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Superconformal indices for $\mathcal{N} = 6$ Chern Simons theories

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Abstract: Aharony, Bergman, Jafferis and Maldacena have recently proposed a dual gravitational description for a family of superconformal Chern Simons theories in three spacetime dimensions. In this note we perform the one loop computation that determines the field theory superconformal index of this theory and compare with the index computed over the Fock space of dual supersymmetric gravitons. In the appropriate limit (large N and large k) we find a perfect match.

Keywords: AdS-CFT Correspondence, Chern-Simons Theories.

Contents

1. Introduction

Aharony, Bergman, Jaffiers and Maldacena (ABJM) have recently proposed that a class of $d = 3$ U(N) $\times$ U(N) $\mathcal{N} = 6$ superconformal Chern Simons field theories admit a dual description in terms of M theory compactified on $AdS_4 \times \frac{S^7}{Z_k}$ [1]. This correspondence has been further studied in [2]. The theories studied by these authors are parameterized by two integers N and k. In dual bulk terms k is the rank of the orbifold action on S^7 while N represents the number of units of 7 form flux that pierce S^7/Z_k. In field theory terms N denotes the rank of each of the U(N) factors of the gauge group and $\pm k$ are the levels of the Chern Simons terms associated with each of these gauge groups.

With an appropriate normalization for fields, the effective 't Hooft coupling constant of any large N gauge theory is given by N times the inverse of the coefficient of the action. As the coefficient of the Chern Simons term is proportional to k, the effective 't Hooft coupling of the ABJM field theory is proportional to N/k. As N and k are both integers this 't Hooft coupling cannot be varied continuously; indeed a shift in k by unity shifts λ by the discrete amount $\delta\lambda = -\frac{\lambda^2}{N}$. Note however that $\delta\lambda \to 0$ in the 't Hooft limit ($N \to \infty$ with λ held fixed). Consequently λ is effectively a continuous parameter in the 't Hooft limit. It follows that the superconformal Witten index, defined for arbitrary 3 dimensional superconformal field theories in [3], must be invariant under deformations of λ in this 't' Hooft scaling regime.[1]

In this note we compute the superconformal index (as defined in [3]) of the ABJM theory in two different regimes. We first use the techniques of [4 – 9] to find an expression for this index at $k = \infty$ (and so $\lambda = 0$ but at arbitrary N) in terms of an integral over two $N \times N$ unitary matrices. Taking a further large N limit we are able to evaluate these unitary integrals explicitly using saddle point techniques.

Next we evaluate the index of this theory at infinite N and large λ, using the ABJM proposal for the dual description of this theory. Effectively, we compute the index over

[1] This is only true of contributions to the index from states whose energy stays finite as N is taken to infinity.

the Fock space of non interacting supersymmetric U(1) neutral (see below) gravitons in $AdS_4 \times \frac{S^7}{Z_k}$. We perform this calculation explicitly in the 't Hooft limit, but also explain the generalization of this calculation to finite values of k, and so to values of λ that scale like N in the large N limit.

We find that our two independent computations of the superconformal Witten index of [3] agree perfectly in the 't Hooft limit. We view this agreement as a test of the ABJM proposal. Indeed, since the index computed in this paper is the most general superconformal index [3] for a $\mathcal{N} = 6$ superconformal field theory[2] our calculation verifies the most detailed matching of supersymmetric states predicted by ABJM conjecture taken together with the requirement of superconformal invariance alone. Of course the agreement of the two independent index computations reported in this paper is closely related to the agreement of the spectrum of chiral operators of the field theory and the spectrum of gravitons in $AdS_4 \times \frac{S^7}{Z_k}$ reported in [1]. Note, however, that while the chiral ring described by ABJM counts states described by at least four supercharges (two supersymmetries and their Hermitian conjugates) the index constructed in this paper receives contributions from all states that are annihilated by a minimum of two supercharges. Note also that the invariance under λ deformations of the index computed in this paper follows from superconformal invariance alone, and uses no dynamical information about the field theory beyond its field content. In particular our field theoretic computation of the index is blind to the nature of the ABJM superpotential whose form played an important role in the construction of the ABJM chiral ring [1].

