Eur. Phys. J. C (2024) 84:1097
https://doi.org/10.1140/epjc/s10052-024-13381-6

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

Gravastars formation with back-reaction effects from extended general relativity

Mauricio Bellini[1,2,a]

[1] Instituto de Investigaciones Físicas de Mar del Plata-IFIMAR, Consejo Nacional de Investigaciones Científicas y Técnicas-CONICET, Mar del Plata, Argentina
[2] Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad Nacional de Mar del Plata, Funes 3350, Mar del Plata C.P. 7600, Argentina

Received: 20 June 2024 / Accepted: 17 September 2024

Abstract Using an extended theory of General Relativity that incorporates normalized relativistic velocities, where the boundary terms in the varied Einstein–Hilbert action are considered. Within this context, I investigate the dynamic evolution of a collapsing spherical system characterized by a metric with spatial curvature and variable time-scale, aiming to describe the process of Gravastar formation. To illustrate the system's dynamics, I analyze a power-law collapsing scenario and derive the equation of state for both cases, with and without geometrical perturbations. In particular, I derive the equation of state, incorporating back-reaction effects related with the boundary terms of the varied Einstein–Hilbert action.

1 Introduction and motivation

Dynamic collapsing systems have a significant relationship with astrophysical objects such as Gravastar [1–6]. When considering Gravastars, the dynamics of collapsing systems become pertinent, particularly during their formation and evolution. These dynamic processes could leave distinct observational imprints, such as unique radiation emissions due to the absence of an event horizon, setting them apart from black holes, because Gravastars are conceived as an hypothetical compact stellar structure that could form through a gravitational collapse process, offering an alternative to black holes. It is characterized by a central region that is not a gravitational singularity but is composed of a condensate of exotic matter. Gravastars present an intriguing possibility in astrophysics, providing a fresh perspective on the nature of compact astrophysical objects and challenging our understanding of gravity and spacetime under extreme conditions. However, further theoretical and observational investigations are required to fully comprehend their properties and potential existence in the universe. In the context of Gravastars, dynamically collapsing systems would be relevant during their formation and evolution. Unlike black holes, the absence of an event horizon in Gravastars could lead to the emission of distinct types of radiation, potentially distinguishable from that emitted by black holes when an accretion disk is present [7]. Although the relationship between Gravastars, black holes, and dark matter remains an active and speculative area of research in theoretical physics, ongoing studies offer new insights into the nature of compact objects in the universe and the potential influence of dark matter on their formation and evolution.

Gravastars are theoretical alternatives to black holes proposed within the framework of general relativity. The theoretical study of Gravastars has been gaining interest, because they offer a solution to some of the paradoxes and limitations associated with black holes and they could explain the existence of various astrophysical phenomena. In particular, it is very interesting studying the dynamical formation of this kind of systems [8]. In this work we aim to study a dynamical model for the formation of Gravastars, though a collapsing isotropic but inhomogeneous system by taking into account the boundary terms in the varied Einstein–Hilbert (EH) action

$$\delta\mathcal{I}=\int d^4x\sqrt{-g}\left[\delta g^{\alpha\beta}\left(G_{\alpha\beta}+\kappa\,T_{\alpha\beta}\right)+g^{\alpha\beta}\delta R_{\alpha\beta}\right]=0, \tag{1}$$

where $\kappa=8\pi G/c^4$ is a constant, R is the scalar curvature of the background spacetime and g is the determinant of

[a] e-mail: mbellini@mdp.edu.ar (corresponding author)

Published online: 22 October 2024

the background metric tensor, with which we define the line element $dS^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$. The Einstein tensor $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}\, R$ represents the background geometry, where $R_{\alpha\beta}$ denotes the Ricci tensor and R is the scalar curvature (or Ricci scalar) of the background, given by $R = g^{\alpha\beta}\, R_{\alpha\beta}$. Furthermore, $T_{\alpha\beta} = -2 \frac{\delta \mathcal{L}_m}{\delta g^{\alpha\beta}} + g_{\alpha\beta} \mathcal{L}_m$, is the stress tensor of the background, such that $\mathcal{L}_m$ denotes the Lagrangian density that takes into account the background physical dynamics. To ensure a well-defined variational principle, it is a common practice to introduce counterterms in the EH action to cancel out the contribution of the last term of the varied action (1) [9–11]. As in previous works, we shall adopt a different approach used in [12,13]. We shall consider the scenario where the variation of the Ricci tensor $\delta R_{\alpha\beta}$ is related with the variation of the metric tensor by the expression:

$$\delta R_{\alpha\beta} = \lambda \left(x^\mu\right)\, \delta g_{\alpha\beta}. \tag{2}$$

Here, $\lambda\left(x^\mu\right)$ is a function of all the coordinates x^α. The expression (2) must be interpreted in the sense that the variations of the metric tensor $\delta g_{\alpha\beta}$, are the source of the varied Ricci tensor. In this case, the boundary terms for the varied action, are

$$g^{\alpha\beta} \delta R_{\alpha\beta} \equiv \delta\Theta. \tag{3}$$

The expression (3) describes the flux of the 4-vector $\delta W^\alpha = b \left[\delta\Gamma^\epsilon_{\beta\epsilon}\, g^{\beta\alpha} - \delta\Gamma^\alpha_{\beta\gamma}\, g^{\beta\gamma}\right]$ through the closed 3D hypersurface ∂M. By employing the proposed equation (2), we can derive the following equation:

$$g^{\alpha\beta}\, \delta R_{\alpha\beta} = \lambda \left(x^\mu\right)\, g^{\alpha\beta}\, \delta g_{\alpha\beta}. \tag{4}$$

On the other hand, using the fact that $\delta \left[g_{\alpha\beta}\, g^{\alpha\beta}\right] = 0$, we obtain that $\delta g^{\alpha\beta}\, g_{\alpha\beta} = -\delta g_{\alpha\beta}\, g^{\alpha\beta}$. From the varied action $\delta\mathcal{I}$ in (1), we obtain the new Einstein's equations

$$G_{\alpha\beta} - \lambda \left(x^\mu\right)\, g_{\alpha\beta} = -\kappa\, T_{\alpha\beta}. \tag{5}$$

These are the background Einstein equations with the boundary terms assimilated.

