JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 30, Number 3, July 2017, Pages 769–798
http://dx.doi.org/10.1090/jams/870
Article electronically published on September 27, 2016

KINK DYNAMICS IN THE ϕ^4 MODEL: ASYMPTOTIC STABILITY FOR ODD PERTURBATIONS IN THE ENERGY SPACE

MICHAŁ KOWALCZYK, YVAN MARTEL, AND CLAUDIO MUÑOZ

1. Introduction

1.1. **Main result.** In this paper we consider a classical nonlinear equation known as the ϕ^4 model, often used in quantum field theory and other areas of physics. We refer the reader for instance to [33, 41, 52, 55, 59]. In one space dimension, this equation reads

$$\partial_t^2\phi - \partial_x^2\phi = \phi - \phi^3, \quad (t,x) \in \mathbb{R}\times\mathbb{R}. \tag{1.1}$$

Recall that

$$H(x) = \tanh\left(\frac{x}{\sqrt{2}}\right), \tag{1.2}$$

is a time-independent solution of (1.1), called the *kink*. Indeed, H is the unique (up to multiplication by -1), bounded, odd solution of the equation

$$-H'' = H - H^3. \tag{1.3}$$

Note also that (1.1) is invariant under time and space translations and under the Lorentz transformation. Written in terms of the pair $(\phi, \partial_t\phi)$, another important property of (1.1) is the fact that the *energy*,

$$E(\phi,\partial_t\phi) := \int \frac{1}{2}|\partial_t\phi|^2 + \frac{1}{2}|\partial_x\phi|^2 + \frac{1}{4}\left(1-|\phi|^2\right)^2, \tag{1.4}$$

is formally conserved along the flow. In particular the energy of the kink $(H,0)$ is finite and $H^1\times L^2$ perturbations of the kink are referred to as *perturbations in the energy space*. By standard arguments, the Cauchy problem for (1.1) is locally well posed for initial data $(\phi(0),\partial_t\phi(0))$ of the form $(H+\varphi_0^{in},\varphi_1^{in})$ where $(\varphi_0^{in},\varphi_1^{in}) \in H^1\times L^2$. Note also that for odd initial data, the corresponding solution of (1.1) is odd.

Received by the editors July 9, 2015 and, in revised form, July 18, 2016.
2010 *Mathematics Subject Classification.* Primary 35L71; Secondary 35Q51, 37K40.
The first author was partially supported by Chilean research grants Fondecyt 1130126, Fondo Basal CMM-Chile and ERC 291214 BLOWDISOL. The author would like to thank Centre de mathématiques Laurent Schwartz at the Ecole Polytechnique and the Université Cergy-Pontoise where part of this work was done.
The second author was partially supported by ERC 291214 BLOWDISOL.
The third author would like to thank the Laboratoire de Mathématiques d'Orsay where part of this work was completed. His work was partly funded by ERC 291214 BLOWDISOL, and Chilean research grants FONDECYT 1150202, Fondo Basal CMM-Chile, and Millennium Nucleus Center for Analysis of PDE NC130017.

As for the long time behavior of solutions of (1.1), we recall that Henry, Perez and Werszinski [22] proved orbital stability of the kink with respect to small perturbations in the energy space (see Proposition 3.1 and its proof for the special case of odd perturbations). For the rest of this paper, we work in this framework, and we consider only odd perturbations.

Set

$$\phi = H + \varphi_1, \quad \partial_t \phi = \varphi_2, \quad \varphi(t) = \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix}. \tag{1.5}$$

Then, φ satisfies

$$\begin{cases} \partial_t \varphi_1 = \varphi_2 \\ \partial_t \varphi_2 = -\mathcal{L}\varphi_1 - (3H\varphi_1^2 + \varphi_1^3), \end{cases} \tag{1.6}$$

where $\mathcal{L}$ is the linearized operator around H:

$$\mathcal{L} = -\partial_x^2 - 1 + 3H^2 = -\partial_x^2 + 2 - 3\,\mathrm{sech}^2\left(\frac{x}{\sqrt{2}}\right). \tag{1.7}$$

Our main result is the asymptotic stability of the kink of the ϕ^4 model with respect to odd perturbations in the energy space.

Theorem 1.1. *There exists $\varepsilon_0 > 0$ such that for any odd $\varphi^{in} \in H^1 \times L^2$ with*

$$\|\varphi^{in}\|_{H^1\times L^2} < \varepsilon_0,$$

the global solution φ of (1.6) with initial data $\varphi(0) = \varphi^{in}$ satisfies

$$\lim_{t\to\pm\infty} \|\varphi(t)\|_{H^1(I)\times L^2(I)} = 0, \tag{1.8}$$

for any bounded interval $I \subset \mathbb{R}$.

To our knowledge, Theorem 1.1 is the first result on the asymptotic stability of the kink for the standard one dimensional ϕ^4 model, which we see as a classical question in the field. Several previous related results, outlined in Section 1.2 below, suggest that there have been several attempts to solve this problem by various techniques. Moreover, a corollary of Theorem 1.1 is that no odd wobbling kinks (periodic in time, topologically nontrivial solutions) close to the kink exist, which partially settles another longstanding open question in the field (see Remark 1.3). Finally, we believe that our approach, elementary and self-contained, is at the same time general and flexible and opens a new way to prove similar results for related models.

Remark 1.1. We comment here on the notion of asymptotic stability introduced in Theorem 1.1. Observe that if a solution φ of (1.6) satisfies $\lim_{t\to+\infty} \|\varphi(t)\|_{H^1\times L^2} = 0$ then by the orbital stability result [22], $\varphi(t) \equiv 0$ for all $t \in \mathbb{R}$. Thus, the notion of "local" asymptotic stability in the energy space as in (1.8) is in some sense optimal.

The statement of Theorem 1.1 does not contain any information on the decay rate of $\|\varphi(t)\|_{H^1(I)\times L^2(I)}$ as $t \to \pm\infty$. To give a precise answer to this question from our proof, we need to introduce a decomposition of $\varphi(t)$ along the discrete and continuous parts of the spectrum of the operator $\mathcal{L}$, which respectively correspond to internal oscillations and radiation. The operator $\mathcal{L}$ is classical and it is well known (see e.g. [39]) that

$$\operatorname{spec}\mathcal{L} = \left\{0, \frac{3}{2}\right\} \cup [2, +\infty).$$

The discrete spectrum consists of simple eigenvalues $\lambda_0 = 0$ and $\lambda_1 = \frac{3}{2}$, with L^2 normalized eigenfunctions, respectively given by

$$Y_0(x) := 2^{-5/4}3^{1/2}\operatorname{sech}^2\left(\frac{x}{\sqrt{2}}\right), \quad \langle Y_0, Y_0\rangle = 1, \tag{1.9}$$

and

$$Y_1(x) := 2^{-3/4}3^{1/2}\tanh\left(\frac{x}{\sqrt{2}}\right)\operatorname{sech}\left(\frac{x}{\sqrt{2}}\right), \quad \langle Y_1, Y_1\rangle = 1. \tag{1.10}$$

(Here and below $\langle F, G\rangle := \int FG$). Note that $Y_0(x) = 2^{-3/4}3^{1/2}H'(x)$ is related to the invariance of equation (1.1) by space translation. Since we restrict ourselves to odd perturbations of the stationary kink, this direction will not be relevant throughout this work. In contrast, the eigenfunction Y_1, usually referred to as the *internal mode* of oscillation of the kink is not related to any invariance. It introduces serious additional difficulties and plays a key role in the analysis of the long time dynamics of φ. We decompose $\varphi(t)$ in the form

$$\begin{aligned} \varphi_1(t,x) &= z_1(t)Y_1(x) + u_1(t,x), \quad \langle u_1(t), Y_1\rangle = 0,\\ \varphi_2(t,x) &= \lambda_1^{1/2}z_2(t)Y_1(x) + u_2(t,x), \quad \langle u_2(t), Y_1\rangle = 0. \end{aligned} \tag{1.11}$$

Theorem 1.1 will be the consequence of the following global estimate.

Theorem 1.2. *Under the assumptions of Theorem* 1.1*,*

$$\int_{-\infty}^{+\infty}\left(|z_1(t)|^4 + |z_2(t)|^4\right)dt + \int_{-\infty}^{+\infty}\int_{\mathbb{R}}\left((\partial_x u_1)^2 + u_1^2 + u_2^2\right)(t,x)e^{-c_0|x|}dxdt \lesssim \|\varphi^{in}\|^2_{H^1\times L^2}, \tag{1.12}$$

for some fixed $c_0 > 0$.

The information given by (1.12) on the solution $\varphi(t)$ may seem rather weak compared to other existing results on asymptotic stability, but it is not clear to us whether a stronger convergence result can be obtained for perturbations in the energy space. Estimate (1.12) follows from the introduction of a new Virial functional for (1.6). This approach based on Virial functionals is similar and inspired by works of Martel and Merle ([34–36]) on the asymptotic stability of solitons for the subcritical generalized Korteweg-de Vries equations and of Merle and Raphaël [38] in the study of the blow-up dynamics for the mass critical nonlinear Schrödinger equation. This remarkable coincidence shows a deep connection between dispersive and wave problems of seemingly different nature, and the generality of such arguments. However, a new, key feature in our approach is to adapt the Virial functional to take into account the internal oscillation mode (z_1, z_2) associated with the direction of Y_1. This mode is expected to have a slower decay rate as $t \to \pm\infty$, as suggested in (1.12) and we believe that in general $\int_{-\infty}^{+\infty}(|z_1(t)|^2 + |z_2(t)|^2)dt = +\infty$.

Remark 1.2. In our opinion, the case of odd perturbations contains the most difficult and at the same time the most original aspect of the problem which is the exchange of energy between the internal oscillations and the radiation, and the discrepancy of decay rates of the different components of the perturbation. To address this issue we have developed new tools both for the linear and the nonlinear parts of the problem. Asymptotic stability of solitons for nonlinear Schrödinger equations or of kinks in the relativistic Ginzburg-Landau equation in the space of odd perturbations has also been considered previously for instance in [9, 25]. From this point of view the oddness hypothesis in Theorem 1.1 is neither new nor artificial. This being said, we conjecture the asymptotic stability result to be true for general perturbations in the energy space. This would require taking into account the translation invariance of the ϕ^4 model by modulation theory, which is standard in this type of problem (see e.g. [7, 10, 34, 58]), and the emergence of an even resonance at $\lambda = 2$, which is a delicate issue. We expect that to treat the general case, a more refined analysis of the Virial functional introduced in Section 4 of this paper will be needed.

Remark 1.3. The sine-Gordon equation,

$$\Box u + \sin u = 0, \tag{1.13}$$

shares some qualitative properties with the ϕ^4 model, as the existence of an explicit kink solution

$$S(x) = 4\arctan(e^x) = 4\mathrm{Arg}\,(1 + ie^x), \qquad x \in \mathbb{R}.$$

As a classical example of an integrable scalar field equation, it also possesses other exceptional solutions, among them a one parameter family of periodic solutions called *wobbling kinks*, given explicitly by (see Theorem 2.6 in [13])

$$W_\alpha(t,x) = 4\mathrm{Arg}\,\big(U_\alpha(t,x) + iV_\alpha(t,x)\big), \tag{1.14}$$

where

$$U_\alpha(t,x) = 1 + \frac{1+\beta}{1-\beta}e^{\,2\beta x} - \frac{2\beta}{1-\beta}e^{\,(1+\beta)x}\cos(\alpha t),$$
$$V_\alpha(t,x) = \frac{1+\beta}{1-\beta}e^{\,x} + e^{\,(1+2\beta)x} - \frac{2\beta}{1-\beta}e^{\,\beta x}\cos(\alpha t),$$

for $\alpha \in (0,1)$, $\beta = \sqrt{1-\alpha^2}$. Taking $t = \frac{\pi}{2\alpha}$ it is not hard to see that

$$\left\| (S,0) - \left(W_\alpha\left(\frac{\pi}{2\alpha}\right), \partial_t W_\alpha\left(\frac{\pi}{2\alpha}\right)\right) \right\|_{H^1\times L^2} = \mathcal{O}(\beta).$$

Since, at the same time W_α is periodic in time, we see that *the sine-Gordon kink is not asymptotically stable in the energy space* in the sense of Theorem 1.1 (take $0 < \beta \ll 1$). Interestingly, note that changing $v = u - \pi$ in (1.13), which gives $\Box v = \sin v$, helps comparing the situation with the ϕ^4 model. Indeed, the kink solution (now $S - \pi$) is an odd function and it is easy to check that $W_\alpha - \pi$ is also odd. Thus, asymptotic stability fails for this model under the oddness condition. This is a remarkable difference between the two models at the nonlinear level.

We note further that

$$S(x) = 2\pi + \mathcal{O}(e^{\,-x}), \qquad W_\alpha\left(\frac{\pi}{2\alpha}, x\right) = 2\pi + \mathcal{O}(e^{\,-x}), \qquad x \to \infty,$$

with similar formulas when $x \to -\infty$. From these facts and the explicit formula for $\partial_t W_\alpha(\frac{\pi}{2\alpha}, x)$ it is not difficult to see that S and $W_\alpha(\frac{\pi}{2\alpha})$ are also close in Sobolev norms with some exponential weight. It follows that even in a stronger topology, asymptotic stability does not hold for the sine-Gordon equation, in contrast to results proven for example in [40] for the generalized Korteweg-de Vries equation, or in [12] for the cubic one dimensional nonlinear Schrödinger equation.

The problem of constructing or proving nonexistence of wobbling kinks for the ϕ^4 model attracted some attention in the past; for an early discussion, we refer the reader to Segur's work [44]. While there was formal and numerical evidence against the existence of wobbling kinks [29], our result provides a rigorous proof of nonexistence (at least in a neighborhood of the kink and under the oddness condition) that, to our knowledge, had been missing.