The matching of the index computed perturbatively in the ABJM theory with the index computed over the spectrum of bulk gravitons is reminiscent of a similar match in the case of $\mathcal{N} = 4$ Yang Mills theory [7]. As explained in that paper, $AdS_5 \times S^5$ hosts a family of 1/16 BPS black holes which the 4 dimensional superconformal index appears to be blind to. In a similar fashion the 3 dimensional superconformal index computed in this paper appears to be blind to the 1/12 BPS black holes presumably hosted by the $AdS_4 \times \frac{S^7}{Z_k}$ dual background (see [10 – 12]).

This note is organized as follows. In § 2 below we review the symmetry algebra of the ABJM theory, the definition of the Witten index of [3] for this theory and the field content of the ABJM Chern Simons theory. We then perform a one loop field theory computation to present a field theoretic formula for this index in terms of an integral over two unitary matrices and evaluate these integrals in the large N limit. In § 3 we present our computation of the same index over the spectrum of U(1) invariant multi gravitons in $AdS_4 \times \frac{S^7}{Z_k}$. In § 4 we end with a discussion of our results and the generalizations they suggest.

2. Field theory computation of the index

The symmetry algebra of the ABJM theory is the $d = 3$ $\mathcal{N} = 6$ superconformal algebra. The structure of this algebra and its unitary representations has been reviewed in detail in [3], and we will use the notation of that paper in what follows.

[2] By a superconformal index we mean a quantity whose invariance under marginal deformations is guaranteed by superconformal invariance alone.

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type of operators	operators	scaling dimension (ϵ_0)	SO(3) highest weight	SO(6) highest weight
dynamical	ϕ_{12}	$\frac{1}{2}$	0	$(\frac{1}{2},\frac{1}{2},-\frac{1}{2})$
	ψ_{12}	1	$\frac{1}{2}$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$
	ϕ_{21}	$\frac{1}{2}$	0	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$
	ψ_{21}	1	$\frac{1}{2}$	$(\frac{1}{2},\frac{1}{2},-\frac{1}{2})$
supersymmetry generators	Q	$\frac{1}{2}$	$\frac{1}{2}$	$(1,0,0)$
derivatives	∂	1	1	$(0,0,0)$

Table 1: A list of the field content of ABJM theory, the supercharges, the derivatives and the representations.

The bosonic subgroup of the $d = 3$ $\mathcal{N} = 6$ superconformal algebra is $SO(3,2) \times SO(6)$. All states and operators in this theory are labeled by their quantum numbers under the maximal compact subalgebra of the superconformal algebra, $SO(3) \times SO(2) \times SO(6)$. In what follows we will denote the eigenvalue of the Cartan generator of SO(3) by j, the eigenvalue under SO(2) (the scaling dimension or global AdS_4 energy) by ϵ_0 and the three Cartan generators of SO(6)- defined as the eigenvalues under the generators of rotations in the three orthogonal two planes - as h_1, h_2, h_3.[3]

The twelve supercharges of the ABJM theory each have $\epsilon_0 = \frac{1}{2}$, and transform in the $j = \frac{1}{2}$ representation of SO(3) algebra and the vector $(h_1, h_2, h_3) = (1, 0, 0)$ of SO(6) algebra. The only propagating fields in the ABJM theory are a set of bi-fundamental and anti bi-fundamental scalars and fermions. All scalars have dimension $\epsilon_0 = \frac{1}{2}$ and are scalars under SO(3), while all fermions have dimension $\epsilon_0 = 1$ and transform in the spin half representation of SO(3). Bi-fundamental scalars/fermions transform in the $(\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2})$ representation of SO(6), while anti bi-fundamental scalars and fermions transform in the $(\frac{1}{2}, \frac{1}{2}, \mp\frac{1}{2})$ representation of SO(6). The symmetry transformation properties of the supersymmetries, propagating fields and derivatives of the ABJM theory are listed in table 1 below. In table 1 and throughout this paper, the symbol ϕ_{12} and ψ_{12} respectively denote scalar and fermionic fields that transform in the fundamental of the first U(N) gauge group and antifundamental of the second U(N) gauge group, while ϕ_{21} and ψ_{21} respectively denote scalar and fermionic fields which transforms in the antifundamental of the first U(N) gauge group and in the fundamental of the second.