1.1 Modified field equations with sources

We consider the stress tensor for a perfect fluid

$$T^{\alpha\beta} = \left(\bar{P}_T + \bar{\rho}_T\right)\, \bar{U}^\alpha\, \bar{U}^\beta - \bar{P}_T\, g^{\alpha\beta}, \tag{6}$$

where $\bar{P}_T$ is the total pressure and $\bar{\rho}_T$ is the total energy density of the system. Hence, from (6) and the field equation

$$\nabla_\beta\, T^{\alpha\beta} = \frac{1}{\kappa}\, g^{\alpha\beta}\, \frac{\partial \lambda(x)}{\partial x^\beta}, \tag{7}$$

we obtain that the background velocities can be described by

$$\left(\bar{P}_T + \bar{\rho}_T\right) \left[\bar{U}^\beta \nabla_\beta\, \bar{U}^\alpha + \bar{U}^\alpha \nabla_\beta \bar{U}^\beta\right] + \frac{\partial\left(\bar{P}_T + \bar{\rho}_T\right)}{\partial x^\beta}\, \bar{U}^\alpha\, \bar{U}^\beta - g^{\alpha\beta}\, \frac{\partial\, \bar{P}_T}{\partial x^\beta} = \frac{1}{\kappa}\, g^{\alpha\beta}\, \frac{\partial\, \lambda(x)}{\partial x^\beta}, \tag{8}$$

The Eq. (8) provides the geodesic equation for a perfect fluid with arbitrary $\bar{P}_T$ and $\bar{\rho}_T$, such that $\bar{P}_T / \bar{\rho}_T = \bar{\omega}$ is not necessarily constant. Summarizing we can write the modified field equations by taking into account the Eq. (7), making $\bar{T}_{\alpha\beta} = T_{\alpha\beta} - g_{\alpha\beta} \left(\frac{\lambda(x)}{\kappa}\right)$, so that the system behaves as conservative

$$\nabla_\beta\, G^{\alpha\beta} = \nabla_\beta\, \bar{T}^{\alpha\beta} = 0. \tag{9}$$

1.2 Extended manifold and covariant derivative of tensors on the extended manifold

In order to describe the geometric fields that produce the flux, and alters the Riemann manifold, we need to consider the varied Ricci tensor, which we will define as an extension of the Palatini expression [14]

$$\delta R_{\alpha\beta} = b \left[\left(\delta\Gamma^\mu_{\alpha\mu}\right)_{\|\beta} - \left(\delta\Gamma^\mu_{\alpha\beta}\right)_{\|\mu}\right]. \tag{10}$$

In this context $(\ldots)_{\|\beta}$ denotes de covariant derivative of $(\ldots)$ on the extended manifold and b is a constant with $1/(length)$-units.

We define the covariant derivative of some vector field ψ^β: $\left[\psi^\beta\right]_{\|\alpha}$

$$\left[\psi^\beta\right]_{\|\alpha} = \nabla_\alpha \psi^\beta + \delta\Gamma^\beta_{\epsilon\alpha} \psi^\epsilon - \eta\, \psi^\beta\, \bar{U}_\alpha, \tag{11}$$

where η describes the interaction of ψ with the extended manifold, $\nabla_\alpha \psi^\beta$ is the covariant derivative on the Riemann manifold and $\delta\Gamma^\beta_{\epsilon\alpha}$ is the displacement of the extended manifold with respect to the Riemann one defined in (13). Notice that in the case where $\psi^\beta \equiv \bar{U}^\beta$ in (11), which is the case with we are dealing, the constant η describes an self-interaction. We can extend this definition for arbitrary tensors. In general, we define the covariant derivative of a (n)-times contravariant and (m)-times covariant mixture tensor $\Upsilon^{\alpha_1\ldots\alpha_n}_{\beta_1\ldots\beta_m}$, on the extended manifold, the $(n+m+1)$-range operator tensor:

$$\Upsilon^{\alpha_1\ldots\alpha_n}_{\beta_1\ldots\beta_m\|\mu} = \nabla_\mu \Upsilon^{\alpha_1\ldots\alpha_n}_{\beta_1\ldots\beta_m} + \sum_{i=1}^{n} \delta\Gamma^{\alpha_i}_{\nu\mu}\, \Upsilon^{\alpha_1..\alpha_{i-1}\nu\alpha_{i+1}..\alpha_n}_{\beta_1\ldots\beta_m} - \sum_{i=1}^{m} \delta\Gamma^{\nu}_{\mu\beta_i}\, \Upsilon^{\alpha_1\ldots\alpha_n}_{\beta_1..\beta_{i-1}\nu\beta_{i+1}.\beta_m} - \eta \sum_{i=1}^{n-1} \left(\Upsilon^{\alpha_1..\alpha_i\alpha_{i+1}..\alpha_n}_{\beta_1\ldots\beta_m}\, \bar{U}_\mu + \bar{U}_\mu\, \Upsilon^{\alpha_1..\alpha_{i+1}\alpha_i..\alpha_n}_{\beta_1\ldots\beta_m}\right)$$

$$+\eta \sum_{i=1}^{m-1} \left(\Upsilon^{\alpha_1 \ldots \alpha_n}_{\beta_1 .. \beta_i \beta_{i+1} .. \beta_m} \bar{U}_\mu + \bar{U}_\mu \Upsilon^{\alpha_1 \ldots \alpha_n}_{\beta_1 .. \beta_{i+1} \beta_i .. \beta_m} \right), \quad (12)$$

where the terms in the last row of (12) take into account the interaction of the tensor $\Upsilon^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_m}$ with the extended manifold and η is a parameter to be determined.

In this work we shall extend the Riemann manifold by using the extended connections [15]

$$\delta\Gamma^{\mu}_{\alpha\beta} = b\, \bar{U}^{\mu}\, g_{\alpha\beta}, \quad (13)$$

where b is some parameter to be determined and $\bar{U}^{\mu} = \frac{dx^{\mu}}{dS}$ are the relativistic 4-velocities calculated with perturbations included.

The manuscript is organized as follows: in Sect. 2 we study a normalized description for the relativistic velocities with boundary terms included and we obtain the metric tensor with these modifications. In Sect. 3 a dynamical model for a collapsing sphere with positive spatial curvature and variable time scale, is introduced. The back-reaction effects due to the scalar field fluctuations are taken into account. In Sect. 4 we illustrate the collapsing approach to a dynamical model for Gravastar formation. In Sect. 5, we focus on a specific scenario and derive a particular family of solutions for that describe different relativistic frames. A particular case is the co-moving frame, for which the function $\lambda(x)$ is a constant: $\lambda(x) \equiv \lambda_*$. Finally, in Sect. 6 we develop some final comments and conclusions.

