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Solenoid Fringe Optics[1]

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1 Solenoidal Magnetic Fields in Paraxial Approximation

Suppose that a charged particle is moving in the z direction and we wish to consider its dynamics under magnetic fields in the z direction. Suppose we are given some $B_z(z)$. $\mathbf{B}$ must satisfy Maxwell's equations. In the absence of charge and time dependent electric fields, this means that the divergence and curl of $\mathbf{B}$ must be 0. Let us consider fields with axial symmetry about the z axis. Thus, we consider

$$\mathbf{B}(z,r) = B_z(z)\hat{\mathbf{z}} + B_r(r,z)\hat{\mathbf{r}}. \tag{1}$$

The divergence condition now says that the radial fields must satisfy

$$\frac{\partial B_r}{\partial r} + \frac{B_r}{r} = -\frac{\partial B_z}{\partial z} \tag{2}$$

This, along with the condition that $B_r(r=0)=0$ fixes the radial fields:

$$B_r(r,z) = -\frac{r}{2}\frac{\partial B_z}{\partial z} \tag{3}$$

Now, $\nabla \times \mathbf{B} = 0$ gives

$$\frac{\partial B_r}{\partial z} = \frac{\partial B_z}{\partial r} \tag{4}$$

or, with (3),

$$\frac{\partial B_z}{\partial r} = -\frac{r}{2}\frac{\partial^2 B_z}{\partial z^2}. \tag{5}$$

This says that B_z must actually have radial dependence as long as its second derivative in the z direction is non-zero. At $r = 0$, however, this equation says that B_z has no radial dependence. This means that if we approximate B_z as having no radial dependence we are ignoring terms of order r^2. This is the paraxial approximation and it leads to linear transfer maps. It is a good approximation for dynamics close to the symmetry axis, i.e. for small r. Later in this paper, we will study piecewise linear B_z's. Eq. (5) shows that in this case, B_z

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is exactly independent of r, except for on the transverse planes where $\frac{\partial B_z}{\partial z}$ changes. If $\frac{\partial B_z}{\partial z}$ increases by Δ_j at z_j, then (3) and (5) show that the exact $\mathbf{B}$ is given by

$$B_z(z,r) = B_z(z, r=0) + \sum_j \frac{-r^2}{4}\Delta_j \delta(z - z_j) \tag{6}$$

and

$$B_r(z,r) = -\frac{r}{2}\frac{\partial B_z}{\partial z}(z \neq z_j) + \sum_j \frac{\Delta_j r^3}{8}\delta'(z - z_j) \tag{7}$$

where the first term in the expression for B_r is 0 at $z = z_j$ for each z_j. One could find the map across these δ fields at each z_j by integrating in z the force due to them across z_j. The $\delta'(z - z_j)$ in the radial fields will integrate to 0. The δ functions in B_z will result in a non-linear map[2]. The changes in x' and y' induced are proportional to r^2. We will ignore these non-linear maps in this paper. For a more general discussion of the non-linear case using Lie algebra techniques, see [1].

Thus, in the paraxial approximation, we have

$$\mathbf{B}(z,r) = B_z(z)\hat{\mathbf{z}} - \frac{r}{2}\frac{\partial B_z}{\partial z}\hat{\mathbf{r}} \tag{8}$$

2 Delta Fringe Solenoid

Now, consider the following model for a solenoid. The solenoid consists of a region of length L with constant B_z of strength B_0 and no field outside of that:

$$B_z(z) = B_0 \Theta(L/2 - |z|). \tag{9}$$

Applying eq. (3) and using the fact that the derivative of a theta function is a delta function, we get radial fringe fields

$$B_r(r,z) = \frac{B_0}{2} r \left(\delta(L/2 - z) - \delta(-L/2 - z)\right) \tag{10}$$

Now, consider a highly relativistic particle of charge e and momentum in the z direction P_0. We use transverse coordinates

$$\mathbf{X}(z) = \begin{pmatrix} x(z) \\ x'(z) \\ y(z) \\ y'(z) \end{pmatrix} \tag{11}$$

[2]One could imagine breaking up an arbitrary B_z into a large number of linear sections and adding in such a non-linear δ map between each one. These maps, in conjunction with the linear maps derived later in this paper could perhaps give insight into the general non-paraxial case.