1.2. **Discussion of related results.** As we have seen, the question of stability of the kink, as a solution of (1.1), with respect to small and odd perturbations reduces to the stability of the zero solution of the nonlinear Klein-Gordon (NLKG) equation (1.6). Similarly, the question of *asymptotic stability* for H, in a suitable topological space, in principle reduces to the problem of "scattering" of small solutions for (1.6). In particular (1.6) presents several well-known difficulties: it is a *variable-coefficients* NLKG equation, with both nontrivial quadratic and cubic nonlinearities, in one space dimension.

The description of the long time behavior for small solutions to NLKG equations has attracted the interest of many researchers during the last 30 years. Klainerman [23, 24] showed global existence of small solutions in $\mathbb{R}^{1+3}$ via the vector field method, assuming that the nonlinearity is quadratic. Similarly, Shatah [45] considered the NLKG equation with quadratic nonlinear terms in $d \geq 3$ space dimensions. By using Poincaré normal norms suitably adapted to the infinite dimensional Hamiltonian system, he showed global existence for small, sufficiently regular initial data in Sobolev spaces. In both of these approaches the main point is to deal with the quadratic nonlinearity, which complicates the analysis even in dimension 3 and higher due to slow rate of decay for linear Klein-Gordon waves. The situation is known to be even more delicate in low dimensions 1 and 2.

In one dimension, fundamental works due to Delort [14, 15] (see also [16] for the two dimensional setting) shows global existence of small solutions not only for semilinear but also for quasilinear Klein-Gordon equations. See also Lindblad and Soffer [30–32] and Sterbenz [50]. The ϕ^4 model serves as one of the motivations of [32] and [50] since the understanding of the asymptotic stability for H is deeply related to the study of the NLKG equations with variable coefficients and quadratic and cubic nonlinearities. However, the ϕ^4 model does not fit the assumptions made in [32] and [50]. Finally, we mention the works by Hayashi and Naumkin [20, 21] on the modified scattering procedure for cubic and quadratic constant coefficients NLKG equations in one dimension.

In addition to the aforementioned difficulties of the NLKG equations with quadratic and cubic nonlinearities, what makes problem (1.6) challenging is the existence of an internal mode of oscillation. The fundamental work of Soffer and Weinstein [49] seems to be the first in which the mechanism of exchange of energy between the internal oscillations and radiation was fully explained for a class of nonlinear Klein-Gordon equations with potential (see also [48] by the same authors). Although the models they considered do not include the ϕ^4 model, they speculated (see page 19 in [49]) that the phenomena of slow radiation for the ϕ^4 model is due to a similar mechanism. Since the two problems are related we will briefly discuss their approach. They study the question of asymptotic stability of the vacuum state (the zero solution) for the following Klein-Gordon equation in $\mathbb{R}^3 \times \mathbb{R}$:

$$\partial_t^2 u = \left(\Delta - V(x) - m^2\right) u + \lambda u^3, \quad \lambda \in \mathbb{R},\ \lambda \neq 0. \tag{1.15}$$

Under some natural hypothesis on the decay of the potential V and assuming that

(i) the operator $L_V = -\Delta + V + m^2$ has: continuous spectrum $\sigma_{\mathrm{cont}} = [m, \infty)$; a single, positive, discrete eigenvalue $\Omega^2 < m^2$; and the bottom of the continuous spectrum is not a resonance, and

(ii) the Fermi golden rule holds (see Remark 3.2 below),

they show that $u = 0$ is asymptotically stable. Moreover, Soffer and Weinstein proved that the internal oscillation mode decays as $\mathcal{O}(|t|^{-1/4})$, while the radiation decays as $\mathcal{O}_{L^8}(|t|^{-3/4})$. From their result we see the anomalously slow time decay rate of the solution is additionally complicated by the existence of different time decay rates of each component of the solution. This discordance seems to hold in general, and it is also a characteristic of our problem, as expressed for instance in (1.12).

The idea of the proof in [49] is first to project (1.15) onto the discrete and the continuous parts of the spectrum of L_V, with the corresponding components η and z satisfying, respectively, a nonlinear dispersive equation and a Hamiltonian system. In the second step the equation for η is solved for a given z and the result then substituted in the ODE for z. The last step is the identification of the equation of the amplitude $A(t) = |z(t)|$ using the Poincaré normal forms. The second step relies on dispersive theory and this is where the assumption (i) is used; the importance of working on $\mathbb{R}^3$ is evident at this point since dispersive estimates improve with the dimension of space which is important to estimate the nonlinear terms.

For nonlinear Schrödinger equations, the study of the asympotic stability of solitons was initiated by Buslaev and Perelman [5, 6], introducing a spectral decomposition of the solution on eigenspaces associated with the discrete and continuous spectrum of the linearized operator near the solitons. We also refer the reader to [3, 7, 12, 28, 43, 53] and references therein for works related to [5, 6] and [49] concerning the asymptotic stability of the solitons for nonlinear Schrödinger equations. See also [2] for Klein-Gordon equations. This list of references in the subject is not exhaustive.

For generalized (KdV) equations, we refer the reader to works concerning asymptotic stability both for solitons [35, 40] and kinks [37] (using the Miura transform).

For Schrödinger equations with Ginzburg-Landau nonlinearity (such as in the ϕ^4 model), the question of asymptotic stability of topologically nontrivial solitons (i.e. solitons satisfying a nontrivial condition at infinity, such as dark and black solitons) was successfully addressed by different techniques in works by Vartanian [54], Bethuel, Gravejat and Smets [4, 19] and Cuccagna and Jenkins [11]. Note that [4, 19] are surprising extensions of [34, 35] to the one dimensional Gross-Pitaevskii equation. We refer the reader to references in those works for orbital stability results in this context.

As for problems more closely related to our result we mention the work of Kopylova and Komech [25] (see also [26]) where the issue of asymptotic stability of the kink in the following relativistic Ginzburg-Landau equation is addressed:

$$\partial_t^2 u = \partial_x^2 u + F(u), \quad \text{in } \mathbb{R} \times \mathbb{R}. \tag{1.16}$$

The form of the nonlinearity $F = -W'$, where W is a smooth double well potential, guarantees existence of a kink U. More specifically it is assumed that, with some

$m > 0$ and $a > 0$:

$$W(u) = \frac{m^2}{2}(u \mp a)^2 + \mathcal{O}(|u \mp a|^{14}), \qquad \text{as } u \to \pm a. \tag{1.17}$$

This assumption, which excludes the ϕ^4 model, is essential to the method of [25] which is based on Poincaré normal forms and dispersive estimates and is inspired by the work of Buslaev and Sulem [7] (see also [5, 6, 43, 53]). Under further hypothesis of the same type as (i) and (ii) above, [25] shows asymptotic stability of the kink U with respect to odd perturbations. The authors obtain explicit rates of decay: $\mathcal{O}(|t|^{-1/2})$ for the internal oscillations, and $\mathcal{O}_{E_{-\sigma}}(|t|^{-1})$, where $E_{-\sigma} = (1+|x|)^{\sigma} H^1 \times (1+|x|)^{\sigma} L^2_{\sigma}$, $\sigma > 5/2$, for the radiation. We note that in these works, because of the slow decay of the solutions of the free equation in one dimension, the perturbation of the quadratic potential has to be taken sufficiently flat near the limit points $\pm a$ in order to close the nonlinear estimates.

It is also important to mention that Cuccagna [8] proved asymptotic stability of planar wave fronts in the ϕ^4 model in $\mathbb{R}^3$ (study of stability of the one dimensional kink subject to three dimensional perturbations). The method used in this paper combines dispersive estimates by Weder [56, 57] (see also [18]), together with Klainerman vectors fields and normal forms. The fact that the space dimension is 3 with better decay estimates for free solutions is essential in order to close the nonlinear estimates.

2. Outline of the proof

The method of the present paper, based on the use of a Virial functional, is inspired by the one introduced for the generalized KdV equation in [34, 35]. This approach is both self-contained and elementary. Below we present the key ideas.

1. *Spectral decomposition.* It is essential to decompose the solution φ of (1.6) to separate the mode of internal oscillations (associated with the eigenfunction Y_1) from the radiation (associated with the continuous part of the spectrum). Indeed, these components have specific asymptotic behavior as $t \to +\infty$. With the notation

$$\langle F, G \rangle := \int FG,$$

we define

$$z_1(t) := \langle \varphi_1(t), Y_1 \rangle, \quad z_2(t) := \frac{1}{\mu}\langle \varphi_2(t), Y_1 \rangle, \quad \mu := \sqrt{\frac{3}{2}}, \tag{2.1}$$

$$u_1(t) := \varphi_1(t) - z_1(t) Y_1, \quad u_2(t) := \varphi_2(t) - \mu z_2(t) Y_1, \tag{2.2}$$

so that

$$\forall t \in \mathbb{R}, \ \langle u_1(t), Y_1(t) \rangle = \langle u_2(t), Y_1(t) \rangle = 0. \tag{2.3}$$

Denote

$$z(t) := \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}, \quad u(t) := \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}. \tag{2.4}$$

Set

$$|z|^2(t) := z_1^2(t) + z_2^2(t), \quad \alpha(t) := z_1^2(t) - z_2^2(t), \quad \beta(t) := 2 z_1(t) z_2(t). \tag{2.5}$$

For the higher order nonlinear terms we will use the notation (we omit the dependence on time)

$$\mathcal{O}_3 := \mathcal{O}(|z|^3, |z| \cdot \|u\|, \|u\|^2),$$

where $\mathcal{O}(\cdot)$ refers to a function which is bounded by a linear combination of its arguments. In this section, we present the argument formally; in particular, we do not specify which norm of u is used. In Section 5, we will give full details on the control of the error terms.

Since $\mathcal{L}Y_1 = \mu^2 Y_1$, we obtain

$$\begin{cases} \dot{z}_1 = \mu z_2 \\ \dot{z}_2 = -\mu z_1 - \dfrac{3}{\mu}\langle HY_1^2, Y_1\rangle\, z_1^2 + F_z, \quad F_z = \mathcal{O}_3. \end{cases} \tag{2.6}$$

In particular,

$$\frac{d}{dt}(|z|^2) = \mathcal{O}_3, \tag{2.7}$$

and

$$\begin{cases} \dot{\alpha} = 2\mu\beta + F_\alpha, \quad F_\alpha = \mathcal{O}_3, \\ \dot{\beta} = -2\mu\alpha + F_\beta, \quad F_\beta = \mathcal{O}_3. \end{cases} \tag{2.8}$$

Moreover, thanks to (2.2) and (1.6), one checks that

$$\begin{cases} \dot{u}_1 = u_2 \\ \dot{u}_2 = -\mathcal{L}u_1 - 2z_1^2 f + F_u, \quad F_u = \mathcal{O}_3; \end{cases} \tag{2.9}$$

where f is an odd Schwartz function given by the expression

$$\begin{aligned} f := \frac{3}{2}\left(HY_1^2 - \langle HY_1^2, Y_1\rangle Y_1\right), \quad &\text{so that } \langle f, Y_1\rangle = 0, \\ &\text{and } \forall x \in \mathbb{R},\ |f(x)| + |f'(x)| \lesssim e^{-\frac{|x|}{\sqrt{2}}}. \end{aligned} \tag{2.10}$$

There is a simple way to replace the term $z_1^2 f$ in the equation of u_2 by a term involving only α without changing the structure of the problem. To this end we introduce a change of unknown

$$\begin{aligned} &v_1(t,x) := u_1(t,x) + |z|^2(t) q(x), \quad \text{where} \quad \mathcal{L}q(x) = f(x); \\ &v_2(t,x) := u_2(t,x), \end{aligned} \tag{2.11}$$

and

$$v(t) := \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}.$$

Note that the existence of a unique odd solution $q \in H^1(\mathbb{R})$ of $\mathcal{L}q = f$ follows from standard ODE arguments. Moreover, q satisfies

$$\forall x \in \mathbb{R}, \quad |q(x)| + |q'(x)| \lesssim e^{-\frac{|x|}{\sqrt{2}}}. \tag{2.12}$$

Then,

$$\begin{cases} \dot{v}_1 = v_2 + F_1, \quad F_1 = \mathcal{O}_3, \\ \dot{v}_2 = -\mathcal{L}v_1 - \alpha f + F_2, \quad F_2 = \mathcal{O}_3. \end{cases} \tag{2.13}$$

Note that

$$0 = \langle f, Y_1 \rangle = \langle \mathcal{L}q, Y_1 \rangle = \langle q, \mathcal{L}Y_1 \rangle = \mu^2 \langle q, Y_1 \rangle$$

and thus $\langle v_1, Y_1 \rangle = \langle v_2, Y_1 \rangle = 0$.

2. *Orbital stability.* Using the stability result of Henry, Perez and Werszinski [22] (see Proposition 3.1 for a short proof of this result for odd perturbations), we infer that if φ^{in} is small enough then $\varphi(t)$ is global in time and uniformly small in $H^1 \times L^2$. It follows that (u_1, u_2), z and thus (v_1, v_2) and α, β are also small, uniformly in time:

$$\forall t \in \mathbb{R}, \quad \|u(t)\|_{H^1 \times L^2} + \|v(t)\|_{H^1 \times L^2} + |z(t)| \lesssim \varepsilon, \quad \varepsilon := \|\varphi^{in}\|_{H^1 \times L^2}.$$

At this point, the system in $(v_1, v_2, \alpha, \beta)$ can be studied by a Virial argument. We present the formal argument, discarding for the moment higher order terms.

3. *Virial type arguments.* The objective of Virial arguments is to prove the following estimate:

$$\int_{-\infty}^{\infty} \left(|z(t)|^4 + \|v(t)\|^2_{H^1_\omega \times L^2_\omega} \right) dt \lesssim \varepsilon^2. \tag{2.14}$$

Here, $H^1_\omega \times L^2_\omega$ means the $H^1 \times L^2$ norm of v with a suitable exponential weight, see (5.3) for a precise definition. The proof of (2.14) requires the use of several functionals which we introduce below.

