^A F(P)_A \,. \tag{4.118}$$

The main difference is that now the field $n - M^A e_A$ transforms as follows

$$\delta_\alpha(n - M^A e_A) = d\theta - d(M^A \xi_A) \,. \tag{4.119}$$

Apart from these equations, one needs to solve for the spin connection $\omega^{AB}$, which is now taken as a flat $SO(d)$ connection. At the moment we do not have a clear way to implement diffeomorphisms with this set of constraints, we will mainly use it for comparison, since it gives essentially equivalent solutions to the consistency conditions.

The main point of the construction above is that, both in option 2. and 3. there exists an "emergent" abelian curvature, constructed out of the "anisotropic" translation curvature $F(\Pi)$ and the Stueckelberg field $M^A$ (or $M$). This will be important since it allows to write explicit solutions to the Wess-Zumino consistency conditions even though the standard definition of Chern-Simons terms is not applicable.

### 4.2.3 Chern-Simons terms and the consistency conditions

Let us now come to the explicit construction of the possible "candidate" anomalies for warped Carrollian theories. Before that, it is important to stress what is usually the problem in constructing such quantities in non-relativistic theories. To understand that, we must return to

Chapter 1, where we had introduced invariant polynomials $P^{D+2,0}(\mathcal{A})$ and used the decomposition into characteristic polynomials $tr\left(\mathcal{F}^n\right)$ to prove their closedness and gauge invariance. The main problem is that most non-relativistic algebras are not semi-simple, they thus (usually) lack a non-degenerate invariant metric to define the trace operation. Some exceptions exist see e.g. [81], if certain central extensions are considered, but they tend to be special to $D = 3$. As the author sees it, there are two ways-out with respect to the consistency conditions:

1. One finds other solutions, which are *not* given by the descent procedure starting with a characteristic polynomial $P^{D+2,0}(\mathcal{A})$.

2. There exists a constrained subset of curvatures $\mathcal{F}'$ for which a characteristic polynomial might be defined, giving a candidate anomaly.

In this instance, we will find that solution 2. will come into play in our problem. However, this might be a nice arena for future studies. The way the story plays out at this point should be clear. We have seen that the gauged-Carroll algebra admits an abelian curvature $F$ on the constraint surface. From this we may construct characteristic polynomials

$$P^{D+2,0}(\mathcal{A}') = F^m \quad 2m = D + 2 \,, \tag{4.120}$$

which will give rise to a chain of descent equation leading a representative of the anomaly. It needs to be stressed that $P^{D+2,0}(\mathcal{A}')$ is a characteristic polynomial only *after* the constraints are imposed, in particular $dP^{D+2,0}(\mathcal{A}) = g(\text{constraints})$ so closure is achieved only on the constraint surface. Before going forward a last comment is in order. Upon introducing a scaling generator in the Carroll algebra on has the following dimensional assignments:

$$[n] = z \,, \quad [f] = -1 + z \,, \quad [e] = 1 \,, \tag{4.121}$$

$$[\theta] = z \,, \quad [\lambda] = -1 + z \,, \quad [\xi] = 1 \,. \tag{4.122}$$

consistency conditions require characteristic polynomials to be dimensionless. A quick check shows that $[F^m] = mz$, so that *only* in the warped limit do such representatives exist. This is the manifestation of a familiar problem in defining torsional contributions to the anomaly polynomial. The story is most known in four dimensions, where a regulator-dependent contribution to the chiral anomaly arises in the presence of torsion. This is proportional to the cutoff scale $\Lambda^2$ times a characteristic polynomial known as the Nieh-Yan density [82]:

$$\Lambda^2 NY[e] = \Lambda^2 \int \left(T^a \wedge T_a - R_{ab} \wedge e^a \wedge e^b\right) \,. \tag{4.123}$$

This is, of course, because torsion is dimension-full according to the standard Weyl scaling and such term should then be regularized away in a well defined QFT[10]. The workaround to this problem is usually that the system has a well defined length scale with which to normalize the torsion (for example in a QFT in AdS space-time one may use the cosmological constant). In our case the solution is that, in the warped scaling regime, the torsion is actually *dimensionless*. Now we can write the possible consistent anomalies[11]

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[10] There are physical exceptions [83, 84] in which one may argue that $\Lambda^2$ should represent a bulk cutoff coming from anomaly inflow, e.g. the mass gap in the bulk theory, then these terms may appear, however their interpretation from the effective field theory perspective is not clear to the author.

[11] Here we do not consider conformal anomalies nor diffeomorphism ones, which were studied by Jensen [77] using a second order formulation. Our aim is to clarify the role played by "torsion" which in that work was not accounted for.

**D=2**

Here the characteristic polynomial is

$$P^{4,0}(\mathcal{A}') = c_\Pi \begin{cases} F \wedge F, & (2.), \\ \bar{F} \wedge \bar{F}, & (3.) \end{cases} \tag{4.124}$$

with either $F = d\hat{n}$ of $\bar{F} = d\left(n - M_A e^A\right)$ depending on the chosen constraints. The Chern-Simons term is

$$P^{3,0}(\mathcal{A}') = c_\Pi \begin{cases} \hat{n} \wedge F, & (2.), \\ \left(n - M_A e^A\right) \wedge \bar{F}, & (3.) \end{cases} . \tag{4.125}$$

Taking the gauge variation leads to the exterior derivative of the consistent anomaly.

$$P^{2,1}(\mathcal{A}') = c_\Pi \begin{cases} \theta F, & (2.), \\ \left(\theta - M_A \xi^A\right) \bar{F}, & (3.) \end{cases} . \tag{4.126}$$

Which reproduces our expectations coming from our previous Fujikawa computation, which of course could have no terms in the boost gauge field since it had to be turned to zero.

This looks like a pure "translational" anomaly in both cases, however we can also analyze the possible Bardeen counter-terms, which are local two-forms in the constrained fields. In this example there is only one such term which one may write:

$$B_{D=2} = x \begin{cases} n \wedge M, & (2.), \\ n \wedge e^A M_A, & (3.) \end{cases} . \tag{4.127}$$

whose variation can give the following contribution to the anomaly polynomial (after possibly partial integration):

$$sB_{D=2} = -x \begin{cases} \theta \, dM + \Sigma \wedge n, & (2.), \\ \theta \, d(e^A M_A) + \Sigma \wedge n, & (3.) \end{cases} \tag{4.128}$$

this can be used to move part of the $\theta$ anomaly into the boost sector, in such a way that, when $dn = 0$, the anomaly is purely given by a functional of the boost parameter $\Sigma$. In the absence of a boost gauge field, this gives back the results of Jensen [77].

**D=4**

The story here, as one may expect, is completely analogous, only that one starts with the characteristic polynomial

$$P^{6,0}(\mathcal{A}') = c_\Pi \begin{cases} F \wedge F \wedge F, & (2.), \\ \bar{F} \wedge \bar{F} \wedge \bar{F}, & (3.) \end{cases} \tag{4.129}$$

which gives the Chern-Simons term

$$P^{5,0}(\mathcal{A}') = c_\Pi \begin{cases} \hat{n} \wedge F \wedge F, & (2.), \\ \left(n - M_A e^A\right) \wedge \bar{F} \wedge \bar{F}, & (3.) \end{cases} , \tag{4.130}$$

and the anomaly

$$P^{4,1}(\mathcal{A}') = c_\Pi \begin{cases} \theta F \wedge F, & (2.), \\ \left(\theta - M_A \xi^A\right) \bar{F} \wedge \bar{F}, & (3.) \end{cases} , \tag{4.131}$$

matching the Fujikawa computation.

In the case (2.) in which the $SO(3)$ curvature was non-vanishing, one can also have the mixed characteristic polynomial

$$\tilde{P}^{6,0}(\mathcal{A}') = c_{SO(3)} \ F \wedge tr\left[F(J) \wedge F(J)\right] , \tag{4.132}$$

which gives a mixed (as we will see momentarily) anomaly between $SO(3)$ and the "an-isotropic translations"

$$\tilde{P}^{4,1}(\mathcal{A}') = c_{SO(3)} \ \theta \ tr\left[F(J) \wedge F(J)\right] . \tag{4.133}$$

However it is not clear whether this possible anomaly is reproduced by the Lifshitz system in the warped limit (it might be also that simply $c_{SO(3)} = 0$).

Here also we may study the possible Bardeen counter-terms In this case there are three possible terms (we only write case (2.) for simplicity):

$$B_{nnM} = x \ n \wedge M \wedge dn , \tag{4.134}$$

$$B_{nMM} = y \ n \wedge M \wedge dM , \tag{4.135}$$

$$B_{SO(3),\hat{n}} = z \ \hat{n} \wedge P^{3,0}(\omega) . \tag{4.136}$$

The first two term essentially do the same thing as in the two dimensional case, making the anomaly appear through boosts, while the third term allows to move the mixed anomaly to the Lorentz sector.

## Matching the dimensional reduction

Recall that, in the previous Section, we has derived an effective description for the warped limit physics by integrating out the fermionic modes in the anisotropic direction, with boundary conditions admitting no zero KK mode. Since we have now derived the form of the consistent anomalies for the warped theory directly, we may apply the inflow arguments of Chapter 2 to check this statement. The argument essentially matches the abelian construction, since one may think as the anisotropic vector $b_\mu$ as describing the holonomy around the anisotropic direction. The co-variant effective action[12] then turns out to be

$$W_{cov}[n] = c_\Pi b \int n , \tag{4.137}$$

and, in three dimensions

$$W_{cov}[n] = 3c_\Pi b \int ndn , \tag{4.138}$$

which match our predictions if we make $n$ dimensionless by a factor of $b$ and give the warped fermion the anomaly of a free chiral fermion.

The careful reader might have noticed that, in the previous Section, we had computed a time-dependent response (viscosity), while the effective theory developed in Chapter 2 only covered

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[12]Let us fix the case 2. geometry for simplicity.

time independent response. The reason this works is that we are dealing with zero temperature observables in the warped limit, then Lorentz symmetry relates the momentum and frequency responses. Indeed one can explicitly computes the momentum dependence of the $\pi^A \pi^B$ correlators in the Lifshitz model and it can be seen to match the $\eta^\pi$ coefficient in the warped limit.

### 4.2.4 Warped fermions

In this final part, we present examples of warped theories and give a physical explanation for the anomalies derived above. Furthermore, we show that, at the classical level, the Lifshitz theory (4.7) indeed admits a Carroll boost generator and identify its warped limit.

**Warped CFTs**

We start by presenting free models of warped conformal theories, these reproduce the two dimensional fermionic theories proposed in the literature [76, 77]. One could ask the question of whether interacting warped CFTs exist. This is not clear, however, at least at the Lagrangian level, we will see that there are no relevant operators.

Let us first start by presenting the Ward identities. One can couple the warped theory to a Carrollian geometry as the one introduced in 3.2 and derive the Ward identities for the currents by standard manipulations:

$$\frac{1}{\sqrt{g}}\partial_\mu \sqrt{g}\pi^\mu = 0 \,, \tag{4.139}$$

$$\frac{1}{\sqrt{g}}D_\mu \sqrt{g}t_a^\mu = 0 \,, \tag{4.140}$$

$$e_{\mu a}\pi^\mu = 0 \,, \tag{4.141}$$

$$\frac{1}{2\sqrt{g}}D_\mu \sqrt{g}S_{ab}^\mu - t_{[ab]} = 0, \tag{4.142}$$

$$t_a^a = 0 \,, \tag{4.143}$$

with the same notation as Chapter 1 and $\pi^\mu$ denoting the translation current coming from variation of $n_\mu$. The third line above is the boost Ward identity which should be fulfilled to promote a Lifshitz scaling theory to a Carrollian one.

To construct free fermionic theories we need to define a Clifford algebra for the Carroll group. An idea is to use the higher dimensional embedding for this theories to define a consistent contraction of the $SO(D, 1)$ Clifford algebra. This indeed may be used, for example, to derive the Levy-Leblond representation for Galileian free-fermions [85], for which one should take

$$(\Gamma^\pm)^2 = 0 \,, \quad \{\Gamma^+, \Gamma^-\} = 2\mathbb{I} \,, \quad \{\Gamma^A, \Gamma^B\} = 2h^{AB}\mathbb{I} \,, \quad \{\Gamma^n, \Gamma^A\} = 0 \,. \tag{4.144}$$

Notice that non-trivial commutators are encoded in twice-contravariant tensors in the non-relativistic geometry (for the Galilei group these are the inverse metric $h^{\mu\nu}$ and the null generator $n^M$) For our simplified purposes we may indulge in lesser generality and use the insight coming from above to work out the Carrollian case. Here the invariant tensor is given by $v^\mu v^\nu$

and we use the further gamma matrix (which is associated to the orthogonal direction in the embedding formulation) to define charge conjugation, the algebra then is spanned by matrices $\Gamma^A$, $\Gamma^n$ satisfying

$$(\Gamma^n)^2 = \mathbb{I}, \quad \{\Gamma^A, \Gamma^B\} = 0, \quad \{\Gamma^n, \Gamma^A\} = 0, \tag{4.145}$$

together with a charge conjugation matrix $\Gamma^{\mathcal{C}}$ which satisfies[13]

$$\mathcal{C}\Gamma^A\mathcal{C}^{-1} = (\Gamma^A)^\dagger \equiv \tilde{\Gamma}^A, \quad \{\Gamma^{\mathcal{C}}, \Gamma^n\} = 0, \quad \left(\Gamma^{\mathcal{C}}\right)^2 = -\mathbb{I}. \tag{4.146}$$

The representations of this algebra admit a block-diagonal splitting as:

$$\Gamma^n = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \quad \Gamma^A = \begin{pmatrix} 0 & \gamma^A \\ 0 & 0 \end{pmatrix}, \quad \Gamma^{\mathcal{C}} = \begin{pmatrix} 0 & \mathcal{C} \\ -\mathcal{C} & 0 \end{pmatrix}, \tag{4.147}$$

with $\gamma^A$ Clifford matrices of $SO(d)$ and $\mathcal{C}$ the charge conjugation matrix of such reduced Clifford algebra. With this we can define the rotation and boosts generators through the products of two gamma matrices:

$$C^A = \frac{1}{4}[\Gamma^A, \Gamma^n] = -\frac{1}{2}\begin{pmatrix} 0 & \gamma^a \\ 0 & 0 \end{pmatrix} = -\frac{1}{2}\Gamma^A, \tag{4.148}$$

$$\Omega^{AB} = \frac{1}{4}[\Gamma^{[A}, \tilde{\Gamma}^{B]}] = \begin{pmatrix} \gamma^{AB} & 0 \\ 0 & -\gamma^{AB} \end{pmatrix}, \tag{4.149}$$

with $\gamma^{AB} = \frac{1}{4}[\gamma^A, \gamma^B]$. Since the representation has a block-diagonal form, as with the representations of the standard Clifford algebra we may have either reduced "Weyl" representations, which are boost invariant and constructed via the projector $P_- = \frac{1}{2}(\mathbb{I} - \Gamma^n)$, or two-component representations, which in the literature are called "BC". We will denote the spinors of the first one by $\varphi$ and the second one by $\Psi = (\chi, \varphi)$. The Lagrangian can be written down by using the Dirac operator $\slashed{D} = \Gamma^A D_A + \Gamma^n D_v$ and the projector $P_-$. The "BC" representation also allows for a Majorana condition, which we will always impose.

