A renormalization approach to the Liouville quantum gravity metric

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Abstract

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This thesis explores metric properties of Liouville quantum gravity (LQG), a random geometry with conformal symmetries introduced in the context of string theory by Polyakov in the 80's. Formally, it corresponds to the Riemannian metric tensor "$e^{\gamma h}(dx^2 + dy^2)$" where h is a planar Gaussian free field and γ is a parameter in $(0,2)$. Since h is a random Schwartz distribution with negative regularity, the exponential $e^{\gamma h}$ only makes sense formally and the associated volume form and distance functions are not well-defined. The mathematical language to define the volume form was introduced by Kahane, also in the 80's. In this thesis, we explore a renormalization approach to make sense of the distance function and we study its basic properties.

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Chapter 1: Introduction

A central theme in statistical mechanics and probability theory is to understand complex systems of a large number of microscopic elements interacting which each others and subject to noise. As the number of constituents of the system increases (or when one zooms out), a new structure emerges, called *scaling limit*. Proving this convergence and understanding the properties of the limit is crucial as it gives information on the large random discrete structures themselves. Interaction rules are typically encoded by various parameters such as the temperature of the system. Macroscopic changes may appear at a *critical state* as those parameters vary, leading to the notion of *phase transition*. In two dimensions, scaling limits of many critical systems become statistically *conformally invariant*.

The renormalization group (RG), pioneered by K. G. Wilson, is a method in theoretical physics to study renormalization, scaling limits and the phase transitions of statistical mechanics models. The method roughly works as follows. The first step is to introduce a map from a model at one scale to another model at a larger scale (the RG map) so that fixed points of the RG map are scale invariant. Different models belong to the same universality class if under iteration of the RG map they converge to the same fixed point. Models in the same universality class share many large scale properties, revealed by the RG map in the neighborhood of their fixed point. This method, though extremely fruitful in physics, is difficult to implement rigorously.

This thesis studies a renormalization approach to define the *Liouville quantum gravity metric*, a distance function associated with canonical random surfaces with conformal symmetries. This metric is expected to describe the scaling limit of distances in discrete random surfaces called random planar maps.

1.1 Random planar maps and planar statistical mechanics models

Uniform planar maps. A planar map is a graph embedded on a surface viewed up to orientation-preserving homeomorphisms and here we only consider the case where the surface is the 2-sphere. Simple families of planar maps are triangulations and more generally p-angulations for which each face has p-edges. Random planar maps (RPM) are planar maps sampled according to a probability measure and when constraining the number of faces to be N, the canonical probability measure is the uniform one. These RPM carry a structure of metric spaces with the graph distance and their universal scaling limit is described by the *Brownian map*, in the sense of the Gromov-Hausdorff convergence of compact metric spaces (or rather isometry classes of compact metric spaces) [51, 73–75, 78].

Such results are possible due to combinatorial observations: there exist bijections between some families of random planar maps and some families of discrete trees whose vertices are assigned integer labels. For these bijections, labels are related to graph distances from a distinguished vertex in the associated planar map. The rescaled tree associated with a large random planar map is an approximation of the Continuous Random Tree introduced by Aldous and the labels on this tree behave as a conditionally independent Brownian motion indexed by the tree: this leads to a natural description, directly in the continuum, of the Brownian map. This metric measure space is rough: its Hausdorff dimension is 4 but it is homeomorphic to the 2-sphere. Also, geodesics that start from the same typical point coincide on a non-trivial interval, contrary to those in Riemannian manifolds.

Planar statistical mechanics models at criticality and conformal invariance. We discuss here some planar statistical mechanics models on deterministic lattices and recent results on their conformal invariance at criticality. These models can be used to construct new laws on planar maps and these are expected to be related with Liouville quantum gravity, which is described below. Before their mathematical resolution, conjectures about these arose in physics and were studied numerically or with the methods of conformal field theory (CFT). Two important models that have been studied in great details are percolation and the Ising model.

We begin with percolation. Vertices, say of the triangular lattice $\mathbb{T}$, are open or closed with probability $p \in (0,1)$ independently of each others. For p small, there is almost surely no infinite cluster formed by open vertices whereas for p large enough there is almost surely a unique one. A phase transition separating this existence result occurs at $p = p_c = \frac{1}{2}$. At this critical probability, several observables become scale invariant. This is the case of the probability of the existence of an open crossing between two marked sides, say (AB) and (CD) of the boundary $\partial\Omega$ of a simply connected domain Ω in the discretization $\varepsilon\mathbb{T} \cap \Omega$: this probability converges when $\varepsilon \to 0$ to a value in $(0,1)$. This was proved by Smirnov in [105] and this value is given by the Cardy formula. A striking feature of the Cardy formula is not the fact that it is an exact formula but rather that the formula is conformally invariant: considering the image of Ω by a conformal map f, as well as the image of the marked arcs and the percolation model associated with this new domain, the limiting crossing probability is the same.

The Ising model is a spin model for which the spins take values in $\{-1, +1\}$. Contrary to the Bernoulli percolation model which possesses exact independence, spins are correlated and tend to be aligned since the Hamiltonian defining the model is proportional to $\beta \sum_{x \sim y} (\sigma_x - \sigma_y)^2$, where β is the inverse temperature of the system. This model exhibits a phase transition which can be phrased as follows: above some temperature, there is a loss of spontaneous magnetization. Mathematically, this translates as follows: when imposing $+1$ boundary condition on a discretization of a domain, the macroscopic effect of this boundary condition disappears as the mesh size of the lattice vanishes. Smirnov [107] and Chelkak and Smirnov [20] proved the conformal invariance of certain observables called "fermionic observables" at criticality. This paved the way to establishing the scaling limit of correlations associated with the spin field and their transformation rules under conformal maps in [19]. Independently and at about the same time, a different approach was taken in [36] using a relation with dimers (in particular, building on [37] and Kenyon's works).

Schramm Loewner Evolutions and Conformal Loop Ensembles. Interfaces between open and closed clusters in percolation or -1 and $+1$ spins in the Ising model turn to be conformally invariant in the scaling limit, in the same sense that the trace of the two-dimensional Brownian motion is conformally invariant. However, it is difficult to show this convergence and this was a challenging problem for some time.

Aizenman and Burchard provided in [2] a sufficient condition to obtain the tightness of the family of random rescaled curves. This condition is implied by the so-called *Russo-Seymour-Welsh* (RSW) estimates which arise in percolation theory or in the study of the Ising model at criticality. They express uniform bounds between the probability of the existence of a left-right crossing path of $[0, aN] \times [0, N]$ for $a < 1$ and the probability of the existence of a left-right crossing path of $[0, bN] \times [0, N]$ for $b > 1$ (a crossing path refers to a path of open sites in percolation and interface between opposite spins for the Ising model, see [94, 95, 98, 110] for percolation and [18, 42] for the Ising model).

The limits are part of a larger one-dimensional family of curves (indexed by $\kappa > 0$), called Schramm-Loewner evolutions (SLE_κ) in the case of non-self-crossing curves joining two marked points in a simply connected domain Ω and Conformal Loop Ensembles (CLE_κ) in the case of nested loops in Ω. Both are characterized by their conformal invariance and domain Markov property [96, 101, 103]. The convergence of critical interfaces towards these curves became accessible after the works of Smirnov on the conformal invariance of critical models (see, e.g., [106]).

Random planar maps weighted by statistical mechanics models. Uniform RPM converge to the Brownian map. This universality class corresponds to "pure gravity" in the sense that it is not decorated by any model of statistical mechanics (or, rather, simply by a non-interactive model such as percolation). Natural other RPM models are obtained by using some interactive models of statistical mechanics such as the Ising model. One gets a probability measure on (map, configuration on this map) and, forgetting about the configuration, the marginal on maps $\mathcal{M}$ is proportional to the partition function $Z(\mathcal{M}, \beta)$ of the model considered on $\mathcal{M}$. When the inverse temperature β is set at criticality, it is believed that the scaling limit of this object is connected to Liouville quantum gravity, a one parameter family of surfaces indexed by $\gamma \in (0, 2)$ where $\gamma = \sqrt{\frac{8}{3}}$ corresponds to "pure gravity". Typical distances in such planar maps are therefore expected to be described by d_γ, the dimension of γ-LQG.

One can also generate maps by favoring some of their geometric properties. Indeed, instead of considering only a random planar map $\mathcal{M}$, one can consider $(\mathcal{M}, \mathcal{T})$ where $\mathcal{T}$ is a spanning tree of $\mathcal{M}$. Forgetting about the tree gives a probability on maps weighted by their number of spanning trees. It is also natural to consider directly the probability measure on maps which is proportional to the number of spanning trees of the map with some power: this gives a ways to favor maps with a large or small number of spanning trees. By Kirchoff's matrix-tree theorem, the number of spanning tree can be expressed by using the determinant of a Laplacian. It is expected that the partition functions of many statistical mechanics models at criticality behave asymptotically like powers of the determinant of the discrete Laplacian (they appear in particular in the partition functions of SLE_κ, themselves allowing couplings of several SLEs with the Gaussian free field [35]).

Scaling limits of conformally embedded random planar maps. One version of Liouville quantum gravity would be to consider the scaling limits of these models: by embedding them in some domain and showing that the associated measure and metric converge with respect to the weak and uniform topologies. However, this direction of research remains wide open, up to one exception: Holden and Sun [64] constructed an embedding (which they called the Cardy embedding and which is related to the Cardy formula mentioned above) and proved such a convergence result towards "pure gravity" in the case of uniform random planar maps.

1.2 Liouville quantum gravity

The version of Liouville quantum gravity we will consider is not the one given by random planar maps weighted by $\det(-\Delta)^{-\mathbf{c}/2}$ but rather a continuum version phrased using only the Gaussian free field as considered mathematically in the work [44] by Duplantier and Sheffield. In this version, one considers the formal Riemannian metric tensor "$e^{\gamma h}(dx^2 + dy^2)$" where h is a planar Gaussian free field and γ is a parameter in $(0, 2)$ (these two different perspectives were considered in the physics literature, see [6] for a recent discussion). The relation between the two approaches is expected to be given by $\mathbf{c} = 25 - 6(2/\gamma + \gamma/2)^2$.

Suppose given a metric tensor ds^2 on a two dimensional Riemannian manifold X. Then, under mild assumption, *locally*, it can be represented using isothermal coordinates by $ds^2 = \rho(du^2 + dv^2)$ for some smooth $\rho > 0$ and the associated conformal factor ϕ is given by $\rho = e^\phi$. Using the complex coordinate $z = u + iv$, the volume form and distance function are locally given by

$$e^{\phi(z)} d^2 z \qquad \text{and} \qquad \inf_{\pi : x \to y} \int_\pi e^{\frac{\phi}{2}} ds.$$

In what follows, we will be interested in the case where the conformal factor ϕ is a random Schwartz distribution with negative regularity, given by γh. The volume form and distance function will be given formally by

$$e^{\gamma h(z)} d^2 z \qquad \text{and} \qquad \inf_{\pi : x \to y} \int_\pi e^{\frac{\gamma h}{d_\gamma}} ds, \tag{2.1}$$

where $d_\gamma > 2$ will be the almost sure Hausdorff and Minkowski dimension of the γ-LQG.

Gaussian free fields. Gaussian free fields (GFF) are a generalization of Brownian motion to a higher dimensional indexing space, appear as the universal scaling limit of various random discrete surfaces [23,68,79,88] and play a fundamental role in mathematical physics, in particular in Quantum Field Theory [50, 104]. Formally, they are measures on fields h defined on a domain D such that

$$\rho(dh) \propto \exp\left(-\sigma^{-2} \int_D |\nabla h|^2 d\lambda\right) \mathcal{D}h \tag{2.2}$$

where $\mathcal{D}h$ is the (formal, infinite dimensional) Lebesgue measure on fields (which does not exist) and σ is a positive number. They can be realized as random Schwartz distribution and their covariance kernel is given by (a multiple of) the Green function associated with the Laplacian. In two dimension, they belong to the class of log-correlated Gaussian fields for which the covariance kernel is given on the diagonal by $\mathbb{E}(h(x)h(y)) = -\log|x - y| + O(1)$ and are conformally invariant measures, as inherited from the Dirichlet energy $\int_D |\nabla h|^2 d\lambda$. Furthermore, the field has an important domain Markov property. (See [35, 100, 113] for more on the GFF.)

Gaussian multiplicative chaos and Liouville measures. Gaussian multiplicative chaos (GMC) is the study of random measures of the form $e^{\gamma\phi}\sigma(dx)$ where $\gamma \in (0, \sqrt{2d})$ is a parameter, ϕ is a log-correlated Gaussian field on a domain D in $\mathbb{R}^d$ and $\sigma(dx)$ is an independent measure on D (in our case, σ will be typically absolutely continuous w.r.t. the Lebesgue measure). Due to the lack of regularity of ϕ, this does not make readily sense. Typically, one consider the approximating measures $\mu_{\phi_\varepsilon}(dz) = e^{\gamma\phi_\varepsilon(z) - \frac{\gamma^2}{2}\operatorname{Var}\phi_\varepsilon(z)}\sigma(dz)$ where $\phi_\varepsilon(x)$ denotes some regularization of ϕ at the space scale ε.

A simple application of Fubini theorem and a Gaussian computation show that the average total mass of these measures is conserved. With slightly additional work, one finds that the family of total masses is uniformly integrable. The renormalizing constants are here explicit functions of the covariance kernel of the log-correlated field. When we will be studying the metric associated with such fields rather than the measure, analog estimates can no longer be obtained in the same way. GMC theory [13, 44, 67, 93, 99] shows that μ_{ϕ_ε} converges in probability towards a Borel measure μ_ϕ on D for the topology of weak convergence and the limit is independent of the approximation scheme. Two properties are clear from the form of the above limit: μ_ϕ is locally determined by ϕ and, for any random continuous function f, $\mu_{\phi+f}(dx) = e^{\gamma f(x)}\mu_\phi(dx)$. This latter property is at the heart of a useful characterization of GMC measures due to Shamov in [99].

When the dimension is two and the field is the GFF, these measures are called *Liouville quantum gravity* measures and were studied by Duplantier and Sheffield in [44] who proved the convergence

$$\mu_h(dz) = \lim_{\varepsilon\to 0} \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(z)}\, dz$$

for the circle-average approximation of the field. Furthermore, they proved that μ_h satisfies a conformal coordinate change formula: if $f : D \to D'$ is a conformal map then, almost surely, $f_*\mu_h = \mu_{h\circ f^{-1} + Q\log|(f^{-1})'|}$ where

$$Q = \frac{2}{\gamma} + \frac{\gamma}{2} \tag{2.3}$$

Two pairs (D, h) and (D', h') which are related by a conformal map as above are considered as being two different parametrizations of the same LQG surface. Thus the coordinate change formula for μ_h says that this measure depends only on the quantum surface, not on the particular choice of parametrization. These measures are singular with respect to the Lebesgue measure and are supported on a set of Hausdorff dimension $2 - \gamma^2/2$. (See [8, 12, 90] for more on this.)

Liouville or GMC measures have been at the core of the definition of LQG *surfaces* (still without distance function). In particular, it paved the way to Liouville Conformal Field Theory (LCFT), beginning with the 2-sphere in [22] and extended to many other Riemann surfaces later on. The reason is that LCFT consists in reweighting the distribution of the Gaussian free field on the 2-sphere (or other surfaces) by the missing terms of the Liouville action functional, which includes the total mass of a GMC measure. This produces a family of (non Gaussian) probability measures on fields. The focus of the theory is on correlation functions, i.e., product of vertex operators $V_\alpha(x) = e^{\alpha h(x)}$ and in particular on the way correlations behave under conformal changes of metrics, differential equations they satisfy and exact formulas [52,69,70]. Beyond the relation with Conformal Field Theory, the importance of these works, in particular in the perspective of this thesis, is to make precise conjectures describing the scaling limits of random planar maps. (See [91, 112] for introductions to this topic.)

Quantum Loewner evolutions and the $\sqrt{\frac{8}{3}}$-LQG metric. Another approach is Sheffield's theory of quantum surfaces decorated by Schramm-Loewner Evolutions, initiated in [102]. In particular, [43] constructed Liouville quantum gravity on the 2-sphere (this construction and the one in LCFT is equivalent, as proved in [9]) together with a space-filling curve and proved that this corresponds to a mating of coupled Continuum Random Trees. This provided a precise geometric understanding of Liouville quantum gravity and played an important role in the series [81–83, 86] which constructed a metric for LQG in the case $\gamma = \sqrt{8/3}$ and proved its equivalence with the Brownian map. A key part in this program was played by the definition of a growth process called *quantum Loewner evolution* (QLE) in [83], whose construction is based on couplings between SLE curves and the GFF and SLE explorations. In particular, they showed that, in a specific case, this process represents growing metric balls of a metric and defined the distance between two points to be the time taken by this process to travel from one point to the other.

1.3 Liouville first passage percolation and the LQG metric

Liouville first passage percolation (LFPP) metrics refer to the distance functions associated with any approximation of the Gaussian free field. This direction of research was initiated by Ding and his collaborators and focused essentially on a discretization of the problem using the discrete Gaussian free field (DGFF). These early works, in the small γ regime, focused on estimating distances and studying qualitative property of the distances such as the fractal behavior of geodesics arising in the scaling limit (see, e.g., [27, 28, 31, 33, 34]).

Ding and Dunlap [25], still using the DGFF, showed that it is possible to renormalize the metrics (when γ is small enough) so as to obtain the existence of subsequential limits. Their approach uses a multiscale analysis to bound inductively a specific measure of dispersion given by the coefficient of variation. A key tool to achieve this is the Efron-Stein inequality, which bounds from above the fluctuation of a random variable of the form $F(X_1, \dots, X_n)$ with independent entries by a sum involving the influence of each variable in F. Along the way, they needed to prove Russo-Seymour-Welsh estimates associated with the side-to-side crossing distances of rectangles with various aspect ratio. Their method to prove these estimates is inspired by the work of Tassion [110]. However, the assumption γ small is already used for these estimates.

A ubiquitous theme in this thesis is the multiscale analysis with the Gaussian free field. The domain Markov property of the field implies the following. Divide a square into four subsquares, then, conditionally on some binding field which is harmonic in each subsquare, the restrictions of the field in each of these are independent and distributed according to a 0-boundary GFF. Repeating this decomposition provides a branching random walk type approximation which, in many situations, is nice enough to develop a multiscale analysis. However, this decomposition introduces a boundary effect throughout the decomposition, the need to control the binding field and each building block is (up to rescaling) associated to the specific choice of the unit square.

$\star$-scale invariant fields. The lack of a priori symmetries of the discrete Gaussian free field becomes a hurdle at the level of metrics. **In Chapter 2**, which is based on a joint work with Julien Dubédat [38], we study Liouville metrics associated with a $\star$-scale invariant Gaussian field with finite range correlation. They provide a simpler framework without binding field but rather with independence between scales and without boundary effects. They admit an ideal scale decomposition which simplifies the multiscale analysis: $\phi = \sum_{k \geq 0} \phi_k$ where the ϕ_k's are independent, smooth and distributed as $\phi_0(2^k \cdot)$ and ϕ_0 has a finite range of dependence, i.e., for some constant c, $\phi_0(x)$ and $\phi_0(x')$ are independent when $|x - x'| \geq c$. These fields have a canonical regularization which is to consider a cutoff at a small scale in their scale decomposition: in particular, set $\phi_{n_0,n} = \sum_{n_0 \leq k \leq n} \phi_k$.

One can also represent the field with a space-time white noise

$$\phi_{0,\infty}(x) := \int_{\mathbb{R}^2} \int_0^1 k\left(\frac{y-x}{t}\right) t^{-3/2} W(dy,dt), \quad \phi_{0,n} = \int_{\mathbb{R}^2} \int_{2^{-n}}^1 k\left(\frac{y-x}{t}\right) t^{-3/2} W(dy,dt),$$

where W is a space-time white noise and k is a bump function. This representation opens the use of Gaussian analysis at the level of the white noise and coupling arguments. Chapter 2 provides in particular Russo-Seymour-Welsh estimates for Liouville metrics associated with this field which hold for every parameter γ. It also investigates properties that should hold in the limit such as tail estimates and the consistence with the Weyl scaling. It also revisits some steps to prove the tightness of the metrics. We provide below some ideas of proof.

Multiplicativity of geodesics. When considering the length metric $e^{\gamma\phi_{0,2n}}ds$, one expects that geodesics satisfy the following: their $\approx 2^{-n}$ coarse grained version is a quasi-geodesic for $\phi_{0,n}$ and on a block of size $\approx 2^{-n}$ they essentially follow geodesics for $e^{\gamma\phi_{n,2n}}$. This is motivated from the decomposition $e^{\gamma\phi_{0,2n}} = e^{\gamma\phi_{0,n}}e^{\gamma\phi_{n,2n}}$, $\phi_{0,n}$ having mild oscillation at the scale 2^{-n} and the restrictions of the field $\phi_{n,2n}$ in two separated blocks at this scale are independent. From such a multiplicativity, one would naturally expects the existence of a scaling limit. In fact, it is difficult to show in great details that coarse grained geodesics are quasi-geodesic of a regularized version of the field. However, the representation of the distance using minimizing paths and planarity arguments are useful in the analysis. To obtain an upper bound on distances, because of the definition of the metric as an infimum over admissible paths, one can pick any them, among which one associated with the previous ansatz. When doing so, one ends up by concatenating together geodesics associated with long rectangle crossing distances (left-right crossing distance of a rectangle isometric to $[0,a] \times [0,b]$ with $a > b$). It is possible to obtain a similar but weaker lower bound, which involve rather a minimum over short rectangle crossing distances. This distinction between thin and long rectangle crossing distances is the reason of the need of some RSW type estimates.

Russo-Seymour-Welsh estimates. We first study the effect of a conformal map on the $\star$-scale invariant field and prove a coupling result between $\phi_{a,b} \circ F$ and $\phi_{a,b}$, where the subscripts represent scales between a and b. Intuitively, $\phi_{a,b} \circ F$ should be approximately distributed like "$\phi_{a/|F'|,b/|F'|}$"

which itself can be decomposed as $\phi_{a/|F'|,a} + \phi_{a,b} - \phi_{b,b/|F'|}$. When $|F'| \geq 1$, we prove such a decompoition where the high frequency field is independent of $\phi_{a,b}$. This relies on the independence of different scales of the field. With this result, we compare quantiles associated with long and thin rectangle crossing distances, uniformly in the approximating scale. A crossing path of thin rectangles implies a crossing path at lower scales of rectangles with the same aspect ratio (important for the hypothesis $|F'| \geq 1$) and the rectangle crossing at a smaller scale implies a crossing of marked sides of a thin ellipse at that scale. Then, one can map the crossing between marked sides of this thin ellipse to a crossing of a longer one at the initial scale. To send arcs from the small ellipse to those of the larger one, one can subdivide the marked sides. Finally, the coupling result when applying the conformal map is the key to compare left and right tails of rectangle crossing distances associated with different aspect ratio.

Percolation arguments and tail estimates. Recall the branching random walk approximation of the GFF. When one forgets about the binding field, then it remains only independent copies of 0-boundary GFF in distinct blocks. One can consider independent events in each of these blocks. When the probability of each event is $1-p$ for p small, then with very high probability, it is possible to find a path from the left to the right of the unit square for which the events occur on each block traversed. To provide estimates for the original problem, one adds back the coarse field and typically use some rough estimate for it. With the $\star$-scale invariant field with finite range of dependence, the coarse field is independent from the fine field and this later one has built in independence properties, thereby offering a nice framework for this type of argument.

Thanks to the Russo-Seymour-Welsh estimates, it is enough to study a single macroscopic length observable, the side-to-side distance of a square, denoted by L_n. Quoting from [109],"the concentration of measure phenomenon roughly states that, if a set A in a product Ω^N of probability spaces has measure at least one half, "most" of the points of Ω^N are "close" to A". Following this principle we prove tail estimates for $\log(L_n/\lambda_n)$, where λ_n is the median of L_n, which are relative to

$$\Lambda_n = \max_{k \leq n} \frac{\bar{\ell}_k(p)}{\ell_k(p)},$$

where $\ell_n(p)$, $\bar{\ell}_n(p)$ are the p-quantile and $(1-p)$-quantile of L_n and p is a fixed constant. The maximum Λ_n is itself a measure of dispersion that we want to bound inductively. The proof relies on this percolation estimate / rough estimate for the coarse field argument by using the following type of events: for right tails, one can glue together long rectangle crossings associated with the fine field for which the distance is small and for the left tails, one can consider blocking paths from the top to bottom on consisting of thin rectangle crossing distance for which the distance is large.

At the time Chapter 2 was about to be completed and following results on Liouville graph distance in [32] (a natural regularized distance function associated with LQG using the LQG measure), Ding and Gwynne [28] showed the existence of an increasing function d_γ called "the fractal dimension of LQG" defined on $(0,2)$. This non-explicit deterministic function arises from a subadditivity argument. They proved in particular that the map $\gamma \mapsto \gamma/d_\gamma$ is increasing and used

$$\xi = \frac{\gamma}{d_\gamma} \tag{3.4}$$

as the parameter associated with the LQG length functional in (2.1). What is crucial in [28] is not that there is an abstract exponent associated with distances but that this exponent has a representation: it is showed that Euclidean macroscopic distances associated with $e^{\xi h_\varepsilon} ds$ are of order $\varepsilon^{1-\xi Q+o(1)}$ where $Q = 2/\gamma + \gamma/2$ and where h_ε denote the GFF circle-average approximation.

In Chapter 3, which is based on a joint work with Jian Ding, Julien Dubédat and Alexander Dunlap, [24], we study the original problem involving the Gaussian free field and we regularize it by using a mollification with the heat kernel. The main result of this chapter is the existence of non-trivial subsequential limits corresponding to "$e^{\gamma h}(dx^2 + dy^2)$", in the range of γ for which a metric bi-Hölder with respect to the Euclidean metric was conjectured to exist.

The proof uses a coupling between two fields: one denoted by ψ with local independence properties (useful for percolation type arguments and in the geometric considerations arising from the Efron-Stein inequality) and another one, denoted by ϕ with better scaling properties (for simple scaling arguments but also when studying the effect of conformal transformations for the RSW estimates). We consider the smoothed Gaussian fields defined for $x \in \mathbb{R}^2$ and $\delta \in (0,1)$ by

$$\varphi_\delta(x) := \sqrt{\pi} \int_{\delta^2}^1 \int_{\mathbb{R}^2} p_{\frac{t}{2}}(x-y) W(dy,dt), \quad \psi_\delta(x) := \sqrt{\pi} \int_{\delta^2}^1 \int_{\mathbb{R}^2} \Phi_{\sigma_t}(x-y) p_{\frac{t}{2}}(x-y) W(dy,dt)$$

where W is a space-time white noise, p_t is the two-dimensional heat kernel, $\sigma_t = r_0\sqrt{t}|\log t|^{\varepsilon_0}$ and $\Phi_{\sigma_t}(\cdot) := \Phi(\cdot/\sigma_t)$ for a bump function Φ. The GFF on a compact domain and mollified by the heat kernel at time $t/2$ is comparable with $\varphi_{\sqrt{t}}$ in the bulk. φ_δ and ψ_δ are also comparable.

Bounds on dispersion and tightness of metrics. To simplify the discussion, we consider only one field $\phi_{0,n}$ as above. The Gaussian Poincaré inequality provides an a priori bound for Λ_K for any base scale K and to study the effect of small scales, we use the Efron-Stein inequality, relying on the product space distribution offered by a block decomposition of the white-noise. The analysis turns to the following condition. Denote by π_n a left-right geodesic of the unit square associated with the field $\phi_{0,n}$ and by π_n^K its $\approx 2^{-K}$ coarse graining. The condition asks for a uniform (in n) exponential decay (in K) of $\|e^{\gamma\phi_{0,K}}\|_{L^2(\pi_n^K)}/\|e^{\gamma\phi_{0,K}}\|_{L^1(\pi_n^K)}$. In words, this is ensured if one can prove that the weight of a geodesic is not essentially supported on a small number of coarse blocks. This Efron-Stein bound at the level of variance transfers at the level of quantiles and provides an inductive inequality for Λ_n which, together with the a priori bound, is enough to conclude. The tightness of the renormalized logarithm of the side-to-side distances is the starting point to study the tightness of metrics by using chaining arguments.

In Chapter 4, which is based on a joint work with Julien Dubédat, Ewain Gwynne, Joshua Pfeffer and Xin Sun [39], we continue to study the properties that the conjectural unique limit should have, as in Chapter 2 with the Weyl scaling and uniform tails. We work with the whole-plane GFF, which offers nice invariance properties. The chapter contains two parts. The first one consists in showing the existence of a distance function associated with the whole-plane GFF that satisfies a specific set of properties. In the second part, we consider an abstract metric satisfying these properties as axioms, which we call a "weak LQG metric" and we derive basic properties of this metric using only these axioms. The list of axioms is a natural one expected to characterize the LQG metric, at the exception of a "tightness across scales" property instead of a scaling property. In particular, we show that such metrics are bi-Hölder w.r.t. the Euclidean metric and derive tails estimates for side-to-side distances, point-to-point distances and for the diameter of a set.

Weak LQG metrics. A random distribution h on $\mathbb{C}$ is a *whole plane GFF plus a continuous function* if there exists a coupling of h with a random continuous function $f : \mathbb{C} \to \mathbb{R}$ such that the law of $h - f$ is that of a whole-plane GFF. The whole-plane GFF is defined only modulo a global additive constant, but this definition does not depend on the choice of additive constant. For $\gamma \in (0,2)$, a *weak γ-LQG metric* is a measurable function $h \mapsto D_h$ from $\mathcal{D}'(\mathbb{C})$ to the space of continuous metrics on $\mathbb{C}$ such that the following is true whenever h is a whole-plane GFF plus a continuous function.

I. **Length space.** Almost surely, $(\mathbb{C}, D_h)$ is a length space, i.e., the D_h-distance between any two points of $\mathbb{C}$ is the infimum of the D_h-lengths of continuous paths between the two points.

II. **Locality.** Fix an open set U. The D_h-internal metric $D_h(\cdot,\cdot;U)$ is determined a.s. by $h|_U$.

III. **Weyl scaling.** For each continuous function $f : \mathbb{C} \to \mathbb{R}$, define

$$(e^{\xi f} \cdot D_h)(z,w) := \inf_{P: z \to w} \int_0^{\mathrm{len}(P;D_h)} e^{\xi f(P(t))}\, dt, \quad \forall z, w \in \mathbb{C},$$

where the infimum is over all continuous paths from z to w parametrized by D_h-length. Then a.s. $e^{\xi f} \cdot D_h = D_{h+f}$ for every continuous function $f : \mathbb{C} \to \mathbb{R}$.

IV. **Translation invariance.** For each deterministic point $z \in \mathbb{C}$, a.s. $D_{h(\cdot+z)} = D_h(\cdot + z, \cdot + z)$.

V. **Tightness across scales.** Suppose that h is a whole-plane GFF and let $\{h_r(z)\}_{r>0, z\in\mathbb{C}}$ be its circle average process. For each $r > 0$, there is a deterministic constant $\mathfrak{c}_r > 0$ such that the set of laws of the metrics $\mathfrak{c}_r^{-1} e^{-\xi h_r(0)} D_h(r\cdot, r\cdot)$ for $r > 0$ is tight (w.r.t. the local uniform topology). Furthermore, the closure of this set of laws w.r.t. the Prokhorov topology on continuous functions $\mathbb{C} \times \mathbb{C} \to [0,\infty)$ is contained in the set of laws on continuous metrics on $\mathbb{C}$ (i.e., every subsequential limit of the laws of the metrics $\mathfrak{c}_r^{-1} e^{-\xi h_r(0)} D_h(r\cdot, r\cdot)$ is supported on metrics which induce the Euclidean topology on $\mathbb{C}$). Finally, there exists $\Upsilon > 1$ such that for each $\delta \in (0,1)$, $r > 0$, $\Upsilon^{-1}\delta^{\Upsilon} \leq \mathfrak{c}_{\delta r}/\mathfrak{c}_r \leq \Upsilon \delta^{-\Upsilon}$.

The existence of weak LQG metrics strongly relies on the tightness of Liouville first passage percolation metrics and the tightness across scale property is fundamental to provide a uniform control on the distribution of observables at scale r in terms of simple functions of the field.

Based on these previous works, Gwynne and Miller completed the construction of the LQG metric. [57] shows that subsequential limits are measurable w.r.t. the free field (and therefore that weak LQG metrics can be taken as measurable w.r.t. the free field). The article [58] studies confluence properties of the geodesics associated with weak LQG metrics. This is an essential input in [59] to prove that weak LQG metrics are unique in law. As a corollary of this uniqueness result, they obtained exact scaling of the metric in [59] and conformal symmetries of the metric on bounded domains in [56]. Altogether, this Liouville quantum gravity metric D_h satisfies the following:

1. D_h is almost surely bi-Hölder with respect to the Euclidean metric.

2. Weyl scaling: if f is a continuous function, then $D_{h+f} = e^{\xi f} \cdot D_h$.

3. Coordinate change: if $f : D \to D'$ is a conformal map, then $f_* D_h = D_{h \circ f^{-1} + Q \log |(f^{-1})'|}$.

4. The Hausdorff dimension of this metric space is almost surely given by d_γ.

5. Confluence: two geodesics that start from the same typical point share a non-trivial arc.

In Chapter 5, which is based on a joint work with Morris Ang and Xin Sun [5], we study the LQG volume of LQG metric balls and prove that d_γ is the Minkowski dimension of LQG. To obtain this result, we prove moment estimates for the volume of metric balls. Namely, if B is a metric ball centered at a fixed point with a given radius, we prove that $\mu_h(B)$ admits finite p-moments for every $p \in \mathbb{R}$. This is different from the volume of an Euclidean ball for which the finiteness of moments only holds for $p < 4/\gamma^2$. We use this estimate to prove that, for any compact $K \subset D$, for $\varepsilon \in (0,1)$, almost surely,

$$\sup_{s\in(0,1)} \sup_{z\in K} \frac{\mu_h\left(\mathcal{B}_s(z; D_h)\right)}{s^{d_\gamma - \varepsilon}} < \infty \quad \text{and} \quad \inf_{s\in(0,1)} \inf_{z\in K} \frac{\mu_h\left(\mathcal{B}_s(z; D_h)\right)}{s^{d_\gamma + \varepsilon}} > 0.$$

This result was known in the case of the Brownian map [74]. The study of moments of GMC measures for Euclidean balls is a classical result and its proof was used in other problems in the field. In our setup, the structure of the volume of metric balls is quite different and our techniques can be used in other setups as well. In particular, this is the case for the first exit time of Liouville Brownian motion (LBM) from metric balls. The Liouville Brownian motion is a diffusion process which is defined as an appropriate time change of the planar Brownian motion. In chapter 5, we prove estimates similar to the volume of metric balls ones for the first exit time of the LBM from a metric ball. Our result says that when starting a LBM at any point z, its exit time from a unitary metric ball has finite p-moments for every $p \in \mathbb{R}$ and its exit time from $\mathcal{B}_s(z; D_h)$ is of order s^{d_γ}.

Positive moments. Denote by $\mathcal{B}_1(0; D_h)$ the unit γ-LQG metric ball and by A_1 the annulus $B_1(0)\backslash\overline{B_{1/2}(0)}$. We explain why $\mathbb{E}[\mu_h(\mathcal{B}_1(0; D_h) \cap A_1)^k] < \infty$ for every $k \geq 1$. The starting point is to rewrite it via a Cameron-Martin shift, as

$$\int_{(A_1)^k} \exp(\gamma^2 \sum_{i<j} \mathrm{Cov}(h(z_i), h(z_j)))\mathbb{P}\left[D_{h+\gamma \sum_j \mathrm{Cov}(h(z_j),h(\cdot))}(0, z_i) < 1, \forall i\right] dz_1 \dots dz_k.$$

Since h is log-correlated, this is bounded from above by the following proxy

$$\int_{A_1^k} \frac{P_{z_1,\dots,z_k}}{\prod_{i<j} |z_i - z_j|^{\gamma^2}} dz_1 \dots dz_k, \quad \text{where} \quad P_{z_1,\dots,z_k} = \mathbb{P}[D_{h+\gamma \sum_j \mathrm{Cov}(h(z_j),h(\cdot))}(z_i, \partial B_{1/2}(z_i)) < 1, \forall i].$$

The volume of Euclidean balls have infinite kth moments when k is large due to the contribution of clusters at mutual distance r whose contribution is $r^{-2+2k-\binom{k}{2}\gamma^2}$ since the sum over dyadic r is finite if and only if $k < 4/\gamma^2$. When $k \geq 4/\gamma^2$, this is counterbalanced by the $P_{z_1,\dots,z_k}$ term. By an annulus crossing distance bound, on the associated event, for any $z \in K = \{z_1, \dots, z_k\}$,

$$1 \gtrsim D_{h+\gamma \sum_{i \leq k} \log|\cdot - z_i|^{-1}}(z, \partial B_{1/2}(z)) \gtrsim r^{\xi Q} e^{\xi h_r(z)} r^{-\xi k\gamma}.$$

Indeed, one can use an annulus centered at z, separating z from $\partial B_{1/2}(z)$ and at distance r of z, whose width is of the same order and the $r^{-\xi k\gamma}$ term comes from the circle average of the log-singularity. This constraint on the coarse field implies $P_{z_1,\dots,z_k} \lesssim \mathbb{P}\left[h_r(z) \leq -c_k \log r^{-1}\right] \approx r^{\frac{1}{2}(k\gamma - Q)^2}$ and the scale r contribution is $r^{\frac{1}{2}Q^2 - 2}$ which is summable for all k since $Q > 2$ for $\gamma \in (0, 2)$.

However, to turn this argument into a proof requires to consider all configurations of clusters $K = \{z_1, \dots, z_k\}$. Our proof works by induction on k: we use a specific splitting procedure of K into two well separated clusters I and J since both $\prod_{i<j} |z_i - z_j|^{\gamma^2}$ and $P_{z_1,\dots,z_k}$ have a nice hierarchical clusters structure (this is clear for the former, less for the later). In our implementation of these ideas, because we have to carry the Euclidean domains associated with the clusters I, J and K, we use $\star$-scale invariant fields and a formalism of random labelled trees to encode the hierarchical decomposition of clusters and the constraints in the scale decomposition of the field.

The short-range correlation of the fine field gives independence between well-separated clusters, and invariance properties of the $\star$-scale invariant field simplifies our multiscale analysis.

Negative moments. To bound from below the volume of metric balls requires significantly less efforts due to two results in the literature: first, it is known [44] and easy to prove that the LQG volume of Eulidean ball has log-normal left tails and second, it is known [60], but requires more work (and in particular a percolation type argument as presented above) to prove that one can find some Euclidean ball within LQG balls, quantitatively. Altogether, the remaining work is to find some Euclidean balls where the coarse field is not too small. Since the coarse field can be read from annulus crossing distances, the bound from below is achieved by finding in the unit LQG ball an Euclidean ball where some nearby annulus crossing distance in not too small.

Chapter 2: Liouville metric of star-scale invariant fields: tails and Weyl scaling

This chapter is adapted from joint work [38] with Julien Dubédat.

2.1 Introduction

In this chapter, the field $\phi_{0,\infty}$ is a log-correlated field with short-range correlations and is approximated by a martingale $\phi_{0,n}$ where each $\phi_{0,n}$ is a smooth field. More precisely, we consider a $\star$-scale invariant field whose covariance kernel is translation invariant and given by $C_{0,\infty}(x) = \int_1^\infty \frac{c(ux)}{u} du$, where $c = k * k$, for a nonnegative, compactly supported and radially symmetric bump function k. We decompose the field $\phi_{0,\infty}$ in a sum of self-similar fields, i.e., $\phi_{0,\infty} = \sum_{n \geq 0} \phi_n$, where the ϕ_n's are smooth independent Gaussian fields, such that ϕ_0 has a finite range of dependence and $(\phi_n(x))_{x \in \mathbb{R}^2}$ has the law of $(\phi_0(x2^n))_{x \in \mathbb{R}^2}$. We then denote by $\phi_{0,n}$ the truncated summation, i.e., $\phi_{0,n} = \sum_{0 \leq k \leq n} \phi_k$. This gives rise to a well-defined random Riemannian metric $e^{\gamma \phi_{0,n}} ds^2$, restricted for technical convenience to $[0,1]^2$, which is the main object studied in this chapter. Here, the length element is given by $e^{\frac{\gamma}{2}\phi_{0,n}} ds$.

In the article [66], the authors proved that any log-correlated field ϕ whose covariance kernel is given by $C(x,y) = -\log|x-y| + g(x,y)$, assuming some regularity on g, can be decomposed as $\phi = \phi_\star + \psi$ where $\phi_\star$ is a $\star$-scale invariant Gaussian field and ψ is a Gaussian field with Hölder regularity. A similar decomposition where the fields are independent can be obtained modulo a weaker property on $\phi_\star$. Let us also mention that $\star$-scale invariant log-correlated fields are natural since they appear in the following characterization (see [3]): if M is a random measure on $\mathbb{R}^d$ such that $\mathbb{E}(M([0,1]^d)^{1+\delta}) < \infty$ for some $\delta > 0$ and satisfying the following cascading rule: for every $\varepsilon \in (0,1)$,

$$(M(A))_{A \in \mathcal{B}(\mathbb{R}^d)} \overset{(d)}{=} \left(\int_A e^{\omega_\varepsilon(x)} M_\varepsilon(dx) \right)_{A \in \mathcal{B}(\mathbb{R}^d)}, \tag{1.1}$$

where $(M_\varepsilon(\varepsilon A))_{A \in \mathcal{B}(\mathbb{R}^d)} \overset{(d)}{=} \varepsilon^d (M(A))_{A \in \mathcal{B}(\mathbb{R}^d)}$ and where ω_ε is a stationary Gaussian field, independent of M_ε, with continuous sample paths, continuous and differentiable covariance kernel on $\mathbb{R}^d\backslash\{0\}$, then, up to some additional technical assumptions, M is the product of a nonnegative random variable $X \in L^{1+\delta}$ and an independent Gaussian multiplicative chaos $e^{\phi}dx$, i.e., $\forall A \in \mathcal{B}(\mathbb{R}^d),\ M(A) = X\int_A e^{\phi(x)-\frac{1}{2}\mathbb{E}(\phi(x)^2)}dx$. Moreover, the covariance kernel of ϕ is given by $C(x) = \int_1^\infty \frac{c(ux)}{u}du$ for some continuous covariance function c such that $c(0) \leq \frac{2d}{1+\delta}$ and notice that we have $C(x) \underset{x\to 0}{\sim} -c(0)\log\|x\|$. A natural question is to consider the metric instead of the measure to construct and characterize metrics on $\mathbb{R}^2$ satisfying a property analogous to (1.1) involving the Weyl scaling (see Section 2.7).

In our approach, we introduce a parameter $\gamma_c > 0$ associated to some observable of the metric and we study the phase where $\gamma < \gamma_c$. More precisely, if $L_{1,1}^{(n)}$ denotes the left-right length of the square $[0,1]^2$ for the random Riemannian metric $e^{\gamma\phi_{0,n}}ds^2$ and μ_n is its median, we then define $\gamma_c := \inf\{\gamma : (\log L_{1,1}^{(n)} - \log\mu_n) \text{ is not tight}\}$. We expect that the set of γ such that $(\log L_{1,1}^{(n)} - \log\mu_n)_{n\geq 0}$ is tight is $(0,\gamma_c)$. We prove that as soon as $\gamma < \gamma_c$, we have the following concentration result: for s large, uniformly in n,

$$ce^{-Cs^2} \leq \mathbb{P}\left(\log L_{1,1}^{(n)} - \log\mu_n \leq -s\right) \leq Ce^{-cs^2},$$
$$ce^{-Cs^2} \leq \mathbb{P}\left(\log L_{1,1}^{(n)} - \log\mu_n \geq s\right) \leq Ce^{-c\frac{s^2}{\log s}}.$$

When $\gamma < \min(\gamma_c, 0.4)$, we obtain the tightness of the metric spaces $([0,1]^2, d_{0,n})_{n\geq 0}$, where $d_{0,n}$ is the geodesic distance associated to the Riemannian metric tensor $e^{\gamma\phi_{0,n}}ds^2$, renormalized by μ_n. The main difference with the proof of Ding and Dunlap is that the RSW estimates do not rely on the method developped by Tassion [110] but follow from an approximate conformal invariance of $\phi_{0,n}$, obtained through a white noise coupling.

We also investigate the Weyl scaling: if $d_{0,\infty}$ is a metric obtained through a subsequential limit associated to the field $\phi_{0,\infty}$ and f is in the Schwartz class, then we prove that the metric associated to the field $\phi_{0,\infty} + f$ is $e^{\frac{\gamma}{2}f}\cdot d_{0,\infty}$, that the couplings $(\phi_{0,\infty} + f, e^{\frac{\gamma}{2}f}\cdot d_{0,\infty})$ and $(\phi_{0,\infty}, d_{0,\infty})$ are mutually absolutely continuous with respect to each other and that their Radon-Nikodým derivative is given by the one of the first marginal. Notice that if the metric $d_{0,\infty}$ is a measurable function of the field $\phi_{0,\infty}$, this property is expected. Here, this property tells us that the metric is not independent of the field $\phi_{0,\infty}$ and is in particular non-deterministic. In fact, this property is fundamental in the work of Shamov [99] on Gaussian multiplicative chaos, where the metric is replaced by the measure. It is used to prove that subsequential limits are measurable with respect to the field, which then implies its uniqueness and that the convergence in law holds in probability.

Shamov [99] takes the following definition of GMC. If ϕ is a Gaussian field on a domain D and M is a random measure on D, measurable with respect to ϕ and hence denoted by $M(\phi, dx)$, which satisfies, for f in the Cameron-Martin space of ϕ, almost surely,

$$M(\phi + f, dx) = e^{f(x)} M(\phi, dx), \tag{1.2}$$

then M is called a Gaussian multiplicative chaos. Furthermore, M is said to to be subcritical if $\mathbb{E}M$ is a σ-finite measure. Note that the left-hand side is well-defined since M is ϕ measurable. It is easy to check that the condition (1.2) implies uniqueness among ϕ-measurable subcritical random measures and we insist that the measurability of M with respect to ϕ is built in the definition. A natural question is thus the following: replace the measure M by the metric $d_{0,\infty}$, assume in a similar way the measurability with respect to ϕ and suppose that in (1.2), the operation is the Weyl scaling defined in Section 2.7, then is there uniqueness?

The chapter is organized as follows. In Section 2, we introduce the fields $\phi_{0,n}$ as well as the definitions and notations that will be used throughout the subsequent sections. Section 3 contains our main theorems. In Section 4, we derive the approximate conformal invariance of $\phi_{0,n}$ together with the RSW estimates. Section 5 is concerned with lognormal tail estimates for crossing lengths, upper and lower bounds. Under the assumption $\gamma < \min(\gamma_c, 0.4)$, we derive the tightness of the metric in Section 6. The Weyl scaling is discussed in Section 7. Section 8 is concerned with $\gamma_c > 0$. Lastly, in Section 9 we prove some independence of γ_c with respect to the bump function k used to define $\phi_{0,n}$. The appendix gathers estimates for the supremum of the field $\phi_{0,n}$ as well as an estimate for a summation which appears when deriving diameter estimates.

2.2 Definitions

2.2.1 Log-correlated Gaussian fields with short-range correlations

A *white noise* on $\mathbb{R}^d$ is a random Schwartz distribution such that for every test function f, $\langle \zeta, f\rangle$ is a centered Gaussian variable with variance $\|f\|^2_{L^2(\mathbb{R}^d)}$. If $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a probability space on which it is defined, we have a natural isometric embedding $L^2(\mathbb{R}^d) \hookrightarrow L^2(\Omega, \mathcal{F}, \mathbb{P})$. By extension, for $f \in L^2(\mathbb{R}^d)$, the pairing $\langle \zeta, f\rangle$ is also a centered Gaussian variable with variance $\|f\|^2_{L^2(\mathbb{R}^d)}$.

Let k be a smooth, radially symmetric and nonnegative bump function supported in $B(0, r_0) \subset \mathbb{R}^2$ and normalized in $L^2(\mathbb{R}^2)$ ($\int_{\mathbb{R}^2} k^2 dx = 1$), where r_0 is a fixed small positive real number. If ζ denotes a standard white noise on $\mathbb{R}^2$, then the convolution $k * \zeta$ is a smooth Gaussian field with covariance kernel $c := k * k$ whose compact support is included in $B(0, 2r_0)$. This can be taken as a starting point to define more general Gaussian fields. Let $\xi(dx, dt)$ be a white noise on $\mathbb{R}^2 \times [0, \infty)$. Then one can define a distributional Gaussian field on $\mathbb{R}^2$ by setting

$$\phi_{0,\infty}(x) := \int_{\mathbb{R}^2} \int_0^1 k\left(\frac{y-x}{t}\right) t^{-3/2} \xi(dy, dt)$$

with covariance kernel given by

$$\begin{aligned}\mathbb{E}\left(\phi_{0,\infty}(x)\phi_{0,\infty}(x')\right) &= \int_{\mathbb{R}^2} \int_0^1 k\left(\frac{x-y}{t}\right) k\left(\frac{y-x'}{t}\right) t^{-3} dy dt = \int_0^1 k * k\left(\frac{x-x'}{t}\right) \frac{dt}{t} \\ &= \int_0^1 c\left(\frac{x-x'}{t}\right) \frac{dt}{t}.\end{aligned}$$

Remark that for $x \neq x'$, the integrand vanishes near 0 since c has compact support, and that if $|x - x'| > 2r_0$, $\mathbb{E}(\phi_{0,\infty}(x)\phi_{0,\infty}(x')) = 0$. Denote $C(r) := \int_0^1 c(r/t) \frac{dt}{t}$. Then

$$C'(r) = \int_0^1 c'(r/t) \frac{dt}{t^2} = \int_0^\infty c'(r/t) \frac{dt}{t^2} - \int_1^\infty c'(r/t) \frac{dt}{t^2} = \frac{\alpha}{r} + f(r)$$

where $\alpha = \int_0^\infty c'(t^{-1}) \frac{dt}{t^2} = -c(0)$ and f is a smooth function. Consequently,

$$C(r) = \alpha \log r + F(r)$$

where F is smooth. By normalizing k in $L^2(\mathbb{R}^2)$, we have $c(0) = k * k(0) = \int_{\mathbb{R}^2} k^2 dx = 1$ and

$$C(r) = -\log r + F(r).$$

2.2.2 Decomposition of $\phi_{0,\infty}$ in a sum of self-similar fields

One can decompose $\phi_{0,\infty}$ as a sum of independent self-similar fields. Indeed, for $0 \leq m \leq n$, set

$$\phi_{m,n}(x) := \int_{\mathbb{R}^2} \int_{2^{-n-1}}^{2^{-m}} k\left(\frac{y-x}{t}\right) t^{-3/2} \xi(dy, dt) \tag{2.3}$$

as well as $\phi_n := \phi_{n,n}$ so that $\phi_{0,n} = \sum_{0 \leq k \leq n} \phi_k$ and $\phi_{0,\infty} = \sum_{n \geq 0} \phi_n$ where the ϕ_n's are independent. Notice also that for $1 \leq m \leq n$, $\phi_{0,n} = \phi_{0,m-1} + \phi_{m,n}$. The covariance kernel of ϕ_n is

$$\mathbb{E}\left(\phi_n(x)\phi_n(x')\right) = \int_{2^{-n-1}}^{2^{-n}} c\left(\frac{x-x'}{t}\right) \frac{dt}{t} =: C_n(\|x-x'\|)$$

so that $C_n(r) = C_0(r2^n)$. We will also denote by $C_{0,n}$ the covariance kernel of $\phi_{0,n}$. The following properties are clear from the construction.

Proposition 2.1. *For every $n \geq 0$,*

1. *ϕ_n is smooth,*

2. *the law of ϕ_n is invariant under Euclidean isometries,*

3. *ϕ_n has finite range dependence with range of dependence $2^{-n} \cdot 2r_0$,*

4. *and $(\phi_n(x))_{x\in\mathbb{R}^2}$ has the law of $(\phi_0(x2^n))_{x\in\mathbb{R}^2}$ (scaling invariance).*

5. *The ϕ_n's are independent Gaussian fields.*

Let us precise that one can see that ϕ_n is smooth from the representation (2.3) since k has compact support and ξ is a distribution (in the sense of Schwartz). This is a deterministic statement.

We will use repeatedly these properties throughout the chapter in particular the independence and scaling ones. Furthermore, one can decompose the field at scale n in spatial blocks. Specifically, we denote by $\mathcal{P}_n$ the set of dyadic blocks at scale n, viz.

$$\mathcal{P}_n := \left\{2^{-n}\left([i,i+1]\times[j,j+1]\right) \ : \ i,j\in\mathbb{Z}^2\right\}.$$

For $P \in \mathcal{P}_n$ we set

$$\phi_{n,P}(x) := \int_P \int_{2^{-n-1}}^{2^{-n}} k\left(\frac{y-x}{t}\right) t^{-3/2}\xi(dy,dt).$$

The following properties are immediate.

Proposition 2.2.

1. *The $\phi_{n,P}$'s are independent Gaussian fields.*

2. *For every $n \geq 0$ and $P \in \mathcal{P}_n$, $\phi_{n,P}$ is smooth and compactly supported in $P + B(0, 2^{-n}\cdot 2r_0)$.*

3. *If $P \in \mathcal{P}_n$, $Q \in \mathcal{P}_m$ and $l : P \to Q$ is an affine bijection, then $\phi_{m,Q}\circ l$ has the same law as $\phi_{n,P}$.*

Finally, we have the decomposition

$$\phi_{0,\infty} = \sum_{n\geq 0}\sum_{P\in\mathcal{P}_n} \phi_{n,P}$$

in which all the summands are independent smooth Gaussian fields, all identically distributed up to composition by an affine map and $\phi_{n,P}$ is supported in a neighborhood of P. In the following sections, we will work with the smooth fields $\phi_{0,n}$, approximations of the field $\phi_{0,\infty}$, and we denote by $\mathcal{F}_{0,n}$ the σ-algebra generated by the ϕ_k's for $0 \leq k \leq n$.

2.2.3 Rectangle lengths and definition of γ_c

For $a, b > 0$ and $0 \leq m \leq n$, we denote by $L_{a,b}^{(m,n)}$ the left-right length of the rectangle $[0,a]\times[0,b]$ for the Riemannian metric $e^{\gamma\phi_{m,n}}ds^2$, where the metric tensor is restricted to $[0,a]\times[0,b]$. When $m = 0$ we simply write $L_{a,b}^{(n)}$. To avoid confusion, let us point out that this is not the Riemannian metric on the full space restricted to the rectangle. In particular, all admissible paths are included in $[0,a]\times[0,b]$. It is clear that the spaces $([0,1]^2, e^{\gamma\phi_{0,n}}ds^2)$ and $([0,1]^2, ds^2)$ are bi-Lipschitz.

Consequently, $([0,1]^2, e^{\gamma\phi_{0,n}} ds^2)$ is a complete metric space and it has the same topology as the unit square with the Euclidean metric. We will denote by $\pi_{m,n}$ a minimizing path associated to $L_{a,b}^{(m,n)}$ and it will be clear depending on the context which a, b are involved. Notice that such a path exists by the Hopf-Rinow theorem and a compactness argument. We will say that a rectangle R is visited by a path π if $\pi \cap R \neq \emptyset$ and crossed by π if a subpath of π connects two opposite sides of R by staying in R.

We recall the *positive association* property and refer the reader to [89] for a proof.

Theorem 2.1. *If f and g are increasing functions of a continuous Gaussian field ϕ with pointwise nonnegative covariance, depending only on a finite-dimensional marginal of ϕ, then $\mathbb{E}(f(\phi)g(\phi)) \geq \mathbb{E}(f(\phi))\,\mathbb{E}(g(\phi))$.*

We will use this inequality several times in situations where the field considered is $\phi_{0,n}$ (since $k \geq 0$) and the functions f and g are lengths associated to different rectangles, without being restricted to a finite-dimensional marginal of $\phi_{0,n}$. If R is a rectangle, denote by $L^{(n)}(R,k)$ the left-right distance of R for the field $\phi_{0,n}^k$, piecewise constant on each dyadic block of size 2^{-k} where it is equal to the value of $\phi_{0,n}$ at the center of this block. We also denote by $L^{(n)}(R)$ the left-right distance of R for the field $\phi_{0,n}$. We have the following comparison,

$$e^{-O(2^{-k})\sup_{P\in\mathcal{P}_k, P\subset R}\|\nabla\phi_{0,n}\|_P} L^{(n)}(R) \leq L^{(n)}(R,k) \leq L^{(n)}(R) e^{O(2^{-k})\sup_{P\in\mathcal{P}_k, P\subset R}\|\nabla\phi_{0,n}\|_P}$$

which gives a.s. $\lim_{k\to\infty} L^{(n)}(R,k) = L^{(n)}(R)$.

If $R_1, \ldots, R_p$ denote $p \geq 2$ fixed rectangles, by an application of Portmanteau theorem and since $(L^{(n)}(R_1), \ldots, L^{(n)}(R_p))$ has a positive density with respect to the Lebesgue measure on $(0,\infty)^d$ (by the argument used in the proof of Proposition 2.9), if $l > 0$, we have, using Theorem 2.1,

$$\begin{aligned}\mathbb{P}\left(L^{(n)}(R_1) > l, \ldots, L^{(n)}(R_p) > l\right) &= \lim_{k\to\infty} \mathbb{P}\left(L^{(n)}(R_1,k) > l, \ldots, L^{(n)}(R_p,k) > l\right) \\ &\geq \lim_{k\to\infty} \mathbb{P}\left(L^{(n)}(R_1,k) > l\right) \ldots \mathbb{P}\left(L^{(n)}(R_p,k) > l\right) \\ &= \mathbb{P}\left(L^{(n)}(R_1) > l\right) \ldots \mathbb{P}\left(L^{(n)}(R_p) > l\right).\end{aligned}$$

Furthermore, if $F, G : (0,\infty)^{[0,1]^2} \to (0,\infty)$ are increasing functions such that

1. a.s. $\lim_{k\to\infty} F(\phi_{0,n}^k) = F(\phi_{0,n})$ and $\lim_{k\to\infty} G(\phi_{0,n}^k) = G(\phi_{0,n})$

2. $\mathbb{E}\left(F(\sup_{[0,1]^2} \phi_{0,n}) G(\inf_{[0,1]^2} \phi_{0,n})^{-1}\right) < \infty$, $\mathbb{E}\left(F(\sup_{[0,1]^2} \phi_{0,n})\right) < \infty$ and $\mathbb{E}\left(G(\inf_{[0,1]^2} \phi_{0,n})^{-1}\right) < \infty$,

then, by dominated convergence theorem and the negative association we have

$$\mathbb{E}(\frac{F(\phi_{0,n})}{G(\phi_{0,n})}) \quad = \quad \lim_{k\to\infty} \mathbb{E}(\frac{F(\phi_{0,n}^k)}{G(\phi_{0,n}^k)}) \quad \leq \quad \lim_{k\to\infty} \mathbb{E}(F(\phi_{0,n}^k))\mathbb{E}(\frac{1}{G(\phi_{0,n}^k)}) \quad = \quad \mathbb{E}(F(\phi_{0,n}))\mathbb{E}(\frac{1}{G(\phi_{0,n})}).$$

We introduce the notations $l_{a,b}^{(n)}(p) := \inf\{l \geq 0 \mid \mathbb{P}(L_{a,b}^{(n)} \leq l) > p\}$ for the p-th quantile associated to $L_{a,b}^{(n)}$ and $\bar{l}_{a,b}^{(n)}(p) := l_{a,b}^{(n)}(1-p)$. Since we will use repetitively $l_{1,3}^{(n)}(\varepsilon)$ and $\bar{l}_{3,1}^{(n)}(\varepsilon)$ for a small fixed ε, we introduce the notation l_n for the first one and $\bar{l}_n$ for the second one. Also, we will be interested by the ratio between these quantiles hence we introduce the notation $\delta_n := \max_{0\leq k\leq n} l_k^{-1}\bar{l}_k$ for $n \geq 0$. Finally, we introduce μ_n for the median of $L_{1,1}^{(n)}$ (note that $L_{1,1}^{(n)}$ has a positive density on $(0,\infty)$ with respect to the Lebesgue measure by the argument used in the proof of Proposition 2.9). We then define the critical parameter γ_c as

$$\gamma_c := \inf\left\{\gamma : \left(\log L_{1,1}^{(n)} - \log \mu_n\right) \text{ is not tight}\right\}$$

and we call *subcriticality* the regime $\gamma < \gamma_c$. Note that anytime we use the assumption $\gamma < \gamma_c$, we use only the tightness of $\log L_{1,1}^{(n)} - \log \mu_n$. However, we expect that the set of γ such that $(\log L_{1,1}^{(n)} - \log \mu_n)_{n\geq 0}$ is tight is the interval $(0, \gamma_c)$.

2.2.4 Compact metric spaces: uniform and Gromov-Hausdorff topologies

We recall first the notion of uniform convergence. A sequence $(d_n)_{n\geq 0}$ of real-valued functions on $[0,1]^2 \times [0,1]^2$ *converges uniformly* to a function d if

$$\sup_{x,x'\in[0,1]^2} \left|d_n(x,x') - d(x,x')\right| \underset{n\to\infty}{\longrightarrow} 0.$$

If d_n are moreover distances on $[0,1]^2$, then d is a priori only a pseudo-distance, i.e., $d(x,y) = 0$ with $x \neq y$ may occur.

Moreover, we recall the definition of the Hausdorff distance. If K_1, K_2 are two compact subsets of a metric space (E, d), the Hausdorff distance d_H between K_1 and K_2 is defined by

$$d_H(K_1, K_2) := \inf \{\varepsilon > 0 : K_1 \subset U_\varepsilon(K_2) \text{ and } K_2 \subset U_\varepsilon(K_1)\}$$

where for $i = 1, 2$, $U_\varepsilon(K_i) := \{x \in E : d(x, K_i) < \varepsilon\}$ is the ε-enlargement of K_i.

We recall now the definition of the Gromov-Hausdorff distance. Let (E_1, d_1) and (E_2, d_2) be two compact metric spaces. The Gromov-Hausdorff distance d_{GH} between E_1 and E_2 is defined as

$$d_{GH}(E_1, E_2) := \inf \{d_H(\phi_1(E_1), \phi_2(E_2))\}$$

where the infimum is over all isometric embeddings $\phi_1 : E_1 \to E$ and $\phi_2 : E_2 \to E$ of E_1 and E_2 into the same metric space (E, d). Here, d_H is the Hausdorff distance associated to the space (E, d). Denote by $\mathbb{M}$ the set of all isometry classes of compact metric spaces (see [51] Section 3.11). The Gromov-Hausdorff distance d_{GH} is a metric on $\mathbb{M}$ and $(\mathbb{M}, d_{GH})$ is a Polish space. We refer the reader to the textbook [17], Section 7 for more details on these topologies.

In our framework, we introduce the sequence of compact metric spaces $(M_n)_{n\geq 0}$ where $M_n := ([0,1]^2, d_{0,n})$ and where $d_{0,n}$ is the geodesic distance induced by the Riemannian metric tensor $\mu_n^{-2} e^{\gamma\phi_{0,n}} ds^2$ restricted to $[0,1]^2$ and we aim to study the convergence in law of M_n to a random metric space M_∞ with respect to the Gromov-Hausdorff topology.

2.2.5 Notation

We will denote by c and C constants whether they should be thought as small or large. They may vary from line to line and depend on the parameters (e.g. the bump function k) or geometry when these are fixed. At the only place of the chapter when we take γ small, but fixed, γ is taken small compared to a constant which does not depend on γ (as soon as we assume that γ is less than an absolute constant, upper bounds like $e^{\gamma\sqrt{k}}$ may be replaced by $e^{C\sqrt{k}}$).

If $F : E \to \mathbb{C}$ is a complex-valued function, we denote by $\|F\|_\infty := \sup_{x\in E} |F(x)|$ and by $\|F\|_{C^\alpha(E)} := \|F\|_\infty + \sup_{x\neq y\in E} \frac{|F(x)-(y)|}{|x-y|^\alpha}$. For $d \geq 1$, $\mathcal{S}(\mathbb{R}^d)$ denotes the space of Schwartz functions and $\mathcal{S}'(\mathbb{R}^d)$ denotes the space of tempered distributions. Our convention for the Fourier transform of a function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ is $\hat{\varphi}(\xi) := \int_{\mathbb{R}^d} \varphi(x)e^{-ix\cdot\xi}dx$. If x is a real number we will denote by x_+ the maximum of x and 0. For two real numbers a and b we denote by $a \vee b := \max(a, b)$ as well as $a \wedge b := \min(a, b)$. Finally, if X is a random variable, $\mathcal{L}(X)$ denotes its law and for $x \in \mathbb{R}$ we set $F_X(x) := \mathbb{P}(X \leq x)$.