First, let

$$\mathcal{I} := \int \psi(\partial_x v_1) v_2 + \frac{1}{2} \int \psi' v_1 v_2, \tag{2.15}$$

$$\mathcal{J} := \alpha \int v_2 g - 2\mu\beta \int v_1 g, \tag{2.16}$$

where ψ and g are functions to be chosen (ψ is bounded, increasing; ψ' and g are Schwartz functions). Using the equations for (v_1, v_2) and α, β, we find

$$-\frac{d}{dt}(\mathcal{I} + \mathcal{J}) = \mathcal{B}(v_1) + \alpha \langle v_1, \tilde{h} \rangle + \alpha^2 \langle f, g \rangle + \varepsilon\, \mathcal{O}\left(|z|^4, \|v\|^2_{H^1_\omega \times L^2_\omega} \right),$$

where $\mathcal{B}$ is a quadratic form and $\tilde{h}$ is a given Schwartz function.

Next, using the orthogonality $\langle v_1, Y_1 \rangle = 0$ and the oddness of v_1, one proves that the following coercivity property holds:

$$\mathcal{B}(v_1) + \alpha \langle v_1, \tilde{h} \rangle + \alpha^2 \langle f, g \rangle \gtrsim \|v_1\|^2_{H^1_\omega} + \alpha^2. \tag{2.17}$$

Estimate (2.17) is the key point of this paper. Note that, as in [34, 35, 38], we rely on the numerical computations of some integrals to prove (2.17) (see Remark 4.2 for more details). We also point out that the choice of a suitable function g is related to the Fermi golden rule, see Section 3.2 for details. From the previous observations, we obtain

$$-\frac{d}{dt}(\mathcal{I}+\mathcal{J}) \gtrsim \alpha^2 + \|v_1\|_{H^1_\omega}^2 + \varepsilon\,\mathcal{O}\left(|z|^4, \|v\|^2_{H^1_\omega\times L^2_\omega}\right). \tag{2.18}$$

Second, let $\gamma = \alpha\beta$. Then,

$$\dot\gamma = 2\mu(\beta^2-\alpha^2) + \varepsilon\,\mathcal{O}\left(|z|^4, \|v\|^2_{H^1_\omega\times L^2_\omega}\right). \tag{2.19}$$

Finally, by direct computations, it can be proven that

$$\frac{d}{dt}\int \operatorname{sech}\left(\frac{x}{2\sqrt{2}}\right) v_1 v_2 \gtrsim \|v_2\|^2_{L^2_\omega} + \mathcal{O}\left(|z|^4, \|v_1\|^2_{H^1_\omega}\right). \tag{2.20}$$

Since $|z|^4 = \alpha^2+\beta^2$, we see that for small enough perturbations, integrating in time a suitable linear combination of (2.18), (2.19) and (2.20) gives (2.14).

4. *Convergence to the zero state for a weighted norm.* From (2.14), one deduces

$$\|v_1(t)\|_{H^1_\omega} + \|v_2(t)\|_{L^2_\omega} + |z(t)| \to 0 \quad \text{as } t\to\infty, \tag{2.21}$$

which implies (1.8).

Indeed, from (2.14) it follows that there exists a sequence $t_n \to +\infty$ such that

$$\lim_{n\to+\infty} \|v(t_n)\|_{H^1_\omega\times L^2_\omega} + |z(t_n)| = 0.$$

For $z(t)$, from (2.7),

$$\left|\frac{d}{dt}|z|^4\right| \lesssim |z|^3\left(|z|^2 + \|v\|^2_{H^1_\omega\times L^2_\omega}\right).$$

Integrating on $[t,t_n]$, passing to the limit $n\to+\infty$ and using (2.14), we deduce that $\lim_{t\to+\infty}|z(t)| = 0$.

For $v(t)$, we consider an energy type quantity (at the linear level)

$$\mathcal{H} = \int \left(|\partial_x v_1|^2 + 2|v_1|^2 + |v_2|^2\right)\operatorname{sech}\left(\frac{x}{2\sqrt{2}}\right),$$

and we show that

$$|\dot{\mathcal{H}}| \lesssim \left(|z|^4 + \|v\|^2_{H^1_\omega\times L^2_\omega}\right).$$

As before, integrating on $[t,t_n]$, and using (2.14), we deduce that $\lim_{t\to+\infty}\mathcal{H}(t) = 0$, which proves (2.21).

From the above sketch it is evident that our approach is different from the ones briefly described in Section 1.2 in that it does not rely on dispersive estimates. In this sense our method allows for minimal hypothesis, is robust and possibly applicable to other type of problems. In particular, it could be applicable, at the cost of a possibly more complicated algebra due to the spectrum of the linearized operator, to generic, analytic, nonlinear perturbations of the sine-Gordon equation of the form $\sin u + h(u)$, $h(u) = \sum_{k=2}^{\infty} a_k u^{2k+1}$, $h(u-\pi) = -h(-u)$, at least in the absence of a resonance (see [1] for the resonance issue). Recall the related fact that breathers in the sine Gordon equation do not persist under generic analytic perturbations (see [17] and the references therein, and our recent work [27]).

3. Preliminaries

3.1. **Orbital stability of the kink.** We recall briefly the proof of the stability result of Henry, Perez and Werszinski [22] restricted to odd perturbations. First, we have the following energy conservation for φ: for all t so that $\varphi(t)$ exists in the energy space it holds ($\varphi(0) = \varphi^{in}$)

$$\mathcal{E}(\varphi(t)) := \int \varphi_2^2(t) + \langle \mathcal{L}\varphi_1(t), \varphi_1(t)\rangle + 2\int H\varphi_1^3(t) + \frac{1}{2}\int \varphi_1^4(t) = \mathcal{E}(\varphi^{in}). \tag{3.1}$$

Second, we have the following consequence of the stability of the zero solution for (1.6).

Proposition 3.1 ([22])**.** *There exist $C > 0$ and $\varepsilon_0 > 0$ such that, for any odd $\varphi^{in} \in H^1 \times L^2$, if $\|\varphi^{in}\|_{H^1\times L^2} < \varepsilon_0$, then the solution φ of (*1.6*) with initial data $\varphi(0) = \varphi^{in}$ is global in time in the energy space and satisfies*

$$\forall t \in \mathbb{R}, \quad \|\varphi(t)\|_{H^1\times L^2} < C\|\varphi^{in}\|_{H^1\times L^2}. \tag{3.2}$$

Proof of Proposition 3.1. We begin with the following simple result.

Claim 3.1. *If $\varphi_1 \in H^1(\mathbb{R})$ satisfies $\langle \varphi_1, Y_0\rangle = 0$, then*

$$\langle \mathcal{L}\varphi_1, \varphi_1\rangle \geq \frac{3}{7}\|\varphi_1\|_{H^1}^2.$$

Proof of Claim 3.1. By the spectral properties of $\mathcal{L}$ and the spectral theorem, we have immediately

$$\langle \mathcal{L}\varphi_1, \varphi_1\rangle \geq \frac{3}{2}\|\varphi_1\|_{L^2}^2.$$

Since $\operatorname{sech}^2\big(\frac{x}{\sqrt{2}}\big) \leq 1$,

$$\begin{aligned}
\langle \mathcal{L}\varphi_1, \varphi_1\rangle &= \int (\partial_x\varphi_1)^2 + 2\int \varphi_1^2 - 3\int \operatorname{sech}^2\left(\frac{x}{\sqrt{2}}\right)\varphi_1^2 \\
&\geq \int (\partial_x\varphi_1)^2 + \frac{5}{7}\int \varphi_1^2 - \frac{12}{7}\int \operatorname{sech}^2\left(\frac{x}{\sqrt{2}}\right)\varphi_1^2 \\
&\geq \frac{3}{7}\int (\partial_x\varphi_1)^2 - \frac{3}{7}\int \varphi_1^2 + \frac{4}{7}\langle \mathcal{L}\varphi_1, \varphi_1\rangle \geq \frac{3}{7}\|\varphi_1\|_{H^1}^2.
\end{aligned}$$

This ends the proof of the Claim. □

Going back to the proof of (3.2), on the one hand,

$$\mathcal{E}(\varphi^{in}) \leq \|\varphi_2^{in}\|_{L^2}^2 + 2\|\varphi_1^{in}\|_{H^1}^2 + C\|\varphi_1^{in}\|_{H^1}^3,$$

and on the other hand,

$$\mathcal{E}(\varphi(t)) \geq \frac{3}{7}\left(\|\varphi_2(t)\|_{L^2}^2 + \|\varphi_1(t)\|_{H^1}^2\right) - C(\|\varphi_1(t)\|_{H^1}^3 + \|\varphi_1(t)\|_{H^1}^4).$$

Combining these estimates with the energy conservation (3.1), for $\varepsilon_0 > 0$ small enough, we get the result. □

3.2. **ODE arguments and the Fermi golden rule.** This section concerns the resolution of the equation $(-\mathcal{L}+4\mu^2)G = F$, where $\mu^2 = \frac{3}{2}$. This will be crucial in choosing the function g in the definition of $\mathcal{J}$ in (2.16).

Lemma 3.1.

(i) *Let $F \in L^1(\mathbb{R}) \cap C^1(\mathbb{R})$ be a real-valued function. The function $G \in L^\infty(\mathbb{R}) \cap C^2(\mathbb{R})$ defined by*

$$G(x) = \frac{1}{12}\operatorname{Im}\left(k(x)\int_{-\infty}^{x} \bar{k}F + \bar{k}(x)\int_{x}^{+\infty} kF\right), \tag{3.3}$$

where

$$k(x) = e^{i2x}\left(1 + \frac{1}{2}\operatorname{sech}^2\left(\frac{x}{\sqrt{2}}\right) + i\sqrt{2}\tanh\left(\frac{x}{\sqrt{2}}\right)\right), \tag{3.4}$$

satisfies

$$(-\mathcal{L}+4\mu^2)G = F. \tag{3.5}$$

(ii) *Assume in addition that $F \in \mathcal{S}(\mathbb{R})$. Then,*

$$G \in \mathcal{S}(\mathbb{R}) \quad \Longleftrightarrow \quad \langle k, F\rangle = 0. \tag{3.6}$$

Remark 3.1. Since $k(-x) = \bar{k}(x)$, if F is odd then G defined by (3.3) is also odd and the orthogonality condition in (3.6) reduces to

$$\langle \operatorname{Im}(k), F\rangle = 0. \tag{3.7}$$

Remark 3.2. Note that $(-\mathcal{L}+4\mu^2)k = (-\mathcal{L}+6)k = 0$ (see the proof of Lemma 3.1). Since $(-\mathcal{L}+\frac{3}{2})Y_1 = 0$, we have $\langle k, Y_1\rangle = 0$. Thus, for f defined in (2.10),

$$\langle k, f\rangle = \frac{3}{2}\langle k, HY_1^2\rangle = \frac{3}{2} i\langle \operatorname{Im}(k), HY_1^2\rangle.$$

The fact that

$$\frac{3}{2}\langle \operatorname{Im}(k), HY_1^2\rangle \neq 0 \tag{3.8}$$

can be easily checked by numerical integration (we find $\frac{3}{2}\langle \operatorname{Im}(k), HY_1^2\rangle \approx -0.333$). This condition is a nonlinear version of the Fermi golden rule (from now on we will refer to it as such) whose origin is in quantum mechanics [47], [42, p. 51]. In a similar nonlinear context, it was identified by Sigal [46] for nonlinear wave and Schrödinger equations and by Soffer and Weinstein [49] (see condition (1.8) in [49]) for a nonlinear Klein-Gordon equation in three dimensions. As was pointed out in [49], this nonzero condition guarantees that the internal oscillations are coupled to radiation and, as a consequence, the energy of the system eventually radiates away from the vicinity of the kink, making it asymptotically stable. This behavior is in deep contrast with the (integrable) sine-Gordon equation, for which the Fermi golden rule is not satisfied (simply because 0 is the only discrete eigenvalue for the associated linear operator), and even worse, explicit periodic solutions (wobblers, see (1.14)) do persist.

In the present paper, the Fermi golden rule (3.8) is the key fact which forbids the existence of a solution g of $(-\mathcal{L}+4\mu^2)g = f$ in the energy space. As we will see, in our particular setting, and for purely algebraical reasons, we will use a modified version of (3.8), which reads

$$\frac{3}{2}\left\langle \operatorname{Im}(k), \left(HY_1^2 - Y_1\langle HY_1^2, Y_1\rangle\right)\operatorname{sech}^2\left(\frac{x}{8\sqrt{2}}\right)\right\rangle \approx -0.327. \tag{3.9}$$

This condition is better adapted to the use of weighted Sobolev spaces which we work with; see Section 4.3 and (4.33). Note finally that we can interpret (3.9) in terms of the distorted Fourier transform associated with $\mathcal{L}$ at the point $\xi = 6$, which is precisely given by $\mathrm{Im}(k)$ for odd data.

Proof of Lemma 3.1.

(i) The explicit expression of k in (3.4) was found by Segur [44]. We easily check by direct computation that

$$(-\mathcal{L} + 4\mu^2)k = (-\mathcal{L} + 6)k = 0.$$

Indeed, let $m(x) := 1 + \frac{1}{2}\operatorname{sech}^2\left(\frac{x}{\sqrt{2}}\right) + i\sqrt{2}\tanh\left(\frac{x}{\sqrt{2}}\right)$. Then,

$$m'(x) = -\frac{\sqrt{2}}{2}\tanh\left(\frac{x}{\sqrt{2}}\right)\operatorname{sech}^2\left(\frac{x}{\sqrt{2}}\right) + i\operatorname{sech}^2\left(\frac{x}{\sqrt{2}}\right),$$
$$m''(x) = -\frac{3}{2}\operatorname{sech}^4\left(\frac{x}{\sqrt{2}}\right) + \operatorname{sech}^2\left(\frac{x}{\sqrt{2}}\right) - i\sqrt{2}\tanh\left(\frac{x}{\sqrt{2}}\right)\operatorname{sech}^2\left(\frac{x}{\sqrt{2}}\right),$$

and thus

$$\begin{aligned}(-\mathcal{L} + 6)k &= [(\partial_x^2 + 4)e^{i2x}]m + 4ie^{i2x}m' + e^{i2x}m'' + 3\operatorname{sech}^2\left(\frac{x}{\sqrt{2}}\right)e^{i2x}m \\ &= e^{i2x}\left(4im' + m'' + 3\operatorname{sech}^2\left(\frac{x}{\sqrt{2}}\right)m\right) = 0.\end{aligned}$$

Since $k(x) \sim (1 + i\sqrt{2})e^{i2x}$ as $x \sim +\infty$, the functions k and $\bar{k}$ form a set of independent solutions of $(-\mathcal{L} + 6)G = 0$ (up to multiplicative constants, k and $\bar{k}$ are the so-called Jost functions). Therefore, for $F \in L^1(\mathbb{R}) \cap C^1(\mathbb{R})$, the following real-valued function G

$$G(x) = -\mathrm{Re}\left[\frac{1}{W(k,\bar{k})}\left(k(x)\int_{-\infty}^{x}\bar{k}F + \bar{k}(x)\int_{x}^{+\infty}kF\right)\right],$$

where $W(k,\bar{k}) = k\bar{k}' - k'\bar{k}$ is the Wronskian of k and $\bar{k}$, solves $(-\mathcal{L}+6)G = F$. Since $W(k,\bar{k}) = -12i$, we obtain (3.3).