2 Geodesic equations and normalized velocities on the extended manifold

The dynamic equations for the 4-velocities $\bar{U}^{\mu}$ can be obtained from Eqs. (1), (2), (3) and (10), by using the extended connections (13) with self-interactions included

$$\left[b\, \left(\bar{U}^{\beta}\, \bar{U}^{\alpha} + \bar{U}^{\alpha} \bar{U}^{\beta}\right) - 2\,\eta\, (\varphi - 1)\, g^{\alpha\beta}\right] \lambda(x^{\mu})\, g_{\alpha\beta} \\ = 3\, b^2 \left[\nabla_{\mu} \bar{U}^{\mu} + (2b + \eta)\, (\varphi - 1)\right], \quad (14)$$

where we shall propose the following normalization condition for the relativistic velocities with sources included, such that summation is realised with a metric tensor without perturbations:

$$\bar{U}_{\mu}\, \bar{U}^{\mu} \equiv g_{\mu\nu}\, \bar{U}^{\mu}\, \bar{U}^{\nu} = (\varphi - 1). \quad (15)$$

Here, φ is a dimensionless parameter to be determined. In this case the left-side bracket is proportional to $\delta g^{\alpha\beta}$

$$\delta g^{\alpha\beta} = \left[2\,\eta\, (\varphi - 1)\, g^{\alpha\beta} - b\, \left(\bar{U}^{\beta}\, \bar{U}^{\alpha} + \bar{U}^{\alpha} \bar{U}^{\beta}\right)\right]. \quad (16)$$

Using the fact that $\delta g_{\alpha\beta}\, g^{\alpha\beta} = -\delta g^{\alpha\beta}\, g_{\alpha\beta}$, we can obtain the covariant variation of the metric tensor

$$\delta g_{\alpha\beta} = \left[b\, \left(\bar{U}_{\beta}\, \bar{U}_{\alpha} + \bar{U}_{\alpha} \bar{U}_{\beta}\right) - 2\,\eta\, (\varphi - 1)\, g_{\alpha\beta}\right]. \quad (17)$$

The dynamics of the relativistic velocities in extended General Relativity are given by the expression $\delta \bar{U}^{\alpha} = 0$, which, written in terms of the covariant derivatives on the extended manifold, takes the form $\bar{U}^{\beta}\, \bar{U}^{\alpha}_{\|\beta} = 0$. This expression can be expanded in terms of the semi-Riemann covariant derivative, and using the fact that $\frac{d}{dS} = \bar{U}^{\beta}\, \frac{\partial}{\partial x^{\beta}}$, we obtain

$$\frac{d\bar{U}^{\alpha}}{dS} + \Gamma^{\alpha}_{\mu\beta}\, \bar{U}^{\mu}\, \bar{U}^{\beta} = [\eta - b]\, (\varphi - 1)\, \bar{U}^{\alpha}, \quad (18)$$

which describes the geodesic equations for $\bar{U}^{\beta}$ with sources on the Riemann manifold. These sources are originated in the extended manifold. We can write it in a compact manner as $\Delta \bar{U}^{\alpha} = [\eta - b]\, (\varphi - 1)\, \bar{U}^{\alpha}$, where the variation of $\bar{U}^{\alpha}$ on the Riemann manifold is expressed as $\Delta \bar{U}^{\alpha} \equiv \bar{U}^{\beta}\, \nabla_{\beta}\, \bar{U}^{\alpha}$. When sources are absent, we obtain $\eta = b$, and the relativistic velocities $\bar{U}^{\alpha}\big|_{b=\eta} \rightarrow U^{\alpha}$, are solution of the homogeneous differential equation without sources

$$\frac{dU^{\alpha}}{dS} + \Gamma^{\alpha}_{\mu\beta}\, U^{\mu}\, U^{\beta} = 0, \quad (19)$$

such that the normalization condition for the relativistic velocity without perturbations is

$$g_{\mu\nu}\, U^{\mu}\, U^{\nu} = \epsilon, \quad (20)$$

with $\epsilon = \pm 1, 0$. On the other hand we must consider the normalization condition for the relativistic velocity with sources in terms of the total metric tensor

$$\bar{g}_{\alpha\beta} = g_{\alpha\beta} + (1/b)\, \delta g_{\alpha\beta}, \qquad \bar{g}^{\alpha\beta} = g^{\alpha\beta} + (1/b)\, \delta g^{\alpha\beta}, \quad (21)$$

which takes into account the perturbations of the system:

$$\bar{g}_{\mu\nu}\, \bar{U}^{\mu}\, \bar{U}^{\nu} = \epsilon. \quad (22)$$

Using Eqs. (17) and (20), with (21), we obtain for the choice $b = 2\eta$ and $\epsilon = 1$, that

$$\bar{g}_{\mu\nu}\, \bar{U}^{\mu}\, \bar{U}^{\nu} = (\varphi - 1) + (\varphi - 1)^2 = 1. \quad (23)$$

which have as solutions

$$\varphi_1 = \frac{1}{2}\left[1 + \sqrt{5}\right] = 1.61803, \\ \varphi_2 = \frac{1}{2}\left[1 - \sqrt{5}\right] = -0.61803 \quad (24)$$

Because we shall consider metric signature $(+, -, -, -)$, we must choose $\varphi = \varphi_1$, which is an irrational number known as the golden number. Using the properties of φ with the fact that $g^{\alpha\beta}\, g_{\alpha\beta} = 4$, we obtain the following relevant equation

$$\bar{g}^{\alpha\beta}\, \bar{g}_{\alpha\beta} = \frac{4}{\varphi_1}, \tag{25}$$

which means that, for $b = 2\eta$, the effective number of dimensions of a relativistic perturbed system is the irrational number $4/\varphi_1 = 8/(1+\sqrt{5})$[1] Therefore, the conditions that normalize the relativistic velocities (for $\varsigma = 2$), within and without perturbations included can be written in an unified manner

$$\varphi_1 \left[g_{\alpha\beta}\, \bar{U}^{\alpha}\, \bar{U}^{\beta}\right] = \bar{g}_{\alpha\beta}\, \bar{U}^{\alpha}\, \bar{U}^{\beta} = g_{\alpha\beta}\, U^{\alpha}\, U^{\beta} = 1, \tag{26}$$

which is a very strong result due to its generality. In this case, the geodesic equations (18) of the perturbed relativistic velocities, take the form

$$\frac{d\bar{U}^{\alpha}}{dS} + \Gamma^{\alpha}_{\mu\beta}\, \bar{U}^{\mu}\, \bar{U}^{\beta} + \eta\,(\varphi_1 - 1)\, \bar{U}^{\alpha} = 0, \tag{27}$$

which take into account the geometric perturbations of the system described on the extended manifold.