The delta fringe map $\mathbb{F}_\delta(k)$ at $z = -L/2$, which takes the particle across the fringe from the more negative z side to the less negative z side, is given by

$$\mathbf{X}(-L/2)_- = \underbrace{\begin{pmatrix} \mathbb{I}_2 & \mathbb{K} \\ -\mathbb{K} & \mathbb{I}_2 \end{pmatrix}}_{\mathbb{F}_\delta(k)} \mathbf{X}(-L/2)_+ \tag{12}$$

where $\mathbb{I}_2$ is the 2×2 unit matrix,

$$\mathbb{K} = \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix} \tag{13}$$

and

$$k = \frac{eB_0}{2P_0}. \tag{14}$$

This map describes a kick of (x', y') in the $-\hat{\phi}$ direction: $x' \to x' + kx$, $y' \to y' - ky$. The fringe map at $z = L/2$ is the same except with $k \to -k$.

The body map $\mathbb{B}(L, k)$ which takes the particle from $-L/2$ to $L/2$ is given by

$$\mathbf{X}(-L/2)_+ = \underbrace{\begin{pmatrix} \mathbb{M}_1 & \mathbb{M}_2 \\ -\mathbb{M}_2 & \mathbb{M}_1 \end{pmatrix}}_{\mathbb{B}(L,k)} \mathbf{X}(L/2)_- \tag{15}$$

with

$$\mathbb{M}_1 = \begin{pmatrix} 1 & \frac{1}{2k}\sin(2kL) \\ 0 & \cos(2kL) \end{pmatrix} \qquad \mathbb{M}_2 = \begin{pmatrix} 0 & \frac{1}{2k}(\cos(2kL) - 1) \\ 0 & -\sin(2kL) \end{pmatrix} \tag{16}$$

This map describes helical motion of radius $\sqrt{x'^2 + y'^2}/2k$. Putting these together, we get the full map for the delta fringe solenoid

$$\mathbb{S}_\delta(L, k) = \mathbb{F}_\delta(k)\mathbb{B}(L, k)\mathbb{F}_\delta(-k) = \begin{pmatrix} \cos(kL)\mathbb{O} & \sin(kL)\mathbb{O} \\ -\sin(kL)\mathbb{O} & \cos(kL)\mathbb{O} \end{pmatrix} \tag{17}$$

where

$$\mathbb{O}(k, L) = \begin{pmatrix} \cos(kL) & \frac{1}{k}\sin(kL) \\ -k\sin(kL) & \cos(kL) \end{pmatrix} \tag{18}$$

Note that $\mathbb{O}(k, L)$ is the transfer map for a harmonic oscillator with strength k through a distance L. We can also write the full map as follows

$$\mathbb{S}_\delta(L, k) = \underbrace{\begin{pmatrix} \cos(kL)\mathbb{I}_2 & \sin(kL)\mathbb{I}_2 \\ -\sin(kL)\mathbb{I}_2 & \cos(kL)\mathbb{I}_2 \end{pmatrix}}_{\mathbb{R}(kL)} \underbrace{\begin{pmatrix} \mathbb{O}(k, L) & 0 \\ 0 & \mathbb{O}(k, L) \end{pmatrix}}_{\mathbb{Q}(k,L)} \tag{19}$$

This result shows that the map for a delta fringe solenoid has two components. The first, $\mathbb{R}(kL)$ is a rotation of the vectors (x, y) and (x', y') by an angle kL and the second, $\mathbb{Q}(k, L)$ is harmonic oscillator evolution via $\mathbb{O}(k, L)$ in both x and y. By noting that we could just as easily have pulled $\mathbb{R}(kL)$ to the right of $\mathbb{Q}(k, L)$, we see that $\mathbb{R}$ and $\mathbb{Q}$ commute. This fact will allow us to find a simple way of expressing the transfer map for an arbitrary solenoidal magnetic field.

