DISPERSION APPROACH TO WARD-TAKAHASHI IDENTITIES

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A dispersion formulation of field theories is presented and is applied to the derivation of various Ward-Takahashi identities.

1. Introduction

This is a brief summary of a series of works on the derivation of various W-T identities carried out in collaboration with R. Sasaki. The conventional field theoretical calculations often lead to ambiguities and also miss anomalies. The S matrix theory, on the other hand, is free from these objections, but we do not have a complete set of dispersion relations so that the theory represents an approximate dynamical scheme.

Thus we choose an approach standing midway between them. This approach is based on unitarity and dispersion relations, but the object is not the S matrix but the whole collection of Green's functions for which we have a complete set of dispersion relations. It provides us with a convenient basis for deriving various W-T identities unambiguously.

2. Unitarity and Dispersion Relations

We shall first introduce unitarity and dispersion relations for Green's functions. We consider the neutral scalar theory and define the following renormalized functions:

$$\tau(x_1,\ldots,x_n) = (-i)^n K_{x_1}\ldots K_{x_n} \langle 0|T[\phi(x_1)\ldots\phi(x_n)]|0\rangle \qquad (2.1)$$

The Fourier transform of the τ function represents the S matrix element when all the four-momenta are on the mass shell.

The unitarity condition for the τ functions follows from the LSZ reduction formalism.

$$\tau(x_1,\ldots,x_n) + \tau^*(x_1,\ldots,x_n)$$
$$+ \sum_{\text{comb}}{}' \sum_{\ell=0}^{\infty} \frac{i^\ell}{\ell!}\int (du)(dv)\tau(x'_1,\ldots,x'_k,u_1,\ldots,u_\ell)$$
$$\times\ \Delta^{(+)}(u_1-v_1)\ldots\Delta^{(+)}(u_\ell-v_\ell)\tau^*(x'_{k+1},\ldots,x'_n,v_1,\ldots,v_\ell)$$
$$= 0\ , \tag{2.2}$$

where $i\Delta^{(+)}$ is the contraction function and the summation has to be taken over all possible divisions of $(x_1,\ldots,x_n)$ into two groups, one entering τ and the other entering τ^* excluding k=0 and n.

Next we consider a local operator A(x) and define

$$\tau_A(x;x_1,\ldots,x_n)=(-i)^{n+1}K_{x_1}\ldots K_{x_n}\langle 0|T[A(x)\phi(x_1)\ldots\phi(x_n)]|0\rangle \tag{2.3}$$

An arbitrary matrix element of A(x) is obtained from (2.3) by taking its Fourier transform with the help of the LSZ reduction formula. The unitarity condition for this set is given by

$$\tau_A(x;x_1,\ldots,x_n) + \tau_A^*(x;x_1,\ldots,x_n)$$
$$+ \sum_{\text{comb}} \sum_{\ell=0}^{\infty} \frac{i^\ell}{\ell!}\int (du)(dv)[\tau_A(x;x'_1,\ldots,x'_k,u_1,\ldots,u_\ell)$$
$$\times\ \Delta^{(+)}(u_1-v_1)\ldots\Delta^{(+)}(u_\ell-v_\ell)\tau^*(x'_{k+1},\ldots,x'_n,v_1,\ldots,v_\ell)$$
$$+ (\tau_A\to\tau,\ \tau^*\to\tau_A^*)]$$
$$= 0 \tag{2.4}$$

This unitarity condition is linear in τ_A and τ_A^* so that it will be referred to as the linear unitarity condition.

Next we proceed to dispersion relations. First, we shall denote the connected parts of τ and of τ_A by ρ and ρ_A, respectively, and shall introduce their Fourier transforms by

$$\rho(x_1,\ldots,x_n) = \frac{-i}{(2\pi)^{4(n-1)}}\int (dp)\delta^4(\sum_j p_j)\exp(i\sum_j p_j x_j)$$
$$\times\ \mathcal{G}(p_1,\ldots,p_n), \tag{2.5}$$

$$\rho_A(x;x_1,\dots,x_n) = \frac{-i}{(2\pi)^{4n}}\int(dp)d^4q\delta^4(q+\sum_j p_j)\exp[i(qx+\sum_j p_jx_j)] \times \mathcal{A}(q;p_1,\dots,p_n) \quad (2.6)$$

These functions, $\mathcal{G}$ and $\mathcal{A}$, are functions of scalar products of four-momenta and may eventually be denoted by $\mathcal{G}(p_\alpha p_\beta)$ and $\mathcal{A}(p_\alpha p_\beta)$, respectively. They are known to satisfy the parametric dispersion relations.