2.3 Statement of main results

Our first main result concerns the relation between lengths of rectangles with different aspect ratio. We want to compare the tails of $L^{(n)}_{a,b}$ for various choices of (a, b). Notice that if $a' \leq a$, $b' \leq b$, a.s.

$$L^{(n)}_{a',b} \leq L^{(n)}_{a,b} \leq L^{(n)}_{a,b'}.$$

In particular, this gives $l^{(n)}_{a',b}(p) \leq l^{(n)}_{a,b}(p) \leq l^{(n)}_{a,b'}(p)$ for every p in $(0, 1)$. The following Russo-Seymour-Welsh estimates give upper bounds of left-right crossing lengths of long rectangles in terms of left-right crossing lengths of short rectangles.

Theorem 2.2. *If $[A, B] \subset (0, \infty)$ there exists $C > 0$ such that for every $(a, b), (a', b') \in [A, B]$ with $a/b < 1 < a'/b'$ and for every $n \geq 0$, $\varepsilon < 1/2$ we have*

$$l^{(n)}_{a',b'}(\varepsilon/C) \leq l^{(n)}_{a,b}(\varepsilon)Ce^{C\sqrt{|\log \varepsilon/C|}}, \tag{3.4}$$

$$\bar{l}^{(n)}_{a',b'}(3\varepsilon^{1/C}) \leq \bar{l}^{(n)}_{a,b}(\varepsilon)Ce^{C\sqrt{|\log \varepsilon/C|}}. \tag{3.5}$$

In the article [25], Ding and Dunlap obtained a related result (see Theorem 5.1 in [25]), inspired by [110]. Their result applies to a rather general setting whereas here we rely on some approximate conformal invariance of the field considered. However the result in [25] holds for γ small and this is a comparison for low quantiles only. Here we obtain comparisons for low, as well as high, quantiles, and there is no assumption on γ. Furthermore, the RSW estimates obtained here are also quantitative: this is instrumental for instance in the proof of left tail estimates.

Theorem 2.3. *If $\gamma < \gamma_c$, the left-right length for various aspect ratio renormalized by μ_n is tight and its tails are quasi-lognormal, i.e., if $[A, B] \subset (0, \infty)$ there exist constants $c > 0$, $C > 0$ such that* *for every* $(a,b) \in [A,B]$*,* $n \geq 0$*,* $s > 1$*:*

$$\mathbb{P}\left(L_{a,b}^{(n)} \geq \mu_n e^{s\sqrt{\log s}}\right) \leq Ce^{-cs^2}, \tag{3.6}$$

$$\mathbb{P}\left(L_{a,b}^{(n)} \leq \mu_n e^{-s}\right) \leq Ce^{-cs^2}. \tag{3.7}$$

These estimates are fundamental ingredients to get:

Theorem 2.4. *Assume that* $\gamma < \min(\gamma_c, 0.4)$*. Then:*

1. *The sequence of compact metric spaces* $(M_n)_{n\geq 0}$ *where* $M_n := \left([0,1]^2, d_{0,n}\right)$ *and where* $d_{0,n}$ *is the geodesic distance induced by the Riemannian metric* $\mu_n^{-2} e^{\gamma\phi_{0,n}} ds^2$ *is tight with respect to the uniform and Gromov-Hausdorff topologies.*

2. *If* (n_k) *is a subsequence along which* $(d_{n_k})_{k\geq 0}$ *converges in law to some* $d_{0,\infty}$*, then for* $f \in \mathcal{S}(\mathbb{R}^2)$*,* $(d_{n_k}, e^{\frac{\gamma}{2}f} \cdot d_{n_k})_{k\geq 0}$ *converges in law to* $(d_{0,\infty}, e^{\frac{\gamma}{2}f} \cdot d_{0,\infty})$ *(see Section 2.7 for a definition of the Weyl scaling).*

3. *Moreover,* $(\phi_{0,\infty} + f, e^{\frac{\gamma}{2}f} \cdot d_{0,\infty})$ *is absolutely continuous with respect to* $(\phi_{0,\infty}, d_{0,\infty})$ *and the associated Radon-Nikodým derivative is the one associated to the first marginal, i.e.,* $\frac{d\mathcal{L}(\phi_{0,\infty}+f)}{d\mathcal{L}(\phi_{0,\infty})}$.

We will also check that $\gamma_c > 0$ which is the content of:

Theorem 2.5. *For every choice of bump function* k*,* $\gamma_c(k) > 0$*.*

The general proof scheme of this result is similar to the one in [25]. The key tool is the Efron-Stein inequality, which was introduced by Kesten in the context of *Euclidean* first passage percolation. It was first used by Ding and Dunlap in a multiscale analysis to study Liouville first passage percolation metrics. Let us mention a few key differences in the implementation of that concentration argument.

In [25], the authors use the Efron-Stein inequality to give an upper bound of $\mathrm{Var}(L_{1,1}^{(n)})$, in order to control inductively the coefficient of variation of $L_{1,1}^{(n)}$, defined as

$$CV^2(L_{1,1}^{(n)}) := \frac{\mathrm{Var}(L_{1,1}^{(n)})}{\mathbb{E}(L_{1,1}^{(n)})^2}.$$

Here, since we expect that the logarithm of the normalized left-right distance is tight, we apply the Efron-Stein inequality to $\log L_{1,1}^{(n)}$ (the underlying product structure is provided naturally by the white noise representation of the field). We recall the notation for quantiles $\bar{l}_{1,1}^{(k)}(p)$, $l_{1,1}^{(k)}(p)$, defined such that $\mathbb{P}(L_{1,1}^{(k)} \geq \bar{l}_{1,1}^{(k)}(p)) = p$ and $\mathbb{P}(L_{1,1}^{(k)} \leq l_{1,1}^{(k)}(p)) = p$, and set

$$\delta_n(p) := \max_{k \leq n} \frac{\bar{l}_{1,1}^{(k)}}{l_{1,1}^{(k)}}(p)$$

which is the quantity we want to bound inductively; p is chosen small enough but fixed so that our tail estimates hold. The starting point of the induction is the inequality

$$\frac{\bar{l}_{1,1}^{(n)}}{l_{1,1}^{(n)}}(p) \leq e^{C_p \sqrt{\operatorname{Var} \log L_{1,1}^{(n)}}}.$$

Here the multiscale analysis, relying in particular on tail estimates (let us point out that instead of quasi-Gaussian bounds, super-exponential bounds would suffice) shows that, for γ small (but which can be quantified) for some $c_\gamma < 1$, we have

$$\operatorname{Var} \log L_{1,1}^{(n)} \leq \gamma^2 \left(C + C\delta_{n-1}(p)^2 \sum_{k=1}^{\infty} c_\gamma^k \right)$$

The absence of an explicit bound on γ_c comes from the fact that we take γ small enough in this inequality to bound inductively $\delta_n(p)$.

Finally, we will work out some independence of the parameter γ_c with respect to the choice of the bump function which is the content of

Theorem 2.6. *If k_1 and k_2 are two bump functions such that $\hat{k}_1(\xi) = e^{-a\|\xi\|^\alpha(1+o(1))}$ and $\hat{k}_2(\xi) = e^{-b\|\xi\|^\alpha(1+o(1))}$, as ξ goes to infinity, for some $\alpha \in (0,1)$ and $a, b > 0$, then $\gamma_c(k_1) = \gamma_c(k_2)$.*

2.4 Russo-Seymour-Welsh estimates: proof of Theorem 2.2

In this section we prove that our approximation $\phi_{0,n}$ of $\phi_{0,\infty}$ is approximately conformally invariant. We will then investigate its consequences on the length of left-right crossings: the RSW estimates, Theorem 2.2, which is a key result of our analysis. Let us already point out that these

RSW estimates eventually lead, as a first corollary, to a lognormal decay of the left tail (inequality (3.7), without assuming $\gamma < \gamma_c$ but with a small quantile instead of the median).

2.4.1 Approximate conformal invariance of $\phi_{0,n}$

Let $F : U \to V$ be a conformal map between two Jordan domains. We wish to compare the laws of $\phi_{0,n}$ and $\phi_{0,n} \circ F$ in U and look for a uniform estimate in n. For this we go back to the defining white noises. We write, for ξ and $\tilde{\xi}$ two standard white noises

$$\phi_{0,n}(x) := \int_{\mathbb{R}^2} \int_{2^{-n-1}}^{1} k\left(\frac{x-y}{t}\right) t^{-3/2} \xi(dy, dt), \quad \tilde{\phi}_{0,n}(x) := \int_{\mathbb{R}^2} \int_{2^{-n-1}}^{1} k\left(\frac{x-y}{t}\right) t^{-3/2} \tilde{\xi}(dy, dt),$$

and we want to couple $\phi_{0,n}$ and $\tilde{\phi}_{0,n} \circ F$, in particular for the high-frequency modes. We couple the defining white noises $\xi, \tilde{\xi}$ in the following way: if $y' \in V$, $y \in U$, $y' = F(y)$, $t' = t|F'(y)|$, then

$$\tilde{\xi}(dy', dt') = \left|F'(y)\right|^{3/2} \xi(dy, dt)$$

i.e., for a test function ϕ compactly supported in $V \times (0, \infty)$,

$$\int \phi(y', t')\tilde{\xi}(dy', dt') = \int \phi(F(y), t|F'(y)|) \left|F'(y)\right|^{3/2} \xi(dy, dt)$$

and both sides have variance $\|\phi\|_{L^2}^2$. The rest of the white noises are chosen to be independent, i.e., $\xi_{|U^c \times (0,\infty)}$, $\xi_{|U \times (0,\infty)}$ and $\xi_{|\tilde{V}^c \times (0,\infty)}$ are jointly independent. Assuming $|F'| \geq 1$ on U, since

$$\int_V \int_{2^{-n-1}}^{1} k\left(\frac{F(x)-y}{t}\right) t^{-3/2} \tilde{\xi}(dy, dt) = \int_U \int_{2^{-n-1}|F'(y)|^{-1}}^{|F'(y)|^{-1}} k\left(\frac{F(x)-F(y)}{t\,|F'(y)|}\right) t^{-3/2} \xi(dy, dt),$$

we can decompose $\phi_{0,n}(x) - \tilde{\phi}_{0,n}(F(x)) = \delta\phi_1(x) + \delta\phi_2(x) + \delta\phi_3(x)$ where

$$\begin{aligned}
\delta\phi_1(x) &= \int_{U^c}\int_{2^{-n-1}}^{1} k\left(\frac{x-y}{t}\right) t^{-3/2}\xi(dy,dt) - \int_{V^c}\int_{2^{-n-1}}^{1} k\left(\frac{F(x)-y}{t}\right) t^{-3/2}\tilde{\xi}(dy,dt) \\
&\quad + \int_{U}\int_{|F'(y)|^{-1}}^{1} k\left(\frac{x-y}{t}\right) t^{-3/2}\xi(dy,dt), \\
\delta\phi_2(x) &= \int_{U}\int_{2^{-n-1}}^{|F'(y)|^{-1}} \left(k\left(\frac{x-y}{t}\right) - k\left(\frac{F(x)-F(y)}{t\,|F'(y)|}\right)\right) t^{-3/2}\xi(dy,dt), \\
\delta\phi_3(x) &= -\int_{U}\int_{2^{-n-1}|F'(y)|^{-1}}^{2^{-n-1}} k\left(\frac{F(x)-F(y)}{t\,|F'(y)|}\right) t^{-3/2}\xi(dy,dt).
\end{aligned}$$

Remark also that $\delta\phi_3$ is independent of $\phi_{0,n}$, $\delta\phi_1$, and $\delta\phi_2$. We will estimate these three terms separately on a convex compact subset K of an open convex set U under the assumption that $\|F'\|_{U,\infty} < \infty$ and $\|F''\|_{U,\infty} < \infty$ and $|F'| \geq 1$ on U.

Lemma 2.7. *$\delta\phi_1$ restricted to K is a smooth field; more precisely there exists $C > 0$ such that for every $n \geq 0$*

$$\mathbb{E}\left(\|\delta\phi_1\|_{C^1(K)}\right) \leq C.$$

Proof. If $x \in K$, since k has compact support included in $B(0, r_0)$ we can write

$$\int_{U^c}\int_{2^{-n-1}}^{1} k\left(\frac{x-y}{t}\right) t^{-3/2}\xi(dy,dt) = \int_{U^c}\int_{(1\wedge d(K,U^c)/r_0)\vee 2^{-n-1}}^{1} k\left(\frac{x-y}{t}\right) t^{-3/2}\xi(dy,dt).$$

The idea is the same for the second term. For the third term, $|F'(y)| \leq \|F'\|_{U,\infty}$ hence

$$\int_{U}\int_{|F'(y)|^{-1}}^{1} k\left(\frac{x-y}{t}\right) t^{-3/2}\xi(dy,dt) = \int_{U}\int_{\|F'\|_{U,\infty}^{-1}}^{1} 1_{1\leq t|F'(y)|} k\left(\frac{x-y}{t}\right) t^{-3/2}\xi(dy,dt)$$

which concludes the proof: the smoothness follows standard results of distribution in the sense of Schwartz. □

Lemma 2.8. *There exists $C > 0$ such that for every $n \geq 0$ and every $x, x' \in K$,*

$$\mathbb{E}\left((\delta\phi_2(x) - \delta\phi_2(x'))^2\right) \leq C\left|x - x'\right|.$$

We also have $\mathbb{E}\left(\delta\phi_2(x)^2\right) \leq C$ uniformly in $x \in K$ and $n \geq 0$.

Proof. Since k is rotationally invariant and has compact support, we will see that

$$k\left(\frac{x-y}{t}\right) = k\left(\frac{F(x)-F(y)}{t\,|F'(y)|}\right) + O(t). \tag{4.8}$$

First, k having a compact support included in $B(0, r_0)$ gives

$$k\left(\frac{x-y}{t}\right) = k\left(\frac{x-y}{t}\right) 1_{\frac{|x-y|}{t}\le r_0} = k\left(\frac{x-y}{t}\right) 1_{t\ge \frac{|x-y|}{r_0}}$$
$$k\left(\frac{F(x)-F(y)}{t\,|F'(y)|}\right) = k\left(\frac{F(x)-F(y)}{t\,|F'(y)|}\right) 1_{\frac{|F(x)-F(y)|}{t|F'(y)|}\le r_0} = k\left(\frac{F(x)-F(y)}{t\,|F'(y)|}\right) 1_{t\ge \frac{|F(x)-F(y)|}{r_0|F'(y)|}}$$

Since $|F'| \geq 1$ on U and $\|F'\|_{U,\infty} < \infty$

$$\frac{|F(x)-F(y)|}{|F'(y)|} \geq \frac{|F^{-1}(F(x)) - F^{-1}(F(y))|}{\|F'\|_{U,\infty}\,\|(F^{-1})'\|_{V,\infty}} = \frac{|x-y|}{C}$$

hence we can directly replace the term $1_{t\ge \frac{|F(x)-F(y)|}{r_0|F'(y)|}}$ by $1_{t\ge \frac{|x-y|}{Cr_0}}$. By Taylor's inequality, $|F(x) - F(y) - F'(y)(x-y)| \leq \frac{1}{2}|x-y|^2\,\|F''\|_{U,\infty}$ thus

$$\left|\frac{F(x)-F(y)}{t\,|F'(y)|} - \frac{x-y}{t}\frac{F'(y)}{|F'(y)|}\right| \leq \frac{|x-y|^2}{2t}\frac{\|F''\|_{U,\infty}}{|F'(y)|}.$$

Using the compact support together with the rotational invariance of k, we get

$$\left|k\left(\frac{F(x)-F(y)}{t\,|F'(y)|}\right) - k\left(\frac{x-y}{t}\right)\right| \leq \|\nabla k\|_\infty \frac{\|F''\|_{U,\infty}}{|F'(y)|}\frac{|x-y|^2}{2t} 1_{t\ge\frac{|x-y|}{Cr_0}} \leq \frac{1}{2}\|\nabla k\|_\infty \left\|F''\right\|_{U,\infty}(Cr_0)^2 t$$

which gives (4.8). Finally, we obtain the following bound

$$\begin{aligned}
&\left(k\left(\frac{x-y}{t}\right) - k\left(\frac{F(x)-F(y)}{t\,|F'(y)|}\right)\right) - \left(k\left(\frac{x'-y}{t}\right) - k\left(\frac{F(x')-F(y)}{t\,|F'(y)|}\right)\right)\\
&= \left(k\left(\frac{x-y}{t}\right) - k\left(\frac{x'-y}{t}\right)\right) - \left(k\left(\frac{F(x)-F(y)}{t\,|F'(y)|}\right) - k\left(\frac{F(x')-F(y)}{t\,|F'(y)|}\right)\right)\\
&= O\left(t \wedge \frac{|x'-x|}{t}\right)
\end{aligned}$$

where in the last equation we both used equation (4.8) and the inequalities, for $x, x' \in K$ and $y \in U$:

$$\left|k\left(\frac{x-y}{t}\right) - k\left(\frac{x'-y}{t}\right)\right| \leq \|\nabla k\|_\infty \frac{|x-x'|}{t}$$

and

$$\left|k\left(\frac{F(x)-F(y)}{t\,|F'(y)|}\right)-k\left(\frac{F(x')-F(y)}{t\,|F'(y)|}\right)\right|\le\|\nabla k\|_\infty\frac{|F(x)-F(x')|}{t\,|F'(y)|}\le\|\nabla k\|_\infty\left\|F'\right\|_{K,\infty}\frac{|x-x'|}{t}.$$

It follows that

$$\begin{aligned}\mathbb{E}\left((\delta\phi_2(x)-\delta\phi_2(x'))^2\right)=&\int_U\int_{2^{-n-1}}^{|F'(y)|^{-1}}\left(\left(k\left(\frac{x-y}{t}\right)-k\left(\frac{F(x)-F(y)}{t\,|F'(y)|}\right)\right)\right.\\&\left.-\left(k\left(\frac{x'-y}{t}\right)-k\left(\frac{F(x')-F(y)}{t\,|F'(y)|}\right)\right)\right)^2t^{-3}dtdy\\\le&\int_0^1O\left(t\wedge\frac{|x-x'|}{t}\right)^2\int_{\mathbb{R}^2}1_{y\in B(x,tCr_0)\cup B(x',tCr_0)}dyt^{-3}dt\\\le&\,C\int_0^1\left(t\wedge\frac{|x-x'|}{t}\right)^2\frac{dt}{t}\end{aligned}$$

But this integral is bounded from above by $C\int_0^{\sqrt{|x-x'|}}tdt+C|x-x'|^2\int_{\sqrt{|x-x'|}}^1t^{-3}dt\le C\,|x-x'|$, where the constant C in the right-hand side is uniform in n. The second assertion directly follows from an analogous computation without keeping track of the x,x'.

□

Proposition 2.3. *There exist $C>0$, $\sigma^2>0$ such that for every $n\ge0$, $x\ge0$,*

$$\mathbb{P}\left(\left\|(\delta\phi_1+\delta\phi_2)_{|K}\right\|_\infty\ge x\right)\le Ce^{-x^2/\sigma^2}.$$

Proof. We have obtained in Lemma 2.8 a bound on the variance of $\delta\phi_2(x)-\delta\phi_2(x')$ which is a centered Gaussian variable, hence it follows that $\mathbb{E}\left((\delta\phi_2(x)-\delta\phi_2(x'))^{2p}\right)=O(|x-x'|^p)$. By the Kolmogorov continuity criterion, for any $\alpha<1/2$, $\mathbb{E}(\|\delta\phi_2\|_{C^\alpha(K)})$ is bounded in n. Together with Lemma 2.7, this shows $\mathbb{E}(\left\|(\delta\phi_1+\delta\phi_2)_{|K}\right\|_\infty)$ is bounded. Consequently by Fernique (see [46]), we have a uniform Gaussian tail estimate in n. □

We are left with the noise $\delta\phi_3$ which is independent of $\phi_{0,n}$, $\delta\phi_1$ and $\delta\phi_2$.

Lemma 2.9. *There exists $C>0$ such that for every $x\in K$, $n\ge0$, $\mathbb{E}\left(\delta\phi_3(x)^2\right)\le C$.*

Proof. Since $|F'(y)|^{-1}\ge\|F'\|_{U,\infty}^{-1}=c>0$ holds for every $y\in U$ and as seen in the proof of Lemma

2.8 we can directly replace the term $1_{t\geq\frac{|F(x)-F(y)|}{r_0|F'(y)|}}$ by $1_{t\geq\frac{|x-y|}{Cr_0}}$. This gives:

$$\begin{aligned}\mathbb{E}\left(\delta\phi_3(x)^2\right) &= \int_U \int_{2^{-n-1}|F'(y)|^{-1}}^{2^{-n-1}} k\left(\frac{F(x)-F(y)}{t\,|F'(y)|}\right)^2 t^{-3}dtdy \\ &\leq \|k\|_\infty^2 \int_{c2^{-n-1}}^{2^{-n-1}} \int_{\mathbb{R}^2} 1_{y\in B(x,tCr_0)} t^{-3}dydt \\ &\leq \|k\|_\infty^2 \int_{c2^{-n-1}}^{2^{-n-1}} Ct^2t^{-3}dt\end{aligned}$$

which concludes the proof. □

In summary, we have seen that along this white noise coupling,

$$\phi_{0,n} - \tilde{\phi}_{0,n} \circ F = \delta\phi_1 + \delta\phi_2 + \delta\phi_3 \tag{4.9}$$

where $\delta\phi_1$ and $\delta\phi_2$ are low frequency noises with uniform Gaussian tails and $\delta\phi_3$ is a high frequency noise with bounded pointwise variance and dependence scale $O(2^{-n})$, which is independent of $\phi_{0,n}$, $\delta\phi_1$ and $\delta\phi_2$.

2.4.2 RSW estimates for crossing lengths

Now we investigate the consequences of the approximate conformal invariance on crossing lengths. More precisely we want to show that the tails of the crossing lengths of rectangles of varying aspect ratios are comparable, uniformly in the roughness of the conformal factor by using (4.9).

Let A, B be two boundary arcs of K and denote by L the distance from A to B in K for the Riemannian metric $e^{\gamma\phi_{0,n}}ds^2$; we denote $A' := F(A)$, $B' := F(B)$, $K' := F(K)$, and L' is the distance from A' to B' in K' for $e^{\gamma\tilde{\phi}_{0,n}}ds^2$.

Proposition 2.4. *(Left tail estimate). If for some $l > 0$ and $\varepsilon < 1/2$, $\mathbb{P}(L \leq l) \geq \varepsilon$,then*

$$\mathbb{P}\left(L' \leq l'\right) \geq \varepsilon/4$$

with $l' = Cle^{\frac{\gamma}{2}\sigma\sqrt{|\log\varepsilon/2C|}}$ and C, σ depend only on the geometry.

Proof. Assume that for some positive l, ε, $\mathbb{P}(L \leq l) \geq \varepsilon$. Setting $x = \sigma\sqrt{|\log(\varepsilon/2C)|}$, we have, using the Proposition 2.3:

$$\mathbb{P}\left(\left\|(\delta\phi_1 + \delta\phi_2)_{|K}\right\|_\infty \geq x\right) \leq \varepsilon/2$$

and

$$\mathbb{P}\left(\left\|(\delta\phi_1 + \delta\phi_2)_{|K}\right\|_\infty \leq x, L \leq l\right) \geq \varepsilon/2.$$

Thus, with probability at least $\varepsilon/2$, the distance from A to B in K for the metric $e^{\gamma(\phi_{0,n}-\delta\phi_1-\delta\phi_2)}ds^2$ is $\leq le^{\frac{\gamma}{2}x}$. On this event, we fix such a path of length $\leq le^{\frac{\gamma}{2}x}$ and average over the independent small scale noise $\delta\phi_3$; the expected length of the path is $\leq le^{\frac{\gamma}{2}x}e^{C\gamma^2}$. With conditional probability at least $1/2$, this length is no more than twice the conditional expectation. Consequently, with probability at least $\varepsilon/4$, the distance from A to B in K for $e^{\gamma\tilde{\phi}_{0,n}\circ F}ds^2$ is less than $2le^{\frac{\gamma}{2}x}e^{C\gamma^2}$. Since F' is bounded on K, we get that $\mathbb{P}(L' \leq l') \geq \varepsilon/4$ where $l' = 2\left\|F'\right\|_{K,\infty} le^{\frac{\gamma}{2}x}e^{C\gamma^2}$. Indeed, since F is holomorphic, if $\pi = (\pi_t)_{t\in[0,1]}$ is a C^1 path and if ϕ is a smooth field, we have:

$$\begin{aligned}
L\left(F\circ\pi, e^{\gamma\phi}ds^2\right) &= \int_0^1 e^{\frac{\gamma}{2}\phi\circ F(\pi(t))}\left|F'(\pi(t))\right|\left|\pi'(t)\right|dt \\
L\left(\pi, e^{\gamma\phi\circ F}ds^2\right) &= \int_0^1 e^{\frac{\gamma}{2}\phi\circ F(\pi(t))}\left|\pi'(t)\right|dt.
\end{aligned}$$

Thus, on the event $\{L(A,B,e^{\gamma\tilde{\phi}_{0,n}\circ F}ds^2) \leq 2le^{\frac{\gamma}{2}x}e^{C\gamma^2}\}$ we have, taking such a path π:

$$\begin{aligned}
L\left(A',B',e^{\gamma\tilde{\phi}_{0,n}}ds^2\right) &\leq L\left(F\circ\pi, e^{\gamma\tilde{\phi}_{0,n}}ds^2\right) \\
&\leq \left\|F'\right\|_{K,\infty} L\left(\pi, e^{\gamma\tilde{\phi}_{0,n}\circ F}ds^2\right) \\
&\leq 2\left\|F'\right\|_{K,\infty} le^{\frac{\gamma}{2}x}e^{C\gamma^2}
\end{aligned}$$

hence $\mathbb{P}(L' \leq l') \geq \varepsilon/4$ with $l' = Cle^{\frac{\gamma}{2}\sigma\sqrt{|\log\varepsilon/2C|}}e^{C\gamma^2} \leq Cle^{\frac{\gamma}{2}\sigma\sqrt{|\log\varepsilon/2C|}}$. □

Proposition 2.5. *(Right tail estimate). If for some $l > 0$ and $\varepsilon < 1/2$, $\mathbb{P}(L \leq l) \geq 1-\varepsilon$ then*

$$\mathbb{P}\left(L' \leq l'\right) \geq 1 - 3\varepsilon$$

with $l' = Cle^{C\gamma\sqrt{|\log\varepsilon/2C|}}$ and C depends only on the geometry.

To prove Proposition 2.5, we will need the following lemma which is a consequence of the moment method and which will be used in the next sections.

Lemma 2.10. *Let μ be a Borel measure on a metric space (X, d). If S is a Borel set such that $\mu(S) \in (0, \infty)$ and ψ is a continuous centered Gaussian field on S, satisfying $\sigma^2 := \sup_{x \in S} \mathrm{Var}(\psi(x)) < \infty$, then for every $s > \sigma^2$ we have*

$$\mathbb{P}\left(\int_S e^{\psi(x)} \mu(dx) \geq \mu(S) e^s\right) \leq e^{-s^2/2\sigma^2}.$$

Proof. By using first Chebychev inequality, then Jensen inequality and finally explicit formula for moment generating function of Gaussian variables, we have for $k > 1/2$:

$$\begin{aligned} \mathbb{P}\left(\int_S e^{\psi(x)} \mu(dx) \geq \mu(S) e^s\right) &\leq e^{-2ks} \mathbb{E}\left(\left(\frac{1}{\mu(S)} \int e^{\psi(x)} \mu(dx)\right)^{2k}\right) \\ &\leq e^{-2ks} \mu(S)^{-1} \int_S \mathbb{E}\left(e^{2k\psi(x)}\right) \mu(dx) \\ &\leq e^{2k^2\sigma^2 - 2ks}. \end{aligned}$$

By setting $k = \frac{s}{2\sigma^2}$, we get the tail estimate for $s > \sigma^2$. □

We are now ready to prove Proposition 2.5.

Proof of Proposition 2.5. Assume that for some positive l, ε, $\mathbb{P}(L \leq l) \geq 1 - \varepsilon$. Setting $x = \sigma\sqrt{|\log(\varepsilon/C)|}$ and using the estimate from Proposition 2.3 we have:

$$\mathbb{P}\left(\left\|(\delta\phi_1 + \delta\phi_2)_{|K}\right\|_\infty \geq x\right) \leq \varepsilon$$

and

$$\mathbb{P}\left(\left\|(\delta\phi_1 + \delta\phi_2)_{|K}\right\|_\infty \leq x, L \leq l\right) \geq 1 - 2\varepsilon.$$

Consequently, with probability at least $1 - 2\varepsilon$, the distance from A to B in K for the metric $e^{\gamma(\phi_{0,n} - \delta\phi_1 - \delta\phi_2)} ds^2$ is $\leq l e^{\frac{\gamma}{2}x}$. On this event, we fix such a path of length $\leq l e^{\frac{\gamma}{2}x}$ and average over the independent small scale noise $\delta\phi_3$. Let μ be the occupation measure of that path, so that $|\mu| \leq l e^{\frac{\gamma}{2}x}$ and $\psi = \frac{\gamma}{2}(\delta\phi_3)$ is independent of μ. Since $\sigma^2 := \sup_{[0,1]^2} \mathrm{Var}\, \psi = O(\gamma^2)$, by using

Lemma 2.10, we note that adding the noise $\delta\phi_3$ increases the length by a factor $\geq e^{C\gamma\sqrt{|\log\varepsilon|}}$ with probability $\leq \varepsilon$. Consequently, with probability $\geq 1-3\varepsilon$, the distance from A to B in K for $e^{\gamma\tilde{\phi}_{0,n}\circ F}ds$ is less than $le^{\frac{\gamma}{2}x}e^{C\gamma\sqrt{|\log\varepsilon|}}$. Using again $L(A',B',e^{\gamma\tilde{\phi}}ds^2) \leq \|F'\|_{K,\infty} L(A,B,e^{\gamma\tilde{\phi}\circ F}ds^2)$ we have $\mathbb{P}(L' \leq l') \geq 1-3\varepsilon$ where $l' = \|F'\|_{K,\infty} le^{\frac{\gamma}{2}x}e^{C\gamma\sqrt{|\log\varepsilon|}}$. □

To prove Theorem 2.2, we will need the following elementary lemma.

Lemma 2.11. *If a and b are two positive real numbers with $a < b$, there exists $j = j(b/a)$ and j rectangles isometric to $[0,a/2] \times [0,b/2]$ such that if π is a left-right crossing of the rectangle $[0,a] \times [0,b]$, at least one of the j rectangles is crossed in the thin direction by a subpath of that crossing.*

Proof. To see it, cover for instance $[0,a/2] \times [0,b]$ by thin rectangles $[0,a/2] \times [0,b/2]$ from bottom to top and spaced by $(b-a)/4$, add also squares of length $a/2$ with the same spacing (see the first two parts on Figure 2.1). Then, starting with a crossing of $[0,a] \times [0,b]$, consider the subpath from the left side to the first hitting point of $\{a/2\} \times [0,b]$, and denote by h is height (max of y - min of y). Consider first the case where $h \leq a/2 + (b-a)/4$ (see the last part on Figure 2.1). Since the bottom part of the path is at distance $\leq (b-a)/4$ of a side of a rectangle of size $[0,a/2] \times [0,b/2]$ the crossing is included in this rectangle of the cover. Now we treat the other case where $h > a/2 + (b-a)/4$ (see the third part on Figure 2.1). Since the bottom part is at distance $\leq (b-a)/4$ of a square which is above, this square of size $a/2$ is then crossed vertically. □

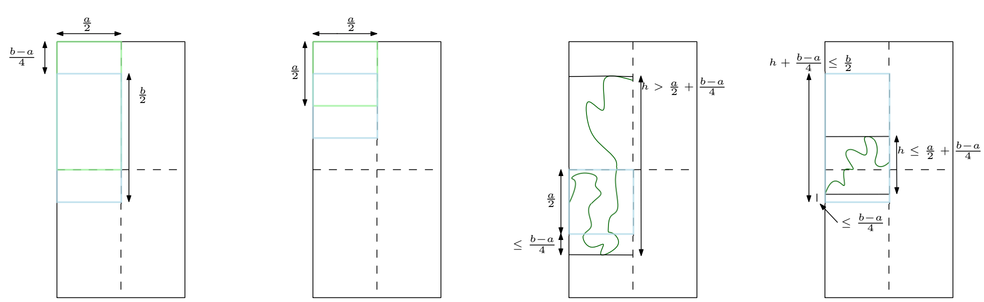

Figure 2.1 – Crossing at a smaller scale.

Now, we want to relate crossings of short rectangles with crossings of long rectangles. Our previous results say that the crossing lengths in K between sides A and B are uniformly (in n) comparable to crossing lengths in $F(K)$ between sides $F(A)$ and $F(B)$. Thus, we would like to take the sides A and B to be those of a short rectangle and to map them to the sides of a long rectangle with a conformal map F such that F' and F'' are bounded and satisfying $|F'| \geq 1$. This cannot be done directly but this is the main idea: to produce a crossing from a short domain to a longer one. In particular, it is enough to consider ellipses and to relate crossings in ellipses with crossings in rectangles and by using the previous lemma one can begin with crossing of sides in a very small domain and then map it to a much larger domain.

Proof of Theorem 2.2. The proof is divided in two steps. First we prove the inequality (3.4) associated with the left tail and then the inequality (3.5) associated to the right one.

Step 1. We study first the left tail under the assumption $\mathbb{P}(L^{(n)}_{a,b} \leq l) \geq \varepsilon$ and we want to obtain a similar estimate for $L^{(n)}_{a',b'}$(in particular if $a/b < 1 < a'/b'$). We assume $a < b$, i.e., $L^{(n)}_{a,b}$ is the length of a crossing in the thin direction.

First, by using Lemma 2.11, we observe that there is an integer $j = j(b/a)$ and j rectangles isometric to $[0, a/2] \times [0, b/2]$ such that on the event $L^{(n)}_{a,b} \leq l$, at least one of the j rectangles is crossed in the thin direction by a subpath of that crossing. Thus, by union bound, we get $\mathbb{P}(L^{(n)}_{a/2,b/2} \leq l) \geq \varepsilon/j$, and by iterating, $\mathbb{P}\left(L^{(n)}_{a/2^p,b/2^p} \leq l\right) \geq \varepsilon/j^p$.

Consider now ellipses E, E', each with two marked arcs, such that: any left-right crossing of $[0, a/2^p] \times [0, b/2^p]$ is a crossing of E, and any crossing of E' is a left-right crossing of $[0, a'] \times [0, b']$.

Divide the marked arcs of E into m subarcs of, say, equal length. With probability at least $\varepsilon/(j^p m^2)$, one of the crossings between pairs of subarcs has length at most l.

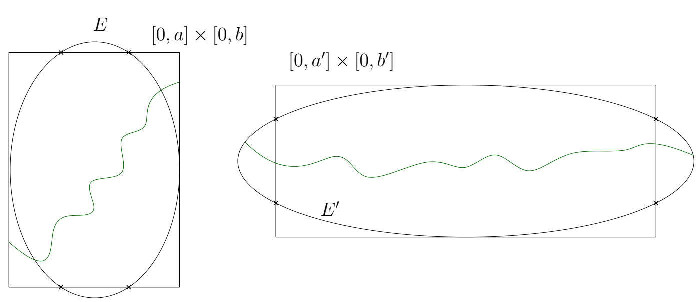

Figure 2.2 – Rectangles and ellipses

For m large enough (depending on E, E'), for any pair of such subsegments (one on each side), there is a conformal equivalence $F : E \to E'$ such that the pair of subarcs is mapped to subarcs of the marked arcs of E'. Remark that ellipses are analytic curves (they are images of circles under the Joukowski map, see [47] Chapter 1 Exercise 15) and consequently (by Schwarz reflection) F extends to a conformal equivalence $U \to V$, where $\bar{E}$ (resp. $\bar{E}'$) is a compact subset of U (resp. V).

By choosing p large enough, $|F'| \geq 1$ on U. By the left tail estimate Proposition 2.4, we obtain that there is $C > 0$ such for all $\varepsilon, l > 0$:

$$\mathbb{P}\left(L_{a,b}^{(n)} \leq l\right) \geq \varepsilon \Rightarrow \mathbb{P}\left(L_{a',b'}^{(n)} \leq Cle^{\frac{\gamma}{2}\sigma\sqrt{|\log \varepsilon/(2Cj^p m^2)|}}\right) \geq \varepsilon/(4j^p m^2)$$

which we rewrite as:

$$\mathbb{P}\left(L_{a,b}^{(n)} \leq l\right) \geq \varepsilon \Rightarrow \mathbb{P}\left(L_{a',b'}^{(n)} \leq Cle^{C\gamma\sqrt{|\log \varepsilon/C|}}\right) \geq \varepsilon/C. \tag{4.10}$$

Step 2. For the right tail we reason similarly: let $a < b$ and take l, ε so that $\mathbb{P}(L_{a,b}^{(n)} \leq l) \geq 1 - \varepsilon$. On the event $\{L_{a,b}^{(n)} \leq l\}$, one of j variables distributed like $L_{a/2,b/2}^{(n)}$ is $\leq l$; moreover these variables have positive association. By the the positive association property (Theorem 2.1) and the square-root trick (see [110] Proposition 4.1), we have $\mathbb{P}(L_{a/2,b/2}^{(n)} \leq l) \geq 1 - \varepsilon^{1/j}$ and then, by iterating, $\mathbb{P}(L_{a/2^p,b/2^p}^{(n)} \leq l) \geq 1 - \varepsilon^{j^{-p}}$.

On the event $\{L_{a/2^p,b/2^p}^{(n)} \leq l\}$, the ellipse E has a crossing of length $\leq l$ between two marked arcs. Again by subdividing each of these arcs into m subarcs, and applying the square-root trick we see that for at least one pair of subarcs, there is a crossing of length $\leq l$ with probability $\geq 1 - \varepsilon^{j^{-p}m^{-2}}$. Combining with the right-tail estimate Proposition 2.5, we get:

$$\mathbb{P}\left(L_{a,b}^{(n)} \leq l\right) \geq 1 - \varepsilon \Rightarrow \mathbb{P}\left(L_{a',b'}^{(n)} \leq Cle^{\gamma C\sqrt{|\log \varepsilon/C|}}\right) \geq 1 - 3\varepsilon^{1/C} \tag{4.11}$$

which completes the proof of Theorem 2.2. □

2.5 Tail estimates for crossing lengths: proof of Theorem 2.3

2.5.1 Concentration: the left tail

Denote by $\tilde{L}_{1,3}^{(n)}$ (resp. $\tilde{L}_{3,1}^{(n)}$) the left-right crossing length of the rectangle $[2,3]\times[0,3]$ (resp. $[0,3]\times[2,3]$). In this subsection we investigate the consequences of the RSW estimates combined with the following inequalities (see Figure 2):

$$L_{1,3}^{(n)} + \tilde{L}_{1,3}^{(n)} \leq L_{3,3}^{(n)} \leq \min\left(L_{3,1}^{(n)}, \tilde{L}_{3,1}^{(n)}\right)$$

which implies the following:

$$L_{3,3}^{(n)} \leq l \Rightarrow \left(L_{1,3}^{(n)} \leq l \text{ and } \tilde{L}_{1,3}^{(n)} \leq l\right)$$
$$L_{3,3}^{(n)} \geq l \Rightarrow \left(L_{3,1}^{(n)} \geq l \text{ and } \tilde{L}_{3,1}^{(n)} \geq l\right).$$

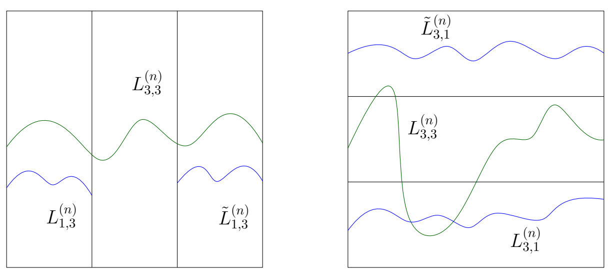

Figure 2.3 – Inequalities between lengths of geodesics associated to different rectangles

The following result is a consequence of the first inequality. It gives lognormal tail estimates on the left tail of crossing lengths renormalized by a small quantile, without any assumption on γ.

Proposition 2.6. *There exists a small $p_0 > 0$ such that for $p \leq p_0$ there exists $c > 0$ so that for every $s > 0$*

$$\mathbb{P}\left(L_{3,3}^{(n)} \leq l_{3,3}^{(n)}(p)\, e^{-s}\right) \leq Ce^{-cs^2},$$

where c, C do not depend on n.

Proof. Our left tail estimate (4.10) gives:

$$\mathbb{P}\left(L_{1,3}^{(n)} \leq l\right) \geq \varepsilon \Rightarrow \mathbb{P}\left(L_{3,3}^{(n)} \leq l'\right) \geq \varepsilon / C \quad \text{with} \quad l' = Cle^{C\gamma\sqrt{|\log \varepsilon/C|}}$$

which can be rewritten as:

$$\mathbb{P}\left(L_{3,3}^{(n)} \leq l\right) \leq \varepsilon \Rightarrow \mathbb{P}\left(L_{1,3}^{(n)} \leq lC^{-1}e^{-C\gamma\sqrt{|\log C\varepsilon|}}\right) \leq C\varepsilon. \tag{5.12}$$

Now, if $L_{3,3}^{(n)}$ is less than l, then both $[0,1]\times[0,3]$ and $[2,3]\times[0,3]$ have a left-right crossing of length $\leq l$ and the field in these two rectangles is independent (if r_0 is small enough). Consequently,

$$\mathbb{P}\left(L_{3,3}^{(n)} \leq l\right) \leq \mathbb{P}\left(L_{1,3}^{(n)} \leq l\right)^2. \tag{5.13}$$

These two results allow us to get the uniform tail bound. Indeed, take ε_0 small, such that $C^2\varepsilon_0 < 1$ and set $r_0^{(n)} := l_{3,3}^{(n)}(\varepsilon_0)$. We define by induction $\varepsilon_{i+1} := (C\varepsilon_i)^2$ (which gives $\varepsilon_i = (\varepsilon_0 C^2)^{2^i} C^{-2}$ as well as $r_{i+1}^{(n)} := r_i^{(n)} C^{-1}\exp(-C\gamma\sqrt{|\log(C\varepsilon_i)|})$. It follows by induction that $\mathbb{P}(L_{3,3}^{(n)} \leq r_i^{(n)}) \leq \varepsilon_i$ for every $i \geq 0$. Indeed, the case $i = 0$ follows by definition and then notice that the RSW estimates under the induction hypothesis implies that

$$\mathbb{P}\left(L_{3,3}^{(n)} \leq r_i^{(n)}\right) \leq \varepsilon_i \Rightarrow \mathbb{P}\left(L_{1,3}^{(n)} \leq r_{i+1}^{(n)}\right) \leq C\varepsilon_i$$

which gives, using the inequality (4.44):

$$\mathbb{P}\left(L_{3,3}^{(n)} \leq r_{i+1}^{(n)}\right) \leq \mathbb{P}\left(L_{1,3}^{(n)} \leq r_{i+1}^{(n)}\right)^2 \leq (C\varepsilon_i)^2 = \varepsilon_{i+1}.$$

Notice that we have the lower bound on $r_i^{(n)}$ for $i \geq 1$:

$$r_i^{(n)} \geq l_{3,3}^{(n)}(\varepsilon_0)C^{-i}e^{-C\gamma\sum_{k=0}^{i-1}\sqrt{|\log(C\varepsilon_k)|}} \geq l_{3,3}^{(n)}(\varepsilon_0)e^{-Ci}e^{-C\gamma\sqrt{|\log \varepsilon_0 C^2|}2^{i/2}}.$$

Our estimate then takes the form, for $i \geq 0$:

$$\mathbb{P}\left(L_{3,3}^{(n)} \leq l_{3,3}^{(n)}(\varepsilon_0)e^{-Ci}e^{-\gamma C\sqrt{|\log \varepsilon_0 C^2|}2^{i/2}}\right) \leq \left(\varepsilon_0 C^2\right)^{2^i} C^{-2}.$$

Which can be rewritten, taking $i = \lfloor 2\log_2 s \rfloor$, with absolute constants, for $s \geq 1$:

$$\mathbb{P}\left(L_{3,3}^{(n)} \leq l_{3,3}^{(n)}(\varepsilon_0) C^{-1} e^{-C\log s} e^{-\gamma s}\right) \leq e^{-cs^2}.$$

Notice that dropping the dependence on γ as we impose it is bounded from above by a large number we get Proposition 2.6. □

Corollary 2.7. *We have a uniform (in n) lognormal tail estimates for the lower bound of thin rectangles, i.e., if ε_0 is small enough for every $n \geq 0$, $s \geq 0$:*

$$\mathbb{P}\left(L_{1,3}^{(n)} \leq l_{1,3}^{(n)}(\varepsilon_0) e^{-s}\right) \leq Ce^{-cs^2},$$

where c, C are absolute constants.

Proof. The proof follows from the RSW estimate (5.12), the bound $l_{1,3}^{(n)}(\varepsilon_0) \leq l_{3,3}^{(n)}(\varepsilon_0)$ and the previous proposition. □

It is tempting to follow the lines of this proof using the second inequality (see also Figure 2.3) in order to derive a right tail estimate. However, this approach cannot be readily extended because of the power $1/C$ in the RSW estimate, inequality (3.5).

2.5.2 Concentration: the right tail

As mentioned in the previous section, we cannot generalize the method used for the left tails to the right one and the following proposition remediates to this. Before stating it, we refer the reader to the definitions of l_n and δ_n in Subsection 2.2.3.

Proposition 2.8. *If ε is small enough we have the following tail estimate:*
For $0 \leq k \leq n$, $s > 1$

$$\mathbb{P}\left(L_{3,1}^{(k)} \geq \delta_n l_k e^{s\sqrt{\log s}}\right) \leq Ce^{-cs^2},$$

where c and C are absolute constants.

Proof. We proceed according to the following steps:

1. Use the RSW estimates to reduce the problem to the case of squares instead of long rectangles.

2. Use a comparison to 1-dependent oriented site percolation to prove that with probability going to one exponentially in k, $L_{k,k}^{(n)}$ is less than $Ck\bar{l}_n$.

3. By scaling and the moment method, obtain a first tail estimate of $L_{1,1}^{(n)}$ with respect to $\bar{l}_{n-m}$: For a constant $\alpha \in (0,1)$, $\mathbb{P}\left(L_{1,1}^{(n)} \geq C\bar{l}_{n-m}e^{\gamma s\sqrt{m}}\right) \leq C\alpha^{2^m} + e^{-\frac{2s^2}{\log 2}}$.

4. Give an upper bound of $\bar{l}_{n-m}$ in terms of l_n.

5. Obtain a tail estimate when the tails are not too large.

6. For the large tails, use a moment method and a lower bound on the quantiles.

Step 1. First, notice by the RSW estimates (4.11) that it is enough to prove that for $0 \leq k \leq n$, $s > 1$

$$\mathbb{P}\left(L_{1,1}^{(k)} \geq \delta_n l_k e^{s\sqrt{\log s}}\right) \leq Ce^{-cs^2}.$$

Step 2. We will see here that taking ε small enough, there exist $C > 0$, $\alpha < 1$ such that for every $k, n \geq 0$:

$$\mathbb{P}\left(L_{k,k}^{(n)} \leq 4k\bar{l}_n\right) \geq 1 - C\alpha^k. \tag{5.14}$$

We consider a graph whose sites x are made by squares of size 3×3 and spaced so that two adjacent squares intersect each other along a rectangle of size $(3,1)$ or $(1,3)$. Denote by $L_{3,1,\text{right}}^{(n)}(x)$ the rectangle crossing length, in the long direction, associated to the rectangle of size $(3,1)$ on the bottom of x and included in x. Similarly, denote by $L_{3,1,\text{up}}^{(n)}(x)$ the rectangle crossing length, in the long direction, associated to the rectangle of size $(1,3)$ on the left of x and included in x. To each site of our graph, we assign the value 0 if the site is closed and 1 if the site is open. A site x is open if the event $\{L_{3,1,\text{up}}^{(n)}(x) + L_{3,1,\text{right}}^{(n)}(x) \leq 2\bar{l}_n\}$ occurs (see Figure 2.4).

We have the following bound on the probability that a site x is open:

$$\mathbb{P}(\omega_x = 1) \geq \mathbb{P}\left(L_{3,1,\text{up}}^{(n)} \leq \bar{l}_n, L_{3,1,\text{right}}^{(n)} \leq \bar{l}_n\right) \geq 2\mathbb{P}\left(L_{3,1}^{(n)} \leq \bar{l}_n\right) - 1 \geq 1 - 2\varepsilon.$$

Therefore, taking ε small gives a highly supercritical 1-dependent percolation model (notice that a

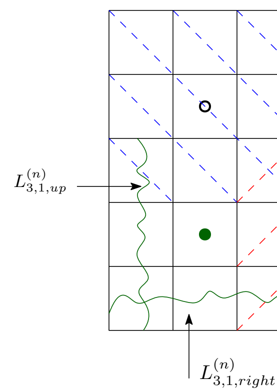

Figure 2.4 – Definition of the model. The green site x is open. Three of its neighbors are drawn, with some colored dashed lines filling their cell and with white vertices at their center.

site x is independent of sites that are not directly weakly adjacent to it). Then, notice that $L^{(n)}_{k,k}$ is smaller than the weight associated to oriented paths from left to right at the percolation level that can go only up or right. Such a path contains at most $2k$ sites. Thus, if there is an open oriented percolation path from left to right, then $L^{(n)}_{k,k} \leq 4k\bar{l}_n$. Hence it is enough to show that the probability that there is such an open oriented path goes to 1 exponentially in k. This follows from a contour argument for highly supercritical 1-dependent percolation model, see for instance [45] Section 10.

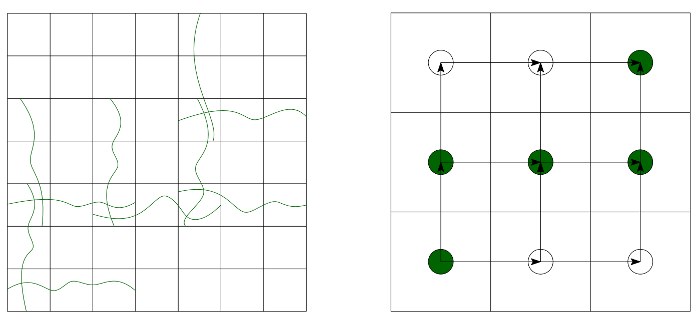

Figure 2.5 – Comparison with 1-Dependent Oriented Site Percolation. The figure on the right is the representation of the figure on the left.

Step 3. In order to obtain an upper bound for $L^{(n)}_{1,1}$, by scaling and the percolation bound (5.14) we see that there exists $\alpha \in (0,1)$ such that for $m \leq n$, we have,

$$\mathbb{P}\left(L^{(m,n)}_{1,1} \leq C\bar{l}_{n-m}\right) = \mathbb{P}\left(L^{(n-m)}_{2^m,2^m} \leq C2^m\bar{l}_{n-m}\right) \geq 1 - C\alpha^{2^m}$$

which can be rewritten in term of $L_{1,1}^{(n)}$ as

$$\begin{aligned}\mathbb{P}\left(L_{1,1}^{(n)} \leq C\bar{l}_{n-m}e^{s}\right) &\geq \mathbb{P}\left(L_{1,1}^{(n)} \leq C\bar{l}_{n-m}e^{s}, L_{1,1}^{(m,n)} \leq C\bar{l}_{n-m}\right)\\ &= \mathbb{P}\left(L_{1,1}^{(m,n)} \leq C\bar{l}_{n-m}\right) - \mathbb{P}\left(L_{1,1}^{(n)} \geq C\bar{l}_{n-m}e^{s}, L_{1,1}^{(m,n)} \leq C\bar{l}_{n-m}\right)\\ &\geq 1 - C\alpha^{2^m} - \mathbb{P}\left(L_{1,1}^{(n)} \geq e^{s}L_{1,1}^{(m,n)}\right).\end{aligned}$$

Now, using that $L_{1,1}^{(n)} \leq \int_{\pi_{m,n}} e^{\frac{\gamma}{2}\phi_{0,m-1}}e^{\frac{\gamma}{2}\phi_{m,n}}ds$ where $\pi_{m,n}$ is a geodesic for $e^{\gamma\phi_{m,n}}ds^2$ and using the bound coming from Lemma 2.10 we have

$$\mathbb{P}\left(L_{1,1}^{(n)} \geq e^{\gamma\sqrt{m}s}L_{1,1}^{(m,n)}\right) \leq \mathbb{E}\left(\mathbb{P}\left(\int_{\pi_{m,n}} e^{\frac{\gamma}{2}\phi_{0,m-1}}e^{\frac{\gamma}{2}\phi_{m,n}} \geq e^{\gamma\sqrt{m}s}L_{1,1}^{(m,n)} \mid \mathcal{F}_{m,n}\right)\right) \leq e^{-\frac{2s^2}{\log 2}}$$

hence for every $0 \leq m \leq n$ and $s \geq 0$

$$\mathbb{P}\left(L_{1,1}^{(n)} \leq C\bar{l}_{n-m}e^{\gamma s\sqrt{m}}\right) \geq 1 - C\alpha^{2^m} - e^{-\frac{2s^2}{\log 2}}. \tag{5.15}$$

Step 4. At this stage we want to replace $\bar{l}_{n-m}$ by l_n. We introduce a notation for a collection of short rectangles that we will use by setting

$$I_k := \{\text{horizontal, vertical rectangles of size } 2^{-k}(1,3) \text{ with corners in } [0,1]\times[0,3]\cap 2^{-k}\mathbb{Z}^2\}. \tag{5.16}$$

It is clear from the definition that $|I_k| \leq C4^k$.Then, notice that a left-right crossing of $[0,1] \times [0,3]$ has to cross at least 2^k rectangles from I_k (by definition of I_k, these are short crossings). For $P \in I_k$, we set

$$L^{(n)}(P) := \text{length of the left-right crossing of the rectangle } P \text{ for } e^{\frac{\gamma}{2}\phi_{0,n}}ds \tag{5.17}$$

and we use similarly the notation $L^{(k,n)}(P)$ when the field considered is $\phi_{k,n}$. We have, almost surely,

$$L_{1,3}^{(n)} \geq 2^k \min_{P\in I_k} L^{(n)}(P) \geq 2^k e^{\frac{\gamma}{2}\inf_{[0,1]^2}\phi_{0,k-1}} \min_{P\in I_k} L^{(k,n)}(P). \tag{5.18}$$

Hence by union bound and scaling, we have, for $s_1 > 0$ and $s_2 > 0$ to be specified

$$\begin{aligned}
\mathbb{P}\left(L_{1,3}^{(n)} \le e^{-\frac{\gamma}{2}s_1} l_{n-k} e^{-s_2}\right) &\le \mathbb{P}\left(e^{\frac{\gamma}{2}\inf_{[0,1]^2}\phi_{0,k-1}} 2^k \min_{P\in I_k} L^{(k,n)}(P) \le e^{-\frac{\gamma}{2}s_1} l_{n-k} e^{-s_2}\right) \\
&\le \mathbb{P}\left(e^{\frac{\gamma}{2}\inf_{[0,1]^2}\phi_{0,k-1}} \le e^{-\frac{\gamma}{2}s_1}\right) + \mathbb{P}\left(\min_{P\in I_k} L^{(k,n)}(P) \le 2^{-k} l_{n-k} e^{-s_2}\right) \\
&\le \mathbb{P}\left(\sup_{[0,1]^2} |\phi_{0,k-1}| \ge s_1\right) + C4^k \mathbb{P}\left(L_{1,3}^{(n-k)} \le l_{n-k} e^{-s_2}\right).
\end{aligned}$$

Using the supremum tail estimate from the appendix (10.40) with $s_1 = k\log 4 + C\sqrt{k} + Cs$ and the lognormal tails from Corollary 2.7 with $s_2 = C\sqrt{k\log 4 + s}$ we have

$$\mathbb{P}\left(L_{1,3}^{(n)} \le l_{n-k} 2^{-\gamma k} e^{-C\sqrt{k}} e^{-Cs} e^{-C\sqrt{s}}\right) \le Ce^{-s},$$

which gives

$$l_n \ge 2^{-\gamma k} e^{-C\sqrt{k}} e^{-C} l_{n-k}, \tag{5.19}$$

hence $\bar{l}_{n-m} \le l_{n-m}\delta_n \le l_n \delta_n 2^{\gamma m} e^{C\sqrt{m}} C$.

Step 5. Using this bound and coming back to our estimate (5.15), for every $m \le n$ and $s \ge 0$

$$\mathbb{P}\left(L_{1,1}^{(n)} \le l_n \delta_n 2^{\gamma m} e^{C\sqrt{m}} C e^{\gamma s\sqrt{m}}\right) \ge 1 - C\alpha^{2^m} - e^{-\frac{2s^2}{\log 2}}.$$

We deal with the range $s \in [1, 2^{n/2}]$, taking m such that $s = 2^{m/2}$, i.e., $m = \lfloor 2\log_2 s\rfloor$ we get:

$$\mathbb{P}\left(L_{1,1}^{(n)} \le l_n \delta_n e^{C\gamma \log s} e^{\gamma s\sqrt{\log s}}\right) \ge 1 - Ce^{-cs^2},$$

which gives, dropping the dependence on γ for $s > 1$:

$$\mathbb{P}\left(L_{1,1}^{(n)} \ge l_n \delta_n e^{s\sqrt{\log s}}\right) \le Ce^{-cs^2}.$$

Step 6. We then treat the case $s \ge 2^{n/2}$. To do it, we use a moment method (Lemma 2.10) to get a right tail estimate on $L_{1,1}^{(n)}$ together with a lower bound on its quantiles. The moment method (taking a straight line) gives:

$$\mathbb{P}\left(L_{1,1}^{(n)} \geq e^{\gamma s}\right) \leq e^{-\frac{2s^2}{(n+1)\log 2}}. \tag{5.20}$$

For the lower bound on quantile, we get a bound by a direct comparison with the supremum of the field $\mathbb{P}(L_{1,3}^{(n)} \leq e^{-\frac{\gamma}{2}x}) \leq \mathbb{P}(\sup_{[0,1]^2} \phi_{0,n} \geq x)$. Using the supremum tails from the appendix (10.40), i.e., taking $x = n\log 4 + C\sqrt{n} + Cs$ gives $l_n \geq e^{-\frac{\gamma}{2}(n\log 4 + C\sqrt{n}+C)} =: e^{-\gamma x_n}$. Since we consider the case $s \geq 2^{n/2}$, $s \geq x_n$ and $n \leq 2\log_2 s$ and coming back to (5.20) leads to

$$\mathbb{P}\left(L_{1,1}^{(n)} \geq l_n e^{\gamma s}\right) \leq \mathbb{P}\left(L_{1,1}^{(n)} \geq e^{\gamma(s-x_n)}\right) \leq e^{-2\frac{(s-x_n)^2}{(n+1)\log 2}} \leq e^{Cs} e^{-\frac{s^2}{\log s}}.$$

Finally, combining the two inequalities ends the proof. □

2.5.3 Quasi-lognormal tail estimates at subcriticality

In this subsection we prove Theorem 2.3. The main idea is the following: the tightness of $\log L_{1,1}^{(n)} - \log \mu_n$ shows that the ratio between low and high quantiles of $L_{1,1}^{(n)}$ is bounded. Using the RSW estimates, it implies that $\delta_\infty < \infty$ which gives, uniformly in n, $\mu_n \leq C l_n$. The tails are then obtained using Corollary 2.7 (with $l_n \geq \mu_n C^{-1}$) and Proposition 2.8 (with $\delta_n l_n \leq \delta_\infty \mu_n$).

Proof of Theorem 2.3. Assuming $\gamma < \gamma_c$ gives the tightness of $(\log L_{1,1}^{(n)} - \log \mu_n)_{n\geq 0}$. Thus, for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for every $n \geq 0$, $\mathbb{P}(L_{1,1}^{(n)} \leq \mu_n e^{-C_\varepsilon}) \leq \varepsilon/C$ and $\mathbb{P}(L_{1,1}^{(n)} \geq \mu_n e^{C_\varepsilon}) \leq \varepsilon^C/3$ which can be rewritten as

$$\mu_n e^{-C_\varepsilon} \leq l_{1,1}^{(n)}(\varepsilon/C) \leq \mu_n \leq \bar{l}_{1,1}^{(n)}(\varepsilon^C/3) \leq \mu_n e^{C_\varepsilon}.$$

Combining with the RSW estimates (2.2), we have

$$\mu_n e^{-C_\varepsilon} \leq l_{1,1}^{(n)}(\varepsilon/C)e^{-C_\varepsilon} \leq l_{1,3}^{(n)}(\varepsilon) \leq l_{1,1}^{(n)}(\varepsilon) \leq \mu_n \leq \bar{l}_{1,1}^{(n)}(\varepsilon) \leq \bar{l}_{3,1}^{(n)}(\varepsilon) \leq \bar{l}_{1,1}^{(n)}(\varepsilon^C/3)e^{C_\varepsilon} \leq \mu_n e^{C_\varepsilon}.$$

In particular, $\delta_n \leq e^{C_\varepsilon}$ holds for every $n \geq 0$ hence $\delta_\infty(\varepsilon) = \sup_{n\geq 0} \delta_n(\varepsilon) < \infty$.

We prove now the lower tail estimates. We have $l_n \geq \mu_n e^{-C_\varepsilon}$ for every $n \geq 0$ hence using Corollary 2.7 we get Theorem 3.7 when $(a,b) = (1,3)$. For the upper tails since $\delta_\infty < \infty$ and $l_n \leq \mu_n$

we can use Proposition 2.8 to get Theorem 3.6 for the case $(a,b)=(3,1)$. The general case follows from the RSW estimates. □

When $\gamma < \gamma_c$, we expect the existence of a $\rho \in (0,1)$ such that $l_n = \rho^{n+o(n)}$ and $\bar{l}_n = \rho^{n+o(n)}$. However, we don't need this level of precision and the following a priori bounds are enough for our analysis.

Lemma 2.12. *If $0 < \varepsilon < 1/2$ we have the following inequalities relating quantiles, for every $0 \leq k \leq n$:*

1. *for the the lower quantiles $l_{n-k} \leq 2^{\gamma k} e^{C\sqrt{k}} l_n$,*
2. *if $\gamma < \gamma_c$, $\bar{l}_n \leq e^{C\sqrt{k}} \bar{l}_{n-k}$,*
3. *and still under the assumption $\gamma < \gamma_c$, $e^{-C}\mu_n \leq l_n \leq \mu_n \leq \bar{l}_n \leq e^C \mu_n$.*

Proof. The first point follows from the proof of Proposition 2.8, see (5.19). For the second point, using Lemma 2.10 gives

$$\begin{aligned}\mathbb{P}\left(L_{1,1}^{(n)} \geq e^{\gamma\sqrt{k}s} L_{1,1}^{(n-k)}\right) &\leq \mathbb{E}\left(\mathbb{P}\left(\int_{\pi_{n-k}} e^{\frac{\gamma}{2}\phi_{0,n-k}} e^{\frac{\gamma}{2}\phi_{n-k,n}} \geq e^{\gamma\sqrt{k}s} L_{1,1}^{(n-k)} \mid \mathcal{F}_{0,n-k}\right)\right) \\ &\leq e^{-\frac{2s^2}{\log 2}}\end{aligned}$$

hence $\mathbb{P}\left(L_{1,1}^{(n)} \geq \bar{l}_{n-k} e^{\gamma\sqrt{k}s} e^s\right) \leq e^{-\frac{2s^2}{\log 2}} + \mathbb{P}\left(L_{1,1}^{(n-k)} \geq \mu_{n-k} e^s\right)$ and the result follows from Theorem 2.3. The last point follows from the previous proof. □

2.5.4 Lower bounds on the tails of crossing lengths

The following result, independent of the value of γ, shows that we cannot expect better than uniform lognormal tails. Its proof is essentially an application of the Cameron-Martin theorem and we see there that the lower bounds are already provided by the low frequencies of the field.

Proposition 2.9. *There exist positive constants c, C such that for every $n \geq 0$, $x > 0$: $\mathbb{P}\left(L_{1,1}^{(n)} \leq \mu_n e^{-x}\right) \geq ce^{-Cx^2}$ and $\mathbb{P}\left(L_{1,1}^{(n)} \geq \mu_n e^{x}\right) \geq ce^{-Cx^2}$.*

Proof. If $x \in [0,1]^2$, for every $t \in (0,1)$, the Euclidean ball centered at x with radius tr_0 is included in the r_0 neighborhood of $[0,1]^2$, denoted by $([0,1]^2)^{r_0}$. Since k has compact support in $B(0,r_0)$,

$$\begin{aligned}\int_{\frac{1}{2}}^{1}\int_{\mathbb{R}^2} k\left(\frac{x-y}{t}\right) t^{-3/2} 1_{y\in([0,1]^2)^{r_0}} dydt &= \int_{\frac{1}{2}}^{1}\int_{B(x,tr_0)} k\left(\frac{x-y}{t}\right) t^{-3/2} dydt \\ &= \int_{\frac{1}{2}}^{1}\int_{B(0,tr_0)} k\left(\frac{y}{t}\right) t^{-3/2} dydt\end{aligned}$$

is independent of x and is equal to some positive real number h.