Another important observation is that both systems allow for "mass terms" $q\bar{\varphi}\varphi$[14] and $q\bar{\Psi}\Psi$. In the Galilei case similar terms indeed appear and may be related to the null momentum by dimensional reduction of the $D + 1$ Dirac operator. Here the mass terms should be interpreted as boundary terms to allow a consistent null embedding.

By expanding in components we find the following two Lagrangians:

$$S_{Weyl} = \int d^d x \sqrt{g} \left( i\bar{\varphi} D_v \varphi + q\bar{\varphi}\varphi \right), \tag{4.150}$$

and

$$S_{BC} = \int \sqrt{g} \left( i\varphi^T \mathcal{C}^{-1} \gamma^A D_A \varphi + 2i\chi^T \mathcal{C}^{-1} D_v \varphi + 2q\chi^T \mathcal{C}^{-1} \varphi \right). \tag{4.151}$$

A first interesting observation regards the dimensions of the spinors and of the mass parameter $q$. For the "Weyl" spinor

$$[\varphi] = (D - 1)/2, \quad [q] = 0, \tag{4.152}$$

---

[13]This is needed since nihilpotent matrices cannot be Hermitian.
[14]The bar here is the one in the $SO(d)$ Clifford algebra.

and, for the "BC" system:

$$[\chi] = D/2\,, \quad [\varphi] = (D-2)/2\,, \quad [q] = 0\,. \tag{4.153}$$

It is important that the boost-invariant component has different dimensions in the two representations. This will allow to identify which of them represents the warped limit of our Lifshitz theory. Further, the mass term is a dimensionless (hence marginal) perturbation, in stark contrast with the general case. Indeed, Jensen [77] pointed out that there exist infinitely many such operators in the form of $\mathcal{O}_n = i\bar{\Psi}\,(iD_v)^n\,\slashed{D}\Psi$. Their presence makes the theory non-local in the $v^\mu$ direction. Recall that, in our discussion of the Lifshitz system, we had imposed a spurionic symmetry involving the parameter $q$. In this warped realization we can do the same. This makes such operator forbidden since they break the spurionic symmetry[15]. Finally, since the bi-linear $\chi^T\varphi$ is already marginal, no higher powers of the fermions can result in relevant operators, which are perturbatively forbidden.

We can now examine the dynamics and the currents of these theories. Let us start with the "Weyl" case. The dynamical equations read:

$$iD_v\varphi + q\varphi = 0\,, \tag{4.154}$$

which basically force $\varphi$ to be a plane wave with fixed momentum. It is interesting to also define the conserved currents, in particular the anisotropic momentum current $\pi^\mu$ which turns out to be

$$\pi^\mu = \frac{1}{2}i\bar{\varphi}v^\mu D_v\varphi\,, \tag{4.155}$$

which satisfies the boost Ward identity $\pi^A = 0$ so that, on-shell

$$\pi^\mu = qv^\mu\bar{\varphi}\varphi\,. \tag{4.156}$$

This fixes how $q$ appears in correlators involving the current. For the "BC" system the story is similar. The dynamical equations

$$iD_v\varphi + q\varphi = 0\,, \tag{4.157}$$
$$i\gamma^A D_A\varphi + (iD_v - q)\,\chi = 0\,, \tag{4.158}$$

determine the plane wave momentum and give an inversion formula to find $\chi$ as a function of $\varphi$, since $\chi$ appeared as a Lagrange multiplier it is not a dynamical degree of freedom. Currents are given by more complex formulas, such as

$$\pi^\mu = v^\mu\left(i\left\{\bar{\chi}D_v\varphi + \bar{\varphi}D_v\chi\right\} - \frac{1}{2}\mathcal{L}\right) + iE_A^\mu\left(\bar{\varphi}\gamma^A D_v\varphi\right)\,, \tag{4.159}$$

which, after substituting the solution to the dynamical equations $\varphi = e^{iqx_v}\tilde{b}\,, \quad \chi = e^{-iqx_v}b - \frac{i}{2q}e^{iqx_v}\gamma^A D_A\tilde{b}$ simplifies to

$$\pi^\mu = \frac{q}{2}v^\mu\,\tilde{b}\,b\,. \tag{4.160}$$

---

[15]Strictly speaking, one may evade this argument by fine-tuning the marginal couplings to be proportional to $q^{n+1}$

Which is also proportional to $q$. This simplification is not a coincidence, in fact, we may use it to define a convenient change of variables for the quantum theory:

$$\varphi = e^{iqx_v}\eta, \quad (Weyl) \tag{4.161}$$

$$\varphi = e^{iqx_v}\tilde{b}, \quad \chi = e^{-iqx_v}b - \frac{i}{2q}e^{iqx_v}\gamma^A D_A\tilde{b}, \quad (BC) \tag{4.162}$$

so that $\eta$ transforms projectively under translations and boosts:

$$\eta \to e^{-iq(\theta + \lambda_A x^A)}\eta, \tag{4.163}$$

while the multiplet $Z = \left(b, \, \tilde{b}\right)$ has a "chiral" transformation

$$Z \to e^{iq\Gamma^n(\theta + \lambda_A x^A)}Z. \tag{4.164}$$

The actions then simplify to

$$S_{Weyl} = i\int \sqrt{g}\bar{\eta}D_v\eta, \tag{4.165}$$

and

$$S_{bc} = i\int \sqrt{g}Z^T \mathcal{C}^{-1}D_v Z, \tag{4.166}$$

explaining the name "BC" representation. From this we may understand why translations in $x_v$ led to anomalous contributions, since they effectively behave as chiral rotations, being $q$ the chiral charge of the fermion. This is not possible for standard translations, since then momentum is a dimension-full quantity. Indeed, direct computations in $D = 2$ show that these systems have a Kac-Moody algebra stemming from the translation current, which is in accordance with our results for the anomaly. Another nice consistency check, using the classical formula for chiral anomalies, is that our co-variant anomaly indeed has the right coefficients to allow the interpretation of $q$ as chiral charge.

**The warped limit of the Lifshitz fermion**

We finally trace the connection between our starting point, the Lifshitz fermion (4.7) and the models introduced above. To do this we will match scaling dimensions and show the presence of the Carrollian Ward identity in the as $z \to 0$. The first point is immediate, since

$$[\varphi]_z = (D - 2 + z)/2 \tag{4.167}$$

we should identify it with the boost invariant component of a "BC" representation. Furthermore in flat space-time

$$i\gamma^A\nabla_A\varphi = sq\left(i\nabla_v/q\right)^{1/z}\varphi. \tag{4.168}$$

Excluding zero modes we may write this as

$$i\nabla_v\varphi = q\left(i\gamma^A\nabla_A/sq\right)^z\varphi, \tag{4.169}$$

which in the warped limit reads

$$i\nabla_v\varphi = q\varphi. \tag{4.170}$$

Then we can inspect the formula for the anisotropic translation current $\pi^\mu$ which we have given in (4.68). Upon plugging in the equation above this reduces to

$$\pi^A \sim q\varphi^T \beta^A \varphi = 0 \,, \tag{4.171}$$

since $\beta^A$ are symmetric and $\varphi$ is a Grassmann variable. We thus have the classical Ward identity

$$\pi^A = 0 \,, \tag{4.172}$$

Which ensures Carrollian boost symmetry to emerge in the warped limit.

## 4.3  Conclusions and future directions

In this Chapter we have studied some interesting and perhaps universal properties of non-relativistic fermions. We have used the tools of Newton-Cartan geometry to compute their response to external perturbations in the absence of interactions at one-loop. We have also argued that, if the dynamical critical exponent $z$ is brought to zero, the features that we have shown are universal. This may be seen as a manifestation of anomalous physics that emerges in such limit.

We finish this Chapter with a list of possible interesting directions for future studies.

1. Full characterization of linear response. This is an interesting but extremely cumbersome task, since computations in anisotropic systems are much more complicated than isotropic ones. However, some ideas such as the analogy between torsion and electromagnetic field may provide still some useful insights.

2. Generic computations of warped anomalies. In this presentation we have characterized warped anomalies through a Fujikawa computation and the Wess-Zumino consistency condition. It would be nice to directly apply the Fujikawa computation to the free warped theory, so that we may directly match the anomaly coefficients, which we only have done through indirect comparison with the chiral anomaly. This is not completely straightforward, as most boost invariant regulators seem to have flat directions. Another possibility would be to compute the triangle diagrams. They however are quite more involved than in the isotropic case and it is not clear how to characterize the possible regularization procedure, indeed, scheme independence in the original ABJ paper rests heavily upon Lorentz invariance.

3. Non-relativistic Clifford algebras. We have suggested studying contractions of $SO(D,1)$ Clifford algebras might be a useful tool to systematically study non-relativistic fermions. Furthermore, it would be usable to define non-relativistic super-symmetry algebras.

4. Chern-Simons terms for non simply-laced algebras. We have shown an instance of a system not admitting a non-degenerate bi-linear form on its algebra still having characteristic polynomials on a constraint surface. It would be nice to establish the generality of such happening.

We hope to address at least some of these questions in future studies.

# Appendices to Chapter 4

## 4.A  Seagull terms

In this appendix we inspect the possible contact (Seagull) terms arising in the Kubo formulas. Seagull terms are found in quantum field theory due to the explicit dependence of the curved space-time operators on the spin (or Christoffel) connection. For our model such a dependence will only arise through differentiation of the spin connection

$$\delta_\mu^\rho \delta_D^A \frac{\delta}{\delta e_\rho^D(x)} \omega_\nu{}^{BC}(y) \equiv Z_{\mu\nu\alpha}^{ABC} \partial^\alpha \delta(x-y) \,, \tag{4.173}$$

where, in the flat space-time limit,

$$Z_{\mu\nu\alpha}^{ABC} = \frac{1}{2} \left( \eta_{\nu\alpha} \delta_\mu^B \delta_A^C + \delta_\alpha^C \delta_A^B \delta_\nu^\mu - \delta_\alpha^B \delta_\mu^C \delta_\nu^A \right) - (B \leftrightarrow C) \,. \tag{4.174}$$

Such terms, when present, may contribute in a finite way to the transport coefficients by a one-loop diagram with no external momenta present. The external momentum is carried by the derivative of the delta function, so that in order to compute viscosities we set $\alpha = 0$. Apart from these, other contact terms may arise by functional differentiation of the vielbein itself. These do not carry any derivative of the vielbein and so do not contribute to the viscosity tensor. We will disregard such contributions.

Let us start from the correlators of two $\tau$. In this case one has to compute the classic contact term of a free fermionic stress tensor. This is a well known computation, see for example [86], the final result gives:

$$C_{ABCD}(x,y) = -\frac{i}{16} \delta_{AB} \chi^T(x) \left( \left\{ \frac{1}{4} [\beta_B, \beta_D], \beta_0 \right\} \right) \chi(x) \partial_0 \delta(x-y) + A \leftrightarrow B \,, C \leftrightarrow D \tag{4.175}$$

which in momentum space gives the contact term integral we will compute in the next section. There are three further cases to be examined. The first is the correlator of two anisotropic momentum currents $\pi_A$, $\pi_B$. Seagull terms in this case arise from the dependence of the anisotropic current on torsion. Since we work with the $SO(1,2)$ connection only, no such dependence arises in the co-variant derivative and the contact term vanishes.

A second contact term may contribute to the $\Sigma_A \Sigma_B$ correlator due to the vielbein dependence of $\Sigma$. To start, recall that in position space this reads

$$C^{AB} = \frac{\partial \Sigma_A}{\partial \omega_\nu{}^{CD}} Z_{\mu\nu\alpha}^{BCD} \partial^\alpha \delta(x-y) l^\mu \,, \tag{4.176}$$

where the last $l^\mu$ projects on the right component of the vielbein variation. We will be interested of the part of said contact term which is proportional to $\epsilon_{AB}$, thus encoding the non-dissipative viscosity. Using the expression above for $Z$ it can be shown that

$$Z_{\mu\nu\alpha}^{BCD} l^\mu = \frac{1}{2} l_\nu \delta_\alpha^D \delta^{BC} - (B \leftrightarrow C) \,, \tag{4.177}$$

thus the only contributions to the contact term will come from derivatives $\nabla_l$ in $\Sigma$. We divide them in two parts, the first stemming from the unimproved strain $\hat{\Sigma}$ and the latter from the improvement term coming from the spin current.

For the first term we have, using the formula for the unimproved $\Sigma_A$

$$
\begin{aligned}
\frac{\partial \hat{\Sigma}_A}{\partial \omega_\nu^{CD}} = {}& \frac{s}{z}\left(\frac{1}{2z}-1\right) l^\nu \chi^T \left[\overleftarrow{\partial}_l \overleftrightarrow{\partial}_A C^{-1}\beta_C C^{-1}\beta_D C^{-1} + \overrightarrow{\partial}_l \overleftrightarrow{\partial}_A \beta_C C^{-1}\beta_D\right] M^{1/2z-2}\chi \\
& + \frac{s}{z}\chi^T\left[\overleftarrow{\partial}_A C^{-1}\beta_C C^{-1}\beta_D C^{-1} + \overrightarrow{\partial}_A \beta_C C^{-1}\beta_D\right] M^{1/2z-1}\chi
\end{aligned}
\tag{4.178}
$$

up to terms orthogonal to $l^\nu$. Going to momentum space and remembering that one of the two $\beta$ matrices is the identity because of (4.177), one is left with an anti-commutator $\beta_D C^{-1} + C^{-1}\beta_D = 0$ if $D$ is spatial. So the whole contribution vanishes.