3 Arbitrary Solenoid from Delta Fringe Solenoid

We now would like to build up the map for arbitrary solenoidal fields from the map derived in the previous section for a delta fringe solenoid. The delta fringe solenoid just considered has two free parameters, the length L and the strength of the longitudinal magnetic field B_0. By lining up many such solenoids back to back with varying lengths and field strengths, we can model any solenoidal field we want. To get the transfer map for such a field, we simply multiply (in order) the transfer maps for all the component delta fringe solenoids with appropriate values of k and L for each. Suppose we have N such delta fringe solenoids with lengths L/N and field strengths k_i. The transfer map for the i^{th} delta fringe solenoid will be $\mathbb{R}(k_i L/N)\mathbb{Q}(k_i, L/N)$. Since $\mathbb{R}$ and $\mathbb{Q}$ commute for all values of their arguments, we can pull all the rotation matrices off to the left and the focusing harmonic oscillator transfer maps to the right. All the rotation matrices combine to a single rotation matrix of an angle equal to the sum of the angles of the N delta fringe solenoids. So we have

$$\mathbb{S} = \mathbb{R}\left(\frac{L}{N}\sum_{i=1}^{N} k_i\right)\prod_j \mathbb{Q}\left(k_j, \frac{L}{N}\right) \tag{20}$$

The product of the harmonic oscillator transfer maps just result in the net transfer map for a particle in a harmonic oscillator potential with z dependent focusing strengths k_i. Now, let us approach the continuum limit, $N \to \infty$. Given $B_z(z)$ this determines $k(z)$ by eq. (14). Let $\mathbb{O}(z_0, z_f)$ be the transfer map corresponding to the differential equation

$$\frac{d^2x}{dz^2} + k^2(z)x = 0. \tag{21}$$

Then the full transfer map in the paraxial approximation for a particle experiencing the magnetic field $B_z(z)$ (and the corresponding $B_r(r,z)$ via eq. (3)) from the position z_0 to the position z_f is

$$\mathbb{S}(z_0, z_f) = \mathbb{R}\left(\frac{e}{2P_0}\int_{z_0}^{z_f} B_z(z)dz\right)\begin{pmatrix} \mathbb{O}(z_0, z_f) & 0 \\ 0 & \mathbb{O}(z_0, z_f)\end{pmatrix} \tag{22}$$

4 Linear Fringe

The only barrier to obtaining explicit maps for various solenoidal fields is solving for the harmonic oscillator transfer map. Here we do so for the case where B_z falls off linearly in the fringes (being constant otherwise).

Let the solenoid be of length L, with longitudinal field in the central region B_0, and linearly decreasing fringe fields of length a. We seek the oscillator map across the fringe. In this region, the field is

$$B_z = \frac{z}{a}B_0 \tag{23}$$

so that

$$k(z) = \frac{eB_0}{2P_0 a}z \equiv k\frac{z}{a} \tag{24}$$

We seek the solution $x(z)$ which satisfies

$$\frac{d^2x}{dz^2} + \frac{k^2}{a^2}z^2x = 0 \tag{25}$$

subject to the initial conditions

$$x(0) = x_0 \tag{26}$$
$$x'(0) = x'_0 \tag{27}$$

The general solution, before imposing the initial conditions is

$$x(z) = \sqrt{z}\left(c_1 J_{-\frac{1}{4}}\left(\frac{kz^2}{2a}\right) + c_2 J_{\frac{1}{4}}\left(\frac{kz^2}{2a}\right)\right) \tag{28}$$

where J_ν is the Bessel function of order ν. Taking a derivative, we get

$$x'(z) = \frac{k}{a}z^{\frac{3}{2}}\left(-c_1 J_{\frac{3}{4}}\left(\frac{kz^2}{2a}\right) + c_2 J_{-\frac{3}{4}}\left(\frac{kz^2}{2a}\right)\right) \tag{29}$$