Let $M \in \mathbb{R}$. By the Cameron-Martin theorem (see [21] Section 2), since $M1_{[\frac{1}{2},1]\times([0,1]^2)^{r_0}}$ is square-integrable, $\xi + M1_{[\frac{1}{2},1]\times([0,1]^2)^{r_0}}$ is absolutely continuous with respect to ξ and its Radon-Nikodým derivative is given by the Cameron-Martin formula:

$$\frac{d\mathcal{L}\left(\xi + M1_{[\frac{1}{2},1]\times([0,1]^2)^{r_0}}\right)}{d\mathcal{L}(\xi)} = \exp\left(M\langle \xi, 1_{[\frac{1}{2},1]\times([0,1]^2)^{r_0}}\rangle - g\frac{M^2}{2}\right)$$

where $g := \frac{1}{2}\mathrm{Leb}(([0,1]^2)^{r_0})$. We introduce the field $\phi_{0,n}^M$ associated to $\xi + M1_{[\frac{1}{2},1]\times([0,1]^2)^{r_0}}$, i.e., for $x \in \mathbb{R}^2$,

$$\phi_{0,n}^M(x) := \int_{2^{-n-1}}^{1}\int_{\mathbb{R}^2} k\left(\frac{x-y}{t}\right) t^{-3/2}\left(\xi(dy,dt) + M1_{[\frac{1}{2},1]\times([0,1]^2)^{r_0}}(t,y)dydt\right)$$

and using the previous remark, we notice that $\phi_{0,n}^M$ is equal to $\phi_{0,n} + Mh$ on $[0,1]^2$. Thus, using the Cameron-Martin theorem, if I is an interval, we have for $n \geq 0$ and $a > 0$:

$$\begin{aligned}\mathbb{P}\left(L_{1,1}^{(n)} \in e^{-\frac{\gamma}{2}hM} I\right) &= \mathbb{P}\left(L_{1,1}\left(\phi_{0,n}^M\right) \in I\right) \\ &= \mathbb{E}\left(1_{L_{1,1}^{(n)}\in I}\exp\left(M\langle \xi, 1_{[\frac{1}{2},1]\times([0,1]^2)^{r_0}}\rangle - g\frac{M^2}{2}\right)\right) \\ &\geq \left(\mathbb{P}\left(L_{1,1}^{(n)} \in I\right) + \mathbb{P}\left(\langle \xi, 1_{[\frac{1}{2},1]\times([0,1]^2)^{r_0}}\rangle \in (-a,a)\right) - 1\right) e^{-a|M|} e^{-\frac{gM^2}{2}}.\end{aligned}$$

Taking $I = (0,\mu_n]$ and $M = x > 0$ gives, with a large enough but fixed,

$$\mathbb{P}\left(L_{1,1}^{(n)} \leq \mu_n e^{-\frac{\gamma}{2}hx}\right) \geq ce^{-ax}e^{-\frac{gx^2}{2}}.$$

Similarly, taking $I = [\mu_n, \infty)$ and $M = -x < 0$ gives, with a large enough but fixed,

$$\mathbb{P}\left(L_{1,1}^{(n)} \geq \mu_n e^{\frac{\gamma}{2}hx}\right) \geq ce^{-ax}e^{-\frac{gx^2}{2}}$$

for every $x > 0$, $n \geq 0$. This completes the proof. □

2.6 Tightness of the metric at subcriticality: proof of Theorem 2.4

2.6.1 Diameter estimates

We focus on the diameter of $[0,1]^2$ for the metric $e^{\gamma\phi_{0,n}}ds^2$. Notice that there may be a gap between it and the left-right length studied in the previous sections since left-right geodesics are between points where the field $\phi_{0,n}$ is small whereas geodesics associated to diameter have their extremities at points where the field $\phi_{0,n}$ may be high. Before going into exponential tail estimates, we start with a first moment estimate.

Proposition 2.10. *If $\gamma < \min(\gamma_c, 1/2)$ then $\left(\log \mathrm{Diam}\left([0,1]^2, \mu_n^{-2}e^{\gamma\phi_{0,n}}ds^2\right)\right)_{n\geq 0}$ is tight.*

Proof. The proof is divided in four steps: in the first step we use a chaining argument to give an upper bound of the diameter in terms of crossing lengths of rectangles at lower scales and in term of the supremum of $\phi_{0,n}$. In the second and third steps, we bound the expected value of the term associated to the crossing lengths of rectangles and the one of term associated to the supremum. By Chebychev inequality, this gives a control of the right tail of $\log \mathrm{Diam}\left([0,1]^2, \mu_n^{-2}e^{\gamma\phi_{0,n}}ds^2\right)$. In the last step, we compare the diameter to the left-right crossing length to obtain a left tail estimate.

Step 1. Let us denote by H_k (resp V_k) the set of horizontal (resp vertical) thin rectangles of size $2^{-k-1}(2,1)$ spaced by 2^{-k-1} and tiling $[0,1]^2$. Each dyadic square of size 2^{-k} in $[0,1]^2$ is split in two thin horizontal rectangles in H_k and two thin vertical rectangles in V_k. For each of these four rectangles, we pick a path minimizing the crossing length in the long direction. We call *system* the union of these four geodesics (on Figure 2.6, the purple and the green sets are systems associated to different squares). At a scale k, there are 4^k systems, each giving rise to four geodesics.

If x and y are two points in $[0,1]^2$, the geodesic distance between x and y is less than the length associated to any path between them. The majorizing path we use is defined as follows: if $P \in \mathcal{P}_n$

is the dyadic block at scale n containing x, we take an Euclidean straight line (red path on Figure 2.6) to join the system of four geodesics (purple set on the Figure 2.6) associated to H_n and V_n in the block P. By following successively systems associated to larger dyadic blocks, we eventually reach to the one associated to $[0,1]^2$. For instance, on Figure 2.6, the path goes from scale n to scale $n-1$ by using the purple and green systems. Proceeding similarly with y gives a path from x to y, constituted by n systems and two Euclidean straight lines.Taking a uniform bound over these gives an upper bound which is uniform for every x and y in $[0,1]^2$, hence a.s.

$$\text{Diam}\left([0,1]^2, e^{\gamma \phi_{0,n}} ds^2\right) \leq 8 \sum_{k=0}^{n} \max_{P \in H_k \cup V_k} L^{(n)}(P) + 2 \times 2^{-n} e^{\frac{\gamma}{2} \sup_{[0,1]^2} \phi_{0,n}}. \tag{6.21}$$

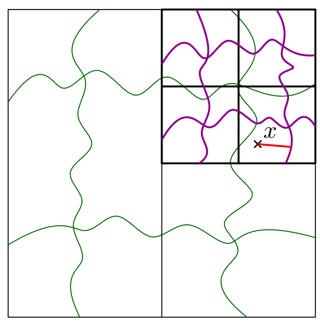

Figure 2.6 – Chaining argument

Step 2. Now, we bound the expected value of the first term in (6.21). We decouple the first scales, a.s. $\max_{P \in H_k \cup V_k} L^{(n)}(P) \leq e^{\frac{\gamma}{2} \sup_{[0,1]^2} \phi_{0,k-1}} \max_{P \in H_k \cup V_k} L^{(k,n)}(P)$ and use independence, $\mathbb{E}(\max_{P \in H_k \cup V_k} L^{(n)}(P)) \leq \mathbb{E}(e^{\frac{\gamma}{2} \sup_{[0,1]^2} \phi_{0,k-1}}) \mathbb{E}(\max_{P \in H_k \cup V_k} L^{(k,n)}(P))$. Then, by using the bound on the exponential moment of the supremum of $\phi_{0,n}$ (Lemma 2.18), we get $\mathbb{E}(e^{\frac{\gamma}{2} \sup_{[0,1]^2} \phi_{0,k-1}}) \leq 2^{\gamma k} e^{C\sqrt{k}}$. By scaling and union bound, the upper tails (3.6) (since $\gamma < \gamma_c$) give the tail estimate $\mathbb{P}(\max_{P \in H_k \cup V_k} L^{(k,n)}(P) \geq 2^{-k} \mu_{n-k} e^{s\sqrt{\log s}}) \leq C 4^k e^{-s^2}$ hence $\mathbb{E}(\max_{P \in H_k \cup V_k} L^{(k,n)}(P)) \leq 2^{-k} \mu_{n-k} e^{C\sqrt{k \log k}}$ by Lemma (2.19). Gathering all the pieces leads to

$$\mathbb{E}\left(\sum_{k=0}^{n} \max_{P \in H_k \cup V_k} L^{(n)}(P)\right) \leq C \sum_{k=0}^{n} 2^{-k} 2^{\gamma k} \mu_{n-k} e^{C\sqrt{k \log k}}.$$

By the bound relating quantiles of different scales (Lemma 2.12) we have

$$\mathbb{E}\left(\sum_{k=0}^{n} \max_{P\in H_k\cup V_k} L^{(n)}(P))\right) \leq C\mu_n \sum_{k=0}^{n} 2^{-k}2^{2\gamma k}e^{C\sqrt{k\log k}}.$$

The series converges for $\gamma < 1/2$.

Step 3. For the second term, using the exponential moment bound for the supremum (Lemma 2.18), the bound $2^{-\gamma n}e^{-C\sqrt{n}} \leq l_n$ for $\gamma < 1/2$ (by comparison with the supremum) we find

$$\mathbb{E}\left(2^{-n}e^{\frac{\gamma}{2}\sup_{[0,1]^2}\phi_{0,n}}\right) \leq 2^{-n}2^{\gamma n}e^{C\sqrt{n}} = 2^{-n}2^{2\gamma n}e^{C\sqrt{n}}2^{-\gamma n}e^{-C\sqrt{n}} \leq Cl_n \leq C\mu_n.$$

Step 4. Since the diameter of the square $[0,1]^2$ is larger than the left-right distance, by using Theorem 2.3 we get

$$\mathbb{P}\left(\text{Diam}([0,1]^2, \mu_n^{-2}e^{\gamma\phi_{0,n}}ds^2) \leq e^{-s}\right) \leq \mathbb{P}\left(L_{1,1}^{(n)} \leq \mu_n e^{-s}\right) \leq Ce^{-cs^2}$$

which completes the proof of Proposition 3.27. □

We now look for exponential tails, when γ is small enough. The following proposition will be used both for the tightness of $d_{0,n}$ and to prove that $\gamma_c > 0$. We refer the reader to the definitions of δ_n and l_n in Subsection 2.2.3.

Proposition 2.11. *If ε is small enough, then for every $c > \frac{\gamma^2}{8(1-2\gamma)}$ there exists $C > 0$ such that for every $n \geq 0$, $s > 0$:*

$$\mathbb{P}\left(\text{Diam}\left([0,1]^2, e^{\gamma\phi_{0,n}}ds^2\right) \geq \delta_n l_n e^{cs}\right) \leq Ce^{-s}.$$

Proof. The proof is divided in three steps. In the two first steps, we give a tail estimate for the first term in (6.21). More precisely, in the first step, we give a tail estimate for $L^{(n)}(P)$ with $P \in H_k \cup V_k$. By union bound, we get one for $\sum_{k=0}^{n} \max_{P\in H_k\cup V_k} L^{(n)}(P)$ in the second step. The third step deals with the second term in (6.21).

Step 1. In order to reuse directly the Proposition 2.8, note first if $P \in H_k \cup V_k$ is fixed, we have a stochastic domination $L^{(n)}(P) \leq L^{(n)}_{2^{-k}(3,1)}$ (since any left-right crossing of $2^{-k}(3,1)$ is a crossing of $2^{-k}(2,1)$) thus we look for a tail estimate for this term. To this end, we decouple the scales by taking a geodesic $\pi_{k,n}$ for the left-right crossing of the rectangle $2^{-k}(1,3)$ for the field $\phi_{k,n}$ and we obtain

$$L^{(n)}_{2^{-k}(3,1)} \leq \int_{\pi_{k,n}} e^{\frac{\gamma}{2}\phi_{0,k-1}} e^{\frac{\gamma}{2}\phi_{k,n}} ds.$$

Therefore, we have the bound

$$\mathbb{P}\left(L^{(n)}(P) \geq 2^{-k}\delta_n l_{n-k} e^{Cs\sqrt{\log s}} e^{\frac{\gamma}{2}s\sqrt{k\log 4}}\right) \\ \leq \mathbb{P}\left(\int_{\pi_{k,n}} e^{\frac{\gamma}{2}\phi_{0,k-1}} e^{\frac{\gamma}{2}\phi_{k,n}} ds \geq 2^{-k}\delta_n l_{n-k} e^{Cs\sqrt{\log s}} e^{\frac{\gamma}{2}s\sqrt{k\log 4}}\right).$$

By union bound, we have

$$\mathbb{P}\left(\int_{\pi_{k,n}} e^{\frac{\gamma}{2}\phi_{0,k-1}} e^{\frac{\gamma}{2}\phi_{k,n}} ds \geq 2^{-k}\delta_n l_{n-k} e^{Cs\sqrt{\log s}} e^{\frac{\gamma}{2}s\sqrt{k\log 4}}\right) \\ \leq \mathbb{P}\left(\int_{\pi_{k,n}} e^{\frac{\gamma}{2}\phi_{0,k-1}} e^{\frac{\gamma}{2}\phi_{k,n}} ds \geq L^{(k,n)}_{2^{-k}(3,1)} e^{\frac{\gamma}{2}s\sqrt{k\log 4}}\right) + \mathbb{P}\left(L^{(k,n)}_{2^{-k}(3,1)} \geq 2^{-k}\delta_n l_{n-k} e^{Cs\sqrt{\log s}}\right).$$

Using Lemma 2.10 for the first term, scaling and the upper tail estimate from Proposition 2.8 for the second term, we get

$$\mathbb{P}\left(\int_{\pi_{k,n}} e^{\frac{\gamma}{2}\phi_{0,k-1}} e^{\frac{\gamma}{2}\phi_{k,n}} ds \geq L^{(k,n)}_{2^{-k}(3,1)} e^{\frac{\gamma}{2}s\sqrt{k\log 4}}\right) + \mathbb{P}\left(L^{(k,n)}_{2^{-k}(3,1)} \geq 2^{-k}\delta_n l_{n-k} e^{Cs\sqrt{\log s}}\right) \leq Ce^{-s^2}.$$

Hence, we get for $P \in H_k \cup V_k$:

$$\mathbb{P}(L^{(n)}(P) \geq 2^{-k}\delta_n l_{n-k} e^{Cs\sqrt{\log s}} e^{\frac{\gamma}{2}s\sqrt{k\log 4}}) \leq Ce^{-s^2}. \tag{6.22}$$

Step 2. In this step we give a tail estimate for $\sum_{k=0}^{n} M_k^{(n)}$ where $M_k^{(n)} := \max_{P\in H_k\cup V_k} L^{(n)}(P)$. By union bound ($|H_k \cup V_k| \leq C4^k$) and by replacing s in (6.22) by $t(s) := \sqrt{k\log(4+\varepsilon)+s^2}$ so that the right-hand side in this inequality becomes $(4+\varepsilon)^{-k}e^{-s^2}$, we get

$$\mathbb{P}\left(M_k^{(n)} \geq \delta_n 2^{-k} l_{n-k} e^{Ct(s)\sqrt{\log t(s)}} e^{\frac{\gamma}{2}t(s)\sqrt{k\log 4}}\right) \leq C\frac{4^k}{(4+\varepsilon)^k} e^{-s^2}$$

Since $\log s \le Cs^{2\delta}$ for some small fixed $\delta > 0$, $t(s)\sqrt{\log t(s)} \le Ct(s)^{1+\delta}$. Moreover, since we have $t(s) \le \sqrt{k\log(4+\varepsilon)} + s$, the convexity of the map $s \mapsto s^{1+\delta}$ gives the bound $Ct(s)\sqrt{\log t(s)} \le Ck^{1/2+\delta/2} + Cs^{1+\delta}$.

Using that $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$ for $a, b > 0$, we have

$$t(s)\sqrt{k\log 4} = \sqrt{k^2\log(4+\varepsilon)\log 4 + s^2 k\log 4} \le a_\varepsilon k\log 4 + s\sqrt{k\log 4}$$

by introducing $a_\varepsilon := \sqrt{\log(4+\varepsilon)/\log 4}$. Therefore, we have $e^{\frac{\gamma}{2}t(s)\sqrt{k\log 4}} \le 2^{a_\varepsilon\gamma k}e^{\frac{\gamma}{2}s\sqrt{k\log 4}}$ and by using the upper bound $l_{n-k} \le l_n 2^{\gamma k}e^{C\sqrt{k}}$ (Lemma 2.12), we get the bound

$$\begin{aligned}2^{-k}l_{n-k}e^{Ct(s)\sqrt{\log t(s)}}e^{\frac{\gamma}{2}t(s)\sqrt{k\log 4}} &\le 2^{-k}(l_n 2^{\gamma k}e^{C\sqrt{k}})(e^{Ck^{1/2+\delta/2}+Cs^{1+\delta}})(2^{a_\varepsilon\gamma k}e^{\frac{\gamma}{2}s\sqrt{k\log 4}})\\ &\le l_n 2^{-k}2^{(1+a_\varepsilon)\gamma k}e^{Ck^{1/2+\delta/2}}e^{Cs^{1+\delta}}e^{\frac{\gamma}{2}s\sqrt{k\log 4}}\end{aligned}$$

which leads to the following tail estimate:

$$\mathbb{P}\left(M_k^{(n)} \ge \delta_n l_n 2^{-k}2^{(1+a_\varepsilon)\gamma k}e^{Ck^{1/2+\delta/2}}e^{Cs^{1+\delta}}e^{\frac{\gamma}{2}s\sqrt{k\log 4}}\right) \le C\frac{4^k}{(4+\varepsilon)^k}e^{-s^2}.$$

We now introduce $F(s) := \sum_{k=0}^{\infty} 2^{-k}2^{\lambda k}e^{Ck^{1/2+\alpha}}e^{\beta s\sqrt{k}}$, where $\lambda := (1+a_\varepsilon)\gamma$, $\alpha := \frac{\delta}{2}$ and $\beta := \frac{\gamma}{2}\sqrt{\log 4}$. We obtain by union bound, $\mathbb{P}(\sum_{k=0}^{n} M_k^{(n)} \ge \delta_n l_n e^{Cs^{1+\delta}}F(s)) \le C\varepsilon^{-1}e^{-s^2}$.

We thus want an upper bound on $F(s)$. To this end, we introduce the function $f_s(t) := -t(1-\lambda)\log 2 + Ct^{1/2+\alpha} + \beta s\sqrt{t}$. We notice that f increases on $[0, t_s]$ and decreases on $[t_s, \infty]$ for some $t_s > 0$. By series/integral comparison we have:

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{[t_s]-1} a_k + a_{[t_s]} + a_{[t_s]+1} + \sum_{k=[t_s]+2}^{\infty} a_k \le \int_0^{[t_s]} a_t dt + 2a_{t_s} + \int_{[t_s]+1}^{\infty} a_t dt \le 2a_{t_s} + \int_0^{\infty} a_t dt,$$

where $a_k := \exp(f_s(k))$.

By introducing $c_\varepsilon := \frac{\gamma^2}{8(1-(1+a_\varepsilon)\gamma)}$, we obtain $F(s) = \sum_{k=0}^{\infty} a_k \le Ce^{c_\varepsilon s^2}e^{Cs^{1+\delta}}$, see the appendix, Subsection 2.10.2 for more details. Thus $\mathbb{P}(\sum_{k=0}^{n} M_k^{(n)} \ge \delta_n l_n e^{c_\varepsilon s^2}e^{Cs^{1+\delta}}) \le Ce^{-s^2}$. Notice that when $\varepsilon \to 0$, $c_\varepsilon = \frac{\gamma^2}{8(1-(1+a_\varepsilon)\gamma)} \to \frac{\gamma^2}{8(1-2\gamma)}$ which is less than 1 if and only if $\gamma < 6\sqrt{2} - 8 \approx 0.485$.

Step 3. Now, we focus on the second term in the chaining inequality (6.21). Since $l_n \geq 2^{-\gamma n}e^{-C\sqrt{n}}$ (Lemma 2.12), we have for $\gamma < 1/2$ and using the tail estimates obtained in Lemma 10.39:

$$\mathbb{P}\left(2^{-n}e^{\frac{\gamma}{2}\sup_{[0,1]^2}|\phi_{0,n}|} \geq l_n e^{\frac{\gamma}{2}s}\right) \leq \mathbb{P}\left(e^{\frac{\gamma}{2}\sup_{[0,1]^2}|\phi_{0,n}|} \geq 2^{\gamma n}e^{C\sqrt{n}}e^{\frac{\gamma}{2}s}\right) \leq Ce^{-s}$$

which concludes the proof. □

2.6.2 Tightness of the metric

We are ready to prove Theorem 2.4, i.e., the tightness of the metric when $\gamma < \gamma_c \wedge 0.4$.

Proof of Theorem 2.4. The proof is divided in two main steps. In the first one, we prove the tightness of the metric in the space of continuous functions by giving a Hölder upper bound. In the second one we prove that the pseudo-metric obtained is a metric. This is done by establishing a Hölder lower bound.

Step 1. We suppose $\gamma < \gamma_c$. We start by proving that for every $0 < h < 1 - 2\gamma - \frac{\gamma^2}{4(1-2\gamma)}$, if $\varepsilon > 0$ there exists a large $C_\varepsilon > 0$ so that for every $n \geq 0$

$$\mathbb{P}\left(\exists x, x' \in [0,1]^2 \ : \ d_{0,n}(x,x') \geq C_\varepsilon \left\|x - x'\right\|^h\right) \leq \varepsilon. \tag{6.23}$$

By union bound we will estimate $\mathbb{P}(\exists x, x' \ \|x - x'\| < 2^{-n}, d_{0,n}(x,x') \geq e^s \left\|x - x'\right\|^h)$ and

$$\sum_{k=0}^{n} \mathbb{P}\left(\exists x, x' : 2^{-k} \leq \left\|x - x'\right\| \leq 2^{-k+1}, d_{0,n}(x,x') \geq e^s \left\|x - x'\right\|^h\right).$$

We start with the term $\mathbb{P}(\exists x, x' : 2^{-k} \leq \|x - x'\| \leq 2^{-k+1}, d_{0,n}(x,x') \geq e^s \left\|x - x'\right\|^h)$. Note that if $2^{-k-1} \leq \|x - x'\| \leq 2^{-k}$, there exists a square P of size 2^{-k+2} among fewer than $C4^k$ fixed such squares such that $x, x' \in P$. Also, for two such x and x', by writing $h = 1 - 2\gamma - c(\gamma) - \delta$ with $c(\gamma) > \frac{\gamma^2}{4(1-2\gamma)}$, $\delta > 0$ we have $\|x - x'\|^h \geq 2^{-k}2^{2\gamma k}2^{c(\gamma)k}2^{\delta k}$. Hence, by union bound, this term is bounded by

$$C4^k\mathbb{P}\left(\mathrm{Diam}\left(P, d_{0,n}\right) \geq 2^{-k}2^{2\gamma k}2^{c(\gamma)k}2^{\delta k}e^s\right).$$

We separate the first k scales of the fields $\phi_{0,n}$ as follows. Recall that $\mathrm{Diam}(P, e^{\gamma\phi_{0,n}}ds^2)$ is larger than $e^{\frac{\gamma}{2}\sqrt{k}t}\mathrm{Diam}(P, e^{\gamma\phi_{k,n}}ds^2)$ with probability less than $e^{-\frac{t^2}{\log 4}}$ (by Lemma 2.10). By taking $t = \sqrt{k}\log 4 + \delta\sqrt{k} + s/\sqrt{k}$, this event has probability less than $4^{-k}e^{-ck}e^{-2s}$. On the complementary event, $\mu_n^{-1}\mathrm{Diam}(P, e^{\gamma\phi_{0,n}}ds^2)$ is less than $\mu_n^{-1}\mathrm{Diam}(P, e^{\gamma\phi_{k,n}}ds^2)2^{\gamma k}2^{\frac{\gamma}{2}\delta k}e^{\frac{\gamma}{2}s}$. Under this event, by scaling the former bound becomes

$$C4^k\mathbb{P}\left(\mathrm{Diam}\left([0,1]^2, d_{n-k}\right) \geq \mu_{n-k}^{-1}\mu_n 2^{\gamma k}2^{c(\gamma)k}2^{(1-\frac{\gamma}{2})\delta k}e^{(1-\frac{\gamma}{2})s}\right).$$

Using Lemma 2.12 we get that $\mu_n \geq \mu_{n-k}2^{-\gamma k}e^{-C\sqrt{k}}$ thus we are left with estimating

$$C4^k\mathbb{P}\left(\mathrm{Diam}\left([0,1]^2, d_{n-k}\right) \geq 2^{c(\gamma)k}2^{(1-\frac{\gamma}{2})\delta k}e^{-C\sqrt{k}}e^{(1-\frac{\gamma}{2})s}\right).$$

We use the diameter estimates obtained in Proposition 2.11: since $2^{c(\gamma)k} = e^{\frac{1}{2}c(\gamma)k\log 4}$ and $\frac{1}{2}c(\gamma) > \frac{\gamma^2}{8(1-2\gamma)}$, taking $\tilde{s}(k,s) = k\log 4 + \delta' k - C\sqrt{k} + c(1-\gamma/2)s$, we have by gathering all the pieces for s large enough, uniformly in n:

$$\sum_{k=0}^{n}\mathbb{P}\left(\exists x, x' : 2^{-k} \leq \|x - x'\| \leq 2^{-k+1}, d_{0,n}(x,x') \geq e^s\|x-x'\|^h\right) \leq Ce^{-cs}.$$

Taking s large enough, the right-hand side is less than ε.

We are left with the term $\mathbb{P}(\exists x, x'\, \|x-x'\| < 2^{-n}, d_{0,n}(x,x') \geq e^s\|x-x'\|^h)$, i.e., with the case of small dyadic blocks where the field is approximately constant. By direct comparison with the supremum of the field, i.e., $d_{0,n}(x,x') \leq \mu_n^{-1}e^{\frac{\gamma}{2}\sup_{[0,1]^2}\phi_{0,n}}\|x-x'\|$ and since on the associated event $\|x-x'\|^{h-1} \geq 2^{n(1-h)}$, this probability is less than the probability $\mathbb{P}(e^{\frac{\gamma}{2}\sup_{[0,1]^2}|\phi_{0,n}|} \geq e^s 2^{n(1-h)}\mu_n)$. Recalling that one can write $h = 1-2\gamma-c(\gamma)$ with $c(\gamma) > \frac{\gamma^2}{4(1-2\gamma)}$ and that we have the lower bound on the median $\mu_n \geq 2^{-\gamma n}e^{-C\sqrt{n}}$ (see the proof of Proposition 2.8, Step 6) the former probability is less than

$$\mathbb{P}\left(\sup_{[0,1]^2}\phi_{0,n} \geq n\log 4 + \frac{\gamma}{4(1-2\gamma)}n\log 4 - \frac{C}{\gamma}\sqrt{n} + s\right)$$

which goes uniformly (in n) to 0 as s goes to infinity according to Lemma 2.17. Altogether we get the intermediate result (5.85). One can check that the interval $(0, 1-2\gamma-\frac{\gamma^2}{4(1-2\gamma)})$ is nonempty if and only if $0 < \gamma < 2/5 = 0.4$.

Hence we obtain the tightness of $(d_{0,n})_{n\geq 0}$ as a random element of $C([0,1]^2\times[0,1]^2, \mathbb{R}^+)$ and every subsequential limit is (by Skorohod's representation theorem) a pseudo-metric.

Step 2. Now we deal with the separation of the pseudo-metric. We prove that if $h > 1+\gamma$ and if $\varepsilon > 0$ there exists a small constant c_ε such that for every $n \geq 0$

$$\mathbb{P}\left(\exists x, x' \in [0,1]^2 \ : \ d_{0,n}(x,x') \leq c_\varepsilon \left\|x - x'\right\|^h\right) \leq \varepsilon. \tag{6.24}$$

As in the proof of (5.85), by union bound it is enough to estimate $\mathbb{P}(\exists x, x' \, \|x-x'\| < 2^{-n}, d_{0,n}(x,x') \leq e^{-s}\left\|x-x'\right\|^h)$ and the term

$$\sum_{k=0}^{n} \mathbb{P}\left(\exists x, x' : 2^{-k} \leq \left\|x - x'\right\| \leq 2^{-k+1}, d_{0,n}(x,x') \leq e^{-s}\left\|x-x'\right\|^h\right).$$

We start with $\mathbb{P}(\exists x, x' : 2^{-k} \leq \|x-x'\| \leq 2^{-k+1}, d_{0,n}(x,x') \leq e^{-s}\left\|x-x'\right\|^h)$. Assume there exists $x, x' \in [0,1]^2$ such that $2^{-k} \leq \|x-x'\| \leq 2^{-k+1}$. Note that any path from x to x' crosses one of the fixed $C4^k$ rectangles of size $2^{-k-1}(1,3)$ that fill vertically and horizontally $[0,1]^2$. Hence $d_{0,n}(x,x') \geq \mu_n^{-1} \min_{C4^k} L^{(n)}_{2^{-k-1}(1,3)}$. By writing $h = 1+\gamma+\delta$ with $\delta > 0$, we can bound the term in the summation above by

$$\mathbb{P}\left(e^{\frac{\gamma}{2}\inf_{[0,1]^2}\phi_{0,k-1}} \min_{C4^k} L^{(k,n)}_{2^{-k-1}(1,3)} \leq \mu_n 2^{-k} 2^{-\gamma k} 2^{-\delta k} e^{-s}\right).$$

By separating the infimum with the term $\mathbb{P}\left(\sup_{[0,1]^2}\phi_{0,n} \geq k\log 4 + \delta' k + s\right)$, by scaling and using the bound $\mu_n \leq l_{n-k} e^{C\sqrt{k}}$ from Lemma 2.12, what is left is

$$\mathbb{P}\left(\min_{C4^k} L^{(n-k)}_{(1,3)} \leq l_{n-k} 2^{-\delta'' k} e^{-(1-\frac{\gamma}{2})s}\right).$$

By union bound, the tail estimates from Corollary 2.7 and gathering all the pieces we get that the summation is less than Ce^{-cs} uniformly in n.

Finally, we control again the second term by comparison with the supremum of the field. On the event $\{\exists x, x' \, \|x-x'\| < 2^{-n}, d_{0,n}(x,x') \leq e^{-\frac{\gamma}{2}s}\left\|x-x'\right\|^h\}$, note that $\exp(\frac{\gamma}{2}\inf_{[0,1]^2}\phi_{0,n}) \leq 2^{-n(h-1)} e^{-\frac{\gamma}{2}s} \leq 2^{-(\gamma+\delta)n} e^{-\frac{\gamma}{2}s}$. The probability of this event is less than $\mathbb{P}(\sup_{[0,1]^2}\phi_{0,n} \geq n\log 4 + \delta' n + s)$ hence the result as before. □

Definition of a metric on $\mathbb{R}^2$. Let us mention here that one can define a random metric associated to $\phi_{0,\infty}$ on the full two-dimensional space. We saw that $(d_{0,n}^{[0,1]^2})_{n\geq 0}$ is tight thus there exists some subsequence that converges in law to $d_{0,\infty}$. The same result remains true for $(d_{0,n}^{[-p,p]^2})_{n\geq 0}$ with $p > 0$. By a diagonal argument, there exists a subsequence (n_k) such that for every $p \in \mathbb{N}$, $(d_{0,n_k}^{[-p,p]^2})_{k\geq 0}$ converges in law to some $d_{0,\infty}^{[-p,p]^2}$. Then, one can define $d_{0,\infty}^{\mathbb{R}^2}$ as the limit of $d_{0,\infty}^{[-p,p]^2}$ when p goes to ∞. Indeed, if we denote by $d_{0,\infty}^{[-p,p]^2}([-1,1]^2)$ the restriction of $d_{0,\infty}^{[-p,p]^2}$ to $[-1,1]^2$, we have

$$\lim_{p_0\to\infty} \mathbb{P}\left(\forall p \geq p_0,\ d_{0,\infty}^{[-p,p]^2}([-1,1]^2) = d_{0,\infty}^{[-p_0,p_0]^2}([-1,1]^2)\right) = 1.$$

Indeed, with high probability, there is a crossing of an annulus around $[0,1]^2$ whose length for $d_{0,n}$ is larger than the diameter of $[0,1]^2$ for $d_{0,n}$, uniformly in n. Also, if we fix $x \in \mathbb{R}^2$ and denote by T_x the map $\phi \mapsto \phi(\cdot - x)$, for a field ϕ and $d \mapsto d(\cdot - x, \cdot - x)$ for a metric d, if the measure on fields is $\phi_{0,\infty}$ and the measure on metrics is $d_{0,\infty}^{\mathbb{R}^2}$, then the transformation T_x is mixing thus ergodic in each case. This ergodic property for the Gaussian multiplicative chaos measure is a useful property to characterize log-normal $\star$-scale invariant random measures. We refer the interested reader to Theorem 4 and the remark following Proposition 5 in [3].

2.7 Weyl scaling

In this section we will see that any limiting metric space is non trivial. In particular, we will show they are not deterministic and not independent of field $\phi_{0,\infty}$.

The main idea of the proof is the following. Take $d_{0,\infty}$ a limiting metric whose existence comes from the previous subsection. Define for some suitable function f the metric $e^{\frac{\gamma}{2}f} \cdot d_{0,\infty}$ associated to the field $\phi_{0,\infty} + f$. Thanks to the approximation procedure together with the Cameron-Martin theorem for Gaussian measures, we will prove that the couplings $P_\infty := \mathcal{L}(\phi_{0,\infty}, d_{0,\infty})$ and $P_\infty^f := \mathcal{L}(\phi_{0,\infty} + f, e^{\frac{\gamma}{2}f} \cdot d_{0,\infty})$ are mutually absolutely continuous and that the associated Radon-Nikodým derivative satisfies $\frac{dP_\infty^f}{dP_\infty} = \frac{d\mathcal{L}(\phi_{0,\infty}+f)}{d\mathcal{L}\phi_{0,\infty}}$, which implies the result we look for: if $\phi_{0,\infty}$ and $d_{0,\infty}$ are independent, it implies $e^{\frac{\gamma}{2}f} \cdot d_{0,\infty} \overset{(d)}{=} d_{0,\infty}$ which leads to a contradiction.

In what follows, we recall some background on metric geometry and we refer the reader to Chapter 2 in [17] for more details. Let (X, d) be a metric space and π be a continuous map from an interval I to X. We define the length $L_d(\pi)$ of π with respect to the metric d by setting

$$L_d(\pi) := \sup \sum_{i=1}^{n} d(\pi(t_{i-1}), \pi(t_i))$$

where the supremum is taken over all $n \geq 1$, $t_0 < t_1 < \cdots < t_n$ in I. If $L_d(\pi) < \infty$, we say that π is *rectifiable*. We also say that π has *constant speed* if there exists a constant $\lambda \geq 0$ such that $L_d(\pi_{|_{[s,t]}}) = \lambda\,|t - s|$ holds for every $s, t \in I$.

Starting with such a length functional $L = L_d$ we can define a metric space (X, d_L) by setting, for every $x, y \in X$,

$$d_L(x, y) := \inf\{L(\pi) \mid \pi \text{ is rectifiable }, \pi(0) = x \text{ and } \pi(1) = y\}.$$

We say that a metric d is *intrinsic* if $d = d_{L_d}$. In this case, (X, d) is called a *length space*. Notice that a Riemannian manifold (M, g) is a length space. Moreover, we say that this metric is *strictly intrinsic* if for any $x, y \in X$ there exists a path π such that $\pi(0) = x$, $\pi(1) = y$ and $d(x, y) = L_d(\pi)$. In this case the path π is called a *shortest path* between x and y.

Let (X, d) be a metric space. A path (π, I) is called a *geodesic* if π has constant speed and if $L_d(\pi_{|_{[s,t]}}) = d(\pi(s), \pi(t))$ for every $s, t \in I$. A path (π, I) is called a *local geodesic* if for every $t \in I$, there exists an $\varepsilon > 0$ such that $\pi_{|_{[t-\varepsilon, t+\varepsilon]}}$ is a geodesic. (X, d) is a *geodesic space* if for every $x, y \in X$, there exists a geodesic $\pi : [0, 1] \to X$ with $\pi(0) = x$, $\pi(1) = y$. It is clear from the definition that every geodesic space is a length space.

For a complete metric space, one can characterize the notion of intrinsic metric using midpoints (see Lemma 2.4.8 and Theorem 2.4.16 in [17] for a reference). A point $z \in (X, d)$ is called a *midpoint* between points x and y if $d(x, z) = d(z, y) = \frac{1}{2}d(x, y)$. The following holds:

1. Assume that (X, d) is a metric space. If d is a strictly intrinsic metric, then for every points x and y in X there exists a midpoint z between them.

2. If (X, d) is a complete metric space and if for every $x, y \in X$ there exists a midpoint z between x and y, then d is strictly intrinsic.

Given a continuous function f and an intrinsic metric d, both defined on $[0,1]^2$, with d homeomorphic to the Euclidean metric on the unit square, we define the metric $e^f \cdot d$ by first describing its length. For a continuous path $\pi : [a,b] \to [0,1]^2$ we define

$$L_d^f(\pi) := \limsup_{n\to\infty} \sum_{i=1}^{n} e^{f(\pi(t_{i-1}^n))} d(\pi(t_{i-1}^n), \pi(t_i^n)),$$

where $a = t_0^n < \cdots < t_n^n = b$ and $\lim_{n\to\infty} \max_{0\le i\le n-1}(t_{i+1}^n - t_i^n) = 0$. Notice that $L_d(\pi) < \infty$ if and only if $L_d^f(\pi) < \infty$. We then define $e^f \cdot d := d_{L_d^f}$. Notice that if f is constant since d is intrinsic we have $e^f \cdot d = e^f d$. Notice also that if ϕ and ψ are smooth functions, then the Riemannian metric associated to the metric tensor $e^{\phi+\psi}ds^2$ is equal to $e^{\frac{1}{2}\phi} \cdot d$ where d is the metric associated to the metric tensor $e^{\psi}ds^2$.

The following lemma will be useful to identify the metric associated to $\phi_{0,\infty} + f$ in terms of the one associated to $\phi_{0,\infty}$.

Lemma 2.13. *Let f be a continuous function on $[0,1]^2$ and $r, R : (0,\infty) \to (0,\infty)$ be continuous increasing functions with $r(0^+) = R(0^+) = 0$. If a sequence of intrinsic metrics $(d_n)_{n\ge 0}$ on $[0,1]^2$ satisfying for every $x, y \in [0,1]^2$, $n \ge 0$ the condition*

$$r(\|x-y\|) \le d_n(x,y) \le R(\|x-y\|),$$

converges uniformly to a metric d_∞ on $[0,1]^2$, then the sequence of metrics $(e^f \cdot d_n)_{n\ge 0}$ converges simply to the metric $e^f \cdot d_\infty$, i.e., for every fixed $x, y \in [0,1]^2$ we have $\lim_{n\to\infty} e^f \cdot d_n(x,y) = e^f \cdot d_\infty(x,y)$.

Proof. We fix $x, y \in [0,1]^2$ and we want to prove that $e^f \cdot d_n(x,y)$ converges to $e^f \cdot d_\infty(x,y)$. We separate the proof in three parts: first we control the oscillation of f over geodesics then the upper bound and finally the lower bound.

By assumption, d_n converges uniformly to d_∞ hence d_∞ is an intrinsic metric (see Exercise 2.4.19 in [17]). Again by assumption, there exists some positive c and C such that for every n

$$r(\|x-y\|) \le d_n(x,y) \le R(\|x-y\|).$$

This condition is then satisfied by d_∞ and since for $n \in \mathbb{N} \cup \{\infty\}$, $e^{-\|f\|_\infty} d_n \leq e^f \cdot d_n \leq e^{\|f\|_\infty} d_n$ this condition is also satisfied by $e^f \cdot d_n$ and $e^f \cdot d_\infty$ by replacing c by $e^{-\|f\|_\infty} c$ and C by $e^{\|f\|_\infty} C$. This tells us that the spaces $([0,1]^2, d_n)$ and $([0,1]^2, e^f \cdot d_n)$ are complete and locally compact for $n \in \mathbb{N} \cup \{\infty\}$. Hence, by Theorem 2.5.23 in [17], these spaces are strictly intrinsic.

Now we look at the oscillation of f over small parts of shortest path associated to the metrics $e^f \cdot d_n$ and d_n for all n's. The first step is to understand that locally $e^{f(x)} d_n(x,y) \approx e^f \cdot d_n(x,y)$. To this end notice the inequality

$$e^{-\operatorname{osc}(f, K^{d_n}_{x,y})} e^{f(x)} d_n(x,y) \leq e^f \cdot d_n(x,y) \leq e^{\operatorname{osc}(f, K^{d_n}_{x,y})} e^{f(x)} d_n(x,y)$$

where $\operatorname{osc}(f,K) := \sup_{x,y \in K} |f(x) - f(y)|$ and where $K^{d_n}_{x,y} := \operatorname{Geo}_{d_n}(x,y) \cup \operatorname{Geo}_{e^f \cdot d_n}(x,y)$. Then notice that if x is close to y then $K^{d_n}_{x,y}$ is small with respect to the Euclidean topology. More precisely, notice that $\operatorname{Geo}_{d_n}(x,y) \subset B(x, r^{-1}(R(\|x-y\|)))$. Indeed, if $z \in \operatorname{Geo}_{d_n}(x,y)$ then

$$r(\|x-z\|) \leq d_n(x,z) \leq d_n(x,y) \leq R(\|x-y\|).$$

For every x and y such that $d_n(x,y) < \delta$, $\operatorname{osc}(f, K^{d_n}_{x,y}) \leq \omega(f, r^{-1}(\delta))$ where $\omega(f,\delta)$ denotes the modulus of continuity of the function f, i.e., $\omega(f,\delta) := \sup\{|f(x) - f(y)| : x, y \in [0,1]^2 \text{ st } |x-y| < \delta\}$. Note that the bound of the oscillation is independent of n.

We start with the upper bound. Since $e^f \cdot d_\infty$ is strictly intrinsic, take by a dichotomy procedure $x = x_0, \ldots, x_N = y$ such that $e^f \cdot d_\infty(x,y) = \sum_{i=0}^{n-1} e^f \cdot d_\infty(x_i, x_{i+1})$ and $d_\infty(x_i, x_{i+1}) < \delta$. For n large enough, for every i, $d_n(x_i, x_{i+1}) < \delta$. Hence, by triangle inequality, for n large enough

$$\begin{aligned} e^f \cdot d_n(x,y) &\leq \sum_{i=0}^{N-1} e^f \cdot d_n(x_i, x_{i+1}) \\ &\leq \sum_{i=0}^{N-1} e^{\operatorname{osc}(f, K^{d_n}_{x_i, x_{i+1}})} e^{f(x_i)} d_n(x_i, x_{i+1}) \\ &\leq e^{\omega(f, C\delta^{1/\alpha})} \sum_{i=0}^{N-1} e^{f(x_i)} d_n(x_i, x_{i+1}). \end{aligned}$$

Hence by taking the lim sup and using the convergence of d_n to d_∞

$$\begin{aligned}\limsup_{n\to\infty} e^f\cdot d_n(x,y) &\le e^{\omega(f,C\delta^{1/\alpha})}\sum_{i=0}^{N-1} e^{f(x_i)} d_\infty(x_i,x_{i+1})\\ &\le e^{\omega(f,C\delta^{1/\alpha})}\sum_{i=0}^{N-1} e^{\operatorname{osc}(f,K^{d_\infty}_{x_i,x_{i+1}})} e^f\cdot d_\infty(x_i,x_{i+1})\\ &\le e^{2\omega(f,C\delta^{1/\alpha})}\sum_{i=0}^{N-1} e^f\cdot d_\infty(x_i,x_{i+1})\\ &= e^{2\omega(f,C\delta^{1/\alpha})} e^f\cdot d_\infty(x,y).\end{aligned}$$

By the uniform continuity of f, we obtain the upper bound by letting δ going to 0.

Now we deal with the lower bound. Up to extracting a subsequence we may assume that $e^f\cdot d_n(x,y)$ converges to its lim inf. Again, since $e^f\cdot d_n$ is strictly intrinsic, take $x_0^n = x,\ldots,x_{N_n}^n = y$, such that

$$e^f\cdot d_n(x,y) = \sum_{i=0}^{N_n-1} e^f\cdot d_n(x_i^n,x_{i+1}^n)$$

and $d_n(x_i^n,x_{i+1}^n) < \delta$. Taking the minimal number N_n (still using the midpoints method) N_n is bounded and up to taking a subsequence, we may assume that N_n converges. In particular, N_n is eventually constant and equal to some N. We may then also assume that the x_i^n's also converges to some x_i's for $0\le i\le N$ and these x_i's satisfy $d_\infty(x_i,x_{i+1})\le\delta$. Then for n large enough

$$\begin{aligned}e^f\cdot d_n(x,y) &\ge \sum_{i=0}^{N-1} e^{-\operatorname{osc}\left(f,K^{d_n}_{x_i^n,x_{i+1}^n}\right)} e^{f(x_i^n)}\cdot d_n(x_i^n,x_{i+1}^n)\\ &\ge e^{-\omega(f,C\delta^{1/\alpha})}\sum_{i=0}^{N-1} e^{f(x_i^n)}\cdot d_n(x_i^n,x_{i+1}^n).\end{aligned}$$

Taking the limit as n goes to ∞ we get by the uniform convergence of d_n to d_∞

$$\left|\sum_{i=0}^{N-1} e^{f(x_i^n)} d_n(x_i^n,x_{i+1}^n) - \sum_{i=0}^{N-1} e^{f(x_i^n)} d_\infty(x_i^n,x_{i+1}^n)\right| \le N e^{\|f\|_\infty}\|d_n-d_\infty\|_\infty \to 0$$

So

$$\begin{aligned}
\liminf_{n\to\infty} e^f \cdot d_n(x,y) &\geq e^{-\omega(f,C\delta^{1/\alpha})} \sum_{i=0}^{N-1} e^{f(x_i)} d_\infty(x_i,x_{i+1}) \\
&\geq e^{-\omega(f,C\delta^{1/\alpha})} \sum_{i=0}^{N-1} e^{-\mathrm{osc}(f,K^{d_\infty}_{x_i,x_{i+1}})} e^f \cdot d_\infty(x_i,x_{i+1}) \\
&\geq e^{-2\omega(f,C\delta^{1/\alpha})} \sum_{i=0}^{N-1} e^f \cdot d_\infty(x_i,x_{i+1}) \\
&\geq e^{-2\omega(f,C\delta^{1/\alpha})} e^f \cdot d_\infty(x,y)
\end{aligned}$$

by the triangle inequality. Letting δ going to 0 we get the result. □

It is easy to see that the same result holds if instead of f, we assume that a sequence of continuous functions $(f_n)_{n\geq 0}$ converges uniformly to f on $[0,1]^2$, then under the same assumptions $(e^{f_n}\cdot d_n)_{n\geq 0}$ converges simply to the metric $e^f \cdot d_{0,\infty}$. This lemma is a key ingredient to prove the following corollary.

Corollary 2.12. *Let (f_n) be a sequence of continuous real-valued functions defined on $[0,1]^2$ and converging uniformly to a function f. If $\gamma < \min(\gamma_c, 0.4)$ then the following statements hold:*

1. *$(d_{0,n}, e^{\frac{\gamma}{2}f_n}\cdot d_{0,n})_{n\geq 0}$ is tight.*
2. *If (n_k) is a subsequence along which $(d_{0,n_k}, e^{\frac{\gamma}{2}f_{n_k}}\cdot d_{0,n_k})_{k\geq 0}$ converges in law to some $(d_{0,\infty}, d'_{0,\infty})$ then $d'_{0,\infty} = e^{\frac{\gamma}{2}f}\cdot d_{0,\infty}$.*
3. *In particular, $(\phi_{0,n_k}, d_{0,n_k})_{k\geq 0}$ converges in law to a coupling $P_\infty := \mathcal{L}(\phi_{0,\infty}, d_{0,\infty})$ and $(\phi_{0,n_k} + f_{n_k}, e^{\frac{\gamma}{2}f_{n_k}}\cdot d_{0,n_k})_{k\geq 0}$ converges in law to a coupling $P^f_\infty := \mathcal{L}(\phi_{0,\infty}+f, e^{\frac{\gamma}{2}f}\cdot d_{0,\infty})$, both couplings are probability measures on the same space.*

Proof. We start with the proof of (i). Since for $n\geq 0$, a.s. $e^{-\frac{\gamma}{2}\sup_{n\geq 0}\|f_n\|_\infty} d_{0,n} \leq e^{\frac{\gamma}{2}f_n}\cdot d_{0,n} \leq e^{\frac{\gamma}{2}\sup_{n\geq 0}\|f_n\|_\infty} d_{0,n}$, the argument giving the tightness of $(d_{0,n})_{n\geq 0}$ then extends to give the one of $(e^{\frac{\gamma}{2}f_n}\cdot d_{0,n})_{n\geq 0}$, see the proof of Theorem 2.4.

We now prove (ii). We first fix $\alpha > 1+\gamma$ and $\beta \in (0, 1-2\gamma - \frac{\gamma^2}{4(1-2\gamma)})$ and we then define $C^n_\alpha := \sup_{x,x'\in[0,1]^2} \frac{\|x-x'\|^\alpha}{d_{0,n}(x,x')}$ and $C^n_\beta := \sup_{x,x'\in[0,1]^2} \frac{d_{0,n}(x,x')}{\|x-x'\|^\beta}$. Using (5.86) and (5.85), $(C^n_\alpha)_{n\geq 0}$ and

$(C_\beta^n)_{n\geq 0}$ are tight. Since $(\phi_{0,n}, \phi_{0,n}+f_n, d_{0,n}, e^{\frac{\gamma}{2}f_n}\cdot d_{0,n}, C_\alpha^n, C_\beta^n)_{n\geq 0}$ is tight, up to extracting a subsequence, we can assume it converges in law. By the Skorohod representation theorem, we obtain an almost sure convergence on a same probability space and we denote by $d_{0,\infty}$ (resp $d'_{0,\infty}$) the limit of $d_{0,n}$ (resp $e^{\frac{\gamma}{2}f_n}\cdot d_{0,n}$). We can thus introduce the random constants $C_\alpha := \sup_{n\geq 0} C_\alpha^n < \infty$ and $C_\beta := \sup_{n\geq 0} C_\beta^n < \infty$. On this probability space, the following condition of Lemma 2.13 is satisfied: a.s. for every $n\geq 0$, $x, x' \in [0,1]^2$,

$$\frac{\|x-x'\|^\alpha}{C_\alpha} \leq \frac{\|x-x'\|^\alpha}{C_\alpha^n} \leq d_{0,n}(x,x') \leq C_\beta^n \left\|x-x'\right\|^\beta \leq C_\beta \left\|x-x'\right\|^\beta.$$

By using Lemma 2.13, we can identify the almost sure limit of $e^{\frac{\gamma}{2}f_n}\cdot d_{0,n}$: $d'_{0,\infty} = e^{\frac{\gamma}{2}f}\cdot d_{0,\infty}$. Finally, notice that (iii) follows from the previous proofs. □

The main result of this subsection is the following proposition. In order to state it, let us recall that the kernel of $\phi_{0,\infty}$ is given by $C_{0,\infty}(x,x') = \int_0^1 c(\frac{x-x'}{t})\frac{dt}{t} = \int_0^1 k*k(\frac{x-x'}{t})\frac{dt}{t}$ and let us make the following remark: the map $C_{0,\infty}: \mathcal{S}(\mathbb{R}^2)\to\mathcal{S}(\mathbb{R}^2)$ defined for $f\in\mathcal{S}(\mathbb{R}^2)$ by $C_{0,\infty}f := C_{0,\infty}*f$ is a bijection. Indeed, notice that $\hat{C}_{0,\infty}(\xi) = \|\xi\|^{-2}\int_0^{\|\xi\|} u\hat{k}(u)^2du$ (see the remark before (9.34) for a proof). In particular, we have $\hat{C}_{0,\infty}(0) = \frac{\hat{k}(0)^2}{2} > 0$ (since $\hat{k}(0) = \int_{B(0,r_0)} k(x)dx$ with k nonnegative and non-identically zero), and $\hat{C}_{0,\infty}(\xi)\sim_\infty \frac{1}{2\pi\|\xi\|^2}$. Thus, the equation $C_{0,\infty}*f = g$ admits the solution f given by $f(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\frac{\hat{g}(\xi)}{\hat{C}_{0,\infty}(\xi)}e^{ix\cdot\xi}$. In particular, if $f\in\mathcal{S}(\mathbb{R}^2)$, $C_{0,\infty}^{-1}f\in\mathcal{S}(\mathbb{R}^2)$ is well-defined.

Proposition 2.13. *For $f\in\mathcal{S}(\mathbb{R}^2)$, the coupling $P_\infty^f = \mathcal{L}(\phi_{0,\infty}+f, e^{\frac{\gamma}{2}f}\cdot d_{0,\infty})$ is absolutely continuous with respect to $P_\infty = \mathcal{L}(\phi_{0,\infty}, d_{0,\infty})$ and its Radon-Nikodým derivative is given by*

$$\frac{dP_\infty^f}{dP_\infty} = \frac{d\mathcal{L}\left(\phi_{0,\infty}+f, e^{\frac{\gamma}{2}f}\cdot d_\infty\right)}{d\mathcal{L}\left(\phi_{0,\infty}, d_\infty\right)} = \frac{d\mathcal{L}(\phi_{0,\infty}+f)}{d\mathcal{L}(\phi_{0,\infty})} = \exp\left(\langle\phi_{0,\infty}, C_{0,\infty}^{-1}f\rangle - \frac{1}{2}\langle f, C_{0,\infty}^{-1}f\rangle\right)$$

In particular, $d_{0,\infty}$ and $\phi_{0,\infty}$ are not independent.

To prove this proposition, we will use the following lemma, whose proof is postponed to the end of the section.

Lemma 2.14. *Fix $g\in\mathcal{S}(\mathbb{R}^2)$ and define for $n\in\mathbb{N}\cup\{\infty\}$, $f_n := C_{0,n}*g$. The following assertions hold:*

1. *For every $n \in \mathbb{N} \cup \{\infty\}$, $\phi_{0,n} + f_n$ is absolutely continuous with respect to $\phi_{0,n}$ and* $\frac{d\mathcal{L}(\phi_{0,n}+f_n)}{d\mathcal{L}(\phi_{0,n})} = \exp(\langle \phi_{0,n}, g\rangle - \frac{1}{2}\langle f_n, g\rangle)$.

2. *$(f_n)_{n\geq 0}$ converges uniformly on $\mathbb{R}^2$ and in $L^2(\mathbb{R}^2)$ to $C_{0,\infty} * g$.*

3. *$(\phi_{0,n})_{n\geq 0}$ converges in law to $\phi_{0,\infty}$ with respect to the weak topology on $\mathcal{S}'(\mathbb{R}^2)$.*

Proof of Proposition 2.13. Take $f \in \mathcal{S}(\mathbb{R}^2)$, set $g := C_{0,\infty}^{-1} f \in \mathcal{S}(\mathbb{R}^2)$ and define $f_n := C_{0,n} * g$. By using Lemma 2.14 assertion (i) for $n = \infty$ we have:

$$D_\infty^f := \frac{d\mathcal{L}(\phi_{0,\infty} + f)}{d\mathcal{L}(\phi_{0,\infty})} = \exp\left(\langle \phi_{0,\infty}, g\rangle - \frac{1}{2}\langle f, g\rangle\right).$$

Using again Lemma 2.14 assertion (i) but for finite n we have:

$$\frac{d\mathcal{L}(\phi_{0,n} + f_n)}{d\mathcal{L}(\phi_{0,n})} = \exp\left(\langle \phi_{0,n}, g\rangle - \frac{1}{2}\langle f_n, g\rangle\right).$$

Now we prove that $\left(\phi_{0,\infty} + f, e^{\frac{\gamma}{2}f} \cdot d_{0,\infty}\right)$ is absolutely continuous with respect to $(\phi_{0,\infty}, d_{0,\infty})$ and that the Radon-Nikodým derivative is given by D_∞^f. By introducing the function G which maps a smooth field ϕ to the Riemannian metric whose metric tensor is $e^{\gamma\phi} ds^2$, we have, for every continuous and bounded functional F:

$$\begin{aligned}
\mathbb{E}\left(F\left(\phi_{0,n} + f_n, e^{\frac{\gamma}{2}f_n} \cdot d_{0,n}\right)\right) &= \mathbb{E}\left(F(\phi_{0,n} + f_n, \mu_n^{-2} G(\phi_{0,n} + f_n))\right) \\
&= \mathbb{E}\left(F\left(\phi_{0,n}, \mu_n^{-2} G(\phi_{0,n})\right) \frac{d\mathcal{L}(\phi_{0,n} + f_n)}{d\mathcal{L}(\phi_{0,n})}\right) \\
&= \mathbb{E}\left(F\left(\phi_{0,n}, d_{0,n}\right) \exp\left(\langle \phi_{0,n}, g\rangle - \frac{1}{2}\langle f_n, g\rangle\right)\right).
\end{aligned}$$

Now we claim that the left-hand side converges to $\mathbb{E}(F(\phi_{0,\infty} + f, e^{\frac{\gamma}{2}f} \cdot d_{0,\infty}))$ and that the right-hand side converges to $\mathbb{E}(F(\phi_{0,\infty}, d_{0,\infty}) D_\infty^f)$.

The first claim follows from the convergence in law from Corollary 2.12 since $(f_n)_{n\geq 0}$ converges uniformly on $[0,1]^2$ and in $L^2(\mathbb{R}^2)$ to f by Lemma 2.14 assertion (ii).

The second one comes from the convergence in law of $(\phi_{0,n}, d_{0,n})_{n\geq 0}$ and from the convergence of $(f_n)_{n\geq 0}$ to f in $L^2(\mathbb{R}^2)$ (Lemma 2.14 assertion (ii)). To be precise, for $M > 0$ the map

$(\phi, d) \mapsto F(\phi, d)\exp(\langle \phi, g\rangle) \wedge M$ is continuous and bounded thus

$$\lim_{n\to\infty} \mathbb{E}\left(F(\phi_{0,n}, d_{0,n})\exp(\langle \phi_{0,n}, g\rangle) \wedge M\right) = \mathbb{E}\left(F(\phi_{0,\infty}, d_{0,\infty})\exp(\langle \phi_{0,\infty}, g\rangle) \wedge M\right).$$

By the triangle inequality and since F is bounded we have

$$\begin{aligned}
&|\mathbb{E}\left(F(\phi_{0,n}, d_{0,n})\exp(\langle \phi_{0,n}, g\rangle)\right) - \mathbb{E}\left(F(\phi_{0,\infty}, d_{0,\infty})\exp(\langle \phi_{0,\infty}, g\rangle)\right)| \\
&\leq |\mathbb{E}\left(F(\phi_{0,n}, d_{0,n})\exp(\langle \phi_{0,n}, g\rangle) \wedge M\right) - \mathbb{E}\left(F(\phi_{0,\infty}, d_{0,\infty})\exp(\langle \phi_{0,\infty}, g\rangle) \wedge M\right)| \\
&\quad + |\mathbb{E}\left(F(\phi_{0,\infty}, d_{0,\infty})\exp(\langle \phi_{0,\infty}, g\rangle) \wedge M\right) - \mathbb{E}\left(F(\phi_{0,\infty}, d_{0,\infty})\exp(\langle \phi_{0,\infty}, g\rangle)\right)| \\
&\quad + C\mathbb{E}\left(\exp(\langle \phi_{0,n}, g\rangle) 1_{\exp(\langle \phi_{0,n}, g\rangle)\geq M}\right).
\end{aligned}$$

Taking the lim sup when n goes to infinity (the first term vanishes) and then letting M goes to infinity (the second term vanishes by uniform integrability), we obtain the result follows by taking the limits $\limsup_{M\to\infty}\limsup_{n\to\infty}\mathbb{E}\left(\exp(\langle \phi_{0,n}, g\rangle) 1_{\exp(\langle \phi_{0,n}, g\rangle)\geq M}\right) = 0$ (easy to check). □

Now, we come back to the proof of Lemma 2.14.

Proof of Lemma 2.14. We will prove successively the assertions (i), (ii) and (iii).

(i). The proof follows from evaluating characteristic functionals. Define for $\phi \in \mathcal{S}(\mathbb{R}^2)$ the functional $F_\varphi : \mathcal{S}'(\mathbb{R}^2) \to \mathbb{R}^+$ such that $F_\varphi(\phi) = \exp(\langle \phi, \varphi\rangle)$. Using the Gaussian characteristic formula, we have $\mathbb{E}(F_\varphi(\phi_{0,n} + f_n)) = e^{\langle f_n, \varphi\rangle}\mathbb{E}(e^{\langle \phi_{0,n}, \varphi\rangle}) = e^{\langle f_n, \varphi\rangle} e^{\frac{1}{2}\mathrm{Var}(\langle \phi_{0,n}, \varphi\rangle)} = e^{\langle f_n, \varphi\rangle} e^{\frac{1}{2}\langle C_{0,n} * \varphi, \varphi\rangle}$ and similarly, since $C_{0,n} * g = f_n$ and $\langle C_{0,n} * \varphi, g\rangle = \langle \varphi, C_{0,n} * g\rangle = \langle \varphi, f_n\rangle = \langle f_n, \varphi\rangle$:

$$\begin{aligned}
\mathbb{E}\left(F_\varphi(\phi_{0,n}) e^{\langle \phi_{0,n}, g\rangle - \frac{1}{2}\langle f_n, g\rangle}\right) &= e^{-\frac{1}{2}\langle f_n, g\rangle}\mathbb{E}\left(e^{\langle \phi_{0,n}, \varphi + g\rangle}\right) \\
&= e^{-\frac{1}{2}\langle f_n, g\rangle} e^{\frac{1}{2}\langle C_{0,n} * (\varphi + g), \varphi + g\rangle} \\
&= e^{-\frac{1}{2}\langle f_n, g\rangle} e^{\frac{1}{2}\langle C_{0,n} * \varphi, \varphi\rangle + \langle C_{0,n} * \varphi, g\rangle + \frac{1}{2}\langle C_{0,n} * g, g\rangle} \\
&= \mathbb{E}\left(F_\varphi(\phi_{0,n} + f_n)\right).
\end{aligned}$$

(ii). First, we prove that $C_{0,n} * f$ converges uniformly to $C_{0,\infty} * f$ on $\mathbb{R}^2$. Notice that

$\|C_{0,n} * f - C_{0,\infty} * f\|_\infty = \|C_{n,\infty} * f\|_\infty \leq \|f\|_\infty \|C_{n,\infty}\|_{L^1(\mathbb{R}^2)}$. Furthermore:

$$\|C_{n,\infty}\|_{L^1(\mathbb{R}^2)} = \int_{\mathbb{R}^2} \int_0^{2^{-n}} c\left(\frac{y}{t}\right) \frac{dt}{t} dy \leq \|c\|_\infty \int_{\mathbb{R}^2} \int_0^{2^{-n}} 1_{y \in B(0, 2r_0 t)} \frac{dt}{t} dy \leq C 2^{-2n}.$$

Now we prove that the convergence holds in $L^2(\mathbb{R}^2)$. By Parseval, we have

$$\|C_{0,n} * g - C_{0,\infty} * g\|_{L^2(\mathbb{R}^2)}^2 = \left\|\hat{C}_{n,\infty} \hat{g}\right\|_{L^2(\mathbb{R}^2)}^2.$$

Moreover, since $\hat{C}_{n,\infty}(\xi) = \|\xi\|^{-2} \int_0^{2^{-n}\|\xi\|} u\hat{k}(u)^2 du$ (see the remark before (9.34) for a proof), we have:

$$\left\|\hat{C}_{n,\infty} \hat{g}\right\|_{L^2(\mathbb{R}^2)}^2 = \int_{\mathbb{R}^2} \left(\|\xi\|^{-2} \int_0^{2^{-n}\|\xi\|} u\hat{k}(u)^2 du\right)^2 |\hat{g}(\xi)|^2 d\xi \leq C 2^{-4n} \left\|\hat{k}\right\|_\infty^4 \|g\|_{L^2(\mathbb{R}^2)}^2$$

and this completes the proof of assertion (ii).