The second term gives

$$
\begin{aligned}
\frac{\partial \Sigma_{\mathrm{imp}}^A}{\partial \omega_\nu^{CD}} = {}& \frac{s}{z} l^\nu \partial_B \chi^T \left[\left(\frac{1}{2z}-1\right) M^{1/2z-2}\overleftrightarrow{\partial}_l\left(\overleftarrow{\partial}_l \gamma^{BA} C^{-1}\gamma^{CD} - \overrightarrow{\partial}_l \gamma^{CD}\gamma^{BA}C^{-1}\right)\right]\chi \\
& + \frac{s}{z} l^\nu \partial_B \chi^T \left[M^{1/2z-1}\left(\gamma^{BA}C^{-1}\gamma^{CD} + \gamma^{CD}\gamma^{BA}C^{-1}\right)\right]\chi\,.
\end{aligned}
\tag{4.179}
$$

This simplifies in a considerable way in momentum space and at zero external frequency

$$
\frac{\partial \Sigma_{\mathrm{imp}}^A}{\partial \omega_\nu^{CD}}(q) = l^\nu \frac{s}{2z^2} q^B \chi^T |q\cdot l|^{1/z-2} X_{AB}^{CD}\chi\,,
\tag{4.180}
$$

with

$$
X_{AB}^{CD} = \gamma^{BA}C^{-1}\gamma^{CD} + \gamma^{CD}\gamma^{BA}C^{-1}\,.
\tag{4.181}
$$

The expression for $X$ is given by either a commutator or an anticommutator depending on whether $CD = 0i$ or $CD = ij$. In our case the relevant part will be

$$
X_{AB}^{CD} = [\gamma^{CD},\gamma_{AB}]C^{-1}\left(\delta_0^C - \delta_0^D\right)\,,
\tag{4.182}
$$

one may now use the Lorentz algebra

$$
[\gamma_{CD},\gamma_{AB}] = \eta_{CA}\gamma_{DB} + (\text{cyclic})\,,
\tag{4.183}
$$

to simplify the expression further. The final result, for the cases of our interest, reads

$$
\frac{\partial \Sigma_{\mathrm{imp}}^A}{\partial \omega_\nu^{CD}}(q) = l^\nu \frac{s}{4z^2}\chi^T |q\cdot l|^{1/z-2}\left(q_0 C^{-1}\beta^A\beta_D + q_D C^{-1}\beta^A\right)\delta_0^C \chi\,.
\tag{4.184}
$$

One last contact term may come from the $\Sigma_A \pi_B$ correlator, and can be seen either through the torsion dependence of $\Sigma_A$ or through the spin connection dependence of $\pi_A$. The first of the two is simpler to compute, since the only dependence on torsion comes from the $G^B \sigma_{BA}$ term in the definition of $\Sigma$, recalling the definition of $G_\mu$ one has

$$
\delta G_\mu = -l^\nu\left(\partial_\nu \delta l_\mu - \partial_\mu \delta l_\nu\right) - \delta l^\nu (dl)_{\nu\mu}\,,
\tag{4.185}
$$

this gives a seagull contribution to $\eta^{\pi\Sigma}$ only if the derivative is in the time direction and $\delta l_\mu$ is in a spatial direction. This is of course not possible, so this last contact term is also zero.

## 4.B  Relevant Feynman graphs and Matsubara sums

In this section of the supplemental material we give the detailed calculations of the 3D Hall viscosity. The main steps of the procedure have already been outlined in the main text.

### 4.B.1  Computation of $\eta^\pi$

We start with the computation of the $\pi_A \pi_B$ correlator. Since we are interested only in the contributions to the Hall viscosity tensor we will always implicitly extract the part of the correlators that goes like the appropriate projector. The $\pi_A \pi_B$ correlator in computed by the Lorentzian continuation of the following Euclidean diagram

$$\langle \pi_A(-\omega, 0)\pi_B(\omega, 0)\rangle = \frac{1}{\beta}\sum_n \int \frac{d^2k dk_3}{(2\pi)^3} \mathrm{tr}\left[S(k, \omega_n)\beta_A S(k, \omega + \omega_n)\beta_B k_3^2\right] \tag{4.186}$$

where $\omega = 2\pi mT$ is a bosonic Matsubara frequency. The discrete sum runs over fermionic frequencies $\omega_n = (2n+1)\pi T$. In Majorana notation the fermionic propagator is

$$S(p) = \left(\beta^A p_A + sM(p)^{1/2z}C^{-1}\right)^{-1}, \quad \beta^A = C^{-1}\gamma^A. \tag{4.187}$$

We begin by evaluating the Dirac trace. It is useful to employ the following identity, which can be checked by representing $\beta_0 = -1$, $\beta_1 = -\sigma_x$, $\beta_2 = \sigma_z$, $C = -i\sigma_y$

$$\mathrm{tr}\left[\beta_A \beta_B C^{-1}\right] = 2\epsilon_{AB}, \tag{4.188}$$

where $\epsilon_{AB} = \epsilon_{ABC}u^C$ and $u^C$ represents the time direction.

In (4.186) the trace can be saturated only in the case in which an $M(k)$ contribution comes from the first propagator and a $\beta_C \omega^C \equiv -\omega$ one from the second. The contribution from the internal Matsubara frequency cancels because of the ordering of the matrices.

Thus we are led to

$$\langle \pi_A(-\omega, 0)\pi_B(\omega, 0)\rangle_H = \epsilon_{AB}\omega \frac{4s}{4\pi^2}\int_0^\infty dk_3 k_3^{1/z+2}\int_0^\infty dkk g(\epsilon, \omega), \tag{4.189}$$

where

$$g(\epsilon, \omega) = \sum_n \frac{1}{\omega_n^2 + \epsilon^2(k, k_3)}\frac{1}{(\omega + \omega_n)^2 + \epsilon^2(k, k_3)}, \quad \epsilon^2(k, k_3) = k^2 + k_3^{2/z}, \tag{4.190}$$

is the Matsubara sum. Since $\omega$ is a bosonic Matsubara frequency it evaluates to

$$g(\epsilon, \omega) = -\frac{\tanh(\beta\epsilon/2)}{8\epsilon(\epsilon^2 + \omega^2/4)}, \tag{4.191}$$

which is half of the Dirac result, as we have no antiparticles.

We'll be eventually interested in continuing the result to the Lorentzian sector to extract the response via the Kubo formula. This is done by the replacement $\omega = 2\pi mT \to i(\omega_L + i0)$, followed by the limit of zero frequency. In our case such limit can be taken quite directly.

This is a consequence of the transport being non-dissipative and follows from the vanishing of the density of states $\rho_{AB}^{\pi\pi}(\omega) = \mathrm{Im}G_{AB}^{\pi\pi}(\omega)$ as the frequency is set to zero. One can explicitly check this by computing the residue of the integrand of $G^{\pi\pi}$, which scales as $\omega^{3z+1}$. A similar reasoning hold for the other integrals. We will then take the $\omega \to 0$ limit inside the integral after performing the Matsubara sums.

At this point we divide vacuum from thermal contributions through the identity

$$\tanh(x/2) = 1 - 2n_F(x), \tag{4.192}$$

where $n_F(x) = 1/(1 + e^x)$ is the Fermi-Dirac distribution. Since the vacuum has no intrinsic Lifshitz scaling parameter, its contribution vanishes in any sensible regulation scheme. On the other hand, the thermal part gives the Hall conductivity to be

$$\eta^\pi = \frac{s}{4\pi^2} \int_0^\infty dk_3 k_3^{1/z} \int_0^\infty dk k \frac{n_F(\beta\epsilon(k, k_3))}{\epsilon(k, k_3)^3}. \tag{4.193}$$

Changing variables to $u = \beta k_3^{1/z}$, $v = \beta k$ and then to polar coordinates $u = \rho\cos(\phi)$, $v = \rho\sin(\phi)$ we find

$$\eta^\pi = \frac{s}{4\pi^2} T^{3z} I_{3z}, \tag{4.194}$$

where

$$I_{3z} = z \int_0^\infty du\, u^{3z} \int_0^\infty dv\, v \frac{n_F(\sqrt{u^2 + v^2})}{(u^2 + v^2)^{3/2}} =$$
$$= z \int_0^\infty d\rho\, \rho^{3z-1} n_F(\rho) \int_0^{\pi/2} d\phi \sin(\phi)\cos(\phi)^{3z} = \frac{z}{3z+1}\Gamma(3z)\eta_D(3z), \tag{4.195}$$

so that

$$\eta^\pi = \frac{s}{4\pi^2} T^{3z} \frac{z}{3z+1}\Gamma(3z)\eta_D(3z). \tag{4.196}$$

The same integration technique will be used all of the other computations.

## 4.B.2  Computation of $\eta^{\pi\Sigma}$

We proceed to compute the Hall viscosity stemming from the correlator between $\pi$ and $\Sigma$. The contribution splits into two parts, one given by the unimproved $\Sigma$

$$\hat{\Sigma}_A = \frac{s}{z}\chi^T M^{1/2z-1}\left(\overleftarrow{\partial}_\nu l^\nu \overrightarrow{\partial}_A + \overleftarrow{\partial}_A l^\nu \overrightarrow{\partial}_\nu\right) C^{-1}\chi \tag{4.197}$$

and one coming from the improvement term $\partial^B\sigma_{BA}$. The first is given by the graph

$$\langle\hat{\Sigma}_A(-\omega, 0)\pi_B(\omega, 0)\rangle = \frac{1}{\beta}\sum_n \int \frac{d^2k\, dk_3}{(2\pi)^3}\mathrm{tr}\left[S(k, \omega_n)C^{-1}k_A\frac{s}{z}|k_3|^{1/z-2}k_3 S(k, \omega+\omega_n)\beta_B k_3\right]. \tag{4.198}$$

The trace is evaluated in a similar way as before, only that now we will need a $\beta_C\omega^C$ and a $\beta_D k^D$ contribution from the propagators. It gives a factor $-2\epsilon_{BD}k^D\omega$. Performing the angular integral over the isotropic momenta we have

$$\langle\hat{\Sigma}_A(-\omega, 0)\pi_B(\omega, 0)\rangle = \frac{2s}{4\pi^2}\epsilon_{AB}\omega \int_0^\infty dk_3 k_3^{1/z} \int_0^\infty dk k^3 g(\epsilon, \omega), \tag{4.199}$$

so that

$$\eta^{\pi\Sigma}(2pf) = \frac{2s}{4z\pi^2}T^{2+z}I_{z+2}\,, \tag{4.200}$$

where

$$
\begin{aligned}
I_{2+z} &= \frac{z}{4}\int_0^\infty du u^z \int_0^\infty dv v^3 \frac{n_F(\sqrt{u^2+v^2})}{(u^2+v^2)^{3/2}} = \\
&= \frac{z}{4}\int_0^\infty d\rho \rho^{z+1}n_F\rho \int_0^{\pi/2}d\phi\sin(\phi)^3\cos(\phi)^z = \frac{1}{2(z+1)(z+3)}\Gamma(z+2)\eta_D(z+2)\,.
\end{aligned}
\tag{4.201}
$$

We thus find

$$\eta^{\pi\Sigma}(2pf) = \frac{s}{4\pi^2}\tau^{z+2}m^3\frac{1}{(z+1)(z+3)}\Gamma(z+2)\eta_D(z+2). \tag{4.202}$$

For the improvement term we need to compute

$$\frac{1}{2}\langle\sigma_{0A}(-\omega)\pi_B(\omega)\rangle = \frac{1}{4}\sum_n\int\frac{d^2kdk_3}{(2\pi)^3}\frac{s|k_3|^{1/z}}{z}\mathrm{tr}\left[S(k,\omega_n)\beta_A C^{-1}S(k,\omega+\omega_n)\beta_B\right]\,, \tag{4.203}$$

The trace gives[16]

$$\mathrm{tr}\left[S(k,\omega_n)\beta_A C^{-1}S(k,\omega+\omega_n)\beta_B\right] = 2\epsilon_{AB}\frac{\omega_n(\omega_n+\omega)+\epsilon(k,k_3)^2}{(\omega_n^2+\epsilon(k,k_3)^2((\omega_n+\omega)^2+\epsilon(k,k_3)^2))}\,. \tag{4.204}$$

This is simplified by employing the identity $\omega_n(\omega_n+\omega) = 1/2(\omega_n^2+(\omega+\omega_n)^2-\omega^2)$ to

$$\epsilon_{AB}\left[\frac{1}{\omega_n^2+\epsilon(k,k_3)^2}+\frac{1}{(\omega_n+\omega)^2+\epsilon(k,k_3)^2}-\frac{\omega^2}{(\omega_n^2+\epsilon(k,k_3)^2)((\omega_n+\omega)^2+\epsilon(k,k_3)^2)}\right]\,. \tag{4.205}$$

The first two sums give the same result $\frac{1}{\beta}\sum_n\frac{1}{(\omega_n+\omega)^2+\epsilon(k,k_3)^2} = -\frac{\tanh\beta\epsilon(k,k_3)/2}{4\epsilon(k,k_3)}$ while the third vanishes in the $\omega\to 0$ limit. Thus

$$
\begin{aligned}
\epsilon^{AB}\frac{1}{2}\langle\sigma_{0A}(-0)\pi_B(0)\rangle &= -\int\frac{d^2kdk_3}{4(2\pi)^3}\frac{s|k_3|^{1/z}}{z}\frac{1}{\epsilon(k,k_3)}\tanh(\beta\epsilon(k,k_3)/2) \\
&= \frac{s}{4\pi^2}T^{2+z}\int_0^\infty du u^z\int_0^\infty dv v\frac{n_F(\sqrt{u^2+v^2})}{(u^2+v^2)^{1/2}} = \\
&= \frac{s}{4\pi^2}T^{2+z}\int_0^\infty d\rho\rho^{z+1}\int_0^{\pi/2}d\phi\sin(\phi)\cos(\phi)^z = \frac{s}{4\pi^2}T^{2+z}\frac{1}{(z+1)}\Gamma(z+2)\eta_D(z+2)\,.
\end{aligned}
\tag{4.206}
$$

Summing the two contributions

$$\eta^{\pi\Sigma} = \frac{s}{4\pi^2}T^{2+z}\frac{(z+4)}{(z+1)(z+3)}\Gamma(z+2)\eta_D(z+2)\,. \tag{4.207}$$

---

[16]As both $A$ and $B$ are spatial, the only way to get an $\epsilon$ tensor is that the matrices from the two propagators contract between each other.