3 Collapsing sphere with spatial curvature

To describe the background collapse of a spherically symmetric system with a variable time scale, we examine the line element where the spatial contributions are written in spherical coordinates [16]

$$ds^2 = e^{-2\int \gamma(t)\, dt}\, c^2\, dt^2 - e^{2\int H(t)\, dt} \left[\frac{dr^2}{1 - K\, r^2} + r^2 \left(d\theta^2 + \sin^2(\theta) d\vartheta^2\right)\right], \tag{28}$$

where c is the light velocity in the vacuum, $H(t) = \dot{a}/a(t) < 0$ and $[a(t) - a_f]/[a_0 - a_f] = e^{\int H(t)\, dt}$ is the radius of the collapsing sphere such that a_0 and a_f are respectively the initial and final radius of the collapsing sphere. The spatial curvature will be considered as positive $K = (H_0/c)^2$, where $H_0 > 0$ is a constant. In the following we shall use natural units: $\hbar = c = 1$. In this framework the dynamic spatial curvature will be $\alpha(t) > 1$, and the valid range for the equation of state (50) will be

$$-1 < \bar{\omega} < -1/3. \tag{29}$$

[1] In general, for $b = \varsigma\,\eta$ we obtain that $\varphi = \frac{1}{4}\left(3\varsigma - 4 + \sqrt{9\,\varsigma^2 - 8\,\varsigma}\right)/(\varsigma - 1)$. The equality $\varsigma = 0$ corresponds to $\varphi = 1$, which is related to light geodesics. It can be seen in Eq. (18) that when $\varphi = 1$, then $\bar{U}^{\alpha} = U^{\alpha}$. For values $\varsigma < 0$ we obtain that $g_{\alpha\beta}\bar{U}^{\alpha}\bar{U}^{\beta}$ take negative values. This is not important because it is not a true invariant. The true invarians are given by Eqs. (20) and (22).

The function $\gamma(t)$ considers the variable timescale along the expansion of the universe [16]

$$\gamma \equiv -\frac{\dot{H}}{H}, \tag{30}$$

where the dot denotes derivative with respect to the time variable.

3.1 Lagrangian formalism and semiclassical expansion for the scalar field

To describe a collapse driven by a quantum scalar field $\hat{\varphi}$, we shall consider a Lagrangian density

$$\mathcal{L}_m = \frac{1}{2} g^{\mu\nu} \hat{\varphi}_{,\mu}\, \hat{\varphi}_{,\nu} - V(\hat{\varphi}), \tag{31}$$

where we denote $\partial\hat{\varphi}/\partial x^{\mu} \equiv \hat{\varphi}_{,\mu}$. Furthermore, $V(\hat{\varphi})$ is the potential that generates the collapse, which we expect that describe a system that departs from the equilibrium. Here, the scalar field is minimally coupled to gravity. Due to the quantum nature of $\hat{\varphi}$, we must adopt a quantization

$$\left[\hat{\varphi}(x^{\mu}), \hat{\Pi}^0(x'^{\mu})\right] = i\, \delta^{(3)}\left(x^{\mu} - x'^{\mu}\right), \tag{32}$$

where the 0-canonical momentum associated to $\varphi(x^{\mu})$, is defined by

$$\hat{\Pi}^0(x^{\mu}) = \frac{\delta}{\delta\hat{\varphi}_{,0}} \left[\sqrt{-g}\, \mathcal{L}_m\right], \tag{33}$$

which, for the metric (28) and using natural units, takes the form

$$\hat{\Pi}^0(x^{\mu}) = \left(\frac{H_0}{H}\right) e^{3\int H(t)\, dt} \frac{r^2\, \sin(\theta)}{\sqrt{1 - H_0^2\, r^2}}\, \dot{\hat{\varphi}} = \left(\frac{H_0}{H}\right) \left(\frac{a(t) - a_f}{a_0 - a_f}\right)^3 \frac{r^2\, \sin(\theta)}{\sqrt{1 - H_0^2\, r^2}}\, \dot{\hat{\varphi}}. \tag{34}$$

In order to describe the background dynamics of $\hat{\varphi}$ and their fluctuations, we shall propose a semiclassical expansion for φ: [17–21]

$$\hat{\varphi}\left(x^{\alpha}\right) = \langle V|\, \hat{\varphi}\left(x^{\alpha}\right) |V\rangle + \hat{\delta\varphi}\left(x^{\alpha}\right). \tag{35}$$

The expectation value of the scalar field $\varphi\left(x^{\alpha}\right)$, is only a function of time: $\langle V|\, \hat{\varphi}\left(x^{\alpha}\right) |V\rangle = \phi_c(t)$, such that, for consistency, the expectation value for the scalar field fluctuations is null:

$$\langle V|\, \delta\varphi\left(x^{\alpha}\right) |V\rangle = 0, \tag{36}$$

where $\left|V\right\rangle$ is a quantum state on the Riemann manifold in the Heisenberg representation, where quantum operators depends on x^μ, but the quantum states are squeezed. In this work we are dealing with a spherically symmetric, but inhomogeneous system, so that we shall assume that the expectation value of a certain quantum operator $\hat{O}(x^\mu)$ can be expressed as the space $3d$-volumetric expectation value $\langle V|\,\hat{O}(x^\mu)\,|V\rangle = O(t)$.

3.2 Scalar field fluctuations and back-reaction effects

The dynamics of the scalar field fluctuations will be given by the differential equation

$$\ddot{\hat{\delta\varphi}} + \left(3H - \frac{\dot{H}}{H}\right)\dot{\hat{\delta\varphi}} - \left(\frac{H}{H_0}\right)^2 e^{-2\int H(t)\,dt}\,\nabla^2\hat{\delta\varphi} + \bar{\Upsilon}''(\phi_c)\,\hat{\delta\varphi} = 0, \tag{37}$$

where $\gamma = -\dot{H}/H > 0$, so that the system dissipates energy during the collapse. Furthermore, we shall use the fact that

$$\nabla^2\hat{\delta\varphi} \equiv \left(1 - K\,r^2\right)\frac{\partial^2\hat{\delta\varphi}}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\hat{\delta\varphi}}{\partial\theta^2} + \frac{1}{r^2\,\sin^2(\theta)}\frac{\partial^2\hat{\delta\varphi}}{\partial\vartheta^2} + \left(\frac{2}{r} - Kr\right)\frac{\partial\hat{\delta\varphi}}{\partial r} + \frac{\cot(\theta)}{r^2}\frac{\partial\hat{\delta\varphi}}{\partial\theta}. \tag{38}$$

The scalar field fluctuations $\hat{\delta\varphi}$ are responsible for the back-reaction effects [22–25] that alter the background dynamics of the system and will be taken into account thought the boundary conditions in the varied action.