(iii). We want to prove here that $(\phi_{0,n})_{n \geq 0}$ converges in law to $\phi_{0,\infty}$ in $\mathcal{S}'(\mathbb{R}^2)$. To this end, take a function $f \in \mathcal{S}(\mathbb{R}^2)$ and notice that:

$$\mathbb{E}\left(\langle \phi_{0,n}, f\rangle^2\right) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) C_{0,n}(x,y) f(y) dx dy = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{C}_{0,n}(\xi) \left|\hat{f}(\xi)\right|^2 d\xi.$$

Since $\hat{C}_{0,n}(\xi) = \|\xi\|^{-2} \int_{2^{-n}\|\xi\|}^{\|\xi\|} u\hat{k}(u)^2 du$ for $n \in \mathbb{N} \cup \{\infty\}$, by monotone convergence, we get that $\mathbb{E}(\langle \phi_{0,n}, f\rangle^2)$ converges to $\mathbb{E}(\langle \phi_{0,\infty}, f\rangle^2)$. Thus, we have the convergence of the characteristic functionals: $\mathbb{E}(e^{i\langle \phi_{0,n}, f\rangle}) = e^{-\frac{1}{2}\mathbb{E}\left(\langle \phi_{0,n}, f\rangle^2\right)} \underset{n\to\infty}{\to} e^{-\frac{1}{2}\mathbb{E}\left(\langle \phi_{0,\infty}, f\rangle^2\right)}$, which is enough to obtain the convergence in law, see for instance [15].

□

2.8 Small noise regime: proof of Theorem 2.5

We want to prove here that $\gamma_c > 0$. To do it, we will show by induction that the ratio between large quantiles and small quantiles is uniformly bounded in n. Recall the notations l_n, $\bar{l}_n$ and δ_n from Subsection 2.2.3. Then $\delta_n \nearrow \delta_\infty$ when n goes to ∞. We start by showing that when ε and γ are small enough, but fixed, then $\delta_\infty < \infty$. By our tail estimates, Corollary 2.7 (with $l_n \geq \mu_n \delta_\infty^{-1}$)

and Proposition 2.8 (with $\delta_n l_n \leq \delta_\infty \mu_n$) this implies the tightness of $\log L_{1,1}^{(n)} - \log \mu_n$.

Proof of Theorem 2.5. We proceed according to the following steps:

1. Relate the ratio δ_n between small quantiles and high quantiles to $\mathrm{Var} \log L_{1,1}^{(n)}$.
2. Give an upper bound on $\mathrm{Var} \log L_{1,1}^{(n)}$ using the Efron-Stein inequality. The bound obtained involves a sum indexed by blocks $P \in \mathcal{P}_k$ for $0 \leq k \leq n$.
3. Get rid of the independent copy term which appears when using the Efron-Stein inequality and see how a small value of γ makes the variance smaller.
4. Give an upper bound on diameter and a lower bound on the left-right distance involving the same quantities at a higher scale.
5. Use the tails estimates obtained for the higher scales and control the ratio of the upper bound over the lower bound using δ_{n-1}.
6. Conclude the induction.

Step 1. To link the quantiles and the variance of a random variable X notice that for $l' \geq l$ we have $2\mathrm{Var}(X) = \mathbb{E}((X' - X)^2) \geq \mathbb{E}(1_{X' \geq l'} 1_{X \leq l}(X' - X)^2) \geq \mathbb{P}(X \geq l')\mathbb{P}(X \leq l)(l' - l)^2$ where X' is an independent copy of X. Together with the RSW estimates obtained in Theorem 2.2 (using (3.5) with $a' = 3$, $b' = 1$, $a = 1$, $b = 1$ and (3.4) with $a' = 1$, $b' = 1$, $a = 1$, $b = 1$), we have, for some constant C_ε depending on ε but not on n:

$$\frac{\bar{l}_{3,1}^{(n)}(\varepsilon)}{l_{1,3}^{(n)}(\varepsilon)} \leq e^{C_\varepsilon} \frac{\bar{l}_{1,1}^{(n)}(\varepsilon^C/3)}{l_{1,1}^{(n)}(\varepsilon/C)} \leq e^{C_\varepsilon} \exp\left(\sqrt{\frac{6C}{\varepsilon^{C+1}} \mathrm{Var}\left(\log L_{1,1}^{(n)}\right)}\right). \tag{8.25}$$

Step 2. The idea is then to bound $\mathrm{Var}(\log L_{1,1}^{(n)})$ by a term involving δ_{n-1} and γ. To do it, we will use the Efron-Stein inequality, see for instance [10] Section 3 where it is used to give an upper bound for the variance of the distance between two points in the model of first passage percolation, which is a similar problem to ours. To this end, note that the variable $L_{1,1}^{(n)}$ can be written as a function of independent fields attached to dyadic blocks: $L_{1,1}^{(n)} = F((\phi_{k,P})_{0 \leq k \leq n, P \in \mathcal{P}_k})$ and only the blocks that intersect $[0,1]^2$ contribute. For $P \in \mathcal{P}_k$, we denote by $L_{1,1}^{(n),P}$ the length obtained by replacing the block field $\phi_{k,P}$ by an independent copy $\phi'_{k,P}$ and keeping all other block fields fixed. The Efron-Stein inequality gives:

$$\operatorname{Var}\log L_{1,1}^{(n)} \leq \sum_{k=0}^{n}\sum_{P\in\mathcal{P}_k}\mathbb{E}\left(\left(\log L_{1,1}^{(n),P}-\log L_{1,1}^{(n)}\right)_+^2\right). \tag{8.26}$$

Step 3. We then focus on the term in the summation. For $0\leq k\leq n$, $P\in\mathcal{P}_k$, $L_{1,1}^{(n),P}$ is bounded from above by

$$\begin{aligned}\int_{\pi_n}\left(e^{\frac{\gamma}{2}\left(\phi_{0,n}-\phi_{k,P}+\phi'_{k,P}\right)}-e^{\frac{\gamma}{2}\phi_{0,n}}\right)ds+L_{1,1}^{(n)} &\leq \int_{\pi_n}e^{\frac{\gamma}{2}\phi_{0,n}}\left(e^{\frac{\gamma}{2}\left(-\phi_{k,P}+\phi'_{k,P}\right)}-1\right)_+ 1_{\pi_n(s)\in P^{2r_0}}ds+L_{1,1}^{(n)}\\ &\leq \gamma\int_{\pi_n}e^{\frac{\gamma}{2}\phi_{0,n}}e^{(1+\frac{\gamma}{2})\left(-\phi_{k,P}+\phi'_{k,P}\right)_+}1_{\pi_n(s)\in P^{2r_0}}ds+L_{1,1}^{(n)}\end{aligned}$$

where $P^{2r_0}:=P+B(0,2^{-k}\cdot 2r_0)$ and where we used in the last inequality the bound

$$(e^{\gamma x}-1)_+\leq e^{\gamma x_+}-1=\sum_{k\geq 1}\frac{(\gamma\ x_+)^k}{k!}\leq \gamma\ x_+\sum_{k\geq 1}\frac{(\gamma\ x_+)^{k-1}}{(k-1)!}\leq \gamma e^{x_+}e^{\gamma x_+}.$$

By setting $S_{k,P}:=\sup_{P^{2r_0}}|\phi_{k,P}|+\sup_{P^{2r_0}}\left|\phi'_{k,P}\right|$, this gives, using $\log(1+x)\leq x$:

$$\begin{aligned}\mathbb{E}((\log L_{1,1}^{(n),P}-\log L_{1,1}^{(n)})_+^2) &\leq \gamma^2\mathbb{E}((L_{1,1}^{(n)})^{-2}(\int_{\pi_n}e^{\frac{\gamma}{2}\phi_{0,n}}e^{(1+\frac{\gamma}{2})(-\phi_{k,P}+\phi'_{k,P})_+}1_{\pi_n(s)\in P^{2r_0}}ds)^2)\\ &\leq \gamma^2\mathbb{E}(e^{CS_{k,P}}(L_{1,1}^{(n)})^{-2}(\int_{\pi_n}e^{\frac{\gamma}{2}\phi_{0,n}}1_{\pi_n(s)\in P^{2r_0}}ds)^2)\end{aligned}$$

which finally gives:

$$\operatorname{Var}\log L_{1,1}^{(n)} \leq \gamma^2\sum_{k=0}^{n}\sum_{P\in\mathcal{P}_k}\mathbb{E}\left(\frac{e^{CS_{k,P}}}{\left(L_{1,1}^{(n)}\right)^2}\left(\int_{\pi_n}e^{\frac{\gamma}{2}\phi_{0,n}}1_{\pi_n(s)\in P^{2r_0}}ds\right)^2\right). \tag{8.27}$$

Notice that for $k=0$ the term in the summation corresponds to $\mathbb{E}(e^{CS_{0,[0,1]^2}})$.

Step 4. We focus now on the case where $k\in\{1,\ldots,n\}$. Since $\mathbb{E}(e^{CS_{k,P}})^{1/2}$ is independent of k

and P by scaling and finite by Fernique, we have by Cauchy-Schwarz:

$$
\begin{aligned}
&\sum_{P\in\mathcal{P}_k} \mathbb{E}\left(e^{CS_{k,P}} \left(L_{1,1}^{(n)}\right)^{-2} \left(\int_{\pi_n} e^{\frac{\gamma}{2}\phi_{0,n}} 1_{\pi_n(s)\in P^{2r_0}} ds \right)^2 \right) \\
&\leq \sum_{P\in\mathcal{P}_k} \mathbb{E}\left(e^{CS_{k,P}}\right)^{1/2} \mathbb{E}\left(\left(L_{1,1}^{(n)}\right)^{-4} \left(\int_{\pi_n} e^{\frac{\gamma}{2}\phi_{0,n}} 1_{\pi_n(s)\in P^{2r_0}} ds \right)^4 \right)^{1/2} \\
&\leq C \sum_{P\in\mathcal{P}_k} \mathbb{E}\left(\left(L_{1,1}^{(n)}\right)^{-4} \left(\int_{\pi_n} e^{\frac{\gamma}{2}\phi_{0,n}} 1_{\pi_n(s)\in P^{2r_0}} ds \right)^4 \right)^{1/2}.
\end{aligned}
$$

Step 4. (a). Upper bound. For $P \in \mathcal{P}_k$, $\int_{\pi_n} e^{\frac{\gamma}{2}\phi_{0,n}} 1_{\pi_n(s)\in P^{2r_0}} ds \leq 9 \max_{Q\sim P} \mathrm{Diam}(Q, e^{\gamma\phi_{0,n}} ds^2)$. Indeed, P^{2r_0} is included in the union of P and its eight neighboring squares (see Figure 2.7). Thus, the length of the parts of π_n included in P^{2r_0} is less than the diameter of this union, which itself is less than the sum of the diameter of all these squares.

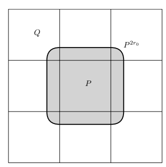

Figure 2.7 – $2r_0$-enlargement of P with its neighbors

Let N_k denote the number of dyadic squares of size 2^{-k} visited by π_n. Since the number of blocks P^{2r_0} (with $P \in \mathcal{P}_k$) visited by π_n is less than $9N_k$, a.s.

$$
\sum_{P\in\mathcal{P}_k} \left(\int_{\pi_n} e^{\frac{\gamma}{2}\phi_{0,n}} 1_{\pi_n(s)\in P^{2r_0}} ds \right)^4 \leq CN_k \sup_{P\in\mathcal{P}_k} \mathrm{Diam}\left(P, e^{\gamma\phi_{0,n}} ds^2\right)^4
$$

and by decoupling the first $k-1$ scales of the field $\phi_{0,n} = \phi_{0,k-1} + \phi_{k,n}$, a.s.

$$
\sum_{P\in\mathcal{P}_k} \left(\int_{\pi_n} e^{\frac{\gamma}{2}\phi_{0,n}} 1_{\pi_n(s)\in P^{2r_0}} ds \right)^4 \leq Ce^{2\gamma \sup_{[0,1]^2} \phi_{0,k-1}} N_k \sup_{P\in\mathcal{P}_k} \mathrm{Diam}\left(P, e^{\gamma\phi_{k,n}} ds^2\right)^4. \tag{8.28}
$$

Step 4. (b). Lower bound. If $\tilde{N}_k$ denotes the maximal number of disjoint left-right rectangle crossings of size $2^{-k}(1,3)$ for π_n, among such rectangles filling vertically and horizontally $[0,1]^2$, spaced by 2^{-k} (this set is denoted by I_k and defined in (5.16)), we have $\tilde{N}_k \geq cN_k$ and $\tilde{N}_k \geq c2^k$ for a small constant $c > 0$. Indeed, if a dyadic square is visited, one of the four rectangles around it is crossed (see Figure 2.8). Considering a fraction of them gives the first claim. It is easy to check the second claim by noticing that π_n crosses each rectangle of size $2^{-k} \times 1$ filling $[0,1]^2$.

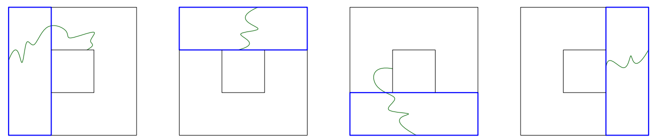

Figure 2.8 – Square visited and associated rectangle crossings

By decoupling the first $k-1$ scales, we get $L_{1,1}^{(n)} \geq c N_k e^{\frac{\gamma}{2}\inf_{[0,1]^2}\phi_{0,k-1}} \inf_{P \in I_k} L^{(k,n)}(P)$ as well as $L_{1,1}^{(n)} \geq c 2^k e^{\frac{\gamma}{2}\inf_{[0,1]^2}\phi_{0,k-1}} \inf_{P \in I_k} L^{(k,n)}(P)$ hence:

$$\left(L_{1,1}^{(n)}\right)^4 \geq c 2^{3k} N_k e^{2\gamma \inf_{[0,1]^2} \phi_{0,k-1}} \left(\inf_{P \in I_k} L^{(k,n)}(P)\right)^4. \tag{8.29}$$

Step 5. Moment estimates and inductive inequality. By concavity of the map $x \mapsto \sqrt{x}$ we have:

$$\begin{aligned} &\sum_{P \in \mathcal{P}_k} \mathbb{E}\left(\left(L_{1,1}^{(n)}\right)^{-4} \left(\int_{\pi_n} e^{\frac{\gamma}{2}\phi_{0,n}} 1_{\pi_n(s) \in P^{2r_0}} ds\right)^4\right)^{1/2} \\ &\leq |\mathcal{P}_k|^{1/2} \mathbb{E}\left(\left(L_{1,1}^{(n)}\right)^{-4} \sum_{P \in \mathcal{P}_k} \left(\int_{\pi_n} e^{\frac{\gamma}{2}\phi_{0,n}} 1_{\pi_n(s) \in P^{2r_0}} ds\right)^4\right)^{1/2}. \end{aligned}$$

Gathering, (8.28) and (8.29),

$$\begin{aligned} &\left(L_{1,1}^{(n)}\right)^{-4} \sum_{P \in \mathcal{P}_k} \left(\int_{\pi_n} e^{\frac{\gamma}{2}\phi_{0,n}} 1_{\pi_n(s) \in P^{2r_0}} ds\right)^4 \\ &\leq C 2^{-3k} e^{4\gamma \sup_{[0,1]^2} |\phi_{0,k-1}|} \sup_{P \in \mathcal{P}_k} \operatorname{Diam}\left(P, e^{\gamma \phi_{k,n}} ds^2\right)^4 \left(\inf_{P \in I_k} L^{(k,n)}(P)\right)^{-4}. \end{aligned}$$

Since $|\mathcal{P}_k| = 4^k$, by independence between scales,

$$\begin{aligned} &\sum_{P \in \mathcal{P}_k} \mathbb{E}\left(\left(L_{1,1}^{(n)}\right)^{-4} \left(\int_{\pi_n} e^{\frac{\gamma}{2}\phi_{0,n}} 1_{\pi_n(s) \in P^{2r_0}} ds\right)^4\right)^{1/2} \\ &\leq C 2^{-\frac{1}{2}k} \mathbb{E}\left(e^{4\gamma \sup_{[0,1]^2} |\phi_{0,k-1}|}\right)^{1/2} \mathbb{E}\left(\sup_{P \in \mathcal{P}_k} \operatorname{Diam}\left(P, e^{\gamma \phi_{k,n}} ds^2\right)^4 \left(\inf_{P \in I_k} L^{(k,n)}(P)\right)^{-4}\right)^{1/2}. \end{aligned}$$

Using Lemma 2.18 to control the exponential moment, the first term is bounded by $2^{4\gamma k} e^{C\sqrt{k}}$. For the second term, notice that the product inside the expectation is between an increasing and a decreasing function of the field. Hence, by the positive association property (Theorem 2.1):

$$\mathbb{E}\left(\sup_{P\in\mathcal{P}_k}\mathrm{Diam}\left(P,e^{\gamma\phi_{k,n}}ds^2\right)^4\left(\inf_{P\in I_k}L^{(k,n)}(P)\right)^{-4}\right)^{1/2}$$
$$\leq \mathbb{E}\left(\sup_{P\in\mathcal{P}_k}\mathrm{Diam}\left(P,e^{\gamma\phi_{k,n}}ds^2\right)^4\right)^{1/2}\mathbb{E}\left(\left(\inf_{P\in I_k}L^{(k,n)}(P)\right)^{-4}\right)^{1/2}.$$

By scaling, the field involved is $\phi_{0,n-k}$. We use our estimates for the diameters, Proposition 2.11, for the first term and Corollary 2.7 for the second one. More precisely, by standard inequality between expected value of positive random variable and integration of tail estimates we have:

$$\mathbb{E}\left(\sup_{P\in\mathcal{P}_k}\mathrm{Diam}\left(P,e^{\gamma\phi_{k,n}}ds^2\right)^4\right)^{1/2}\leq 2^{-2k}\delta_{n-k}^2 l_{n-k}^2 e^{c\gamma k}\leq \delta_{n-1}^2 2^{-2k} l_{n-k}^2 e^{c\gamma k}$$

and

$$\mathbb{E}\left(\left(\inf_{P\in I_k}L^{(k,n)}(P)\right)^{-4}\right)^{1/2}\leq 2^{2k} l_{n-k}^{-2} e^{C\sqrt{k}}.$$

Altogether, we get for $1\leq k\leq n$:

$$\sum_{P\in\mathcal{P}_k}\mathbb{E}\left(\frac{e^{CS_{k,P}}}{L_{1,1}^{(n)2}}\left(\int_{\pi_n}e^{\frac{\gamma}{2}\phi_{0,n}}1_{\pi_n(s)\in P^{2r_0}}ds\right)^2\right)\leq \delta_{n-1}^2 2^{-\frac{1}{2}k}e^{c\gamma k}e^{C\sqrt{k}} \tag{8.30}$$

for some constant $c>0$.

Step 6. Combining (8.27) and (8.30) we get

$$\mathrm{Var}\log L_{1,1}^{(n)}\leq \gamma^2\delta_{n-1}^2\sum_{k=0}^{n}2^{-\frac{1}{2}k}e^{c\gamma k}e^{C\sqrt{k}}\leq \gamma^2\delta_{n-1}^2\sum_{k=0}^{\infty}2^{-\frac{1}{2}k}e^{c\gamma k}e^{C\sqrt{k}}. \tag{8.31}$$

Hence for γ small enough the series in the right-hand side of (8.31) converges and we have the bound $\mathrm{Var}\log L_{1,1}^{(n)}\leq\gamma^2\left(C+C\delta_{n-1}^2\right)$. Coming back to (8.25), if $\delta_{n-1}<M$ then $\delta_n< e^{C_\varepsilon}\exp(C\gamma\delta_{n-1})<e^{C_\varepsilon}\exp(C\gamma M)$. Hence, if $M>e^{C_\varepsilon}$ and γ is small enough so $e^{C_\varepsilon}\exp(C\gamma M)<M$ shows that there exists γ_0 (which depends on ε) such that if $\gamma<\gamma_0$, $\delta_\infty<\infty$. Finally, we can conclude that $\gamma_c>0$ by use of Corollary 2.7 and Proposition 2.8. $\square$

2.9 Independence of γ_c with respect to k: proof of Theorem 2.6

We want to prove that γ_c is independent of k, i.e., if we have two bump functions k_1, k_2 then $\gamma_c(k_1) = \gamma_c(k_2)$. We will prove that if $\log L_{1,1}(\phi^1_{0,n}) - \log \mu^1_n$ is tight then $\log L_{1,1}(\phi^2_{0,n}) - \log \mu^2_n$ is also tight, where the superscripts corresponds to the bump function k_i for $i \in \{1, 2\}$. The proof presented here relies on the assumption that $\hat{k}_1$ and $\hat{k}_2$ have similar tails.

Main lines of the proof. The main idea of the proof is to couple $\phi^1_{0,n}$ and $\phi^2_{0,n}$ up to some additive noises that don't affect too much the lengths. To control the perturbation due to the noises, note that if $\delta\phi$ is a low frequency noise, the length $L_{1,1}(\phi)$ is comparable to the length $L_{1,1}(\phi + \delta\phi)$ by a uniform bound a.s.:

$$e^{\inf_{[0,1]^2} \delta\phi} L_{1,1}(\phi) \leq L_{1,1}(\phi + \delta\phi) \leq e^{\sup_{[0,1]^2} \delta\phi} L_{1,1}(\phi) \tag{9.32}$$

and if $\delta\phi$ is a high frequency noise with bounded pointwise variance we have a one-sided bound on high and low quantiles given by the following lemma.

Lemma 2.15. *If Φ is a continuous field and $\delta\Phi$ is an independent continuous centered Gaussian field with variance bounded by C then*

1. $l^{\Phi+\delta\Phi}_{1,1}(\varepsilon) \leq \varepsilon^{-1} e^{\frac{1}{2}C} l^{\Phi}_{1,1}(2\varepsilon)$,
2. $\bar{l}^{\Phi+\delta\Phi}_{1,1}(2\varepsilon) \leq \varepsilon^{-1} e^{\frac{1}{2}C} \bar{l}^{\Phi}_{1,1}(\varepsilon)$.

Proof. To bound from above $L^{\Phi+\delta\Phi}_{1,1}$, we take a geodesic for Φ and use a moment estimate on $\delta\Phi$. We start with the lower tail. For $s > 0$ we have

$$\begin{aligned}
\mathbb{P}\left(L^{\Phi}_{1,1} \leq l^{\Phi+\delta\Phi}_{1,1}(\varepsilon)e^{-s}\right) &\leq \mathbb{P}\left(L^{\Phi+\delta\Phi}_{1,1} \leq e^s L^{\Phi}_{1,1}, L^{\Phi}_{1,1} \leq l^{\Phi+\delta\Phi}_{1,1}(\varepsilon)e^{-s}\right) + \mathbb{P}\left(L^{\Phi+\delta\Phi}_{1,1} > e^s L^{\Phi}_{1,1}\right) \\
&\leq \mathbb{P}\left(L^{\Phi+\delta\Phi}_{1,1} \leq l^{\Phi+\delta\Phi}_{1,1}(\varepsilon)\right) + \mathbb{P}\left(\int_{\pi^\Phi} e^{\Phi+\delta\Phi} ds > e^s L^{\Phi}_{1,1}\right) \\
&\leq \varepsilon + e^{\frac{1}{2}\sup \mathrm{Var}(\delta\Phi) - s}
\end{aligned}$$

where we used Chebychev inequality and the independence between the field Φ and $\delta\Phi$ in the last inequality. Taking then $s = \frac{1}{2}\sup \mathrm{Var}(\delta\Phi) - \log\varepsilon$ completes the proof of (i). For the upper tails taking the same s gives

$$\begin{aligned}\mathbb{P}\left(L_{1,1}^{\Phi+\delta\Phi} \geq \bar{l}_{1,1}^{\Phi}(\varepsilon)e^{s}\right) &\leq \mathbb{P}\left(L_{1,1}^{\Phi+\delta\Phi} \geq \bar{l}_{1,1}^{\Phi}(\varepsilon)e^{s}, \bar{l}_{1,1}^{\Phi}(\varepsilon) \geq L_{1,1}^{\Phi}\right) + \mathbb{P}\left(L_{1,1}^{\Phi} \geq \bar{l}_{1,1}^{\Phi}(\varepsilon)\right) \\ &\leq \mathbb{P}\left(L_{1,1}^{\Phi+\delta\Phi} \geq e^{s} L_{1,1}^{\Phi}\right) + \varepsilon \\ &\leq 2\varepsilon\end{aligned}$$

which concludes the proof of the lemma. □

Note that if $\delta\phi$ is a high frequency noise, with scale dependence 2^{-n}, say an approximation of 4^n i.i.d. standard Gaussian variables, its supremum is of order $\sqrt{n}$ and the inequality (9.32) is inappropriate compared to Lemma 3.9 which gives a bound of order one, but one-sided. However, for a low frequency noise $\delta\phi$, independent of n, the bound (9.32) gives two-sided bounds on quantiles.

If (X_n) and (Y_n) denote two sequences of positive random variables, with positive density with respect to the Lebesgue measure on $(0,\infty)$, we write $X_n \lesssim Y_n$ if there exists a constant C independent of n such that for every $\varepsilon > 0$ small, there exists C_ε, independent of n, such that $F_{X_n}^{-1}(\varepsilon/C) \leq C_\varepsilon F_{Y_n}^{-1}(\varepsilon)$ and $F_{X_n}^{-1}(1-C\varepsilon) \leq C_\varepsilon F_{Y_n}^{-1}(1-\varepsilon)$, where $F_X(x) := \mathbb{P}(X \leq x)$ for a random variable X. A direct corollary of Lemma 3.9 is the following: if $(\phi_n)_{n\geq 0}$ and $(\delta\phi_n)_{n\geq 0}$ are two sequences of independent centered continuous Gaussian fields, and that the pointwise variance of $\delta\phi_n$ is bounded, then $L_{1,1}(\phi_n + \delta\phi_n) \lesssim L_{1,1}(\phi_n)$. Similarly, a direct consequence of (9.32) is that, under the same assumptions for $(\phi_n)_{n\geq 0}$, if ψ is a continuous centered Gaussian field, then $L_{1,1}(\phi_n) \lesssim L_{1,1}(\phi_n + \psi) \lesssim L_{1,1}(\phi_n)$.

Now that the notations and the key tools are settled, let us explain the main idea of the proof. Let us assume for now that we have the following couplings, for a fixed k:

1. $\left(\phi_{0,n}^1(x) + \delta_n^1(x)\right)_{x\in\mathbb{R}^2} \overset{(d)}{=} \left(\phi_{0,n}^2(x) + \delta_n^2(x)\right)_{x\in\mathbb{R}^2}$
2. $\left(\phi_{n,n+k}^1(x) + \psi(x)\right)_{x\in\mathbb{R}^2} \overset{(d)}{=} \left(\delta_n^1(x) + r_n^1(x)\right)_{x\in\mathbb{R}^2}$
3. $\left(\phi_{n,n+k}^2(x) + \psi(x)\right)_{x\in\mathbb{R}^2} \overset{(d)}{=} \left(\delta_n^2(x) + r_n^2(x)\right)_{x\in\mathbb{R}^2}$

where fields in the same side of an equality are independent and all fields are centered, continuous and Gaussian. Let us also assume that ψ is a fixed continuous Gaussian field, independent of n and thus a low frequency noise. Notice that if such couplings hold, it is clear that the δ_n^i's and r_n^i's have bounded pointwise variance since this is the case for the fields in the left-hand sides of (ii) and (iii). We then have, since ψ is a low frequency noise, by using (ii) and Lemma 3.9:

$$L_{1,1}\left(\phi^1_{0,n+k}\right) \lesssim L_{1,1}\left(\phi^1_{0,n}+\delta^1_n+r^1_n\right) \lesssim L_{1,1}\left(\phi^1_{0,n}+\delta^1_n\right) \lesssim L_{1,1}\left(\phi^1_{0,n}\right)$$

which gives, using (i):

$$L_{1,1}\left(\phi^1_{0,n+k}\right) \lesssim L_{1,1}(\phi^2_{0,n}+\delta^2_n) \lesssim L_{1,1}(\phi^1_{0,n}). \tag{9.33}$$

If we suppose that $\log L_{1,1}(\phi^1_{0,n}) - \log \mu^1_n$ is tight, then $((\mu^1_n)^{-1}\mu^1_{n+k})_{n\geq 0}$ is bounded by Lemma 2.12. But then, using (9.33), $\log L_{1,1}(\phi^2_{0,n}+\delta^2_n) - \log \mu^1_n$ is tight. Furthermore, this implies the tightness of $\log L_{1,1}(\phi^2_{0,n}) - \log \mu^1_n$ since

$$L_{1,1}\left(\phi^2_{0,n+k}+\delta^2_{n+k}\right) \lesssim L_{1,1}\left(\phi^2_{0,n+k}\right) \lesssim L_{1,1}\left(\phi^2_{0,n}+\delta^2_n\right).$$

Finally, the tightness of $\log L_{1,1}(\phi^2_{0,n}) - \log \mu^2_n$ follows from the fact that if X is random variable and $\mu(X)$ is its median, then for every $a \in \mathbb{R}$, $\mu(X+a) = \mu(X)+a$. This concludes the proof up to the results we claimed on the couplings.

All the fields in the couplings will be defined by using the following standard result:

Lemma 2.16. *If f is a continuous, symmetric and nonnegative function on $\mathbb{R}^d$ such that $\|\xi\|\, f(\xi) \in L^1(\mathbb{R}^d)$, then one can define a continuous stationary centered Gaussian field with covariance given by:*

$$C(x,y) := \frac{1}{(2\pi)^2}\int_{\mathbb{R}^d} f(\xi)e^{i(x-y)\cdot\xi}d\xi.$$

Proof. Since $f \in L^1(\mathbb{R}^d)$, C is well-defined. Then, since f is symmetric, a change of variables gives that C is real-valued and $C(x,y) = C(y,x)$. Moreover, notice that $(C(x,y))_{x,y\in\mathbb{R}^2}$ is positive semidefinite: for every $(a_k)_{1\le k\le n}$ and $(x_k)_{1\le k\le n}$ in $(\mathbb{R}^d)^n$ we have

$$\begin{aligned}\sum_{k,l=1}^{n} a_k C(x_k,x_l)a_l &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^d} f(\xi)\left(\sum_{k=1}^{n} a_k e^{ix_k\cdot\xi}\right)\left(\sum_{l=1}^{n} a_l e^{-ix_l\cdot\xi}\right)d\xi\\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^d} f(\xi)\left|\sum_{k=1}^{n} a_k e^{ix_k\cdot\xi}\right|^2 d\xi \ge 0.\end{aligned}$$

By a standard result on Gaussian processes (see [1] Section 1), there exists a centered Gaussian process $(h(x))_{x\in\mathbb{R}^d}$ whose covariance is given by $\mathbb{E}(h(x)h(y)) = C(x,y)$. Finally, since we have the Lipschitz bound $\mathbb{E}((h(x)-h(y))^2) \le 2\|x-y\|\int_{\mathbb{R}^d} f(\xi)\|\xi\|\,d\xi$ and $\|\xi\|\,f(\xi)\in L^1(\mathbb{R}^d)$, by the Kolmogorov continuity criterion there exists a modification of h which is continuous. □

We also recall that $C_{0,n}(x) = \int_{2^{-n}}^{1} c\left(\frac{x}{t}\right)\frac{dt}{t} = \int_{2^{-n}}^{1} c_t(x)\frac{dt}{t}$ with $c_t(\cdot) = c(\cdot/t)$ thus its Fourier transform satisfies $\hat{C}_{0,n}(\xi) = \int_{2^{-n}}^{1}\hat{c}_t(\xi)\frac{dt}{t} = \int_{2^{-n}}^{1} t\hat{c}(t\xi)dt$ and since $c = k*k$, $\hat{c} = \hat{k}^2$ and then $\hat{C}_{0,n}(\xi) = \int_{2^{-n}}^{1} t\hat{k}(t\xi)^2 dt = \|\xi\|^{-2}\int_{2^{-n}\|\xi\|}^{\|\xi\|} u\hat{k}(u)^2 du$.

Coupling $\phi^1_{0,n}$ and $\phi^2_{0,n}$. First we define δ^1_n and δ^2_n such that

$$\left(\phi^1_{0,n}(x) + \delta^1_n(x)\right)_{x\in\mathbb{R}^2} \overset{(d)}{=} \left(\phi^2_{0,n}(x) + \delta^2_n(x)\right)_{x\in\mathbb{R}^2} \tag{9.34}$$

where δ^1_n (resp δ^2_n) is a noise independent of $\phi^1_{0,n}$ (resp $\phi^2_{0,n}$). The covariance kernel of $\phi^i_{0,n}$ is given by $C^i_{0,n}(x,y) = \int_{2^{-n}}^{1} c_i\left(\frac{x-y}{t}\right)\frac{dt}{t}$ where $c_i = k_i * k_i$. We recall also that these kernels are isotropic, i.e., $C^i_{0,n}(x,y) = C^i_{0,n}(\|x-y\|)$. By Fourier inversion (of Schwartz function) we can write

$$C^i_{0,n}(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\hat{C}^i_{0,n}(\xi)e^{i\xi\cdot x}d\xi.$$

We define R^1_n by replacing the term $\hat{C}^i_{0,n}(\xi)$ in the integrand by $f^1_n(\xi) := \hat{C}^1_{0,n}(\xi)\vee\hat{C}^2_{0,n}(\xi) - \hat{C}^1_{0,n}(\xi) \ge 0$ and similarly R^2_n associated with $f^2_n(\xi) := \hat{C}^2_{0,n}(\xi)\vee\hat{C}^1_{0,n}(\xi) - \hat{C}^2_{0,n}(\xi) \ge 0$ so that $C^1_{0,n} + R^1_n = C^2_{0,n} + R^2_n$. By using Lemma 2.16, the covariance kernels R^1_n and R^2_n correspond to some continuous Gaussian fields δ^1_n and δ^2_n so that (9.34) holds and for $i\in\{1,2\}$, $\phi^i_{0,n}$ is independent of δ^i_n.

Coupling the remaining noise with the lower scales. We now prove the second coupling:

$$\left(\phi^1_{n,n+k}(x) + \psi(x)\right)_{x\in\mathbb{R}^2} = \left(\delta^1_n(x) + r^1_n(x)\right)_{x\in\mathbb{R}^2}. \tag{9.35}$$

The goal is to show that the Fourier transform of the kernel of $\phi^1_{n,n+k} + \psi$ (for ψ to be specified) is larger than the one of δ^1_n in order to define, in a similar way as before, the continuous Gaussian field r^1_n, independent of δ^1_n.

To be precise, recall first that the spectrum of δ^1_n and $\phi^1_{n,n+k}$ are given respectively by $f^1_n(\xi) = (\hat{C}^2_{0,n}(\xi) - \hat{C}^1_{0,n}(\xi))1_{\hat{C}^2_{0,n}(\xi) \geq \hat{C}^1_{0,n}(\xi)}$ with $\hat{C}^i_{0,n}(\xi) = \|\xi\|^{-2} \int_{2^{-n}\|\xi\|}^{\|\xi\|} u\hat{k}_i(u)^2 du$ and $\hat{C}^1_{n,n+k}(\xi) = \|\xi\|^{-2} \int_{2^{-n-k}\|\xi\|}^{2^{-n}\|\xi\|} u\hat{k}_1(u)^2 du$. If the spectrum of ψ is given by $\|\xi\|^{-2} g(\xi)$, we look for the inequality $f^1_n(\xi) \leq \hat{C}^1_{n,n+k}(\xi) + \|\xi\|^{-2} g(\xi)$ which is equivalent to

$$\left(\int_{2^{-n}\|\xi\|}^{\|\xi\|} u\hat{k}_2(u)^2 du - \int_{2^{-n}\|\xi\|}^{\|\xi\|} u\hat{k}_1(u)^2 du\right)_+ \leq \int_{2^{-(n+k)}\|\xi\|}^{2^{-n}\|\xi\|} u\hat{k}_1(u)^2 du + g(\xi). \tag{9.36}$$

If the left-hand side is 0, the inequality trivially holds. Otherwise, we want to get:

$$\int_{2^{-n}\|\xi\|}^{\|\xi\|} u\hat{k}_2(u)^2 du \leq \int_{2^{-(n+k)}\|\xi\|}^{\|\xi\|} u\hat{k}_1(u)^2 du + g(\xi).$$

Our analysis of this inequality will be separated in three steps, corresponding respectively to the low frequencies $[0, c2^n]$, the high ones $[C2^n, \infty)$ and the remaining part of the spectrum $[c2^n, C2^n]$, for c and C to be specified. The field ψ in (9.35) is defined in the first step. An additional step is devoted to the conclusion.

Step 1. We start with the low frequencies $\|\xi\| \leq c2^n$. Since $\hat{k}_1$ and $\hat{k}_2$ are radially symmetric with the same L^2 normalization, $\int_{(0,\infty)} u\hat{k}_1(u)^2 du = \int_{(0,\infty)} u\hat{k}_2(u)^2 du$ and

$$\left(\int_{2^{-n}\|\xi\|}^{\|\xi\|} u\hat{k}_2(u)^2 du - \int_{2^{-n}\|\xi\|}^{\|\xi\|} u\hat{k}_1(u)^2 du\right)_+ \leq \left(\int_0^{2^{-n}\|\xi\|} u\hat{k}_1(u)^2 du - \int_0^{2^{-n}\|\xi\|} u\hat{k}_2(u)^2 du\right)_+ + \left(\int_{\|\xi\|}^{\infty} u\hat{k}_1(u)^2 du - \int_{\|\xi\|}^{\infty} u\hat{k}_2(u)^2 du\right)_+ .$$

We define the continuous Gaussian field ψ (independent of n), whose covariance kernel has Fourier transform defined by $\|\xi\|^{-2} g(\xi) := \|\xi\|^{-2} \left|\int_{\|\xi\|}^{\infty} u\hat{k}_1(u)^2 du - \int_{\|\xi\|}^{\infty} u\hat{k}_2(u)^2 du\right|$.

Since we want to show that the Fourier transform of the kernel of $\phi^1_{n,n+k} + \psi$ is larger than the one of δ_n^1, we want to prove that for $\|\xi\| \le c2^n$ (c to be specified, small):

$$\left(\int_0^{2^{-n}\|\xi\|} u\hat{k}_1(u)^2 du - \int_0^{2^{-n}\|\xi\|} u\hat{k}_2(u)^2 du\right)_+ \le \int_{2^{-(n+k)}\|\xi\|}^{2^{-n}\|\xi\|} u\hat{k}_1(u)^2 du.$$

By setting $r = 2^{-n}\|\xi\|$, we want to prove that for r small enough ($r \le c$), and k large enough but fixed:

$$\left(\int_0^r u\hat{k}_1(u)^2 du - \int_0^r u\hat{k}_2(u)^2 du\right)_+ \le \int_{2^{-k}r}^r u\hat{k}_1(u)^2 du. \tag{9.37}$$

Notice that when r goes to 0, $\int_0^r u(\hat{k}_1(u)^2 du - \int_0^r u\hat{k}_2(u)^2 du \sim \frac{1}{2}r^2(\hat{k}_1(0)^2 - \hat{k}_2(0)^2)$. If the left-hand side is 0, there is nothing to prove. Thus we can restrict to the case where it is > 0 i.e when $\hat{k}_1(0)^2 > \hat{k}_2(0)^2$ (notice that $\hat{k}(0) = \int_{B(0,r_0)} k(u)du > 0$ since k is non-negative and $\int_{B(0,r_0)} k(x)^2 dx = 1$). The asymptotic of the right-hand side is given by $\int_{2^{-k}r}^r u\hat{k}_1(u)^2 du \sim \frac{1}{2}r^2\hat{k}_1(0)^2(1-2^{-2k})$. Thus as soon as $\hat{k}_1(0)^2 - \hat{k}_2(0)^2 < \hat{k}_1(0)^2(1-2^{-2k})$, there exists $r(k)$ such that for $r \le r(k)$, the inequality (9.37) is satisfied.

Step 2. We now deal with the large frequencies, i.e., $\|\xi\| \ge C2^n$. Again, we look for the inequality (9.36). Since we added the field ψ and the following inequality holds,

$$\begin{aligned}\left(\int_{2^{-n}\|\xi\|}^{\|\xi\|} u\hat{k}_2(u)^2 du - \int_{2^{-n}\|\xi\|}^{\|\xi\|} u\hat{k}_1(u)^2 du\right)_+ \le &\left(\int_{2^{-n}\|\xi\|}^{\infty} u\hat{k}_2(u)^2 du - \int_{2^{-n}\|\xi\|}^{\infty} u\hat{k}_1(u)^2 du\right)_+ \\ &+ \left(\int_{\|\xi\|}^{\infty} u\hat{k}_1(u)^2 du - \int_{\|\xi\|}^{\infty} u\hat{k}_2(u)^2 du\right)_+\end{aligned}$$

we look for the inequality:

$$\left(\int_{2^{-n}\|\xi\|}^{\infty} u\hat{k}_2(u)^2 du - \int_{2^{-n}\|\xi\|}^{\infty} u\hat{k}_1(u)^2 du\right)_+ \le \int_{2^{-(n+k)}\|\xi\|}^{2^{-n}\|\xi\|} u\hat{k}_1(u)^2 du.$$

By setting $r = 2^{-n}\|\xi\|$, we want to prove that for r large enough ($r \ge C$), and k large enough but fixed:

$$\int_r^\infty u\hat{k}_2(u)^2 du \le \int_{2^{-k}r}^\infty u\hat{k}_1(u)^2 du. \tag{9.38}$$

Since $\hat{k}_1(u) = e^{-bu^\alpha(1+o(1))}$ and $\hat{k}_2(u) = e^{-au^\alpha(1+o(1))}$, we may assume that $0 < a \le b$ (otherwise $k = 0$ would be fine). Notice that there exists some $R > 0$ such that for every $r \ge R$, $\int_r^\infty u\hat{k}_2(u)^2 du \le$

e^{-br^α} and $e^{-3ar^\alpha} \leq \int_r^\infty u\hat{k}_2(u)^2 du$. Then, by taking k large enough so that $b > 3a2^{-k\alpha}$, for $r \geq 2^k R$ the inequality (9.38) is satisfied.

Step 3. Take k_0 such that $\hat{k}_1(0)^2 - \hat{k}_2(0)^2 < \hat{k}_1(0)^2(1 - 2^{-2k_0})$ and $b > 3a2^{-k_0\alpha}$ are satisfied. Set $c := r(k_0)$ and $C := 2^{k_0}R$, keeping the notations of Step 1 and Step 2. We proved there that (9.36) holds for $\|\xi\| \leq c2^n$ and $\|\xi\| \geq C2^n$ and this inequality still holds by taking k larger, with the same c and C. We are left with the frequencies $c2^n \leq \|\xi\| \leq C2^n$. First, fix $k \geq k_0$ such that $\int_{2^{-k}C}^\infty u\hat{k}_1(u)^2 du > \int_c^\infty u\hat{k}_2(u)^2 du$ (since $\int_{2^{-k}C}^\infty u\hat{k}_1(u)^2 du \to \int_0^\infty u\hat{k}_2(u)^2$). Then, fix n_0 such that $\int_{2^{-k}C}^{2^{n_0}c} u\hat{k}_1(u)^2 du \geq \int_c^\infty u\hat{k}_2(u)^2 du$. Thus, for every $n \geq n_0$, $\|\xi\| \in [c2^n, C2^n]$ we have:

$$\int_{2^{-(n+k)}\|\xi\|}^{\|\xi\|} u\hat{k}_1(u)^2 du \geq \int_{2^{-k}C}^{2^n c} u\hat{k}_1(u)^2 du \geq \int_c^\infty u\hat{k}_2(u)^2 du \geq \int_{2^{-n}\|\xi\|}^{\|\xi\|} u\hat{k}_2(u)^2 du.$$

Step 4. We have proved that if k is large enough, but fixed, for every $n \geq n_0$ the inequality (9.36) holds for all $\xi \in \mathbb{R}^2$. Also, our arguments prove that the same result is true by exchanging the subscripts 1 and 2 in (9.36). Therefore, we can define for $i \in \{1, 2\}$, r_n^i whose covariance kernel has Fourier transform given by the positive difference in the inequality (9.36), multiplied by $\|\xi\|^{-2}$. In particular, we get the couplings (ii) and (iii) with the desired properties on the fields. This completes the proof of the existence of the couplings, therefore the proof of Theorem 2.6.

2.10 Appendix

2.10.1 Tail estimates for the supremum of $\phi_{0,n}$

We derive in the following lemma some tail estimates for the field $\phi_{0,n}$. The tail estimates are obtained by controlling a discretization of $\phi_{0,n}$ (by union bound and Gaussian tail estimates) and its gradient.

Lemma 2.17. *The supremum of the field $\phi_{0,n}$ satisfies the following tails estimates*

$$\mathbb{P}\left(\sup_{[0,1]^2} |\phi_{0,n}| \geq \alpha(n + C\sqrt{n})\right) \leq C4^n e^{-\frac{\alpha^2}{\log 4}n} \tag{10.39}$$

as well as

$$\mathbb{P}\left(\sup_{[0,1]^2} |\phi_{0,n}| \geq n\log 4 + C\sqrt{n} + Cs\right) \leq Ce^{-s}. \tag{10.40}$$

Proof. First we bound a discretization of the field $\phi_{0,n}$. Since the variance of $\phi_{0,n}(x)$ is equal to $(n+1)\log 2$, by union bound and classical Gaussian tail estimates we have $\mathbb{P}(\max_{[0,1]^2\cap 2^{-n}\mathbb{Z}^2} |\phi_{0,n}(x)| \geq x) \leq 4^n e^{-\frac{x^2}{(n+1)\log 4}}$ hence by introducing $x_n := \sqrt{n+1}\sqrt{n}$ we get

$$\mathbb{P}\left(\max_{x\in[0,1]^2\cap 2^{-n}\mathbb{Z}^2} |\phi_{0,n}(x)| \geq \alpha x_n\right) \leq 4^n e^{-\frac{\alpha^2}{\log 4}n}. \tag{10.41}$$

Now we want to bound $\sup_{[0,1]^2} |\phi_{0,n}(x)|$ for which we want an equivalent of the bound (10.41). By Fernique's theorem, we have a tail estimate for the gradient of ϕ_0, i.e., there exists some $C > 0$ so that for every $x > 0$, $\mathbb{P}(\sup_{[0,1]^2} |\nabla\phi_0| \geq x) \leq Ce^{-x^2/2C}$. Then, by scaling, for any dyadic cube $P \in \mathcal{P}_k$, $\mathbb{P}(\sup_P |\nabla\phi_k| \geq 2^k x) \leq Ce^{-x^2/2C}$ thus, by union bound $\mathbb{P}(\sup_{[0,1]^2} |\nabla\phi_k| \geq 2^k x) \leq C4^k e^{-x^2/2C}$. We can now work out the gradient field $\nabla\phi_{0,n}$: $\mathbb{P}(\sup_{[0,1]^2} |\nabla\phi_{0,n}| \geq 2^{n+1}x) \leq \mathbb{P}(\sum_{k=0}^n \sup_{[0,1]^2} |\nabla\phi_k| \geq \sum_{k=0}^n 2^k x) \leq C4^n e^{-x^2/2C}$ hence $\mathbb{P}(2^{-n}\sup_{[0,1]^2} |\nabla\phi_{0,n}| \geq x) \leq C4^n e^{-x^2/2C}$. This inequality can be rewritten by introducing $y_n := C\sqrt{n}$ as:

$$\mathbb{P}\left(2^{-n}\sup_{[0,1]^2} |\nabla\phi_{0,n}| \geq \alpha y_n\right) \leq C4^n e^{-\frac{\alpha^2}{\log 4}n}. \tag{10.42}$$

Using the discrete bound (10.41) and the gradient one (10.42), since

$$\sup_{[0,1]^2} |\phi_{0,n}| \leq \max_{[0,1]^2\cap 2^{-n}\mathbb{Z}^2} |\phi_{0,n}| + 2^{-n}\sup_{[0,1]^2} |\nabla\phi_{0,n}|,$$

we get the result (10.39) by union bound. Indeed, with $z_n := x_n + y_n$. $\mathbb{P}(\sup_{[0,1]^2} |\phi_{0,n}| \geq \alpha z_n) \leq \mathbb{P}(X_n \geq \alpha x_n) + \mathbb{P}(Y_n \geq \alpha Y_n) \leq C4^n e^{-\frac{\alpha^2}{\log 4}n}$. Taking $\alpha = \log 4\sqrt{1 + \frac{s}{n\log 4}} \leq \log 4 + \frac{s}{n}$ gives the second part (10.40). $\square$

The following lemma is a corollary of the previous one: using the tail estimates we control exponential moments.

Lemma 2.18. *We have the following upper bounds for the exponential moments of the field $\phi_{0,n}$: for $\gamma < 2$ and $n \geq 0$, $\mathbb{E}\left(e^{\gamma\sup_{[0,1]^2}|\phi_{0,n}|}\right) \leq C4^{\gamma n(1+o(1))}$, where $o(1)$ is of the form $O(n^{-1/2})$.*

Proof. Fix $0 < \gamma < 2$. We use the bound (10.39) as follows. By introducing $s_n := n + C\sqrt{n}$ we have, by using the elementary bound $\mathbb{E}(e^{\gamma X}) \leq e^{\gamma x} + \int_x^\infty \gamma e^{\gamma t} \mathbb{P}(X \geq t) dt$ and for α to be specified:

$$\mathbb{E}\left(e^{\gamma \sup\limits_{[0,1]^2} |\phi_{0,n}|}\right) \leq e^{\gamma \alpha s_n} + \gamma \int_{\alpha s_n}^\infty e^{\gamma t} \mathbb{P}\left(\sup_{[0,1]^2} |\phi_{0,n}| \geq t\right) dt.$$

Setting $t = s_n u$, $\int_{\alpha s_n}^\infty e^{\gamma t} \mathbb{P}(\sup_{[0,1]^2} |\phi_{0,n}| \geq t) dt = s_n \int_\alpha^\infty e^{\gamma s_n u} \mathbb{P}(\sup_{[0,1]^2} |\phi_{0,n}| \geq s_n u) du$ and by using the bound (10.39)

$$\int_\alpha^\infty e^{\gamma s_n u} \mathbb{P}\left(\sup_{[0,1]^2} |\phi_{0,n}| \geq s_n u\right) du \leq C 4^n \int_\alpha^\infty e^{\gamma s_n u} e^{-\frac{u^2}{\log 4} n} du.$$

By introducing $r_n := n^{-1} s_n$, by a change of variables we obtain:

$$\int_\alpha^\infty e^{\gamma s_n u} e^{-\frac{u^2}{\log 4} n} du \leq 4^{\frac{\gamma^2 r_n^2}{4} n} \int_{\alpha - \gamma r_n \frac{\log 4}{2}}^\infty e^{-\frac{n}{\log 4} u^2} du.$$

Taking $\alpha := r_n \log 4$, the integral in the right-hand side becomes

$$\int_{\alpha - \gamma r_n \frac{\log 4}{2}}^\infty e^{-\frac{n}{\log 4} u^2} du = \int_{(1-\gamma/2) r_n \log 4}^\infty e^{-\frac{n}{\log 4} u^2} du \leq \frac{4^{-n\left(1-\frac{\gamma}{2}\right)^2 r_n^2}}{(2-\gamma)\, n r_n},$$

by using the inequality $\int_a^\infty e^{-bx^2} dx \leq (2ab)^{-1} e^{-ba^2}$ valid for $a > 0$ and $b > 0$. Gathering the pieces we get $\mathbb{E}(e^{\gamma \sup_{[0,1]^2} |\phi_{0,n}|}) \leq (1 + C\frac{\gamma}{2-\gamma}) 4^{\gamma r_n^2 n}$ hence the result. □

We add here a Lemma which is in the same vein as the previous one.

Lemma 2.19. *Suppose that we have the following tail estimate on a sequence of positive random variables $(X_k)_{k \geq 0}$: for $k \geq 0$ and $s > 2$,*

$$\mathbb{P}(X_k \geq e^s) \leq 4^k e^{-c\frac{s^2}{\log s}}.$$

Then, we have the following moment estimate: there exists $C > 0$ depending only on c such that for k large,

$$\mathbb{E}(X_k) \leq e^{C\sqrt{k \log k}}.$$

Proof. Fix $x_k > 2$ to be specified. We can rewrite $\mathbb{E}(X_k) - e^{x_k}$ as

$$\int_{e^{x_k}}^{\infty} \mathbb{P}\left(X_k \geq x\right) dx = \int_{x_k}^{\infty} \mathbb{P}\left(X_k \geq e^s\right) e^s ds \leq 4^k \int_{x_k}^{\infty} e^{-c\frac{s^2}{\log s}} e^s ds \leq 4^k e^{x_k} \int_{x_k}^{\infty} e^{-c\frac{s^2}{\log x_k}} ds.$$

By using $\int_a^\infty e^{-bx^2} dx \leq (2ab)^{-1} e^{-ba^2}$, we get $\mathbb{E}(X_k) \leq e^{x_k} + 4^k e^{x_k} (2x_k \frac{c}{\log x_k})^{-1} e^{-c\frac{x_k^2}{\log x_k}}$. Taking x_k such that $k \log 4 = c\frac{x_k^2}{\log x_k}$ gives $\log k \sim 2 \log x_k$ and $x_k \sim C\sqrt{k \log k}$. □

2.10.2 Upper bound for $F(s)$

In this subsection, we derive two lemmas that allow us to bound the term $F(s)$ which appears in the proof of Proposition 2.11. The first one corresponds to a_{t_s}, the second one to $\int_0^\infty a_t dt$.

Lemma 2.20. *If $a, b, c > 0$ and $\alpha \in (0, 1/2)$ then the function $f_s(t) := -at + bt^{1/2+\alpha} + cs\sqrt{t}$ in increasing on $[0, t_s]$, decreasing on $[t_s, \infty]$ for some $t_s > 0$ which satisfy $at_s^{1/2} = \frac{1}{2}cs + O(s^{2\alpha})$. In particular, we have: $\exp(f_s(t_s)) \leq e^{\frac{c^2 s^2}{4a} + Cs^{1+2\alpha}}$.*

Proof. First, notice that $f_s'(t) = -a + (\frac{1}{2} + \alpha)bt^{-1/2+\alpha} + \frac{1}{2}cst^{-1/2}$. Since $f_s'(t_s) = 0$ we obtain $a = (\frac{1}{2} + \alpha)bt_s^{-1/2+\alpha} + \frac{1}{2}cst_s^{-1/2}$ which we write:

$$at_s^{1/2} = \frac{cs}{2} + (\frac{1}{2} + \alpha)bt_s^{\alpha}. \tag{10.43}$$

Thus $at_s^{1/2} \geq cs/2$. In particular, $\lim_{s\to\infty} t_s = +\infty$. Using (10.43), we obtain $at_s^{1/2} \sim_{s\to\infty} \frac{1}{2}cs$. Using again (10.43), we have $at_s^{1/2} = \frac{1}{2}cs + O(s^{2\alpha})$. Using again (10.43) we conclude by noticing that: $f_s(t_s) = -at_s + bt_s^{1/2+\alpha} + cst_s^{1/2} = at_s - 2b\alpha t_s^{1/2+\alpha}$. □

Lemma 2.21. *Let $\alpha, a, b > 0$ with $\alpha < 1/2$. For every $s > 0$ the following inequality holds*

$$\int_0^\infty e^{-t + at^{1/2+\alpha} + bs\sqrt{t}} dt \leq C_{\alpha,a}(2 + bs) e^{\frac{(bs)^2}{4}} e^{C_\alpha (bs)^{1+2\alpha}},$$

where $C_{\alpha,a} < \infty$ just depends on a and C_α just depends on α.

Proof. By writing $-t+bs\sqrt{t}=\frac{(bs)^2}{4}-(\sqrt{t}-\frac{bs}{2})^2$ and the change of variable $u=\sqrt{t}$,

$$\int_0^\infty e^{-t+at^{1/2+\alpha}+bs\sqrt{t}}dt = e^{\frac{(bs)^2}{4}}\int_0^\infty e^{-(u-\frac{bs}{2})^2+au^{1+2\alpha}}2udu.$$

Now, by the change of variables $v=u-bs/2$, we get

$$\int_0^\infty e^{-(u-\frac{bs}{2})^2+au^{1+2\alpha}}2udu = \int_{-\frac{bs}{2}}^\infty e^{-v^2+a(v+\frac{bs}{2})^{1+2\alpha}}(2v+bs)dv.$$

Finally, by Jensen's inequality, $(v+\frac{bs}{2})^{1+2\alpha}\le C_\alpha(|v|^{1+2\alpha}+(bs)^{1+2\alpha})$ thus

$$\begin{aligned}\int_{-\frac{bs}{2}}^\infty e^{-v^2+a(v+\frac{bs}{2})^{1+2\alpha}}(2v+bs)dv &\le e^{C_\alpha a(bs)^{1+2\alpha}}\int_{-\frac{bs}{2}}^\infty e^{-v^2+C_\alpha a|v|^{1+2\alpha}}(2v+bs)dv\\ &\le e^{C_\alpha a(bs)^{1+2\alpha}}(2+bs)\int_{-\infty}^\infty e^{-v^2+C_\alpha a|v|^{1+2\alpha}}\left(1+|v|\right)dv.\end{aligned}$$

□

Now, we bound $F(s)$. Recall first that $F(s)\le 2a_{t_s}+\int_0^\infty a_t dt$ where $a_t=\exp(f_s(t))$, $f_s(t):=-t(1-\lambda)\log 2+Ct^{1/2+\alpha}+\beta s\sqrt{t}$, $\lambda:=(1+a_\varepsilon)\gamma$, $\alpha:=\frac{\delta}{2}$ and $\beta:=\frac{\gamma}{2}\sqrt{\log 4}$. By Lemma 2.20, $a_{t_s}\le e^{\frac{\beta^2s^2}{4(1-\lambda)\log 2}+Cs^{1+2\alpha}}=e^{\frac{\gamma^2\log 4s^2}{16(1-(1+a_\varepsilon)\gamma)\log 2}+Cs^{1+\delta}}=e^{\frac{\gamma^2s^2}{8(1-(1+a_\varepsilon)\gamma)}+Cs^{1+\delta}}$. By the change of variable $u=t(1-\lambda)\log 2$ and Lemma 2.21, we obtain the integral bound $\int_0^\infty a_t dt\le Ce^{\frac{\gamma^2s^2}{8(1-(1+a_\varepsilon)\gamma)}}e^{Cs^{1+\delta}}$. Altogether we get $F(s)\le Ce^{\frac{\gamma^2s^2}{8(1-(1+a_\varepsilon)\gamma)}}e^{Cs^{1+\delta}}$.

Chapter 3: Tightness of Liouville first passage percolation for $\gamma \in (0,2)$

This chapter is based on joint work with Julien Dubédat, Jian Ding and Alexander Dunlap [24] .

3.1 Introduction

The present study concerns the tightness of Liouville first-passage percolation (LFPP) metrics associated with a regularization of the Gaussian free field. This proves the existence of subsequential limiting metrics. Given this, it remains to show that such limiting metrics are unique in law for each $\gamma \in (0,2)$ in order to complete the construction of the LQG metric in this regime. The latter task was carried out in the series of works [39, 56–59], thus completing the construction. The present study follows three main tightness results for discretized or smoothed LQG metrics. In [25], tightness of LFPP metrics (on a discrete lattice) was proved in the small noise regime for which γ is very small. In [38], tightness was shown for metrics arising in the same way from $\star$-scale invariant fields, still in the small noise regime. In [26], tightness was shown for all $\gamma < 2$ for the Liouville graph distance, which is a graph metric equal to the least number of Euclidean balls of a given LQG measure necessary to cover a path between a pair of points.

We consider a smoothed Gaussian field

$$\phi_\delta(x) := \sqrt{\pi} \int_{\delta^2}^{1} \int_{\mathbb{R}^2} p_{\frac{t}{2}}(x-y) W(dy, dt) \tag{1.1}$$

for $x \in \mathbb{R}^2$ and $\delta \in (0,1)$, where $p_t(x-y) := \frac{1}{2\pi t} e^{-\frac{|x-y|^2}{2t}}$ and W is a space-time white noise. This approximation is natural since it can be uniformly compared on a compact domain with a Gaussian free field h mollified by the heat kernel defined on a slightly larger domain, viz. $\phi_{\sqrt{t}}$ and $p_{t/2} * h$ (where $*$ denotes the convolution operator) are comparable. Furthermore, this approximation provides some nice invariance and scaling properties on the full plane.

For $\gamma \in (0,2)$, recall the notation ξ

$$\xi := \gamma/d_\gamma \tag{1.2}$$

where d_γ is the "Liouville quantum gravity dimension" defined in [28]. It is known (see Theorem 1.2 and Proposition 1.7 in [28]) that the function $\gamma \mapsto \gamma/d_\gamma$ is strictly increasing and continuous on $(0,2)$. Therefore, in this chapter we will be interested in the range $\xi \in (0,(2/d_2)^-)$, where $(2/d_2)^- = \lim_{\gamma\uparrow 2}\gamma/d_\gamma$.

We consider the length metric $e^{\xi\phi_\delta}ds$ (equivalently, the metric whose Riemannian metric tensor is given by $e^{2\xi\phi_\delta}ds^2$), restricted to the unit square $[0,1]^2$. We recall that a length metric is a metric such that the distance between two points is given by the infimum over the arc lengths of paths connecting the two points. We denote by λ_δ the median of the left-right distance of $[0,1]^2$ for the metric $e^{\xi\phi_\delta}ds$. Our main theorem is the following.

Theorem 3.1. *1. If $\gamma \in (0,2)$, then $\left(\lambda_\delta^{-1}e^{\xi\phi_\delta}ds\right)_{\delta\in(0,1)}$ is tight with respect to the uniform topology on the space of continuous functions $[0,1]^2 \times [0,1]^2 \to \mathbb{R}^+$. Furthermore, any subsequential limit is almost surely bi-Hölder with respect to the Euclidean metric on $[0,1]^2$.*

2. Let $K = [0,1]^2$. If h is a Gaussian free field with zero boundary conditions on a bounded open domain D containing K (extended to zero outside of D), then the internal metrics $(\lambda_{\sqrt{\delta}}^{-1}e^{\xi p_{\frac{\delta}{2}}*h}ds)_{\delta\in(0,1)}$ on K are tight with respect to the uniform topology of continuous functions $K\times K \to \mathbb{R}^+$.

Furthermore, the normalizing constants $(\lambda_\delta)_{\delta\in(0,1)}$ satisfy

$$\lambda_\delta = \delta^{1-\xi Q}e^{O\left(\sqrt{|\log\delta|}\right)} \tag{1.3}$$

where $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$.

A year after the article [24] corresponding to this chapter was posted, the subsequent work [29] proved a similar result to ours when $\xi \geq (2/d_2)^-$. However, in that case the tightness does not hold in the uniform topology and the Beer topology on lower semicontinuous functions was used.

In order to establish the tightness of renormalized metrics $(d_{\phi_\delta})_{\delta\in(0,1)} := (\lambda_\delta^{-1}e^{\xi\phi_\delta}ds)_{\delta\in(0,1)}$, we prove a number of uniform estimates for that family (which also hold when the approximation is the GFF mollified by the heat kernel). Such estimates that are closed under weak convergence also apply to subsequential limits. Let us summarize these properties. Let $\mathcal{D}$ denote the family of laws of d_{ϕ_δ}, $\delta \in (0,1)$ (i.e. seen as random continuous functions on $([0,1]^2)^2$), and $\overline{\mathcal{D}}$ denotes its closure under weak convergence (i.e., $\overline{\mathcal{D}}$ also includes the laws of all subsequential limits).

1. Under any $\mathbb{P} \in \overline{\mathcal{D}}$, d is $\mathbb{P}$-a.s. a length metric. This is clear for the renormalized metrics d_{ϕ_δ} by definition, and the property of being a length metric extends to limits. (See [17, Exercise 2.4.19].)

2. If d is a metric on $\mathbb{R}^2$ and R is a rectangle, we denote by $d(R)$ the left-right length of R for d. We have the following tail estimates. There exists $c, C > 0$ such that for $s > 2$, uniformly in $\mathbb{P} \in \overline{\mathcal{D}}$ we have

$$ce^{-Cs^2} \leq \mathbb{P}\left(d(R) \leq e^{-s}\right) \leq Ce^{-cs^2}, \tag{1.4}$$

$$ce^{-Cs^2} \leq \mathbb{P}\left(d(R) \geq e^{s}\right) \leq Ce^{-c\frac{s^2}{\log s}}. \tag{1.5}$$

   The upper bounds are proved in Section 3.4, while the lower bounds are consequences of the Cameron–Martin theorem, considering shifts of the field at the coarsest scale as in [38, Section 5.4].

3. If d is a metric on $\mathbb{R}^2$ and R is a rectangle, we denote by $\mathrm{Diam}(R, d)$ the diameter of R for d. We have the following uniform first moment bound:

$$\sup_{\mathbb{P} \in \overline{\mathcal{D}}} \mathbb{E}\left(\mathrm{Diam}(R, d)\right) < \infty. \tag{1.6}$$

   This is shown in the course of the proof of Proposition 3.27 below.

4. Under any $\mathbb{P} \in \overline{\mathcal{D}}$, d is $\mathbb{P}$-a.s. bi-Hölder with respect to the Euclidean metric and we have the following bounds for exponents: for $\alpha < \xi(Q-2)$, $\beta > \xi(Q+2)$, and R a rectangle, the families

$$\left(\sup_{x,x' \in R} \frac{|x - x'|^\alpha}{d(x, x')}\right)_{\mathcal{L}(d) \in \overline{\mathcal{D}}} \quad \text{and} \quad \left(\sup_{x,x' \in R} \frac{d(x, x')}{|x - x'|^\beta}\right)_{\mathcal{L}(d) \in \overline{\mathcal{D}}} \tag{1.7}$$

   are tight. Here $\mathcal{L}(d)$ means the law of d. These properties are shown in Proposition 3.28

below.