### 4.B.3   Computation of $\eta^\tau$

We now compute the intrinsic $2 + 1$ dimensional thermal Hall viscosity, for which one should compute both the two point function $\tau\tau$ and the seagull term $C_{ABCD}$. The first is given by the integral

$$\langle \tau_{AB}(-\omega, 0)\tau_{CD}(\omega, 0)\rangle = \frac{1}{\beta}\sum_n \int \frac{d^2k \, dk_3}{(2\pi)^3}\text{tr}\left[S(k, \omega_n)\beta_{(A}S(k, \omega + \omega_n)\beta_{(C}k_{B)}k_{D)}\right] . \quad (4.208)$$

We are interested in the contribution proportional to $P_{ABCD}$ of this correlator. To get the right factors it is sufficient to work with one combination of indices, symmetrization takes care of recovering the full structure. Computing the trace we find

$$\langle \tau_{AB}(-\omega, 0)\tau_{CD}(\omega, 0)\rangle_H = \omega P_{ABCD}\frac{2s}{4\pi^2}\int_0^\infty dk_3 k^{1/z}\int_0^\infty dk k^3 g(\epsilon, \omega) = z P_{ABCD}\eta^{\pi\Sigma}(\text{2pf}) . \quad (4.209)$$

Confronting this with $\eta^{\pi\Sigma}$ we deduce that

$$\eta^\tau(\text{2pf}) = z\eta^{\pi\Sigma}(\text{2pf}) = \frac{s}{4\pi^2}T^{z+2}\frac{z}{(z+1)(z+3)}\Gamma(z+2)\eta_D(z+2) . \quad (4.210)$$

In momentum space the seagull term is given by

$$C_{ABCD}(\omega) = \frac{\delta_{AC}}{16}\frac{1}{\beta}\sum_n \int \frac{d^2k \, dk_3}{(2\pi)^3}\text{tr}\left[\beta_{[A}C^{-1}\beta_{B]}\omega S(k, \omega_n)\right] + A \leftrightarrow B\,, C \leftrightarrow D\,, \quad (4.211)$$

the trace is computed as before and the index structure organizes to give a projector, so

$$\begin{aligned}
C_{ABCD}(\omega) &= \omega P_{ABCD}\frac{s}{4\pi^2}T^{2+z}z\int_0^\infty du\, u^z \int_0^\infty dv\, v \frac{n_F(\sqrt{u^2 + v^2})}{(u^2 + v^2)^{1/2}} \\
&= \omega P_{ABCD}\frac{s}{4\pi^2}T^{2+z}\frac{1}{(z+1)}\Gamma(z+2)\eta(z+2) .
\end{aligned} \quad (4.212)$$

Summing all up we get the relation

$$\eta^\tau = z\eta^{\pi\Sigma} = \frac{s}{4\pi^2}T^{2+z}\frac{z(z+4)}{(z+1)(z+3)}\Gamma(z+2)\eta_D(z+2) . \quad (4.213)$$

### 4.B.4   Computation of $\eta^\Sigma$

Finally we compute the value of $\eta^\Sigma$. We can divide the problem in three parts: the first coming from the correlators of the unimproved strains $\hat{\Sigma}$, the second coming from the correlator of one of these with the improvement term and the last one stemming from the contact terms. The unimproved correlator vanishes, since it contains a term $k_A k_B$ which should be antisymmetrized..

The improvement term gives a contribution

$$\eta^\Sigma_{\text{imp}} = \lim_{\omega\to 0}\langle\sigma_{0A}(-\omega)\hat{\Sigma}_B(\omega)\rangle\epsilon^{AB} , \quad (4.214)$$

which reads in terms on Feynman diagrams

$$\eta_{\text{imp}}^{\Sigma} = \lim_{\omega \to 0} \frac{1}{\beta} \sum_n \int \frac{d^2k dk_3}{(2\pi)^3} \text{tr} \left[ |k_3|^{1/z} k_3^{-1} k_B C^{-1} S(k, \omega_n) |k_3|^{1/z} k_3^{-1} \frac{1}{2} \beta_A C^{-1} S(k, \omega_n + \omega) \right] .$$
(4.215)

The odd part of the trace may be computed by bringing up one term with the an-isotropic momentum and one $\beta$ matrix

$$\begin{aligned}
\eta_{\text{imp}}^{\Sigma} &= \lim_{\omega \to 0} \frac{1}{z^2} \frac{s}{2\pi^2} \frac{1}{\beta} \sum_n \int_0^\infty dk_3 \int_0^\infty dk k^3 k_3^{3/z - 2} g(\epsilon, \omega) \\
&= \frac{s}{8\pi^2 z} T^{4-z} \int d\rho \rho^{3-z} n_F(\rho) \int_0^{\pi/2} d\phi \sin(\phi)^3 \cos(\phi) \\
&= \frac{s}{4z\pi^2} T^{4-z} \frac{\Gamma(4-z) \eta_D(4-z)}{(5-z)(3-z)} .
\end{aligned}$$
(4.216)

Finally one has to evaluate the contributions from the contact term through the Feynman diagram

$$\eta_{\text{ct}}^{\Sigma} = \frac{1}{2z^2} \frac{1}{\beta} \sum_n \int \frac{d^2k dk_3}{(2\pi)^3} \text{tr} \left[ |k_3|^{1/z - 2} \frac{1}{2} \left( \omega_n C^{-1} \beta_A \beta_B + k_A C^{-1} \beta_B \right) S(k, \omega_n) \right] \epsilon^{AB} .$$
(4.217)

The trace gives

$$\text{tr} \left[ |k_3|^{1/z - 2} \frac{1}{2} \left( \omega_n C^{-1} \beta_A \beta_B + k_B C^{-1} \beta_A \right) S(k, \omega_n) \right] \epsilon^{AB} = 1 - \frac{|k_3|^{2/z}}{\omega_n^2 + \epsilon(k, k_3)^2} ,$$
(4.218)

the first term is a vacuum contribution which we regulate to zero. The second Matsubara sum can be easily computed so that

$$\eta_{\text{ct}}^{\Sigma} = \frac{s}{4z\pi^2} \int_0^\infty d\rho \rho^{3-z} n_F(\rho) \int_0^{\pi/2} d\phi \sin(\phi) \cos(\phi)^{2-z} = \frac{s}{4z\pi^2} T^{4-z} \frac{\Gamma(4-z) \eta_D(4-z)}{(3-z)} .$$
(4.219)

Putting everything together

$$\eta^{\Sigma} = \frac{s}{4z\pi^2} T^{4-z} \frac{(6-z)}{(5-z)(3-z)} \Gamma(4-z) \eta_D(4-z) .$$
(4.220)

The three viscosities $\eta^\tau$, $\eta^\Sigma$ and $\eta^{\pi\Sigma}$ can be compactly re-expressed (provided we redefine $\Sigma \to z\Sigma$) as functions of their scaling dimension $\xi$

$$\eta(\xi)/z = \frac{s}{4\pi^2} T^\xi \frac{(\xi + 2)}{(\xi + 1)(\xi - 1)} \Gamma(\xi) \eta_D(\xi) .$$
(4.221)

## 4.C  Expansion of the regulated Jacobian

In this first Appendix we give the expansion of the regulator and the computation of the gravitational contributions to the anomaly in four dimensions. Recall that we have defined

$$\mathcal{R} = A^\dagger A, \quad A = i\tau_1 \gamma^a \nabla_a / q + s\tau_2 \left( i\nabla_v / q \right)^{1/z} ,$$
(4.222)

in this appendix we will drop factors of $q$ for ease of notation, they are straightforward to re-establish The regulator then can be computed by expanding

$$\tau_1^2 \gamma^a \gamma^b \nabla_a \nabla_b + is\tau_1\tau_2[\gamma^a \nabla_a, (i\nabla_v)^{1/z}] + \tau_2^2 (i\nabla_v)^{2/z} , \qquad (4.223)$$

the last term does not give any interesting space-time dependence and is the source for the anisotropic Gaussian term. We then need to compute various commutators, these are given by the formula

$$[\nabla_\mu, \nabla_\nu] = -T^\rho_{\mu\nu}\nabla_\rho + R^{ab}_{\mu\nu}J_{ab} , \qquad (4.224)$$

with $J_{ab}$ the $SO(d)$ rotation generator.

$$\gamma^a\gamma^b\nabla_a\nabla_b = \nabla_\perp^2 + \frac{i}{2}\epsilon^{abc}\gamma_c[\nabla_a, \nabla_b] =$$
$$= \nabla_\perp^2 - i\frac{i}{2}\epsilon^{abc}\gamma_c T_{ab}\nabla_v + \frac{1}{4}\epsilon^{abc}\epsilon_{efg}\gamma_c\gamma^g R_{ab}{}^{ef} , \qquad (4.225)$$

the first term will contribute the the Gaussian integral, the second to the torsional anomaly while the third may be further massaged into

$$\frac{1}{4}\epsilon^{abc}\epsilon_{efg}\gamma_c\gamma^g R_{ab}{}^{ef} = \frac{1}{2}R - \frac{i}{2}\epsilon^{abc}R_{abcf}\gamma^f . \qquad (4.226)$$

We also have the commutator

$$\gamma^a[\nabla_a, \nabla_v] = -\gamma^a G_a \nabla_v + \frac{i}{2}\epsilon_{efg}\gamma^a\gamma^g R_{av}{}^{ef} . \qquad (4.227)$$

which appears in $is[\gamma^a\nabla_a, (i\nabla_v)^{1/z}]$ repeatedly. The leading term in the plane wave expansion is given by $1/z$ times such commutator, multiplied by the plane wave momentum $k_v^{1/z-1}$.

Passing to a plane wave basis for the computation of the trace and rescaling the momenta as $k_a = \Lambda_1 u_a, k_v = \Lambda_2^z v$ one is led to the following integral expansion

$$J(\theta) = \frac{\theta}{(2\pi)^d}\tau_1^{1-d}\tau_2^{-z}\int d^{d-1}u_a \int dx_v \left(\nabla_v + i\tau_2^{-z}v\right)\exp\left(-u_a u^a - v^{2/z}\right)\times$$
$$\times \exp\left(i\tau_1 u^a\nabla_a - \tau_1^2\left(\nabla_\perp^2 - R\right) + i\tau_1^2\epsilon^{abc}\gamma_a T_{bc}(\nabla_v + iv\tau_2^{-z})+\right.$$
$$+s\tau_1\tau_2\sum_k^{1/z} c_k\nabla_v^k\gamma^a\left\{R_{av}^{ef}\gamma_{ef} + G_a(\nabla_v + iv\tau_2^{-z})\right\}\left(\nabla_v + iv\tau_2^{-z}\right)^{1/z-k}$$
$$\left.\tau_2^2\left\{(\nabla_v + iv\tau_2^{-z})^{2/z} - \tau_2^{-2}v^{2/z}\right\}\right) . \qquad (4.228)$$

Notice that all terms with indices $a, b, c...$ in the non-Gaussian part decay at least as $1\tau_1$ or $\tau_2$ so that our expansion will terminate at the third order in four dimensions an at the second order in two dimensions. The torsional part is particularly simple, since the $T_{ab}$ contribution already comes in as $\tau_1^2$. In this case the only contribution comes together with the highest weight term in $G_a$ to give the integral in the main text. Recall also that, in order to preserve the spurionic symmetry, $\tau_2 = q^{1/z-1}\tilde\tau_2$.

In the absence of torsion, one could hope to find further contributions from the Riemann tensor in four dimensions, however there seems to be no such contribution from our system. In any case the evaluation of gravitational anomalies using the Fujikawa technique is known to be cumbersome and this question should be further studied by Super-symmetric methods.

# Chapter 5

# Anomalous transport & higher spin symmetry

## 5.1 Introduction

We conclude this thesis with a somewhat lighter and more speculative chapter, which concerns the possible interplay between 't Hooft anomalies and higher spin symmetry.

The underlying idea is that, in theories which have an (infinite) tower of higher spin conserved currents, one may use the higher spin symmetry to constrain the "anomalous" effective actions introduced in Chapter one. The end result is that the full tower of higher spin currents, when evaluated e.g. in a thermal state, should gain non-trivial one point functions akin to those presented in Chapter 2 for their lower spin relatives, with coefficients related by the higher spin symmetry.

While it is by now established that, in dimensions bigger than $d = 2$ and in flat space-time, higher spin theories need to be free [87], there are of course many interacting CFTs in two dimensions displaying interesting higher spin algebras. Our examples will mainly concern free realizations, with the idea that such relations may be used to detect the departure from the free-field fixed point, since interactions will softly break the higher spin symmetry [88].

The main part of this Chapter will include examples of this phenomenon which are computed in perturbation theory, which are given by free fermions and, surprisingly, Maxwell theory (or a self-dual p-form field in $d = 2p$ dimensions). This will also allow to tell an interesting story about the duality current in Maxwell and the Zilch [89, 90].

In the two dimensional case we will see that the algebraic nature of such response coefficients is efficiently encoded in the Jonquierre relations for the polylogarithm. In four dimensional a similar interpretation can be made for some of them by compactifying the theory on an $S^2$ with $n$ units of magnetic flux.

Two dimensional theories also allow for a direct interpretation of this occurrence in terms of the transformation laws of the (quasi)-primary conserved currents under e.g. diffeomorphisms and their mixing, which in this case can be found rather explicitly for finite transformations.

Finally, there has been work during the last years [91, 92, 93] in defining higher spin effective actions for free theories, making the higher spin symmetry manifest. We review such formulation and comment on its possible uses in view of the results presented.

At this point it is useful to introduce some basic formalism, so that the presentation in the rest of the Chapter will be smoother. Most of the material is taken from the papers cited above and the lectures [94]. First, we will call higher-spin currents of spin $s$ traceless symmetric tensors $J^{\mu_1...\mu_s}$:

$$J_s^{\mu_1...\mu_i...\mu_j...\mu_s} = J_s^{\mu_1...\mu_j...\mu_i...\mu_s} \,, \quad J_s^{\mu_1...\mu_i...\mu_s}{}_{\mu_i} = 0 \,, \tag{5.1}$$

which are also conserved:

$$\partial_\mu J_s^{\mu\mu_2...\mu_s} = 0 \,. \tag{5.2}$$

A practical way to define these operators is to take symmetric conserved currents $j_s^{\mu_1...\mu_s}$ and take a linear combination of them which imposes the tracelessness condition:

$$J_s^{\mu_1...\mu_s} = j_s^{\mu_1...\mu_s} + \alpha_2 \left( P^{(2)}_{\mu_1\mu_2}(\partial) j_{s-2}^{\mu_3...\mu_s} + \text{permutations} \right) + ... \,, \tag{5.3}$$

with $P^{(2)}$ a symmetric polynomial in the derivatives. The sum eventually ends with the lower spin current of the multiplet $s = 2$ $s = 1$, even though in some case one may need to include scalars and improvement terms, as is familiar in the discussion of e.g. the stress tensor of a free scalar field.