Now, the Bessel functions with $\nu > 0$ are 0 at $z = 0$ while for $\nu < 0$, $J_\nu(z) \to \infty$ as $z \to 0$. However,

$$\lim_{z\to 0}\left\{\sqrt{z}J_{-\frac{1}{4}}\left(\frac{kz^2}{2a}\right)\right\} = \frac{-4\sqrt{2}}{\Gamma(-\frac{1}{4})}\left(\frac{k}{a}\right)^{-\frac{1}{4}} = \frac{x_0}{c_1} \tag{30}$$

and

$$\lim_{z\to 0}\left\{(z)^{\frac{3}{2}}J_{-\frac{3}{4}}\left(\frac{kz^2}{2a}\right)\right\} = \frac{2\sqrt{2}}{\Gamma(\frac{1}{4})}\left(\frac{k}{a}\right)^{\frac{1}{4}} = \frac{x'_0}{c_2} \tag{31}$$

We can therefore write down the solution to the oscillator part of the fringe map. We have

$$\mathbb{O}_{\mathbb{F}}(k,z) = \begin{pmatrix} \frac{-\Gamma(\frac{-1}{4})}{4\sqrt{2}}\left(\frac{k}{a}\right)^{\frac{1}{4}}\sqrt{z}J_{-\frac{1}{4}}\left(\frac{kz^2}{2a}\right) & \frac{\Gamma(\frac{1}{4})}{2\sqrt{2}}\left(\frac{k}{a}\right)^{-\frac{1}{4}}\sqrt{z}J_{\frac{1}{4}}\left(\frac{kz^2}{2a}\right) \\ \frac{\Gamma(-\frac{1}{4})}{4\sqrt{2}}\left(\frac{k}{a}\right)^{\frac{5}{4}}z^{\frac{3}{2}}J_{\frac{3}{4}}\left(\frac{kz^2}{2a}\right) & \frac{\Gamma(\frac{1}{4})}{2\sqrt{2}}\left(\frac{k}{a}\right)^{\frac{3}{4}}z^{\frac{3}{2}}J_{-\frac{3}{4}}\left(\frac{kz^2}{2a}\right) \end{pmatrix} \tag{32}$$

One can check the following properties of this transfer map:

$$\det\mathbb{O}_{\mathbb{F}}(k,z) = 1 \tag{33}$$

and

$$\lim_{a\to 0}\mathbb{O}_{\mathbb{F}}(k,a) = \mathbb{I}_2. \tag{34}$$

The full fringe map for a linear field is then given by

$$\mathbb{F}_l(k,a) = \begin{pmatrix} \cos(ka/2)\mathbb{O}_{\mathbb{F}}(k,a) & \sin(ka/2)\mathbb{O}_{\mathbb{F}}(k,a) \\ -\sin(ka/2)\mathbb{O}_{\mathbb{F}}(k,a) & \cos(ka/2)\mathbb{O}_{\mathbb{F}}(k,a) \end{pmatrix} \tag{35}$$

and the full solenoid map is

$$\mathbb{S}_l(L,a,B_0) = \mathbb{F}_l(k,a)\mathbb{S}_\delta(L,k)\mathbb{F}_l^{-1}(k,a) \tag{36}$$

or,

$$\mathbb{S}_l(L,a,B_0) = \begin{pmatrix} \cos(k(L+a))\mathbb{N} & \sin(k(L+a))\mathbb{N} \\ -\sin(k(L+a))\mathbb{N} & \cos(k(L+a))\mathbb{N} \end{pmatrix} \tag{37}$$

where

$$\mathbb{N} = \mathbb{O}_{\mathbb{F}}(k,a)\mathbb{O}(kL)\mathbb{O}_{\mathbb{F}}^{-1}(k,a) \tag{38}$$

5 Conclusions and Further Work

The linear fringe field model of a solenoid given here should be an improvement over the delta fringe model. By measuring B_z at various places in the fringe field of an actual solenoid, one could find the values of a and L which fit best. One could also model an asymmetric solenoid by choosing different values for a at the two ends and making the obvious changes in the maps above.

Given that real solenoidal fields are generally produced by a cylindrical current configuration, one would like to find the map across the resulting fringe fields. A possibility for $B_z(z)$ in the fringe fields would be fields of the form $\tanh(z/\lambda)$ (see [1]) where λ is related to the radius of the cylindrical current. If exact harmonic oscillator transfer maps are not available for this B_z then numerical maps could be calculated.

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References

[1] A. Dragt, Nucl. Instr. Meth. A298 (1990)441-459