In this note we will compute the Witten index

$$I^W = \mathrm{Tr}\left((-1)^F x^{\epsilon_0 + j} y_1^{h_2} y_2^{h_3}\right) \tag{2.1}$$

[3] SO(6) may be thought of as the group of rotations about the origin in R^6 parameterized by x^i, $i = 1 \ldots 6$. h_1, h_2 and h_3 are simply the generators of rotations in the two planes (12), (34) and (56) respectively. Throughout this paper we will label representations of SO(6) by their highest weights under (h_1, h_2, h_3). In our conventions an SO(6) weight is positive if h_1 is positive, or if $h_1 = 0$ and h_2 is positive or if $h_1 = h_2 = 0$ and h_3 is positive. We use a similar conventions for SO(8) representations below.

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for the ABJM theory quantized on $S^2 \times R$. As explained in [3] this index receives contributions only from states that are annihilated by a special supercharge Q together with its Hermitian conjugate $Q^\dagger$. Q has quantum numbers $\epsilon_0 = \frac{1}{2}$, $j = -\frac{1}{2}$, $(h_1, h_2, h_3) = (1, 0, 0)$. It follows from the superconformal algebra that

$$\{Q, Q^\dagger\} = (\epsilon_0 - j - h_1) \equiv \Delta.$$

As a consequence, a state is annihilated by both Q and $Q^\dagger$ if and only if $\Delta = 0$. Consequently the index (2.1) receives contributions only from states with $\Delta = 0$. In table 2 below we list all $\Delta = 0$ propagating fields and derivatives of the ABJM theory. We also list the partition function over all $\Delta = 0$ bosonic fields and their derivatives ($f_{12}^{\rm bosonic}$ and $f_{21}^{\rm bosonic}$), the partition function over $\Delta = 0$ fermionic fields and their derivatives ($f_{12}^{\rm fermionic}$ and $f_{21}^{\rm fermionic}$), and the Witten index (f_{12} and f_{21}) over these fields.

Following [4, 5] it was demonstrated in [8] (see equation(2.7)) that the free superconformal index of a Yang-Mills theory with the field content listed above is given by,

$$I^W = \int dU_1 dU_2 \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} [f_{12}(x^n, y_1^n, y_2^n) \mathrm{Tr} U_1^n \mathrm{Tr} U_2^{-n} + f_{21}(x^n, y_1^n, y_2^n) \mathrm{Tr} U_1^{-n} \mathrm{Tr} U_2^n] \right) \tag{2.2}$$

The unitary integrals described in (2.2) may be evaluated in the large N limit. Let $\rho_n = \frac{TrU_1^n}{N}$ and $\chi_n = \frac{TrU_2^n}{N}$. In the large N limit the various ρ_n may be treated as independent variables (modulo a positivity constraint - see for instance [5] - that will turn out to be irrelevant for our considerations below) and

$$DU_1 = \prod_n d\rho_n \exp\left(-N^2 \sum_n \frac{\rho_n \rho_{-n}}{n} \right), \quad DU_2 = \prod_n d\rho_n \exp\left(-N^2 \sum_n \frac{\chi_n \chi_{-n}}{n} \right)$$

so that

$$I^W = \int \prod_{n \neq 0} d\rho_n d\chi_n \exp\Bigg(N^2 \sum_{n=1}^{\infty} \frac{1}{n} \Big(- |\rho_n|^2 - |\chi_n|^2 + f_{12}(x^n, y_1^n, y_2^n) \rho_n \chi_{-n} + f_{21}(x^n, y_1^n, y_2^n) \rho_{-n} \chi_n \Big) \Bigg). \tag{2.3}$$