Since the stress tensor $J_2^{\mu\nu} = T^{\mu\nu}$ is also part of the multiplet (and as such it is traceless) such higher spin theories tend to be conformal. From the point of view of conformal field theory the higher spin currents are conformal (quasi)-primaries which obey conservation laws. As such they all must have protected dimension and saturate the unitarity bound:

$$\Delta_s = d - 2 + s \,. \tag{5.4}$$

Once the current $J_s$ are obtained, they may be repackaged conveniently in a generating function $J(x, z)$ with $x$ the space-time coordinates and $z^\mu$ is an auxiliary vector field which is null $z^\mu z_\mu = 0$. Then one may define:

$$J_s(x, z) = J_s(x)^{\mu_1...\mu_s} z_{\mu_1}...z_{\mu_s} \,, \quad J(x, z) = \sum_s \frac{1}{s!} J_s(x, z) \,, \tag{5.5}$$

so that the higher spin currents may be extracted from Taylor expansion of $J(x, z)$ in the $z$s. A similar generating functional, which does not require $z$ to be null, can be set up for the $j_s$. The conservation and tracelessness conditions can then be expressed as a single equation for the functional $J(x, z)$. To this end one defined a differential operator $D^\mu$ which frees indices in real space:

$$D_\mu = \left( \frac{d}{2} - 1 + z \cdot \partial_z \right) \frac{\partial}{\partial z^\mu} - \frac{1}{2} z_\mu \, \Box_z \,. \tag{5.6}$$

The conservation law reads

$$\partial_\mu D^\mu J(x, z) = 0 \,, \tag{5.7}$$

while tracelessness is guaranteed by the choice of $D_\mu$. One can alternatively free indices with normal derivatives $\partial/\partial z^\mu$, then the tracelessness equation reads

$$\Box_z J(x, z) = 0 \,. \tag{5.8}$$

A simple examples comes from two dimensional theories. In this case on the plane with metric $ds^2 = 2dz d\bar{z}$ null vectors have only $z$ ( or $\bar{z}$) non-vanishing component. The generating function

then automatically projects currents onto their purely holomorphic (or anti-holomorphic) parts. As it should be, since in two dimensions the unitarity bound is saturated by zero twist operators which have $(h, \bar{h}) = (0, s)$, $(s, 0)$. While in higher dimensions such factorization in two towers needs not to happen, in our cases it will be enforced by the presence of the chirality projector $(1 \pm \gamma_{d+1})$ for Dirac fields and $(1 \pm s \star)/2$ for Maxwell fields[1].

The improvement procedure to define traceless symmetric currents may be streamlined if one considers writing an Ansatz:

$$J_s(x, z) = \Phi^\dagger f_s \left( z \cdot \overleftarrow{\partial}, z \cdot \overrightarrow{\partial} \right) \Phi, \tag{5.9}$$

with $\Phi$ the free particle field. The idea is to impose the equation $\partial_\mu D^\mu J_s$ as a differential equation for $f_s(u, v)$, to do this one first acts directly with the operator $D_\mu$, whose explicit action will depend also on the type of field $\Phi$ used, then substitutes by hand $z \cdot \overleftarrow{\partial} = u$, $z \cdot \overrightarrow{\partial} = v$. For example, for a scalar $\phi$ one finds

$$\left( \frac{d}{2} - 1 \right) (\partial_u + \partial_v) f_s^\phi(u, v) + \left( u \partial_u^2 + v \partial_v^2 \right) f_s^\phi(u, v) = 0, \tag{5.10}$$

for a scalar and

$$\frac{d}{2} (\partial_u + \partial_v) f_s^\psi(u, v) + \left( u \partial_u^2 + v \partial_v^2 \right) f_s^\psi(u, v) = 0, \tag{5.11}$$

for a (Dirac) fermion. The solutions to these equations are degree $s$ polynomials in $u$ and $v$ which can be found by an Ansatz of the type:

$$f_s(u, v) = \sum_{k=0}^{s} c_{k,s} u^k v^{s-k}, \quad c_{k,s} = (-)^s c_{s-k,s}. \tag{5.12}$$

One can use radial coordinates to simplify the equation further. The final solution is given in terms of Gegenbauer polynomials, for example:

$$f_s^\phi(u, v) = \sum_{k=0}^{s} \frac{(-)^k}{k!(k + (d-4)/2)!(s - k + (d - 4/2))!(s - k)!} u^k v^{s-k}, \tag{5.13}$$

and

$$f^\psi(u, v) = \sum_{k=0}^{s} \frac{(-)^k}{k!(k + (d-2)/2)!(s - k + (d - 2)/2)!(s - k)!} u^k v^{s-k}. \tag{5.14}$$

In some case (for example the Maxwell field) these equations further simplify and are given essentially by $f_s(u, v) \approx (u - v)^s$.

## 5.2 Some direct computations

In this Section we present some direct computations giving support for the ideas presented in the introduction. Since we will be dealing with free theories most of the observables we will be interested in are computable either by statistical sums or by simple one-loop diagrams.

---

[1]s depends on the signature chosen and the dimension, as does the square of the Hodge operator

The main difficulty which may arise is efficiently packing the computation for all values of the spin and regularizing (possible) divergences in Feynman graphs. We will work mostly in an Euclidean setup at finite temperature, that is on $\mathbb{R}^{d-1} \times S^1$.

More to the point, we will compute $d/2$-point functions of particular components of the higher-spin currents

$$J_s^i = J_s^{0\ldots 0i} \,, \tag{5.15}$$

with $i$ a spatial index, to extract their dependence on external magnetic and gravito-magnetic fields as well as temperature and chemical potentials. This is of course only a partial study, in the sense that one may also think of using Generalized-Gibbs-Ensembles of higher spin conserved charges, using the higher (spatial)-spin components etc. We could not find a straightforward way to incorporate these generalizations at the present time, one problem being that higher spin (odd) chemical potentials lead to diverging contributions to the partition function from antiparticle states.

### 5.2.1   2d and 4d Fermions

We start by examining the most notable case of free fermionic theories. Since we work in $d = 2, 4$ we will always be able to split a Dirac fermion $\Psi$ in its Weyl components $(\chi_L, \chi_R)$. Since the observables we are interested in show up in chiral (higher spin) currents, we might as well restrict to a single Weyl fermion $\chi$ to simplify the discussion. Our first task is to write down the generating function for higher spin currents in these models. It is important to remember that the definition of higher spin currents for free fields is rather simple in flat space-time and in the absence of external fields, however it is not simple nor clearly possible in arbitrary curved metrics. The expressions given here are thus valid in flat space-time. It is also possible to extend them in the case of linearized perturbations, which is useful in constructing contact terms for perturbative computations.

In dealing with fermions it is actually useful to treat the $\gamma$-matrix index in a slightly different way as the others to simplify the resulting expressions. Then we will define the lowest component of the current multiplet as

$$J^a = \chi^\dagger \sigma^a \chi \,, \tag{5.16}$$

and higher spin currents $J_s^{a\mu_1\ldots\mu_s}$ roughly as

$$J_s^{a\mu_1\ldots\mu_s} = \chi^\dagger \sigma^a \overset{\leftrightarrow{\mu_1}}{\partial} \ldots \overset{\leftrightarrow{\mu_s}}{\partial} \chi + \text{lower spin} \,, \tag{5.17}$$

with $\chi^\dagger \overset{\leftrightarrow{\mu}}{\partial} \chi = \frac{1}{2} \left( \chi^\dagger \partial^\mu \chi - \partial^\mu \chi^\dagger \chi \right)$. These will satisfy:

$$\partial_\mu J_s^{a\mu\mu_2\ldots\mu_s} = 0 \,, \quad E_a^\mu \partial_\mu J^{a\mu_1\ldots\mu_s} = 0 \,, \tag{5.18}$$

as a consequence of the Dirac equation. Is is also simple to prove that the following holds

$$E_a^{\mu_i} J^{a\mu_1\ldots\mu_i\ldots\mu_s} = 0 \,, \tag{5.19}$$

due to the Dirac equation. The tracelessness condition is harder to impose, it is however simplified in $d = 2, 6, 10\ldots$ due to the possibility of making the Weyl fermions real. For our examples we won't need it directly.

**2d fermions and Jonquierre relations**

The simplest case is given by two dimensional fermions. since $d = 2$ we will be interested in one-point functions of the higher spin currents in the presence of chemical potentials and temperature. Since we are working in a free-particle picture, these might be computed through standard thermal sums involving the Fermi-Dirac distributions. These sums at first sight look like very complicated functions of temperature and chemical potential. However it turns out that, with higher spin currents, they simplify considerably upon employing the Jonquierre relations for the $Li_s$ function:

$$Li_s(e^{2\pi ix}) + (-)^s Li_s(e^{-2\pi ix}) = \frac{(2\pi i)^s}{s!}B_s(x), \tag{5.20}$$

with $B_s(x)$ the Bernoulli polynomials. This is the higher spin analogue of what happens in the case of chiral transport, e.g. in the lowest Landau level [16].

The higher spin currents we consider are holomorphic operators constructed from the stress tensor $T(w) = \psi \overset{\leftrightarrow}{\partial} \psi$. Here we denote with $\omega$ the holomorphic variable (in Euclidean signature), $\partial = \partial/\partial\omega$ and $\psi(\omega)$ the holomorphic component of the Fermi field. We will work with complex fermions, so that we may turn on a $U(1)$ chemical potential. One may also construct a tower of "flavor" currents, which generate the stress tensor via the Sugawara construction. The explicit expression for the non-vanishing components of the currents in question is

$$J_s = c_s \psi^\dagger \left(\overset{\leftrightarrow}{\partial}\right)^{s-1} \psi. \tag{5.21}$$

Since in $d = 2$ there are only two null vectors (up to rescalings), the expressions introduced in the previous Section would lead to two independent components for the higher spin currents, which are the holomorphic and anti-holomorphic parts. The currents $J^{0\ldots0i}$ which we are interested in will be (minus) the difference between the two[2]. We should fix a normalization $c_s$ for the currents. In this case it is convenient to fix it so that the obey the $W_{1+\infty}$ algebra [95]. In the holomorphic coordinates this just means $c_s = 1$.

To compute the one-point functions it is convenient to go to Lorentizan signature in the light-cone coordinates

$$w_\pm = t \pm x, \tag{5.22}$$

and perform the mode expansion for the free fields:

$$\psi(w_\pm) = \int_{\omega\geq0} \frac{d\omega}{2\pi} \left(e^{i\omega w_\pm}\alpha_\pm + e^{-i\omega 2_\pm}\beta^\dagger\right), \tag{5.23}$$

with $\alpha$ and $\beta$ satisfying canonical commutation relations. The derivatives just act on this expansion by bringing down factors of $\omega_\pm$. The one point functions are defined as

$$\langle J_s \rangle = \text{tr}\left[e^{-\beta(H-\mu Q)}J_s\right], \tag{5.24}$$

and may be computed by plugging in the mode expansions. Here one may use the explicit expression (5.21) of use (5.11). The final result is written an an integral of the Fermi-Dirac distribution over particle and anti-particle states of fixed $\omega$, which reads

$$\langle J_s \rangle = \int_0^\infty \frac{d\omega}{2\pi} \left(\frac{\omega^{s-1}}{e^{\beta(k-\mu)} + 1} + (-)^s \frac{\omega^{s-1}}{e^{\beta(k+\mu)} + 1}\right) \tag{5.25}$$

---

[2]For chiral fermions one of the two will vanish of course.

where the first term comes from the particle sum and the second from the anti-particle sum. Recalling that

$$Li_s(-z) = -\frac{1}{(s-1)!} \int_0^\infty dt \frac{t^{s-1}}{1 + e^t/z} \,, \tag{5.26}$$

and the Jonquierre inversion formulas (5.20), this leads to

$$\langle J_s \rangle = -\frac{(s-1)!}{2\pi s!} \left(2\pi i \beta^{-1}\right)^s B_s\left(\frac{1}{2} + \frac{\mu\beta}{2\pi i}\right) \,, \tag{5.27}$$

so that, for a chiral species

$$\langle J_s^{0...0i} \rangle = \frac{(s-1)!}{2\pi s!} \left(2\pi i \beta^{-1}\right)^s B_s\left(\frac{1}{2} + \frac{\mu\beta}{2\pi i}\right) \equiv F_s(\mu, \beta) \,, \tag{5.28}$$

the expressions reported do not seem real, however upon expansion one finds, setting $\beta^{-1} = T$:

$$F_1 = \mu \tag{5.29}$$

$$F_2 = \frac{\mu^2}{2} + \frac{\pi^2 T^2}{6} \tag{5.30}$$

$$F_3 = \frac{\mu^3}{3} + \frac{1}{3}\pi^2 \mu T^2 \tag{5.31}$$

$$F_4 = \frac{\mu^4}{4} + \frac{7\pi^4 T^4}{60} + \frac{1}{2}\pi^2 \mu^2 T^2 \tag{5.32}$$

$$F_5 = \frac{\mu^5}{5} + \frac{7}{15}\pi^4 \mu T^4 + \frac{2}{3}\pi^2 \mu^3 T^2 \tag{5.33}$$

$$F_6 = \frac{\mu^6}{6} + \frac{31\pi^6 T^6}{126} + \frac{7}{6}\pi^4 \mu^2 T^4 + \frac{5}{6}\pi^2 \mu^4 T^2 \tag{5.34}$$

$$F_7 = \frac{\mu^7}{7} + \frac{31}{21}\pi^6 \mu T^6 + \frac{7}{3}\pi^4 \mu^3 T^4 + \pi^2 \mu^5 T^2 \tag{5.35}$$

$$F_8 = \frac{\mu^8}{8} + \frac{127\pi^8 T^8}{120} + \frac{31}{6}\pi^6 \mu^2 T^6 + \frac{49}{12}\pi^4 \mu^4 T^4 + \frac{7}{6}\pi^2 \mu^6 T^2 \tag{5.36}$$

$$F_9 = \frac{\mu^9}{9} + \frac{127}{15}\pi^8 \mu T^8 + \frac{124}{9}\pi^6 \mu^3 T^6 + \frac{98}{15}\pi^4 \mu^5 T^4 + \frac{4}{3}\pi^2 \mu^7 T^2 \tag{5.37}$$

$$F_{10} = \frac{\mu^{10}}{10} + \frac{511\pi^{10} T^{10}}{66} + \frac{381}{10}\pi^8 \mu^2 T^8 + 31\pi^6 \mu^4 T^6 + \frac{49}{5}\pi^4 \mu^6 T^4 + \frac{3}{2}\pi^2 \mu^8 T^2 \tag{5.38}$$

We see that the first two lines indeed coincide with known results from two dimensions, while the other lines are predictions of higher spin symmetry. The purely temperature dependent pieces can be traced back to the transformations property of the $W_s$ currents under diffeomorphisms. Upon matching conventions, these results coincide with those of [96] where these quantities were computed using the a nontrivial black-hole background.