The physical origin of $\lambda(x)$ is due to the back-reaction effects produced by the quantum field fluctuations $\hat{\delta\varphi} = \hat{\varphi} - \left\langle\hat{\varphi}\right\rangle$ [21], which are related a massive quantum scalar field $\hat{\varphi}$

$$\begin{aligned}\lambda(x) &= \kappa\,\left\langle B\left|\hat{\rho}_T(\hat{\varphi})\right|B\right\rangle - \kappa\,\bar{\rho}_T\left[\phi_c(t)\right]\\ &= \kappa\,\left\langle B\left|\frac{1}{2}g^{00}\,\left(\dot{\hat{\delta\varphi}}\right)^2 - \frac{1}{2}g^{ij}\,\left(\nabla_i\hat{\delta\varphi}\nabla_j\hat{\delta\varphi}\right)\right.\right.\\ &\quad\left.\left. + \sum_{n=1}^{\infty}\frac{1}{n!}\left.\frac{\delta^{(n)}\bar{\Upsilon}(\hat{\varphi})}{\hat{\delta\varphi}^{(n)}}\right|_{\phi_c}\hat{\delta\varphi}^n\right|B\right\rangle,\end{aligned} \tag{39}$$

such that $\left.\frac{\delta^{(n)}\bar{\Upsilon}(\hat{\varphi})}{\hat{\delta\varphi}^{(n)}}\right|_{\phi_c}$ is the n-th variation of the effective potential $\bar{\Upsilon}$ with respect to the field $\hat{\varphi}$, evaluated on the background, with a field expectation value $\phi_c(t)$. This effect is represented by a cosmological parameter [see Eq. (39)], whose effects are proportional to $\left\langle B\left|\hat{\rho}_T(\varphi)\right|B\right\rangle - \bar{\rho}_T\left[\phi_c(t)\right]$, where the expectation value of the total energy density is given by

$$\left\langle B\left|\hat{\rho}_T(\hat{\varphi})\right|B\right\rangle = \left\langle B\left|\frac{1}{2}g^{00}\,\left(\dot{\hat{\varphi}}\right)^2 - \frac{1}{2}g^{ij}\,\left(\nabla_i\hat{\varphi}\nabla_j\hat{\varphi}\right) + \bar{\Upsilon}(\hat{\varphi})\right|B\right\rangle, \tag{40}$$

and $\bar{\rho}_T\left[\phi_c(t)\right]$ is the background energy density, given by Eq. (46).

3.3 Background dynamics of the scalar field with external flow included

The dynamics of $\langle V|\,\varphi\,(x^\alpha)\,|V\rangle = \phi_c(t)$ is described by the equation

$$\ddot{\phi}_c + \left[3\,H - \frac{\dot{H}}{H}\right]\dot{\phi}_c + \left.\frac{\delta\bar{\Upsilon}(\varphi)}{\delta\varphi}\right|_{\varphi\equiv\phi_c} = 0, \tag{41}$$

where the effective potential $\bar{\Upsilon}(\phi_c)$, which also takes into account back-reaction effects, is

$$\begin{aligned}\bar{\Upsilon}(\phi_c) &= \left(\frac{H}{H_0}\right)^2\left[V(\phi_c) - b\,\frac{\langle V|\,\delta\hat{\Theta}\,|V\rangle}{\kappa}\right]\\ &= \left(\frac{H}{H_0}\right)^2\left[V(\phi_c) + \frac{\lambda(x)}{\kappa}\right],\end{aligned} \tag{42}$$

where we have considered $\lambda(x)$ as a general function of the coordinates. The background Einstein equations with the boundary terms included, are given by

$$3H(t)^2\left[1 + \alpha(t)\right] = \kappa\left[\frac{\dot{\phi}_c^2}{2} + \bar{\Upsilon}(\phi_c)\right], \tag{43a}$$

$$-H(t)^2\left[3 + \alpha(t)\right] = \kappa\left[\frac{\dot{\phi}_c^2}{2} - \bar{\Upsilon}(\phi_c)\right], \tag{43b}$$

where the time dependent contribution of the spatial curvature is

$$\alpha(t) = (K/H_0^2)\,e^{-2\int H(t)\,dt}. \tag{44}$$

The background pressure and energy densities, will be defined as

$$\bar{P}_T = \left(\frac{H}{H_0}\right)^2\left[P - \frac{\lambda(x)}{\kappa}\right], \tag{45}$$

$$\bar{\rho}_T = \left(\frac{H}{H_0}\right)^2\left[\rho + \frac{\lambda(x)}{\kappa}\right], \tag{46}$$

where $P = \frac{\dot{\phi_c}^2}{2} - V(\phi_c)$ and $\rho = \frac{\dot{\phi_c}^2}{2} + V(\phi_c)$, are the pressure and energy density in absence of the external flux, for $\gamma = 0$. The kinetic and potential contributions of the energy density can be obtained from Eqs. (43)

$$\frac{\dot{\phi_c}^2}{2} = \frac{H^2(t)}{\kappa}\,\alpha(t), \tag{47}$$

$$\bar{\Upsilon}[\phi_c(t)] = \frac{H^2(t)}{\kappa}\left(3 + 2\alpha(t) + \frac{\lambda(x)}{H_0^2}\right). \tag{48}$$

Using the Eq. (47) with (44), hence we obtain the dynamics for $\phi_c(t)$

$$\phi_c(t) = \sqrt{\frac{2\alpha(t)}{\kappa}}. \tag{49}$$

For $K > 0$, we obtain that $\alpha(t) > 0$ and $\phi_c(t)$ is a real function. On the other hand, from Eqs. (45) and (46), we can obtain the equation of state for the system when $\lambda(x) = 0$: $\bar{\omega}|_{\lambda=0} = \left.\frac{\bar{P}_T}{\bar{\rho}_T}\right|_{\lambda=0}$, in terms of the dynamical spatial curvature $\alpha(t)$

$$\bar{\omega}|_{\lambda=0} = -1 + \frac{2\alpha(t)}{3\,[1+\alpha(t)]}, \tag{50}$$

where the dynamical curvature $\alpha(t)$ is the relevant function that determines the background equation of state of the system. The Eq. (50) give us the equation of state of the system in absence of external flow.

4 A model for Gravastar formation

Gravastars are compact objects formed from the gravitational collapse of matter. They are postulated as objects where the collapse stops before forming a singularity, substituting it with a state of exotic matter. This makes Gravastars a theoretical alternative to black holes. In this section we shall consider a collapsing system which initial with a ratio a_0 and collapses until an asymptotic ratio a_f. To describe the rate of collapsing, we can consider a negative function $H(t)$, which is given by

$$H(t) = -\frac{\beta}{t}, \tag{51}$$

where $1 > \beta = H_0\, t_0 > 0$, $t \geq t_0$ and H_0 is the value when the collapse begins: $H_0 \equiv H(t = t_0)$. Using the fact that $\gamma(t) = -\dot{H}/H$, it is easy to obtain

$$\gamma(t) = \frac{1}{t}. \tag{52}$$

Using the Eq. (30), hence the metric (28) holds

$$ds^2 = \left(\frac{H(t)}{H_0}\right)^2 dt^2 - e^{2\int H(t)\,dt}\left[\frac{dr^2}{1-K\,r^2} + r^2\left(d\theta^2 + \sin^2(\theta)\,d\vartheta^2\right)\right], \tag{53}$$

where $e^{\int H(t)\,dt}$ is the radius $a(t)$ of the collapsing sphere, such that

$$\frac{a(t)-a_f}{a_0-a_f} = e^{\int H(t)\,dt} = \left(\frac{t}{t_0}\right)^{-\beta}, \tag{54}$$

describes the evolution of such radius as a function of time, or as a function of ϕ_c

$$\frac{a[\phi_c(t)]-a_f}{a_0-a_f} = \left(\frac{\phi_c(t)}{\phi_0}\right)^{-1}. \tag{55}$$