(ii) By the definition of G in (3.3), we have the following asymptotics at $\pm\infty$:

$$G(x) \sim \frac{1}{12}\mathrm{Im}\left(k(x)\langle\bar{k}, F\rangle\right) \text{ as } x \to +\infty, \quad G(x) \sim \frac{1}{12}\mathrm{Im}\left(\bar{k}(x)\langle k, F\rangle\right) \text{ as } x \to -\infty.$$

Thus, $\lim_{x\to\pm\infty} G = 0$ if and only if $\langle\bar{k}, F\rangle = 0$. Moreover, if $\langle\bar{k}, F\rangle = 0$ and $F \in \mathcal{S}(\mathbb{R})$, it follows directly from (3.3) that $G \in \mathcal{S}(\mathbb{R})$.

□

4. Virial type arguments

Recall the set of coupled equations (2.8) and (2.13), which can be rewritten as a mixed system in $(v_1, v_2, \alpha, \beta)$:

$$\begin{cases}\dot{v}_1 = v_2 + F_1, \\ \dot{v}_2 = -\mathcal{L}v_1 - \alpha f + F_2, \\ \dot{\alpha} = 2\mu\beta + F_\alpha, \\ \dot{\beta} = -2\mu\alpha + F_\beta,\end{cases} \tag{4.1}$$

where $\mathcal{L} = -\partial_x^2 + 2 - 3(1-H^2)$, $\mu = \sqrt{3/2}$, and where f was introduced in (2.10). In this section, we concentrate on the main terms of this system and thus we do not compute explicitly the nonlinear error terms F_1, F_2, F_α and F_β, which will be considered in detail in Section 5.

4.1. **Virial type identities.** For a smooth and bounded function ψ to be chosen later, let

$$\mathcal{I} := \int \psi(\partial_x v_1) v_2 + \frac{1}{2}\int \psi' v_1 v_2 = \int \left(\psi \partial_x v_1 + \frac{1}{2}\psi' v_1\right) v_2. \tag{4.2}$$

First, using (4.1) and integrating by parts,

$$\begin{aligned}
\frac{d}{dt}\int \psi(\partial_x v_1) v_2 &= \int \psi(\partial_x \dot{v}_1) v_2 + \int \psi(\partial_x v_1)\dot{v}_2 \\
&= \int \psi(\partial_x v_2) v_2 + \int \psi(\partial_x v_1)(\partial_x^2 v_1 - 2v_1 + 3(1-H^2)v_1) \\
&\quad - \alpha \int \psi(\partial_x v_1) f + \int \psi\left((\partial_x F_1) v_2 + (\partial_x v_1) F_2\right) \\
&= -\frac{1}{2}\int \psi'\left(v_2^2 + (\partial_x v_1)^2 - 2v_1^2\right) - \frac{3}{2}\int \left(\psi(1-H^2)\right)' v_1^2 \\
&\quad + \alpha \int v_1 (\psi f)' + \int \psi\left((\partial_x F_1) v_2 + (\partial_x v_1) F_2\right).
\end{aligned} \tag{4.3}$$

Second,

$$\begin{aligned}
\frac{d}{dt}\int \psi' v_1 v_2 &= \int \psi' \dot{v}_1 v_2 + \int \psi' v_1 \dot{v}_2 \\
&= \int \psi' v_2^2 + \int \psi' v_1\left(\partial_x^2 v_1 - 2v_1 + 3(1-H^2)v_1\right) - \alpha \int \psi' v_1 f \\
&\quad + \int \psi'\left(F_1 v_2 + v_1 F_2\right) \\
&= \int \psi'\left(v_2^2 - (\partial_x v_1)^2 - 2v_1^2\right) + \frac{1}{2}\int \psi''' v_1^2 + 3\int \psi'(1-H^2)v_1^2 \\
&\quad - \alpha \int \psi' v_1 f + \int \psi'\left(F_1 v_2 + v_1 F_2\right).
\end{aligned} \tag{4.4}$$

Therefore,

$$\begin{aligned}
\frac{d}{dt}\mathcal{I} = -\mathcal{B}(v_1) &+ \alpha \int v_1 \left(\psi f' + \frac{1}{2}\psi' f\right) + \int v_2 \left(\psi \partial_x F_1 + \frac{1}{2}\psi' F_1\right) \\
&- \int v_1 \left(\psi \partial_x F_2 + \frac{1}{2}\psi' F_2\right),
\end{aligned} \tag{4.5}$$

where

$$\mathcal{B}(v_1) := \int \psi'(\partial_x v_1)^2 - \frac{1}{4}\int \psi''' v_1^2 - 3\int \psi H H' v_1^2. \tag{4.6}$$

For a smooth function g to be chosen later, let

$$\mathcal{J} := \alpha \int v_2 g - 2\mu\beta \int v_1 g, \tag{4.7}$$

Then, using (4.1)

$$\begin{aligned}\frac{d}{dt}\mathcal{J} &= \dot\alpha\int v_2 g + \alpha\int \dot v_2 g - 2\mu\dot\beta\int v_1 g - 2\mu\beta\int \dot v_1 g\\ &= \alpha\int\left(-\mathcal{L}g + 4\mu^2 g\right)v_1 - \alpha^2\int fg \\ &\quad + F_\alpha\int v_2 g - 2\mu F_\beta\int v_1 g + \alpha\int F_2 g - 2\mu\beta\int F_1 g.\end{aligned} \tag{4.8}$$

Adding (4.5) and (4.8), we obtain

$$\frac{d}{dt}\left(\mathcal{I}+\mathcal{J}\right) = -\mathcal{D}(v_1,\alpha) + \mathcal{R}_{\mathcal{D}}, \tag{4.9}$$

where

$$\mathcal{D}(v_1,\alpha) := \mathcal{B}(v_1) - \alpha\int v_1\left(\psi f' + \frac{1}{2}\psi' f - \mathcal{L}g + 4\mu^2 g\right) + \alpha^2\int fg, \tag{4.10}$$

and the error term is

$$\begin{aligned}\mathcal{R}_{\mathcal{D}} &:= \int g\left(\alpha F_2 - 2\mu\beta F_1\right) + \int v_2\left(\psi\partial_x F_1 + \frac{1}{2}\psi' F_1 + gF_\alpha\right)\\ &\quad - \int v_1\left(\psi\partial_x F_2 + \frac{1}{2}\psi' F_2 + 2\mu g F_\beta\right).\end{aligned} \tag{4.11}$$

4.2. **Coercivity of the bilinear form $\mathcal{B}$.** Now we choose a specific function ψ and we consider the question of the coercivity of the bilinear form $\mathcal{B}$ given in (4.6). Let $\lambda > 1$ be chosen (we anticipate that in the sequel we set $\lambda = 8$) and in the definition of $\mathcal{I}$, let

$$\psi(x) := \lambda\sqrt{2}H\left(\frac{x}{\lambda}\right) = \lambda\sqrt{2}\tanh\left(\frac{x}{\lambda\sqrt{2}}\right); \quad \zeta(x) := \sqrt{\psi'(x)} = \operatorname{sech}\left(\frac{x}{\lambda\sqrt{2}}\right). \tag{4.12}$$

Note that $\zeta > 0$ everywhere. Let w be the following auxiliary function:

$$w := \zeta v_1. \tag{4.13}$$

First, note that by integration by parts,

$$\begin{aligned}\int w_x^2 &= \int(\zeta\partial_x v_1 + \zeta' v_1)^2 = \int\psi'(\partial_x v_1)^2 + 2\int\zeta\zeta' v_1(\partial_x v_1) + \int(\zeta')^2 v_1^2\\ &= \int\psi'(\partial_x v_1)^2 - \int\zeta\zeta'' v_1^2 = \int\psi'(\partial_x v_1)^2 - \int\frac{\zeta''}{\zeta}w^2.\end{aligned}$$

Thus,

$$\int\psi'(\partial_x v_1)^2 = \int w_x^2 + \int\frac{\zeta''}{\zeta}w^2. \tag{4.14}$$

Second,

$$\int\psi''' v_1^2 = \int\frac{(\zeta^2)''}{\zeta^2}w^2 = 2\int\left(\frac{\zeta''}{\zeta} + \frac{(\zeta')^2}{\zeta^2}\right)w^2.$$

Therefore,

$$\begin{aligned}\mathcal{B}(v_1) &= \int\psi'(\partial_x v_1)^2 - \frac{1}{4}\int\psi''' v_1^2 - 3\int\psi HH' v_1^2\\ &= \int w_x^2 + \frac{1}{2}\int\left(\frac{\zeta''}{\zeta} - \frac{(\zeta')^2}{\zeta^2}\right)w^2 - 3\int\frac{\psi}{\psi'}HH' w^2.\end{aligned} \tag{4.15}$$

Set

$$\mathcal{B}^\sharp(w) := \int \left(w_x^2 - Vw^2\right), \quad \text{where} \quad V := -\frac{1}{2}\left(\frac{\zeta''}{\zeta} - \frac{(\zeta')^2}{\zeta^2}\right) + 3\frac{\psi}{\psi'}HH',$$

so that

$$\mathcal{B}^\sharp(w) = \mathcal{B}(v_1). \tag{4.16}$$

Note that by (4.12) and direct computations,

$$\frac{\zeta''}{\zeta} = \frac{1}{2\lambda^2}(1-2\zeta^2), \quad \frac{(\zeta')^2}{\zeta^2} = \frac{1}{2\lambda^2}(1-\zeta^2),$$
$$\frac{\zeta''}{\zeta} - \frac{(\zeta')^2}{\zeta^2} = -\frac{\zeta^2}{2\lambda^2} = -\frac{1}{2\lambda^2}\operatorname{sech}^2\left(\frac{x}{\lambda\sqrt{2}}\right).$$

Moreover,

$$\frac{\psi}{\psi'} = \lambda\sqrt{2}\tanh\left(\frac{x}{\lambda\sqrt{2}}\right)\cosh^2\left(\frac{x}{\lambda\sqrt{2}}\right).$$

Therefore, $\mathcal{B}^\sharp(w) = \int \left(w_x^2 - Vw^2\right)$, where

$$\begin{aligned} V(x) &= \frac{1}{4\lambda^2}\operatorname{sech}^2\left(\frac{x}{\lambda\sqrt{2}}\right) \\ &\quad + 3\lambda\tanh\left(\frac{x}{\lambda\sqrt{2}}\right)\cosh^2\left(\frac{x}{\lambda\sqrt{2}}\right)\tanh\left(\frac{x}{\sqrt{2}}\right)\operatorname{sech}^2\left(\frac{x}{\sqrt{2}}\right). \end{aligned} \tag{4.17}$$

Recall Y_1 from (1.10). Let

$$Z_1 := Y_1\cosh\left(\frac{x}{\lambda\sqrt{2}}\right) \quad \text{so that} \quad \langle v_1, Y_1\rangle = 0 \iff \langle w, Z_1\rangle = 0. \tag{4.18}$$

Note that Z_1 is odd. In the following we claim the coercivity of the quadratic form $\mathcal{B}^\sharp$ on a space of odd functions which are orthogonal to Z_1. From now on, κ will stand for a generic positive constant whose value may change from line to line.

Lemma 4.1. *Let $\lambda = 8$. There exists $\kappa > 0$ such that, for any odd function $w \in H^1$,*

$$\langle w, Z_1\rangle = 0 \implies \mathcal{B}^\sharp(w) \geq \kappa\int w_x^2. \tag{4.19}$$

Proof. First, note that

$$\mathcal{B}^\sharp(w) = \frac{1}{4}\mathcal{B}_1^\sharp(w) + \mathcal{B}_2^\sharp(w), \tag{4.20}$$

where

$$\mathcal{B}_1^\sharp(w) = \int w_x^2 - \frac{1}{\lambda^2}\operatorname{sech}^2\left(\frac{x}{\lambda\sqrt{2}}\right)w^2;$$
$$\mathcal{B}_2^\sharp(w) = \int\left(\frac{3}{4}w_x^2 - V_2w^2\right),$$
$$V_2 = 3\lambda\tanh\left(\frac{x}{\lambda\sqrt{2}}\right)\cosh^2\left(\frac{x}{\lambda\sqrt{2}}\right)\tanh\left(\frac{x}{\sqrt{2}}\right)\operatorname{sech}^2\left(\frac{x}{\sqrt{2}}\right).$$

The idea is to show that $\mathcal{B}_1^\sharp(w)$ and $\mathcal{B}_2^\sharp(w)$ are both nonnegative.

Step 1. We claim

Claim 4.1. *For all $\lambda_0 > 0$, for any odd function $w \in H^1$,*

$$\int w_x^2 - \frac{1}{\lambda_0^2}\operatorname{sech}^2\left(\frac{x}{\lambda_0\sqrt{2}}\right) w^2 \geq 0. \tag{4.21}$$

Proof of Claim 4.1. By a change of variables, we can restrict ourselves to the case $\lambda_0\sqrt{2} = 1$. Note that the classical operator $-\partial_x^2 - 2\operatorname{sech}^2(x)$ has a continuous spectrum $[0, +\infty)$ and only one negative eigenvalue -1, associated with the even eigenfunction $\operatorname{sech}(x)$ (see Titchmarsh [51, Section 4.18]). Thus, by the spectral theorem, for any odd function $w \in H^1$,

$$\int w_x^2 - 2\operatorname{sech}^2(x) w^2 \geq 0. \qquad \square$$

The rest of the proof is devoted to showing that the second term is nonnegative as well.