Let us also mention that subsequential limits are consistent with the Weyl scaling: for a function f in the Cameron-Martin space of the Gaussian free field h, for any coupling (h, d) associated to a subsequential limit of the sequence of laws of $((h, \lambda_{\sqrt{\delta}}^{-1} e^{\xi p_{\frac{\delta}{2}} * h} ds))_{\delta>0}$, the couplings (h, d) and $(h + f, e^{\xi f} \cdot d)$ are mutually absolutely continuous with respect to each other and the associated Radon-Nikodým derivative is the one of the first marginal. This can be proved using similar arguments to those of [38, Section 7]. An analogue of this property for the Liouville measure together with the conservation of the Liouville volume average is enough to characterize the Liouville measure, as seen by Shamov in [99].

Furthermore, in our setting where the metrics are on a compact subset of $\mathbb{C}$, we can directly use the uniform topology instead of working with the Gromov-Hausdorff topology (note that the former is stronger than the latter). In this chapter, we show tightness for the full subcritical range $\gamma \in (0, 2)$ of renormalized side-to-side crossing lengths, point-to-point distance and metrics. Limiting metrics are bi-Hölder with respect to the Euclidean metric.

3.1.1 Strategy of the proof and comparison with previous works

In contrast with previous works on the LQG measure, the variational problem defining the LQG metric means that most direct computations are impossible, and in particular most of techniques used in the theory of Gaussian multiplicative chaos and LQG measure are unavailable. This necessitates the more intricate multiscale geometric arguments that we employ.

Our tightness proof relies on two key ingredients, a Russo-Seymour-Welsh argument and multiscale analysis. In both parts we extend and refine many arguments used in the previous works [25, 26, 38] on the tightness of various types of LQG metrics.

Russo-Seymour-Welsh. The RSW argument relates, to within a constant factor, quantiles of the left–right LFPP crossing distances of a "portrait" rectangle and of a "landscape" rectangle. (By a crossing distance we simply mean the distance between two opposite sides of a rectangle.) In [25, 26], these crossings are referred to as "easy" and "hard" respectively. The utility of such a result is that crossings of larger rectangles necessarily induce easy crossings of subrectangles, while hard crossings of smaller rectangles can be glued together to create crossings of larger rectangles. Thus, multiscale analysis arguments can establish lower bounds in terms of easy crossings and upper bounds in terms of hard crossings. RSW arguments then allow these bounds to be compared.

RSW arguments originated in the works [94, 95, 98] for Bernoulli percolation, and have since been adapted to many percolation settings. The work [25] introduced an RSW result for LFPP in the small noise regime based on an RSW result for Voronoi percolation devised by Tassion [110]. Tassion's result is beautiful but intricate, and becomes quite complex when it is adapted to take into account the weights of crossing in the first-passage percolation setting, as was done in [25].

The RSW approach of this chapter is based on the much simpler approach introduced in the first chapter, (corresponding to [38]), which relies on an approximate conformal invariance of the field. (We recall that the Gaussian free field is exactly conformally invariant in dimension 2, and that the LQG measure enjoys an exact conformal covariance.) Roughly speaking, the conformal invariance argument relies on writing down a conformal map between the portrait and landscape rectangles, and analyzing the effect of such a map on crossings of the rectangle. We note that the approximate conformal invariance used in this chapter relies in an important way on the exact independence of different "scales" of the field, which is manifest in the independence of the white noise at different times in the expression (1.1). Thus, the argument we use here is not immediately applicable to mollifications of the Gaussian free field by general mollifiers (for example, the common "circle-average approximation" of the GFF). The RSW argument of [38] was also adapted in [26] to the Liouville graph distance case.

Tail estimates. Once the RSW result is established, we derive tail estimates with respect to fixed quantiles. The lower tail estimate is unconditional, while the upper tail estimate depends on a quantity Λ_n measuring the concentration at the current scale, which will later be uniformly bounded by an inductive argument.

Multiscale analysis. With RSW and tail estimates in hand, we turn to the multiscale analysis part of the chapter. This argument turns on the Condition (T) formulated in (3.5.1) below, which, informally, states that the arclength of the crossing is not concentrated on a small number of subarcs of small Euclidean diameter. The argument of [26] requires similar input, which is a key role of the subcriticality $\gamma < 2$. While [26] relies directly on certain scaling symmetries of the Liouville graph distance to use subcriticality, the present work relies on the characterization of the Hausdorff dimension d_γ obtained in [28], along with some weak multiplicativity arguments and concentration obtained from percolation arguments.

Condition (T). Our formulation of Condition (T), which has not appeared in previous works, precisely captures the property of the metric needed to obtain the tightness of the left–right crossing distances, the existence of the exponent, and the tail estimates (via a uniform bound on the Λ_n).

Condition (T) makes sense for LFPP with any underlying field and any parameter ξ. In particular, this condition or a variant thereof could possibly hold for LFPP for some $\xi > 2/d_2$. Therefore, a byproduct of the present work is a simple criterion (that implies, as noted above, tightness of the crossing distances, existence of exponents, and tail estimates) that may be applicable more generally.

The utility of Condition (T) is that it allows us to use an Efron–Stein argument to obtain a contraction in an inductive bound on the crossing distance logarithm variance. Informally, since the crossing distance feels the effect of many different subboxes, the subbox crossing distances are effectively being averaged to form the overall crossing distance. This yields a contraction in variance. (Of course, the coarse scales also contribute to the variance, and hence the variance of the crossing distance does not decrease as the discretization scale decreases but rather stays bounded.)

The way we verify Condition (T) is quite rough: we bound the field uniformly over a coarse grained geodesic by the supremum of the field over the unit square. It turns out that this bound together with the identification of the exponent $1 - \xi Q$ is enough to establish the condition.

Tightness of the metrics. Once the tightness of the left–right crossing distance is established, we turn to the tightness of the diameter and of the metric itself. This is done by a chaining argument, and requires again $\xi < 2/d_2$. The diameter is not expected to be tight when $\xi > 2/d_2$, since there are points that become infinitely distant from the bulk of the space as the discretization scale goes to 0.

3.2 Description and comparison of approximations

We recall that a white noise W on $\mathbb{R}^d$ is a random Schwartz distribution such that for every smooth and compactly supported test function f, $\langle W, f\rangle$ is a centered Gaussian variable with variance $\|f\|_{L^2(\mathbb{R}^d)}$ (see e.g. [21]). The main approximation of the Gaussian free field that we consider in this chapter is defined for $\delta \in (0,1)$ by

$$\phi_\delta(x) := \sqrt{\pi}\int_{\delta^2}^1 \int_{\mathbb{R}^2} p_{\frac{t}{2}}(x-y) W(dy, dt) \tag{2.8}$$

where $p_t(x-y) := \frac{1}{2\pi t}e^{-\frac{|x-y|^2}{2t}}$ and W is a space-time white noise on $[0,1]\times\mathbb{R}^2$. This approximation is different than the one considered in [38] which is

$$\tilde{\phi}_\delta(x) := \int_\delta^1 \int_{\mathbb{R}^2} k\left(\frac{x-y}{t}\right) t^{-3/2} W(dy, dt)$$

for a smooth nonnegative bump function k, radially symmetric and with compact support. Up to a change of variable in t, the difference is essentially replacing p_1 by k. Both fields are normalized in such a way that $\mathbb{E}(\phi_0(x)\phi_0(y)) = -\log|x-y| + g(x,y)$ with g continuous (see e.g. Section 2 in [38]): this is the reason for the factor $\sqrt{\pi}$ in (2.8).

Let us mention that $\star$-scale invariant Gaussian fields with compactly-supported bump function k

1. are invariant under Euclidean isometries,
2. have finite-range correlation at each scale,
3. and have convenient scaling properties.

The Gaussian field ϕ_δ introduced above satisfies 1 and 3 but not 2. Because of the lack of finite-range correlation, we will also use a field ψ_δ (defined in the next section) which satisfies 1 and 2 such that $\sup_{n\geq 0}\|\phi_{0,n} - \psi_{0,n}\|_{L^\infty([0,1]^2)}$ has Gaussian tails, where we use the notation $\phi_{0,n}$ for ϕ_δ with $\delta = 2^{-n}$.

3.2.1 Basic properties of ϕ_δ and ψ_δ

Scaling property of ϕ_δ. We use the scale decomposition

$$\phi := \sum_{n\geq 0} \phi_n \quad \text{where} \quad \phi_n(x) = \sqrt{\pi}\int_{2^{-2(n+1)}}^{2^{-2n}} \int_{\mathbb{R}^2} p_{\frac{t}{2}}(x-y)W(dy,dt)$$

If we denote by C_n the covariance kernel of ϕ_n, so $C_n(x,x') = \mathbb{E}(\phi_n(x)\phi_n(x'))$, then we have

$$C_n(x,x') = \int_{2^{-2(n+1)}}^{2^{-2n}} \frac{1}{2t} e^{-\frac{|x-x'|^2}{2t}} dt = C_0(2^n x, 2^n x').$$

Therefore, the law of $(\phi_n(x))_{x\in[0,1]^2}$ is the same as $(\phi_0(2^n x))_{x\in[0,1]^2}$. Because of the $\frac{1}{2t}$ above, we choose δ^2 and not δ in (2.8) so that the pointwise variance ϕ_δ is $\log \delta^{-1}$. Similarly, for $0 < a < b$ and $x \in \mathbb{R}^2$, set

$$\phi_{a,b}(x) := \sqrt{\pi}\int_{a^2}^{b^2} \int_{\mathbb{R}^2} p_{\frac{t}{2}}(x-y)W(dy,dt) \tag{2.9}$$

and note that we have the scaling identity $\phi_{a,b}(r\cdot) \stackrel{(d)}{=} \phi_{a/r,b/r}(\cdot)$. Indeed, $\mathbb{E}(\phi_{a,b}(rx)\phi_{a,b}(rx'))$ is given by

$$\pi\int_{a^2}^{b^2}\int_{\mathbb{R}^2} p_{\frac{t}{2}}(rx-y)p_{\frac{t}{2}}(y-rx')dydt = \pi\int_{a^2}^{b^2} p_t(r(x-x'))dt = \int_{a^2}^{b^2}\frac{1}{2t}e^{-\frac{r^2|x-x'|^2}{2t}}dt,$$

and by the change of variable $t = r^2 u$, this gives

$$\int_{a^2}^{b^2}\frac{1}{2t}e^{-\frac{r^2|x-x'|^2}{2t}}dt = \int_{(a/r)^2}^{(b/r)^2}\frac{1}{2t}e^{-\frac{|x-x'|^2}{2t}}dt = \mathbb{E}(\phi_{a/r,b/r}(x)\phi_{a/r,b/r}(x')).$$

We will use the notation $\phi_{k,n}$ when $a = 2^{-n}$ and $b = 2^{-k}$ for $0 \leq k \leq n$.

Maximum and oscillation of ϕ_δ. We have the same estimates for the supremum of the field $\phi_{0,n}$ as those for the $\star$-scale invariant case considered in [38] (it is essentially a union bound combined with a scaling argument). The following proposition corresponds to Lemma 10.1 and Lemma 10.2 in [38].

Proposition 3.2 (Maximum bounds). *We have the following tail estimates for the supremum of*

$\phi_{0,n}$ *over the unit square: for* $a > 0$, $n \geq 0$,

$$\mathbb{P}\left(\max_{[0,1]^2} |\phi_{0,n}| \geq a(n + C\sqrt{n})\right) \leq C4^n e^{-\frac{a^2}{\log 4}n} \tag{2.10}$$

as well as the following moment bound: if $\gamma < 2$, *then*

$$\mathbb{E}(e^{\gamma \max_{[0,1]^2} |\phi_{0,n}|}) \leq 4^{\gamma n + O(\sqrt{n})} \tag{2.11}$$

We will also need some control on the oscillation of the field $\phi_{0,n}$. We introduce the following notation for the L^∞-norm on a subset of $\mathbb{R}^d$. If A is a subset of $\mathbb{R}^d$ and $f : A \to \mathbb{R}^m$, we set

$$\|f\|_A := \sup_{x \in A} |f(x)| \tag{2.12}$$

We introduce the following notation to describe the oscillation of a smooth field ϕ: if $A \subset \mathbb{R}^2$ we set

$$\text{osc}_A(\phi) := \text{diam}(A) \|\nabla \phi\|_A , \tag{2.13}$$

so that if A is convex then $\sup_{x,y \in A} |\phi(x) - \phi(y)| \leq \text{osc}_A(\phi)$ and

$$\max_{P \in \mathcal{P}_n, P \subset [0,1]^2} \text{osc}_P(\phi_{0,n}) \leq C2^{-n} \|\nabla \phi_{0,n}\|_{[0,1]^2} ,$$

where $\mathcal{P}_n$ denotes the set of dyadic blocks at scale n, viz.

$$\mathcal{P}_n := \{2^{-n}([i, i+1] \times [j, j+1]) : i, j \in \mathbb{Z}\}. \tag{2.14}$$

In order to simplify the notation $P \in \mathcal{P}_n, P \subset [0,1]^2$ later on, we also set

$$\mathcal{P}_n^1 := \{P \in \mathcal{P}_n : P \subset [0,1]^2\}. \tag{2.15}$$

Proposition 3.3 (Oscillation bounds). *We have the following tail estimates for the oscillation of*

$\phi_{0,n}$: *there exists* $C > 0$, $\sigma^2 > 0$, *so that, for all* $x, \varepsilon > 0$, $n \geq 0$,

$$\mathbb{P}\left(2^{-n} \left\|\nabla \phi_{0,n}\right\|_{[0,1]^2} \geq x\right) \leq C 4^n e^{-\frac{x^2}{2\sigma^2}} \tag{2.16}$$

as well as the following moment bound: for $a > 0$, *there exists* $c_a > 0$ *so that for* $n \geq 0$,

$$\mathbb{E}\left(e^{a n^{\varepsilon} 2^{-n} \|\nabla \phi_{0,n}\|_{[0,1]^2}}\right) \leq e^{c_a n^{\frac{1}{2}+\varepsilon} + O(n^{2\varepsilon})} \tag{2.17}$$

Proof. Inequality (2.16) was obtained between Equation (10.3) and Equation (10.4) in [38]. Now, we prove (2.17). Set $a_n := a n^{\varepsilon}$, $O_n = 2^{-n} \left\|\nabla \phi_{0,n}\right\|_{[0,1]^2}$, and take $x_n = a_n \sigma^2 + \alpha \sigma \sqrt{n}$ with $\alpha > 0$ so that $\frac{\alpha^2}{2} = \log 4$. We have, using (2.16),

$$\int_{e^{a_n x_n}}^{\infty} \mathbb{P}\left(e^{a_n O_n} \geq x\right) dx = \int_{a_n x_n}^{\infty} \mathbb{P}\left(e^{a_n O_n} \geq e^s\right) e^s ds \leq C 4^n \int_{a_n x_n}^{\infty} e^{-\frac{s^2}{2 a_n^2 \sigma^2}} e^s ds$$

By a change of variable ($s \leftrightarrow a_n \sigma s + (a_n \sigma)^2$), we get

$$\int_{a_n x_n}^{\infty} e^{-\frac{s^2}{2 a_n^2 \sigma^2}} e^s ds = a_n \sigma e^{\frac{1}{2} a_n^2 \sigma^2} \int_{\frac{x_n}{\sigma} - a_n \sigma}^{\infty} e^{-\frac{s^2}{2}} ds = a_n \sigma e^{\frac{1}{2} a_n^2 \sigma^2} \int_{\alpha \sqrt{n}}^{\infty} e^{-\frac{s^2}{2}} ds$$

since $x_n = a_n \sigma^2 + \alpha \sigma \sqrt{n}$. Using that $\int_a^{\infty} e^{-bx^2} dx \leq (2ab)^{-1} e^{-ba^2}$, we get $\int_{e^{a_n x_n}}^{\infty} \mathbb{P}\left(e^{a_n O_n} \geq x\right) dx \leq e^{O(n^{2\varepsilon})}$. The result follows from writing $\mathbb{E}(e^{a_n O_n}) \leq e^{a_n x_n} + \int_{e^{a_n x_n}}^{\infty} \mathbb{P}(e^{a_n O_n} \geq x) dx$. $\square$

Definition of ψ_δ. We fix a smooth, nonnegative, radially symmetric bump function Φ such that $0 \leq \Phi \leq 1$ and Φ is equal to one on $B(0,1)$ and to zero outside $B(0,2)$. We also fix small constants $r_0 > 0$ and $\varepsilon_0 > 0$. We will specify these constants later on. In particular, ε_0 appears in the main proof in (5.60) and its final effect is in (5.65). All other constants C, c will implicitly depend on r_0 and ε_0. Then, we introduce for each $\delta \in [0,1]$, the field

$$\psi_\delta(x) := \int_{\delta^2}^{1} \int_{\mathbb{R}^2} \Phi_{\sigma_t}(x-y) p_{\frac{t}{2}}(x-y) W(dy, dt) = \int_{\delta^2}^{1} \int_{\mathbb{R}^2} p_{\frac{t}{2}}^{\mathrm{Tr}}(x-y) W(dy, dt)$$

$$\text{where} \quad \sigma_t = r_0 \sqrt{t} |\log t|^{\varepsilon_0}, \quad \Phi_{\sigma_t}(\cdot) := \Phi(\cdot/\sigma_t) \quad \text{and} \quad p_{\frac{t}{2}}^{\mathrm{Tr}} := p_{\frac{t}{2}} \Phi_{\sigma_t}. \tag{2.18}$$

Thanks to the truncation, the fields $(\psi_\delta)_{\delta \in [0,1]}$ have finite correlation length $8 r_0 \sup_{t \in [0,1]} \sqrt{t} |\log t|^{\varepsilon_0}$.

Decomposition in scales and blocks of ψ_δ. We have the scale decomposition

$$\psi(x) := \int_0^1 \int_{\mathbb{R}^2} p^{\mathrm{Tr}}_{\frac{t}{2}}(x-y) W(dy,dt) = \sum_{k=1}^{\infty} \sum_{P \in \mathcal{P}_k} \int_{2^{-2k}}^{2^{-2k+2}} \int_P p^{\mathrm{Tr}}_{\frac{t}{2}}(x-y) W(dy,dt) = \sum_{k \geq 1} \sum_{P \in \mathcal{P}_k} \psi_{k,P}(x) \tag{2.19}$$

where $\psi_{k,P}$ is defined for $P \in \mathcal{P}_k$ by $\psi_{k,P}(x) := \int_{2^{-2k}}^{2^{-2k+2}} \int_P p^{\mathrm{Tr}}_{\frac{t}{2}}(x-y) W(dy,dt)$ and thus has correlation length less than $Ck^{\varepsilon_0}2^{-k}$. In particular, a fixed block field is only correlated with fewer than $Ck^{2\varepsilon_0}$ other block fields at the same scale. In fact, when we apply the Efron-Stein inequality (see (5.58)) we will use the following decomposition:

$$\psi_{0,n} = \psi_{0,K} + \sum_{P \in \mathcal{P}_K} \psi_{K,n,P}(x) \quad \text{where} \quad \psi_{K,n,P}(x) := \int_{2^{-2n}}^{2^{-2K+2}} \int_P p^{\mathrm{Tr}}_{\frac{t}{2}}(x-y) W(dy,dt). \tag{2.20}$$

We note that there is a formal conflict in notation between (2.9) and (2.20), but it will always be clear from context whether the second subscript is a number or an element of $\mathcal{P}_k$ (a set), so confusion should not arise.

Variance bounds for ϕ_δ and ψ_δ. Later on we will need the following lemma.

Lemma 3.4. *There exists $C > 0$ so that for $\delta \in [0,1]$ and $x, x' \in \mathbb{R}^2$, we have*

$$\mathrm{Var}\left(\phi_\delta(x) - \phi_\delta(x')\right) + \mathrm{Var}\left(\psi_\delta(x) - \psi_\delta(x')\right) \leq C\frac{|x-x'|}{\delta}. \tag{2.21}$$

Proof. We start by estimating the first term. Using the inequality $1 - e^{-z} \leq z \leq \sqrt{z}$ for $z \in [0,1]$ and $1 - e^{-z} \leq 1 \leq \sqrt{z}$ for $z \geq 1$ we get

$$\begin{aligned}
&\mathrm{Var}\left(\phi_\delta(x) - \phi_\delta(x')\right) = C\int_{\delta^2}^1 \left(p_{\frac{t}{2}} * p_{\frac{t}{2}}(0) - p_{\frac{t}{2}} * p_{\frac{t}{2}}(x-x')\right) dt \\
&\quad = C\int_{\delta^2}^1 \left(p_t(0) - p_t(x-x')\right) dt = C\int_{\delta^2}^1 \frac{1}{t}(1 - e^{-\frac{|x-x'|^2}{2t}}) dt \leq C|x-x'| \int_{\delta^2}^1 \frac{dt}{t^{3/2}} = C\frac{|x-x'|}{\delta}.
\end{aligned}$$

Similarly, for the second term, we have

$$\mathrm{Var}\left(\psi_\delta(x) - \psi_\delta(x')\right) = C\int_{\delta^2}^1 \left(p^{\mathrm{Tr}}_{\frac{t}{2}} * p^{\mathrm{Tr}}_{\frac{t}{2}}(0) - p^{\mathrm{Tr}}_{\frac{t}{2}} * p^{\mathrm{Tr}}_{\frac{t}{2}}(x-x')\right) dt.$$

Set $p^{\mathrm{Tr}}_{\frac{t}{2}} * p^{\mathrm{Tr}}_{\frac{t}{2}} =: p_t(x)q_t(x)$. Using the identity $p_{t/2}(y)p_{t/2}(x-y) = p_t(x)p_{t/4}(y-x/2)$ we get

$$q_t(x) = \int_{\mathbb{R}^2} \frac{p_{t/2}(y)p_{t/2}(x-y)}{p_t(x)} \Phi_{\sigma_t}(y)\Phi_{\sigma_t}(x-y)dy = \int_{\mathbb{R}^2} p_{t/4}(y-x/2)\Phi_{\sigma_t}(y)\Phi_{\sigma_t}(x-y)dy.$$

We rewrite the variance in terms of q_t: replacing $x - x'$ by z we look at

$$\begin{aligned}\mathrm{Var}\left(\psi_\delta(x) - \psi_\delta(x')\right) &= C\int_{\delta^2}^1 (p_t(0)q_t(0) - p_t(z)q_t(z))dt \\ &= C\int_{\delta^2}^1 p_t(0)(q_t(0) - q_t(z))dt + C\int_{\delta^2}^1 q_t(z)(p_t(0) - p_t(z))dt.\end{aligned}$$

We deal with these two terms separately. For the second one, since $0 \le \Phi \le 1$, we have $0 \le q_t \le 1$. Therefore, following what we did for ϕ_δ above we directly have $0 \le \int_{\delta^2}^1 q_t(z)(p_t(0) - p_t(z))dt \le C\frac{|z|}{\delta}$. For the first term, since $p_t(0) = Ct^{-1}$, it is enough to get the bound $\sqrt{t}|q_t(0) - q_t(z)| \le C|z|$ to complete the proof of the lemma. Changing variables, we have

$$q_t(z) = C\int_{\mathbb{R}^2} e^{-2|y|^2}\Phi_{\sigma_t}(\sqrt{t}y + z/2)\Phi_{\sigma_t}(\sqrt{t}y - z/2)dy.$$

Therefore, using that $0 \le \Phi \le 1$,

$$\begin{aligned}|q_t(z) - q_t(0)| &\le C\int_{\mathbb{R}^2} e^{-2|y|^2}|\Phi_{\sigma_t}(\sqrt{t}y + z/2) - \Phi_{\sigma_t}(\sqrt{ty})|dy \\ &\quad + C\int_{\mathbb{R}^2} e^{-2|y|^2}|\Phi_{\sigma_t}(\sqrt{t}y - z/2) - \Phi_{\sigma_t}(\sqrt{ty})|dy \\ &\le C|z|\int_{\mathbb{R}^2} e^{-2|y|^2}\left\|\nabla\Phi_{\sigma_t}\right\|_{\mathbb{R}^2} dy \le C\frac{|z|}{\sigma_t}\left\|\nabla\Phi\right\|_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-2|y|^2}dy.\end{aligned}$$

Since $\sigma_t = r_0\sqrt{t}|\log t|^{\varepsilon_0}$, we see that $\sup_{t\in[0,1]}\frac{\sqrt{t}}{\sigma_t} < \infty$, and the result follows. □

3.2.2 Comparison between ϕ_δ and ψ_δ

The following proposition justifies the introduction of the field ψ_δ.

Proposition 3.5. *There exist $C > 0$ and $c > 0$ such that for all $x > 0$, we have*

$$\mathbb{P}\left(\sup_{n\ge 0}\|\phi_{0,n} - \psi_{0,n}\|_{[0,1]^2} \ge x\right) \le Ce^{-cx^2}. \tag{2.22}$$

Proof. For $k \geq 1$, we introduce the quantity $D_k(x) := \phi_{k-1,k}(x) - \psi_{k-1,k}(x)$. The proof follows from an adaptation of Lemma 2.7 in [32] as soon as we have the estimates

$$\mathrm{Var}\, D_k(x) \leq Ce^{-ck^{2\varepsilon_0}} \tag{2.23}$$

and

$$\mathrm{Var}\,(\phi_k(x) - \phi_k(y)) + \mathrm{Var}\,(\psi_k(x) - \psi_k(y)) \leq 2^k |x-y|. \tag{2.24}$$

(The estimate (2.23) is weaker than that used in [32, Lemma 2.7] but still much stronger than required for the proof given there.) Note that (2.24) follows from Lemma 3.4 and for (2.23) we proceed as follows: first note that

$$\mathbb{E}\left((\phi_{k-1,k}(x) - \psi_{k-1,k}(x))^2\right) = \int_{2^{-2k}}^{2^{-2k+2}} \int_{\mathbb{R}^2} p_{\frac{t}{2}}(y)^2 (1 - \Phi_{\sigma_t}(y))^2 dy dt.$$

For every y, we have $p_{t/2}(y)(1 - \Phi_{\sigma_t}(y)) \leq (2\pi t)^{-1} e^{-\sigma_t^2/t}$ since $0 \leq \Phi_{\sigma_t} \leq 1$ and $\Phi_{\sigma_t}(y) = 1$ for $|y| \leq \sigma_t$. Therefore,

$$\mathbb{E}\left((\phi_{k-1,k}(x) - \psi_{k-1,k}(x))^2\right) \leq \int_{2^{-2k}}^{2^{-2k+2}} \frac{e^{-\frac{\sigma_t^2}{t}}}{2\pi t} \int_{\mathbb{R}^2} p_{\frac{t}{2}}(y) dy dt \leq Ce^{-ck^{2\varepsilon_0}}. \quad \square$$

Let us point out that in fact $\sum_{n\geq 0} \mathbb{E}(\|\phi_{n,n+1} - \psi_{n,n+1}\|_{[0,1]^2}) < \infty$ holds but we won't use it. Since we will be working with two different approximations of the Gaussian free field, we introduce here some notation, referring to one field or the other. We will denote by $R_{a,b} := [0,a] \times [0,b]$ the rectangle of size (a,b). We define

$$X_{a,b} := \sup_{n \geq 0} \|\phi_{0,n} - \psi_{0,n}\|_{R_{a,b}} \tag{2.25}$$

and $X_a := X_{a,a}$ for the supremum norm of the difference between the two fields on various rectangles.

3.2.3 Length observables

The symbol $L^{(n)}_{a,b}(\phi)$ (and similarly $L^{(n)}_{a,b}(\psi)$) will refer to the left-right distance of the rectangle $R_{a,b}$ for the length functional $e^{\xi\phi_{0,n}}ds$:

$$L^{(n)}_{a,b}(\phi) := \inf_{\pi} \int_{\pi} e^{\xi\phi_{0,n}} ds, \tag{2.26}$$

where ds refers to the Euclidean length measure and the infimum is taken over all smooth curves π connecting the left and right sides of $R_{a,b}$. We will sometimes consider a geodesic associated to this variational problem. Such a path exists by the Hopf-Rinow theorem and a compactness argument.

We introduce some notation for the quantiles associated to this observable: $\ell^{(n)}_{a,b}(\phi,p)$ (similarly $\ell^{(n)}_{a,b}(\psi,p)$) is such that $\mathbb{P}\left(L^{(n)}_{a,b}(\phi) \le \ell^{(n)}_{a,b}(\phi)\right) = p$. For high quantiles, we introduce $\bar{\ell}^{(n)}_{a,b}(\phi,p) := \ell^{(n)}_{a,b}(\phi,1-p)$. Note that $\ell^{(n)}_{a,b}(\phi,p)$ is increasing in p whereas $\bar{\ell}^{(n)}_{a,b}(\phi,p)$ is decreasing in p. Note that both are well-defined, i.e., there are no Dirac deltas in the law of $L^{(n)}_{a,b}$. This follows from an application of the Cameron–Martin formula. We will also need the notation

$$\Lambda_n(\phi,p) := \max_{k\le n} \frac{\bar{\ell}_k(\phi,p)}{\ell_k(\phi,p)} \quad \text{where } \ell_k(\phi,p) := \ell^{(k)}_{1,1}(\phi,p) \quad \text{and} \quad \bar{\ell}_k(\phi,p) := \bar{\ell}^{(k)}_{1,1}(\phi,p). \tag{2.27}$$

The following inequalities are straightforward:

$$e^{-\xi X_{a,b}} L^{(n)}_{a,b}(\psi) \le L^{(n)}_{a,b}(\phi) \le e^{\xi X_{a,b}} L^{(n)}_{a,b}(\psi) \tag{2.28}$$

Therefore, using Proposition 3.5 (and a union bound, if necessary), we obtain that for some $C > 0$ (depending only on a and b), for any $\varepsilon > 0$ we have

$$e^{-\xi C\sqrt{|\log \varepsilon/C|}} \bar{\ell}^{(n)}_{a,b}(\psi, p+\varepsilon) \le \bar{\ell}^{(n)}_{a,b}(\phi,p) \le e^{\xi C\sqrt{|\log \varepsilon/C|}} \bar{\ell}^{(n)}_{a,b}(\psi, p-\varepsilon)$$
$$e^{-\xi C\sqrt{|\log \varepsilon/C|}} \ell^{(n)}_{a,b}(\psi, p-\varepsilon) \le \ell^{(n)}_{a,b}(\phi,p) \le e^{\xi C\sqrt{|\log \varepsilon/C|}} \ell^{(n)}_{a,b}(\psi, p+\varepsilon)$$

In particular, there exists $C_p > 0$ such that, uniformly in n,

$$\ell_n(\psi, p/2) \geq \sqrt{C_p}^{-1} \ell_n(\phi, p), \quad \bar{\ell}_n(\psi, p/2) \leq \sqrt{C_p} \bar{\ell}_n(\phi, p) \quad \text{and} \quad \Lambda_n(\psi, p/2) \leq C_p \Lambda_n(\phi, p). \tag{2.29}$$

Now, we discuss how the scaling property of the field ϕ translates at the level of lengths. We will use the following equality in law: for $a, b > 0$ and $0 \leq m \leq n$,

$$L_{a,b}^{(m,n)}(\phi) \stackrel{(d)}{=} 2^{-m} L_{a \cdot 2^m, b \cdot 2^m}^{(n-m)}(\phi). \tag{2.30}$$

Finally, for a rectangle P with two marked opposite sides, we define $L^{(n)}(P, \phi)$ to be the crossing distance between the two marked sides under the field $e^{\xi \phi_{0,n}}$. The marked sides will be clear from context: if we call P a "long rectangle," then we mean that the marked sides are the two shorter sides, so that $L^{(n)}(P, \phi)$ is the distance across P "the long way."

3.2.4 Outline of the proof and roles of ϕ_δ and ψ_δ

The key idea of the proof is to obtain a self-bounding estimate associated to a measure of concentration of some observables, say rectangle crossing lengths. This is naturally expected because of the tree structure of our model. We introduce a general condition, which we call Condition (T), (see (3.5.1)) which ensures a contraction in the self-bounding estimate (5.68), which relates a measure of concentration at scale n, the variance, with the measure of concentration that we inductively bound, Λ_{n-K} (see (2.27)), which is at a smaller scale.

We then prove that this condition, which depends only on ξ and on the field considered, is satisfied when $\xi \in (0, (2/d_2)^-)$. This proof uses a result taken from [28] about the existence of an exponent for circle average Liouville first passage percolation and this is the reason we don't consider the simpler $\star$-scale invariant field with compactly-supported kernel but the field ϕ_δ, which can be compared to the circle average process by a result obtained in [27].

The roles of ϕ_δ and ψ_δ in the proof are the following.

1. Prove Russo-Seymour-Welsh estimates for ϕ.

2. Prove tail estimates w.r.t low and high quantiles for both ϕ and ψ:

   (a) Lower tails: Use directly the RSW estimates together with a Fernique-type argument for the field ψ with local independence properties.

   (b) Upper tails: use a percolation/scaling argument, percolation using ψ and scaling using ϕ.

3. Concentration of the log of the left-right distance: use Efron-Stein for the field ψ (because of the local independence properties at each scale). This gives the same result for ϕ.

4. To conclude for the concentration of diameter and metric, this is essentially a chaining/scaling argument using only the field ϕ.

3.3 Russo-Seymour-Welsh estimates

3.3.1 Approximate conformal invariance

In order to establish our RSW result, we first show an approximate conformal invariance property of the field. The arguments in this section are similar to those of [38, Section 3.1]. The difference is that the Gaussian kernel has infinite support.

Recall that $\phi_\delta(x) = \int_{\delta^2}^1 \int_{\mathbb{R}^2} p_{\frac{t}{2}}(x-y)W(dy,dt)$ where $p_t(x-y) = \frac{1}{2\pi t}e^{-\frac{|x-y|^2}{2t}}$. Consider a conformal map F between two bounded, convex, simply-connected open sets U and V such that $|F'| \geq 1$ on U, $\|F'\|_U < \infty$ and $\|F''\|_U < \infty$. (We point out here that the assumption $|F'| \geq 1$ will be obtained later on by starting from a very small domain; this is exactly the content of Lemma 3.11.) We consider another field $\tilde{\phi}_\delta(x) = \int_{\delta^2}^1 \int_{\mathbb{R}^2} p_{\frac{t}{2}}(x-y)\tilde{W}(dy,dt)$ where $\tilde{W}$ is a white noise that we will couple with W in order to compare ϕ_δ and $\tilde{\phi}_\delta \circ F$. The coupling goes as follows: for $y \in U$, $t \in (0,\infty)$, let $y' = F(y) \in V$ and $t' = t|F'(y)|^2$ and set $\tilde{W}(dy',dt') = |F'(y)|^2 W(dy,dt)$. That is, for every L^2 function ω on $V \times (0,\infty)$,

$$\int \omega(y',t')\tilde{W}(dy',dt') = \int \omega(F(y), t|F'(y)|^2) \left|F'(y)\right|^2 W(dy,dt)$$

and both sides have variance $\|\omega\|_{L^2}^2$. The rest of the white noises are chosen to be independent, i.e., $W_{|U^c \times (0,\infty)}$, $W_{|U \times (0,\infty)}$ and $\tilde{W}_{|V^c \times (0,\infty)}$ are jointly independent.

Lemma 3.6. *Under this coupling, we can compare the two fields $\tilde{\phi}_\delta(F(x))$ and $\phi_\delta(x)$ on a compact, convex subset K of U as follows,*

$$\tilde{\phi}_\delta(F(x)) - \phi_\delta(x) = \phi_{\mathrm{L}}^{(\delta)}(x) + \phi_{\mathrm{H}}^{(\delta)}(x), \tag{3.31}$$

where $\phi_{\mathrm{L}}^{(\delta)}$ (L for low frequency noise) is a smooth Gaussian field whose L^∞-norm on K has uniform Gaussian tails, and $\phi_{\mathrm{H}}^{(\delta)}$ (H for high frequency noise) is a smooth Gaussian field with uniformly bounded pointwise variance (in δ and $x \in K$). Furthermore, $\phi_{\mathrm{H}}^{(\delta)}$ is independent of $(\phi_\delta, \phi_{\mathrm{L}}^{(\delta)})$.

This aforementioned independence property will be crucial for our argument.

Proof. Step 1: Decomposition. For fixed F and small δ, we decompose $\phi_\delta(x) - \tilde{\phi}_\delta(F(x)) = \phi_1^{(\delta)}(x) + \phi_2^{(\delta)}(x) + \phi_3^{(\delta)}(x)$, where

$$\begin{aligned}
\phi_1^{(\delta)}(x) &= \int_U \int_{\delta^2}^{|F'(y)|^{-2}} \left(p_{\frac{t}{2}}(x-y) - p_{\frac{t}{2}|F'(y)|^2}\left(F(x)-F(y)\right) \left|F'(y)\right|^2 \right) W(dy,dt) \\
&= \int_U \int_{\delta^2}^{|F'(y)|^{-2}} \left(p_{\frac{t}{2}}(x-y) - p_{\frac{t}{2}}\left(\frac{F(x)-F(y)}{F'(y)} \right) \right) W(dy,dt) \\
\phi_2^{(\delta)}(x) &= \int_{U^c} \int_{\delta^2}^1 p_{\frac{t}{2}}(x-y) W(dy,dt) - \int_{V^c} \int_{\delta^2}^1 p_{\frac{t}{2}}\left(F(x)-y\right) \tilde{W}(dy,dt) \\
&\quad + \int_U \int_{|F'(y)|^{-2}}^1 p_{\frac{t}{2}}(x-y) W(dy,dt) \\
\phi_3^{(\delta)}(x) &= -\int_U \int_{\delta^2|F'(y)|^{-2}}^{\delta^2} p_{\frac{t}{2}}\left(\frac{F(x)-F(y)}{F'(y)} \right) W(dy,dt)
\end{aligned}$$

Remark also that $\phi_3^{(\delta)}$ is independent of ϕ_δ, $\phi_1^{(\delta)}$, and $\phi_2^{(\delta)}$.

Step 2: Conclusion, assuming uniform estimates. We will estimate $\phi_i^{(\delta)}$, $i = 1,2,3$, over K. In what follows, we take $x, x' \in K$. We assume first the following uniform estimates:

$$\mathbb{E}((\phi_1^{(\delta)}(x) - \phi_1^{(\delta)}(x'))^2) \le C\left|x-x'\right|, \quad \mathbb{E}((\phi_2^{(\delta)}(x) - \phi_2^{(\delta)}(x'))^2) \le C\left|x-x'\right|, \quad \mathbb{E}\left(\phi_3^{(\delta)}(x)\right) \le C.$$

An application of Kolmogorov's continuity criterion and Fernique's theorem give uniform Gaussian tails for $\phi_1^{(\delta)}$ and $\phi_2^{(\delta)}$. We then set $\phi_{\mathrm{H}}^{(\delta)} := \phi_3^{(\delta)}$ and $\phi_{\mathrm{L}}^{(\delta)} := \phi_1^{(\delta)} + \phi_2^{(\delta)}$.

Step 3: Uniform estimates.

First term. We prove that $\mathbb{E}((\phi_1^{(\delta)}(x)-\phi_1^{(\delta)}(x'))^2) \le C\,|x-x'|$ by controlling

$$\int_0^1\int_U \left(p_{\frac{t}{2}}(x-y) - p_{\frac{t}{2}}\left(\frac{F(x)-F(u)}{F'(y)}\right) - p_{\frac{t}{2}}(x'-y) + p_{\frac{t}{2}}\left(\frac{F(x')-F(u)}{F'(y)}\right)\right)^2 dydt$$

By introducing $p(x) = e^{-\frac{|x|^2}{2}}$ and by a change of variable $t \leftrightarrow 2t^2$, it is equivalent (up to a multiplicative constant) to bound from above the quantity

$$\int_0^1 \frac{dt}{t^3}\int_U \left(p\left(\frac{x-y}{t}\right) - p\left(\frac{F(x)-F(y)}{tF'(y)}\right) - p\left(\frac{x'-y}{t}\right) + p\left(\frac{F(x')-F(y)}{tF'(y)}\right)\right)^2 dy. \tag{3.32}$$

We will estimate this term by considering the case where $t \le \sqrt{|x-x'|}$ and the case where $t \ge \sqrt{|x-x'|}$.

Step 3.(A): Case $t \ge \sqrt{|x-x'|}$. Using the identity $|x-y|^2 + |x'-y|^2 = \frac{1}{2}|x-x'|^2 + 2|y - \frac{x+x'}{2}|^2$ and the inequality $1 - e^{-z} \le z$, we get

$$\int_U \left(p\left(\frac{x-y}{t}\right) - p\left(\frac{x'-y}{t}\right)\right)^2 dy \le Ct^2(1 - e^{-\frac{|x-x'|^2}{4t^2}}) \le C\left|x-x'\right|^2. \tag{3.33}$$

Similarly,

$$\int_U \left(p\left(\frac{F(x)-F(y)}{tF'(y)}\right) - p\left(\frac{F(x')-F(y)}{tF'(y)}\right)\right)^2 dy \le C\left|F(x)-F(x')\right|^2 \le C\left|x-x'\right|^2, \tag{3.34}$$

where the constant C depends on $\|F'\|_U$. Then the corresponding part in (3.32) is bounded from above by $|x-x'|^2\int_{\sqrt{|x-x'|}}^1 \frac{dt}{t^3} \le C|x-x'|$.

Step 3.(B): For $t \le \sqrt{|x-x'|}$, using the Taylor inequality $|F(x)-F(y)-F'(y)(x-y)| \le \frac{1}{2}\|F''\|_U\,|x-y|^2$ and the mean value inequality (as we have assumed that K is convex),

$$\begin{aligned}&\left|p\left(\frac{x-y}{t}\right) - p\left(\frac{F(x)-F(y)}{tF'(y)}\right)\right| \\ &\qquad \le C\frac{|x-y|^2}{t}\left(\frac{|x-y|}{t} + \frac{|x-y|^2}{t}\right)e^{-\frac{1}{2t^2}\inf_{\alpha\in(0,1)}\left|\alpha(x-y)+(1-\alpha)\frac{F(x)-F(y)}{F'(y)}\right|^2}.\end{aligned} \tag{3.35}$$

Step 3.(B): case (a). If $y \in B(x,\varepsilon)$ for ε small enough (depending only on $\|F''\|_U$), we have, using again $|F(x)-F(y)-F'(y)(x-y)| \le \frac{1}{2}\|F''\|_U |x-y|^2$, uniformly in $\alpha \in (0,1)$,

$$\left|\alpha(x-y)+(1-\alpha)\frac{F(x)-F(y)}{F'(y)}\right| \ge |x-y| - \frac{1}{2}\|F''\|_U |x-y|^2 \ge \frac{1}{2}|x-y|.$$

Therefore, for such y's we have, coming back to (3.35),

$$\left|p\left(\frac{x-y}{t}\right)-p\left(\frac{F(x)-F(y)}{tF'(y)}\right)\right| \le C\frac{|x-y|^3}{t^2}e^{-\frac{|x-y|^2}{4t^2}}.$$

For this case we get the bound

$$\int_{B(x,\varepsilon)}\left(p\left(\frac{x-y}{t}\right)-p\left(\frac{F(x)-F(y)}{tF'(y)}\right)\right)^2 dy \le C\int_{B(x,\varepsilon)}\frac{|x-y|^6}{t^4}e^{-\frac{|x-y|^2}{2t^2}} = Ct^{-2}\mathbb{E}(|B_{t^2}|^6) \le Ct^4.$$

where B_t denotes a two-dimensional Gaussian variable with covariance matrix t times the identity. This term contributes to (3.32) as $C\int_0^{\sqrt{|x-x'|}}\frac{dt}{t^3}t^4 \le C|x-x'|$.

Step 3.(B): case (b). Now, for $t \le \sqrt{|x-x'|}$ and $y \in U \setminus B(x,\varepsilon)$ we write

$$\int_0^{\sqrt{|x-x'|}}\frac{dt}{t^3}\int_{U\setminus B(x,\varepsilon)} p\left(\frac{x-y}{t}\right)^2 dy$$
$$\le C\int_0^{\sqrt{|x-x'|}}\frac{dt}{t}\mathbb{P}(|B_{t^2}|>\varepsilon) \le C\int_0^{\sqrt{|x-x'|}}\frac{dt}{t}e^{-\frac{\varepsilon^2}{2t^2}} \le C\left|x-x'\right|,$$

and similarly

$$\int_0^{\sqrt{|x-x'|}}\frac{dt}{t^3}\int_{U\setminus B(x,\varepsilon)} p\left(\frac{F(x)-F(y)}{tF'(y)}\right)^2 dy \le C\left|x-x'\right|,$$

where the constant C depends on $\|F'\|_U$ and $\|(F^{-1})'\|_U$.

Applying Step 3.(A) and then Step 3.(B) twice (once for x and then again for x') to (3.32), we get $\mathbb{E}((\phi_1^{(\delta)}(x)-\phi_1^{(\delta)}(x'))^2) \le C|x-x'|$.

Second term. We want to prove here that $\mathbb{E}((\phi_2^{(\delta)}(x)-\phi_2^{(\delta)}(x'))^2) \le C|x-x'|$. Note that three terms contribute to $\delta\phi_2$. The third one is a nice Gaussian field independent of δ. The first two terms are similar, so we will just focus on the first one, namely $\phi_{2,1}^{(\delta)}(x) := \int_{U^c}\int_{\delta^2}^1 p_{\frac{t}{2}}(x-y)W(dy,dt)$.

We have, similarly to (3.32) and (3.33),

$$\mathbb{E}\left(\left(\phi_{2,1}^{(\delta)}(x)-\phi_{2,1}^{(\delta)}(x')\right)^2\right)=\int_{\delta^2}^{1}\int_{U^c}\left(p_{\frac{t}{2}}(x-y)-p_{\frac{t}{2}}(x'-y)\right)^2 dydt$$
$$\le C\int_0^1\frac{dt}{t^3}\int_{U^c}\left(p\left(\frac{x-y}{t}\right)-p\left(\frac{x'-y}{t}\right)\right)^2 dy$$
$$\le C\int_0^{\sqrt{|x-x'|}}\frac{dt}{t^3}\int_{U^c}p\left(\frac{x-y}{t}\right)+p\left(\frac{x'-y}{t}\right)dy+C|x-x'|.$$

The remaining term can be controlled as follows (noting the symmetry between x and x'):

$$\int_0^{\sqrt{|x-x'|}}\frac{dt}{t}\int_{U^c}\frac{1}{t^2}e^{-\frac{|x-y|^2}{2t^2}}dy\le C\int_0^{\sqrt{|x-x'|}}\frac{dt}{t}\mathbb{P}(|B_{t^2}|>d)\le C\int_0^{\sqrt{|x-x'|}}\frac{dt}{t}e^{-\frac{d^2}{2t^2}}\le C\left|x-x'\right|.$$

where $d=d(K,U^c)$. Thus $\mathbb{E}((\phi_2^{(\delta)}(x)-\phi_2^{(\delta)}(x'))^2)\le C\,|x-x'|$.

Third term. We give here a bound on the pointwise variance of $\phi_3^{(\delta)}$. By using $\left|\frac{F(x)-F(y)}{F'(y)}\right|\ge\frac{|x-y|}{C}$ we get $\mathbb{E}(\phi_3^{(\delta)}(x)^2)\le\int_{c\delta^2}^{\delta^2}\frac{dt}{t}\int_{\mathbb{R}^2}\frac{e^{-\frac{|x-y|^2}{Ct}}}{t}dy\le C$. □

3.3.2 Russo-Seymour-Welsh estimates

The main result of this section is the following RSW estimate. It shows that appropriately-chosen quantiles of crossing distances of "long" and "short" rectangles at the same scale can be related by a multiplicative factor that is uniform in the scale. This is the equivalent of Theorem 3.1 from [38] but with the field mollified by the heat kernel instead of a compactly-supported kernel. It holds for any fixed $\xi>0$.

Proposition 3.7 (RSW estimates for ϕ_δ)**.** *If $[A,B]\subset(0,\infty)$, there exists $C>0$ such that for (a,b), $(a',b')\in[A,B]$ with $\frac{a}{b}<1<\frac{a'}{b'}$, for $n\ge0$ and $\varepsilon<1/2$, we have,*

$$\ell_{a',b'}^{(n)}(\phi,\varepsilon/C)\le C\ell_{a,b}^{(n)}(\phi,\varepsilon)e^{C\sqrt{\log|\varepsilon/C|}};\tag{3.36}$$

$$\bar{\ell}_{a',b'}^{(n)}(\phi,3\varepsilon^C)\le C\bar{\ell}_{a,b}^{(n)}(\phi,\varepsilon)e^{C\sqrt{\log|\varepsilon/C|}}.\tag{3.37}$$

The following corollary then follows from Propositions 3.5 and 3.7.

Corollary 3.8 (RSW estimates for ψ_δ)**.** *Under the same assumptions as used in Proposition 3.7,* *we have*

$$\ell^{(n)}_{a',b'}(\psi,\varepsilon/C) \le C\ell^{(n)}_{a,b}(\psi,\varepsilon)e^{C\sqrt{\log|\varepsilon/C|}} \tag{3.38}$$

and

$$\bar{\ell}^{(n)}_{a',b'}(\psi,3\varepsilon^C) \le C\bar{\ell}^{(n)}_{a,b}(\psi,\varepsilon)e^{C\sqrt{\log|\varepsilon/C|}}. \tag{3.39}$$

We point out that the constants C in (3.38) and (3.39) are not equal to those in (3.36) and (3.37). The remaining parts of the section will only deal with approximations associated with ϕ so we will omit this dependence in the various observables.

We describe below the main lines of the argument. Consider $R_{a,b}$ and $R_{a',b'}$, two rectangles with respective side lengths (a,b) and (a',b') satisfying $\frac{a}{b} < 1 < \frac{a'}{b'}$. Suppose that we could take a conformal map $F : R_{a,b} \to R_{a',b'}$ mapping the long left and right sides of $R_{a,b}$ to the short left and right sides of $R_{a',b'}$. (This is not in fact possible since there are only three degrees of freedom in the choice of a conformal map, but for the sake of illustration we will consider this idealized setting first.) Then the proof goes as follows.

Take a geodesic $\tilde{\pi}$ for $\tilde{\phi}_{0,n}$ for the left-right crossing of $R_{a,b}$. Then, using the coupling (3.31), we have

$$\begin{aligned} L^{\phi_{0,n}}(R_{a',b'}) \le L^{\phi_{0,n}}(F(\tilde{\pi})) &= \int_0^T e^{\xi\phi_{0,n}(F(\tilde{\pi}(t)))}|F'(\tilde{\pi}(t))|\cdot|\tilde{\pi}'(t)|dt \\ &\le \|F'\|_{R_{a,b}} \int_{\tilde{\pi}} e^{\xi(\tilde{\phi}_{0,n}+\delta\phi_{\mathrm{L}}+\delta\phi_{\mathrm{H}})}ds \\ &\le \|F'\|_{R_{a,b}} e^{\xi\|\delta\phi_{\mathrm{L}}\|_{R_{a,b}}} \int_{\tilde{\pi}} e^{\xi\tilde{\phi}_{0,n}}e^{\xi\delta\phi_{\mathrm{H}}}ds. \end{aligned}$$

It is essential that $\tilde{\pi}$ is $\tilde{\phi}_{0,n}$ measurable and $\tilde{\phi}_{0,n}$ is independent of $\delta\phi_{\mathrm{H}}$. Then, we can use the following lemma.

Lemma 3.9. *If Γ is a continuous field and Ψ is an independent continuous centered Gaussian field with pointwise variance bounded above by $\sigma^2 > 0$, then we have, as long as ε is sufficiently small compared to σ^2,*

1. $\ell_{1,1}(\Gamma+\Psi,\varepsilon) \le e^{\sqrt{2\sigma^2\log\varepsilon^{-1}}}\ell_{1,1}(\Gamma,2\varepsilon)$;
2. $\bar{\ell}_{1,1}(\Gamma+\Psi,2\varepsilon) \le e^{\sqrt{2\sigma^2\log\varepsilon^{-1}}}\bar{\ell}_{1,1}(\Gamma,\varepsilon)$.

Proof. Fix $s := \sqrt{2\sigma^2 \log \varepsilon^{-1}}$ throughout the proof. Let $\pi(\Gamma)$ be a geodesic associated with the left–right crossing length for the field Γ, and define the measure μ on $\pi(\Gamma)$ by $\mu(ds) = L_{1,1}(\Gamma)^{-1} e^{\Gamma} ds$, so $\int_{\pi(\Gamma)} e^{\Gamma} ds = 1$. Conditionally on Γ, using Jensen's inequality with $\alpha = \frac{s}{2\sigma^2} = \sqrt{(\log \varepsilon^{-1})/(2\sigma^2)}$, which is greater than 1 for small enough ε, and Chebyshev's inequality, we have

$$\mathbb{P}\left(\int_{\pi(\Gamma)} e^{\Gamma+\Psi} ds > e^s L_{1,1}(\Gamma) \mid \Gamma \right) \leq \mathbb{P}\left(\int_{\pi(\Gamma)} e^{\alpha\Psi} d\mu \geq e^{\alpha s} \mid \Gamma \right) \leq e^{\frac{1}{2}\alpha^2\sigma^2} e^{-\alpha s} = e^{-\frac{s^2}{2\sigma^2}} = \varepsilon. \tag{3.40}$$

To bound from above $L_{1,1}(\Gamma + \Psi)$, we take a geodesic for Γ and use the moment estimate (3.40). We start with the left tail. Still with $s := \sqrt{2\sigma^2 \log \varepsilon^{-1}}$, we have

$$\begin{aligned}
\mathbb{P}\left(L_{1,1}(\Gamma) \leq \ell_{1,1}(\Gamma+\Psi,\varepsilon)e^{-s}\right) &\leq \mathbb{P}\left(L_{1,1}(\Gamma+\Psi) \leq e^s L_{1,1}(\Gamma), L_{1,1}(\Gamma) \leq \ell_{1,1}(\Gamma+\Psi,\varepsilon)e^{-s}\right) \\
&\quad + \mathbb{P}\left(L_{1,1}(\Gamma+\Psi) > e^s L_{1,1}(\Gamma)\right) \\
&\leq \mathbb{P}\left(L_{1,1}(\Gamma+\Psi) \leq \ell_{1,1}(\Gamma+\Psi,\varepsilon)\right) + \mathbb{P}\left(\int_{\pi(\Gamma)} e^{\Gamma+\Psi} ds > e^s L_{1,1}(\Gamma) \right)
\end{aligned}$$

which is bounded from above by 2ε. For the right tail, we have similarly that

$$\begin{aligned}
&\mathbb{P}\left(L_{1,1}(\Gamma+\Psi) \geq \bar{\ell}_{1,1}(\Gamma,\varepsilon)e^s\right) \\
&\quad \leq \mathbb{P}\left(L_{1,1}(\Gamma+\Psi) \geq \bar{\ell}_{1,1}(\Gamma,\varepsilon)e^s, \bar{\ell}_{1,1}(\Gamma,\varepsilon) \geq L_{1,1}(\Gamma)\right) + \mathbb{P}\left(L_{1,1}(\Gamma) \geq \bar{\ell}_{1,1}(\Gamma,\varepsilon)\right) \\
&\quad \leq \mathbb{P}\left(L_{1,1}(\Gamma+\Psi) \geq e^s L_{1,1}(\Gamma)\right) + \varepsilon \leq 2\varepsilon,
\end{aligned}$$

which concludes the proof of the lemma. □

The previous reasoning does not apply directly to rectangle crossing lengths but provides the following proposition. Recall that K is a compact subset of U. Let A, B be two boundary arcs of K and denote by L the distance from A to B in K for the metric $e^{\xi\phi_{0,n}} ds$; we denote $A' := F(A)$, $B' := F(B)$, $K' := F(K)$, and L' is the distance from A' to B' in K' for $e^{\xi\tilde{\phi}_{0,n}} ds$. Recall that we have $|F'| \geq 1$ on U. In the application we will achieve this by scaling U to be sufficiently small.

Proposition 3.10. *We have the following comparisons between quantiles. There exists $C > 0$ such that*

1. *if for some $l > 0$ and $\varepsilon < 1/2$, $\mathbb{P}(L \leq l) \geq \varepsilon$, then $\mathbb{P}(L' \leq l') \geq \varepsilon/4$ with $l' = \|F'\|_K e^{C\sqrt{|\log \varepsilon/2C|}}$.*

2. *if for some* $l > 0$ *and* $\varepsilon < 1/2$, $\mathbb{P}(L \leq l) \geq 1 - \varepsilon$, *then* $\mathbb{P}(L' \leq l') \geq 1 - 3\varepsilon$ *with* $l' = \|F'\|_K e^{C\sqrt{|\log \varepsilon/2C|}}$.

Now, we want to prove a similar result for rectangle crossing lengths. We will need the three following lemmas that were used in [38]. The first one is a geometrical construction, the second one is a complex analysis result and the last one comes essentially from [89] together with an approximation argument. In these lemmas, by "crossings" we mean continuous path from marked sides to marked sides.

Lemma 3.11 (Lemma 4.8 of [38])**.** *If* a *and* b *are two positive real numbers with* $a < b$, *there exists* $j = j(b/a)$ *and* j *rectangles isometric to* $[0, a/2] \times [0, b/2]$ *such that if* π *is a left-right crossing of the rectangle* $[0, a] \times [0, b]$, *at least one of the* j *rectangles is crossed in the thin direction by a subpath of that crossing.*

Lemma 3.12 (Step 1 in the proof of Theorem 3.1 in [38])**.** *If* $a/b < 1$ *and* $a'/b' > 1$, *there exists* $m, p \geq 1$ *and two ellipses* E_p, E' *with marked arcs* (AB), (CD) *for* E_p *and* $(A'B')$, $(C'D')$ *for* E' *such that:*

1. *Any left-right crossing of* $[0, a/2^p] \times [0, b/2^p]$ *is a crossing of* E_p.
2. *Any crossing of* E' *is a left-right crossing of* $[0, a'] \times [0, b']$.
3. *When dividing the marked sides of* E_p *into* m *subarcs of equal length, for any pair of such subarcs (one on each side), there exists a conformal map* $F : E_p \to E'$ *and the pair of subarcs is mapped to subarcs of the marked sides of* E'.
4. *For each pair, the associated map* F *extends to a conformal equivalence* $U \to V$ *where* $\overline{E_p} \subset U$, $\overline{E'} \subset V$ *and* $|F'| \geq 1$ *on* U.

We refer the reader to Figure 3.1 for an illustration.

Lemma 3.13 (Positive association and square-root-trick)**.** *If* $k \geq 2$ *and* $(R_1, \ldots, R_k)$ *denote a collection of* k *rectangles, then, for* $(x_1, \ldots, x_k) \in (0, \infty)^k$, *we have*

$$\mathbb{P}\left(L^{(n)}(R_1) > x_1, \ldots, L^{(n)}(R_k) > x_k\right) \geq \mathbb{P}\left(L^{(n)}(R_1) > x_1\right) \cdots \mathbb{P}\left(L^{(n)}(R_k) > x_k\right).$$

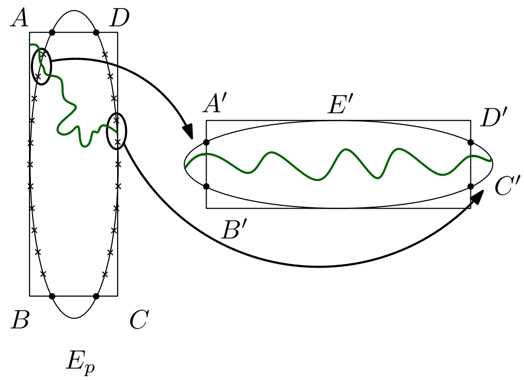

Figure 3.1 – Illustration of Lemma 3.12.

An easy consequence of this positive association is the so-called "square-root-trick":

$$\max_{i\le k}\mathbb{P}\left(L^{(n)}(R_i)\le x_i\right)\ge 1-\left(1-\mathbb{P}\left(\exists i\le k : L^{(n)}(R_i)\le x_i\right)\right)^{1/k}.$$

The main result of this section, Proposition 3.7, is a rephrasing of the following one.

Proposition 3.14. *We have the following comparisons between quantiles. If $a/b<1$ and $a'/b'>1$, there exists $C>0$ such that, for any $\varepsilon\in(0,1/2)$,*

1. *if* $\mathbb{P}\left(L^{(n)}_{a,b}\le l\right)\ge\varepsilon$, *then* $\mathbb{P}\left(L^{(n)}_{a',b'}\le Cle^{C\sqrt{|\log\varepsilon/C|}}\right)\ge\varepsilon/C$,
2. *and if* $\mathbb{P}\left(L^{(n)}_{a,b}\le l\right)\ge 1-\varepsilon$, *then* $\mathbb{P}\left(L^{(n)}_{a',b'}\le Cle^{C\sqrt{|\log\varepsilon/C|}}\right)\ge 1-3\varepsilon^{1/C}$.

Proof. We provide first a comparison between low quantiles and then a comparison between high quantiles.

Step 1: Comparison of small quantiles. Suppose $\mathbb{P}(L^{(n)}_{a,b}\le l)\ge\varepsilon$. By Lemma 3.11 and union bound, $\mathbb{P}(L^{(n)}_{a/2,b/2}\le l)\ge\varepsilon/j$. Furthermore, by iterating, we have $\mathbb{P}(L^{(n)}_{a/2^p,b/2^p}\le l)\ge\varepsilon/j^p$. Under this event, by Lemma 3.12, there exists a crossing of E_p between two subarcs of E_p (one on each side) hence with probability at least $\varepsilon/(j^pm^2)$, one of these crossings has length at most l. By the left tail estimate Proposition 3.10 and Lemma 3.12, we obtain a $C>0$ (depending also on $\|F'\|_{\overline{E_p}}$) such that for all $\varepsilon,l>0$:

$$\mathbb{P}\left(L^{(n)}_{a,b}\le l\right)\ge\varepsilon\Rightarrow\mathbb{P}\left(L^{(n)}_{a',b'}\le Cle^{C\sqrt{|\log\varepsilon/(2Cj^pm^2)|}}\right)\ge\varepsilon/(4j^pm^2),$$

hence the first assertion.

Step 2: Comparison of high quantiles. Now suppose $\mathbb{P}(L^{(n)}_{a,b} \le l) \ge 1 - \varepsilon$. By Lemma 3.11 (to start with a crossing at a lower scale) and Lemma 3.13 (square-root-trick), we have $\mathbb{P}(L^{(n)}_{a/2,b/2} \le l) \ge 1 - \varepsilon^{1/j}$. Furthermore, by iterating, we have $\mathbb{P}(L^{(n)}_{a/2^p,b/2^p} \le l) \ge 1 - \varepsilon^{1/j^p}$. On the event $\{L^{(n)}_{a/2^p,b/2^p} \le l\}$, the ellipse E_p from Lemma 3.12 has a crossing of length $\le l$ between two marked arcs. Again by subdividing each its marked arcs into m subarcs and applying the square-root trick, we see that for at least one pair of subarcs, there is a crossing of length $\le l$ with probability $\ge 1 - \varepsilon^{j^{-p}m^{-2}}$. Combining with the right-tail estimate Proposition 3.10 and Lemma 3.12, we get:

$$\mathbb{P}\left(L^{(n)}_{a,b} \le l\right) \ge 1 - \varepsilon \Rightarrow \mathbb{P}\left(L^{(n)}_{a',b'} \le Cle^{C\sqrt{|\log \varepsilon/C|}}\right) \ge 1 - 3\varepsilon^{1/C}, \tag{3.41}$$

which completes the proof. □

Remark 3.15. *The importance of the Russo-Seymour-Welsh estimates comes from the following: percolation arguments/estimates work well when taking small quantiles associated with short crossings and high quantiles associated with long crossings. Thanks to the RSW estimates, we can instead keep track only of low and high quantiles associated to the unit square crossing, $\ell_n(p)$ and $\bar{\ell}_n(p)$.*

3.4 Tail estimates with respect to fixed quantiles

Lower tails. This is where we take r_0 small enough (recall the definition (2.18)) to obtain some small range of dependence of the field ψ so that a Fernique-type argument works.