The integral in (2.3) takes the form

$$I^W = \int \prod_{n \neq 0} d\rho_n d\chi_n \exp\left(-N^2 \sum_{n=1}^{\infty} \frac{1}{n} \left((C^n)_i (M^n)_{ij} (C^n)_j \right) \right), \tag{2.4}$$

where the column C^n and the matrix $(M^n)_{ij}$ are given by

$$C^n = \begin{pmatrix} \chi_n \\ \rho_n \\ \chi_{-n} \\ \rho_{-n} \end{pmatrix}, \quad M^n = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 & -f_{21} \\ 0 & 0 & -f_{12} & 1 \\ 1 & -f_{12} & 0 & 0 \\ -f_{21} & 1 & 0 & 0 \end{pmatrix}. \tag{2.5}$$

letter	ϵ_0	SO(3) (j)	SO(6) weights	ϵ_0+j	partition function	index
bi-fundamental						
$(\phi_{12})_1$	$\frac{1}{2}$	0	$(\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$\frac{1}{2}$	$f_{12}^{\text{bosonic}} = \frac{x^{\frac{1}{2}}}{1-x^2}\left(\frac{y_1}{y_2}+\frac{y_1}{y_2}\right)$	$f_{12} = \frac{x^{\frac{1}{2}}}{1-x^2}\left(\frac{y_1}{y_2}+\frac{y_1}{y_2}\right) - \frac{x^{\frac{3}{2}}}{1-x^2}\left(y_1y_2+\frac{1}{y_1y_2}\right)$
$(\phi_{12})_2$	$\frac{1}{2}$	0	$(\frac{1}{2},-\frac{1}{2},\frac{1}{2})$	$\frac{1}{2}$		
$(\psi_{12})_1$	1	$\frac{1}{2}$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$\frac{3}{2}$	$f_{12}^{\text{fermionic}} = \frac{x^{\frac{3}{2}}}{1-x^2}\left(y_1y_2+\frac{1}{y_1y_2}\right)$	
$(\psi_{12})_2$	1	$\frac{1}{2}$	$(\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$	$\frac{3}{2}$		
bi-fundamental						
$(\phi_{21})_1$	$\frac{1}{2}$	0	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$\frac{1}{2}$	$f_{21}^{\text{bosonic}} = \frac{x^{\frac{1}{2}}}{1-x^2}\left(y_1y_2+\frac{1}{y_1y_2}\right)$	$f_{21} = \frac{x^{\frac{1}{2}}}{1-x^2}\left(y_1y_2+\frac{1}{y_1y_2}\right) - \frac{x^{\frac{3}{2}}}{1-x^2}\left(\frac{y_1}{y_2}+\frac{y_1}{y_2}\right)$
$(\phi_{21})_2$	$\frac{1}{2}$	0	$(\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$	$\frac{1}{2}$		
$(\psi_{21})_1$	1	$\frac{1}{2}$	$(\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$\frac{3}{2}$	$f_{21}^{\text{fermionic}} = \frac{x^{\frac{3}{2}}}{1-x^2}\left(\frac{y_1}{y_2}+\frac{y_1}{y_2}\right)$	
$(\psi_{21})_2$	1	$\frac{1}{2}$	$(\frac{1}{2},-\frac{1}{2},\frac{1}{2})$	$\frac{3}{2}$		
derivative						
∂	1	+1	$(0,0,0)$	2		

Table 2: List of the supersymmetric ($\Delta = 0$) fields or 'letters' of the theory over which we calculate the index