Here, a_f is the final radius of the collapsing sphere, and $a_0 > a_f$ is the initial radius of the sphere. During the collapse, $a[\phi_c(t)]$ decreases from the value a_0 to the asymptotic value $\lim_{t\to\infty} a[\phi_c(t)] \to a_f$. The dynamical curvature of the sphere $\alpha(t)$, is a dimensionless function given by

$$\alpha(t) = \alpha_0 \left(\frac{t}{t_0}\right)^{2\beta}. \tag{56}$$

From (50), we obtain the equation of state during the collapse

$$\bar{\omega}(t) = -\frac{1}{3}\frac{\left[3 + \left(\frac{t}{t_0}\right)^{2\beta}\right]}{\left[1 + \left(\frac{t}{t_0}\right)^{2\beta}\right]}. \tag{57}$$

Notice that at the beginning of the collapse, when $t = t_0$, the equation of state for the system is

$$\bar{\omega}(t_0) = -\frac{2}{3}. \tag{58}$$

As the collapse evolves $\lim_{t\to\infty} \bar{\omega}(t) \to -1/3$. Therefore, the system is always accelerated during the collapse. This is because the effective pressure remains null during the collapse.

Since $\dot{\phi_c}^2 = \frac{2\alpha(t)}{\kappa}\, H(t)^2$, from (51) and (56) we can obtain the time evolution for $\phi_c(t)$:

$$\phi_c(t) = \phi_0 \left(\frac{t}{t_0}\right)^{\beta}, \tag{59}$$

where $t \geq t_0$ and $\phi_c(t_0) = \phi_0$. The parameter H can be written in terms of ϕ_c

$$H(\phi_c) = -H_0 \left(\frac{\phi_c}{\phi_0}\right)^{-1/\beta}. \tag{60}$$

Similarly, the dynamics curvature can also be described in terms of ϕ_c:

$$\alpha(\phi_c) = \left(\frac{\phi_c}{\phi_0}\right)^2, \tag{61}$$

so that, using the Eqs. (48) and (60), it is possible to obtain the effective potential as a function of the background scalar field

$$\bar{\Upsilon}(\phi_c) - \bar{\Upsilon}_0 = H_0^2\, \phi_0^2 \left(\frac{\phi_c}{\phi_0}\right)^{\frac{2\beta-2}{\beta}}, \tag{62}$$

where $\bar{\Upsilon}_0$ is the value of the effective potential at the end of the collapse:

$$\lim_{t\to\infty} \bar{\Upsilon}[\phi_c(t)] \to \bar{\Upsilon}_0, \qquad \lim_{t\to\infty} \phi_c(t) \to \infty, \tag{63}$$

for $0 < \beta < 1$. The function γ also can be expressed in terms of the background classical field

$$\gamma(\phi_c) = t_0^{-1} \left(\frac{\phi_c}{\phi_0}\right)^{-1/\beta}. \tag{64}$$

In the following section we shall study a particular case with $\beta = 1/2$, which could be relevant to study the dynamics formation of Gravastars.

5 Example: particular case with $\beta = 1/2$

To illustrate the model described in Sect. 4, we can consider the case where $\beta = 1/2$, which means that $H_0 = 1/(2\,t_0)$. The metric (28) for $\beta = 1/2$ will be valid for $r < 2\,t_0$ in order for describe the interior of the collapsing sphere. In this case $\phi_c(t) = \phi_0 \left(\frac{t}{t_0}\right)^{1/2}$, and therefore we have that

$$H(\phi_c) = -H_0 \left(\frac{\phi_c}{\phi_0}\right)^{-2}, \tag{65}$$

$$\gamma(\phi_c) = t_0^{-1} \left(\frac{\phi_c}{\phi_0}\right)^{-2}, \tag{66}$$

$$\bar{\Upsilon}(\phi_c) - \bar{\Upsilon}_0 = H_0^2\, \phi_0^2 \left(\frac{\phi_c}{\phi_0}\right)^{-2}, \tag{67}$$

meanwhile $\alpha(\phi_c)$ and $a[\phi_c(t)]$ do not depend on the particular β-value and they are given respectively by (61) and (55). Notice that $\alpha(\phi_c)$ increases as t and the radius of the sphere $a[\phi_c(t)]$, decay as $t^{-1/2}$. The curious thing is that $H(\phi_c)$, $\gamma(\phi_c)$ and $\bar{\Upsilon}(\phi_c) - \bar{\Upsilon}_0$, have the same dependency on ϕ_c [see Eqs. (65), (66) and (67)]. Therefore, the absolute value of all of them decay with time as t^{-1}.

5.1 Geodesic solutions for $\beta = 1/2$ and $\epsilon = 1$ and perturbed metric tensor

We shall consider solutions for the geodesic differential equations (18) for $\epsilon = 1$, which corresponds to the metric signature $(+, -, -, -)$. Because we are considering a isotropic but inhomogeneous spacetime described by (28), we must impose that $\bar{U}^2 = \bar{U}^3 = 0$, which means that the angles will be constants: $d\theta = d\phi = 0$. The relevant geodesic equations to be solved will be those for $\bar{U}^0$ and $\bar{U}^1$:

$$\frac{d\bar{U}^0}{dS} + \Gamma^0_{\mu\beta}\, \bar{U}^\mu\, \bar{U}^\beta + \varpi\, \bar{U}^0 = 0, \tag{68}$$

$$\frac{d\bar{U}^1}{dS} + \Gamma^1_{\mu\beta}\, \bar{U}^\mu\, \bar{U}^\beta + \varpi\, \bar{U}^1, \tag{69}$$

where $\varpi = \eta\,(\varphi_1 - 1)$. For an relativistic observer that moves in a radial direction, the 4-length is given by

$$S(t,r) = \sqrt{g_{\alpha\beta}\, x^\alpha x^\beta}\Big|_{\theta=\phi=0} = \sqrt{t_0^2 - \left(\frac{t_0}{t}\right)\frac{r^2}{(1-K\,r^2)}}. \tag{70}$$