Proposition 3.16 (Lower tail estimates for ψ)**.** *We have the following lower tail estimate: for p small enough, but fixed, there is a constant C so that for all $s > 0$,*

$$\mathbb{P}\left(L^{(n)}_{1,3}(\psi) \le e^{-s}\ell_n(\psi, p)\right) \le Ce^{-cs^2}. \tag{4.42}$$

Proof. The RSW estimate (3.38) gives

$$\mathbb{P}\left(L^{(n)}_{3,3}(\psi) \le l\right) \le \varepsilon \Rightarrow \mathbb{P}\left(L^{(n)}_{1,3}(\psi) \le lC^{-1}e^{-C\xi\sqrt{|\log C\varepsilon|}}\right) \le C\varepsilon \tag{4.43}$$

Now, if $L_{3,3}^{(n)}(\psi)$ is less than l, then both $[0,1] \times [0,3]$ and $[2,3] \times [0,3]$ have a left-right crossing of length $\leq l$ and the restrictions of the field to these two rectangles are independent (if r_0 defined in (2.18) is small enough). Consequently,

$$\mathbb{P}\left(L_{3,3}^{(n)}(\psi) \leq l\right) \leq \mathbb{P}\left(L_{1,3}^{(n)}(\psi) \leq l\right)^2 \tag{4.44}$$

Take p_0 small, such that $C^2 p_0 < 1$ where C is the constant in (4.43) and set $r_0^{(n)} := \ell_{3,3}^{(n)}(\psi, p_0)$. (This is not related to r_0, defined previously.) For $i \geq 0$, set

$$p_{i+1} := (Cp_i)^2 \tag{4.45}$$

$$r_{i+1}^{(n)} := r_i^{(n)} C^{-1} \exp(-C\xi\sqrt{|\log(Cp_i)|}) \tag{4.46}$$

By induction we get, for $i \geq 0$,

$$\mathbb{P}(L_{3,3}^{(n)}(\psi) \leq r_i^{(n)}) \leq p_i \tag{4.47}$$

Indeed, the case $i = 0$ follows by definition and then notice that the RSW estimate (4.43) under the induction hypothesis implies that $\mathbb{P}(L_{3,3}^{(n)}(\psi) \leq r_i^{(n)}) \leq p_i \Rightarrow \mathbb{P}(L_{1,3}^{(n)}(\psi) \leq r_{i+1}^{(n)}) \leq Cp_i$ which gives, using (4.44), $\mathbb{P}(L_{3,3}^{(n)}(\psi) \leq r_{i+1}^{(n)}) \leq \mathbb{P}(L_{1,3}^{(n)}(\psi) \leq r_{i+1}^{(n)})^2 \leq (Cp_i)^2 = p_{i+1}$.

From (4.45) we get $p_i = (p_0 C^2)^{2^i} C^{-2}$ and from (4.46) we have the lower bound, for $i \geq 1$,

$$r_i^{(n)} \geq \ell_{3,3}^{(n)}(\psi, p_0) C^{-i} e^{-C\xi \sum_{k=0}^{i-1} \sqrt{|\log(Cp_k)|}} \geq \ell_{3,3}^{(n)}(\psi, p_0) e^{-Ci} e^{-C\xi\sqrt{|\log p_0 C^2|} 2^{i/2}}.$$

Our estimate (4.47) then takes the form, for $i \geq 0$,

$$\mathbb{P}\left(L_{3,3}^{(n)}(\psi) \leq \ell_{3,3}^{(n)}(\psi, p_0) e^{-Ci} e^{-\xi C\sqrt{|\log p_0 C^2|} 2^{i/2}}\right) \leq \left(p_0 C^2\right)^{2^i} C^{-2}.$$

This can be rewritten, taking $i = \lfloor 2 \log_2 s \rfloor$, as

$$\mathbb{P}\left(L_{3,3}^{(n)}(\psi) \leq \ell_{3,3}^{(n)}(\psi, p_0) C^{-1} e^{-C\log s} e^{-\xi s}\right) \leq e^{-cs^2}$$

for $s > 2$ with absolute constants. We obtain the statement of the proposition by using again the RSW estimates. $\square$

Using the comparison result between ϕ and ψ (Proposition 3.5), we get the following corollary.

Corollary 3.17 (Lower tail estimates for ϕ)**.** *For p small enough, but fixed, for all $s > 0$ we have a constant $C < \infty$ so that*

$$\mathbb{P}\left(L_{1,3}^{(n)}(\phi) \leq e^{-s}\ell_n(\phi, p)\right) \leq Ce^{-cs^2}. \tag{4.48}$$

Upper tails. The proof for the upper tails is similar to the one of Proposition 5.3 in [38]. The main difference is that we have to switch between ϕ and ψ, so that we can use the independence properties of ψ together with the scaling properties of ϕ. Before stating the proposition, we refer the reader to (2.27) for the definition of $\Lambda_n(\phi, p)$. In constract with the lower tails estimates which are relative to $\ell_n(\phi, p)$, we do not know how to prove (at least a priori) the analogous result for the upper tails with $\bar{\ell}_n(\phi, p)$ only. However, we can prove it by replacing $\bar{\ell}_n(\phi, p)$ by $\Lambda_n(\phi, p)\ell_n(\phi, p)$ and this is the content of the following proposition.

Proposition 3.18 (Upper tail estimates for ϕ)**.** *For p small enough, but fixed, we have a constant $C < \infty$ so that for all $n \geq 0$ and $s > 2$,*

$$\mathbb{P}\left(L_{3,1}^{(n)}(\phi) \geq e^{s}\Lambda_n(\phi, p)\ell_n(\phi, p)\right) \leq Ce^{c\frac{s^2}{\log s}}. \tag{4.49}$$

Proof. The proof uses percolation and scaling arguments. A percolation argument is used to build a crossing of a larger rectangle from smaller annular circuits, and then a scaling argument is used to relate quantiles of these annular crossings to crossing quantiles of the larger rectangle.

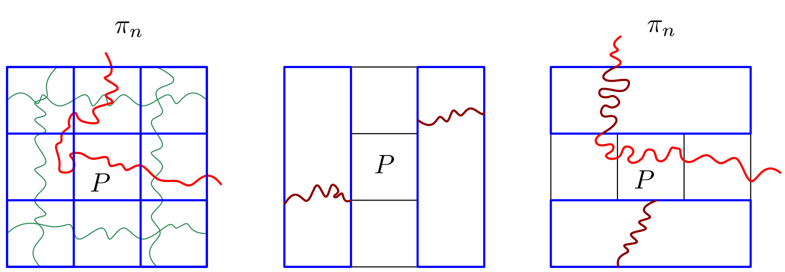

Figure 3.2 – Four blue rectangles are surrounding the square P. Left-right geodesics associated to the long and short rectangles surrounding P are drawn in green and brown respectively. Any geodesic π_n, here in red, which intersects P has to cross the green circuit and to induce a short crossing of one of the four rectangles.

Step 1: Percolation argument. To each unit square P of $\mathbb{Z}^2$, we associate the four crossings of long rectangles of size $(3, 1)$ surrounding P, each comprising three squares on one side of the eight-square annulus surrounding P, as illustrated in Figure 3.2. We define $S^{(n)}(P,\psi)$ to be the sum of the four crossing lengths, and declare the site P to be open when the event $\{S^{(n)}(\psi,P)\le 4\bar{\ell}^{(n)}_{3,1}(\psi,p)\}$ occurs. This occurs with probability at least $1-\varepsilon(p)$, where $\varepsilon(p)$ goes to zero as p goes to zero (recall that $\mathbb{P}(L^{(n)}_{3,1}(\psi)\le \bar{\ell}^{(n)}_{3,1}(p))=1-p$). Using a highly supercritical finite-range site percolation estimate to obtain exponential decay of the probability of a left–right crossing (which is standard technique in classical percolation theory [41]; see also for example the proof of Proposition 4.2 in [26]) together with the Russo-Seymour-Welsh estimates (to come back to $\bar{\ell}_n(\psi,p)$), we have

$$\mathbb{P}\left(L^{(n)}_{3k,k}(\psi)\ge Ck^2\bar{\ell}_n(\psi,p)\right)\le Ce^{-ck}.$$

Therefore, using this bound together with Proposition 3.5 to bound $X_{3k,k}$ (recalling the definition (2.25)),

$$\begin{aligned}\mathbb{P}\left(L^{(n)}_{3k,k}(\phi)\ge e^{\xi C\sqrt{k}}C_pCk^2\bar{\ell}_n(\phi,p/2)\right)&\le \mathbb{P}\left(e^{\xi X_{3k,k}}L^{(n)}_{3k,k}(\psi)\ge e^{\xi C\sqrt{k}}C_pCk^2\bar{\ell}_n(\phi,p/2)\right)\\ &\le \mathbb{P}\left(X_{3k,k}\ge C\sqrt{k}\right)+\mathbb{P}\left(L^{(n)}_{3k,k}(\psi)\ge C_pCk^2\bar{\ell}_n(\phi,p/2)\right)\\ &\le Ce^{-ck}+\mathbb{P}\left(L^{(n)}_{3k,k}(\psi)\ge Ck^2\bar{\ell}_n(\psi,p)\right)\le Ce^{-ck}.\end{aligned}$$

Note that we used the bound $\bar{\ell}_n(\psi,p)\le C_p\bar{\ell}_n(\phi,p/2)$ from (2.29) in the third inequality; here C_p is defined as in (2.29).

Step 2: Decoupling and scaling. In this step, we give a rough bound of the coarse field $\phi_{0,m}$, to obtain spatial independence of the remaining field between blocks of size 2^{-m}. When an event occurs on one block with high enough probability, the percolation argument of Step 1 then provides, with very high probability, a left-right path of such events occuring simultaneously. Since $L^{(n)}_{3,1}(\phi)\le e^{\xi\max_{R_{3,1}}\phi_{0,m}}L^{(m,n)}_{3,1}(\phi)$, the scaling property of the field ϕ, i.e. $L^{(m,n)}_{3,1}(\phi)\overset{(d)}{=}2^{-m}L^{(n-m)}_{3\cdot 2^m,2^m}(\phi)$, gives

$$\begin{aligned}&\mathbb{P}\left(L^{(n)}_{3,1}(\phi)\ge e^{\xi s\sqrt{m}}e^{c\sqrt{2^m}}\bar{\ell}_{n-m}(\phi,p)\right)\\ &\quad\le \mathbb{P}\left(\max_{R_{3,1}}\phi_{0,m}\ge Cm+s\sqrt{m}\right)+\mathbb{P}\left(2^{-m}L^{(n-m)}_{3\cdot 2^m,2^m}(\phi)\ge e^{c\sqrt{2^m}}\bar{\ell}_{n-m}(\phi,p)\right)\le Ce^{-cs^2}+Ce^{-c2^m},\end{aligned}$$

where the first term of the second expression is bounded by taking $a=C+sm^{-1/2}$ in Proposition 3.2 and the second bound follows from the result obtained in Step 1 with $k=2^m$, taking a slightly larger c in $\exp(c\sqrt{2^m})$ to absorb the factor e^{Cm}.

Step 3: We derive an a priori bound $\ell_n(\phi,p) \geq 2^{-2\xi k}\ell_{n-k}(\phi,p)e^{-C\sqrt{k}}$. (Note that the argument below will be optimized in (5.80).) For each dyadic block of size 2^{-k} visited by $\pi_n(\phi)$, one of the four rectangles of size $2^{-k}(1,3)$ around P has to be crossed by $\pi_n(\phi)$. Therefore, since $\pi_n(\phi)$ has to visit at least 2^k dyadic blocks of size 2^{-k}, we have

$$L_{1,1}^{(n)}(\phi) \geq 2^k e^{\xi \inf_{[0,1]^2} \phi_{0,k}} \min_{P\in\mathcal{P}_k, P\cap\pi_n(\phi)\neq\emptyset} \min_{1\leq i\leq 4} L^{(k,n)}(R_i^S(P),\phi),$$

where $(R_i^S(P))_{1\leq i\leq 4}$ denote the four long rectangles of size $2^{-k}(1,3)$ surrounding P. Using the supremum tail estimate (2.10) and the left tail estimates (4.48), we get $\ell_n(\phi,p) \geq 2^{-2\xi k}\ell_{n-k}(\phi,p)e^{-C\sqrt{k}}$. Indeed,

$$\mathbb{P}\left(e^{\xi \inf_{[0,1]^2} \phi_{0,k}} \min_{P\in\mathcal{P}_k, P\cap\pi_n(\phi)\neq\emptyset} \min_{1\leq i\leq 4} 2^k L^{(k,n)}(R_i^S(P),\phi) \leq 2^{-2\xi k}\ell_{n-k}(\phi,p)e^{-C\sqrt{k}}\right) \leq$$
$$\mathbb{P}(\inf_{[0,1]^2} \phi_{0,k} \leq -k\log 4 - C\sqrt{k}) + \mathbb{P}\left(\min_{P\in\mathcal{P}_k, P\cap\pi_n(\phi)\neq\emptyset} \min_{1\leq i\leq 4} 2^k L^{(k,n)}(R_i^S(P),\phi) \leq \ell_{n-k}(\phi,p)e^{-C\sqrt{k}}\right)$$

and each term is less than $p/2$ if C is large enough, depending on p. Therefore, we have

$$\bar{\ell}_{n-m}(\phi,p) \leq \Lambda_{n-m}(\phi,p)\ell_{n-m}(\phi,p) \leq 2^{2\xi m}e^{C\sqrt{m}}\Lambda_{n-m}(\phi,p)\ell_n(\phi,p).$$

Now, by coming back to the partial result obtained in Step 2 and by taking $s^2 = 2^m$ for $s \in [1, 2^{n/2}]$, we get

$$\mathbb{P}\left(L_{3,1}^{(n)}(\phi) \geq e^{cs\sqrt{\log s}}e^{cs}\Lambda_n(\phi,p)\ell_n(\phi,p)\right) \leq e^{-cs^2}.$$

Step 4: Now we consider large tails, so we assume $s \geq 2^{\frac{n}{2}}$. By a direct comparison with the supremum, we have $\ell_n(\phi,p) \geq 2^{-\xi(2n+C\sqrt{n})}$ (later on we will use a more precise estimate from [28], see (5.54)). Moreover, bounding from above the left-right distance by taking a straight path from left to right and then using a moment method analogous to the one in (3.40), we get $\mathbb{P}\left(L_{1,1}^{(n)}(\phi) \geq e^{\xi s}\right) \leq e^{-\frac{s^2}{2(n+1)\log 2}}$. Altogether,

$$\mathbb{P}\left(L_{1,1}^{(n)}(\phi) \geq \ell_n(\phi,p)\Lambda_n(\phi,p)e^{\xi s}\right) \leq \mathbb{P}\left(L_{1,1}^{(n)}(\phi) \geq \ell_n(\phi,p)e^{\xi s}\right) \leq e^{-\frac{(s-n\log 4-C\sqrt{n})^2}{2(n+1)\log 2}} \leq e^{Cs}e^{-c\frac{s^2}{\log s}},$$

where we used $\Lambda_n(\phi, p) \geq 1$ in the first inequality and the bound $\ell_n(\phi, p) \geq 2^{-\xi(2n+C\sqrt{n})}$ together with the tail estimate $\mathbb{P}\left(L_{1,1}^{(n)}(\phi) \geq e^{\xi s}\right) \leq e^{-\frac{s^2}{2(n+1)\log 2}}$ in the second one. The last inequality follow since $s \geq 2^{\frac{n}{2}}$.

Combining the tail estimate of Step 3, valid for $s \in [1, 2^{n/2}]$, and the one of Step 4, valid for $s \geq 2^{n/2}$, completes the proof. □

Using again the comparison between ϕ and ψ given in Proposition 3.5, we get the following corollary.

Corollary 3.19 (Upper tail estimates for ψ)**.** *For p small enough, but fixed, we have, for all $n \geq 0$ and $s > 2$,*

$$\mathbb{P}\left(L_{3,1}^{(n)}(\psi) \geq e^s \Lambda_n(\psi, p)\ell_n(\psi, p)\right) \leq Ce^{c\frac{s^2}{\log s}}. \tag{4.50}$$

3.5 Concentration

3.5.1 Concentration of the log of the left-right crossing length

Condition (T). Denote by $\pi_n(\psi)$ the left-right geodesic of the unit square associated to the field $\psi_{0,n}$. If there are multiple such geodesics, let $\pi_n(\psi)$ be chosen among them in some measurable way, for example by taking the uppermost geodesic. By $\pi_n^K(\psi)$ its K-coarse graining which we define as

$$\pi_n^K(\psi) := \{P \in \mathcal{P}_K \ : \ P \cap \pi_n(\psi) \neq \varnothing\}, \tag{5.51}$$

recalling the definition (2.14) of $\mathcal{P}_K$. Let $\psi_{0,n}(P)$ denote the value of the field $\psi_{0,n}$ taken at the center of a block P. We introduce the following condition: there exist constants $\alpha > 1$, $c > 0$ so that for K large we have

$$\sup_{n \geq K} \mathbb{E}\left(\left(\frac{\sum_{P \in \pi_n^K(\psi)} e^{2\xi\psi_{0,K}(P)}}{\left(\sum_{P \in \pi_n^K(\psi)} e^{\xi\psi_{0,K}(P)}\right)^2}\right)^\alpha\right)^{1/\alpha} \leq e^{-cK}. \qquad \text{(Condition (T))}$$

The importance of Condition (T) comes from the following theorem.

Theorem 3.20. *If ξ is such that Condition (T) above is satisfied, then* $(\log L_{1,1}^{(n)}(\phi) - \log \lambda_n(\phi))_{n\geq 0}$ *is tight, where $\lambda_n(\phi)$ denotes the median of $L_{1,1}^{(n)}$.*

It is not expected that the weight is approximately constant over the crossing (since there may be some large level lines of the field that the crossing must cross). Condition (T), however, roughly requires that the length of the crossing is supported by a number of coarse blocks that grows at least like some small but positive power of the total number of coarse blocks. Note that the fraction in Condition (T) is the ℓ^2 norm of the vector of crossing weights on each block divided by the square of the ℓ^1 norm of the same, and thus controlling it amounts to an anticoncentration condition for this vector.

The core of this section is the proof of Theorem 3.20. Before proving it, let us already jump to the important following proposition. Here we use the assumption that $\xi \in (0, 2/d_2)$, although the formulation of Condition (T) is designed so that it could also hold for larger ξ.

Proposition 3.21. *If $\gamma \in (0,2)$, then $\xi := \frac{\gamma}{d_\gamma}$ satisfies Condition (T).*

Proof. Step 1: Supremum bound. Taking the supremum over all blocks of size 2^{-K} in $[0,1]^2$, we get

$$\frac{\sum_{P\in\pi_n^K(\psi)} e^{2\xi\psi_{0,K}(P)}}{\left(\sum_{P\in\pi_n^K(\psi)} e^{\xi\psi_{0,K}(P)}\right)^2} \leq \frac{e^{\xi \max_{P\in\mathcal{P}_K} \psi_{0,K}(P)}}{\sum_{P\in\pi_n^K(\psi)} e^{\xi\psi_{0,K}(P)}} \leq \frac{e^{\xi \max_{P\in\mathcal{P}_K} \phi_{0,K}(P)}}{\sum_{P\in\pi_n^K(\psi)} e^{\xi\psi_{0,K}(P)}} e^{\xi X_1},$$

recalling the definition of X_1 below (2.25).

Step 2: We give a lower bound of the denominator of the right-hand side. By taking the concatenation of straight paths in each box of $\pi_n^K(\psi)$, we get a left-right crossing of $[0,1]^2$. Denote this crossing by $\Gamma_{n,K,\psi}$. We have,

$$\sum_{P\in\pi_n^K(\psi)} e^{\xi\psi_{0,K}(P)} \geq e^{-\xi X_1} \sum_{P\in\pi_n^K(\psi)} e^{\xi\phi_{0,K}(P)} \tag{5.52}$$

$$\geq e^{-\xi X_1} \exp(-\xi \max_{P\in\mathcal{P}_K^1} \mathrm{osc}_P(\phi_{0,K})) 2^K L^{(K)}(\phi, \Gamma_{n,K,\psi}) \geq e^{-\xi X_1} \exp(-\xi \max_{P\in\mathcal{P}_K^1} \mathrm{osc}_P(\phi_{0,K})) 2^K L_{1,1}^{(K)}(\phi),$$

where osc_P was defined in (2.13) and $\mathcal{P}_K^1$ was defined in (2.15).

Step 3: Combining the two previous steps, we have

$$\frac{\sum_{P\in\pi_n^K(\psi)} e^{2\xi\psi_{0,K}(P)}}{\left(\sum_{P\in\pi_n^K(\psi)} e^{\xi\psi_{0,K}(P)}\right)^2} \leq \frac{e^{\xi \max_{P\in\mathcal{P}_K^1} \phi_{0,K}(P)}}{2^K L_{1,1}^{(K)}(\phi)} e^{2\xi X_1} e^{\xi \max_{P\in\mathcal{P}_K^1} \mathrm{osc}_P(\phi_{0,K})}.$$

Now, we take $\alpha > 1$ close to 1. Using Hölder's inequality with $\frac{1}{r}+\frac{1}{s}=1$ and r close to 1, together with Cauchy-Schwarz, we get

$$\begin{aligned}
&\mathbb{E}\left(\left(\frac{\sum_{P\in\pi_n^K(\psi)} e^{2\xi\psi_{0,K}(P)}}{\left(\sum_{P\in\pi_n^K(\psi)} e^{\xi\psi_{0,K}(P)}\right)^2}\right)^\alpha\right)^{1/\alpha} \\
&\leq 2^{-K}\mathbb{E}\left(\frac{e^{\alpha\xi \max_{P\in\mathcal{P}_K} \phi_{0,K}(P)}}{(L_{1,1}^{(K)}(\phi))^\alpha} e^{2\alpha\xi X_1} e^{\alpha\xi \max_{P\in\mathcal{P}_K^1} \mathrm{osc}_P(\phi_{0,K})}\right)^{1/\alpha} \\
&\leq 2^{-K}\mathbb{E}\left(e^{\alpha r\xi \max_{P\in\mathcal{P}_K^1} \phi_{0,K}(P)}\right)^{1/\alpha r} \mathbb{E}\left(\left(L_{1,1}^{(K)}(\phi)\right)^{-2\alpha s}\right)^{1/2\alpha s} \\
&\quad \times \mathbb{E}\left(e^{8\alpha s\xi X_1}\right)^{1/4\alpha s} \mathbb{E}\left(e^{4\alpha s\xi \max_{P\in\mathcal{P}_K^1} \mathrm{osc}_P(\phi_{0,K})}\right)^{1/4\alpha s}.
\end{aligned}$$

Therefore, using (2.11) for the maximum, (4.48) for the left-right crossing, Proposition 3.5 to bound X_1 and (2.17) for the maximum of oscillations, we finally get, when $\alpha r\xi < 2$ (recall that αr can be taken arbitrarily close to 1),

$$\mathbb{E}\left(\left(\frac{\sum_{P\in\pi_n^K(\psi)} e^{2\xi\psi_{0,K}(P)}}{\left(\sum_{P\in\pi_n^K(\psi)} e^{\xi\psi_{0,K}(P)}\right)^2}\right)^\alpha\right)^{1/\alpha} \leq 2^{-K}2^{2\xi K}\ell_{1,1}^{(K)}(\phi,p)^{-1}e^{C\sqrt{K}}. \tag{5.53}$$

Step 4: Lower bound on quantiles. For $\gamma\in(0,2)$, $Q := \frac{2}{\gamma}+\frac{\gamma}{2} > 2$. Using Proposition 3.17 from [28] (circle average LFPP) and Proposition 3.3 from [27] (comparison between ϕ_δ and circle average), we have, if p is fixed and $\varepsilon\in(0,Q-2)$, for K large enough,

$$\ell_{1,1}^{(K)}(\phi,p) \geq 2^{-K(1-\xi Q+\xi\varepsilon)}. \tag{5.54}$$

Step 5: Conclusion. Using the results from the two previous steps, we finally get

$$\mathbb{E}\left(\left(\frac{\sum_{P\in\pi_n^K(\psi)} e^{2\xi\psi_{0,K}(P)}}{\left(\sum_{P\in\pi_n^K(\psi)} e^{\xi\psi_{0,K}(P)}\right)^2}\right)^{\alpha}\right)^{1/\alpha} \leq 2^{-\xi(Q-2-\varepsilon)K} e^{C\sqrt{K}},$$

which completes the proof. □

Now, we come back to the proof of Theorem 3.20. We first derive a priori estimates on the quantile ratios.

Lemma 3.22. *Let Z be a random variable with finite variance and $p \in (0,1/2)$. If a pair $(\bar{\ell}(Z,p), \ell(Z,p))$ satisfies $\bar{\ell}(Z,p) \geq \ell(Z,p)$, $\mathbb{P}(Z \geq \bar{\ell}(Z,p)) \geq p$ and $\mathbb{P}(Z \leq \ell(Z,p)) \geq p$, then, we have:*

$$(\bar{\ell}(Z,p) - \ell(Z,p))^2 \leq \frac{2}{p^2} \operatorname{Var} Z. \tag{5.55}$$

Proof. If Z' is an independent copy of Z, notice that for $l' \geq l$ we have $2\mathrm{Var}(Z) = \mathbb{E}((Z'-Z)^2) \geq \mathbb{E}(1_{Z'\geq l'} 1_{Z \leq l}(Z'-Z)^2) \geq \mathbb{P}(Z \geq l')\mathbb{P}(Z \leq l)(l'-l)^2$. □

In the following lemma, we derive an a priori bound on the variance of $\log L_{1,1}^{(n)}(\phi)$.

Lemma 3.23. *For all $n \geq 0$ we have the bound*

$$\operatorname{Var}\log L_{1,1}^{(n)}(\phi) \leq \xi^2(n+1)\log 2$$

Proof. Denote by $L_{1,1}^{(n)}(\mathrm{D}_k)$ the left-right distance of $[0,1]^2$ for the length metric $e^{\xi\phi_{0,n}^k} ds$, where $\phi_{0,n}^k$ is piecewise constant on each dyadic block of size 2^{-k} where it is equal to the value of $\phi_{0,n}$ at the center of this block. (We do not assign an independent meaning to the notation D_k.) Note that we have

$$e^{-C2^{-k}\|\nabla\phi_{0,n}\|_{[0,1]^2}} L_{1,1}^{(n)} \leq L_{1,1}^{(n)}(\mathrm{D}_k) \leq L_{1,1}^{(n)} e^{C2^{-k}\|\nabla\phi_{0,n}\|_{[0,1]^2}},$$

which gives almost surely that $L_{1,1}^{(n)}(\phi) = \lim_{k\to\infty} L_{1,1}^{(n)}(\mathrm{D}_k)$. By dominated convergence we have

$$\operatorname{Var}\log L_{1,1}^{(n)}(\phi) = \lim_{k\to\infty} \operatorname{Var}\log L_{1,1}^{(n)}(\mathrm{D}_k).$$

Now, $\log L_{1,1}^{(n)}(\mathrm{D}_k)$ is a ξ-Lipschitz function of $p = 4^k$ Gaussian variables denoted by $Y = (Y_1, ..., Y_p)$, where on $\mathbb{R}^p$ we use the supremum metric. We can write $Y = AN$ for some symmetric positive semidefinite matrix A and standard Gaussian vector N on $\mathbb{R}^{4^k}$. Then $\log L_{1,1}^{(n)}(\mathrm{D}_k) = f(Y) = f(AN)$ which is $\xi\sigma$-Lipschitz as a function of N where $\sigma = \max(|A_1|, ..., |A_p|)$. By the Gaussian concentration inequality of [32, Lemma 2.1], applied as in [26, Lemma 5.8], since the pointwise variance of the field is $(n+1)\log 2$ we have

$$\operatorname{Var}\log L_{1,1}^{(n)}(\mathrm{D}_k) \leq \max(\operatorname{Var}(Y_1), ..., \operatorname{Var}(Y_p)) = \xi^2(n+1)\log 2. \qquad \square$$

Before stating the following lemma, we refer the reader to the definition of quantile ratios in (2.27).

Lemma 3.24 (A priori bound on the quantile ratios)**.** *Fix $p \in (0, 1/2)$. There exists a constant C_p depending only on p such that for all $n \geq 1$,*

$$\Lambda_n(\psi, p) \leq e^{C_p\sqrt{n}}. \tag{5.56}$$

Proof. By using Lemma 3.23 we get $\operatorname{Var}(\log L_{1,1}^{(k)}(\psi)) \leq Ck$ for all $1 \leq k \leq n$ and an absolute constant $C > 0$. This implies the same bound for ψ by Proposition 3.5. Using then Lemma 3.22 with $Z_k = \log L_{1,1}^{(k)}(\psi)$ for $k \leq n$, we finally get the bound $\max_{k\leq n} \frac{\bar{\ell}_k(\psi,p)}{\ell_k(\psi,p)} \leq e^{C_p\sqrt{n}}$. $\square$

Proof of Theorem 3.20. The proof is divided in five steps. K will denote a large positive number to be fixed at the last step.

Step 1. Quantiles-variance relation / setup. We aim to get an inductive bound on $\Lambda_n(\psi, p)$. We will therefore bound $\frac{\bar{\ell}_n(\psi,p/2)}{\ell_n(\psi,p/2)}$ in term of Λ's at lower scales. p will be fixed from now on, small enough so that we have the tail estimates from Section 3.4 for ϕ with p and for ψ with $p/2$. The starting point is the bound

$$\frac{\bar{\ell}_n(\psi, p/2)}{\ell_n(\psi, p/2)} \leq e^{C_p\sqrt{\operatorname{Var}\log L_{1,1}^{(n)}(\psi)}}. \tag{5.57}$$

Step 2. Efron-Stein. Using the Efron-Stein inequality with the block decomposition of $\psi_{0,n}$ introduced in (2.20), defining the length with respect to the unresampled field $L_n(\psi) = L_{1,1}^{(n)}(\psi)$, we get

$$\operatorname{Var}\log L_{1,1}^{(n)}(\psi) \leq \mathbb{E}\left(\left(\log L_n^K(\psi) - \log L_n(\psi)\right)_+^2\right) + \sum_{P\in\mathcal{P}_K} \mathbb{E}\left(\left(\log L_n^P(\psi) - \log L_n(\psi)\right)_+^2\right), \quad (5.58)$$

where in the first term (resp. second term) we resample the field $\psi_{0,K}$ (resp. $\psi_{K,n,P}$) to get an independent copy $\tilde{\psi}_{0,K}$ (resp. $\tilde{\psi}_{K,n,P}$) and we consider the left-right distance $L_n^K(\psi)$ (resp. $L_n^P(\psi)$) of the unit square associated to the field $\psi_{0,n} - \psi_{0,K} + \tilde{\psi}_{0,K}$ (resp. $\psi_{0,n} - \psi_{K,n,P} + \tilde{\psi}_{K,n,P}$).

Step 3. Analysis of the first term. For the first term, using Gaussian concentration as in the proof of Lemma 3.23, we get

$$\mathbb{E}((\log L_n^K(\psi) - \log L_n(\psi))^2) = 2\mathbb{E}(\operatorname{Var}(\log L_n(\psi)|\psi_{0,n} - \psi_{0,K})) \leq CK. \quad (5.59)$$

Step 4. Analysis of the second term. For $P \in \mathcal{P}_K$, if $L_n^P(\psi) > L_n(\psi)$, the block P is visited by the geodesic $\pi_n(\psi)$ associated to $L_n(\psi)$. Define

$$P^K := \{Q \in \mathcal{P}_K \ : \ d(P,Q) \leq CK^{\varepsilon_0}2^{-K}\}. \quad (5.60)$$

where we recall that ε_0 is associated with the range of dependence of the resampled field $\tilde{\psi}_{K,n,P}$ through (2.18) (see also the subsection following this definition). Here, $d(P,Q)$ is the L^∞-distance between the sets P and Q.

We upper-bound $L_n^P(\psi)$ by taking the concatenation of the part of $\pi_n(\psi)$ outside of P^K together with four geodesics associated to long crossings in rectangles comprising a circuit around P^K (for the field $\psi_{0,n}$ which coincides with the field $\psi_{0,n}^P$ outside of P^K). We get, introducing the rectangles $(Q_i(P))_{1\leq i\leq 4}$ of size $2^{-K}(CK^{\varepsilon_0}, 3)$ surrounding P^K (P^K and its $3\cdot 2^{-K}$ neighborhood form an annulus, and gluing the four crossings gives a circuit in this annulus) and using the inequality $\log x \leq x - 1$,

$$\left(\log L_n^P(\psi) - \log L_n(\psi)\right)_+ \leq \frac{(L_n^P(\psi) - L_n(\psi))_+}{L_{1,1}^{(n)}(\psi)} \leq 4\frac{\max_{1\leq i\leq 4} L^{(n)}(Q_i(P),\psi)}{L_{1,1}^{(n)}(\psi)}. \quad (5.61)$$

- We recall the notation $\phi_{0,K}(P)$ to denote the value of the field $\phi_{0,K}$ at the center of P. We bound from above each term in the maximum of (5.61) as follows:

$$\begin{aligned}L^{(n)}(Q_i(P),\psi) &\leq e^{\xi X} L^{(n)}(Q_i(P),\phi)\\ &\leq e^{\xi X} e^{\xi \phi_{0,K}(P)} e^{\xi \operatorname{osc}_{P^K}(\phi_{0,K})} L^{(K,n)}(Q_i(P),\phi)\\ &\leq e^{2\xi X} e^{\xi \psi_{0,K}(P)} e^{\xi \operatorname{osc}_{P^K}(\phi_{0,K})} L^{(K,n)}(Q_i(P),\phi),\end{aligned}$$

where the oscillation osc is defined in (2.13) and P^K is defined in (5.60).

For a rectangle Q of size 2^{-K}, with corners in $2^{-K}\mathbb{Z}^2$, we denote by $(R_i^L(Q))_{1\leq i\leq 4}$ the four long rectangles of size $2^{-K}(3,1)$ surrounding Q. We can upper-bound the rectangle crossing lengths associated to the $Q_i(P)$'s by gluing $O(K^{\varepsilon_0})$ rectangle crossings of size $2^{-K}(3,1)$, which include an annulus around each block Q of size $2^{-K}(1,1)$ (with corners in $2^{-K}\mathbb{Z}^2$) in the shaded region A^K of Figure 3.3. We get

$$\max_{1\leq i\leq 4} L^{(K,n)}(Q_i(P),\phi) \leq CK^{\varepsilon_0} \max_{Q\in A^K, 1\leq i\leq 4} L^{(K,n)}(R_i^L(Q),\phi)$$

and we end up with the following upper bound:

$$\left(\log L_n^P(\psi) - \log L_n(\psi)\right)_+ \leq e^{2\xi X} \frac{e^{\xi\psi_{0,K}(P)}}{L_{1,1}^{(n)}(\psi)} e^{\xi \operatorname{osc}_{P^K}(\phi_{0,K})} CK^{\varepsilon_0} \max_{Q\in A^K, 1\leq i\leq 4} L^{(K,n)}(R_i^L(Q),\phi). \tag{5.62}$$

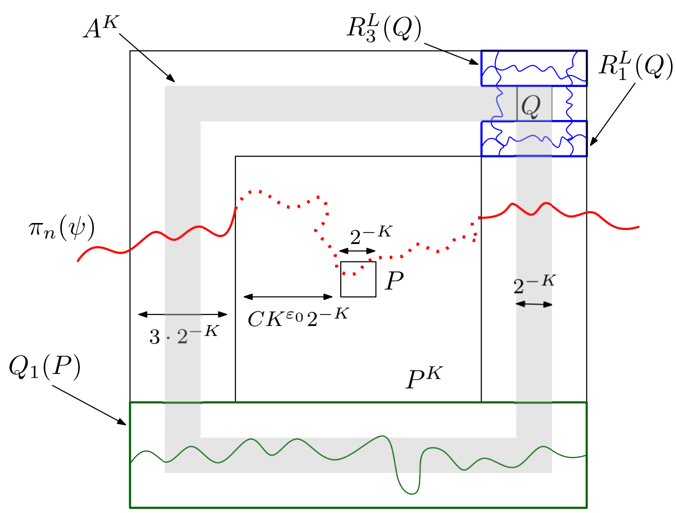

Figure 3.3 – Illustration of the geodesics used in the upper bound of Step 4.

• We lower-bound the denominator of (5.62) as follows. If $P \in \mathcal{P}_K$ is visited by a $\pi_n(\psi)$ geodesic, then there are at least two short disjoint rectangle crossings among the four surrounding P. Therefore, if we denote by $\hat{P}$ the box containing P at its center whose size is three times that of P,

$$\begin{aligned}\int_{\pi_n(\psi)\cap\hat{P}} e^{\xi\psi_{0,n}}ds &\geq 2\min_{1\leq i\leq 4} L^{(n)}(R_i^S(P),\psi) \geq e^{-\xi X}\min_{1\leq i\leq 4} L^{(n)}(R_i^S(P),\phi)\\ &\geq e^{-\xi X}e^{\xi\phi_{0,K}(P)}e^{-\xi\,\mathrm{osc}_{\hat{P}}(\phi_{0,K})}\min_{1\leq i\leq 4} L^{(K,n)}(R_i^S(P),\phi)\\ &\geq e^{-2\xi X}e^{\xi\psi_{0,K}(P)}e^{-\xi\,\mathrm{osc}_{\hat{P}}(\phi_{0,K})}\min_{1\leq i\leq 4} L^{(K,n)}(R_i^S(P),\phi),\end{aligned}$$

where $(R_i^S(P))_{1\leq i\leq 4}$ denote the four short rectangles of size $2^{-K}(1,3)$ surrounding P. Summing over all P's and taking uniform bounds for the rectangle crossings at higher scales,

$$\begin{aligned}L_{1,1}^{(n)}(\psi) &= \sum_{P\in\mathcal{P}_K}\int_{P\cap\pi_n(\psi)} e^{\xi\psi_{0,n}}ds \geq \frac{1}{9}\sum_{P\in\mathcal{P}_K}\int_{\hat{P}\cap\pi_n(\psi)} e^{\xi\psi_{0,n}}ds\\ &\geq \frac{1}{9}e^{-2\xi X}\left(\min_{P\in\mathcal{P}_K^1}\min_{1\leq i\leq 4} L^{(K,n)}(R_i^S(P),\phi)\right)\left(\sum_{P\in\mathcal{P}_K,P\cap\pi_n(\psi)\neq\emptyset} e^{\xi\psi_{0,K}(P)}e^{-\xi\,\mathrm{osc}_{\hat{P}}(\phi_{0,K})}\right).\end{aligned}$$

Therefore, taking a uniform bound for the oscillation, we get

$$L_{1,1}^{(n)}(\psi) \geq \frac{1}{9}e^{-2\xi X}\left(\sum_{P\in\pi_n^K(\psi)} e^{\xi\psi_{0,K}(P)}e^{-\xi\,\mathrm{osc}_{\hat{P}}(\phi_{0,K})}\right)\min_{P\in\mathcal{P}_K^1,1\leq i\leq 4} L^{(K,n)}(R_i^S(P),\phi) \tag{5.63}$$

$$\geq \frac{1}{9}e^{-2\xi X}e^{-\xi\max_{P\in\mathcal{P}_K^1}\mathrm{osc}_{\hat{P}}(\phi_{0,K})}\min_{P\in\mathcal{P}_K^1,1\leq i\leq 4} L^{(K,n)}(R_i^S(P),\phi)\sum_{P\in\pi_n^K(\psi)} e^{\xi\psi_{0,K}(P)}. \tag{5.64}$$

• We recall that $(R_i^L(P))_{1\leq i\leq 4}$ denote the four rectangles of size $2^{-K}(3,1)$ surrounding P. Gathering inequalities (5.62) and (5.64), we see that $\sum_{P\in\mathcal{P}_K}\mathbb{E}\left(\left(\log L_n^P(\psi)-\log L_n(\psi)\right)_+^2\right)$ is bounded from above by

$$K^{2\varepsilon_0}\mathbb{E}\frac{\sum_{P\in\pi_n^K(\psi)} e^{2\xi\psi_{0,K}(P)}}{\left(\sum_{P\in\pi_n^K(\psi)} e^{\xi\psi_{0,K}(P)}\right)^2}\left(\frac{\max_{P\in\mathcal{P}_K^1,1\leq i\leq 4} L^{(K,n)}(R_i^L(P),\phi)}{\min_{P\in\mathcal{P}_K^1,1\leq i\leq 4} L^{(K,n)}(R_i^S(P),\phi)}\right)^2 e^{C\xi\max_{P\in\mathcal{P}_K^1}\mathrm{osc}_{P^K}(\phi_{0,K})}e^{8\xi X}$$

• Condition (T) gives us a $\alpha > 1$ and $c > 0$ so that for K large enough, for $n \geq K$,

$$\mathbb{E}\left(\left(\frac{\sum_{P\in\pi_n^K(\psi)} e^{2\xi\psi_{0,K}(P)}}{\left(\sum_{P\in\pi_n^K(\psi)} e^{\xi\psi_{0,K}(P)}\right)^2}\right)^{\alpha}\right)^{1/\alpha} \leq e^{-cK}.$$

Then, by using the gradient estimate (2.17) and recalling the definition of $\mathcal{P}^K$ in (5.60), we have

$$\mathbb{E}\left(e^{C\max_{P\in\mathcal{P}_K^1}\operatorname{osc}_{P^K}(\phi_{0,K})}\right) \leq \mathbb{E}\left(e^{CK^{\varepsilon_0}2^{-K}\left\|\nabla\phi_{0,K}\right\|_{[0,1]^2}}\right) \leq e^{CK^{\frac{1}{2}+\varepsilon_0}}. \tag{5.65}$$

It is for the second inequality that in (2.18) we take ε_0 to be small in the definition of ψ; $\varepsilon_0 < 1/2$ is sufficient. Furthermore, using our tail estimates with regard to upper and lower quantiles for ϕ (see (4.48) and (4.49), and the scaling property (2.30), for $\beta > 1$ so that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, we get

$$\mathbb{E}\left(\left(\frac{\max_{P\in\mathcal{P}_K^1,1\leq i\leq 4} L^{(K,n)}(R_i^L(P),\phi)}{\min_{P\in\mathcal{P}_K^1,1\leq i\leq 4} L^{(K,n)}(R_i^S(P),\phi)}\right)^{2\beta}\right)^{\frac{1}{\beta}} \leq \Lambda_{n-K}^2(\phi,p)e^{CK^{\frac{1}{2}+\varepsilon_0}}. \tag{5.66}$$

Note that we could have a $\log K$ term instead of the K^{ε_0} in (5.66). Altogether, by applying Hölder inequality and Cauchy-Schwarz, we get

$$\sum_{P\in\mathcal{P}_K} \mathbb{E}\left(\left(\log L_n^P(\psi) - \log L_n(\psi)\right)_+^2\right) \leq e^{-cK}e^{CK^{\frac{1}{2}+\varepsilon_0}}\Lambda_{n-K}^2(\phi,p) \leq e^{-cK}e^{CK^{\frac{1}{2}+\varepsilon_0}}C_p\Lambda_{n-K}^2(\psi,p/2), \tag{5.67}$$

where we used (2.29) in the last inequality to get $\Lambda_{n-K}^2(\phi,p) \leq C_p\Lambda_{n-K}^2(\psi,p/2)$.

Step 5. Conclusion. Gathering the bounds obtained in Step 3 (inequality (5.59)) and Step 4 (inequality (5.67)), we get, coming back to the inequality (5.58), for K large enough,

$$\operatorname{Var}\log L_{1,1}^{(n)}(\psi) \leq C_1 K + e^{-C_2 K}\Lambda_{n-K}^2(\psi,p/2). \tag{5.68}$$

Now, we will show that this bound together with the a priori bound on the quantile ratios (Lemma 3.24) is enough to conclude first that $\Lambda_\infty(\psi,p/2) < \infty$ and then that $\sup_{n\geq 0}\operatorname{Var}\log L_{1,1}^{(n)}(\psi) < \infty$, using the tail estimates (4.48) and (4.50).

Coming back to Step 1 (equation (5.57)) and using (5.68), we get the inductive inequality (5.69) below for K large enough and $n \geq K$, and (5.70) below by the a priori bound on the quantile ratios Lemma 3.24:

$$\frac{\bar{\ell}_n(\psi, p/2)}{\ell_n(\psi, p/2)} \leq e^{C_p\sqrt{\operatorname{Var}\log L_{1,1}^{(n)}(\psi)}} \leq e^{C_p\sqrt{C_1K + e^{-C_2K}\Lambda^2_{n-K}(\psi,p/2)}}; \tag{5.69}$$

$$\Lambda_K(\psi, p/2) \leq e^{\tilde{C}_p\sqrt{K}}. \tag{5.70}$$

From now on, we take K large enough but fixed so that

$$e^{-C_2K}(e^{\tilde{C}_p\sqrt{K}} + e^{C_p\sqrt{2C_1K}})^2 \leq C_1K. \tag{5.71}$$

Set

$$\Lambda_{\text{Rec}} := \Lambda_K(\psi, p/2) \vee e^{C_p\sqrt{2C_1K}}. \tag{5.72}$$

so that $\Lambda_K(\psi, p/2) \leq \Lambda_{\text{Rec}}$. This is the initialization of the induction. Now, assume that $\Lambda_{n-1}(\psi, p/2) \leq \Lambda_{\text{Rec}}$. In particular, $\Lambda_{n-K}(\psi, p/2) \leq \Lambda_{\text{Rec}}$ and using (5.69)

$$\frac{\bar{\ell}_n(\psi, p/2)}{\ell_n(\psi, p/2)} \leq e^{C_p\sqrt{C_1K + e^{-C_2K}\Lambda^2_{\text{Rec}}}}$$

The right-hand side is smaller than $e^{C_p\sqrt{2C_1K}}$ and therefore than Λ_{Rec}. Indeed, by (5.72), (5.70) and (5.71),

$$e^{-C_2K}\Lambda^2_{\text{Rec}} \leq e^{-C_2K}(\Lambda_K(\psi, p/2) + e^{C_p\sqrt{2C_1K}})^2 \leq e^{-C_2K}(e^{\tilde{C}_p\sqrt{K}} + e^{C_p\sqrt{2C_1K}})^2 \leq C_1K.$$

Therefore,

$$\Lambda_n(\psi, p/2) = \Lambda_{n-1}(\psi, p/2) \vee \frac{\bar{\ell}_n(\psi, p/2)}{\ell_n(\psi, p/2)} \leq \Lambda_{\text{Rec}}.$$

Therefore, $\Lambda_\infty(\psi, p/2) < \infty$ thus $\Lambda_\infty(\phi, p) < \infty$ and by the tail estimates (4.48) and (4.49), the sequence $(\log L_{1,1}^{(n)}(\phi) - \log \lambda_n(\phi))_{n \geq 0}$ is tight. □

3.5.2 Weak multiplicativity of the characteristic length and error bounds

Henceforth, we will only consider the case $\xi = \frac{\gamma}{d_\gamma}$ for $\gamma \in (0,2)$ and the field $\phi_{0,n}$. All observables will be assumed to be taken with respect to ϕ and we will drop the additional notation used to differ between ϕ and ψ. In this case, we saw that there exists a fixed constant $C > 0$ so that for all $n \geq 0$, $\bar{\ell}^{(n)}_{3,1}(p) \leq C\bar{\ell}^{(n)}_{1,3}(p)$, $C^{-1}\ell^{(n)}_{3,1}(p) \leq \ell^{(n)}_{1,3}(p)$ and with the tail estimates, $\mathbb{E}(L^{(n)}_{3,1}) \leq C\mathbb{E}(L^{(n)}_{1,3})$. All these characteristic lengths are uniformly comparable. We will take λ_n to denote one of them, say the median of $L^{(n)}_{1,1}$.

In the next elementary lemma, we prove that a sequence satisfying a certain quantitative weak multiplicative property has an exponent, and we quantify the error.

Lemma 3.25. *Consider a sequence of positive real numbers $(\lambda_n)_{n\geq 1}$. If there exists $C > 0$ such that for all $n \geq 1$, $k \geq 1$ we have*

$$e^{-C\sqrt{k}}\lambda_n\lambda_k \leq \lambda_{n+k} \leq e^{C\sqrt{k}}\lambda_n\lambda_k, \tag{5.73}$$

then there exists $\rho > 0$ such that $\lambda_n = \rho^{n+O(\sqrt{n})}$.

Proof. We introduce the sequence $(a_n)_{n\geq 0}$ such that $\lambda_{2^{n+1}} = (\lambda_{2^n})^2 e^{a_n}$. By iterating, we get

$$\lambda_{2^{n+1}} = (\lambda_{2^n})^2 e^{a_n} = (\lambda_{2^{n-1}})^4 e^{2a_{n-1}+a_n} = \cdots = \lambda_1^{2^{n+1}} e^{2^n a_0 + 2^{n-1}a_1 + \cdots + 2a_{n-1} + a_n}.$$

The condition (5.73) gives that the sequence $\left(2^{-n/2}a_n\right)_{n\geq 0}$ is bounded, therefore the series $\sum_{k\geq 0}\frac{a_k}{2^k}$ converges and $|\sum_{k\geq n}\frac{a_k}{2^k}| \leq 2\ (\sup_{k\geq 0} 2^{-k/2}|a_k|)\ 2^{-n/2}$. In particular there exists $\rho > 0$ such that

$$\lambda_{2^{n+1}} = e^{2^{n+1}\left(\log\lambda_1 + \frac{1}{2}\sum_{k=0}^{n}\frac{a_k}{2^k}\right)} = e^{2^{n+1}\left(\log\lambda_1 + \frac{1}{2}\sum_{k=0}^{\infty}\frac{a_k}{2^k}\right)} e^{-2^n\sum_{k\geq n+1}\frac{a_k}{2^k}} = \rho^{2^n} e^{O(2^{n/2})}.$$

Now that we have the existence of an exponent, we prove the upper bound of Lemma 3.25. There exist $C_1, C_2 > 0$ such that we have the following upper bounds:

$$\lambda_{2^k} \leq \rho^{2^k} e^{C_1 2^{k/2}}, \tag{5.74}$$

$$\lambda_{n+k} \leq \lambda_n\lambda_k e^{C_2\sqrt{k}}. \tag{5.75}$$

Take C_3 large enough so that $(C_1+C_2)^2+(C_1+C_2)C_3 \le C_3^2$ and $\lambda_1 \le \rho e^{C_3}$. We want to prove by induction that for all $n \ge 1$, $\lambda_n \le \rho^n e^{C_3\sqrt{n}}$. The assumption on C_3 implies that this holds for $n=1$. By induction (in a dyadic fashion), take $n \in [2^k, 2^{k+1})$. We decompose n as $n = 2^k + n_k$ with $n_k \in [0, 2^k)$. We have, by using (5.75), (5.74) and the induction hypothesis,

$$\lambda_n \le \lambda_{2^k}\lambda_{n_k} e^{C_2 2^{k/2}} \le (\rho^{2^k} e^{C_1 2^{k/2}})(\rho^{n_k} e^{C_3\sqrt{n_k}}) e^{C_2 2^{k/2}} = \rho^n e^{(C_1+C_2)2^{k/2}+C_3\sqrt{n_k}} \le \rho^n e^{C_3\sqrt{n}},$$

since by the assumption on C_3 we have

$$\left((C_1+C_2)2^{k/2}+C_3\sqrt{n_k}\right)^2 = (C_1+C_2)^2 2^k + (C_1+C_2)C_3 2^{k/2}\sqrt{n_k} + C_3^2 n_k \le C_3^2(2^k+n_k) = C_3^2 n.$$

The proof of the lower bound is similar. □

In the next proposition we prove that the characteristic length λ_n satisfies the weak multiplicativity property (5.73) and we identify the exponent by using the results of [28].

Proposition 3.26. *For ξ satisfying Condition (T), there exists $C > 0$ such that for all $n \ge 1$, $k \ge 1$ we have*

$$e^{-C\sqrt{k}}\lambda_n\lambda_k \le \lambda_{n+k} \le e^{C\sqrt{k}}\lambda_n\lambda_k. \tag{5.76}$$

Furthermore, when $\gamma \in (0,2)$ and $\xi = \gamma/d_\gamma$, we have

$$\lambda_n = 2^{-n(1-\xi Q)+O(\sqrt{n})}. \tag{5.77}$$

Proof. Let us assume first that (5.76) holds. Then, by using Lemma 3.25, there exists $\rho > 0$ such that we have $\lambda_n = \rho^{n+O(\sqrt{n})}$. Similarly to (5.54), for each fixed small $\delta > 0$, for k large enough we have,

$$\lambda_k \le 2^{-k(1-\xi Q-\delta)}. \tag{5.78}$$

The proof of (5.78) follows the same lines as the one of (5.54). Combining (5.78) and (5.54) we get $\rho = 2^{-(1-\xi Q)}$. Now, we prove that the characteristic length satisfies (5.76).

Step 1: Weak submultiplicativity. Let π_k be such that $L^{(k)}(\pi_k) = L^{(k)}_{1,1}$. If $P \in \mathcal{P}_k$ is visited by π_k, consider the concatenation $S^{(k,n+k)}(P)$ of four geodesics for $e^{\xi\phi_{k,n+k}}ds$ associated to the rectangles of size $2^{-k}(3,1)$ surrounding P. Each geodesic is in the long direction of its rectangle so that this concatenation is a circuit. By scaling, $\mathbb{E}(L^{(k,n+k)}(S^{(k,n+k)}(P))) = 2^{-k+2}\mathbb{E}(L^{(n)}_{3,1})$. Note that the collection $\pi^k_k(\phi) = \{P \in \mathcal{P}_k : P \cap \pi_k \neq \varnothing\}$ is measurable with respect to $\phi_{0,k}$, which is independent of $\phi_{k,n+k}$. Set $\Gamma_{k,n} := \bigcup_{P\in\pi^k_k(\phi)} S^{(k,n+k)}(P)$. Note that $\Gamma_{k,n}$ contains a left-right crossing of $[0,1]^2$ whose length is bounded above by

$$L^{(n+k)}(\Gamma_{k,n}) = \sum_{P\in\pi^k_k(\phi)} L^{(n+k)}(S^{(k,n+k)}(P)) \leq \sum_{P\in\pi^k_k(\phi)} L^{(k,n+k)}(S^{(k,n+k)}(P))e^{\xi\phi_{0,k}(P)}e^{\xi \operatorname{osc}_{\hat{P}}(\phi_{0,k})},$$

where $\hat{P}$ denotes the box containing P at its center whose side length is three times that of P. Since $L^{(n+k)}_{1,1} \leq L^{(n+k)}(\Gamma_{k,n})$, by independence we have

$$\mathbb{E}(L^{(n+k)}_{1,1}) \leq 4\mathbb{E}(L^{(n)}_{3,1})\mathbb{E}\left(\sum_{P\in\pi^k_k(\phi)} 2^{-k}e^{\xi\phi_{0,k}(P)}e^{\xi \operatorname{osc}_{\hat{P}}(\phi_{0,k})}\right).$$

If P is visited, then one of the four rectangles of size $2^{-k}(1,3)$ in $\hat{P}$ surrounding P contains a short crossing, denoted by $\tilde{\pi}_k(P)$ and we have

$$\int_{\pi_k} e^{\xi\phi_{0,k}}1_{\pi_k\cap\hat{P}}ds \geq L^{(k)}(\tilde{\pi}_k(P)) \geq 2^{-k}e^{\xi\inf_{\hat{P}}\phi_{0,k}} \geq 2^{-k}e^{\xi\phi_{0,k}(P)}e^{-\xi \operatorname{osc}_{\hat{P}}(\phi_{0,k})},$$

hence

$$\sum_{P\in\pi^k_k(\phi)} 2^{-k}e^{\xi\phi_{0,k}(P)}e^{\xi \operatorname{osc}_{\hat{P}}(\phi_{0,k})} \leq \sum_{P\in\pi^k_k(\phi)} e^{2\xi \operatorname{osc}_{\hat{P}}(\phi_{0,k})}\int_{\pi_k} e^{\xi\phi_{0,k}}1_{\pi_k\cap\hat{P}}ds.$$

Taking the supremum of the oscillation over all blocks,

$$\sum_{P\in\pi^k_k(\phi)} e^{2\xi \operatorname{osc}_{\hat{P}}(\phi_{0,k})}\int_{\pi_k} e^{\xi\phi_{0,k}}1_{\pi_k\cap\hat{P}}ds \leq 9e^{2\xi 2^{-k}\|\nabla\phi_{0,k}\|_{[0,1]^2}}L^{(k)}_{1,1}.$$

Altogether, by Cauchy-Schwarz,

$$\mathbb{E}(L^{(n+k)}_{1,1}) \leq 36\mathbb{E}(L^{(n)}_{3,1})\mathbb{E}((L^{(k)}_{1,1})^2)^{1/2}\mathbb{E}(e^{4\xi 2^{-k}\|\nabla\phi_{0,k}\|_{[0,1]^2}})^{1/2}.$$

When ξ satisfies Condition (T), by using the uniform bounds for quantile ratios together with the upper tail estimates (4.49) and the gradient estimate (2.17) we get $\lambda_{n+k} \leq e^{C\sqrt{k}}\lambda_n\lambda_k$.

Step 2: Weak supermultiplicativity. We argue here that

$$\lambda_{n+k} \geq e^{-C\sqrt{k}}\lambda_n\lambda_k. \tag{5.79}$$

Using a slightly easier argument than (5.64) (since we just have the field ϕ here), we have

$$L_{1,1}^{(n+k)} \geq e^{-\xi \max_{P\in\mathcal{P}_k^1} \mathrm{osc}_{\hat{P}}(\phi_{0,k})} \left(\min_{P\in\mathcal{P}_k^1, 1\leq i\leq 4} L^{(k,k+n)}(R_i^S(P)) \right) \sum_{P\in\pi_{n+k}^k} e^{\xi\phi_{0,k}(P)},$$

where π_{n+k}^k denotes the k-coarse grained approximation of π_{n+k}, the left-right geodesic of $[0,1]^2$ for the field $\phi_{0,n+k}$, and where we recall that $(R_i^S(P))_{1\leq i\leq 4}$ denote the four rectangles of size $2^{-k}(1,3)$ surrounding P. Furthermore, by using a similar argument to (5.52), we have

$$\sum_{P\in\pi_{n+k}^k} e^{\xi\phi_{0,k}(P)} \geq e^{-\xi\max_{P\in\mathcal{P}_k} \mathrm{osc}_P(\phi_{0,k})} 2^k L_{1,1}^{(k)}.$$

Altogether, we get the following weak supermultiplicativity,

$$L_{1,1}^{(n+k)} \geq L_{1,1}^{(k)} \left(\min_{P\in\mathcal{P}_k^1, 1\leq i\leq 4} 2^k L^{(k,k+n)}(R_i^S(P)) \right) e^{-2\xi\max_{P\in\mathcal{P}_k}\mathrm{osc}_P(\phi_{0,k})} \tag{5.80}$$

When ξ satisfies Condition (T), by scaling and the tail estimates (4.48), we obtain the inequality $\mathbb{P}(\min_{P\in\mathcal{P}_k^1, 1\leq i\leq 4} 2^k L^{(k,k+n)}(R_i^S(P)) \geq \lambda_n e^{-C\sqrt{k}}) \geq 1 - e^{-ck}$. Furthermore, using the gradient estimates (2.16), we get $\mathbb{P}(2^{-k}\|\nabla\phi_{0,k}\|_{[0,1]^2} \geq C\sqrt{k}) \geq 1 - e^{-ck}$ for C large enough. Therefore, with probability $\geq 1/2$, $L_{1,1}^{(n)} \leq e^{-C\sqrt{k}}\lambda_n\lambda_k$ hence the bound $\lambda_{n+k} \geq e^{-C\sqrt{k}}\lambda_k\lambda_n$. □

3.5.3 Tightness of the log of the diameter

Proposition 3.27. *If $\gamma \in (0,2)$ and $\xi = \gamma/d_\gamma$ then $\left(\log \mathrm{Diam}\left([0,1]^2, \lambda_n^{-1} e^{\xi\phi_{0,n}} ds\right)\right)_{n\geq 0}$ is tight.*

Proof. Step 1: Chaining. By a standard chaining argument, (see (6.1) in [38] for more details), we have

$$\text{Diam}\left([0,1]^2, e^{\xi\phi_{0,n}}ds^2\right) \le C\sum_{k=0}^{n}\max_{P\in\mathcal{C}_k} L^{(n)}(P) + C\times 2^{-n}e^{\xi\sup_{[0,1]^2}\phi_{0,n}}, \tag{5.81}$$

where $\mathcal{C}_k$ is a collection of no more than $C4^k$ long rectangles of side length $2^{-k}(3,1)$.

Using the bound for the maximum (2.11), when $\xi < 2$, we have $\mathbb{E}(2^{-n}e^{\xi\sup_{[0,1]^2}\phi_{0,n}}) \le 2^{-n}2^{2\xi n}e^{C\sqrt{n}}$.

Fix $0 \le k \le n$ and $P \in \mathcal{C}_k$. We can bound $L^{(n)}(P)$ by taking a left-right geodesic $\pi_{k,n}$ for $\phi_{k,n}$. Therefore,

$$L^{(n)}(P) \le L^{(n)}(\pi_{k,n}) \le e^{\xi\max_{[0,1]^2}\phi_{0,k}}L^{(k,n)}(P),$$

and consequently,

$$\max_{P\in\mathcal{C}_k} L^{(n)}(P) \le e^{\xi\max_{[0,1]^2}\phi_{0,k}}\max_{P\in\mathcal{C}_k} L^{(k,n)}(P). \tag{5.82}$$

Using independence, the maximum bound (2.11), scaling of the field ϕ and the tail estimates (4.49), we get

$$\mathbb{E}\left(e^{\xi\max_{[0,1]^2}\phi_{0,k}}\max_{P\in\mathcal{C}_k} L^{(k,n)}(P)\right) \le 2^{-k}2^{2\xi k}e^{C\sqrt{k}}\lambda_{n-k}e^{Ck^{\frac{1}{2}+\varepsilon}} \tag{5.83}$$

for some fixed small $\varepsilon > 0$ (again, the term k^ε could in fact be $\log k$). Taking the expectation in (5.81), using (5.82) and (5.83), we obtain the following bound for the expected value of the diameter,

$$\mathbb{E}(\text{Diam}([0,1]^2, e^{\xi\phi_{0,n}}ds)) \le C\sum_{k=0}^{n}2^{-k}2^{2\xi k}\lambda_{n-k}e^{Ck^{\frac{1}{2}+\varepsilon}}. \tag{5.84}$$

Step 2: Right tail. By Proposition 3.26, $\lambda_{n-k} \le \lambda_n\frac{e^{C\sqrt{k}}}{\lambda_k} \le \lambda_n 2^{k(1-\xi Q)}e^{C\sqrt{k}}$. Together with (5.84), this implies that

$$\mathbb{E}(\text{Diam}([0,1]^2, e^{\xi\phi_{0,n}}ds)) \le C\sum_{k=0}^{n}2^{-k}2^{2\xi k}\lambda_{n-k}e^{Ck^{\frac{1}{2}+\varepsilon}} \le \lambda_n C\sum_{k=0}^{\infty}2^{-k\xi(Q-2)}e^{Ck^{\frac{1}{2}+\varepsilon}}.$$

Since $Q > 2$, Markov's inequality gives $\mathbb{P}\left(\text{Diam}([0,1]^2, \lambda_n^{-1}e^{\xi\phi_{0,n}}ds) \ge e^s\right) \le Ce^{-s}$.

Step 3: Left tail. Finally, since the diameter of the square $[0,1]^2$ is larger than the left-right distance, by our tail estimates (4.48), we get $\mathbb{P}\left(\text{Diam}([0,1]^2, \lambda_n^{-1}e^{\xi\phi_{0,n}}ds) \le e^{-s}\right) \le \mathbb{P}\left(L_{1,1}^{(n)} \le \lambda_n e^{-s}\right) \le Ce^{-cs^2}$. $\square$

3.5.4 Tightness of the metrics

Proposition 3.28. *If $\gamma \in (0,2)$ and $\xi = \gamma/d_\gamma$ then the sequence of metrics $\left(\lambda_n^{-1} e^{\xi\phi_{0,n}} ds\right)_{n\geq 0}$ is tight. Moreover, if we define*

$$C_\alpha^n := \sup_{x,x'\in[0,1]^2} \frac{|x-x'|^\alpha}{d_{0,n}(x,x')} \quad \textit{and} \quad C_\beta^n := \sup_{x,x'\in[0,1]^2} \frac{d_{0,n}(x,x')}{|x-x'|^\beta}$$

then, for $\alpha > \xi(Q+2)$ and $\beta < \xi(Q-2)$, the sequence $(C_\alpha^n, C_\beta^n)_{n\geq 0}$ is tight.

Henceforth, we use the notation $d_{0,n}$ for the renormalized metric $\lambda_n^{-1}e^{\xi\phi_{0,n}}ds$ restricted to $[0,1]^2$.

Proof. The proof has two parts. In the first part we show the tightness of the metrics in the space of continuous function from $[0,1]^2 \times [0,1]^2 \to \mathbb{R}^+$ and in the second part we show that subsequential limits are metrics. A byproduct result of the argument is explicit bi-Hölder bounds.