Step 2. Let $\lambda = 8$. We observe the following elementary inequality:

$$\forall x \in \mathbb{R}, \quad V_2(x) = 3\lambda \tanh\left(\frac{x}{\lambda\sqrt{2}}\right)\cosh^2\left(\frac{x}{\lambda\sqrt{2}}\right)\tanh\left(\frac{x}{\sqrt{2}}\right)\operatorname{sech}^2\left(\frac{x}{\sqrt{2}}\right) < \frac{21}{10}\operatorname{sech}^2\left(\frac{x}{2}\right). \tag{4.22}$$

(This is easily checked by numerics.) In particular, combining (4.21) with $\lambda_0 = 8$ and (4.22) in (4.20), we have, for any odd function w,

$$\mathcal{B}_2^\sharp(w) \geq \int \left(\frac{3}{4}w_x^2 - \frac{21}{10}\operatorname{sech}^2\left(\frac{x}{2}\right) w^2\right). \tag{4.23}$$

Step 3. Finally, we claim the following coercivity property: there exists $\kappa > 0$ such that

$$\langle w, Z_1\rangle = 0 \quad \Longrightarrow \quad \int \left(\frac{3}{4}w_x^2 - \frac{21}{10}\operatorname{sech}^2\left(\frac{x}{2}\right)w^2\right) \geq \kappa \int w_x^2. \tag{4.24}$$

Note that in view of (4.23), this statement completes the proof of the lemma.

Proof of (4.24). Let

$$\tilde{Y}_1(x) := \sqrt{\frac{15}{8}}\frac{\sinh\left(\frac{x}{2}\right)}{\cosh^3\left(\frac{x}{2}\right)}, \quad \langle \tilde{Y}_1, \tilde{Y}_1\rangle = 1. \tag{4.25}$$

We claim that for any odd function w,

$$\int \left(w_x^2 - 3\operatorname{sech}^2\left(\frac{x}{2}\right) w^2\right) + \langle \tilde{Y}_1, w\rangle^2 \geq 0. \tag{4.26}$$

Indeed, the operator $-\partial_x^2 - 3\operatorname{sech}^2\left(\frac{x}{2}\right)$ is a classical operator which has exactly three negative eigenvalues (see Titchmarsh [51, Section 4.18]) $-\frac{9}{4}$, -1, and $-\frac{1}{4}$, and continuous spectrum $[0, +\infty)$. The eigenvalues $-\frac{9}{4}$ and $-\frac{1}{4}$ are associated with the even eigenfunctions

$$\operatorname{sech}^3\left(\frac{x}{2}\right) \quad \text{and} \quad \operatorname{sech}^3\left(\frac{x}{2}\right) - \frac{4}{5}\operatorname{sech}\left(\frac{x}{2}\right),$$

respectively; on the other hand, -1 is associated with the odd eigenfunction $\tilde{Y}_1$ defined in (4.25). □

For w odd, we use the following (orthogonal) decomposition: $w = w_1 + \langle \tilde{Y}_1, w\rangle \tilde{Y}_1$, where $\langle w_1, \tilde{Y}_1\rangle = 0$. Then, since $\tilde{Y}_1$ is an eigenfunction of $-\partial_x^2 - 3\operatorname{sech}^2\left(\frac{x}{2}\right)$ with eigenvalue -1, we obtain

$$\int w_x^2 - 3\operatorname{sech}^2\left(\frac{x}{2}\right) w^2 = \int (w_1)_x^2 - 3\operatorname{sech}^2\left(\frac{x}{2}\right) w_1^2 - \langle \tilde{Y}_1, w\rangle^2.$$

Additionally, since w_1 is odd and orthogonal to $\tilde{Y}_1$, by the spectral theorem, we obtain

$$\int (w_1)_x^2 - 3\operatorname{sech}^2\left(\frac{x}{2}\right) w_1^2 \geq 0,$$

and (4.26) follows.

Let us come back to the proof of (4.24). We note that, using (4.26),

$$\begin{aligned}\int \left(\frac{3}{4}w_x^2 - \frac{21}{10}\operatorname{sech}^2\left(\frac{x}{2}\right) w^2\right) &= \frac{1}{20}\int w_x^2 + \frac{7}{10}\int \left(w_x^2 - 3\operatorname{sech}^2\left(\frac{x}{2}\right) w^2\right)\\ &\geq \frac{1}{20}\left(\int w_x^2 - 14\langle \tilde{Y}_1, w\rangle^2\right).\end{aligned}$$

Now, we estimate $\langle \tilde{Y}_1, w\rangle^2$ using the orthogonality condition $\langle w, Z_1\rangle = 0$. Let $\nu \in \mathbb{R}$ and

$$\xi_\nu := \tilde{Y}_1 - \nu Z_1.$$

Since $\tilde{Y}_1$ and Z_1 are odd, there exists an even function $\upsilon_\nu \in \mathcal{S}(\mathbb{R})$ such that $\upsilon_\nu' = \xi_\nu$. In particular,

$$\langle \tilde{Y}_1, w\rangle = \langle \xi_\nu, w\rangle = -\langle \upsilon_\nu, w_x\rangle,$$

and using Cauchy-Schwarz's inequality,

$$\langle \tilde{Y}_1, w\rangle^2 \leq \left(\int w_x^2\right)\left(\min_{\nu\in\mathbb{R}}\int \upsilon_\nu^2\right).$$

Note that $\tilde{Y}_1$ in (4.25) and Z_1 in (4.18) are explicit functions, so that ξ_ν and υ_ν are easily computable. A numerical computation gives that

$$\min_{\nu\in\mathbb{R}}\int \upsilon_\nu^2 \approx 0.04 < \frac{1}{14} \approx 0.071.$$

Thus, (4.24) and (4.19) are proved. □

Remark 4.1. In this proof, we have used numerical computations of elementary integrals. We present briefly an alternative proof of the fact that (4.26) implies (4.24) relying more strongly on numerics. Let

$$\mathcal{L}^\sharp := -\partial_x^2 - V \tag{4.27}$$

be the linear operator representing $\mathcal{B}^\sharp$. From the arguments of the proof in [38, Lemma 11], there exists a function $Z_1^\sharp \in L^\infty$ such that $\int ((Z_1^\sharp)_x)^2 < \infty$ and $\mathcal{L}^\sharp Z_1^\sharp = Z_1$. Using numerical computations, we observe

$$\langle \mathcal{L}^\sharp Z_1^\sharp, Z_1^\sharp\rangle = \langle Z_1, Z_1^\sharp\rangle \approx -2.53 < 0. \tag{4.28}$$

We find the function $Z_1^\sharp$ by a shooting method (in particular, we obtain $Z_1'(0) \approx -0.386$), and then $\langle Z_1, Z_1^\sharp\rangle$ by numerical integration. See Remark 4.2 for more details on the method. Using the arguments of the proof of Lemma 13 in [38], (4.28) and (4.26) imply (4.24).

Corollary 4.1. *Let* $\lambda = 8$. *There exists* $\kappa > 0$ *such that for any odd function* $w \in L^\infty$ *with* $w_x \in L^2$,

$$\langle w, Z_1\rangle = 0 \implies \mathcal{B}^\sharp(w) \geq \kappa \int w_x^2. \tag{4.29}$$

Proof. First note that a standard consequence of (4.19) is the following estimate: there exists a constant $C > 0$, depending on $\|Z_1\|_{H^1}$ and the constant in estimate (4.19) of Lemma 4.1, and a constant $\kappa > 0$ (recall that κ is generic), such that for any odd function $w \in H^1$,

$$\mathcal{B}^\sharp(w) + C|\langle w, Z_1\rangle|^2 \geq \kappa \int w_x^2. \tag{4.30}$$

Let χ be a smooth even function such that $\chi(x) = 1$ for $|x| < 1$, $\chi(x) = 0$ for $|x| > 2$ and $0 \leq \chi \leq 1$. For $A > 1$, let $\chi_A(x) = \chi(x/A)$. Let $w \in L^\infty$, odd and such that $w_x \in L^2$, $\langle w, Z_1\rangle = 0$. Set $w_A = w\chi_A$. Then, we claim

$$\|w_x\|_{L^2}^2 - \|\partial_x w_A\|_{L^2}^2 = o_A(1), \quad \int Vw^2 - \int Vw_A^2 = o_A(1), \quad \langle w_A, Z_1\rangle = o_A(1), \tag{4.31}$$

where $o_A(1)$ denotes a function such that $\lim_{A\to+\infty} o_A(1) = 0$ (possibly depending on w). Indeed, from direct computations

$$\left|\int w_x^2 - \int(\partial_x w_A)^2\right| \leq \left|\int w_x^2(1-\chi_A^2)\right| + \left|\int w^2\chi_A''\chi_A\right| \lesssim \int_{|x|>A} w_x^2 + \frac{1}{A}\|w\|_{L^\infty}^2;$$

$$\left|\int Vw^2 - \int Vw_A^2\right| \leq \|w\|_{L^\infty}^2 \int_{|x|>A} V;$$

$$|\langle w_A, Z_1\rangle| \leq \left|\int wZ_1\right| + \left|\int Z_1(w-w_A)\right| \leq \left|\int Z_1(w-w_A)\right| \leq \|w\|_{L^\infty}\int_{|x|>A}|Z_1|.$$

Applying (4.30) to $w_A \in H^1$ (which is odd), we obtain

$$\mathcal{B}^\sharp(w_A) + C|\langle w_A, Z_1\rangle|^2 \geq \kappa \int (w_A)_x^2.$$

Thus, by (4.31),

$$\mathcal{B}^\sharp(w) \geq \kappa \int w_x^2 + o_A(1).$$

and we obtain the result passing to the limit $A \to +\infty$. □

4.3. **Coercivity of the bilinear form $\mathcal{D}$.** Let us come back to the modified quadratic form $\mathcal{D}$ introduced in (4.10). Written in terms of w (see (4.13) and (4.16)), we have

$$\mathcal{D}(v_1,\alpha) = \mathcal{D}^\sharp(w,\alpha) := \mathcal{B}^\sharp(w) - \alpha\int w\frac{1}{\zeta}\left(\psi f' + \frac{1}{2}\psi' f - \mathcal{L}g + 4\mu^2 g\right) + \alpha^2\int fg. \tag{4.32}$$

Now we use a modified version of the Fermi golden rule to choose a particular g. Let k be the function defined in (3.4). Let

$$a := -\frac{\langle \psi f' + \frac{1}{2}\psi' f, \mathrm{Im}(k)\rangle}{\langle \psi' f, \mathrm{Im}(k)\rangle}, \quad \text{so that} \quad \left\langle \psi f' + \left(a + \frac{1}{2}\right)\psi' f, \mathrm{Im}(k)\right\rangle = 0. \tag{4.33}$$

Numerically, $a \approx 0.687$. From condition (3.9) in Remark 3.2, which ensures that $\langle \psi' f, \mathrm{Im}(k)\rangle \neq 0$, and Lemma 3.1, there exists a unique real Schwartz solution g of

$$\mathcal{L}g - 4\mu^2 g = \psi f' + \left(a + \frac{1}{2}\right)\psi' f. \tag{4.34}$$

Moreover, in view of the decay of Y_1 (1.10), the decay of f (2.10), and the explicit formula in (3.3), the function g satisfies

$$\forall x \in \mathbb{R}, \quad |g(x)| + |g'(x)| \lesssim e^{-\frac{|x|}{\sqrt{2}}}. \tag{4.35}$$

Consequently, (4.34) leads to the simplified expression

$$\mathcal{D}^\sharp(w, \alpha) = \mathcal{B}^\sharp(w) + \alpha a \int w\zeta f + \alpha^2 \int fg.$$

Recall that from (4.19) we already know that, under $\langle w, Z_1\rangle = 0$, $\mathcal{B}^\sharp(w)$ is bounded below by $\int w_x^2$. Numerical computations (see Remark 4.2), using (3.3) show that

$$\langle f, g\rangle \approx 0.0163 > 0. \tag{4.36}$$

Now we prove

Lemma 4.2. *Let $\lambda = 8$. There exists $\kappa > 0$ such that for any odd H^1 function w satisfying $\langle w, Z_1\rangle = 0$,*

$$\mathcal{D}^\sharp(w, \alpha) \geq \kappa\left(\alpha^2 + \int w_x^2\right). \tag{4.37}$$

Proof. Denote $h := a\zeta f + bZ_1$, where b is chosen so that (see (4.28))

$$\langle h, Z_1^\sharp\rangle = 0.$$

Here $Z_1^\sharp$ solves $\mathcal{L}^\sharp Z_1^\sharp = Z_1$ and is such that $Z_1^\sharp \in L^\infty$, $(Z_1^\sharp)_x \in L^2$ (see Remark 4.1). Note that h is odd. Let $h^\sharp \in L^\infty$ be the odd function such that

$$\mathcal{L}^\sharp h^\sharp = h \quad \text{and} \quad \int (h_x^\sharp)^2 < \infty. \tag{4.38}$$

Since $\langle w, Z_1\rangle = 0$, we have $a\int w\zeta f = \langle w, h\rangle$, and thus

$$\mathcal{D}^\sharp(w, \alpha) = \mathcal{B}^\sharp(w) + \alpha\langle w, h\rangle + \alpha^2\langle f, g\rangle.$$

Furthermore we observe

$$\langle h^\sharp, Z_1\rangle = \langle h^\sharp, \mathcal{L}^\sharp Z_1^\sharp\rangle = \langle h, Z_1^\sharp\rangle = 0,$$

hence because $h^\sharp$ is odd, by Corollary 4.1,

$$0 < \mathcal{B}^\sharp(h^\sharp) = \langle \mathcal{L}^\sharp h^\sharp, h^\sharp\rangle = \langle h^\sharp, h\rangle.$$