**4d fermions and reduction**

This computation gives us also some mileage regarding the situation for four dimensional fermions in a magnetic field. Indeed, recall that a system of fermions on $\mathbb{R}^2 \times S^2$ with $n$ units of magnetic flux on the $S^2$

$$\frac{1}{2\pi} \int_{S^2} F = n \,, \tag{5.39}$$

gives rise, in the infrared, to $n$ chiral model which behaves as 2d chiral fermions. All other modes, which live in higher Landau levels, are gapped by a factor proportional to $\sqrt{n}$. In the limit of big $n$ (i.e. big magnetic field) we may thus effectively compactify the system to two dimensions and read the magnetic response for the higher spin currents from the lines above. This fixes the coefficients such as

$$J_s^{0\ldots 0i} = d_s(\mu, T) B^i \quad B^i = \epsilon^{i0z\bar{z}} F_{S^2} \,, \tag{5.40}$$

in the hydrodynamic expansion for the higher spin currents. Other response functions need to be computed explicitly via the Kubo formalism, in a way very similar to the one exposed in the next example.

### 5.2.2 Maxwell fields and Zilches

Another interesting application regards Maxwell fields (as well as $p$ forms in $d = 2p$ dimensions, for which Maxwell fields are the first non-trivial case after free scalars in $d = 2$). In this case the story is less known so we will give a short review. We start with Maxwell theory, without matter, in $d = 4$. The Maxwell equations and the Bianchi identities read

$$d \star F = 0 \tag{5.41}$$

$$dF = 0 \,, \tag{5.42}$$

or, in components

$$\partial_\mu F^{\mu\nu} = 0 \,, \tag{5.43}$$

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \,, \quad \tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \,. \tag{5.44}$$

These couple of equations are invariant under the electromagnetic duality transformation which rotates $F$ and $\tilde{F}$ between each other. At the classical level such a transformation may be generated by a "duality current"

$$j_D^\mu \sim \epsilon^{\mu\nu\rho\sigma} \left( A_\nu F_{\rho\sigma} - C_\nu \tilde{F}_{\rho\sigma} \right) \,, \tag{5.45}$$

where $dC = \star F$. This is not a well defined operator on the Hilbert space due to lack of gauge invariance, nor is local in the fields $A_\mu$. At the quantum level the transformations generated by the duality action are broken in nontrivial curved backgrounds since one finds [97]

$$F \wedge F = c_M \mathrm{tr} \left[ R \wedge R \right] \,. \tag{5.46}$$

This breaks the duality symmetry since the left hand side is duality-invariant. This idea can be generalized [98] to treat the divergence of the current $j_D^\mu$, so that a similar equation hold

$$d \star j_D = \tilde{c}_M \text{tr} [R \wedge R] \,, \tag{5.47}$$

which looks strikingly similar to the chiral anomaly equation for Dirac fermions (see for example [99, 100] for studies on this matter). One may indeed ask whether the presence of an "anomalous" divergence also gives rise to non-trivial one point functions in e.g. gravito-magnetic potential backgrounds. One may try to do a direct computation for the duality current [101, 102], however it is not clear whether this has a well defined, gauge invariant meaning. Another approach, initiated in [89] and followed up in [90] is to realize that one may build a tower of (odd) higher-spin currents over the duality current which starts at spin 3 with the Zilch [103, 104]

$$Z_{\mu\nu\rho} = F^\alpha_{(\mu} \overleftrightarrow{\partial}_\nu \tilde{F}_{\rho)\alpha} \,, \tag{5.48}$$

which is a local, gauge-invariant operator. At this level the similarity with chiral currents may be further appreciated by decomposing the field strength in self-dual and anti-self-dual components $F = F_+ + F_-$ and $\tilde{F} = F_+ - F_-$. Then indeed one sees that the Zilch separates into two currents with only (anti)self-dual components, much like the chiral current of a fermionic system. In fact, one may work with chiral two forms directly, which we have avoided since it turns out to be more cumbersome.

The expectation from our intuition based on chiral fields turns out to be consistent: quantizing Maxwell theory in a cylinder [89] one finds

$$\langle Z_{00i} \rangle \frac{8}{45} \pi^2 T^4 \Omega_i \quad \Omega^i = \epsilon^{ijk} \partial_j a_k \,, \tag{5.49}$$

with $a_i$ the gravito-magnetic potential field. Introducing as in Chapter 1 the one-form $u_\mu$ this comes from a constitutive relation of the type

$$\langle Z_{\mu\nu\rho} \rangle = \sigma_Z u_{(\mu} u_\nu \Omega_{\rho)} \,, \tag{5.50}$$

so that

$$\langle Z_{00i} \rangle = \frac{\sigma_Z}{3} \Omega_i \,. \tag{5.51}$$

The coefficient $\sigma_Z$ may also be computed by utilizing the linear response formalism to relate it to a two-point function of the Zilch with the Maxwell stress tensor

$$T_{\mu\nu} = F_{\mu\alpha} F_\nu{}^\alpha - \frac{1}{4} g_{\mu\nu} F^2 \,, \tag{5.52}$$

by

$$\sigma_Z = 6 \lim_{\vec{p} \to 0} \frac{-i}{2p_k} \epsilon^{ijk} \big( G_{00i,0j}(p) + C_{00i,0j}(p) \big) \,, \tag{5.53}$$

where

$$G_{\mu\nu\rho,\alpha\beta}(x - y) = -i \big\langle [Z_{\mu\nu\rho}(x), T_{\alpha\beta}(y)] \big\rangle \Theta(t - t') \,, \tag{5.54}$$

$$C_{\mu\nu\rho,\alpha\beta}(x - y) = 2i \left\langle \frac{\delta Z_{\mu\nu\rho}(x)}{\delta g_{\alpha\beta}(y)} \right\rangle \,, \tag{5.55}$$

with $C_{\mu\nu\rho,\alpha\beta}$ a seagull term. The computations of such correlators is similar in spirit to what already done in Chapter 3. One goes to Euclidean signature to perform the thermal Matsubara sums and then continues the frequency dependence into the Lorentzian realm. Also, since the only quantity that enters the correlators is the field strength, it is convenient to employ the following gauge-invariant rules for the Wick contractions:

$$\overbrace{F_{\mu\nu}(p)F}^{}{}_{\rho\sigma}(q) = (2\pi)^{-4}\delta(p+q)L_{\mu\nu\rho\sigma}(p)\,, \tag{5.56}$$

$$L_{\mu\nu\rho\sigma}(q) = -\frac{4}{q^2}q_{[\mu}g_{\nu][\sigma}q_{\rho]}\,. \tag{5.57}$$

Here we are interested in generalizing these methods to the full higher spin tower of Zilches. We show in appendix 5.B that, although we work in higher dimension, they can be written explicitly as

$$Z^{(s)}_{\mu_1\dots\mu_s} = F_{(\mu_1}{}^{\alpha}\overset{\leftrightarrow}{\partial}_{\mu_2}\dots\overset{\leftrightarrow}{\partial}_{\mu_{s-1}}\tilde{F}_{\mu_s)\alpha}\,. \tag{5.58}$$

The key point which allows such simplification, as opposed to the Dirac case, is the fact that Maxwell fields are real. The currents above can be verified to be conserved and traceless on-shell.

Now one must find a convenient way to repackage the result for all the higher spin tower by examining the correlators for arbitrary spin. In Appendix (5.B) we construct the explicit form of the contact term by coupling the system to linearized gravity. After this has been done one finds:

$$C_{\mu_1\dots\mu_s,\alpha\beta}(x-y) = -\frac{(s-2)}{2}\left(F_{(\mu_1\gamma}\overset{\leftrightarrow}{\partial}_{\mu_2}\dots\overset{\leftrightarrow}{\partial}_{\mu_{s-2}}\tilde{F}_{\mu_{s-1}}{}^{\delta} - F_{(\mu_1}{}^{\delta}\overset{\leftrightarrow}{\partial}_{\mu_2}\dots\overset{\leftrightarrow}{\partial}_{\mu_{s-2}}\tilde{F}_{\mu_{s-1}\gamma}\right)\frac{\delta\Gamma^{\gamma}{}_{\mu_s)\delta}}{\delta g^{\alpha\beta}(y)} + \mathcal{O}(\partial\Gamma)\,. \tag{5.59}$$

To be precise, these are the parts of the contact terms which do not vanish in the momentum space integral due to the identity:

$$\epsilon^{\nu\alpha\beta\gamma}L_{\mu\alpha\beta\gamma}(q) = 0\,. \tag{5.60}$$

After performing the needed Wick contractions the correlators read:

$$G_{\mu_1\dots\mu_s,\alpha\beta}(p) = \frac{-i}{2\beta}\sum_n\int\frac{d^3\vec{k}}{(2\pi)^3}(-)^{(s-3)/2}\epsilon_{\mu_1}{}^{\sigma\tau\gamma}\left(\frac{p}{2}-k\right)_{\mu_2}\dots\left(\frac{p}{2}-k\right)_{\mu_{s-1}}\left[L_{\mu_s\sigma\alpha\xi}(k)L_{\tau\gamma\beta}{}^{\xi}(p-k)\right.$$
$$\left. + L_{\mu_s\sigma\beta\xi}(k)L_{\tau\gamma\alpha}{}^{\xi}(p-k)\right]\,, \tag{5.61}$$

while:

$$C_{\mu_1\dots\mu_s,\alpha\beta}(p) = (s-2)ip^{\gamma}\frac{1}{\beta}\sum_n\int\frac{d^3\vec{k}}{(2\pi)^3}(-)^{(s-3)/2}k_{\mu_2}\dots k_{\mu_{s-2}}\left[\epsilon_{\mu_1(\alpha}{}^{\sigma\tau}\delta_{\beta)\mu_{s-1}}L_{\mu_s\gamma\sigma\tau}(k)\right.$$
$$\left. - \epsilon_{\mu_1\gamma}{}^{\delta\tau}\delta_{\mu_{s-1}(\alpha}L_{\mu_s\beta)\delta\tau}(k)\right]\,. \tag{5.62}$$

At this point we set most indices equal to 0 apart from one, which has the effect of multiplying the result by powers of the internal Matsubara frequency. To give the results in a concise way it is very useful to introduce the following integrals:

$$I_D^{(a,b,c)} = \frac{1}{\beta} \sum_n \int \frac{d^D k}{(2\pi)^D} \frac{|\vec{k}|^{2a} \omega_n^{2c}}{(\omega_n^2 + |\vec{k}|^2)^b} \, , \tag{5.63}$$

where $\omega_n = 2\pi n T$ are the bosonic Matsubara frequencies. Depending on the way the integration is performed we may encounter divergences. We follow [46, 105] and start by using dimensional regularization to do the momentum space integral first. Since $D = d - 1$ is odd the divergences are automatically canceled. Afterwards we perform the Matsubara sums, which can be done by zeta function regularization. The details can be found in Appendix 5.A. The final result is rather compact and reads:

$$I_D^{(a,b,c)} = T^{D+1+2(a-b+c)} 2^{-D/2+1} (2\pi)^{D/2+2(a-b+c)} \frac{\Gamma(a+D/2)\Gamma(b-a-D/2)}{\Gamma(D/2)\Gamma(b)} \zeta(-D-2(a-b+c)) \, . \tag{5.64}$$

We now write

$$G_{00\ldots 0i,0j}(p) = i\epsilon_{ijk} p^k \mathcal{I}_s^G \, , \tag{5.65}$$

$$C_{00\ldots 0i,0j}(p) = i\epsilon_{ijk} p^k \mathcal{I}_s^C \, , \tag{5.66}$$

where, by inspection:

$$\mathcal{I}_s^G = \frac{2}{s}\left[ \frac{s-2}{3}\left( I_3^{(1,1,n)} - 2I_3^{(1,2,n+1)} \right) \right.$$
$$\left. - \frac{4}{3} I_3^{(1,2,n+1)} - I_3^{(0,1,n+1)} + 2I_3^{(0,2,n+2)} \right] \, , \tag{5.67}$$

$$\mathcal{I}_s^C = -\frac{2}{s}(s-2)\left( \frac{1}{3} I_3^{(1,1,n)} - I_3^{(0,1,n+1)} \right) \, . \tag{5.68}$$

for odd values of the spin. Even values of $s$ lead to vanishing contributions. Let us first look at the case $s = 3$, which was already computed by other methods in [89]:

$$\mathcal{I}_3^G = \frac{2}{3}\left( \frac{1}{3} I_3^{(1,1,0)} - 2I_3^{(1,2,1)} - I_3^{(0,1,1)} + 2I_3^{(0,2,2)} \right)$$
$$= \frac{8}{135}\pi^2 T^4 \, , \tag{5.69}$$

$$\mathcal{I}_3^C = -\frac{2}{3}\left( \frac{1}{3} I_3^{(1,1,0)} - I_3^{(0,1,1)} \right) = \frac{4}{135}\pi^2 T^4 \, . \tag{5.70}$$

Hence:

$$\sigma_Z = \frac{8}{15}\pi^2 T^4 \, , \tag{5.71}$$

which reproduces the known result once the factor 3 difference in our conventions is taken into account. For generic spin one may expect a very complicated answer, however using the identity:

$$\zeta(-s) = (-)^s \frac{B_{s+1}}{s+1} \, , \tag{5.72}$$

which is nontrivial for odd spins, together with the representation of $I_3^{a,b,c}$ one finds a remarkably simple result:

$$\sigma_Z^{(s)} = \frac{4}{\pi^2} \, (2\pi T)^{s+1} \, \frac{|B_{s+1}|}{s+1} \,. \tag{5.73}$$

Notice how this resembles very closely the structure shown by the two dimensional computation, since the Bernoulli numbers are the lowest order terms in the expansion of the Bernoulli polynomials.

## 5.3 Effective actions with HS couplings

We have seen through various examples that an underlying organizing principle given by the higher spin symmetry may emerge for the tower of currents constructed from a representative which suffers 't Hooft anomalies. In this Section we make some steps toward answering two questions:

1. Are the transport coefficients extracted from the perturbative computations a function of the lower spin 't Hooft anomalies?

2. Is there a compact way to formulate them through an effective action treatment?

The answer to the first question is probably yes. This can be most easily appreciated in two dimensions, where a clear characterization of higher spin algebras exists for CFTs. Take for example the $W_{1+\infty}$ algebra spanned by the fermionic currents. This schematically reads in mode expansion [95] ($W_m^s$ are the Fourier modes of $J^s$, the Virasoro generators are $W_m^0$ in this convention):

$$[W_n^s, W_m^{s'}] = \sum_{s''=0}^{s+s'} g_{s''}{}^{m+n} W_{m+n}^{s+s'-s''} + \delta^{ss'} \delta_{m+n,0} c_s(m) \,, \tag{5.74}$$

there are two important facts for our discussion

1. The central charges $c_s(m) = f_s(m)c$ are proportional to the central charge of the underlying CFT.

2. There is central extension only for generators of equal spin.

The first point tells us that higher spin "anomalies" are not independent from the diffeomorphism anomaly, since they are uniquely determined by the central charge. The second point gives a tentative explanation for the emergence of nontrivial one-point functions. Indeed on the right-hand side, the lower spin term signals the mixing of the $W^s$ currents with the Virasoro generators under a diffeomorphism. Such mixing means that, one the cylinder, the $W^s$ will pick up a nontrivial one-point function due to the Schwartzian transformation of the Virasoro generators. This line of reasoning should be generalized to higher dimensions.