$$\mathrm{Re}\begin{pmatrix}\mathcal{G}(p_\alpha p_\beta\cdot\xi)\\ \mathcal{A}(p_\alpha p_\beta\cdot\xi)\end{pmatrix} = \frac{P}{\pi}\int_{\infty}^{\infty}\frac{d\xi'}{\xi'-\xi}\varepsilon(\xi')\mathrm{Im}\begin{pmatrix}\mathcal{G}(p_\alpha p_\beta\cdot\xi')\\ \mathcal{A}(p_\alpha p_\beta\cdot\xi')\end{pmatrix}, \quad (2.7)$$

where ξ is a common scaling parameter to be multiplied into all the scalar products of the form $p_\alpha p_\beta$. Now we have a complete set of dispersion relations in the sense that Eq.(2.7) is valid for any n except we need subtractions in some cases.

3. Subtractions and Power Counting

Subtractions in dispersion relations are so introduced as to reproduce the renormalized perturbation theory. This is our guiding principle to settle the subtraction conditions. As a simple example we shall consider the theory corresponding to the following Lagrangian:

$$\mathcal{L} = -\frac{1}{2}[(\partial_\mu\phi)^2 + m^2\phi^2] - \frac{\lambda}{24}\phi^4 \quad (3.1)$$

The subtraction conditions for the two-point function $\mathcal{G}^{(2)}(p)=\mathcal{G}^{(2)}(p,-p)$ follow from $\mathcal{G}^{(2)}(p)=-(p^2+m^2)^2\Delta_F'(p)$ and the Lehmann representation.

$$\mathcal{G}^{(2)}(p) = 0, \qquad \frac{\partial}{\partial p^2}\mathcal{G}^{(2)}(p) = -1 \quad \text{for } p^2+m^2=0. \quad (3.2)$$

The four-point function, in the lowest non-vanishing order, is given by

$$\mathcal{G}^{(4)} = \lambda \quad (3.3)$$

suggesting one subtraction for the four-point function. Thus we assume

$$\mathrm{Re}\,\mathcal{G}^{(4)}(p_\alpha p_\beta\cdot\xi) = \mathcal{G}^{(4)}(0)+\frac{\xi}{\pi}\int\frac{d\xi'}{\xi'(\xi'-\xi)}\varepsilon(\xi')\mathrm{Im}\,\mathcal{G}^{(4)}(p_\alpha p_\beta\cdot\xi') \quad (3.4)$$

When higher order corrections are taken into account we may fix the subtraction condition as

$$\mathcal{G}^{(4)}(0) = \lambda \tag{3.5}$$

corresponding to the subtraction point $p_1=p_2=p_3=p_4=0$.

In order to study the subtraction problem in perturbation theory we shall consider the ultraviolet asymptotic behavior of $\mathcal{G}(p_\alpha p_\beta \cdot \xi)$ for large values of ξ. We shall assume a power law:

$$\mathcal{G}^{(n)}(p_\alpha p_\beta \cdot \xi) \sim \xi^{c(n)/2} \quad \text{for } \xi \to \infty \tag{3.6}$$

except possibly for logarithmic factors. In perturbation theory the powers $c(n)$ are determined by combining unitarity with the assumed dispersion relations. Namely, the power of the absorptive part of a $\mathcal{G}$ function can never exceed that of the full $\mathcal{G}$ function so that we obtain, by rewriting the absorptive part by means of the unitarity condition, the following set of inequalities:

$$d(n) \geqq \underset{\substack{k+\ell>2 \\ n-k+\ell>2}}{\mathrm{Max}}[d(k+\ell)+d(n-k+\ell)] \tag{3.7}$$

where $d(n)=c(n)+n-4$. From the subtraction conditions (3.2) and (3.5) we find

$$c(2) \geqq 2, \qquad c(4) \geqq 0 \tag{3.8}$$

The only solution of (3.7) and (3.8) is given by

$$d(n) = 0, \text{ or } c(n) = 4 - n \tag{3.9}$$

This result can be extended to more general cases.

When we have both spinless and spinor fields the τ functions are defined

$$\begin{aligned} &\tau(x_1,\ldots,x_n;y_1,\ldots,y_\ell;z_1,\ldots,z_\ell) \\ &= (-i)^n K_{x_1}\ldots K_{x_n} D_{y_1}\ldots D_{y_\ell} \overleftarrow{D}_{z_1}\ldots \overleftarrow{D}_{z_\ell} \\ &\times \langle 0|T[\phi(x_1)\ldots\phi(x_n)\psi(y_1)\ldots\psi(y_\ell)\bar{\psi}(z_1)\ldots\bar{\psi}(z_\ell)]|0\rangle, \end{aligned} \tag{3.10}$$

where $D=\gamma\partial+M$, $\tilde{D}=\gamma^T\partial-M$, and M is the fermion mass. In this case we introduce the power $c(n,m)$ with $m=2\ell$ and