Given w odd, we decompose it as follows:

$$w = w^\perp + c\, h^\sharp \quad \text{where} \quad \langle w^\perp, h\rangle = \langle w^\perp, \mathcal{L}^\sharp h^\sharp\rangle = 0 \quad \text{and} \quad \langle w^\perp, Z_1\rangle = \langle w, Z_1\rangle = 0. \tag{4.39}$$

Thus, by orthogonality

$$\mathcal{B}^\sharp(w) = \mathcal{B}^\sharp(w^\perp) + c^2\mathcal{B}^\sharp(h^\sharp),$$

and

$$\mathcal{D}^\sharp(w, \alpha) = \mathcal{B}^\sharp(w^\perp) + \left(c^2 + c\alpha\right)\langle h^\sharp, h\rangle + \alpha^2\langle f, g\rangle.$$

Numerical computations (see Remark 4.2) show that

$$\langle h^\sharp, h\rangle \approx 0.0161. \tag{4.40}$$

Now we perform a standard argument to *complete the square* above. Since

$$0 < \langle h^\sharp, h\rangle < 4\langle f, g\rangle$$

(see (4.36)), we obtain for all $\alpha \in \mathbb{R}$, $c \in \mathbb{R}$,

$$\left(c^2 + c\alpha\right)\langle h^\sharp, h\rangle + \alpha^2\langle f, g\rangle \gtrsim \alpha^2 + c^2.$$

Combined with Corollary 4.1 and (4.39), this gives

$$\mathcal{D}^\sharp(w,\alpha) \gtrsim \alpha^2 + c^2 + \int (w_x^\perp)^2 \gtrsim \alpha^2 + \int w_x^2,$$

as required. □

Remark 4.2. The approximate value of a defined in (4.33) and the numerical check of (3.9) follows by numerical integration of explicit functions (as f and k) that decays exponentially at ∞. The values can be checked using any standard software with numerical integration (we have used independently Maple, *Mathematica* and Scilab). In contrast, the values in (4.36) and (4.40) involve the functions g and $h^\sharp$ that are not explicit but defined as solutions of simple linear second order ordinary differential equations (see (4.34) and (4.38)) with a decay condition at ∞. To compute the integrals in (4.36) and (4.40), we first determine numerically the functions g and $h^\sharp$ by a shooting argument, i.e. adjusting the initial data $g'(0) \approx -0.333$ and $(h^\sharp)'(0) \approx 0.0297$ (as odd functions, we set $g(0) = h^\sharp(0) = 0$). Observe that the functions considered here or their derivatives are exponentially decaying, which guarantees the high accuracy of the computations. We use three codes with three different softwares (Maple, *Mathematica* and Scilab) and obtain the same values with high accuracy. Note that similar numerical computations were used in [34] and [38] to check spectral conditions.

5. End of the proof of Theorem 1.1

As in the statement of Theorem 1.1, we consider an odd function $\varphi^{in} \in H^1 \times L^2$ satisfying $\|\varphi^{in}\|_{H^1\times L^2} \le \varepsilon$ for some small $\varepsilon > 0$ to be chosen. By Proposition 3.1, the corresponding solution $\varphi(t)$ of (1.6) is global in $H^1 \times L^2$ and satisfies, for all $t \in \mathbb{R}$,

$$\|\varphi(t)\|_{H^1\times L^2} \lesssim \varepsilon. \tag{5.1}$$

Now we use the decomposition and computations of Section 2, introducing in particular the functions $u(t)$, $z(t)$, $\alpha(t)$, $\beta(t)$ and $v(t)$ as in (2.1), (2.2), (2.4), (2.5) and (2.11). Note that from (5.1), it holds

$$\forall t \in \mathbb{R}, \quad \|u(t)\|_{H^1\times L^2} + \|v(t)\|_{H^1\times L^2} + \|u_1(t)\|_{L^\infty} + \|v_1(t)\|_{L^\infty} + |z(t)| \lesssim \varepsilon. \tag{5.2}$$

In order to simplify some estimates, we define

$$\|v_1\|_{H^1_\omega}^2 := \int \left(|\partial_x v_1|^2 + v_1^2\right) \operatorname{sech}\left(\frac{x}{2\sqrt{2}}\right), \quad \|v_2\|_{L^2_\omega}^2 := \int v_2^2 \operatorname{sech}\left(\frac{x}{2\sqrt{2}}\right), \tag{5.3}$$

and

$$\|v\|_{H^1_\omega\times L^2_\omega}^2 := \|v_1\|_{H^1_\omega}^2 + \|v_2\|_{L^2_\omega}^2. \tag{5.4}$$

5.1. **Control of the error terms and conclusion of the Virial argument.** The key ingredient of the proof of asymptotic stability in the energy space is the following result.

Proposition 5.1. *For $\varepsilon > 0$ small enough,*

$$\int_{-\infty}^{+\infty} \left(|z(t)|^4 + \|v(t)\|^2_{H^1_\omega \times L^2_\omega} \right) dt \lesssim \varepsilon^2. \tag{5.5}$$

Proof. Let

$$\gamma(t) := \alpha(t)\beta(t), \tag{5.6}$$

and recall the virial-type quantities $\mathcal{I}(t)$, $\mathcal{J}(t)$, already defined in (4.2), (4.7) for ψ as in (4.12) for $\lambda = 8$ and g introduced in (4.34).

The proof of (5.5) is based on a suitable combination of the following three estimates, which hold for some fixed constants $\kappa_0, C > 0$:

$$\frac{d}{dt}\gamma \geq 2\mu(\beta^2 - \alpha^2) - C\varepsilon \left(|z(t)|^4 + \|v_1\|^2_{H^1_\omega} \right), \tag{5.7}$$

$$-\frac{d}{dt}(\mathcal{I} + \mathcal{J}) \geq \kappa_0 \left(\alpha^2 + \|v_1\|^2_{H^1_\omega} \right) - C\varepsilon \left(|z(t)|^4 + \|v_2\|^2_{L^2_\omega} \right), \tag{5.8}$$

$$2\frac{d}{dt} \int \operatorname{sech}\left(\frac{x}{2\sqrt{2}} \right) v_1 v_2 \geq \|v_2\|^2_{L^2_\omega} - C \left(|z(t)|^4 + \|v_1\|^2_{H^1_\omega} \right). \tag{5.9}$$

Step 4. Proof of (5.5) assuming (5.7), (5.8) and (5.9). For $\sigma > 0$ small to be chosen, let

$$\mathcal{K} := \frac{\kappa_0}{4\mu}\gamma - (\mathcal{I} + \mathcal{J}) + 2\sigma \int \operatorname{sech}\left(\frac{x}{2\sqrt{2}} \right) v_1 v_2.$$

From (5.7), (5.8) and (5.9), we obtain

$$\frac{d}{dt}\mathcal{K} \geq \frac{\kappa_0}{2}(\alpha^2+\beta^2)+\kappa_0\|v_1\|^2_{H^1_\omega}+\sigma\|v_2\|^2_{L^2_\omega}-C\,(\sigma+\varepsilon)\left(|z(t)|^4 + \|v_1\|^2_{H^1_\omega} \right)-C\varepsilon\|v_2\|^2_{L^2_\omega}.$$

Note that from (2.5) we have $\alpha^2 + \beta^2 = |z|^4$. Thus, choosing $\sigma > 0$ sufficiently small, and then $\varepsilon > 0$ small enough, we obtain

$$\frac{d}{dt}\mathcal{K} \gtrsim |z(t)|^4 + \|v\|^2_{H^1_\omega \times L^2_\omega}. \tag{5.10}$$

By the expressions of $\mathcal{I}$, $\mathcal{J}$, γ and (5.2), we easily see that

$$\forall t \in \mathbb{R}, \quad |\mathcal{K}(t)| \lesssim \|v(t)\|^2_{H^1 \times L^2} + |z(t)|^4 \lesssim \varepsilon^2. \tag{5.11}$$

Therefore, integrating (5.10) on $[-t_0, t_0]$ and passing to the limit as $t_0 \to +\infty$, we find (5.5).

To finish the proof, we only have to prove (5.7), (5.8) and (5.9).

Step 5. Preliminary computations and estimates. First, from the computations of Section 2, we give the expressions of F_α, F_β, F_1 and F_2 in (4.1). Note that from (2.1), (1.6) and the fact that $\mathcal{L}Y_1 = \mu^2 Y_1$,

$$\begin{cases} \dot{z}_1 = \langle \dot{\varphi}_1, Y_1 \rangle = \langle \varphi_2, Y_1 \rangle = \mu z_2, \\ \dot{z}_2 = \frac{1}{\mu}\langle \dot{\varphi}_2, Y_1 \rangle = -\frac{1}{\mu}\langle \mathcal{L}\varphi_1 + (3H\varphi_1^2 + \varphi_1^3), Y_1 \rangle = -\mu z_1 - \frac{1}{\mu}\langle 3H\varphi_1^2 + \varphi_1^3, Y_1 \rangle. \end{cases} \tag{5.12}$$

(Recall that $\varphi_1 = z_1 Y_1 + u_1$ and $\langle u_1, Y_1\rangle = 0$, see (2.3).) In particular, (2.5) leads to

$$\begin{aligned}\dot\alpha &= 2z_1\dot z_1 - 2z_2\dot z_2 = 2\mu\beta + \frac{2}{\mu} z_2 \langle 3H\varphi_1^2 + \varphi_1^3, Y_1\rangle,\\ \dot\beta &= 2z_1\dot z_2 + 2\dot z_1 z_2 = -2\mu\alpha - \frac{2}{\mu} z_1 \langle 3H\varphi_1^2 + \varphi_1^3, Y_1\rangle.\end{aligned}$$

Thus, we obtain in (4.1),

$$F_\alpha := \frac{2}{\mu} z_2 \langle 3H\varphi_1^2 + \varphi_1^3, Y_1\rangle, \quad F_\beta := -\frac{2}{\mu} z_1 \langle 3H\varphi_1^2 + \varphi_1^3, Y_1\rangle.$$

Next, since

$$\frac{d}{dt}|z|^2 = 2z_1\dot z_1 + 2\dot z_2 z_2 = -\frac{2}{\mu} z_2 \langle 3H\varphi_1^2 + \varphi_1^3, Y_1\rangle = -F_\alpha, \tag{5.13}$$

we deduce, from (2.11), (2.2), (1.6) and (5.12), that

$$\begin{aligned}\dot v_1 = \dot u_1 + \frac{d}{dt}|z|^2 q &= \dot\varphi_1 - \dot z_1 Y_1 - qF_\alpha\\ &= \varphi_2 - \mu z_2 Y_1 - qF_\alpha = u_2 - qF_\alpha\\ &= v_2 - qF_\alpha,\end{aligned}$$

and that in (2.13) and (4.1),

$$F_1 := -qF_\alpha. \tag{5.14}$$

We have by direct computations

$$\begin{aligned}\dot u_2 = \dot\varphi_2 - \mu\dot z_2 Y_1 &= -\mathcal{L}\varphi_1 - (3H\varphi_1^2 + \varphi_1^3) + \mu^2 z_1 Y_1 + \langle 3H\varphi_1^2 + \varphi_1^3, Y_1\rangle Y_1\\ &= -\mathcal{L}u_1 - (3H(u_1 + z_1Y_1)^2 + \varphi_1^3) + \langle 3H(u_1 + z_1Y_1)^2 + \varphi_1^3, Y_1\rangle Y_1\\ &= -\mathcal{L}u_1 - 3z_1^2(HY_1^2 - \langle HY_1^2, Y_1\rangle Y_1) - (3H(u_1^2 + 2z_1u_1Y_1) + \varphi_1^3)\\ &\quad + \langle 3H(u_1^2 + 2z_1u_1Y_1) + \varphi_1^3, Y_1\rangle Y_1\\ &= -\mathcal{L}u_1 - 2z_1^2 f + F_u,\end{aligned}$$

so that in (2.9),

$$F_u := -\left[3H(u_1^2 + 2u_1z_1Y_1) + \varphi_1^3 - \langle 3H(u_1^2 + 2u_1z_1Y_1) + \varphi_1^3, Y_1\rangle Y_1\right].$$

We also observe that

$$\dot v_2 = \dot u_2 = -\mathcal{L}v_1 - \alpha f + F_u,$$

and thus

$$F_2 = F_u = -\left[3H(u_1^2 + 2u_1z_1Y_1) + \varphi_1^3 - \langle 3H(u_1^2 + 2u_1z_1Y_1) + \varphi_1^3, Y_1\rangle Y_1\right].$$

Second, we prove

$$\|v_1\|_{H^1_\omega} \lesssim \|\partial_x w\|_{L^2}, \tag{5.15}$$

where we recall from Section 4 the notation $w(t,x) = v_1(t,x)\zeta(x) = v_1(t,x)$ $\operatorname{sech}\left(\frac{x}{8\sqrt{2}}\right)$ (see (4.13)).