It is possible to demonstrate that the real part of the quadratic form $(C^n)_i(M^n)_{ij}(C^n)j$ in (2.4) is positive whenever the chemical potentials obey the inequalities

$$x < \min\{y_1y_2,\ \frac{y_1}{y_2},\ \frac{y_2}{y_1},\ \frac{1}{y_1y_2}\} \tag{2.6}$$

a condition that it necessary for the index to be well defined in the first place.[4] As a consequence it appears that the integral (2.3) is always dominated by the saddle point at $\rho_n = \chi_n = 0$ for all $n \geq 0$. It thus appears that, in perfect analogy with the situation

[4]In order that the index be well defined it is necessary that every $\Delta = 0$ letter contribute to the partition function with a weight less than unity, leading to (2.6).

for $\mathcal{N}=4$ Yang Mills, the Witten index (2.3) never undergoes the phase transition into a black hole like phase.

As the saddle point contribution to the integral (2.3) vanishes, the first nonzero contribution to this integral is given by the inverse square root of the determinant

$$I^W = \prod_{n=1}^{\infty} \frac{1}{\sqrt{16 \det M^n}} \tag{2.7}$$

where the normalization in (2.7) is fixed by the requirement that I^W tends to unity when $x=0$ (at which point only the vacuum contributes to the Witten index).

The determinant is given by,

$$\begin{aligned} \det(M^n) &= \frac{1}{16}(1 - f_{12}(x^n, y_1^n, y_2^n) f_{21}(x^n, y_1^n, y_2^n))^2 \\ &= \frac{1}{16} \frac{\left(1-\frac{x^n}{y_1^n}\right)^2 \left(1-\frac{x^n}{y_2^n}\right)^2 (1-x^n y_1^n)^2 (1-x^n y_2^n)^2}{(1-x^{2n})^4} \end{aligned} \tag{2.8}$$

so that,

$$I^W = \prod_n \frac{\left(1-x^{2n}\right)^2}{\left(1-\frac{x^n}{y_1^n}\right)\left(1-\frac{x^n}{y_2^n}\right)(1-x^n y_1^n)(1-x^n y_2^n)}. \tag{2.9}$$

As the large N unitary integrals in (2.3) never undergo a large N Gross-Witten-Wadia transition [15, 16], it follows that the Witten index I^W receives contributions only from states of finite energy (and charge) at finite values of the chemical potential. In particular, (2.9) is blind to states whose energy is of order N^a where a is any positive power.

In order to get a feel for (2.9) it is useful to set $y_1 = y_2 = 1$. If we define the Indicial entropy $S_{\rm ind}(E)$ by the formula $I^w(x) = \int dE e^{S_{\rm ind}(E)} x^E$ then it is easy to show that $S_{\rm ind}(E) \approx \sqrt{2}\pi\sqrt{E}$ at high energies. This is the growth of states of a two dimensional massless gas; a similar growth in density of states was captured by the four dimensional index (see [7]). This growth is slower than the $E^{\frac{2}{3}}$ growth demonstrated by the index of the M2 brane and M5 brane world volume thoeries [3].

3. Gravity computation of the index

Gravitons[5] in $AdS_4 \times S^7$ may be organized into representations of the $d=3$ $\mathcal{N}=8$ superconformal algebra. Working in conventions in which the $M2$ brane world volume scalar, fermion and supersymmetries respectively transform in the $(\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2})$, $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ and $(1,0,0,0)$ representations of SO(8), the highest weight states of the representations that occur in this decomposition each have $j=0$, $\epsilon_0 = \frac{n}{2}$ and SO(8) highest weight charges $(n/2, n/2, n/2, -n/2)$. See [3], Table 1 for more details.

[5] In this section we use the word 'graviton' for any field on AdS_4 obtained upon compactification from a field in the 11 dimensional gravity multiplet.

Gravitons on $AdS_4 \times \frac{S^7}{Z_k}$ are those graviton states on $AdS_4 \times S^7$ whose charge under the generator $2h_4$ is 0 mod k.[6] In the large k 't Hooft limit under study in this note, all projected in gravitons are simply neutral under the U(1) charge h_4. Consequently, the superconformal index over single gravitons in $AdS_4 \times \frac{S^7}{Z_k}$ is simply the projection of the same quantity in $AdS_4 \times S^7$ to the sector of zero h_4 charge.