$$d(n,m) = c(n,m) + n + \tfrac{3}{2}m - 4 \qquad (3.11)$$

Then we find that renormalizable theories are characterized by

$$d(n,m) = 0 \qquad (3.12)$$

For the function $\mathcal{A}^{(n,m)}$ we define the power $a(n,m)$ just as we introduced $c(n,m)$ for $\mathcal{G}^{(n,m)}$, and define

$$b(n,m) = a(n,m) + n + \tfrac{3}{2}m - 4 \qquad (3.13)$$

Then from the linear unitarity condition and (3.12) we find

$$b(n,m) = b_0, \text{ constant.} \qquad (3.14)$$

This parameter b_0 is referred to as the index of the set $\{\mathcal{A}\}$. As an example we shall consider

$$A(x) = \tfrac{1}{2}\phi^2(x). \qquad (3.15)$$

In the free field approximation we have

$$\mathcal{A}^{(2)}(q;p_1,p_2) = 1, \text{ all others} = 0 \qquad (3.16)$$

This suggests $a(2)=0$ so that $b_0=-2$. When higher order corrections are included we formulate the subtraction condition as

$$\mathcal{A}^{(2)}(0;p,-p) = 1 \quad \text{for } p^2 + m^2 = 0 \qquad (3.17)$$

The relation (3.15) is rather symbolic and our $\mathcal{A}$ functions are regularized through subtracted dispersion relations.

4. Linear Identities

In this section we study linear relationships among different sets of $\mathcal{A}$ functions. The W-T identities fall into this category. We present a basic theorem on the linear relationship in the scalar theory.

Theorem: Let $\{\mathcal{A}\}$, $\{\mathcal{B}\}$, $\{\mathcal{C}\}$ and $\{\mathcal{D}\}$ be sets of functions of indices 0 or -2, transforming as scalar functions and satisfying the linear unitarity condition, then they are linearly dependent.

As a consequence of this theorem we find a linear relationship of the form:

$$a\,\mathcal{A}^{(n)} + b\,\mathcal{B}^{(n)} + c\,\mathcal{C}^{(n)} + d\,\mathcal{D}^{(n)} = 0 \qquad (4.1)$$

When all the subtraction conditions for the sets are given, these sets of functions are uniquely determined at least in the sense of perturbation theory. Now take a linear combination

$$\mathcal{X}^{(n)} = a\,\mathcal{A}^{(n)} + b\,\mathcal{B}^{(n)} + c\,\mathcal{C}^{(n)} + d\,\mathcal{D}^{(n)} \qquad (4.2)$$

The index of the set $\{\mathcal{X}\}$ is assumed to be equal to 0 without loss of generality. The set $\{\mathcal{X}\}$ satisfies the linear unitarity condition and is determined uniquely when the following three subtraction constants are given:

$$\mathcal{X}^{(2)}(0;p,-p) \quad \text{and} \quad \frac{\partial}{\partial p^2}\mathcal{X}^{(2)}(0;p,-p) \quad \text{for } p^2 + m^2 = 0, \qquad (4.3)$$
$$\mathcal{X}^{(4)}(0;0,0,0,0)$$

These three constants are linear in a,b,c and d, so that we can always make them vanish by choosing the four coefficients appropriately, and then we have

$$\mathcal{X}^{(n)} = 0 \qquad \text{for all } n \qquad (4.4)$$

In this way the relationship (4.1) is obtained.

5. Ward-Takahashi Identities

The theorem given in the preceding section can trivially be generalized and we shall give some examples.

First, we shall derive anomalous trace identities and the Callan-Symanzik equations. For this purpose we introduce the energy-momentum tensor $T_{\mu\nu}$ and the corresponding set of functions.

$$\{\mathcal{T}_{\mu\nu}(q;p_1,\ldots,p_n)\} \qquad (5.1)$$

This set satisfies the following W-T identities in the scalar case:

$$q_\mu \mathcal{T}_{\mu\nu}(q;p_1,\ldots,p_n) = \sum_j \frac{p_j^2+m^2}{(p_j+q)^2+m^2-i\varepsilon}(p_j+q)_\nu \mathcal{G}(p_1,\ldots,p_j+q,\ldots,p_n) \tag{5.2}$$