What we have said above essentially answers most of our second question, however let us conclude this Chapter by presenting a way to justify this in higher dimensions and to uncover some possible interesting phenomena which should be studied.

First however, we must introduce another bit of formalism to represent geometrically linearized higher spin symmetries.

Such representations work well for free fields in flat space-time, which is what we are interested in at the moment, and have been developed starting from the paper [91]. The basic idea is to interpret the $2d$ coordinates appearing in the higher spin generating functions for the currents as defining the proper geometry in which the HS theory lives. This will turn out to be a non-commutative space-time

This representation is simpler to develop for symmetric-conserved, but *not* traceless currents $j_s$, since in this case the auxiliary variables are simply flat space coordinates. The starting point is the introduction of higher spin gauge fields $A^s_{\mu_1...\mu_s}$ and the minimal coupling

$$S[\Phi, A^s] = S[\Phi] + \sum_s \int d^d x \; j_s^{\mu_1...\mu_s} A^s_{\mu_1...\mu_s} \, . \tag{5.75}$$

We also introduce a generating function for the $A^s$:

$$A(x, z) = \sum_s \frac{1}{s!} A^s_{\mu_1...\mu_s} z^{\mu_1} ... z^{\mu_s} \, . \tag{5.76}$$

Conservation and tracelessness of the currents translates in invariance of the effective action under the following gauge transformations:

$$\delta_\alpha A^s_{\mu_1....\mu_s} = \partial_{\mu_1} \alpha^{s-1}_{\mu_2...\mu_s} \, , \tag{5.77}$$

for conservation and

$$\delta_\epsilon A^s_{\mu_1...\mu_s} = \delta_{\mu_1 \mu_2} \epsilon^{s-2}_{\mu_3...\mu_s} \, , \tag{5.78}$$

for tracelessness. Of course symmetrization is left implicit. These gauge transformations form the linearized higher spin symmetry of the system. It will also be useful to throw away the spin 0 part of the minimal coupling, which does not correspond to any invariance, and view $A^s$ as a one-form which is in a spin s representation of the rotation group.

Our aim is to rewrite the minimal coupling in terms of the generating functions $A(x, z)$ and $J(x, z)$. This is done as in [91] by introducing the Wigner transform of currents and sources. The final answer is that the minimal coupling can be written as an ordinary (spin one) coupling in a non-commutative $2d$ dimensional space-time with coordinates $(x^\mu, z_\alpha)$ and nontrivial commutators

$$[x^\mu, z_\alpha] = i \delta^\mu_{\;\alpha} \, . \tag{5.79}$$

As it is well known, this may be represented as flat $2d$ dimensional space-time endowed with the non-commutative Moyal product $\star_M$, which treats $x^\mu$, $z_\alpha$ as canonically conjugated variables and acts as

$$f(x, z) \star_M g(x, z) = f(x, z) e^{\frac{1}{2}\left(\overleftarrow{\partial}_x \cdot \overrightarrow{\partial}_z - \overleftarrow{\partial}_z \cdot \overrightarrow{\partial}_x\right)} g(x, z) \, . \tag{5.80}$$

The nice property about this construction is that (linearized) higher spin transformations are naturally realized by considering a one-form $A^3$ on this non-commutative space and the gauge

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[3]To be precise, all forms from here on will have no $dz^\alpha$ components. It would be nice to understand how to make this formulation look more co-variant.

transformations[4]:

$$\delta_\alpha A = d_x \alpha + [A, \alpha]_{\star_M} \,, \tag{5.81}$$

with the commutator constructed from the Moyal product and $\alpha$ is just the generating function for the gauge transformations $\alpha^s$ on the non-commutative space-time. It is thus, if not certain, at least plausible, the HS effective actions may be constructed by imposing gauge invariance under (5.81). This brings us in a natural way to the concept of consistent anomalies for these theories by imposing the non-commutative Wess-Zumino consistency condition:

$$\left(\delta_{\alpha_1}\delta_{\alpha_2} - \delta_{\alpha_2}\delta_{\alpha_1}\right) W[A] = \delta_{[\alpha_1,\alpha_2]_{\star_M}} W[A] \,. \tag{5.82}$$

it was proposed in [106] that one can recycle the usual construction of Chern-Simons terms to give formal solution to the *integrated* descent equations, that is one needs to integrate over $d^d x d^d z$ to show the consistency conditions. This is because the integration over the non-commutative space-time is cyclic under the Moyal product. This gives rise to non-commutative Chern-Simons terms in $2(d+1)$ dimensions

$$\ll P^{d+1,0}(A) \gg = \frac{(d+2)}{2} \int d^{d+1}x \int d^{d+1}z \; P^{d+2}(A, F_t, ..., F_t)_{\star_M} \,, \tag{5.83}$$

upon gauge variation this give a representative for possible higher spin anomalies. Such representative, however, cannot explain the nontrivial one point functions of higher spin currents, since is only comes with $(d+2/2)$ powers of the gauge field, while higher spin currents one-point functions can have s powers of the spin one chemical potentials. These terms should be rather investigated as a way to describe the higher spin central extensions for the gauge current algebra. One can in principle apply directly the procedure developed in Chapter 2 to define the effective action. Its evaluation on non-commutative space, however, is a very complicated task.

We have anyhow presented such a construction for the Chern-Simons term to make a stronger argument for the explanation given at the start of the Section.

In this language, the idea is that in the presence of nontrivial holonomies (or temperature) there may be mixing between the spin $s$ current operator $J_s^{\mu_1...\mu_s}$ and the lower spin currents $J_r^{\nu_1...\nu_r}$, $r < s$, with coefficients proportional to the external gauge field. In particular the higher spin gauge symmetry implies (through the Ward identity) that there may be a term $A^{\mu_1} J_{s-1}^{\mu_2...\mu_{s-1}}$ in such mixing. Since we know that $J_1^\mu$ has nontrivial one-point functions due to the $U(1)$ anomaly, these are "exported", through the mixing, to the whole higher spin tower. Notice that, although powers of $\mu^s$ appear, the proportionality coefficient will always be the spin one anomaly.

## 5.4   Closing remarks

In this Chapter we have seen how, in free theories, the diffeomorphism and $U(1)$ anomalies give rise to non-trivial one-point functions for the whole higher spin tower. We have given some

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[4]Here we do not talk about the scaling transformations, although they also have an expression in terms of non-commutative geometry.

practical examples in which they can be computed in both two and four dimensions and argued that they should be explained through the mixing between the operators of the tower upon the introduction of non-trivial holonomies or non-zero temperature. We have also reviewed a possible formulation of consistent anomalies for higher spin theories, arguing that it should give rise to a non-trivial extension of the effective action for the spin 1 and 2 currents derived in Chapter 2. The precise form of this extension is however very hard to determine apart from some simple terms.

# Appendices to Chapter 5

## 5.A  Regularization of $I^{(a,b,c)}$

Here we briefly show how to regulate the various divergent integrals appearing in the thermal computations. We will use a mix of $\zeta$-function and dimensional regularization. This is quite suitable, since the spatial integral happens to be in an odd number of dimensions and thus it will get automatically regulated. We wish to compute

$$I_D^{(a,b,c)} = \frac{1}{\beta} \sum_n \int \frac{d^D k}{(2\pi)^D} \frac{|\vec{k}|^{2a} \omega_n^{2c}}{(\omega_n^2 + |\vec{k}|^2)^b} \,. \tag{5.84}$$

We start with the spatial part, which can be expressed in terms of

$$I_D^{(a,b)}(\Delta) = \int \frac{d^D \vec{k}}{(2\pi)^D} \frac{|\vec{k}|^{2a}}{(\Delta + |\vec{k}|^2)^b} \,. \tag{5.85}$$

Changing to spherical coordinates and using the integral representation of Euler's beta function

$$B(u,v) = \int_0^\infty dy\, y^{u-1}(1+y)^{-v-u} \,, \tag{5.86}$$

gives immediately

$$I^{(a,b)}(\Delta) = \frac{\Delta^{D/2+a-b}}{(4\pi)^{D/2}\Gamma(D/2)} \frac{\Gamma(a+D/2)\Gamma(b-a-D/2)}{\Gamma(b)} \,. \tag{5.87}$$

Our initial integral has now become

$$I_D^{(a,b,c)} = T(2\pi T)^{D+2(a-b+c)} \left(1+(-)^{2c}\right) \sum_{n=0}^\infty n^{D+2(a-b+c)} I_D^{(a,b)}(1) \,. \tag{5.88}$$

The final sum is regulated by using zeta function regularization $\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$. Notice that the whole expression vanishes when $c$ is half integer, which is the case for even spin Zilches. Algebraic simplifications then give

$$I_D^{(a,b,c)} = T^{D+1+2(a-b+c)} 2^{-D/2+1} (2\pi)^{D/2+2(a-b+c)} \frac{\Gamma(a+D/2)\Gamma(b-a-D/2)}{\Gamma(D/2)\Gamma(b)} \zeta(-D-2(a-b+c)) \,, \tag{5.89}$$

which is perfectly well defined for $D = 3$.
In order to obtain the final expression in the main text, one uses that

$$\zeta(-s) = (-)^s \frac{B_{s+1}}{s+1} \,, \tag{5.90}$$

where $B_n$ are the Bernoulli numbers, e.g. $B_2 = 1/6$, $B_4 = -1/30$ etc. Direct computation shows that all the relevant integrals are proportional to each other with proportionality constants independent of the powers $c$ of the frequency.

## 5.B  Higher Zilches and contact terms

In this section we explicitly verify the conservation for the higher Zilch currents and derive from these currents the contact terms for the Kubo formula computation. We omit for simplicity unimportant normalization factors. We start by showing that the current is conserved $\partial^\mu Z^{(s)}_{\mu\mu_2...\mu_s} = 0$. In order to do this, one expands

$$Z^{(s)}_{\mu\mu_2...\mu_s} = \frac{1}{s}\left(F_\mu{}^\alpha \overset{\leftrightarrow}{\partial}_{(\mu_2}...\overset{\leftrightarrow}{\partial}_{\mu_{s-1}}\tilde{F}_{\mu_s)\alpha} + F_{(\mu_2}{}^\alpha\overset{\leftrightarrow}{\partial}_{\mu_3}...\overset{\leftrightarrow}{\partial}_{\mu_s)}\tilde{F}_{\mu\alpha}\right) + \frac{s-2}{s}F_{(\mu_2}{}^\alpha\overset{\leftrightarrow}{\partial}_\mu\overset{\leftrightarrow}{\partial}_{\mu_3}...\overset{\leftrightarrow}{\partial}_{\mu_{s-1}}\tilde{F}_{\mu_s)\alpha}. \tag{5.91}$$

When applying the divergence, the last term vanishes due to the equation of motion $\Box F_{\mu\nu} = \Box\tilde{F}_{\mu\nu} = 0$. This happens since its contraction with the two sided derivative reads $\overset{\leftrightarrow+}{\partial}_\mu\overset{\leftrightarrow\mu}{\partial} = \overset{\leftrightarrow}{\Box}$, where $\overset{\leftrightarrow+}{\partial}_\mu = \overset{\rightarrow}{\partial}_\mu + \overset{\leftarrow}{\partial}_\mu$. The first two terms give a contribution

$$\partial^\mu Z^{(s)}_{\mu\mu_2...\mu_s} = \frac{1}{s}\left(F_\mu{}^\alpha\overset{\leftrightarrow}{\partial}_{(\mu_2}...\overset{\leftrightarrow}{\partial}_{\mu_{s-1}}\partial^\mu\tilde{F}_{\mu_s)\alpha} + \partial^\mu F_{(\mu_2}{}^\alpha\overset{\leftrightarrow}{\partial}_{\mu_3}...\overset{\leftrightarrow}{\partial}_{\mu_s)}\tilde{F}_{\mu\alpha}\right), \tag{5.92}$$

where we have already dropped the other combination which vanishes due to Maxwell's equation $\partial_\mu F^{\mu\nu} = \partial_\mu\tilde{F}^{\mu\nu} = 0$.

The remaining terms have to be manipulated a bit in order to show that they cancel. To do this one uses the Bianchi identity and the antisymmetry in $\alpha\mu$ to substitute $\partial_\mu F_{\mu_2\alpha}$ by $-\frac{1}{2}\partial_{\mu_2}F_{\mu\alpha}$ and the same for $\tilde{F}$. This results in

$$\partial^\mu Z^{(s)}_{\mu\mu_2...\mu_s} = -\frac{1}{2s}\partial_{(\mu_2}\left(F^{\mu\alpha}\overset{\leftrightarrow}{\partial}_{\mu_3}...\overset{\leftrightarrow}{\partial}_{\mu_s)}\tilde{F}_{\mu\alpha}\right). \tag{5.93}$$

Since the number of double sided derivatives is odd, the expression is both symmetric and anti-symmetric in $F \leftrightarrow \tilde{F}$ so it vanishes.

Tracelessness follows in a similar way. In fact, using the equation of motion we can rewrite the trace of the Zilches as

$$Z^{(s)\mu}{}_{\mu\mu_3...\mu_s} = \frac{2}{s(s-1)}F^{\mu\alpha}\overset{\leftrightarrow}{\partial}_{(\mu_3}...\overset{\leftrightarrow}{\partial}_{\mu_s)}\tilde{F}_{\mu\alpha} - \frac{(s-2)(s-3)}{4s(s-1)}\partial_\mu F_{(\mu_3}{}^\alpha\overset{\leftrightarrow}{\partial}_{\mu_4}...\overset{\leftrightarrow}{\partial}_{\mu_{s-1}}\partial^\mu\tilde{F}_{\mu_s)\alpha}$$
$$+ \frac{s-2}{s(s-1)}\left(F_\mu{}^\alpha\overset{\leftrightarrow}{\partial}_{(\mu_3}...\overset{\leftrightarrow}{\partial}_{\mu_{s-1}}\partial^\mu\tilde{F}_{\mu_s)\alpha} - \partial^\mu F_{(\mu_3}{}^\alpha\overset{\leftrightarrow}{\partial}_{\mu_4}...\overset{\leftrightarrow}{\partial}_{\mu_s)}\tilde{F}_{\mu\alpha}\right), \tag{5.94}$$

which is immediately seen to vanish term by term once the Bianchi identity is used to simplify the second line. Notice that is is critical for the spin to be odd in order for the computation to work out. Having constructed a conserved spin-s Zilch in flat space-time we wish to extend it to the curved case to extrapolate the contact terms relevant to our calculation.