The index

$$I^W = \mathrm{Tr}[(-1)^F x^{\epsilon_0+j} y_1^{h_2} y_2^{h_3} y_3^{h_4}] \tag{3.1}$$

over single gravitions in $AdS_4 \times S^7$ was evaluated in [3] (see equation 2.17 in that paper). For the convenience of the reader we reproduce the formula here,

$$I^W_{AdS_4\times S^7}(x, y_1, y_2, y_3) = \frac{\text{Numerator}}{\text{Denominator}}, \tag{3.2}$$

where,

$$\begin{aligned}
\text{Numerator} = &- \sqrt{y_1}\sqrt{y_2}\sqrt{y_3}(y_2y_3y_1 + y_1 + y_2 + y_3)x^{7/2} \\
&+ \left(y_2y_3y_1^2 + \left(y_3y_2^2 + y_3^2y_2 + y_2 + y_3\right)y_1 + y_2y_3\right)x^3 \\
&- \left(y_2y_3y_1^2 + \left(y_3y_2^2 + y_3^2y_2 + y_2 + y_3\right)y_1 + y_2y_3\right)x \\
&+ \sqrt{y_1}\sqrt{y_2}\sqrt{y_3}(y_2y_3 + y_1(y_2 + y_3) + 1)\sqrt{x} \\
\text{Denominator} = &\left(1 - x^2\right)\left(\sqrt{x}\sqrt{y_1}\sqrt{y_2} - \sqrt{y_3}\right)\left(\sqrt{x}\sqrt{y_1}\sqrt{y_3} - \sqrt{y_2}\right) \\
&\left(\sqrt{y_1} - \sqrt{x}\sqrt{y_2}\sqrt{y_3}\right)\left(\sqrt{y_1}\sqrt{y_2}\sqrt{y_3} - \sqrt{x}\right)
\end{aligned} \tag{3.3}$$

The index over single gravitons of zero h_4 charge is given by

$$\int \frac{d\theta}{2\pi i} I^W_{AdS_4\times S^7}(x, y_1, y_2, e^{i\theta}) = \int_C \frac{dy_3}{2\pi y_3 i} I^W_{AdS_4\times S^7}(x, y_1, y_2, y_3)$$

where the contour C surrounds the poles at $y_3 = 0$, $y_3 = xy_1y_2$ and $\frac{x}{y_1y_2}$.[7]

Performing this sum of residues we find that the index over U(1) neutral gravitons on $AdS_4 \times \frac{S^7}{Z_k}$ is given by,

$$I^W_{\text{Single Particle}} = \frac{x}{y_1 - x} + \frac{1}{1 - xy_1} + \frac{x}{y_2 - x} + \frac{1}{1 - xy_2} - \frac{2}{1 - x^2}. \tag{3.4}$$

a result that is significantly simpler than (3.3).

[6] As for SO(6) we think of SO(8) as the group of rotations in R^8 parameterized by x^i, $i = 1 \ldots 8$. h_1, h_2, h_3, h_4 are the eigenvalues of the generators of rotations in the (12), (34), (56) and (78) planes respectively. We label representations of SO(8) by their highest weights under (h_1, h_2, h_3, h_4); our positivity convention for weights is the obvious generalization of that for SO(6). Note that the Z_k orbifolding described in [1] is, in our conventions, simply a rotation by the angle $4\pi/k$ in the (78) 2 plane. The subgroup of SO(8) that commutes with this rotation is SO(6) × SO(2). The SO(2) factor is simply rotations in the (78) 2 plane itself while the SO(6) factor describes rotations among the remaining orthogonal 2 planes and is the R symmetry of the surviving $\mathcal{N} = 6$ supersymmetry algebra. Note that the supersymmetry of the parent theory decomposes into $6_0 + 1_2 + 1_{-2}$ under this decomposition, while the scalar decomposes into the $4_1 + \bar{4}_{-1}$.