The set $\{\mathcal{T}_{\mu\nu}\}$ as the solution of the above equations is uniquely determined for $q=0$. Then we introduce a new set corresponding to $T=m^2\phi^2$:

$$\{\mathcal{T}(q;p_1,\ldots,p_n)\} \tag{5.3}$$

We further introduce two other sets:

$$\{\sum_j \frac{p_j^2+m^2}{(p_j+q)^2+m^2-i\varepsilon} \mathcal{G}(p_1,\ldots,p_j+q,\ldots,p_n)\}, \tag{5.4}$$

$$\{\frac{\partial}{\partial\lambda} \mathcal{G}(p_1,\ldots,p_n)\} \tag{5.5}$$

The indices of these sets are equal to 0 except that the index of $\{\mathcal{T}\}$ is equal to -2. Application of the theorem leads to the relation

$$\mathcal{T}_{\mu\mu}(0;p_1,\ldots,p_n) - \mathcal{T}(0;p_1,\ldots,p_n) - nd_\phi(\lambda)\,\mathcal{G}(p_1,\ldots,p_n)$$
$$= \beta(\lambda)\frac{\partial}{\partial\lambda}\mathcal{G}(p_1,\ldots,p_n) \tag{5.6}$$

These are precisely the anomalous trace identities.

Now let us denote the Green's functions defined by dropping the Klein-Gordon operators in Eqs.(2.1) and (2.3) by printed letters instead of script letters, then (5.6) clearly holds also for the printed letters. Furthermore, we have for the latter

$$T_{\mu\mu}(0;p_1,\ldots,p_n) = (n - m\frac{\partial}{\partial m})G(p_1,\ldots,p_n) \tag{5.7}$$

Substituting (5.7) into the printed version of (5.6) we arrive at the Callan-Symanzik equations:

$$[m\frac{\partial}{\partial m} + \beta(\lambda)\frac{\partial}{\partial\lambda} + n\gamma_\phi(\lambda)]G(p_1,\ldots,p_n) = -T(0;p_1,\ldots,p_n), \tag{5.8}$$

where $\gamma_\phi=d_\phi-1$ is the anomalous dimension of the field ϕ and is defined as the following expansion coefficient:

$$\mathcal{T}(0;p,-p) = -2m^2 + 2\gamma_\phi(p^2 + m^2) + O((p^2 + m^2)^2) \qquad (5.9)$$

Next, we shall apply a similar argument to the derivation of the Adler anomaly for the axial vector current in QED.

$$A_\lambda = i\bar{\psi}\gamma_\lambda\gamma_5\psi, \quad P = i\bar{\psi}\gamma_5\psi, \quad C = EH = \frac{i}{8}\varepsilon_{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta} \qquad (5.10)$$

Application of the theorem then leads to the following W-T identities:

$$iq_\lambda \mathcal{A}_\lambda = 2m\mathcal{P} + \mathcal{W} - \frac{2\alpha}{\pi}\mathcal{C} \qquad (5.11)$$

The last term is the so-called Adler anomaly, and the set of functions $\{\mathcal{W}\}$ is defined by

$$\mathcal{W}(q;k_1,\ldots,k_n;p_1,\ldots,p_\ell;\bar{p}_1,\ldots,\bar{p}_\ell)$$
$$= \sum_{j=1}^{\ell}[(ip_j\gamma + m)(-i(p_j + q)\gamma + m)^{-1}(i\gamma_5)_j$$
$$\times \mathcal{G}(k_1,\ldots,k_n,p_1,\ldots,p_j+q,\ldots p_\ell,\bar{p}_1,\ldots,\bar{p}_\ell)$$
$$+ \mathcal{G}(k_1,\ldots,k_n,p_1,\ldots,p_\ell,\bar{p}_1,\ldots,\bar{p}_j+q,\ldots,\bar{p}_\ell)$$
$$\times (i\gamma_5)_j(i(\bar{p}_j+q)\gamma+m)^{-1}(-i\bar{p}_j\gamma+m)]. \qquad (5.12)$$

It is characteristic of this set $\{\mathcal{W}\}$ that all members vanish identically when all the momenta $p_1,\ldots,p_\ell,\bar{p}_1,\ldots,\bar{p}_\ell$ are on the mass shell.

There are further applications of the theorem and we shall mention only the results.

(1) The W-T identities in quantum electrodynamics have no anomalous terms and the derivation is straightforward.

(2) It is possible to prove that the Schwinger term in spinor electrodynamics is a c number.

(3) R. Sasaki has succeeded in formulating the σ model entirely in terms of renormalized expressions alone.

References

1. K. Nishijima, Progr. Theor. Phys. 51, 1193 (1974)
2. K. Nishijima and R. Sasaki, Progr. Theor. Phys. 53, No.1 (1975)

Discussion

Slavnov (question): Can you make some comments concerning the applicability of your procedure to the gauge theories. It seems that your original arguments are not strictly applicable to these theories. I mean the positivity of the scalar product.

Nishijima (answer): It is formally possible to incorporate the indefinite metric into our scheme as we are doing in the case of quantum electrodynamics, but we need an extra proof that the S matrix is unitary.