Part 1. Upper bound on the modulus of continuity. We suppose $\gamma \in (0,2)$. We start by proving that for every $0 < \beta < \xi(Q-2)$, if $\varepsilon > 0$, there exists a large $C_\varepsilon > 0$ so that for every $n \geq 0$

$$\mathbb{P}\left(\exists x, x' \in [0,1]^2 \ : \ d_{0,n}(x,x') \geq C_\varepsilon |x-x'|^\beta\right) \leq \varepsilon, \tag{5.85}$$

i.e. $\left(\|d_{0,n}\|_{C^\beta([0,1]^2\times[0,1]^2)}\right)_{n\geq 0}$ is tight, where the C^β-norm is defined for $f : [0,1]^2 \times [0,1]^2 \to \mathbb{R}$ as

$$\|f\|_{C^\beta([0,1]^2\times[0,1]^2)} := \|f\|_{[0,1]^2\times[0,1]^2} + \sup_{(x,y)\neq(x',y')\in[0,1]^2\times[0,1]^2} \frac{|f(x,y)-f(x',y')|}{|(x,y)-(x',y')|^\beta}.$$

By a union bound it suffices to estimate $\mathbb{P}(\exists x, x' : \ |x-x'| < 2^{-n}, d_{0,n}(x,x') \geq e^s|x-x'|^\beta)$ and

$$\sum_{k=0}^{n} \mathbb{P}\left(\exists x, x' : 2^{-k} \leq |x-x'| \leq 2^{-k+1}, d_{0,n}(x,x') \geq e^s|x-x'|^\beta\right).$$

Step 1: We start with the term $\mathbb{P}(\exists x, x' : 2^{-k} \leq |x-x'| \leq 2^{-k+1}, d_{0,n}(x,x') \geq e^s|x-x'|^\beta)$. We use the chaining argument (5.81) at scale k which gives

$$\sup_{2^{-k}\leq|x-x'|\leq 2^{-k+1}} d_{0,n}(x,x') \leq C\lambda_n^{-1} \sum_{i=k}^{n} \max_{P\in\mathcal{C}_i} L^{(n)}(P) + C\lambda_n^{-1} \times 2^{-n} e^{\xi \sup_{[0,1]^2} \phi_{0,n}}.$$

Taking the expected value and using the same bounds as those obtained in the proof of Proposition 3.27, we get

$$\mathbb{E}\left(\sup_{2^{-k}\le|x-x'|\le 2^{-k+1}} d_{0,n}(x,x')\right) \le \sum_{i=k}^{n} 2^{-i\xi(Q-2)} e^{Ci^{\frac{1}{2}+\varepsilon}} \le C 2^{-k\xi(Q-2)} e^{Ck^{\frac{1}{2}+\varepsilon}}.$$

Therefore, using Markov's inequality we get the bound

$$\sum_{k=0}^{n} \mathbb{P}\left(\exists x,x' : 2^{-k} \le |x-x'| \le 2^{-k+1}, d_{0,n}(x,x') \ge e^{s}|x-x'|^{\beta}\right)$$
$$\le \sum_{k=0}^{n} \mathbb{P}\left(\sup_{2^{-k}\le|x-x'|\le 2^{-k+1}} d_{0,n}(x,x') \ge e^{s} 2^{-k\beta}\right) \le e^{-s}\sum_{k=0}^{n} 2^{k\beta} 2^{-k\xi(Q-2)}.$$

The series is convergent since $\xi(Q-2)-\beta>0$.

Step 2: We bound from above $\mathbb{P}(\exists x,x'\ |x-x'|<2^{-n}, d_{0,n}(x,x')\ge e^{s}|x-x'|^{\beta})$ using a bound on the supremum of the field. Indeed, for such x and x', note that

$$e^{s}|x-x'|^{\beta} \le d_{0,n}(x,x') \le \lambda_n^{-1} e^{\xi \sup_{[0,1]^2}\phi_{0,n}}|x-x'|$$

Writing $\beta = \xi(Q-2)-\varepsilon\xi$ for some $\varepsilon>0$, it follows that $1-\beta = (1-\xi Q+2\xi)+\varepsilon\xi>0$ since the LFPP exponent $1-\xi Q\ge -2\xi$ by a simple uniform bound. Therefore, $|x-x'|^{\beta-1}\ge 2^{n(1-\beta)}$ and $\lambda_n^{-1}2^{n(1-\beta)} = 2^{n(2\xi+\varepsilon\xi+o(1))}$. Altogether, this probability is bounded from above by $\mathbb{P}(\sup_{[0,1]^2}\phi_{0,n}\ge n\log 4+\varepsilon n\log 2+o(n)+\xi^{-1}s)$ and using (2.10) gives a uniform tail estimate.

Therefore, we obtain the tightness of $(d_{0,n})_{n\ge 0}$ as a random element of $C([0,1]^2\times[0,1]^2,\mathbb{R}^+)$ and every subsequential limit is (by Skorohod's representation theorem) a pseudo-metric.

Part 2. Lower bound on the modulus of continuity. We prove that if $\alpha>\xi(Q+2)$ and $\varepsilon>0$ then there exists a small constant $c_\varepsilon>0$ such that for every $n\ge 0$,

$$\mathbb{P}\left(\exists x,x'\in[0,1]^2\ :\ d_{0,n}(x,x')\le c_\varepsilon|x-x'|^{\alpha}\right)\le\varepsilon. \tag{5.86}$$

Similarly as before, by union bound it is enough to estimate the term

$$\mathbb{P}(\exists x,x'\in[0,1]^2 : |x-x'|<2^{-n}, d_{0,n}(x,x')\le e^{-\xi s}|x-x'|^{\alpha}) \tag{5.87}$$

and the term

$$\sum_{k=0}^{n} \mathbb{P}\left(\underbrace{\exists x, x' : 2^{-k} \le |x-x'| \le 2^{-k+1}, d_{0,n}(x,x') \le e^{-\xi s}|x-x'|^{\alpha}}_{:=E_{k,n,s}}\right). \tag{5.88}$$

Step 1: We give an upper bound for (5.88). Fix $x, x' \in [0,1]^2$ such that $2^{-k} \le |x-x'| \le 2^{-k+1}$. Note that any path from x to x' crosses one of the rectangles in the collection $\{R_i^S(P) : P \in \mathcal{P}_{k+2}^1, 1 \le i \le 4\}$. Hence, under the event $E_{k,n,s}$, there exists x, x' such that

$$2^{-k\alpha} \ge d_{0,n}(x,x') \ge \lambda_n^{-1} 2^{-k} e^{\xi \inf_{[0,1]^2} \phi_{0,k}} \left(\min_{P \in \mathcal{P}_{k+2}^1, 1 \le i \le 4} 2^k L^{(k,n)}(R_i^S(P)) \right). \tag{5.89}$$

Since $\alpha = \xi(Q+2) + \xi\delta$ for a small $\delta > 0$, by using Proposition 3.26 we get

$$2^{-k\alpha}\lambda_n 2^k \le 2^{-k\alpha}\lambda_k \lambda_{n-k} e^{C\sqrt{k}} \le 2^{-k(\alpha - \xi Q)}\lambda_{n-k} e^{C\sqrt{k}} = 2^{-k(2+\delta)\xi}(\lambda_{n-k} 2^{-\xi\delta k} e^{C\sqrt{k}}) \tag{5.90}$$

Now, using (5.89), (5.90) and scaling, we get

$$\begin{aligned}
\mathbb{P}(E_{k,n,s}) &\le \mathbb{P}\left(e^{\xi \inf_{[0,1]^2} \phi_{0,k}} \left(\min_{P \in \mathcal{P}_{k+2}^1, 1 \le i \le 4} 2^k L^{(k,n)}(R_i^S(P)) \right) \le 2^{-k\alpha}\lambda_n 2^k e^{-\xi s} \right) \\
&\le \mathbb{P}(\sup_{[0,1]^2} |\phi_{0,k}| \ge k \log 4 + k\delta \log 2 + s/2) \\
&\quad + \mathbb{P}(\min_{P \in \mathcal{P}_{k+2}^1, 1 \le i \le 4} L^{(n-k)}(R_i^S(P)) \le \lambda_{n-k} 2^{-k\delta\xi} e^{C\sqrt{k}} e^{-\xi s/2}) \\
&\le C e^{-ck} e^{-cs},
\end{aligned}$$

where we used in the last inequality the supremum bounds (2.10) and the left tail estimate (4.48).

Step 2: Finally, we control (5.87). $\mathbb{P}(\exists x, x' : |x-x'| < 2^{-n}, d_{0,n}(x,x') \le e^{-\xi s}|x-x'|^{\alpha})$ is bounded from above by

$$\mathbb{P}\left(\inf_{|x-x'| \le 2^{-n}} \frac{d_{0,n}(x,x')}{|x-x'|^{\alpha}} \le e^{-\xi s} \right) \le \mathbb{P}\left(\lambda_n^{-1} e^{\xi \inf_{[0,1]^2} \phi_{0,n}} \inf_{|x-x'| \le 2^{-n}} |x-x'|^{1-\alpha} \le e^{-\xi s} \right).$$

We recall that $\alpha > \xi Q + 2\xi$, and in particular $\alpha > 1$: indeed, $1 - \xi Q \le 2\xi$ follows from a comparison with the infimum of the field. In this case, $\inf_{|x-x'| \le 2^{-n}} |x-x'|^{1-\alpha} = 2^{-n(1-\alpha)}$, and by Proposition

3.26,

$$2^{-n(1-\alpha)}\lambda_n^{-1} \geq 2^{-n(1-\alpha)}2^{n(1-\xi Q)}e^{-C\sqrt{n}} = 2^{n(\alpha-\xi Q)}e^{-C\sqrt{n}}$$

Therefore, since $\alpha - \xi Q = 2\xi + \delta\xi$ for some $\delta > 0$, we have for n large that

$$\mathbb{P}\left(\lambda_n^{-1}e^{\xi \inf_{[0,1]^2}\phi_{0,n}} \inf_{|x-x'|\leq 2^{-n}} |x-x'|^{1-\alpha} \leq e^{-\xi s}\right) \leq \mathbb{P}\left(\sup_{[0,1]^2} |\phi_{0,n}| \geq n\log 4 + n\frac{\delta}{2}\log 2 + s\right)$$

Using (2.10) completes the proof. □

3.6 Appendix

3.6.1 Comparison with the GFF mollified by the heat kernel

Let h be a GFF with Dirichlet boundary condition on a domain D and $U \subset\subset D$ be a subdomain of D. We recall that we denote by p_t the two-dimensional heat kernel at time t i.e. $p_t(x) = \frac{1}{2\pi t}e^{-\frac{|x|^2}{2t}}$. The goal of this section is to obtain a uniform estimate to conclude on the tightness of the renormalized metric associated to $p_{\frac{t}{2}} * h$ assuming the one associated to $\phi_{\sqrt{t}}$. In particular, the second assertion of Theorem 3.1 is a corollary of the following proposition.

Proposition 3.29. *There exist constants $C, c > 0$ such that for all $t \in (0, 1/2)$, there is a coupling of h and $\varphi_t \overset{(d)}{=} \phi_{\sqrt{t}}$ such that for all $x \geq 0$, we have*

$$\mathbb{P}\left(\left\|\varphi_t - p_{\frac{t}{2}} * h\right\|_U \geq x\right) \leq Ce^{-cx^2}.$$

Mollification of the GFF by the heat kernel. The covariance of the Gaussian field $p_{\frac{t}{2}} * h$ is given for $x, x' \in U$ by

$$\mathbb{E}\left(p_{\frac{t}{2}} * h(x)\ p_{\frac{t}{2}} * h(x')\right) = \int_D\int_D p_{\frac{t}{2}}(x-y)G_D(y,y')p_{\frac{t}{2}}(y'-x')dydy',$$

where G_D is the Green function associated to the Laplacian operator on D. For an open set A, we denote by $p_t^A(x,y)$ the transition probability density of a Brownian motion killed upon exiting A.

White noise representation. Take a space-time white noise W and define the field η_t on U by

$$\eta_t(x) := \int_0^\infty \int_D p_{\frac{t}{2}} * p^D_{\frac{s}{2}}(x,y) W(dy,ds) \quad \text{where} \quad p_{\frac{t}{2}} * p^D_{\frac{s}{2}}(x,y) := \int_D p_{\frac{t}{2}}(x-y') p^D_{\frac{s}{2}}(y',y) dy', \tag{6.91}$$

so that $(\eta_t(x))_{x\in U} \overset{(d)}{=} (p_{\frac{t}{2}} * h(x))_{x\in U}$. Indeed, by Fubini, we have

$$\begin{aligned}
\mathbb{E}(\eta_t(x)\eta_t(x')) &= \int_0^\infty \int_D p_{\frac{t}{2}} * p^D_{\frac{s}{2}}(x,y)\ p_{\frac{t}{2}} * p^D_{\frac{s}{2}}(x',y) dy ds \\
&= \int_0^\infty \int_D \int_D \int_D p_{\frac{t}{2}}(x-y') p^D_{\frac{s}{2}}(y',y)\ p_{\frac{t}{2}}(x'-y'')\ p^D_{\frac{s}{2}}(y'',y) dy dy' dy'' ds \\
&= \int_D \int_D p_{\frac{t}{2}}(x-y') \left(\int_0^\infty \int_D p^D_{\frac{s}{2}}(y',y) p^D_{\frac{s}{2}}(y,y'') dy ds \right) p_{\frac{t}{2}}(x'-y'') dy' dy'' \\
&= \int_D \int_D p_{\frac{t}{2}}(x-y') G_D(y',y'') p_{\frac{t}{2}}(y''-x') dy' dy''.
\end{aligned}$$

Coupling. Note that for $t \in (0,1/2)$ $\phi_{\sqrt{t}}(x) = \int_t^1 \int_{\mathbb{R}^2} p_{\frac{s}{2}}(x-y) W(dy,ds) \overset{(d)}{=} \varphi_t(x)$, where we set

$$\varphi_t(x) := \int_0^{1-t} \int_{\mathbb{R}^2} p_{\frac{t+s}{2}}(x-y) W(dy,ds).$$

Furthermore, we can decompose $\varphi_t(x) = \varphi_t^1(x) + \varphi_t^2(x)$, where

$$\varphi_t^1(x) := \int_0^{1-t} \int_D p_{\frac{t+s}{2}}(x-y) W(dy,ds); \tag{6.92}$$

$$\varphi_t^2(x) := \int_0^{1-t} \int_{D^c} p_{\frac{t+s}{2}}(x-y) W(dy,ds). \tag{6.93}$$

Recalling the definition of η in (6.91), we introduce η_t^1 and η_t^2 so that

$$\eta_t(x) = \int_0^{1-t} \int_D p_{\frac{t}{2}} * p^D_{\frac{s}{2}}(x,y) W(dy,ds) + \int_{1-t}^\infty \int_D p_{\frac{t}{2}} * p^D_{\frac{s}{2}}(x,y) W(dy,ds) =: \eta_t^1(x) + \eta_t^2(x). \tag{6.94}$$

Therefore, under this coupling (viz. using the same white noise W), we have

$$\varphi_t^1(x) - \eta_t^1(x) = \int_0^{1-t} \int_D \left(p_{\frac{t+s}{2}}(x-y) - p_{\frac{t}{2}} * p^D_{\frac{s}{2}}(x,y) \right) W(dy,ds). \tag{6.95}$$

Comparison between kernels. We will consider $x, y \in U$, subdomain of D. Set $d := d(U, D^c) > 0$.

$$p_{\frac{t}{2}} * p^D_{\frac{s}{2}}(x,y) := \int_D p_{\frac{t}{2}}(x-y')p^D_{\frac{s}{2}}(y',y)dy' = \int_D p_{\frac{t}{2}}(x-y')p_{\frac{s}{2}}(y'-y)q^D_{\frac{s}{2}}(y',y)dy',$$

where $q^D_t(x,x')$ is the probability that a Brownian bridge between x and x' with lifetime t stays in D. Therefore, using Chapman-Kolmogorov,

$$p_{\frac{t}{2}} * p^D_{\frac{s}{2}}(x,y) - p_{\frac{t+s}{2}}(x,y) = -\int_{D^c} p_{\frac{t}{2}}(x-y')p_{\frac{s}{2}}(y'-y)dy' + \int_D p_{\frac{t}{2}}(x-y')p_{\frac{s}{2}}(y'-y)(q^D_{\frac{s}{2}}(y',y)-1)dy'.$$

Note that the first term can be bounded by using that $|y-y'| \geq d$ for $y \in U$ and $y' \in D^c$. For the second term, we can split the integral over D in two parts: one over the ε-neighborhood of ∂D (within D), denoted by $(\partial D)^\varepsilon$, and one over its complement. To give an upper bound on the first, we use that for $y \in U$ and $y' \in (\partial D)^\varepsilon$, $|y-y'| \geq d(U, (\partial D)^\varepsilon)$. Finally, we bound the second part by using a uniform estimate on the probability that a Brownian bridge between a point in U and a point $D \setminus (\partial D)^\varepsilon$ exits D in time less than $s/2$. (Note that $1 - q^D_{\frac{s}{2}}(y,y')$ is the probability that a Brownian bridge between y and y' with time length $s/2$ exits D.) Therefore, we get that uniformly in $x, y \in U$ and t,

$$|p_{\frac{t}{2}} * p^D_{\frac{s}{2}}(x,y) - p_{\frac{t+s}{2}}(x,y)| \leq Ce^{-\frac{c}{s}}. \tag{6.96}$$

Comparison between φ_t and $p_{\frac{t}{2}} * h$. By the triangle inequality,

$$\left\| \varphi_t - p_{\frac{t}{2}} * h \right\|_U \leq \left\| \varphi^1_t - \eta^1_t \right\|_U + \left\| \varphi^2_t \right\|_U + \left\| \eta^2_t \right\|_U. \tag{6.97}$$

We look for a uniform right tail estimate (in t) of each term in the right-hand side of (6.97). In order to do so, we will use the Kolmogorov continuity criterion. Therefore, we derive below some pointwise and difference estimates.

First term. We derive first a pointwise estimate. For $x \in U$, using the kernel comparison (6.96), there exists some $C' > 0$ such that, uniformly in t,

$$\mathrm{Var}\left(\left(\eta^1_t(x) - \varphi^2_t(x)\right)^2 \right) = \int_0^{1-t} \int_D \left(p_{\frac{t}{2}} * p^D_{\frac{s}{2}}(x,y) - p_{\frac{t+s}{2}}(x,y) \right)^2 dyds \leq C \int_0^{1-t} e^{-\frac{c}{s}} ds \leq C'.$$

We now give a difference estimate: introducing $\Delta_t(x) := \varphi_t^1(x) - \eta_t^1(x)$, for $x, x' \in U$,

$$\mathbb{E}\left(\left(\Delta_t(x) - \Delta_t(x')\right)^2\right)$$
$$= \int_0^{1-t} \int_D \left(\left(p_{\frac{t+s}{2}}(x-y) - p_{\frac{t}{2}} * p^D_{\frac{s}{2}}(x,y)\right) - \left(p_{\frac{t+s}{2}}(x'-y) - p_{\frac{t}{2}} * p^D_{\frac{s}{2}}(x',y)\right)\right)^2 dyds,$$

which is uniformly bounded in $t \in (0, 1/2)$ by a quantity of size $O(|x-x'|)$. (By splitting the integral at $\sqrt{|x-x'|}$, one can use (6.96) for the small values of s and gradient estimates for both kernels for larger values of s.)

Second term. We recall here that $\varphi_t^2(x)$ is defined for $x \in U$ by

$$\varphi_t^2(x) = \int_0^{1-t} \int_{D^c} p_{\frac{t+s}{2}}(x-y) W(dy, ds) \overset{(d)}{=} \int_t^1 \int_{D^c} p_{\frac{s}{2}}(x-y) W(dy, ds).$$

We have, for $x, x' \in U$, with $d \coloneqq d(U, D^c)$,

$$\mathbb{E}\left(\left(\varphi_t^2(x) - \varphi_t^2(x')\right)^2\right) \le \int_t^1 \int_{D^c} \left(p_{\frac{s}{2}}(x-y) - p_{\frac{s}{2}}(x'-y)\right)^2 dyds$$
$$\le \int_{\sqrt{|x-x'|}}^1 \int_{\mathbb{R}^2} \left(p_{\frac{s}{2}}(x-y) - p_{\frac{s}{2}}(x'-y)\right)^2 dyds + \int_0^{\sqrt{|x-x'|}} \int_{D^c} \left(p_{\frac{s}{2}}(x-y) - p_{\frac{s}{2}}(x'-y)\right)^2 dyds$$
$$\le 2\int_{\sqrt{|x-x'|}}^1 \left(p_s(0) - p_s(x-x')\right) ds + 4\int_0^{\sqrt{|x-x'|}} p_{\frac{s}{2}}(d) ds \le C|x-x'|,$$

where we use $1 - e^{-z} \le z$ in the last inequality. Similarly, we can prove that there exists $C > 0$ independent of t such that $\mathbb{E}(\phi_t(x)^2) \le C$.

Third term. We recall here that $\eta_t^2(x)$ is defined for $x \in U$ by $\eta_t^2(x) = \int_{1-t}^{\infty} \int_D p_{\frac{t}{2}} * p^D_{\frac{s}{2}}(x,y) W(dy, ds)$. Similarly, there exists $C > 0$ such that for $t \in (0, 1/2)$, $x, x' \in U$, we have

$$\mathbb{E}\left(\left(\eta_t^2(x) - \eta_t^2(x')\right)^2\right) \le \int_{1/2}^{\infty} \int_D \left(p_{\frac{t}{2}} * p^D_{\frac{s}{2}}(x,y) - p_{\frac{t}{2}} * p^D_{\frac{s}{2}}(x',y)\right)^2 dyds \le C|x-x'|.$$

Furthermore, the pointwise variance is uniformly bounded.

Result. Altogether, coming back to (6.97) and combining Kolmogorov continuity criterion with Fernique's theorem (see Section 1.3 in [46]), we get the following tail estimate on the above coupling:

there exist $C, c > 0$ such that for all $t \in (0, 1/2)$, $x \geq 0$, we have

$$\mathbb{P}\left(\left\|\varphi_t - p_{\frac{t}{2}} * h\right\|_U \geq x\right) \leq Ce^{-cx^2}.$$

3.6.2 Approximations for $\delta \in (0, 1)$

We explain here how results obtained along the sequence $\{2^{-n} : n \geq 0\}$ can be extended to $\delta \in (0, 1)$. For each $\delta \in (0, 1)$, let $n \geq 0$ and $r \in [0, 1]$ such that $\delta = 2^{-(n+r)}$. Then by decoupling the field $\phi_{0,r}$, using a uniform estimate for $r \in [0, 1]$ and a scaling argument, we generalize our previous results obtained along the sequence 2^{-n} to $\delta \in (0, 1)$.

Decoupling low frequency noise. Note that there exists $C > 0$ such that for $n \geq 0$ and $r \in [0, 1]$ we have

$$e^{-C}\lambda_n \leq \lambda_{n+r} \leq \lambda_n e^C. \tag{6.98}$$

Indeed, note that a.s. $e^{-\xi \inf_{[0,1]^2} \phi_{0,r}} L_{1,1}^{(r,n+r)} \leq L_{1,1}^{(n+r)} \leq e^{\xi \sup_{[0,1]^2} \phi_{0,r}} L_{1,1}^{(r,n+r)}$. Furthermore, with high probability $\sup_{[0,1]^2} |\phi_{0,r}| \leq C_r \leq C$. Then, note that $L_{1,1}^{(r,n+r)} \overset{(d)}{=} 2^{-r} L_{2^r,2^r}^{(n)}$ and a.s. $L_{1,2}^{(n)} \leq L_{2^r,2^r}^{(n)} \leq L_{2,1}^{(n)}$. By the tightness result, there exists a constant $C > 0$ such that uniformly in n, with high probability, $L_{1,2}^{(n)} \geq e^{-C}\lambda_n$ and $L_{2,1}^{(n)} \leq e^C \lambda_n$, therefore, with high probability, $e^{-C}\lambda_n \leq L_{1,1}^{(r,n+r)} \leq e^C \lambda_n$, hence (6.98).

Weak multiplicativity. In this paragraph, we will use the notation λ_δ from the introduction. We recall that writing λ_n instead of $\lambda_{2^{-n}}$ was an abuse of notation. Now we prove that there exists $C > 0$ such that for $\delta, \delta' \in (0, 1)$ we have

$$C^{-1} e^{-C\sqrt{|\log \delta \vee \delta'|}} \lambda_\delta \lambda_{\delta'} \leq \lambda_{\delta\delta'} \leq C e^{C\sqrt{|\log \delta \vee \delta'|}} \lambda_\delta \lambda_{\delta'}. \tag{6.99}$$

Similarly as (6.98), there exists $C > 0$ such that for $r, r' \in [0, 1]$, $n, n' \geq 0$,

$$e^{-C} \lambda_{2^{-n-n'}} \leq \lambda_{2^{-n-r-n'-r'}} \leq \lambda_{2^{-n-n'}} e^C. \tag{6.100}$$

For $\delta, \delta' \in (0,1)$, let $n, n' \geq 0$ and $r, r' \in [0,1]$ such that $\delta = 2^{-(n+r)}$, $\delta' = 2^{-(n'+r')}$. Note that $n = [-\log_2 \delta]$. Using the weak multiplicativity for powers of 2, we have

$$e^{-C\sqrt{n \wedge n'}} \lambda_{2^{-n}} \lambda_{2^{-n'}} \leq \lambda_{2^{-n-n'}} \leq \lambda_{2^{-n}} \lambda_{2^{-n'}} e^{C\sqrt{n \wedge n'}}. \tag{6.101}$$

Without loss of generality, we consider just the upper bound in (6.99). The lower bound follows along the same lines. By using first (6.100) and then (6.101) we get

$$\lambda_{\delta\delta'} = \lambda_{2^{-n-r-n'-r'}} \leq \lambda_{2^{-n-n'}} e^C \leq \lambda_{2^{-n}} \lambda_{2^{-n'}} e^{C\sqrt{n \wedge n'}} e^C.$$

Now, the result follows by using (6.98):

$$\lambda_{2^{-n}} \lambda_{2^{-n'}} e^{C\sqrt{n \wedge n'}} \leq \lambda_{2^{-n-r}} \lambda_{2^{-n'-r'}} e^{C\sqrt{n+r \wedge n'+r'}} e^{2C} = \lambda_\delta \lambda_{\delta'} e^{C\sqrt{\log|\delta \vee \delta'|}} e^{2C}.$$

Tail estimates and tightness of metrics. Using the same argument as in the two previous paragraphs and the tail estimates obtained along the sequence $\{2^{-n} : n \geq 1\}$, we have the following tail estimates for crossing lengths of the rectangles $[0,a] \times [0,b]$: there exists $c, C > 0$ (depending only on a, b and γ) such that for $s > 2$, uniformly in $\delta \in (0,1)$, we have

$$\mathbb{P}\left(\lambda_\delta^{-1} L_{a,b}^{(\delta)} \geq e^s\right) \leq C e^{-c\frac{s^2}{\log s}}; \tag{6.102}$$

$$\mathbb{P}\left(\lambda_\delta^{-1} L_{a,b}^{(\delta)} \leq e^{-s}\right) \leq C e^{-cs^2}. \tag{6.103}$$

Furthermore, the sequence of metrics $(\lambda_\delta^{-1} e^{\xi\phi_\delta} ds)_{\delta \in (0,1)}$ on $[0,1]^2$ is tight.

Chapter 4: Weak LQG metrics and Liouville first passage percolation

This chapter corresponds to the joint work [39] with Julien Dubédat, Ewain Gwynne, Joshua Pfeffer and Xin Sun.

4.1 Introduction

For $\gamma \in (0,2)$, we define a *weak γ-Liouville quantum gravity (LQG) metric* to be a function $h \mapsto D_h$ which takes in an instance of the planar Gaussian free field (GFF) and outputs a metric on the plane satisfying a certain list of natural axioms. We show that these axioms are satisfied for any subsequential limits of Liouville first passage percolation. Such subsequential limits were proven to exist in the previous chapter, namely in [24]. It is also known that these axioms are satisfied for the $\sqrt{8/3}$-LQG metric constructed by Miller and Sheffield in [81–83, 86].

For any weak γ-LQG metric, we obtain moment bounds for diameters of sets as well as point-to-point, set-to-set, and point-to-set distances. We also show that any such metric is locally bi-Hölder continuous with respect to the Euclidean metric and compute the optimal Hölder exponents in both directions. Finally, we show that LQG geodesics cannot spend a long time near a straight line or the boundary of a metric ball. These results are used in subsequent work by Gwynne and Miller which proves that the weak γ-LQG metric is unique for each $\gamma \in (0,2)$, which in turn gives the uniqueness of the subsequential limit of Liouville first passage percolation. However, most of our results are new even in the special case when $\gamma = \sqrt{8/3}$. We remark that versions of some of the estimates for weak LQG metrics which are proven in this chapter (including tail estimates for the distance across a rectangle, the first moment bound for diameters, and Hölder continuity) were previously proven for subsequential limits of LFPP in [24], namely the second chapter of this thesis. However, it is important to have these estimates for general weak γ-LQG metrics: indeed, such estimates will be used in [59] to show the uniqueness of the weak γ-LQG metric (which is a stronger statement than just the uniqueness of the subsequential limit for the variant of LFPP considered in [24]). Many of our estimates are also new for subsequential limits of LFPP, e.g., the optimality of the Hölder exponents in Theorem 4.7, the moment bounds in Theorems 4.8, 4.10, and 4.11, and the estimates for geodesics in Section 4.4.

Due to our axiomatic approach, our proofs do not require any outside input besides the existence of LFPP subsequential limits from [24] and a general theorem about local metrics from [57] (both of which can be taken as black boxes).

4.1.1 Weak LQG metrics and subsequential limits of LFPP

We will primarily focus on the whole-plane case. We say that a random distribution h on $\mathbb{C}$ is a *whole plane GFF plus a continuous function* if there exists a coupling of h with a random continuous function $f : \mathbb{C} \to \mathbb{R}$ such that the law of $h - f$ is that of a whole-plane GFF. If such a coupling exists for which f is bounded, then we say that h is a *whole-plane GFF plus a bounded continuous function.*[1] Note that the whole-plane GFF is defined only modulo a global additive constant, but these definitions do not depend on the choice of additive constant.

If h is a whole-plane GFF, or more generally a whole-plane GFF plus a bounded continuous function, we define a mollified version of the GFF by

$$h_\varepsilon^*(z) := (h * p_{\varepsilon^2/2})(z) = \int_{\mathbb{C}} h(w) p_{\varepsilon^2/2}(z, w)\, dw, \tag{1.1}$$

where $p_s(z, w) = \frac{1}{2\pi s} \exp\left(-\frac{|z-w|^2}{2s}\right)$ is the heat kernel on $\mathbb{C}$ and where the integral is interpreted in the sense of distributional pairing. For $z, w \in \mathbb{C}$ and $\varepsilon > 0$, we define the *ε-LFPP metric* by

$$D_h^\varepsilon(z, w) := \inf_{P : z \to w} \int_0^1 e^{\xi h_\varepsilon^*(P(t))} |P'(t)|\, dt \tag{1.2}$$

where the infimum is over all piecewise continuously differentiable paths from z to w.

Remark 4.1. The reason why we define LFPP using h_ε^* instead of some other continuous approximation of the GFF is that this is the approximation for which tightness is proven in [24]. If we had a tightness result similar to those in [24] for LFPP defined using a different approximation (such as

[1]The reason why we sometimes restrict to bounded continuous functions is that it ensures that the convolution with the whole-plane heat kernel is finite (so D_h^ε is defined) and it makes parts of the proof of Theorem 4.2 simpler.

the circle average process of [44, Section 3.1] or the convolution of h with $\varepsilon^{-1}\phi(|z-w|/\sqrt{\varepsilon})$, where ϕ is a continuous non-negative radially symmetric function with total integral one), then similar arguments to those in Section 4.2 would show that the subsequential limits are also weak LQG metrics. Together with the uniqueness of weak LQG metrics proven in [59], this means that in order to show that such approximations converge to the γ-LQG metric one only needs to prove tightness.

For $\varepsilon > 0$, let $\mathfrak{a}_\varepsilon$ be the median of the D_h^ε-distance between the left and right boundaries of the unit square along paths which stay in the unit square. It follows from results in [24] (see Lemma 4.17 below) that the laws of the metrics $\{\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon\}_{\varepsilon>0}$ are tight with respect to the local uniform topology on $\mathbb{C} \times \mathbb{C}$ and every subsequential limit induces the Euclidean topology on $\mathbb{C}$.

Building on this, we will prove that in fact the metrics $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ admit subsequential limits in probability and that every subsequential limit satisfies a certain natural list of axioms. To state these axioms, we need some preliminary definitions. Let (X, D) be a metric space.

For a curve $P : [a, b] \to X$, the *D-length* of P is defined by

$$\mathrm{len}\,(P; D) := \sup_T \sum_{i=1}^{\#T} D(P(t_i), P(t_{i-1}))$$

where the supremum is over all partitions $T : a = t_0 < \cdots < t_{\#T} = b$ of $[a, b]$. Note that the D-length of a curve may be infinite.

For $Y \subset X$, the *internal metric of D on Y* is defined by

$$D(x, y; Y) := \inf_{P \subset Y} \mathrm{len}\,(P; D)\,, \quad \forall x, y \in Y \tag{1.3}$$

where the infimum is over all paths P in Y from x to y. Then $D(\cdot, \cdot; Y)$ is a metric on Y, except that it is allowed to take infinite values.

We say that (X, D) is a *length space* if for each $x, y \in X$ and each $\varepsilon > 0$, there exists a curve of D-length at most $D(x, y) + \varepsilon$ from x to y.

A *continuous metric* on a domain $U \subset \mathbb{C}$ is a metric D on U which induces the Euclidean topology on U, i.e., the identity map $(U, |\cdot|) \to (U, D)$ is a homeomorphism. We equip the space of continuous metrics on U with the local uniform topology for functions from $U \times U$ to $[0, \infty)$ and the associated

Borel σ-algebra. We allow a continuous metric to have $D(u,v)=\infty$ if u and v are in different connected components of U. In this case, in order to have $D^n \to D$ w.r.t. the local uniform topology we require that for large enough n, $D^n(u,v)=\infty$ if and only if $D(u,v)=\infty$.

Let $\mathcal{D}'(\mathbb{C})$ be the space of distributions (generalized functions) on $\mathbb{C}$, equipped with the usual weak topology. For $\gamma \in (0,2)$, a *weak γ-LQG metric* is a measurable function $h \mapsto D_h$ from $\mathcal{D}'(\mathbb{C})$ to the space of continuous metrics on $\mathbb{C}$ such that the following is true whenever h is a whole-plane GFF plus a continuous function.

I. **Length space.** Almost surely, $(\mathbb{C}, D_h)$ is a length space, i.e., the D_h-distance between any two points of $\mathbb{C}$ is the infimum of the D_h-lengths of D_h-continuous paths (equivalently, Euclidean continuous paths) between the two points.

II. **Locality.** Let $U \subset \mathbb{C}$ be a deterministic open set. The D_h-internal metric $D_h(\cdot,\cdot;U)$ is determined a.s. by $h|_U$.

III. **Weyl scaling.** Let ξ be as in (3.4) and for each continuous function $f:\mathbb{C}\to\mathbb{R}$, define

$$(e^{\xi f}\cdot D_h)(z,w) := \inf_{P:z\to w}\int_0^{\operatorname{len}(P;D_h)} e^{\xi f(P(t))}\,dt, \quad \forall z,w\in\mathbb{C}, \tag{1.4}$$

where the infimum is over all continuous paths from z to w parametrized by D_h-length. Then a.s. $e^{\xi f}\cdot D_h = D_{h+f}$ for every continuous function $f:\mathbb{C}\to\mathbb{R}$.

IV. **Translation invariance.** For each deterministic point $z\in\mathbb{C}$, a.s. $D_{h(\cdot+z)} = D_h(\cdot+z,\cdot+z)$.

V. **Tightness across scales.** Suppose that h is a whole-plane GFF and let $\{h_r(z)\}_{r>0,z\in\mathbb{C}}$ be its circle average process. For each $r>0$, there is a deterministic constant $\mathfrak{c}_r>0$ such that the set of laws of the metrics $\mathfrak{c}_r^{-1}e^{-\xi h_r(0)}D_h(r\cdot,r\cdot)$ for $r>0$ is tight (w.r.t. the local uniform topology). Furthermore, the closure of this set of laws w.r.t. the Prokhorov topology on continuous functions $\mathbb{C}\times\mathbb{C}\to[0,\infty)$ is contained in the set of laws on continuous metrics on $\mathbb{C}$ (i.e., every subsequential limit of the laws of the metrics $\mathfrak{c}_r^{-1}e^{-\xi h_r(0)}D_h(r\cdot,r\cdot)$ is supported on metrics which induce the Euclidean topology on $\mathbb{C}$). Finally, there exists $\Lambda>1$ such that for each $\delta\in(0,1)$,

$$\Lambda^{-1}\delta^{\Lambda} \le \frac{\mathfrak{c}_{\delta r}}{\mathfrak{c}_r} \le \Lambda\delta^{-\Lambda}, \quad \forall r>0. \tag{1.5}$$

We emphasize that the definition of a weak γ-LQG metric depends on γ only via the parameter ξ in Axiom III. We will therefore sometimes say that a metric satisfying the above axioms is a *weak LQG metric with parameter* ξ.

It is easy to see, at least heuristically, why Axioms I through V should be satisfied for subsequential limits of LFPP, although there is some subtlety involved in checking these axioms rigorously. The first main result of this chapter is the following statement, whose proof builds on results from [24,57].

Theorem 4.2. *Let* $\gamma \in (0,2)$*. For every sequence of* ε*'s tending to zero, there is a weak* γ*-LQG metric* D *and a subsequence* $\{\varepsilon_n\}_{n\in\mathbb{N}}$ *for which the following is true. Let* h *be a whole-plane GFF, or more generally a whole-plane GFF plus a bounded continuous function. Then the re-scaled LFPP metrics* $\mathfrak{a}_{\varepsilon_n}^{-1} D_h^{\varepsilon_n}$ *from* (1.2) *converge in probability to* D_h*.*

We will explain why we get convergence in probability, instead of just in law, in Theorem 4.2 just below. Let us first discuss the axioms for a weak LQG metric. Axioms I through IV are natural from the perspective that γ-LQG is a “random two-dimensional Riemannian manifold” obtained by exponentiating h. Axiom V is a substitute for exact scale invariance of the metric. To explain this, it is expected (and will be proven in [56,59]) that the γ-LQG metric, like the γ-LQG measure, is invariant under coordinate changes of the form (2.3). In particular, it should be the case that for any $a \in \mathbb{C}\setminus\{0\}$, a.s.

$$D_h(a\cdot, a\cdot) = D_{h(a\cdot)+Q\log|a|}(\cdot,\cdot), \quad \text{for} \quad Q = \frac{2}{\gamma} + \frac{\gamma}{2}. \tag{1.6}$$

Under Axiom III, the formula (1.6) together with the scale invariance of the law of h, modulo an additive constant, implies Axiom V with $\mathfrak{c}_r = r^{\xi Q}$. We define a *strong LQG metric* to be a mapping $h \mapsto D_h$ which satisfies Axioms I through IV as well as (1.6).

A similar definition of a strong LQG metric has appeared in earlier literature. Indeed, the paper [80] proved several properties of geodesics for any metric associated with γ-LQG which satisfies a similar list of axioms to the ones in our definition of a strong LQG metric; however, at that point such a metric had only been constructed for $\gamma = \sqrt{8/3}$.

It far from obvious that subsequential limits of LFPP satisfy (1.6). The reason for this is that scaling space results in scaling the value of ε in (1.2), which in turn changes the subsequence which we are working with. It will eventually be proven in [59] that every weak LQG metric satisfies (1.6), i.e., every weak LQG metric is a strong LQG metric, but the proof requires all of the results of the present chapter as well as those of [57, 58].

Nevertheless, Axiom V can be used in place of (1.6) in many situations. Basically, this axiom allows us to compare distance quantities at the same Euclidean scale. For example, Axiom V implies that if $U \subset \mathbb{C}$ is open and $K \subset U$ is compact, then the laws of

$$\left(\mathfrak{c}_r^{-1} e^{-\xi h_r(0)} D_h(rK, r\partial U)\right)^{-1} \quad \text{and} \quad \mathfrak{c}_r^{-1} e^{-\xi h_r(0)} \sup_{u,v \in rK} D_h(u,v; rU) \tag{1.7}$$

as r varies are tight.

Part of the proof of Theorem 4.2 is to show that for any joint subsequential limit (h, D_h) of the laws of the pairs $(h, \mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon)$, the limiting metric D_h is a measurable function of h. This is not obvious since convergence in law does not in general preserve measurability. In our setting, we will prove that D_h is determined by h by checking the conditions of [57, Corollary 1.8], which gives a list of conditions under which a random metric coupled with the GFF is determined by the GFF. The reason why we have convergence in probability, instead of convergence in law, in Theorem 4.2 is the following elementary probabilistic lemma (see e.g. [97, Lemma 4.5]).[2]

Lemma 4.3. *Let (Ω_1, d_1) and (Ω_2, d_2) be complete separable metric spaces. Let X be a random variable taking values in Ω_1 and let $\{Y^n\}_{n\in\mathbb{N}}$ and Y be random variables taking values in Ω_2, all defined on the same probability space, such that $(X, Y^n) \to (X, Y)$ in law. If Y is a.s. determined by X, then $Y^n \to Y$ in probability.*

Theorem 4.2 will be proven in Section 4.2. Once this is done, throughout the rest of the chapter we will only ever work with a weak γ-LQG metric — we will not need to make explicit reference to LFPP. An important advantage of this approach is that the Miller-Sheffield $\sqrt{8/3}$-LQG metric from [81, 82, 86] is known to satisfy the axioms for a weak $\sqrt{8/3}$-LQG metric. See [60, Section 2.4] for a careful explanation of why this is the case. Note that [60, Section 2.4] checks the coordinate change relation (1.6) for the Miller-Sheffield metric which (as discussed above) implies Axiom V.

[2]Since the space of continuous metrics is not complete w.r.t. any natural choice of metric which induces the local uniform topology, we apply the lemma with (Ω_2, d_2) equal to the larger space of continuous functions $\mathbb{C} \times \mathbb{C} \to [0, \infty)$ equipped with the local uniform topology, which is completely metrizable.

Hence all of our results for weak γ-LQG metrics apply to both this $\sqrt{8/3}$-LQG metric and to subsequential limits of LFPP.

Remark 4.4 (Liouville graph distance)**.** Besides LFPP, there is another natural scheme for approximating LQG metrics called *Liouville graph distance* (LGD). The ε-LGD distance between two points in $\mathbb{C}$ is defined to be the minimum number of Euclidean balls with LQG mass ε whose union contains a path between the two points. It has been proven in [26] that for each $\gamma \in (0,2)$, the ε-LGD metric, appropriately renormalized, admits subsequential limiting metrics as $\varepsilon \to 0$ which induce the Euclidean topology. In the contrast to LFPP, for subsequential limits of LGD the coordinate change relation (1.6) is easy to verify but Weyl scaling (Axiom III) appears to be very difficult to verify, so these subsequential limits are *not* known to be weak LQG metrics in the sense of this chapter. It is still an open problem to establish uniqueness of the scaling limit for LGD. Similar considerations apply to variants of LGD defined using embedded planar maps (such as maps constructed from LQG square subdivision [44, 53] or mated-CRT maps [54, 61]) instead of Euclidean balls, although for these variants tightness has not been checked.

4.1.2 Quantitative properties of weak LQG metrics

In what follows, we assume that D is a weak γ-LQG metric and h is a whole-plane GFF. Perhaps surprisingly, the axioms for a weak LQG metric imply much sharper bounds on the scaling constants $\mathfrak{c}_r$ than (1.5).

Theorem 4.5. *Let ξ be as in* (3.4) *and let $Q = 2/\gamma + \gamma/2$. Then for $r > 0$, the scaling constants satisfy*

$$\frac{\mathfrak{c}_{\delta r}}{\mathfrak{c}_r} = \delta^{\xi Q + o_\delta(1)} \quad \text{as } \delta \to 0, \tag{1.8}$$

at a rate which is uniform over all $r > 0$.

The definition of a weak LQG metric uses only the parameter ξ. Theorem 4.5 connects this definition to the coordinate change parameter Q. This will be important for the proof in [59] that any weak LQG metric satisfies the coordinate change formula (1.6). Theorem 4.5 will be proven in Section 4.3.2 by comparing D_h-distances to LFPP distances and using the fact that the δ-LFPP distance between two fixed points is typically of order $\delta^{1-\xi Q + o_\delta(1)}$ [28, Theorem 1.5] (for convenience, for this argument we will work with a variant of LFPP which is defined in a slightly different manner than the version in (1.2)).

Remark 4.6. Theorem 4.5 gives a proof purely in the continuum that the exponent $d_{\sqrt{8/3}}$ of [28,32] is equal to 4. Previously, this was proven in [28] (building on [55]) using the known ball volume growth exponent for random triangulations [7]. To see why Theorem 4.5 implies that $d_{\sqrt{8/3}} = 4$, we observe that the $\sqrt{8/3}$-LQG metric of [81, 82, 86] satisfies the axioms for a weak LQG metric with parameter $\xi = 1/\sqrt{6}$. Moreover, by the LQG coordinate change formula for the $\sqrt{8/3}$-LQG metric, Axiom V holds for this metric with with $\mathfrak{c}_r = r^{5/6}$. Theorem 4.5 therefore implies that if $\gamma \in (0,2)$ is chosen so that $\gamma/d_\gamma = 1/\sqrt{6}$, then the associated parameter $Q = 2/\gamma + \gamma/2$ satisfies $Q/\sqrt{6} = 5/6$, i.e., $Q = 5/\sqrt{6}$ which is equivalent to $\gamma = \sqrt{8/3}$. Hence $\gamma/d_\gamma = 1/\sqrt{6}$ when $\gamma = \sqrt{8/3}$, so $d_{\sqrt{8/3}} = 4$.

Our next main result gives the optimal Hölder exponents for D_h with respect to the Euclidean metric.

Theorem 4.7 (Optimal Hölder exponents)**.** *Let $U \subset \mathbb{C}$ be open and bounded. Almost surely, the identity map from U, equipped with the Euclidean metric, to (U, D_h) is locally Hölder continuous with any exponent smaller than $\xi(Q-2)$ and is not locally Hölder continuous with any exponent larger than $\xi(Q-2)$. Furthermore, the inverse of this map is a.s. locally Hölder continuous with any exponent smaller than $\xi^{-1}(Q+2)^{-1}$ and is not locally Hölder continuous with any exponent larger than $\xi^{-1}(Q+2)^{-1}$.*

For $\gamma = \sqrt{8/3}$, one has $\xi = 1/\sqrt{6}$ and $Q = 5/\sqrt{6}$, so the optimal Hölder exponents are given by

$$\xi(Q-2) = \frac{1}{6}(5 - 2\sqrt{6}) \approx 0.0168 \quad \text{and} \quad \xi^{-1}(Q+2)^{-1} = 30 - 12\sqrt{6} \approx 0.6061. \tag{1.9}$$

The intuitive reason why Theorem 4.7 is true is as follows. If z is an α-thick point for h, i.e., the circle average satisfies $h_\varepsilon(z) = (\alpha + o_\varepsilon(1)) \log \varepsilon^{-1}$ as $\varepsilon \to 0$, then we can show that the D_h-distance from z to $\partial B_\varepsilon(z)$ behaves like $\varepsilon^{\xi(Q-\alpha)+o_\varepsilon(1)}$ as $\varepsilon \to 0$. Indeed, this is an easy consequence of the estimates in Section 4.3.4. Almost surely, α-thick points exist for $\alpha \in (-2,2)$ but not for $|\alpha| > 2$ [65].

We next state some basic moment estimates for distances which are metric analogues of the well-known fact that the γ-LQG measure has finite moments of all orders in $(-\infty, 4/\gamma^2)$ [90, Theorems 2.11 and 2.12].

Theorem 4.8 (Moment bounds for diameters)**.** *Let $U \subset \mathbb{C}$ be open and let $K \subset U$ be a compact connected set with more than one point. Then the U-internal diameter of K satisfies*

$$\mathbb{E}\left[\left(\sup_{z,w\in K} D_h(z,w;U)\right)^p\right] < \infty, \quad \forall p \in \left(-\infty, \frac{4d_\gamma}{\gamma^2}\right). \tag{1.10}$$

For $\gamma = \sqrt{8/3}$, we get finite moments up to order 6. We also have the following bound for distances between sets. In this case, we get finite moments of all orders.

Theorem 4.9 (Distance between sets)**.** *Let $U \subset \mathbb{C}$ be an open set (possibly all of $\mathbb{C}$) and let $K_1, K_2 \subset U$ be connected, disjoint compact sets which are not singletons. Then*

$$\mathbb{E}\left[(D_h(K_1,K_2;U))^p\right] < \infty, \quad \forall p \in \mathbb{R}. \tag{1.11}$$

The results of [24] show that if D_h is a subsequential scaling limit of the LFPP metrics (1.2), then one has the following slightly stronger version of Theorem 4.9:

$$\mathbb{P}\left[A^{-1} \leq \mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon(K_1,K_2;U) \leq A\right] \geq 1 - c_0 e^{-c_1(\log A)^2/\log\log A}, \quad \forall A > 2e^e \tag{1.12}$$

for constants $c_0, c_1 > 0$ allowed to depend on K_1, K_2, U. *A posteriori*, one gets (1.12) for every weak LQG metric since [59] proves that the weak LQG metric is unique for each $\gamma \in (0,2)$, so in particular it is the limit of LFPP.

We now turn our attention to point-to-point distances. These estimates also work if we allow the field to have a log singularity. To make sense of the metric in this case, we note that since $\log|\cdot|$ is continuous away from 0, we can define $D_{h-\alpha\log|\cdot|}$ as a continuous length metric on $\mathbb{C}\setminus\{0\}$ by $D_{h-\alpha\log|\cdot|} = |\cdot|^{-\alpha\xi}\cdot D_h$, in the notation (1.4). We can then extend $D_{h-\alpha\log|\cdot|}$ to a metric defined on all of $\mathbb{C}$ which is allowed to take the value ∞ by taking the infima of the $D_{h-\alpha\log|\cdot|}$-lengths of paths. We can similarly define the metric associated with fields with two or more log singularities.

Theorem 4.10 (Distance from a point to a circle)**.** *Let $\alpha \in \mathbb{R}$ and let $h^\alpha := h - \alpha\log|\cdot|$. If*

$\alpha \in (-\infty, Q)$, *then*

$$\mathbb{E}\left[(D_{h^\alpha}(0, \partial\mathbb{D}))^p\right] < \infty, \quad \forall p \in \left(-\infty, \frac{2d_\gamma}{\gamma}(Q-\alpha)\right). \tag{1.13}$$

If $\alpha > Q$, *then a.s.* $D_{h^\alpha}(0, z) = \infty$ *for every* $z \in \mathbb{C} \setminus \{0\}$.

For example, if $\gamma = \sqrt{8/3}$ and $\alpha = 0$, we get finite moments up to order 10. If instead $\gamma = \sqrt{8/3}$ and $\alpha = \gamma$ (which corresponds to the case when 0 is a "quantum typical" point, see, e.g., [44, Proposition 3.4]) we only get finite moments up to order 2. In the critical case when $\alpha = Q$, our estimates at this point are not sufficiently sharp to determine whether $D_{h^Q}(0, \partial\mathbb{D})$ is finite. However, once we know that every weak LQG metric is a strong LQG metric (which is proven in [59]) it is not hard to check that a.s. $D_{h^Q}(0, z) = \infty$ for every $z \in \mathbb{C} \setminus \{0\}$. Similar comments apply in the case when $\alpha = Q$ or $\beta = Q$ in Theorem 4.11 just below.

Theorem 4.11 (Distance between two points)**.** *Let* $\alpha, \beta \in \mathbb{R}$, *let* $z, w \in \mathbb{C}$ *be distinct, and let* $h^{\alpha,\beta} := h - \alpha \log|\cdot - z| - \beta \log|\cdot - w|$. *If* $\alpha, \beta \in (-\infty, Q)$, *then*

$$\mathbb{E}\left[\left(D_{h^\alpha}\left(z, w; B_{4|z-w|}(z)\right)\right)^p\right] < \infty, \quad \forall p \in \left(-\infty, \frac{2d_\gamma}{\gamma}(Q - \max\{\alpha, \beta\})\right). \tag{1.14}$$

If either $\alpha > Q$ *or* $\beta > Q$, *then a.s.* $D_{h^{\alpha,\beta}}(z, w) = \infty$.

As applications of our main results, in Section 4.4 we will also prove some estimates which constrain the behavior of D_h-geodesics and which will be important in [59]. To be more precise, the first main estimate of Section 4.4 is Proposition 4.57, which gives an upper bound for the amount of time that a D_h-geodesic can spend in a small neighborhood of a line segment or a circular arc. Intuitively, one expects that this amount of time is small since LQG geodesics should be fractal and hence should look very different from smooth curves. The particular bound given in Proposition 4.57 is used in [59, Section 3] to prevent a geodesic from spending a long time in an annulus with a small aspect ratio; and in [59, Section 5] in order to force a geodesic to enter a "good" region of the plane in which certain distance bounds hold.

The other main estimate in Section 4.4 is Proposition 4.59, which is an upper bound for how much time an LQG geodesic can spend near the boundary of an LQG metric ball centered at its starting point. Intuitively, this amount of time should be small since if P is a D_h-geodesic, then $D_h(P(0), P(t)) = t$ but $D_h(P(0), \cdot)$ is constant on the boundary of a D_h-ball centered at $P(0)$. The bound given in Proposition 4.59 is used in [59, Lemma 4.7].

Remark 4.12 (The case when $\xi > 2/d_2$)**.** Throughout this chapter, we focus on the case of weak γ-LQG metrics. Since $\gamma \mapsto \gamma/d_\gamma$ is increasing [28, Proposition 1.7], weak γ-LQG metrics have parameter $\xi \in (0, 2/d_2)$ (here, $d_2 := \lim_{\gamma \to 2^-} d_\gamma$). It is natural to wonder whether one can say anything about weak LQG metrics which satisfy the same axioms but with a parameter $\xi \geq 2/d_2$. In the critical case when $\xi = 2/d_2$ (i.e., $\gamma = 2$), we expect that a weak LQG metric still exists and is the scaling limit of LFPP with parameter $2/d_2$. This metric should be the γ-LQG metric with $\gamma = 2$ (the $\gamma = 2$ metric should also be the limit as $\gamma \nearrow 2$ of the γ-LQG metrics, appropriately renormalized). We expect that all of the theorem statements in this section still hold for $\xi = 2/d_2$, except that the metric D_h is not Hölder continuous w.r.t. the Euclidean metric.

For $\xi > 2/d_2$, we do not expect that any weak LQG metrics with parameter ξ exist. However, there should be metrics which satisfy a similar list of properties except that such metrics no longer induce the Euclidean topology. Instead, there should be an uncountable, dense set of points $z \in \mathbb{C}$ such that $D_h(z, w) = \infty$ for every $w \in \mathbb{C} \setminus \{z\}$. More precisely, let $\lambda(\xi)$ be the exponent for the typical LFPP distance between the left and right sides of $[0,1]^2$ and let $Q(\xi) = (1 - \lambda(\xi))/\xi$. By [28, Theorem 1.5], $Q(\gamma/d_\gamma) = 2/\gamma + \gamma/2 > 2$. By [63, Lemma 4.1] and [30, Theorem 1.1], $Q(\xi) \in (0, 2)$ for $\xi > 2/d_2$. For $\xi > 2/d_2$, the points $z \in \mathbb{C}$ which lie at infinite D_h-distance from every other point should correspond to so-called *thick points* of h (as defined in [65]) with thickness $\alpha > Q$.

It is shown in [29] that LFPP with parameter $\xi > 2/d_2$ admits subsequential scaling limits in law w.r.t. the topology on lower semicontinuous functions. We expect that the subsequential limit is unique, satisfies the properties discussed in the preceding paragraph, and is related to LQG with matter central charge $\mathbf{c} \in (1, 25)$ (LQG with $\gamma \in (0, 2]$ corresponds to $\mathbf{c} \in (-\infty, 1]$). In particular, with $Q(\xi)$ as above, the central charge should be related to ξ by $\mathbf{c} = 25 - 6Q(\xi)^2$. See [6,29,30,53,63] for further discussion of this extended phase of LQG and some justification for the above predictions.

4.1.3 Outline

In Section 4.2, we prove Theorem 4.2, which says that subsequential limits of LFPP are weak γ-LQG metrics, taking [24] as a starting point. Throughout the rest of the chapter, we work with an arbitrary weak γ-LQG metric (not necessarily assumed to arise as a subsequential limit of LFPP). Section 4.3 contains the proofs of the results stated in Section 4.1.2. In fact, for most of these results, we will prove more quantitative versions which are required to be uniform over all Euclidean scales. At this point, these statements are not implied by the statements in Section 4.1.2 since we are working with a weak γ-LQG metric, which is only known to be "tight across scales" (Axiom V) instead of exactly scale invariant.

The first result that we prove for a weak γ-LQG metric is the estimate for the distance between two sets from Theorem 4.9; this is the content of Section 4.3.1. In Section 4.3.2, we use this estimate to relate D_h-distances to LFPP distances and thereby prove Theorem 4.5. Once Theorem 4.5 is established, we have some ability to compare D_h-distances at different Euclidean scales. This allows us to prove the moment estimate (1.10) of Theorem 4.8 in Section 4.3.3 as well as the moment estimates of Theorems 4.10 and 4.11 in Section 4.3.4. Using these moment estimates, we then prove Theorem 4.7 in Section 4.3.5.

In Section 4.4, we apply the estimates of Section 4.1.2 to prove some bounds for D_h-geodesics.

4.1.4 Notation

We write $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

For $a < b$, we define the discrete interval $[a, b]_{\mathbb{Z}} := [a, b] \cap \mathbb{Z}$.

If $f : (0, \infty) \to \mathbb{R}$ and $g : (0, \infty) \to (0, \infty)$, we say that $f(\varepsilon) = O_\varepsilon(g(\varepsilon))$ (resp. $f(\varepsilon) = o_\varepsilon(g(\varepsilon))$) as $\varepsilon \to 0$ if $f(\varepsilon)/g(\varepsilon)$ remains bounded (resp. tends to zero) as $\varepsilon \to 0$. We similarly define $O(\cdot)$ and $o(\cdot)$ errors as a parameter goes to infinity.

If $f, g : (0, \infty) \to [0, \infty)$, we say that $f(\varepsilon) \preceq g(\varepsilon)$ if there is a constant $C > 0$ (independent from ε and possibly from other parameters of interest) such that $f(\varepsilon) \leq C g(\varepsilon)$. We write $f(\varepsilon) \asymp g(\varepsilon)$ if $f(\varepsilon) \preceq g(\varepsilon)$ and $g(\varepsilon) \preceq f(\varepsilon)$.

Let $\{E^\varepsilon\}_{\varepsilon>0}$ be a one-parameter family of events. We say that E^ε occurs with

- *polynomially high probability* as $\varepsilon \to 0$ if there is a $p > 0$ (independent from ε and possibly from other parameters of interest) such that $\mathbb{P}[E^\varepsilon] \geq 1 - O_\varepsilon(\varepsilon^p)$.
- *superpolynomially high probability* as $\varepsilon \to 0$ if $\mathbb{P}[E^\varepsilon] \geq 1 - O_\varepsilon(\varepsilon^p)$ for every $p > 0$.

We similarly define events which occur with polynomially or superpolynomially high probability as a parameter tends to ∞.

We will often specify any requirements on the dependencies on rates of convergence in $O(\cdot)$ and $o(\cdot)$ errors, implicit constants in $\preceq$, etc., in the statements of lemmas/propositions/theorems, in which case we implicitly require that errors, implicit constants, etc., appearing in the proof satisfy the same dependencies.

For $z \in \mathbb{C}$ and $r > 0$, we write $B_r(z)$ for the Euclidean ball of radius r centered at z. We also define the open annulus

$$\mathbb{A}_{r_1,r_2}(z) := B_{r_2}(z) \setminus \overline{B_{r_1}(z)}, \quad \forall 0 < r_r < r_2 < \infty. \tag{1.15}$$

We write $\mathbb{S} = (0,1)^2$ for the open Euclidean unit square.

4.2 Subsequential limits of LFPP are weak LQG metrics

The goal of this section is to deduce Theorem 4.2 from the tightness result of [24]. We start in Section 4.2.1 by introducing a "localized" variant of LFPP, defined using the convolution of h with a truncated version of the heat kernel, which (unlike the ε-LFPP metric D_h^ε defined in (1.2)) depends locally on h. We then show that this localized variant of LFPP is a good approximation for D_h^ε (Lemma 4.13). In Section 4.2.2, we explain why the results of [24] imply that the re-scaled LFPP metrics $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ as well as the associated internal metrics on certain domains in $\mathbb{C}$ are tight w.r.t. the local uniform topology and that every subsequential limit is a continuous length metric on $\mathbb{C}$. In Sections 4.2.3, 4.2.4, and 4.2.5, respectively, we will prove versions of Weyl scaling, tightness across scales, and locality for the subsequential limits (i.e., Axioms III, V, and II). In Section 4.2.6, we use a theorem from [57] to show that subsequential limits of LFPP can be realized as measurable functions of h. We then conclude the proof of Theorem 4.2.

Throughout this section, we will frequently need to switch between working with a whole-plane GFF and working with a whole-plane GFF plus a continuous function. As such, we will always write h for a whole-plane GFF (with some choice of additive constant, specified as needed) and h for a whole-plane GFF plus a continuous function (usually, this will be a whole-plane GFF plus a bounded continuous function). Note that this differs from the convention elsewhere in the chapter, where h is sometimes used to denote a whole-plane GFF plus a continuous function.

4.2.1 A localized version of LFPP

Let h be a whole-plane GFF plus a bounded continuous function. The mollified field $\mathsf{h}_\varepsilon^*(z)$ of (1.1) does not depend on h in a local manner, and hence $D_{\mathsf{h}}^\varepsilon$-distances do not depend on h in a local manner. However, as $\varepsilon \to 0$ the heat kernel $p_{\varepsilon^2/2}(z,w)$ concentrates around the diagonal, so we expect that $\mathsf{h}_\varepsilon^*(z)$ "almost" depends locally on h when ε is small. To quantify this, we will introduce an approximation $\widehat{\mathsf{h}}_\varepsilon^*$ of h_ε^* which depends locally on h and prove a lemma (Lemma 4.13) to the effect that $\widehat{\mathsf{h}}_\varepsilon^*$ and h_ε^* are close when ε are small. This will be useful at several places in this section, especially for the proof of locality (essentially, Axiom II) in Section 4.2.5.

For $\varepsilon > 0$, let $\psi_\varepsilon : \mathbb{C} \to [0,1]$ be a deterministic, smooth, radially symmetric bump function which is identically equal to 1 on $B_{\varepsilon^{1/2}/2}(0)$ and vanishes outside of $B_{\varepsilon^{1/2}}(0)$ (in fact, the power $1/2$ could be replaced by any $p \in (0,1)$). We can choose ψ_ε in such a way that $\varepsilon \mapsto \psi_\varepsilon$ is a continuous mapping from $(0,\infty)$ to the space of continuous functions on $\mathbb{C}$, equipped with the uniform topology. Recalling that $p_s(z,w)$ denotes the heat kernel, we define

$$\widehat{\mathsf{h}}_\varepsilon^*(z) := \int_{\mathbb{C}} \psi_\varepsilon(z-w)\mathsf{h}(w)p_{\varepsilon^2/2}(z,w)\,dw, \tag{2.16}$$

with the integral interpreted in the sense of distributional pairing. Since ψ_ε vanishes outside of $B_{\varepsilon^{1/2}}(0)$, we have that $\widehat{\mathsf{h}}_\varepsilon^*(z)$ is a.s. determined by $\mathsf{h}|_{B_{\varepsilon^{1/2}}(z)}$. It is easy to see that $\widehat{\mathsf{h}}_\varepsilon^*$ a.s. admits a continuous modification (see Lemma 4.13 below). We henceforth assume that $\widehat{\mathsf{h}}_\varepsilon^*$ is replaced by such a modification.

As in (1.2), we define the localized LFPP metric

$$\widehat{D}_{\mathsf{h}}^{\varepsilon}(z,w) := \inf_{P:z\to w} \int_0^1 e^{\xi \widehat{\mathsf{h}}_\varepsilon^*(P(t))} |P'(t)|\, dt, \tag{2.17}$$

where the infimum is over all piecewise continuously differentiable paths from z to w. By the definition of $\widehat{\mathsf{h}}_\varepsilon^*$,

$$\text{for any open } U \subset \mathbb{C}, \text{ the internal metric } \widehat{D}_{\mathsf{h}}^{\varepsilon}(\cdot,\cdot;U) \text{ is a.s. determined by } \mathsf{h}|_{B_{\varepsilon^{1/2}}(U)}. \tag{2.18}$$

Lemma 4.13. *Let* h *be a GFF plus a bounded continuous function. Then a.s.* $(z,\varepsilon) \mapsto \widehat{\mathsf{h}}_\varepsilon^*(z)$ *is continuous. Furthermore, for each bounded open set* $U \subset \mathbb{C}$*, a.s.*

$$\lim_{\varepsilon\to 0} \sup_{z\in\overline{U}} |\mathsf{h}_\varepsilon^*(z) - \widehat{\mathsf{h}}_\varepsilon^*(z)| = 0. \tag{2.19}$$

In particular, a.s.

$$\lim_{\varepsilon\to 0} \frac{\widehat{D}_{\mathsf{h}}^{\varepsilon}(z,w;U)}{D_{\mathsf{h}}(z,w;U)} = 1, \quad \textit{uniformly over all } z,w \in U \textit{ with } z \neq w. \tag{2.20}$$

To prove Lemma 4.13, we will need the following elementary estimate for the circle average process, whose proof we postpone until after the proof of Lemma 4.13.