[7] The index (3.3) is well defined only when $x < y_1^a y_2^b y_3^c$ where (a, b, c) run over the values of $(2h_1, 2h_2, 2h_3)$ for the antichiral spinor of SO(6). It follows that the contour of our integral must exclude the poles at $y_3 = \frac{y_1}{xy_2}$ and $y_3 = \frac{y_2}{xy_1}$.

Range of n	Scaling dimension ϵ_0	SO(3) weight (j)	SO(6) highest weight (orthogonal basis)	$\epsilon_0 + j$	statistics
$n \geq 1$	n	0	$(n, n, 0)$	n	bosonic
$n \geq 1$	$n + \frac{1}{2}$	$\frac{1}{2}$	$(n, n, 1)$	$n + 1$	fermionic
$n \geq 1$	$n + \frac{1}{2}$	$\frac{1}{2}$	$(n, n, -1)$	$n + 1$	fermionic
$n \geq 1$	$n + \frac{1}{2}$	$\frac{1}{2}$	$(n, n-1, 0)$	$n + 1$	fermionic
$n \geq 1$	$n + 1$	1	$(n, n, 0)$	$n + 2$	bosonic
$n \geq 2$	$n + 1$	1	$(n, n-1, 1)$	$n + 2$	bosonic
$n \geq 2$	$n + 1$	1	$(n, n-1, -1)$	$n + 2$	bosonic
$n \geq 2$	$n + \frac{3}{2}$	$\frac{3}{2}$	$(n, n-1, 0)$	$n + 3$	fermionic

Table 3: The supersymmetric ($\Delta = 0$) graviton spectrum in $AdS_4 \times \frac{S^7}{Z_k}$. Here n is an integer greater than or equal to 1.

We have also verified (3.4) more directly. As explained in [13, 1] the U(1) neutral gravitons in $AdS_4 \times \frac{S^7}{Z_k}$ appear in a direct sum representations of the $\mathcal{N} = 6$ superconformal algebra labeled by the highest weight states with $\epsilon_0 = n$, $j = 0$ and $(h_1, h_2, h_3) = (n, n, 0)$ for $n = 1 \ldots \infty$. It is not difficult to decompose every such representation of the superconformal algebra into irreducible representations of the $d = 3$ conformal algebra (using, for instance, the techniques described in [3]). This decomposition could also be read off from the table 1 in [13]. In table 3 we list those conformal representations that have states with $\Delta = 0$. Only states with maximum values of h_1 and j in the representations listed in table 3 have $\Delta = 0$ and contribute to the index. Consequently, the contribution of any of the representations listed below, to the index is simply given by

$$I^W_{\epsilon_0, j, h_2, h_3} = (-1)^j \frac{x^{\epsilon_0 + j}}{1 - x^2} \chi^{(h_2, h_3)}_{\mathrm{SO}(4)}(y_1, y_2)$$

where ϵ_0 and j are respectively the dimension and SO(3) charge of the highest weight state in the representation, the factor of $\frac{1}{1-x^2}$ is the contribution from the supersymmetric derivatives and $\chi^{(h_2,h_3)}_{\mathrm{SO}(4)}(y_1, y_2)$ is the SO(4) character with highest weights h_2, h_3. Summing this quantity over all the representations listed in table 3 for all $n \geq 1$ we recover (3.4).

In order to compute the index over multi gravitons we use the single partice index (3.4) and the formulas of Bose statistics to obtain[8]

$$I^W = \prod_n \frac{\left(1 - x^{2n}\right)^2}{\left(1 - \frac{x^n}{y_1^n}\right)\left(1 - \frac{x^n}{y_2^n}\right)\left(1 - x^n y_1^n\right)\left(1 - x^n y_2^n\right)}. \tag{3.5}$$

in perfect agreement with (2.9).