Now we move on to compute the contact term, by making the partial derivatives co-variant. We will work only at linear level in the curved metric and, in order to do this, it is expedient to rewrite the currents as

$$Z^{(s)}_{\mu_1...\mu_s} = \sum_{k=0}^{s-2} c_{s,k}\partial_{(\mu_2}...\partial_{\mu_k}F_{\mu_1}{}^\alpha\partial_{\mu_{k+1}}...\partial_{\mu_{s-1-k}}\tilde{F}_{\mu_s)\alpha}, \tag{5.95}$$

where $c_{s,k} = \frac{(-)^k}{2^{s-2}}\binom{s-2}{k}$. To co-variantize we simply replace partial derivatives with co-variant ones, and to linear order we only have to worry of a single co-variant derivative at a time, so that we may write

$$Z^{(s)}_{\mu_1...\mu_s} = \sum_{k=0}^{s-2} c_{s,k} \sum_{i=0}^{k} \partial_{(\mu_2}...\nabla_{\mu_i}...\partial_{\mu_k}F_{\mu_1}{}^{\alpha}\partial_{\mu_{k+1}}...\partial_{\mu_{s-1-k}}\tilde{F}_{\mu_s)\alpha} - \partial_{(\mu_{k+1}}...\partial_{\mu_{s-1-k}}F_{\mu_1}{}^{\alpha}\partial_{\mu_2}...\nabla_{\mu_i}...\partial_{\mu_k}\tilde{F}_{\mu_s)\alpha} ,$$
(5.96)

where the minus sign is a consequence of the odd number of derivatives.

The metric dependence of the above expression comes from three different places. The first, which will not contain external momenta when we perform the integral in momentum space, is through the contraction of the $\alpha$ indices between $F$ and $\tilde{F}$. The second contribution can be obtained by expanding the co-variant derivatives acting on the $\mu_j$ indices in terms of the Christoffel symbols. Those terms with derivatives acting on the Christoffel symbols will involve higher orders of the external momenta when we integrate and thus can be dropped. The remaining ones will be of the form

$$\Gamma^{\gamma}_{(\mu_i\mu_j}\partial_{\mu_2}...\partial_{\gamma}...\partial_{\mu_k}F_{\mu_1}{}^{\alpha}\partial_{\mu_{k+1}}...\partial_{\mu_{s-1-k}}\tilde{F}_{\mu_s)\alpha} ,$$
(5.97)

and

$$\Gamma^{\gamma}_{(\mu_i\mu_1}\partial_{\mu_2}...\partial_{\mu_k}F_{\gamma}{}^{\alpha}\partial_{\mu_{k+1}}...\partial_{\mu_{s-1-k}}\tilde{F}_{\mu_s)\alpha} ,$$
(5.98)

and the same with $F \leftrightarrow \tilde{F}$. These terms come into various combinations in the complete sum but for our purposes it is enough to argue that they will cancel term by term. To see this, one takes the functional derivative with respect to the external metric and goes to Fourier space. After performing Wick contractions, what remains is an integral of the form

$$\frac{1}{\beta}\sum_n \int \frac{d^3\vec{q}}{(2\pi)^3}(...) \epsilon_{\mu_i}{}^{\alpha\beta\gamma}L_{\mu_j\alpha\beta\gamma}(q) ,$$
(5.99)

where the dots stand for a combination of momenta and the rightmost part comes from the Wick contraction. The point is that such formula vanishes identically since

$$\epsilon_{\mu}{}^{\alpha\beta\gamma}L_{\nu\alpha\beta\gamma} = \frac{1}{q^2}\epsilon_{\mu}{}^{\alpha\beta\gamma}(q_\nu q_\beta\delta_{\alpha\gamma} - q_\alpha q_\beta\delta_{\nu\gamma} - q_\nu q_\gamma\delta_{\alpha\beta} + q_\alpha q_\gamma\delta_{\nu\beta}) = 0.$$
(5.100)

Finally, the last source of contact terms are those cases in which one acts with the co-variant derivative on the contracted index $\alpha$. In position space, they give a contribution

$$\sum_{k=0}^{s-2}c_{s,k}\sum_{i=0}^{k}\Gamma^{\gamma}_{\mu_i\alpha}\left(\partial_{(\mu_2}......\partial_{\mu_k}F_{\mu_1\gamma}\partial_{\mu_{k+1}}...\partial_{\mu_{s-1-k}}\tilde{F}_{\mu_s)}{}^{\alpha} - \partial_{(\mu_{k+1}}...\partial_{\mu_{s-1-k}}F_{\mu_1}{}^{\alpha}\partial_{\mu_2}...\partial_{\mu_k}\tilde{F}_{\mu_s)\gamma}\right) + \mathcal{O}(\partial\Gamma) ,$$
(5.101)

where the $\mu_i$-th derivative is missing. Since the indices are all symmetrized, the sum over $i$ just gives a factor of $k$. Manipulating the binomial coefficient, one can recast the whole expression as

$$-\frac{s-2}{2}\Gamma^{\gamma}_{(\mu_1\alpha}\left(F_{\mu_2\gamma}\overset{\leftrightarrow}{\partial}_{\mu_3}...\overset{\leftrightarrow}{\partial}_{\mu_{s-1}}\tilde{F}_{\mu_s)}{}^{\alpha} - F_{\mu_2}{}^{\alpha}\overset{\leftrightarrow}{\partial}_{\mu_3}...\overset{\leftrightarrow}{\partial}_{\mu_{s-1}}\tilde{F}_{\mu_s)\gamma}\right) + \mathcal{O}(\partial\Gamma) .$$
(5.102)

Finally using

$$\frac{\delta\Gamma^{\gamma}_{\mu\nu}(x)}{\delta g_{\alpha\beta}(y)}\Big|_{g=\delta} = \frac{1}{2}\left[-\delta_{\mu}^{(\alpha}\delta_{\nu}^{\beta)}\partial^{\gamma}\delta(x-y) + \delta_{\nu}^{(\alpha}\delta^{\beta)\gamma}\partial_{\mu}\delta(x-y) + \delta_{\mu}^{(\alpha}\delta^{\beta)\gamma}\partial_{\nu}\delta(x-y)\right] ,$$
(5.103)

one arrives at the momentum space expression for the contact terms.

# Discussion and Outlook

We dedicate this last Section to a small overview of relevant results and of open problems which may be of future interest.

**In Chapter 2** we have studied how the 't Hooft anomalies of a Quantum Field Theory fix univocally the form of certain Chern-Simons terms in the effective action on curved backgrounds. We saw that, in the gravitational case, the full perturbative anomaly contributes to such terms ans the general substitution rule of [55] can be seen as a consistency condition for the Callan-Harvey inflow mechanism. This prompts the natural question of whether this philosophy of "group extension" may be used to derive equally interesting results elsewhere. In order to do so, however, we need first to develop more formal methods to characterize the phenomenon in full generality.

Another line of research which has seen growing interest is about the fate of such Chern-Simons terms when the effective action is computed in a time-dependent (e.g. quenched) state. In this case even perturbative field theory computations are very hard, since they must use a real time approach and only few results have been found at strong coupling using holography see e.g. [107, 108]. It is not clear whether the inflow arguments can give any hint about what happens in these cases. Perhaps there is some hope for some particular class of states which might be represented geometrically, such as boundary states for CFTs.

Finally there might be still something to say about 't Hooft anomalies for higher form symmetries [10], even though our results at the moment only apply to $U(1)$ higher form symmetry and mixed anomalies with non-abelian 0-form ones (our formalism is not apt to treat discrete symmetries). Nevertheless these Abelian symmetries have been used in the last years in interesting constructions for hydrodynamic theories involving magnetic one-form symmetries [109, 110, 111, 112].

**In Chapter 3** We have introduced some basic notation and well known realizations of non-relativistic symmetry groups. This Chapter was instrumental for the presentation of specific results in Chapter 4.

**In Chapter 4** we have examined interesting transport phenomena that arise in fermionic Lifshitz systems and their warped limit explanation in terms of torsional anomalies. While the discussion for generic $z$ did not show any universality, it did allow to develop a parallel between the usual hydrodynamic expansion and the introduction of torsion in non-relativistic geometries: torsion works as an external gauge field as far as transport is concerned. This should be studied further, although precise computation can be quite challenging as already mentioned.

In the second part of the Chapter we have shown an example of a system in which torsional anomalies are actually well defined (i.e. regulator independent objects). This is still a topic of debate in the condensed matter literature due to the often alluring possibility of treating torsion as an external electromagnetic field. We also have shown that, in the case of Carrollian systems, nontrivial solutions to the consistency conditions do exist, albeit only when curvature constraints are imposed. This is in contrast to the case of Galileian theories, where flavor and diffeomorphism anomalies are absent, at least in even dimensionality. The construction of Non-Relativistic Chern-Simons terms and their connection with consistency conditions for

non-relativistic groups should be studied further.

**In Chapter 5**, finally, we have examined the way in which the anomalous one-point functions derived in Chapter 2 are reflected in the cases in which the systems has an underlying higher spin symmetry. We have shown interesting examples in which this phenomenon appears, including Maxwell theory, which does not have a local, gauge-invariant chiral current of spin one, but does have an higher spin tower of well defined chiral currents (Zilches). We have argued that the higher spin symmetry determines the temperature and chemical potential dependence of the transport coefficients since it fixes the way in which the higher spin currents mix upon going from flat space to a non-trivial state. Thus such transport coefficients are not independent of the lower spin anomalies. Furthermore, even for higher spin chemical potential, it is plausible that Chern-Simons terms might be defined which incorporate the whole higher spin tower, thus rendering all of the higher spin one-point functions in principle computable from the lower spin ones. While this was an interesting topic to research, we do not see the reason to study it further.

# Conclusiones

Dedicamos esta última sección a una pequeña descripción general de los resultados relevantes y de los problemas abiertos que pueden ser de interés futuro.

**En el Capítulo 2** hemos estudiado cómo las anomalías de 't Hooft de una teoría de campo cuántico fijan unívocamente la forma de ciertos términos de Chern-Simons en la acción efectiva sobre fondos curvos. Vimos que, en el caso gravitacional, la anomalía perturbativa completa contribuye a tales términos y la regla general de sustitución de [55] puede verse como una condición de consistencia para el mecanismo de inflow de Callan-Harvey. Esto lleva a la pregunta natural de si esta filosofía de "extensión de grupo" puede usarse para obtener resultados igualmente interesantes en otros lugares. Para hacerlo, sin embargo, primero necesitamos desarrollar métodos más formales para caracterizar el fenómeno en general.

Otra línea de investigación que ha visto un interés creciente es sobre el destino de dichos términos de Chern-Simons cuando la acción efectiva se calcula en un estado dependiente del tiempo (por ejemplo, un "quench"). En este caso, incluso los cálculos de la teoría de campo perturbativo son muy difíciles, ya que deben usar un formalismo en tiempo real y solo se han encontrado pocos resultados en acoplamiento fuerte usando holografía, ver p. Ej. [107, 108]. No está claro si los argumentos de "inflow" pueden dar alguna pista sobre lo que sucede en estos casos. Quizás haya alguna esperanza para una clase particular de estados que podrían representarse geométricamente, como los estados "boundary" para CFT.

Finalmente, aún podría haber algo que decir sobre las anomalías 't Hooft para simetrías de forma superior [10], a pesar de que nuestros resultados en este momento solo se aplican a la simetrías $U(1)$ de forma más alta y anomalías mixtas con no-abelianas de forma 0 (nuestro formalismo no es apto para tratar simetrías discretas). Sin embargo, estas simetrías abelianas se han utilizado en los últimos años en construcciones interesantes para teorías hidrodinámicas que involucran simetrías magnéticas de uno forma [109, 110, 111, 112].

**En el Capítulo 3** Hemos introducido algunas notaciones básicas y realizaciones bien conocidas de grupos de simetría no relativistas. Este Capítulo fue instrumental para la presentación de resultados específicos en el Capítulo 4.

**En el Capítulo 4** hemos examinado interesantes fenómenos de transporte que surgen en los sistemas fermiónicos de Lifshitz y su explicación en el límite "warped" en términos de anomalías torsionales. Si bien la discusión sobre $z$ genérico no mostró ninguna universalidad, permitió desarrollar un paralelo entre la expansión hidrodinámica habitual y la introducción de la torsión en geometrías no relativistas: la torsión funciona como un campo electromagnetico externo en lo que respecta al transporte . Esto debería estudiarse más a fondo, aunque el cálculo preciso puede ser bastante difícil como ya se mencionó.

En la segunda parte del Capítulo, hemos mostrado un ejemplo de un sistema en el que las anomalías torsionales están realmente bien definidas (es decir, son objetos independientes del regulador). Esto sigue siendo un tema de debate en la literatura de materia condensada debido a la posibilidad a menudo atractiva de tratar la torsión como un campo electromagnético externo. También hemos demostrado que, en el caso de los sistemas Carrollianos, existen soluciones no triviales a las condiciones de consistencia, aunque solo cuando se imponen restricciones de curvatura. Esto contrasta con el caso de las teorías galileanas, donde las anomalías de sabor

y difeomorfismo están ausentes, al menos en una dimensionalidad par. La construcción de términos de Chern-Simons no relativistas y su conexión con las condiciones de consistencia para grupos no relativistas deben estudiarse más a fondo.

**En el Capítulo 5**, finalmente, hemos examinado la forma en que las funciones anómalas de un punto derivadas en el Capítulo 2 se reflejan en los casos en que los sistemas tengan una simetría de espín más alta subyacente. Hemos mostrado ejemplos interesantes en los que aparece este fenómeno, incluida la teoría de Maxwell, que no tiene una corriente quiral local, invariante de gauge, de espín uno, pero tiene una torre de espín más alto de corrientes quirales bien definidas (Zilches).

Hemos argumentado que la mayor simetría de espín determina la dependencia de la temperatura y el potencial químico de los coeficientes de transporte, ya que fija la forma en que las corrientes de espín más altas se mezclan entre ellas al pasar del espacio plano a un estado no trivial. Por lo tanto, dichos coeficientes de transporte no son independientes de las anomalías de espín inferiores. Además, incluso para un potencial químico de espín superior, es posible que se definan los términos de Chern-Simons que incorporan la torre de espín superior completa, haciendo que todas las funciones de un punto de espín superior sean en principio computables a partir de las de espín inferior. Si bien este fue un tema interesante para la investigación, no vemos la razón para estudiarlo más a fondo.

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