Indeed, we observe that from (4.21) (with the choice $\lambda_0 = 100$),

$$\begin{aligned}\|\partial_x w\|_{L^2}^2 &\gtrsim \int \operatorname{sech}^2\left(\frac{x}{100}\right) w^2 = \int \operatorname{sech}^2\left(\frac{x}{100}\right)\operatorname{sech}^2\left(\frac{x}{8\sqrt{2}}\right) v_1^2\\ &\gtrsim \int \operatorname{sech}\left(\frac{x}{2\sqrt{2}}\right) v_1^2.\end{aligned} \tag{5.16}$$

Next, we have

$$\begin{aligned}\|\partial_x w\|_{L^2}^2 &\gtrsim \int \operatorname{sech}^2\left(\frac{x}{100}\right)|\partial_x w|^2 = \int \operatorname{sech}^2\left(\frac{x}{100}\right)|\zeta\partial_x v_1 + \zeta' v_1|^2\\ &\gtrsim \int \operatorname{sech}^2\left(\frac{x}{100}\right)\operatorname{sech}^2\left(\frac{x}{8\sqrt{2}}\right)|\partial_x v_1|^2 + 2\int \operatorname{sech}^2\left(\frac{x}{100}\right)\zeta\zeta'(\partial_x v_1)v_1\\ &\quad + \int \operatorname{sech}^2\left(\frac{x}{100}\right)(\zeta')^2 v_1^2\\ &\gtrsim \int \operatorname{sech}\left(\frac{x}{2\sqrt{2}}\right)|\partial_x v_1|^2\\ &\quad + \int v_1^2\left(-\left(\operatorname{sech}^2\left(\frac{x}{100}\right)\zeta\zeta'\right)' + \operatorname{sech}^2\left(\frac{x}{8\sqrt{2}}\right)(\zeta')^2\right).\end{aligned}$$

Thus, using (5.16),

$$\int \operatorname{sech}\left(\frac{x}{2\sqrt{2}}\right)|\partial_x v_1|^2 \lesssim \|\partial_x w\|_{L^2}^2 + \int \operatorname{sech}^2\left(\frac{x}{100}\right)\operatorname{sech}^2\left(\frac{x}{8\sqrt{2}}\right)v_1^2 \lesssim \|\partial_x w\|_{L^2}^2,$$

which completes the proof of (5.15).

Step 6. Proof of (5.7). From (4.1) we find

$$\dot\gamma = \dot\alpha\beta + \alpha\dot\beta = 2\mu\left(\beta^2 - \alpha^2\right) + \mathcal{R}_\gamma, \quad \text{where} \quad \mathcal{R}_\gamma = \beta F_\alpha + \alpha F_\beta. \tag{5.17}$$

Replacing $\varphi_1 = u_1 + z_1 Y_1 = v_1 - |z|^2 q + z_1 Y_1$ in the expression of F_α and F_β, then using the explicit decay of Y_1 (see (1.10)) and (5.2), we have

$$|F_\alpha| + |F_\beta| \lesssim |z|\left(|z|^2 + \|v_1\|_{L^2_\omega}^2\right). \tag{5.18}$$

From the definition of α, β and (5.2) we obtain finally

$$|\mathcal{R}_\gamma| \lesssim |z|^3\left(|z|^2 + \|v_1\|_{L^2_\omega}^2\right) \lesssim \varepsilon\left(|z|^4 + \|v_1\|_{L^2_\omega}^2\right).$$

Step 7. Proof of (5.8). From (4.9), (4.37) and (5.15), it is sufficient to prove the following estimate:

$$|\mathcal{R}_{\mathcal{D}}| \lesssim \varepsilon\left(|z(t)|^4 + \|\partial_x w\|_{L^2}^2 + \|v_2\|_{L^2_\omega}^2\right), \tag{5.19}$$

where $\mathcal{R}_{\mathcal{D}}$ is defined in (4.11).

Using (5.18), (4.35) and Cauchy-Schwarz inequality, we have

$$\left|F_\alpha \int v_2 g\right| + \left|F_\beta \int v_1 g\right| \lesssim |z|\left(|z|^2 + \|v_1\|_{L^2_\omega}^2\right)\left(\|v_1\|_{L^2_\omega} + \|v_2\|_{L^2_\omega}\right). \tag{5.20}$$

From (4.35), (5.14), (5.18) and (2.12), we get

$$\left|\beta \int g F_1\right| + \left|\int v_2\left(\psi\partial_x F_1 + \frac{1}{2}\psi' F_1\right)\right| \lesssim |z|\left(|z|^2 + \|v_1\|_{L^2_\omega}^2\right)\left(|z|^2 + \|v_2\|_{L^2_\omega}\right). \tag{5.21}$$

By (4.35) and (5.2), we obtain

$$\left|\alpha \int g F_2\right| \lesssim |z|^2\left(|z|^3 + |z|\|v_1\|_{L^2_\omega} + \|v_1\|_{L^2_\omega}^2\right). \tag{5.22}$$

Now, we address the only remaining term in (4.11), which is $\int v_1\left(\psi\partial_x F_2 + \frac{1}{2}\psi' F_2\right)$. We decompose

$$F_2 = \widetilde{F}_2 - 3Hv_1^2 - v_1^3.$$

Observe that $\widetilde{F}_2$ contains only terms with an explicit decay $e^{-\frac{|x|}{\sqrt{2}}}$ (coming from $Y_1(x)$, $q(x)$ and their derivatives). Thus, by similar arguments as before,

$$\left|\int v_1 \left(\psi \partial_x \widetilde{F}_2 + \frac{1}{2}\psi' \widetilde{F}_2\right)\right| \lesssim \|v_1\|_{L^2_\omega} \left(|z|^3 + |z|\|v_1\|_{L^2_\omega} + \|v_1\|^2_{L^2_\omega}\right). \tag{5.23}$$

We are now reduced to controlling the term

$$-\int v_1\left(\psi\partial_x(3Hv_1^2+v_1^3)+\frac{1}{2}\psi'\left(3Hv_1^2+v_1^3\right)\right) = \int\left(\psi\partial_x v_1+\frac{1}{2}\psi' v_1\right)\left(3Hv_1^2+v_1^3\right).$$

Integrating by parts,

$$3\int\left(\psi\partial_x v_1 + \frac{1}{2}\psi' v_1\right) H v_1^2 = \int\left(\frac{1}{2}\psi' H - \psi H'\right) v_1^3,$$

hence

$$\left|3\int\left(\psi\partial_x v_1 + \frac{1}{2}\psi' v_1\right) H v_1^2\right| \lesssim \int (\psi' + H')|v_1|^3.$$

We claim

$$\int (\psi' + H')|v_1|^3 \lesssim \|v_1\|_{L^\infty}\|\partial_x w\|^2_{L^2} \lesssim \varepsilon\|\partial_x w\|^2_{L^2}. \tag{5.24}$$

Indeed, by parity, the definition of ψ (4.12) and w (4.13), and the decay property of H', we have (with $\lambda = 8$)

$$\int (\psi' + H')|v_1|^3 \lesssim \int_0^{+\infty} e^{-\frac{2x}{\lambda\sqrt{2}}}|v_1|^3 \lesssim \int_0^{+\infty} e^{\frac{x}{\lambda\sqrt{2}}}|w|^3.$$

Then, integrating by parts, using $w(0) = 0$ (the function w is odd)

$$\begin{aligned}\int_0^{+\infty} e^{\frac{x}{\lambda\sqrt{2}}}|w|^3 &= -\lambda\sqrt{2}\int_0^{+\infty} e^{\frac{x}{\lambda\sqrt{2}}}\partial_x(|w|^3) = -3\lambda\sqrt{2}\int_0^{+\infty} e^{\frac{x}{\lambda\sqrt{2}}}(\partial_x w)w|w| \\ &\le 6\lambda\|v_1\|^{\frac{1}{2}}_{L^\infty}\int_0^{+\infty} e^{\frac{x}{2\lambda\sqrt{2}}}|\partial_x w||w|^{\frac{3}{2}} \le 18\lambda^2\|v_1\|_{L^\infty}\int_0^{+\infty}|\partial_x w|^2 \\ &\quad + \frac{1}{4}\int_0^{\infty} e^{\frac{x}{\lambda\sqrt{2}}}|w|^3.\end{aligned}$$

Thus,

$$\int_0^{+\infty} e^{\frac{x}{\lambda\sqrt{2}}}|w|^3 \lesssim \|v_1\|_{L^\infty}\|\partial_x w\|^2_{L^2},$$

and (5.24) is proved.

Finally, we have by integration by parts

$$\int\left(\psi\partial_x v_1 + \frac{1}{2}\psi' v_1\right) v_1^3 = \frac{1}{4}\int \psi' v_1^4 \ge 0.$$

This term happens to have the right sign for estimate (5.8) (this is related to the fact that equation (1.1) is defocusing), but we can also bound this term in absolute value since by (5.24),

$$\int \psi' v_1^4 \lesssim \varepsilon\int \psi'|v_1|^3 \lesssim \varepsilon^2\|\partial_x w\|^2_{L^2}. \tag{5.25}$$

In conclusion, (5.19) is a consequence of (5.15), (5.20), (5.21), (5.22), (5.23), (5.24) and (5.25).

Step 8. Proof of (5.9). We use (4.4) with $\psi'(x)=\operatorname{sech}\left(\frac{x}{2\sqrt{2}}\right)$. Note that

$$\int \operatorname{sech}\left(\frac{x}{2\sqrt{2}}\right)\left(|\partial_x v_1|^2+v_1^2\right)+\int \left|\operatorname{sech}''\left(\frac{x}{2\sqrt{2}}\right)\right| v_1^2 \lesssim \|v_1\|_{H^1_\omega}^2,$$

$$\left|\alpha\int \operatorname{sech}\left(\frac{x}{2\sqrt{2}}\right) v_1 f\right| \lesssim |z|^2\|v_1\|_{H^1_\omega},$$

and

$$\left|\int \operatorname{sech}\left(\frac{x}{2\sqrt{2}}\right)(|F_1v_2|+|v_1F_2|)\right| \lesssim |z|^3\|v_2\|_{L^2_\omega}+\|v_1\|_{H^1_\omega}^2.$$

Using these estimates, we obtain from (4.4)

$$\frac{d}{dt}\int \operatorname{sech}\left(\frac{x}{2\sqrt{2}}\right) v_1v_2 \geq \|v_2\|_{L^2_\omega}^2 - C\|v_1\|_{H^1_\omega}^2 - C|z|^3\|v_2\|_{L^2_\omega}+C|z|^4,$$

and (5.9) follows. □

5.2. **Conclusion. Proof of (1.8).** Let

$$\mathcal{H} := \int \left((\partial_x v_1)^2+2v_1^2+v_2^2\right)\operatorname{sech}\left(\frac{x}{2\sqrt{2}}\right). \tag{5.26}$$

Then, using (4.1), we have

(5.27)

$$\begin{aligned}
\dot{\mathcal{H}} &= 2\int \operatorname{sech}\left(\frac{x}{2\sqrt{2}}\right)\left((\partial_x \dot v_1)(\partial_x v_1)+2\dot v_1 v_1+\dot v_2 v_2\right)\\
&= 2\int \operatorname{sech}\left(\frac{x}{2\sqrt{2}}\right)\left((\partial_x v_2)(\partial_x v_1)+2v_2v_1-(\mathcal{L}v_1)v_2-\alpha f v_2\right)\\
&\quad + 2\int \operatorname{sech}\left(\frac{x}{2\sqrt{2}}\right)\left((\partial_x F_1)(\partial_x v_1)+2F_1v_1+F_2v_2\right)\\
&= -\frac{1}{\sqrt{2}}\int \operatorname{sech}'\left(\frac{x}{2\sqrt{2}}\right) v_2(\partial_x v_1) + 2\int \operatorname{sech}\left(\frac{x}{2\sqrt{2}}\right)\left(3(1-H^2)v_1v_2-\alpha f v_2\right)\\
&\quad + 2\int \operatorname{sech}\left(\frac{x}{2\sqrt{2}}\right)\left((\partial_x F_1)(\partial_x v_1)+2F_1v_1+F_2v_2\right).
\end{aligned}$$

Note that

$$\left|\int \operatorname{sech}'\left(\frac{x}{2\sqrt{2}}\right) v_2(\partial_x v_1)\right| \lesssim \int \left((\partial_x v_1)^2+v_2^2\right)\operatorname{sech}\left(\frac{x}{2\sqrt{2}}\right). \tag{5.28}$$

From (5.5) there exists a sequence $t_n\to+\infty$ such that $\mathcal{H}(t_n)+z(t_n)\to 0$. From (5.27), (5.28) and the estimates on F_1 and F_2, we have

$$|\dot{\mathcal{H}}| \lesssim |z(t)|^4+\|v(t)\|_{H^1_\omega\times L^2_\omega}^2.$$

Let $t\in\mathbb{R}$. Integrating on $[t,t_n]$ and passing to the limit as $n\to+\infty$ we obtain

$$\mathcal{H}(t) \lesssim \int_t^{+\infty}\left(|z(t)|^4+\|v(t)\|_{H^1_\omega\times L^2_\omega}^2\right)dt.$$

From (5.5) it follows that $\lim_{t\to+\infty}\mathcal{H}(t)=0$. The same holds for $t\to-\infty$. Thus, $\lim_{t\to\pm\infty}\|v\|_{H^1_\omega\times L^2_\omega}=0$. Moreover, from (5.13) and (5.18), we have

$$\left|\frac{d}{dt}|z|^4\right| = 2|\alpha F_\alpha+\beta F_\beta| \lesssim |z|^3\left(|z|^2+\|v_1\|_{L^2_\omega}^2\right) \lesssim |z|^4+\|v_1\|_{L^2_\omega}^2. \tag{5.29}$$

Using (5.5) and similar arguments as before, we obtain $\lim_{t\to\pm\infty}|z(t)|=0$. Therefore, (1.8) follows from (2.11).

Acknowledgments

We would like to thank S. Cuccagna, F. Merle and W. Schlag for several enlightening discussions and comments on a preliminary version of this paper and Y. Liu and M. Ward for pointing to us some key references. We are also indebted to the anonymous referee for useful comments.

References

[1] Juan A. Barceló, Alberto Ruiz, and Luis Vega, *Some dispersive estimates for Schrödinger equations with repulsive potentials*, J. Funct. Anal. **236** (2006), no. 1, 1–24, DOI 10.1016/j.jfa.2006.03.012. MR2227127

[2] Dario Bambusi and Scipio Cuccagna, *On dispersion of small energy solutions to the nonlinear Klein Gordon equation with a potential*, Amer. J. Math. **133** (2011), no. 5, 1421–1468, DOI 10.1353/ajm.2011.0034. MR2843104

[3] Marius Beceanu, *A centre-stable manifold for the focussing cubic NLS in $\mathbb{R}^{1+3}$*, Comm. Math. Phys. **280** (2008), no. 1, 145–205, DOI 10.1007/s00220-008-0427-3. MR2391193

[4] Fabrice Bethuel, Philippe Gravejat, and Didier Smets, *Asymptotic stability in the energy space for dark solitons of the Gross-Pitaevskii equation* (English, with English and French summaries), Ann. Sci. Éc. Norm. Supér. (4) **48** (2015), no. 6, 1327–1381. MR3429470

[5] Vladimir Buslaev and Galina Perelman, *Scattering for the nonlinear Schrödinger equations: States close to a soliton*, St. Petersburgh Math. J. **4** (1993), no. 6, 1111–1142.