[8] According to the rules of Bose Statistics a single particle index of the form $\sum_n c_n x^n$ translates into a multi particle index $\prod_n (1 - x^n)^{-c_n}$ where c_n are integers that could be either negative or positive, and x schematically represents all chemical potentials.

4. Discussion

In this note we have computed the supersymmetric index of [3] for the ABJM theory in two different ways. We first performed a one loop field theory computation to evaluate this index in the free theory field at large N; this calculation was performed at $\lambda = \frac{N}{k} = 0$. We then evaluated the same index over the Fock space of U(1) neutral gravitons in $AdS_4 \times \frac{S^7}{Z_k}$. This calculation was valid in the large N limit with λ fixed at a large value. The results of our two calculations match perfectly, providing a check of the ABJM conjecture.

It would be easy to generalize the gravitational calculation presented in this note to apply the large N limit with k held fixed. All one needs to do is to project (3.3) onto the sector with $2h_4 = 0 \mod k$ (rather than simply to zero), before applying the formulas of Bose statistics. For instance when $k = 1$ this simply amounts to setting y_3 to unity in (3.3). It may be possible to reproduce the full finite k gravitational index from an (almost) free field theory calculation after summing over flux sectors on S^2, as suggested by the discussion in [1]. It would be very interesting to try to carry this through.

Another direction that would be interesting to explore would be the determination of the full supersymmetric partition function (rather than the supersymmetric index) of the ABJM theory, with varying amounts of supersymmetry. For instance a formula for finite N partition function over the chiral ring (states that preserve 4 supercharges) has been proposed in [14] at $k = 1$. It would be interesting to verify this formula and to generalize it to other values of k. More ambitiously one could attempt to determine the full partition function (in contrast to the index computed in this paper) over all supersymmetric of in the ABJM theory; note however that this programme has not yet been completed even for $\mathcal{N} = 4$ Yang Mills (see [18] for a recent status report).

In this connection note that like $AdS_5 \times S^5$, the ABJM gravitational background presumably hosts supersymmetric black holes that preserve 2 supersymmetries (see [10 – 12] for relevant work). This is exactly the minimum amount of supersymmetry that a state needs to preserve to contribute to the index described in this paper. However the index computed in this paper sees no sign of these states. Indeed a naive estimate suggests that the entropy of these supersymmetric black holes scale like $N^2/\sqrt{\lambda}$ times functions of chemical potentials. As a result, the entropy of these black holes appears to be a function of λ at large λ, and so cannot be captured by any quantity like an index that is independent of λ. The smooth dependence of the entropy of a supersymmetric configuration on a continuous coupling constant appears non intuitive at first sight, and it would be interesting to understand how this comes about. Perhaps the states that make up the entropy of the black hole receive important contributions from the nontrivial flux sectors (these sectors, whose energy scales like N, could in principle contribute to the entropy of a black hole - whose energy scales like N^2, even in the 't Hooft limit).[9]

Turning to the spectrum of nonsupersymmetric states in this theory, it seems possible that the integrability of the spin chain spectrum of chiral operators could carry over to the ABJM theory. It would be interesting to study this possibility in more detail. Indeed, one of the exciting aspects of the ABJM proposal (in our opinion) are the prediction that

[9] We thank R. Gopakumar for this suggestion.

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the effective string that describes spin chain dynamics metamorphoses into a membrane at $\lambda = \mathcal{O}(N)$. It would be very interesting to attempt to get a concrete handle on this.

Finally, of course the ABJM duality permits the computation of all correlators (not just the spectrum) of all the chiral operators in the theory at strong coupling. A simple scaling estimate reveals that k point functions of these operators scale like $(\lambda^{\frac{1}{4}}/N)^{k-2}$ at strong coupling. In particular, three point functions are functions of λ and cannot enjoy the nonrenormalization properties of 3 point functions of chiral operators in $\mathcal{N} = 4$ Yang Mills theory in $d = 4$ [17].

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