In our case, the nonzero Levi-Civita connections are

$$\Gamma^0_{00} = -t^{-1}, \qquad \Gamma^0_{11} = -2\,t0\left(1 - K\,r^2\right), \tag{71}$$

$$\Gamma^1_{11} = -\frac{K\,r}{(K\,r^2-1)}, \qquad \Gamma^1_{01} = \Gamma^1_{10} = -\frac{t^{-1}}{2}. \tag{72}$$

To solve the system (68)–(69), we can introduce the redefined 4-velocity components

$$\tilde{U}^0 = \bar{U}^0 \left(\frac{t_0}{t}\right), \qquad \tilde{U}^1 = \bar{U}^1 \left(\frac{t_0}{t}\right)^{1/2} \left[1 - K\,r^2\right]^{-1/2}, \tag{73}$$

so that the system of differential equations (68)–(69), after taking $\tilde{U}^0 = \cosh(S/S_0)$ and $\tilde{U}^1 = \sinh(S/S_0)$, are

$$\frac{d\tilde{U}^0}{dS} - \frac{1}{2t_0}\left[\left(\tilde{U}^0\right)^2 - 1\right] + \varpi\, \tilde{U}^0 = 0, \tag{74}$$

$$\frac{d\tilde{U}^1}{dS} - \frac{1}{2t_0}\, \tilde{U}^0\, \tilde{U}^1 + \varpi\, \tilde{U}^1 = 0, \tag{75}$$

where we have used the redefinitions (73) for $\beta = 1/2$ and $\epsilon = 1$. For $\eta = 2\,b$, the solutions for $\tilde{U}^0(S)$ and $\tilde{U}^1(S)$, are

$$\tilde{U}^0(S) = \tilde{U}^0(S_0) + \sqrt{1 + [\varpi\, t_0]^2}\, \tanh\left[\frac{\sqrt{1+[\varpi\, t_0]^2}}{2\,t_0}(S - S_0)\right], \tag{76}$$

$$\tilde{U}^1(S) = \tilde{U}^1(S_0)\, \frac{e^{-(3/2)\varpi\,(S-S_0)}}{\sqrt{1 - \tanh^2\left[\frac{\sqrt{1+[\varpi\, t_0]^2}}{2\,t_0}(S - S_0)\right]}}, \tag{77}$$

where $t_0 > (S - S_0) \geq 0$, $\tilde{U}^0(S_0) = -\varpi\, t_0 \geq 0$, $S_f \equiv S(t \to \infty, r \to 0) \to t_0$. Finally, using the fact that $S(t,r)$ is given by (70), we obtain the nonzero relativistic velocity components in terms of $t \geq t_0$ and $r > 1/\sqrt{K} \equiv 2\,t_0$:

$$
\begin{aligned}
\bar{U}^0(t,r) &= \left(\frac{t}{t_0}\right)\Bigg\{-\varpi\, t_0\\
&+\sqrt{1+[\varpi\, t_0]^2}\,\tanh\left[\frac{\sqrt{1+[\varpi\, t_0]^2}}{2\,t_0}\right.\\
&\left.\times\left[\sqrt{t_0^2-\left(\frac{t_0}{t}\right)\frac{r^2}{(1-K\,r^2)}}-S_0\right]\right]\Bigg\}, \qquad (78)
\end{aligned}
$$

$$
\begin{aligned}
&\bar{U}^1(t,r)\\
&= \tilde{U}^1(S_0)\left(\frac{t}{t_0}\right)^{1/2}\left[1-K\,r^2\right]\\
&\frac{e^{-(3/2)\varpi\left[\sqrt{t_0^2-\left(\frac{t_0}{t}\right)\frac{r^2}{(1-K\,r^2)}}-S_0\right]}}{\sqrt{1-\tanh^2\left[\frac{\sqrt{1+[\varpi\, t_0]^2}}{2\,t_0}\left[\sqrt{t_0^2-\left(\frac{t_0}{t}\right)\frac{r^2}{(1-K\,r^2)}}-S_0\right]\right]}}.
\end{aligned}
\qquad (79)
$$

We must require that $\varpi < 0$ in order for $0 < \tilde{U}^0(S_0) \leq 1$. Furthermore, in order for $S(t,r) - S(t_0, r_0) \geq 0$, we must require that $t \geq \frac{1}{t_0}\frac{r_0^2}{1-K\,r_0^2}$, where $r_0 < 1/\sqrt{K}$ is the initial position of the relativistic observer inside the collapsing Gravastar. We can define $S_0 \equiv S(t_0, r_0)$, as the initial 4-length value

$$
S_0 = \sqrt{t_0^2 - \frac{r_0^2}{(1-K\,r_0^2)}}. \qquad (80)
$$

Once obtained the relativistic velocity components, we can calculate the components for the perturbed metric tensor $\bar{g}_{\alpha\beta} = g_{\alpha\beta} + (1/b)\,\delta g_{\alpha\beta}$, by using the expression (17):

$$
\bar{g}_{00} = \left(\frac{t_0}{t}\right)^2\left\{1+\left[2\left(\frac{t_0}{t}\right)^2\left[\bar{U}^0(t,r)\right]^2-\frac{1}{2}\right]\right\}, \qquad (81)
$$

$$
\begin{aligned}
\bar{g}_{11} &= -\left(\frac{t}{t_0}\right)^{-1}\frac{1}{(1-K\,r^2)}\{1\\
&+\left[\left(\frac{t}{t_0}\right)^{-1}\frac{2}{(1-K\,r^2)}\left[\bar{U}^1(t,r)\right]^2-\frac{1}{2}\right]\Bigg\}, \qquad (82)
\end{aligned}
$$

$$
\bar{g}_{01} = \left(\frac{t_0}{t}\right)^3\frac{1}{(1-K\,r^2)}\,\bar{U}^0(t,r)\,\bar{U}^1(t,r), \qquad (83)
$$

$$
\bar{g}_{22} = g_{22}, \qquad (84)
$$

$$
\bar{g}_{33} = g_{33}, \qquad (85)
$$

where $\bar{U}^0(t,r)$ and $\bar{U}^1(t,r)$ are given respectively by the expressions (78) and (79). Notice that when $\eta \rightarrow 0$, the geometric perturbations of the background are negligible and $\bar{g}_{\alpha\beta} \rightarrow g_{\alpha\beta}$. In order to assure a non-perturbative formalism, we must preserve the invariant (25), in the perturbed metric tensor:

$$
g_{\alpha\beta}\,g^{\alpha\beta} = \varphi_1\,\bar{g}_{\alpha\beta}\,\bar{g}^{\alpha\beta} = 4, \qquad (86)
$$

where φ_1 is given by (24). Notice that the conditions (86) preserves the number of dimensions. We can consider an interaction parameter

$$
\eta = \mp\frac{m}{r_0\,(m+M)}, \qquad (87)
$$

where m and M are respectively the masses of $\hat{\delta\varphi}$ and $\phi_c(t)$

$$
M^2 \equiv \left.\frac{\delta^2 V(\hat{\varphi})}{\delta\hat{\varphi}^2}\right|_{\hat{\varphi}=\phi_c(t)}, \qquad m^2 \equiv \left\langle B\left|\left.\frac{\delta^2 V(\hat{\varphi})}{\delta\hat{\varphi}^2}\right|_{\hat{\varphi}=\hat{\delta\varphi}}\right|B\right\rangle. \qquad (88)
$$