[6] Vladimir Buslaev and Galina Perelman, *On the stability of solitary waves for nonlinear Schrödinger equations*, Nonlinear evolution equations, 75–98, Amer. Math. Soc. Transl. Ser. 2, 164, Amer. Math. Soc., Providence, RI, 1995.

[7] Vladimir S. Buslaev and Catherine Sulem, *On asymptotic stability of solitary waves for nonlinear Schrödinger equations* (English, with English and French summaries), Ann. Inst. H. Poincaré Anal. Non Linéaire **20** (2003), no. 3, 419–475, DOI 10.1016/S0294-1449(02)00018-5. MR1972870

[8] Scipio Cuccagna, *On asymptotic stability in 3D of kinks for the ϕ^4 model*, Trans. Amer. Math. Soc. **360** (2008), no. 5, 2581–2614, DOI 10.1090/S0002-9947-07-04356-5. MR2373326

[9] Scipio Cuccagna, *The Hamiltonian structure of the nonlinear Schrödinger equation and the asymptotic stability of its ground states*, Comm. Math. Phys. **305** (2011), no. 2, 279–331, DOI 10.1007/s00220-011-1265-2. MR2805462

[10] Scipio Cuccagna, *On asymptotic stability of moving ground states of the nonlinear Schrödinger equation*, Trans. Amer. Math. Soc. **366** (2014), no. 6, 2827–2888, DOI 10.1090/S0002-9947-2014-05770-X. MR3180733

[11] S. Cuccagna and R. Jenkins, *On asymptotic stability of N-solitons of the Gross-Pitaevskii equation*, preprint 2012, arXiv:1410.6887.

[12] Scipio Cuccagna and Dmitry E. Pelinovsky, *The asymptotic stability of solitons in the cubic NLS equation on the line*, Appl. Anal. **93** (2014), no. 4, 791–822, DOI 10.1080/00036811.2013.866227. MR3180019

[13] Sara Cuenda, Niurka R. Quintero, and Angel Sánchez, *Sine-Gordon wobbles through Bäcklund transformations*, Discrete Contin. Dyn. Syst. Ser. S **4** (2011), no. 5, 1047–1056, DOI 10.3934/dcdss.2011.4.1047. MR2754104

[14] Jean-Marc Delort, *Existence globale et comportement asymptotique pour l'équation de Klein-Gordon quasi linéaire à données petites en dimension 1* (French, with English and French summaries), Ann. Sci. École Norm. Sup. (4) **34** (2001), no. 1, 1–61, DOI 10.1016/S0012-9593(00)01059-4. MR1833089

[15] Jean-Marc Delort, *Semiclassical microlocal normal forms and global solutions of modified one-dimensional KG equations* (English, with English and French summaries), Ann. Inst. Fourier (Grenoble) **66** (2016), no. 4, 1451–1528. MR3494176

[16] Jean-Marc Delort, Daoyuan Fang, and Ruying Xue, *Global existence of small solutions for quadratic quasilinear Klein-Gordon systems in two space dimensions*, J. Funct. Anal. **211** (2004), no. 2, 288–323, DOI 10.1016/j.jfa.2004.01.008. MR2056833

[17] Jochen Denzler, *Nonpersistence of breather families for the perturbed sine Gordon equation*, Comm. Math. Phys. **158** (1993), no. 2, 397–430. MR1249601

[18] M. Goldberg and W. Schlag, *Dispersive estimates for Schrödinger operators in dimensions one and three*, Comm. Math. Phys. **251** (2004), no. 1, 157–178, DOI 10.1007/s00220-004-1140-5. MR2096737

[19] Philippe Gravejat and Didier Smets, *Asymptotic stability of the black soliton for the Gross-Pitaevskii equation*, Proc. Lond. Math. Soc. (3) **111** (2015), no. 2, 305–353, DOI 10.1112/plms/pdv025. MR3384514

[20] Nakao Hayashi and Pavel I. Naumkin, *The initial value problem for the cubic nonlinear Klein-Gordon equation*, Z. Angew. Math. Phys. **59** (2008), no. 6, 1002–1028, DOI 10.1007/s00033-007-7008-8. MR2457221

[21] Nakao Hayashi and Pavel I. Naumkin, *Quadratic nonlinear Klein-Gordon equation in one dimension*, J. Math. Phys. **53** (2012), no. 10, 103711, 36, DOI 10.1063/1.4759156. MR3050628

[22] Daniel B. Henry, J. Fernando Perez, and Walter F. Wreszinski, *Stability theory for solitary-wave solutions of scalar field equations*, Comm. Math. Phys. **85** (1982), no. 3, 351–361. MR678151

[23] Sergiu Klainerman, *Global existence for nonlinear wave equations*, Comm. Pure Appl. Math. **33** (1980), no. 1, 43–101, DOI 10.1002/cpa.3160330104. MR544044

[24] Sergiu Klainerman, *Global existence of small amplitude solutions to nonlinear Klein-Gordon equations in four space-time dimensions*, Comm. Pure Appl. Math. **38** (1985), no. 5, 631–641, DOI 10.1002/cpa.3160380512. MR803252

[25] E. Kopylova and A. I. Komech, *On asymptotic stability of kink for relativistic Ginzburg-Landau equations*, Arch. Ration. Mech. Anal. **202** (2011), no. 1, 213–245, DOI 10.1007/s00205-011-0415-1. MR2835867

[26] E. A. Kopylova and A. I. Komech, *On asymptotic stability of moving kink for relativistic Ginzburg-Landau equation*, Comm. Math. Phys. **302** (2011), no. 1, 225–252, DOI 10.1007/s00220-010-1184-7. MR2770013

[27] M. Kowalczyk, Y. Martel and C. Muñoz, *Nonexistence of small odd breathers for a class of nonlinear wave equations*, preprint 2016.

[28] J. Krieger and W. Schlag, *Stable manifolds for all monic supercritical focusing nonlinear Schrödinger equations in one dimension*, J. Amer. Math. Soc. **19** (2006), no. 4, 815–920, DOI 10.1090/S0894-0347-06-00524-8. MR2219305

[29] Harvey Segur and Martin D. Kruskal, *Nonexistence of small-amplitude breather solutions in ϕ^4 theory*, Phys. Rev. Lett. **58** (1987), no. 8, 747–750, DOI 10.1103/PhysRevLett.58.747. MR879720

[30] Hans Lindblad and Avy Soffer, *A remark on long range scattering for the nonlinear Klein-Gordon equation*, J. Hyperbolic Differ. Equ. **2** (2005), no. 1, 77–89, DOI 10.1142/S0219891605000385. MR2134954

[31] Hans Lindblad and Avy Soffer, *A remark on asymptotic completeness for the critical nonlinear Klein-Gordon equation*, Lett. Math. Phys. **73** (2005), no. 3, 249–258, DOI 10.1007/s11005-005-0021-y. MR2188297

[32] Hans Lindblad and Avy Soffer, *Scattering for the Klein-Gordon equation with quadratic and variable coefficient cubic nonlinearities*, (2013) preprint arXiv:1307.5882.

[33] Nicholas Manton and Paul Sutcliffe, *Topological solitons*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 2004. MR2068924

[34] Yvan Martel and Frank Merle, *A Liouville theorem for the critical generalized Korteweg-de Vries equation*, J. Math. Pures Appl. (9) **79** (2000), no. 4, 339–425, DOI 10.1016/S0021-7824(00)00159-8. MR1753061

[35] Yvan Martel and Frank Merle, *Asymptotic stability of solitons for subcritical generalized KdV equations*, Arch. Ration. Mech. Anal. **157** (2001), no. 3, 219–254, DOI 10.1007/s002050100138. MR1826966

[36] Yvan Martel and Frank Merle, *Asymptotic stability of solitons of the subcritical gKdV equations revisited*, Nonlinearity **18** (2005), no. 1, 55–80, DOI 10.1088/0951-7715/18/1/004. MR2109467

[37] F. Merle and L. Vega, *L^2 stability of solitons for KdV equation*, Int. Math. Res. Not. **13** (2003), 735–753, DOI 10.1155/S1073792803208060. MR1949297

[38] Frank Merle and Pierre Raphael, *The blow-up dynamic and upper bound on the blow-up rate for critical nonlinear Schrödinger equation*, Ann. of Math. (2) **161** (2005), no. 1, 157–222, DOI 10.4007/annals.2005.161.157. MR2150386

[39] Arnold F. Nikiforov and Vasilii B. Uvarov, *Special functions of mathematical physics*, Birkhäuser Verlag, Basel, 1988. A unified introduction with applications; Translated from the Russian and with a preface by Ralph P. Boas; With a foreword by A. A. Samarskiĭ. MR922041

[40] Robert L. Pego and Michael I. Weinstein, *Eigenvalues, and instabilities of solitary waves*, Philos. Trans. Roy. Soc. London Ser. A **340** (1992), no. 1656, 47–94, DOI 10.1098/rsta.1992.0055. MR1177566

[41] Michael E. Peskin and Daniel V. Schroeder, *An introduction to quantum field theory*, Addison-Wesley Publishing Company, Advanced Book Program, Reading, MA, 1995. Edited and with a foreword by David Pines. MR1402248

[42] Michael Reed and Barry Simon, *Methods of Modern mathematical Physics IV: Analysis of Operators*, Academic Press, New York, 1978.

[43] I. Rodnianski, W. Schlag, and A. Soffer, *Asymptotic stability of N-soliton states of NLS*, preprint, arXiv:math/0309114.

[44] Harvey Segur, *Wobbling kinks in φ^4 and sine-Gordon theory*, J. Math. Phys. **24** (1983), no. 6, 1439–1443, DOI 10.1063/1.525867. MR708660

[45] Jalal Shatah, *Normal forms and quadratic nonlinear Klein-Gordon equations*, Comm. Pure Appl. Math. **38** (1985), no. 5, 685–696, DOI 10.1002/cpa.3160380516. MR803256

[46] I. M. Sigal, *Nonlinear wave and Schrödinger equations. I. Instability of periodic and quasiperiodic solutions*, Comm. Math. Phys. **153** (1993), no. 2, 297–320. MR1218303

[47] Barry Simon, *Resonances in n-body quantum systems with dilatation analytic potentials and the foundations of time-dependent perturbation theory*, Ann. of Math. (2) **97** (1973), 247–274. MR0353896

[48] A. Soffer and M. I. Weinstein, *Time dependent resonance theory*, Geom. Funct. Anal. **8** (1998), no. 6, 1086–1128, DOI 10.1007/s000390050124. MR1664792

[49] A. Soffer and M. I. Weinstein, *Resonances, radiation damping and instability in Hamiltonian nonlinear wave equations*, Invent. Math. **136** (1999), no. 1, 9–74, DOI 10.1007/s002220050303. MR1681113

[50] Jacob Sterbenz, *Dispersive decay for the 1D Klein-Gordon equation with variable coefficient nonlinearities*, Trans. Amer. Math. Soc. **368** (2016), no. 3, 2081–2113, DOI 10.1090/tran/6478. MR3449234

[51] E. C. Titchmarsh, *Eigenfunction Expansions Associated with Second-Order Differential Equations* (German), Oxford, at the Clarendon Press, 1946. MR0019765

[52] Tanmay Vachaspati, *Kinks and domain walls*, Cambridge University Press, New York, 2006. An introduction to classical and quantum solitons. MR2282481

[53] Tai-Peng Tsai and Horng-Tzer Yau, *Asymptotic dynamics of nonlinear Schrödinger equations: resonance-dominated and dispersion-dominated solutions*, Comm. Pure Appl. Math. **55** (2002), no. 2, 153–216, DOI 10.1002/cpa.3012. MR1865414

[54] A. H. Vartanian, *Long-time asymptotics of solutions to the Cauchy problem for the defocusing nonlinear Schrödinger equation with finite-density initial data. II. Dark solitons on continua*, Math. Phys. Anal. Geom. **5** (2002), no. 4, 319–413, DOI 10.1023/A:1021179311172. MR1942685

[55] A. Vilenkin and E. P. S. Shellard, *Cosmic strings and other topological defects*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 1994. MR1446491

[56] Ricardo Weder, *The $W_{k,p}$-continuity of the Schrödinger wave operators on the line*, Comm. Math. Phys. **208** (1999), no. 2, 507–520, DOI 10.1007/s002200050767. MR1729096

[57] Ricardo Weder, *L^p-$L^{\dot{p}}$ estimates for the Schrödinger equation on the line and inverse scattering for the nonlinear Schrödinger equation with a potential*, J. Funct. Anal. **170** (2000), no. 1, 37–68, DOI 10.1006/jfan.1999.3507. MR1736195

[58] Michael I. Weinstein, *Lyapunov stability of ground states of nonlinear dispersive evolution equations*, Comm. Pure Appl. Math. **39** (1986), no. 1, 51–67, DOI 10.1002/cpa.3160390103. MR820338

[59] Edward Witten, *From superconductors and four-manifolds to weak interactions*, Bull. Amer. Math. Soc. (N.S.) **44** (2007), no. 3, 361–391 (electronic), DOI 10.1090/S0273-0979-07-01167-6. MR2318156

Departamento de Ingeniería Matemática and Centro de Modelamiento Matemático (UMI 2807 CNRS), Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile
E-mail address: kowalczy@dim.uchile.cl

CMLS, École polytechnique, CNRS, Université Paris-Saclay, 91128 Palaiseau Cedex, France
E-mail address: yvan.martel@polytechnique.edu

CNRS and Departamento de Ingeniería Matemática and Centro de Modelamiento Matemático (UMI 2807 CNRS), Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile
E-mail address: claudio.munoz@math.u-psud.fr, cmunoz@dim.uchile.cl