Furthermore, from the normalized velocities evaluated at $S = S_0$: $\bar{g}_{\alpha\beta}\bar{U}^\alpha\bar{U}^\beta\big|_{S_0} = 1$, we obtain that

$$
\bar{U}^1(S_0) \geq 0. \qquad (89)
$$

Notice that the case with $\bar{U}^1(S_0) = 0$, give us $\tilde{U}^0(S_0) = 1$ that corresponds to a observer co-moving with the collapse. Finally, we can obtain $\lambda(x)$ by using the Eqs. (14), (16):

$$
\lambda(x) = 3\eta\left[\nabla_\mu\bar{U}^\mu + (5\,\eta)\,(\varphi_1 - 1)\right]. \qquad (90)
$$

where $\nabla_\mu\,\bar{U}^\mu = \partial_\mu\,\bar{U}^\mu + \Gamma^\mu_{\nu\mu}\,\bar{U}^\nu$. We can calculate $\lambda(x)$ for the more simple case, where observers fall with the collapse. In this framework, can be demonstrated that observers that move on hypersurfaces

$$
r(t) = \frac{1}{\sqrt{K - \left(\frac{t_0}{t}\right)\frac{1}{(S_0^2 - t_0^2)}}}, \qquad (91)
$$

with $S_0^2 > t_0^2, t > t_0$ and $\tilde{U}^1(S_0) = 0$, will observe a constant parameter [φ_1 is given by Eq. (24)]

$$
\begin{aligned}
\lambda_* &= \left[\frac{9}{2} + 15(\varphi_1 - 1)\right]\eta^2\\
&= \left[\frac{9}{2} + 15(\varphi_1 - 1)\right]\left(\frac{m}{r_0\,(m+M)}\right)^2, \qquad (92)
\end{aligned}
$$

which corresponds to observers which are co-moving with the collapse. Here, r_0 is given by $r(t_0)$:

$$
r_0 \equiv r(t)|_{t=t_0} = \frac{1}{\sqrt{K - \frac{1}{(S_0^2 - t_0^2)}}} > 0, \qquad (93)
$$

and the final radius is

$$
r_f \equiv r(t)|_{t\rightarrow\infty} \rightarrow \frac{1}{\sqrt{K}}, \qquad (94)
$$

The parameter λ_* is related with back-reaction effects of the scalar field fluctuations [see Eq. (39)]

$$\lambda_* = \kappa \left\langle B \left| \hat{\rho}_T(\hat{\varphi}) \right| B \right\rangle - \kappa\, \bar{\rho}_T\left[\phi_c(t)\right] = \left[\frac{9+30(\varphi_1-1)}{2}\right]\eta^2 = \left[\frac{9+30(\varphi_1-1)}{2}\right]\left(\frac{m}{(m+M)}\right)^2 \times \left(K - \frac{1}{(S_0^2-t_0^2)}\right) > 0, \tag{95}$$

that is related to the flow $\langle V|\,\delta\Theta\,|V\rangle = -\lambda_* \left(\delta g^{\alpha\beta}\, g_{\alpha\beta}\right)$. For a co-moving observer with η given by (87), such flow results

$$\langle V|\,\delta\hat{\Theta}\,|V\rangle = \mp 2(\varphi_1-1)\left[9+30(\varphi_1-1)\right]\left(\frac{m}{(m+M)}\right)^3 \left[K - \frac{1}{S_0^2-t_0^2}\right]^{3/2}. \tag{96}$$

Using Eqs. (90) and (87), we obtain the flow of the relativistic velocities:

$$\nabla_\mu \bar{U}^\mu = \left[\frac{\varphi_1-1}{6}\right]\left[30(\varphi_1-2)+9\right]\eta = \pm\left[\frac{m}{r_0\,(m+M)}\right]\left[\frac{\varphi_1-1}{6}\right]\left[30(\varphi_1-2)+9\right], \tag{97}$$

which is a relativistic invariant. This means that the result (97) is valid for any solutions of Eq. (18), for $\bar{U}^\mu$. In this framework, we can rewrite the equation of state for the collapsing system for a co-moving observer that see a parameter (92):

$$\bar{\omega}|_{\lambda_*} = -1 + \frac{2\alpha(t)}{3\left[1+\alpha(t)+\frac{\lambda_*}{3}\right]}, \tag{98}$$

which is the corrected equation of state (50), with the geometric perturbations included for a co-moving observer that falls with the collapse. However, due to the relativistic invariance of $\langle V|\,\delta\hat{\Theta}\,|V\rangle$, and the result (96), we can assure that the equation of state (98) is valid for any relativistic frame.

6 Final comments

The study of Gravastars represents an intriguing avenue in astrophysics, offering a fresh perspective on the nature of compact astrophysical objects and posing challenges to our understanding of gravity and spacetime under extreme conditions when the system avoids the final singularity. In this work, we investigate a model wherein collapse is driven by a scalar field, described through a semiclassical expansion [see Eq. (35)]. Notice that, during the collapse, back-reaction effects play a crucial role in characterizing its dynamics. Employing an extended formalism of General Relativity, we incorporate boundary terms in the varied Einstein–Hilbert action to capture the flux of relativistic velocity components $\bar{U}^\alpha$. These velocities arise from geodesic equations with sources and can be consistently normalized in a relativistic framework where invariants are preserved [see Eqs. (25) and (26)]. The flow of $\bar{U}^\alpha$ reflects the geometric response to back-reaction effects induced by scalar field fluctuations $\delta\hat{\varphi}$, such that the function $\lambda(x)$ is given by the Eq. (39). We have demonstrated that since $\lambda(x)$ is a relativistic invariant given by the constant λ_*; it do not depends on the motion of the relativistic observer [see Eqs. (90) and (92)].

Acknowledgements The author acknowledges CONICET, Argentina (PIP 11220200100110CO), and UNMdP (EXA1156/24) for financial support.

Data Availability Statement This manuscript has no associated data. [Authors' comment: Data sharing not applicable to this article as no datasets were generated or analysed during the current study.]

Code Availability Statement This manuscript has no associated code/software. [Authors' comment: Code/Software sharing not applicable to this article as no code/software was generated or analysed during the current study.]

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