Lemma 4.14. *Let* h *be a whole-plane GFF (with any choice of additive constant) and let* $\{h_r\}_{r\geq 0}$ *be its circle average process. For each* $R > 0$ *and* $\zeta > 0$*, a.s.*

$$\sup_{z\in B_R(0)} \sup_{r>0} \frac{|h_r(z)|}{\max\{(2+\zeta)\log(1/r), (\log r)^{1/2+\zeta}, 1\}} < \infty. \tag{2.21}$$

Proof of Lemma 4.13. We first consider the case when $\mathsf{h} = h$ is a whole-plane GFF normalized so that $h_1(0) = 0$. The functions $w \mapsto \psi_\varepsilon(z-w)$ and $w \mapsto p_{\varepsilon^2/2}(z,w)$ are each radially symmetric about z, i.e., they depend only on $|z-w|$. Using the circle average process $\{h_r\}_{r>0}$, we may therefore write in polar coordinates

$$h_\varepsilon^*(z) = \frac{2}{\varepsilon^2}\int_0^\infty r h_r(z) e^{-r^2/\varepsilon^2}\, dr \quad \text{and} \quad \widehat{h}_\varepsilon^*(z) = \frac{2}{\varepsilon^2}\int_0^{\varepsilon^{1/2}} r h_r(z)\psi_\varepsilon(r) e^{-r^2/\varepsilon^2}\, dr. \tag{2.22}$$

From this representation and the continuity of the circle average process, we infer that $(z,\varepsilon) \mapsto \widehat{h}^*_\varepsilon(z)$ a.s. admits a continuous modification.

Since $\psi_\varepsilon \equiv 1$ on $B_{\varepsilon^{1/2}/2}(z)$ and ψ_ε takes values in $[0,1]$,

$$|h^*_\varepsilon(z) - \widehat{h}^*_\varepsilon(z)| \leq \frac{2}{\varepsilon^2}\int_{\varepsilon^{1/2}/2}^\infty r|h_r(z)|e^{-r^2/\varepsilon^2}\,dr. \tag{2.23}$$

By Lemma 4.14 (applied with $\zeta = 1/2$, say), there is a random constant $C = C(U) > 0$ such that $|h_r(z)| \leq C\max\{\log(1/r), \log r, 1\}$ for each $z \in U$ and $r > 0$. Plugging this into (2.23) shows that a.s.

$$\sup_{z\in U}|h^*_\varepsilon(z) - \widehat{h}^*_\varepsilon(z)| \leq \frac{2C}{\varepsilon^2}\int_{\varepsilon^{1/2}}^\infty r\max\{\log(1/r), \log r, 1\}e^{-r^2/\varepsilon^2}\,dr, \tag{2.24}$$

which tends to zero exponentially fast as $\varepsilon \to 0$. This gives (2.19) in the case of a whole-plane GFF with $h_1(0) = 0$.

If $f : \mathbb{C} \to \mathbb{R}$ is a bounded continuous function, we similarly obtain a.s. $\lim_{\varepsilon\to0}\sup_{z\in U}|f^*_\varepsilon(z) - \widehat{f}^*_\varepsilon(z)| = 0$, using the notation (1.1) and (2.16) with f in place of h or h. This gives (2.19) in the case of a whole-plane GFF plus a bounded continuous function. The relation (2.20) is immediate from (2.17) and the definition of LFPP. □

To conclude the proof of Lemma 4.13 we still need to prove Lemma 4.14. To deal with large values of r, we will use the following lemma.

Lemma 4.15. *Let h be a whole-plane GFF. For each $R > 0$ and $\zeta > 0$, a.s.*

$$\lim_{r\to\infty}\sup_{z\in B_R(0)}\frac{|h_r(z)|}{(\log r)^{1/2+\zeta}} = 0. \tag{2.25}$$

Proof. The process $\{h_r(z) - h_r(0) : z \in B_R(0), r \in [1/2, 1]\}$ is centered Gaussian with variances bounded above by a constant depending only on R. Furthermore, this process a.s. admits a continuous modification [44, Proposition 3.1], so if we replace it by such a modification then a.s. $\sup_{z\in B_R(0)}\sup_{r\in[1/2,1]}|h_r(z) - h_r(0)| < \infty$. By the Borel-TIS inequality [16, 108] (see, e.g., [1, Theorem 2.1.1]), we have $\mathbb{E}\left[\sup_{z\in B_R(0)}\sup_{r\in[1/2,1]}|h_r(z) - h_r(0)|\right] < \infty$ and there are constants

$c_0, c_1 > 0$ depending only on R such that for each $A > 0$,

$$\mathbb{P}\left[\sup_{z\in B_R(0)}\sup_{r\in[1/2,1]}|h_r(z)-h_r(0)|>A\right]\leq c_0 e^{-c_1A^2}. \tag{2.26}$$

Note that we absorbed the R-dependent constant $\mathbb{E}\left[\sup_{z\in B_R(0)}\sup_{r\in[1/2,1]}|h_r(z)-h_r(0)|\right]$ into c_0.

By the scale invariance of the law of h, viewed modulo an additive constant, we infer from (2.26) that for each $k \in \mathbb{N}_0$ and $A > 0$,

$$\mathbb{P}\left[\sup_{z\in B_{R2^k}(0)}\sup_{r\in[2^{k-1},2^k]}|h_r(z)-h_r(0)|>A\right]\leq c_0 e^{-c_1A^2}. \tag{2.27}$$

By applying this with A equal to a universal constant times $k^{1/2+\zeta/2}$, say, then using the Borel-Cantelli lemma, we get that a.s.

$$\lim_{k\to\infty}\sup_{z\in B_{R2^k}(0)}\sup_{r\in[2^{k-1},2^k]}\frac{|h_r(z)-h_r(0)|}{(\log r)^{1/2+\zeta}}=0. \tag{2.28}$$

Each $z \in K$ is contained in $B_{R2^k}(0)$ for each $k \in \mathbb{N}$ and each $r \geq 1/2$ is contained in $[2^{k-1}, 2^k]$ for some $k \in \mathbb{N}$. Hence, (2.28) implies that a.s.

$$\lim_{r\to\infty}\sup_{z\in B_R(0)}\frac{|h_r(z)-h_r(0)|}{(\log r)^{1/2+\zeta}}=0. \tag{2.29}$$

Since $t \mapsto h_{e^t}(0)$ is a standard two-sided linear Brownian motion [44, Section 3], it follows that a.s. $|h_r(0)|/(\log r)^{1/2+\zeta} \to 0$ as $r \to \infty$. Combining this with (2.29) yields (2.25). □

Proof of Lemma 4.14. Standard estimates for the maximum of the circle average process (see, e.g., the proof of [65, Lemma 3.1]) show that a.s.

$$\sup_{z\in B_R(0)}\sup_{r\in(0,1/2]}\frac{|h_r(z)|}{(2+\zeta)\log(1/r)}<\infty. \tag{2.30}$$

By the continuity of the circle average process, a.s. for any $r_0 > 1/2$, $\sup_{z\in B_R(0)}\sup_{r\in[1/2,r_0]}|h_r(z)| <$

∞. By Lemma 4.15, it is a.s. the case that for each large enough $r_0 > 0$,

$$\sup_{z \in B_R(0)} \sup_{r \geq r_0} \frac{|h_r(z)|}{(\log r)^{1/2+\zeta}} < \infty. \tag{2.31}$$

Combining these estimates gives (2.21). □

4.2.2 Subsequential limits

In this subsection we explain why the results of [24] imply that the laws of the re-scaled LFPP metrics $\mathfrak{a}_\varepsilon^{-1} D_{\mathsf{h}}^\varepsilon$ are tight (this is not entirely immediate since [24] considers a slightly different class of fields and only looks at metrics on bounded domains). We will in fact obtain a stronger convergence statement which also includes the convergence of internal metrics of $\mathfrak{a}_\varepsilon^{-1} D_{\mathsf{h}}^\varepsilon$ on a certain class of sub-domains of $\mathbb{C}$.

Definition 4.16 (Dyadic domain)**.** A closed square $S \subset \mathbb{C}$ is *dyadic* if S has side length 2^k and corners in $2^k \mathbb{Z}^2$ for some $k \in \mathbb{Z}$. We say that $W \subset \mathbb{C}$ is a *dyadic domain* if there exists a finite collection of dyadic squares $\mathcal{S}$ such that W is the interior of $\bigcup_{S \in \mathcal{S}} S$. Note that a dyadic domain is a bounded open set.

Lemma 4.17. *Let* h *be a whole-plane GFF plus a bounded continuous function.*

A. The laws of the metrics $\mathfrak{a}_\varepsilon^{-1} D_{\mathsf{h}}^\varepsilon$ *are tight w.r.t. the local uniform topology on* $\mathbb{C} \times \mathbb{C}$ *and any subsequential limit of these laws is supported on continuous length metrics on* $\mathbb{C}$*.*

B. Let $\mathcal{W}$ *be the (countable) set of all dyadic domains. For any sequence of positive* ε*'s tending to zero, there is a subsequence* $\mathcal{E}$ *and a coupling of a continuous length metric* D_{h} *on* $\mathbb{C}$ *and a length metric* $D_{\mathsf{h},W}$ *on* $\overline{W}$ *for each* $W \in \mathcal{W}$ *which induces the Euclidean topology on* $\overline{W}$ *such that the following is true. Along* $\mathcal{E}$*, we have the convergence of joint laws*

$$\left(\mathfrak{a}_\varepsilon^{-1} D_{\mathsf{h}}^\varepsilon, \{\mathfrak{a}_\varepsilon^{-1} D_{\mathsf{h}}^\varepsilon(\cdot,\cdot;\overline{W})\}_{W \in \mathcal{W}}\right) \to \left(D_{\mathsf{h}}, \{D_{\mathsf{h},W}\}_{W \in \mathcal{W}}\right) \tag{2.32}$$

where the first coordinate is given the local uniform topology on $\mathbb{C} \times \mathbb{C}$ *and each element of the collection in the second coordinate is given the uniform topology on* $\overline{W} \times \overline{W}$*. Furthermore, for each* $W \in \mathcal{W}$ *we have the a.s. equality of internal metrics* $D_{\mathsf{h},W}(\cdot,\cdot;W) = D_{\mathsf{h}}(\cdot,\cdot;W)$*.*

In the setting of Assertion A, we note that the space of continuous functions $\mathbb{C} \times \mathbb{C} \to \mathbb{R}$, equipped with the local uniform topology, is separable and completely metrizable, which means that we can apply Prokhorov's theorem in this space. Assertion B of Lemma 4.17 does *not* give that $D_{\mathsf{h}}^{\varepsilon}(\cdot,\cdot;\overline{W}) \to D_{\mathsf{h}}(\cdot,\cdot;\overline{W})$ in law along $\mathcal{E}$ for each $W \in \mathcal{W}$. The reason why we do not prove this statement is to avoid worrying about possible pathologies near ∂W (see Lemma 4.23). We now proceed with the proof of Lemma 4.17. At several places in this section, we will use the following elementary scaling relation for LFPP.

Lemma 4.18. *Let h be a whole-plane GFF normalized so that $h_1(0) = 0$. Let $r > 0$ and let $h^r := h(r\cdot) - h_r(0)$, so that $h^r \overset{d}{=} h$. The LFPP metrics defined as in* (1.2) *for h and h^r are related by*

$$D_{h^r}^{\varepsilon/r} \overset{d}{=} D_h^{\varepsilon/r} \quad \text{and} \quad D_{h^r}^{\varepsilon/r}(z,w) = r^{-1} e^{-\xi h_r(0)} D_h^{\varepsilon}(rz,rw), \quad \forall \varepsilon > 0, \quad \forall z,w \in \mathbb{C}. \tag{2.33}$$

Proof. Using the notation (1.1), we get from a standard change of variables that the convolutions of h^r and h with the heat kernel satisfy $h_{\varepsilon/r}^{r,*}(z) = h_\varepsilon^*(rz) - h_r(0)$ for each $\varepsilon > 0$ and $z \in \mathbb{C}$. Using the definition (1.2) of LFPP, we now compute

$$\begin{aligned} e^{-\xi h_r(0)} D_h^\varepsilon(rz,rw) &= \inf_{P : rz \to rw} \int_0^1 e^{\xi(h_\varepsilon^*(P(t)) - h_r(0))} |P'(t)|\, dt \\ &= \inf_{P : rz \to rw} \int_0^1 e^{\xi h_{\varepsilon/r}^{r,*}(P(t)/r)} |P'(t)|\, dt \\ &= r \inf_{\widetilde{P} : z \to w} \int_0^1 e^{\xi h_{\varepsilon/r}^{r,*}(\widetilde{P}(t))} |\widetilde{P}'(t)|\, dt \quad (\text{set } \widetilde{P} = P/r) \\ &= r D_{h^r}^{\varepsilon/r}(z,w) \end{aligned}$$

and this completes the proof. ☐

To check that our limiting metrics are length metrics, we will need the following standard fact from metric geometry.

Lemma 4.19. *Let X be a compact topological space and let $\{D^n\}_{n \in \mathbb{N}}$ be a sequence of length metrics on X which converge uniformly to a metric D on X. Then D is a length metric on X.*

Proof. This is [17, Exercise 2.4.19], which in turn is an easy consequence of [17, Corollary 2.4.17]. ☐

Let us now record what we get from [24].

Lemma 4.20. *Let $S \subset \mathbb{C}$ be a closed square and let h be a whole-plane GFF plus a bounded continuous function. The laws of the internal metrics $\mathfrak{a}_\varepsilon^{-1} D_{\mathsf{h}}^\varepsilon(\cdot,\cdot;S)$ for $\varepsilon \in (0,1)$ are tight w.r.t. the uniform topology on $S \times S$ and any subsequential limit of these laws is supported on length metrics which induce the Euclidean topology on S.*

Proof. We first consider the case when $S = [0,1]^2$ is the Euclidean unit square and $\mathsf{h} = h$ is a whole-plane GFF normalized so that $h_1(0) = 0$. Let $\mathring{h}$ be a zero-boundary GFF on $(-1,2)^2$. By the Markov property of the whole-plane GFF, we can couple h and $\mathring{h}$ in such a way that $h - \mathring{h}$ is a.s. harmonic, hence continuous, on $(-1,2)^2$.

Recall the heat kernel $p_s(z,w) = \frac{1}{2\pi s} e^{-|z-w|/(2s)}$. For $z \in [0,1]^2$ and $\varepsilon \in (0,1)$, we define the convolution $\mathring{h}_\varepsilon^* = \mathring{h} * p_{\varepsilon^2/2}$ as in (1.1). For $z, w \in (-1,2)^2$, define $D_{\mathring{h}}^\varepsilon(z,w)$ as in (1.2) with $\mathring{h}_\varepsilon^*$ in place of h_ε^*. It is shown in [24, Theorem 1] (see also [24, Section 6.1]) that there are constants $\{\lambda_\varepsilon\}_{\varepsilon>0}$ such that the internal metrics $\lambda_\varepsilon^{-1} D_{\mathring{h}}^\varepsilon\left(\cdot,\cdot;[0,1]^2\right)$ are tight w.r.t. the uniform topology on $[0,1]^2 \times [0,1]^2$ and any subsequential limit of these laws is supported on length metrics which induce the Euclidean topology on $[0,1]^2$.

We now want to compare $D_{\mathring{h}}^\varepsilon$ and D_h^ε using the fact that $(h - \mathring{h})|_{(-1,2)^2}$ is a continuous function. However, we cannot do this directly since we only have a uniform bound for $h - \mathring{h}$ on compact subsets of $(-1,2)^2$ and the convolution (1.1) does not depend locally on the field. To this end, we define the localized LFPP metrics $\widehat{D}_h^\varepsilon$ and $\widehat{D}_{\mathring{h}}^\varepsilon$ as in (2.17) with $\mathsf{h} = h$ and with $\mathring{h}$ in place of h, respectively. Then Lemma 4.13 remains true with $D_{\mathring{h}}^\varepsilon$ and $\widehat{D}_{\mathring{h}}^\varepsilon$ in place of D_h^ε and $\widehat{D}_h^\varepsilon$ and with U any open set satisfying $\overline{U} \subset (-1,2)^2$, with the same proof (actually, the proof is simpler since one does not need Lemma 4.15). Therefore, a.s. $\widehat{D}_{\mathring{h}}^\varepsilon(z,w;U)/D_{\mathring{h}}^\varepsilon(z,w;U) \to 1$ uniformly over all distinct $z, w \in U$ and the conclusion of the preceding paragraph is true with $\widehat{D}_{\mathring{h}}^\varepsilon$ in place of $D_{\mathring{h}}^\varepsilon$.

Since $h - \mathring{h}$ is a.s. equal to a continuous function on a neighborhood of $[0,1]^2$, we infer from (2.18) that a.s. the metrics $\widehat{D}_{\mathring{h}}^\varepsilon(\cdot,\cdot;[0,1]^2)$ and $\widehat{D}_h^\varepsilon(\cdot,\cdot;[0,1]^2)$ are bi-Lipschitz equivalent with (random) ε-independent Lipschitz constants. By combining this with the conclusion of the preceding paragraph and Lemma 4.19, we get that the laws of the internal metrics $\lambda_\varepsilon^{-1} D_h^\varepsilon(\cdot,\cdot;S)$ for $\varepsilon \in (0,1)$ are tight w.r.t. the uniform topology on $[0,1]^2 \times [0,1]^2$ and any subsequential limit of these laws is supported on length metrics which induce the Euclidean topology on S. In particular, this implies that λ_ε is bounded above and below by ε-independent constants times the median $\widehat{D}_h^\varepsilon$-distance between the left and right sides of $[0,1]^2$. By Lemma 4.13 (for h), we now get that $\{\mathfrak{a}_\varepsilon/\lambda_\varepsilon\}_{\varepsilon\in(0,1)}$ is bounded above and below by positive, finite constants and the statement of the lemma holds in the special case when $\mathsf{h} = h$ and $S = [0,1]^2$.

By Lemma 4.18 and the scale and translation invariance of the law of h, modulo additive constant, this implies the statement of the lemma for a general choice of S, but still with $\mathsf{h} = h$. If h is a whole-plane GFF and f is a bounded continuous function, then the metrics D_{h+f}^ε and D_h^ε are bi-Lipschitz equivalent, with Lipschitz constants $e^{\pm\xi\|f\|_\infty}$. Hence the case of a whole-plane GFF implies the case of a whole-plane GFF plus a continuous function. □

We now upgrade from internal metrics on closed squares to internal metrics on closures of dyadic domains.

Lemma 4.21. *Let $W \subset \mathbb{C}$ be a dyadic domain. The laws of the internal metrics $\mathfrak{a}_\varepsilon^{-1} D_{\mathsf{h}}^\varepsilon(\cdot,\cdot;\overline{W})$ for $\varepsilon \in (0,1)$ are tight w.r.t. the uniform topology on $\overline{W}\times\overline{W}$ and any subsequential limit of these laws is supported on length metrics which induce the Euclidean topology on $\overline{W}$.*

Proof. If W is a dyadic domain, then $\overline{W}$ has finitely many connected components and these connected components are the closures of dyadic domains which lie at positive Euclidean distance from each other. By considering each connected component separately, we can assume without loss of generality that $\overline{W}$ is connected.

For a connected set $X \subset \mathbb{C}$, a collection $\mathcal{D}$ of random metrics on X is tight w.r.t. the local uniform topology if and only if for each $\zeta > 0$, there exists $\delta > 0$ such that for each $d \in \mathcal{D}$, it holds with probability at least $1-\zeta$ that

$$d(z,w) \leq \zeta, \quad \forall z,w \in X \quad \text{such that} \quad |z-w| \leq \delta. \tag{2.34}$$

Indeed, this is an easy consequence of the Arzéla-Ascoli theorem, the Prokhorov theorem, and the triangle inequality.

For any closed square $S \subset \overline{W}$, the restriction of $D_{\mathsf{h}}^\varepsilon(\cdot,\cdot;\overline{W})$ to S is bounded above by the internal metric of $D_{\mathsf{h}}^{\varepsilon}(\cdot,\cdot;\overline{W})$ on S, which equals $D_{\mathsf{h}}^{\varepsilon}(\cdot,\cdot;S)$. By Lemma 4.20 and the above tightness criterion, the laws of the restrictions of $\{\mathfrak{a}_{\varepsilon}^{-1}D_{\mathsf{h}}^{\varepsilon}(\cdot,\cdot;\overline{W})\}_{\varepsilon\in(0,1)}$ to S are tight. Since W is a dyadic domain, we can choose a finite collection $\mathcal{S}$ of closed squares such that $\bigcup_{S\in\mathcal{S}} S = \overline{W}$.

By the above tightness criterion applied to each square in $\mathcal{S}$, for each $\zeta > 0$, there exists $\delta > 0$ such that for each $\varepsilon \in (0,1)$, it holds with probability at least $1-\zeta$ that

$$\mathfrak{a}_{\varepsilon}^{-1}D_{\mathsf{h}}^{\varepsilon}(z,w;\overline{W}) \le \zeta, \quad \forall z,w\in\overline{W} \quad \text{s.t.} \quad |z-w|\le\delta \quad \text{and} \quad z,w\in S \text{ for some } S\in\mathcal{S}. \tag{2.35}$$

Now assume that (2.35) holds and consider points $z,w\in\overline{W}$ such that $|z-w|\le\delta/2$ but z and w do not lie in the same square of $\mathcal{S}$. If δ is sufficiently small (depending only on the collection of squares $\mathcal{S}$), then we can find squares $S,S'\in\mathcal{S}$ such that $z\in S, w\in S'$, and $S\cap S'\neq\emptyset$. Since S and S' are closed squares, geometric considerations show that there is a $u\in S\cap S'$ such that $|z-u|\le\delta$ and $|w-u|\le\delta$. By (2.35) and the triangle inequality this implies that $\mathfrak{a}_{\varepsilon}^{-1}D_{\mathsf{h}}^{\varepsilon}(z,w;\overline{W})\le 2\zeta$. Therefore, $\forall\varepsilon\in(0,1)$ it holds with probability at least $1-\zeta$ that

$$\mathfrak{a}_{\varepsilon}^{-1}D_{\mathsf{h}}^{\varepsilon}(z,w;\overline{W}) \le 2\zeta, \quad \forall z,w\in\overline{W} \quad \text{such that} \quad |z-w|\le\delta/2.$$

Since ζ is arbitrary, the above tightness criterion applied on all of $\overline{W}$ now shows that the laws of the metrics $\mathfrak{a}_{\varepsilon}^{-1}D_{\mathsf{h}}^{\varepsilon}(\cdot,\cdot;W)$ for $\varepsilon\in(0,1)$ are tight w.r.t. the uniform topology on $\overline{W}\times\overline{W}$.

Let $\widetilde{D}$ be a subsequential limit of $\mathfrak{a}_{\varepsilon}^{-1}D_{\mathsf{h}}^{\varepsilon}(\cdot,\cdot;W)$ in law w.r.t. the local uniform topology. A priori $\widetilde{D}$ might be a pseudometric, not a metric. We need to show that $\widetilde{D}$ is in fact a length metric and that it induces the Euclidean topology on $\overline{W}$. To this end, consider two squares (not necessarily dyadic) $S_1\subset S_2\subset\overline{W}$ such that S_1 lies at positive Euclidean distance from $\partial S_2\setminus\partial W$. For each $\varepsilon>0$, we have $D_{\mathsf{h}}^{\varepsilon}(S_1,W\setminus S_2;\overline{W}) = D_{\mathsf{h}}^{\varepsilon}(S_1,\partial S_2\setminus\partial W;S_2)$ and $D_{\mathsf{h}}^{\varepsilon}(S_1,W\setminus S_2;\overline{W})\to\widetilde{D}(S_1,W\setminus S_2)$ in law. From this and Lemma 4.20, we infer that a.s. $\widetilde{D}(S_1,W\setminus S_2)>0$. By considering an appropriate countable collection of such square annuli whose inner squares S_1 cover $\overline{W}$, we infer that a.s. $\widetilde{D}(u,v)>0$ whenever $u,v\in\overline{W}$ with $u\neq v$. This implies that $\widetilde{D}$ is a metric. Since $\overline{W}$ is compact, it follows that $\widetilde{D}$ induces the Euclidean topology on $\overline{W}$. By Lemma 4.19, $\widetilde{D}$ is a length metric. $\square$

The following lemma will allow us to extract tightness of $\mathfrak{a}_\varepsilon^{-1} D_{\mathsf{h}}^\varepsilon$ from tightness of $\mathfrak{a}_\varepsilon^{-1} D_{\mathsf{h}}^\varepsilon(\cdot,\cdot;S)$ for squares $S \subset \mathbb{C}$.

Lemma 4.22. *For $r > 0$, let $S_r(0)$ be the closed square of side length r centered at zero. Let h be a whole-plane GFF plus a bounded continuous function. For each $p \in (0,1)$ and each $C > 0$, there exists $R = R(p,C) > 1$ (depending on p, C and the law of h) such that for each fixed $r > 0$,*

$$\liminf_{\varepsilon \to 0} \mathbb{P}\left[\sup_{u,v \in S_r(0)} D_{\mathsf{h}}^\varepsilon(u,v) < \frac{1}{C} D_{\mathsf{h}}^\varepsilon\left(S_r(0), \partial S_{Rr}(0)\right)\right] \geq p. \tag{2.36}$$

Proof. We first consider the case when $\mathsf{h} = h$ is a whole-plane GFF normalized so that $h_1(0) = 0$. By Lemma 4.20 applied with $\overline{W} = S_1(0)$, there exists $R = R(p,C) > 1$ such

$$\liminf_{\varepsilon \to 0} \mathbb{P}\left[\sup_{u,v \in S_{1/R}(0)} D_h^\varepsilon(u,v) < \frac{1}{C} D_h^\varepsilon\left(S_{1/R}(0), \partial S_1(0)\right)\right] \geq p. \tag{2.37}$$

The occurrence of the event in (2.37) is unaffected by re-scaling D_h^ε by a constant factor. By Lemma 4.18 applied with Rr in place of r, we see that (2.37) implies that for each fixed $r > 0$,

$$\liminf_{\varepsilon \to 0} \mathbb{P}\left[\sup_{u,v \in S_r(0)} D_h^\varepsilon(u,v) < \frac{1}{C} D_h^\varepsilon\left(S_r(0), \partial S_{Rr}(0)\right)\right] \geq p. \tag{2.38}$$

Now suppose that $\mathsf{h} = h + f$ is a whole-plane GFF plus a bounded continuous function. If f is a (possibly random) bounded continuous function, then D_{h+f}^ε and D_h^ε are a.s. bi-Lipschitz equivalent with Lipschitz constants $e^{-\xi\|f\|_\infty}$ and $e^{\xi\|f\|_\infty}$. Furthermore, since f is a.s. bounded exists a deterministic $A > 1$ such that $\mathbb{P}\left[e^{\xi\|f\|_\infty} \leq A\right] \geq p$. By (2.38) with $A^2 C$ in place of C, we get (2.36) but with $1 - 2(1-p)$ in place of p. Since p can be made arbitrarily close to 1, this yields (2.36). ∎

The last lemma we need for the proof of Lemma 4.17 is the following deterministic compatibility statement for limits of internal metrics, which is used to get the relationship between internal metrics in assertion B of Lemma 4.17.

Lemma 4.23. *Let $V \subset U \subset \mathbb{C}$ be open. Let $\{D^n\}_{n \in \mathbb{N}}$ be a sequence of continuous length metrics on U which converges to a continuous length metric D (w.r.t. the local uniform topology on $U \times U$).*

Suppose also that $D^n(\cdot,\cdot;\overline{V})$ converges to a continuous length metric $\widetilde{D}$ w.r.t. the uniform topology on $\overline{V}\times\overline{V}$. Then $D(\cdot,\cdot;V)=\widetilde{D}(\cdot,\cdot;V)$.

In the setting of Lemma 4.23, we do not necessarily have $D(\cdot,\cdot;\overline{V})=\widetilde{D}$. The reason is that it could be, e.g., that paths of near-minimal $\widetilde{D}$-length spend a positive fraction of their time in ∂V.

Proof of Lemma 4.23. Let $u,v\in V$ such that $D(u,v)<D(u,\partial V)$. Since D is a length metric, $D(u,v)=D(u,v;V)=D(u,v;\overline{V})$. Furthermore, for large enough $n\in\mathbb{N}$ we have $D^n(u,v)<D^n(u,\partial V)$ which implies that $D^n(u,v)=D^n(u,v;V)=D^n(u,v;\overline{V})$. Therefore, $D^n(u,v)$ converges to both $D(u,v)=D(u,v;V)$ and $\widetilde{D}(u,v)$. Furthermore, we have $\widetilde{D}(u,v)<\widetilde{D}(u,v;\partial V)$ which implies that $\widetilde{D}(u,v)=\widetilde{D}(u,v;V)$. Consequently, $D(u,v;V)=\widetilde{D}(u,v;V)$ for each $u,v\in V$ with $D(u,v)<D(u,\partial V)$. This implies that the D-length of any path in V which lies at positive Euclidean distance from ∂V is the same as its $\widetilde{D}$-length. Since $D(\cdot,\cdot;V)$ and $\widetilde{D}(\cdot,\cdot;V)$ are length metrics, we conclude that $D(\cdot,\cdot;V)=\widetilde{D}(\cdot,\cdot;V)$. □

Proof of Lemma 4.17. For $r>0$, let $S_r(0)$ be the closed square of side length r centered at zero, as in Lemma 4.22. Let $p\in(0,1)$ and let $R=R(p)>1$ be as in Lemma 4.22 with $C=2$ and with $(1+p)/2$, say, in place of p. Then for each fixed $r>0$ and each small enough $\varepsilon>0$, it holds with probability at least p that

$$\sup_{u,v\in S_r(0)} D_{\mathsf{h}}^{\varepsilon}(u,v)\leq\frac{1}{2}D_{\mathsf{h}}^{\varepsilon}(S_r(0),\partial S_{Rr}(0))$$
$$\text{which implies}\quad D_{\mathsf{h}}^{\varepsilon}(u,v)=D_{\mathsf{h}}^{\varepsilon}(u,v;S_{Rr}(0)),\quad\forall u,v\in S_r(0). \tag{2.39}$$

We now apply Lemma 4.20 with $S=S_{Rr}(0)$ and use that p can be made arbitrarily close to 1 to get that the laws of $\mathfrak{a}_\varepsilon^{-1}D_{\mathsf{h}}^{\varepsilon}|_{S_r(0)}$ are tight w.r.t. the local uniform topology on $S_r(0)$. Furthermore, any subsequential limit in law of these metrics a.s. induces the Euclidean topology on $S_r(0)$. Since r can be made arbitrarily large, we get that the metrics $\mathfrak{a}_\varepsilon^{-1}D_{\mathsf{h}}^{\varepsilon}$ are tight w.r.t. the local uniform topology on $\mathbb{C}\times\mathbb{C}$ and any subsequential limit in law is a.s. a continuous metric on $\mathbb{C}$.

To prove assertion A, it remains to check that if D_{h} is a subsequential limit in law of the metrics $\mathfrak{a}_\varepsilon^{-1}D_{\mathsf{h}}^{\varepsilon}$, then a.s. D_{h} is a length metric. To this end, let $p\in(0,1)$ and let $R=R(p)>1$ be as above. By Lemma 4.20, if we are given $r>0$ then by possibly passing to a further subsequence we can arrange that along our subsequence, the joint law of $(\mathfrak{a}_\varepsilon^{-1} D_{\mathsf{h}}^\varepsilon, \mathfrak{a}_\varepsilon^{-1} D_{\mathsf{h}}^\varepsilon(\cdot,\cdot; S_{Rr}(0)))$ converges to a coupling $(D_{\mathsf{h}}, \widetilde{D})$ where $\widetilde{D}$ is a length metric on $S_{Rr}(0)$. By passing to the (subsequential) limit in (2.39), we get that with probability at least p,

$$\sup_{u,v\in S_r(0)} D_{\mathsf{h}}(u,v) \leq \frac{1}{2} D_{\mathsf{h}}(S_r(0), \partial S_{Rr}(0)) \quad \text{and} \quad D_{\mathsf{h}}(u,v) = \widetilde{D}(u,v), \quad \forall u,v \in S_r(0). \tag{2.40}$$

By Lemma 4.23, a.s. the internal metrics of D_{h} and $\widetilde{D}$ on the interior of $S_{Rr}(0)$ coincide. Hence (2.39) implies that with probability at least p, $D_{\mathsf{h}}(u,v)$ is equal to the infimum of the D_{h}-lengths of all continuous paths from u to v which are contained in the interior of $S_{Rr}(0)$, which (by the first condition in (2.39)) is equal to the infimum of the D_{h}-lengths of all continuous paths from u to v. Since p can be made arbitrarily close to 1 and r can be made arbitrarily large, we get that a.s. D_{h} is a length metric.

To get the joint convergence (2.32), we first apply Lemma 4.21 and the Prokhorov theorem to get that the joint law of the metrics on the left side of (2.32) is tight. Moreover any subsequential limit of these joint laws is a coupling of a continuous length metric D_{h} on $\mathbb{C}$ and a length metric $D_{\mathsf{h},W}$ on $\overline{W}$ for each $W \in \mathcal{W}$ which induces the Euclidean topology on $\overline{W}$. We then apply Lemma 4.23 to say that $D_{\mathsf{h},W}(\cdot,\cdot;W) = D_{\mathsf{h}}(\cdot,\cdot;W)$ for each $W \in \mathcal{W}$. □

4.2.3 Weyl scaling

The following lemma will be used to check Axiom III.

Lemma 4.24. *Let* h *be a whole-plane GFF plus a bounded continuous function and consider a sequence* $\varepsilon_n \to 0$ *along which* $\mathfrak{a}_{\varepsilon_n}^{-1} D_{\mathsf{h}}^{\varepsilon_n}$ *converges in law to some metric* D_{h} *w.r.t. the local uniform topology. Suppose we have, using the Skorokhod theorem, coupled so this convergence occurs a.s. Then, a.s., for every sequence of bounded continuous functions* $f^n : \mathbb{C} \to \mathbb{R}$ *such that* f^n *converges to a bounded continuous function* f *uniformly on compact subsets of* $\mathbb{C}$, *we have the local uniform convergence* $D_{\mathsf{h}+f^n}^{\varepsilon_n} \to e^{\xi f} \cdot D_{\mathsf{h}}$, *where here* $D_{\mathsf{h}+f^n}^{\varepsilon}$ *is defined as in* (1.2) *with* $\mathsf{h} + f^n$ *in place of* h *and* $e^{\xi f} \cdot D_{\mathsf{h}}$ *is defined as in* (1.4).

As a consequence of Lemma 4.24, if h is a whole-plane GFF plus a bounded continuous function and $\varepsilon_n \to 0$ is a sequence along which $\mathfrak{a}_{\varepsilon_n}^{-1} D_{\mathsf{h}}^{\varepsilon_n} \to D_{\mathsf{h}}$ in law, then whenever h' is another whole-plane

GFF plus a bounded continuous function, we have $\mathfrak{a}_{\varepsilon_n}^{-1} D_{\mathsf{h}'}^{\varepsilon_n} \to D_{\mathsf{h}'}$ in law for some limiting metric $D_{\mathsf{h}'}$. Furthermore, $(\mathsf{h}, \mathsf{h}', D_{\mathsf{h}}, D_{\mathsf{h}'})$ can be coupled together in such a way that $\mathsf{h}' - \mathsf{h}$ is a bounded continuous function and $D_{\mathsf{h}'} = e^{\xi(\mathsf{h}'-\mathsf{h})} \cdot D_{\mathsf{h}}$. Consequently, any subsequence along which $\mathfrak{a}_{\varepsilon_n}^{-1} D_{\mathsf{h}}^{\varepsilon_n}$ converges in law gives us a way to define a metric associated with any whole-plane GFF plus a bounded continuous function.

Proof of Lemma 4.24. Let $f_{\varepsilon_n}^{*,n} = f^n * p_{\varepsilon_n^2/2}$ be defined as in (1.1) with with f^n in place of h. Then $f_{\varepsilon_n}^{*,n} \to f$ uniformly on compact subsets of $\mathbb{C}$. By the definition (1.2) of LFPP, we have $D_{\mathsf{h}+f^n}^{\varepsilon_n} = e^{\xi f_{\varepsilon_n}^{*,n}} \cdot D_{\mathsf{h}}^{\varepsilon_n}$.

We now want to apply an argument as in the proof of [38, Lemma 7.1] to say that $D_{\mathsf{h}+f^n}^{\varepsilon_n} \to e^{\xi f} \cdot D_{\mathsf{h}}$ w.r.t. the local uniform topology. That lemma only applies for metrics defined on squares, so we need to localize. We do this by means of Lemma 4.22. By taking a limit as $\varepsilon \to 0$ in the estimate of Lemma 4.22, then sending $p \to 1$, we find that a.s. for each $r > 0$ and each $C > 1$, there exists $r' = r'(r, C) > 0$ (random) such that

$$\sup_{u,v \in S_r(0)} D_{\mathsf{h}}(u, v) \le \frac{1}{2C} D_{\mathsf{h}}(S_r(0), \partial S_{r'}(0)). \tag{2.41}$$

Furthermore, the uniform convergence $\mathfrak{a}_{\varepsilon_n}^{-1} D_{\mathsf{h}}^{\varepsilon_n} \to D_{\mathsf{h}}$, we get that (2.41) is a.s. true with $\mathfrak{a}_{\varepsilon_n}^{-1} D_{\mathsf{h}}^{\varepsilon_n}$ in place of D_{h} for large enough $n \in \mathbb{N}$, but with C instead of $2C$. This implies that each path of near-minimal D_{h}-length between two points of $S_r(0)$ is contained in $S_{r'}(0)$, and the same is true with $\mathfrak{a}_{\varepsilon_n}^{-1} D_{\mathsf{h}}^{\varepsilon_n}$ in place of D_{h} for large enough $n \in \mathbb{N}$. If we choose $C > \sup_{n\in\mathbb{N}} \|f^n\|_\infty$, then from (2.41) we deduce that each path of near-minimal $e^{\xi f} \cdot D_{\mathsf{h}}$-length between two points of $S_r(0)$ is contained in $S_{r'}(0)$, and the same is true with $\mathfrak{a}_{\varepsilon_n}^{-1} D_{\mathsf{h}+f^n}^{\varepsilon_n}$ in place of D_{h} for large enough $n \in \mathbb{N}$. With these conditions in hand, the lemma now follows from the same proof as in [38, Lemma 7.1]. □

4.2.4 Tightness across scales

In this section we check that subsequential limits of LFPP satisfy Axiom V. For the statement, we note that we can take a subsequential limit of the joint laws of $(\mathsf{h}, \mathfrak{a}_\varepsilon^{-1} D_{\mathsf{h}}^\varepsilon)$ due to Lemma 4.17 and the Prokhorov theorem.

Lemma 4.25. *Let h be a whole-plane GFF normalized so that $h_1(0) = 0$. Let (h, D_h) be any subsequential limit of the laws of the field/metric pairs $(h, \mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon)$. There are deterministic constants $\{\mathfrak{c}_r\}_{r\geq 0}$, depending on the law of D_h, such that the laws of the metrics $\{\mathfrak{c}_r^{-1} e^{-\xi h_r(0)} D_h(r\cdot, r\cdot)\}_{r>0}$ are tight w.r.t. the local uniform topology. Furthermore, the closure of this set of laws w.r.t. the Prokhorov topology for probability measures on continuous functions $\mathbb{C}\times\mathbb{C} \to [0,\infty)$ is contained in the set of laws on continuous metrics on $\mathbb{C}$. Finally, there exists $\Lambda > 1$ such that for each $\delta \in (0,1)$,*

$$\Lambda^{-1}\delta^{\Lambda} \leq \frac{\mathfrak{c}_{\delta r}}{\mathfrak{c}_r} \leq \Lambda\delta^{-\Lambda}, \quad \forall r > 0. \tag{2.42}$$

We first produce the scaling constants $\mathfrak{c}_r$ appearing in Axiom V.

Lemma 4.26. *Consider a sequence $\mathcal{E} \subset (0,1)$ converging to zero along which $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ converges in law to a limiting metric D_h. For each $r > 0$, the limit*

$$\mathfrak{c}_r := \lim_{\mathcal{E}\ni\varepsilon\to 0} \frac{r\mathfrak{a}_{\varepsilon/r}}{\mathfrak{a}_\varepsilon} \tag{2.43}$$

exists and satisfies the relation (2.42) for some choice of $\Lambda > 1$ depending only on $\mathcal{E}$ and γ.

Proof. Let $h^r := h(r\cdot) - h_r(0)$ be as in Lemma 4.18, so that $h^r \stackrel{d}{=} h$. By our choice of subsequence $\mathcal{E}$ and Lemma 4.18,

$$\mathfrak{a}_\varepsilon^{-1} D_{h^r}^{\varepsilon/r} = r^{-1} e^{-\xi h_r(0)} \mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon(r\cdot, r\cdot) \xrightarrow{\mathcal{E}\ni\varepsilon\to 0} r^{-1} e^{-\xi h_r(0)} D_h(r\cdot, r\cdot) \tag{2.44}$$

in law w.r.t. the local uniform topology on $\mathbb{C}\times\mathbb{C}$. Let m_r be the median distance between the left and right boundaries of $[0,1]^2$ w.r.t. the metric on the right side of (2.44). Since $h^r \stackrel{d}{=} h$,

$$\underbrace{\mathfrak{a}_{\varepsilon/r}^{-1} D_h^{\varepsilon/r}}_{\text{tight}} \stackrel{d}{=} \mathfrak{a}_{\varepsilon/r}^{-1} D_{h^r}^{\varepsilon/r} = \frac{\mathfrak{a}_\varepsilon}{\mathfrak{a}_{\varepsilon/r}} \underbrace{\mathfrak{a}_\varepsilon^{-1} D_{h^r}^{\varepsilon/r}}_{\substack{\text{convergent} \\ \text{by (2.44)}}}. \tag{2.45}$$

If we consider a subsequence $\mathcal{E}'$ of $\mathcal{E}$ along which the joint law of $\mathfrak{a}_{\varepsilon/r}^{-1} D_h^{\varepsilon/r}$ and $\mathfrak{a}_\varepsilon^{-1} D_{h^r}^{\varepsilon/r}$ converges, then (2.45) shows that along this subsequence, $\mathfrak{a}_{\varepsilon/r}/\mathfrak{a}_\varepsilon$ converges to some number $s_r(\mathcal{E}') > 0$ (we know the limit is strictly positive since the limits of $\mathfrak{a}_{\varepsilon/r}^{-1} D_h^{\varepsilon/r}$ and $\mathfrak{a}_\varepsilon^{-1} D_{h^r}^{\varepsilon/r}$ are metrics). By the definitions of $\mathfrak{a}_\varepsilon$ and of m_r and Portmanteau's lemma, the median distance between the left and right boundaries of $[0,1]^2$ w.r.t. the metric on the left (resp. right) side of (2.45) is 1 (resp. $m_r/s_r(\mathcal{E}')$). Hence $s_r(\mathcal{E}') = m_r$, i.e., the limit does not depend on the choice of subsequence $\mathcal{E}' \subset \mathcal{E}$. This shows the convergence of $\mathfrak{a}_{\varepsilon/r}/\mathfrak{a}_\varepsilon$ along the subsequence $\mathcal{E}$, which in turn implies the existence of the limit (2.43). The bounds (2.42) (in fact, substantially stronger bounds) are immediate from [24, Theorem 1, Equation (1.3)] and the fact the ratio of our $\mathfrak{a}_\varepsilon$ and the scaling factor λ_ε from [24] is bounded above and below by deterministic, ε-independent constants (see the proof of Lemma 4.20). $\square$

Proof of Lemma 4.25. Define $\mathfrak{c}_r$ for $r > 0$ as in Lemma 4.26. Let $h^r := h(r\cdot) - h_r(0)$, as in Lemma 4.18, so that $h^r \stackrel{d}{=} h$ and the metrics $D_{h^r}^{\varepsilon/r}$ and D_h^ε are related as in (2.33). We know from Lemma 4.17 that the laws of the metrics $\{\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon\}_{0<\varepsilon<1}$ are tight, and every element of the closure of this set of laws is supported on continuous metrics on $\mathbb{C}$. It follows that the same is true for the laws of the metrics $\{\mathfrak{a}_{\varepsilon/r}^{-1} D_{h^r}^{\varepsilon/r}\}_{0<\varepsilon<r}$. By combining this with (2.33), we get that the laws of the metrics

$$e^{-\xi h_r(0)} \left(\frac{r\mathfrak{a}_{\varepsilon/r}}{\mathfrak{a}_\varepsilon}\right)^{-1} \mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon(r\cdot, r\cdot) = \mathfrak{a}_{\varepsilon/r}^{-1} D_{h^r}^{\varepsilon/r}, \quad \forall r > 0, \quad \forall \varepsilon \in (0,r) \tag{2.46}$$

are tight and every element of the closure of this set of laws w.r.t. the Prokhorov topology is supported on continuous metrics on $\mathbb{C}$.

Now consider a subsequence $\mathcal{E} \subset (0,1)$ along which $(h, \mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon) \to (h, D_h)$ in law. By the definition (2.43) of $\mathfrak{c}_r$,

$$e^{-\xi h_r(0)} \left(\frac{r\mathfrak{a}_{\varepsilon/r}}{\mathfrak{a}_\varepsilon}\right)^{-1} \mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon(r\cdot, r\cdot) \to e^{-\xi h_r(0)} \mathfrak{c}_r^{-1} D_h(r\cdot, r\cdot), \quad \text{in law along } \mathcal{E}.$$

Therefore, the metrics $e^{-\xi h_r(0)} \mathfrak{c}_r^{-1} D_h(r\cdot, r\cdot)$ for $r > 0$ are all subsequential limits as $\varepsilon \to 0$ of the family of random metrics (2.46). It follows that the laws of the metrics $e^{-\xi h_r(0)} \mathfrak{c}_r^{-1} D_h(r\cdot, r\cdot)$ are tight and every element of the closure of this set of laws is supported on continuous metrics on $\mathbb{C}$. $\square$

4.2.5 Locality

In this section, we will prove a variant of Axiom II for subsequential limits of LFPP, restricted to the case of a whole-plane GFF (locality for a whole-plane GFF plus a continuous function will be checked in Section 4.2.6). At this point, we have not yet established that such subsequential limits can be realized as measurable functions of the field, so we will actually check a somewhat different condition. In what follows, if $K \subset \mathbb{C}$ is closed we define the σ-algebra generated by $h|_K$ to be $\bigcap_{\delta>0} h|_{B_\delta(K)}$. With this definition it makes sense to condition on $h|_K$. The following definitions first appeared in [57].

Definition 4.27 (Local metric)**.** Let $U \subset \mathbb{C}$ be a connected open set and let (h, D) be a coupling of a GFF on U and a random continuous length metric on U. We say that D is a *local metric* for h if for any open set $V \subset U$, the internal metric $D(\cdot,\cdot;V)$ is conditionally independent from the pair $(h, D(\cdot,\cdot;U \setminus \overline{V}))$ given $h|_{\overline{V}}$.

Definition 4.27 is formulated in a slightly different way than [57, Definition 1.2]; the equivalence of the definitions is proven in [57, Lemma 2.3]. The following is [57, Definition 1.5].

Definition 4.28 (Additive local metric)**.** Let $U \subset \mathbb{C}$ be a connected open set and let (h, D) be a coupling of a GFF on U and a random continuous length metric on U which is local for h. For $\xi \in \mathbb{R}$, we say that D is ξ*-additive* for h if for each $z \in U$ and each $r > 0$ such that $B_r(z) \subset U$, the metric $e^{-\xi h_r(z)} D$ is local for $h - h_r(z)$.

Lemma 4.29. *Let h be a whole-plane GFF. Let (h, D_h) be any subsequential limit of the laws of the pairs $(h, \mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon)$. Then D_h is a ξ-additive local metric for h. That is, suppose $z \in \mathbb{C}$ and $r > 0$ and that h is normalized so that the circle average $h_r(z)$ is zero. Also let $V \subset \mathbb{C}$ be an open set. Then the internal metric $D_h(\cdot,\cdot;V)$ is conditionally independent from the pair $\big(h, D_h(\cdot,\cdot;\mathbb{C} \setminus \overline{V})\big)$ given $h|_{\overline{V}}$.*

There are two main difficulties in the proof of Lemma 4.29.

1. The mollified GFF $h_\varepsilon^*(z)$ of (1.1) does not exactly depend locally on h (since the heat kernel $p_{\varepsilon^2/2}(z,\cdot)$ does not have compact support), so the D_h^ε-lengths of paths are not locally determined by h.

2. Conditional independence does not in general behave nicely under taking limits in law.

Difficulty 1 will be resolved by means of the localization results for LFPP in Section 4.2.1. To resolve Difficulty 2, we will use the Markov property of the GFF (see Lemma 4.30) and Weyl scaling (Lemma 4.24) in order to reduce to working with metrics which are actually independent, not just conditionally independent. The use of the Markov property is the reason why we restrict to a whole-plane GFF, not a whole-plane GFF plus a bounded continuous function, in Lemma 4.29.

For the proof of Lemma 4.29 we will need the following version of the Markov property of the whole-plane GFF, which is proven in [62, Lemma 2.2]. We note that the statement of this Markov property is slightly more complicated than in the case of the zero-boundary GFF due to the need to fix the additive constant for h.

Lemma 4.30 ([62])**.** *Let $z \in \mathbb{C}$ and $r > 0$ and let h be a whole-plane GFF with the additive constant chosen so that $h_r(z) = 0$. For each open set $V \subset \mathbb{C}$ which is non-polar (i.e., Brownian motion started in V a.s. hits ∂V in finite time), we have the decomposition*

$$h = \mathfrak{h} + \mathring{h} \tag{2.47}$$

where $\mathfrak{h}$ is a random distribution which is harmonic on V and is determined by $h|_{\mathbb{C}\setminus V}$ and $\mathring{h}$ is independent from $\mathfrak{h}$ and has the law of a zero-boundary GFF on V minus its average over $\partial B_r(z) \cap V$. If V is disjoint from $\partial B_r(z)$, then $\mathring{h}$ is a zero-boundary GFF and is independent from $h|_{\mathbb{C}\setminus V}$.

The following lemma will allow us to apply Lemma 4.30 to study $h|_{\mathbb{C}\setminus\overline{V}}$.

Lemma 4.31. *It suffices to prove Lemma 4.29 in the case when $B_r(z) \subset V$.*

Proof. Assume that we have proven Lemma 4.29 in the case when $B_r(z) \subset V$. Fix $z_0 \in \mathbb{C}$ and $r_0 > 0$ such that $B_{r_0}(z_0) \subset V$ and assume that h is normalized so that $h_{r_0}(z_0) = 0$. By assumption, $D_h(\cdot,\cdot;V)$ is conditionally independent from the pair $\big(h, D_h(\cdot,\cdot;\mathbb{C}\setminus\overline{V})\big)$ given $h|_{\overline{V}}$.

Now let $z \in \mathbb{C}$ and $r > 0$ and define $\widetilde{h} := h - h_r(z)$, so that $\widetilde{h}$ is a whole-plane GFF normalized so that $\widetilde{h}_r(z) = 0$. Lemma 4.24 implies that $D^\varepsilon_{\widetilde{h}} \to e^{-\xi h_r(z)} D_h =: D_{\widetilde{h}}$ in law along the same subsequence for which $D^\varepsilon_h \to D_h$ in law, so $D_{\widetilde{h}}$ is unambiguously defined. We need to show that the conclusion of the first paragraph remains true with $(\widetilde{h}, D_{\widetilde{h}})$ in place of (h, D_h).

The key fact which allows us to show this is that $\widetilde{h}_{r_0}(z_0) = -h_r(z)$. Since $B_{r_0}(z_0) \subset V$, this means that $h_r(z) \in \sigma\left(\widetilde{h}|_{\overline{V}}\right)$. In particular, $h|_{\overline{V}} = \widetilde{h}|_{\overline{V}} + h_r(z)$ is determined by $\widetilde{h}|_{\overline{V}}$. Therefore, our assumption implies that $D_h(\cdot,\cdot;V)$ is conditionally independent from the pair $\left(h, D_h(\cdot,\cdot;\mathbb{C}\setminus\overline{V})\right)$ given $\widetilde{h}|_{\overline{V}}$ (instead of just $h|_{\overline{V}}$).

We have $D_{\widetilde{h}}(\cdot,\cdot;V) = e^{-\xi h_r(z)} D_h(\cdot,\cdot;V)$, so $D_{\widetilde{h}}(\cdot,\cdot;V)$ is determined by $\widetilde{h}|_{\overline{V}}$ and $D_h(\cdot,\cdot;V)$. Similarly, $D_{\widetilde{h}}(\cdot,\cdot;\mathbb{C}\setminus\overline{V})$ is determined by $\widetilde{h}|_{\overline{V}}$ and $D_h(\cdot,\cdot;\mathbb{C}\setminus\overline{V})$. Obviously, h and $\widetilde{h}$ determine the same information. Therefore, $D_{\widetilde{h}}(\cdot,\cdot;V)$ is conditionally independent from the pair $\left(\widetilde{h}, D_{\widetilde{h}}(\cdot,\cdot;\mathbb{C}\setminus\overline{V})\right)$ given $\widetilde{h}|_{\overline{V}}$, as required. $\square$

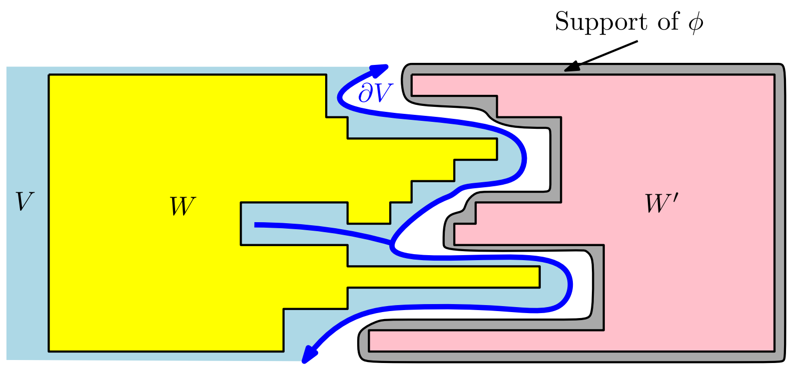

Figure 4.1 – Illustration of the sets used in the proof of Lemma 4.29. The set $\phi^{-1}(1)$ is not shown; it contains the closure of the pink set W' and is contained in the grey set $\operatorname{supp}\phi$.

Proof of Lemma 4.29. Step 1: reductions. By Lemma 4.13, for any sequence of ε's tending to zero along which $(h, \mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon) \to (h, D_h)$ in law, we also have $(h, \mathfrak{a}_\varepsilon^{-1} \widehat{D}_h^\varepsilon) \to (h, D_h)$ in law. This allows us to work with $\widehat{D}_h^\varepsilon$ instead of D_h^ε throughout the proof. The reason why we want to do this is the locality property (2.18) of $\widehat{D}_h^\varepsilon$.

The statement of the lemma is vacuous if $\overline{V} = \mathbb{C}$, so we can assume without loss of generality that $\overline{V} \neq \mathbb{C}$, which implies that $\mathbb{C}\setminus\overline{V}$ is non-polar. By Lemma 4.31, we can also assume without loss of generality that $B_r(z) \subset V$. These assumptions together with Lemma 4.30 applied with $\mathbb{C}\setminus\overline{V}$ in place of V allows us to write

$$h|_{\mathbb{C}\setminus\overline{V}} = \mathfrak{h} + \mathring{h} \tag{2.48}$$

where $\mathfrak{h}$ is a random harmonic function on $\mathbb{C}\setminus\overline{V}$ which is determined by $h|_{\mathbb{C}\setminus\overline{V}}$ and $\mathring{h}$ is a zero-boundary GFF in $\mathbb{C}\setminus\overline{V}$ which is independent from $h|_{\mathbb{C}\setminus\overline{V}}$.

Step 2: independence for LFPP. We want to apply the convergence of internal metrics given in Lemma 4.17, so we fix dyadic domains (Definition 4.16) W, W' with $\overline{W} \subset V$ and $\overline{W}' \subset \mathbb{C} \setminus \overline{V}$ (we will eventually let W and W' increase to all of V and $\mathbb{C} \setminus \overline{V}$, respectively). Let ϕ be a deterministic, smooth, compactly supported bump function which is identically equal to 1 on a neighborhood of $\overline{W}'$ and which vanishes outside of a compact subset of $\mathbb{C} \setminus \overline{V}$. See Figure 4.1 for an illustration of these objects.

The restrictions of the fields $h - \phi\mathfrak{h}$ and $\mathring{h}$ to the set $\phi^{-1}(1) \supset \overline{W}'$ are identical. By the locality property (2.18) of $\widehat{D}_h^\varepsilon$, if $\varepsilon > 0$ is small enough that $B_\varepsilon(W') \subset \phi^{-1}(1)$, then the ε-LFPP metric for $h - \phi\mathfrak{h}$ satisfies

$$\widehat{D}^\varepsilon_{h-\phi\mathfrak{h}}(\cdot,\cdot;\overline{W}') \in \sigma\left(\mathring{h}\right). \tag{2.49}$$

Similarly, for small enough $\varepsilon > 0$ the metric $\widehat{D}_h^\varepsilon(\cdot,\cdot;\overline{W})$ is a.s. determined by $h|_V$. Since $h|_V$ and $\mathring{h}$ are independent, we obtain

$$\left(h|_V, \mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon(\cdot,\cdot;\overline{W})\right) \quad \text{and} \quad \left(\mathring{h}, \mathfrak{a}_\varepsilon^{-1} D^\varepsilon_{h-\phi\mathfrak{h}}(\cdot,\cdot;\overline{W}')\right) \quad \text{are independent.} \tag{2.50}$$

Step 3: passing to the limit. We now want to pass the independence (2.50) through to the (subsequential) scaling limit. To this end, consider a sequence $\mathcal{E}$ of positive ε's tending to zero along which $(h, \mathfrak{a}_\varepsilon^{-1}\widehat{D}_h^\varepsilon) \to (h, D_h)$ in law. By possibly passing to a further deterministic subsequence, we can arrange that in fact $(h, \mathfrak{h}, \mathfrak{a}_\varepsilon^{-1}\widehat{D}_h^\varepsilon) \to (h, \mathfrak{h}, D_h)$ in law along $\mathcal{E}$, where here the second coordinate is given the local uniform topology on $\mathbb{C} \setminus \overline{V}$. By the analog of Lemma 4.24 with $\widehat{D}^\varepsilon_\cdot$ in place of $D^\varepsilon_\cdot$ (which is proven in an identical manner), if we set $D_{h-\phi\mathfrak{h}} = e^{-\xi\phi\mathfrak{h}} \cdot D_h$, then along this same subsequence we have the convergence of joint laws

$$\left(h, \mathfrak{h}, \mathfrak{a}_\varepsilon^{-1}\widehat{D}_h^\varepsilon, \mathfrak{a}_\varepsilon^{-1}\widehat{D}^\varepsilon_{h-\phi\mathfrak{h}}\right) \to (h, \mathfrak{h}, D_h, D_{h-\phi\mathfrak{h}})\,. \tag{2.51}$$

By assertion B of Lemma 4.17, applied once to each of h and $h - \phi\mathfrak{h}$, by possibly replacing $\mathcal{E}$ with a further deterministic subsequence we can find a coupling $(h, D_h, D_{h,W}, D_{h-\phi\mathfrak{h},W'})$ of (h, D_h) with length metrics on $\overline{W}$ and $\overline{W}'$, respectively, which induce the Euclidean topology and which satisfy

$$D_{h,W}(\cdot,\cdot;W) = D_h(\cdot,\cdot;W) \quad \text{and} \quad D_{h-\phi\mathfrak{h},W'}(\cdot,\cdot;W') = D_{h-\phi\mathfrak{h}}(\cdot,\cdot;W') \tag{2.52}$$

such that the following is true. Along $\mathcal{E}$, we have the convergence of joint laws

$$\begin{aligned}&\left(h, \mathfrak{h}, \mathfrak{a}_\varepsilon^{-1}\widehat{D}_h^\varepsilon, \mathfrak{a}_\varepsilon^{-1}\widehat{D}_{h-\phi\mathfrak{h}}^\varepsilon, \mathfrak{a}_\varepsilon^{-1}\widehat{D}_h^\varepsilon(\cdot,\cdot;\overline{W}), \mathfrak{a}_\varepsilon^{-1}\widehat{D}_{h-\phi\mathfrak{h}}^\varepsilon(\cdot,\cdot;\overline{W}')\right)\\ &\qquad \to \left(h,, \mathfrak{h}, D_h, D_{h-\phi\mathfrak{h}}, D_{h,W}, D_{h-\phi\mathfrak{h},W'}\right)\end{aligned} \tag{2.53}$$

where the last two coordinates are given the uniform topology on $\overline{W}\times\overline{W}$ and on $\overline{W}'\times\overline{W}'$, respectively. Since independence is preserved under convergence in law, we obtain from (2.50) and (2.53) that $(h|_V, D_{h,W})$ and $(\mathring{h}, D_{h-\phi\mathfrak{h},W'})$ are independent. By (2.52), this means that

$$(h|_V, D_h(\cdot,\cdot;W)) \quad \text{and} \quad (\mathring{h}, D_{h-\phi\mathfrak{h}}(\cdot,\cdot;W')) \quad \text{are independent.} \tag{2.54}$$

Step 4: adding back in the harmonic part. By (2.54), $D_h(\cdot,\cdot;W)$ is conditionally independent from $(\mathring{h}, D_{h-\phi\mathfrak{h}}(\cdot,\cdot;W'))$ given $h|_V$. We now argue that $(h, D_h(\cdot,\cdot;W'))$ is a measurable function of $(\mathring{h}, D_{h-\phi\mathfrak{h}}(\cdot,\cdot;W'))$ and $h|_V$, so that $D_h(\cdot,\cdot;W)$ is conditionally independent from $(h, D_h(\cdot,\cdot;W'))$ given $h|_V$. Indeed, by Lemma 4.24, a.s. $D_h(\cdot,\cdot;W') = (e^{\xi\phi\mathfrak{h}} \cdot D_{h-\phi\mathfrak{h}})(\cdot,\cdot;W')$. Hence $D_h(\cdot,\cdot;W')$ is a measurable function of $\mathfrak{h} \in \sigma(h|_{\overline{V}})$ and $D_{h-\phi\mathfrak{h}}(\cdot,\cdot;W')$. Since $h|_{\mathbb{C}\setminus\overline{V}} = \mathring{h} + \mathfrak{h}$, we get that h is a measurable function of $\mathring{h}$ and $h|_{\overline{V}}$. It therefore follows that $D_h(\cdot,\cdot;W)$ is conditionally independent from $(h, D_h(\cdot,\cdot;W'))$ given $h|_{\overline{V}}$. Letting W increase to V and W' increase to $\mathbb{C}\setminus\overline{V}$ now concludes the proof. □

4.2.6 Measurability

We have not yet established that subsequential limits of LFPP can be realized as measurable functions of the corresponding field. We will accomplish this in this subsection using a result from [57].

Lemma 4.32. *Let h be a whole-plane GFF normalized so that $h_1(0) = 0$ and let (h, D_h) be any subsequential limit of the laws of the pairs $(h, \mathfrak{a}_\varepsilon^{-1}D_h^\varepsilon)$. Then D_h is a.s. determined by h. In particular, $\mathfrak{a}_\varepsilon^{-1}D_h^\varepsilon \to D_h$ in probability along the given subsequence.*

The following theorem is a special case of [57, Corollary 1.8].

Theorem 4.33 ([57])**.** *There is a universal constant $p \in (0,1)$ such that the following is true. Let $\xi \in \mathbb{R}$, let h be a whole-plane GFF normalized so that $h_1(0) = 0$, and let (h, D) be a coupling of h with a random continuous length metric satisfying the following properties.*

1. *D is a ξ-additive local metric for h (Definition 4.28).*
2. *Condition on h and let D and $\widetilde{D}$ be conditionally i.i.d. samples from the conditional law of D given h. There is a deterministic constant $C > 0$ such that*

$$\mathbb{P}\left[\sup_{u,v\in\partial B_r(z)} \widetilde{D}\left(u,v; B_{2r}(z)\setminus\overline{B_{r/2}(z)}\right) \leq C D(\partial B_{r/2}(z), \partial B_r(z))\right] \geq p, \quad \forall z\in\mathbb{C}, \quad \forall r>0. \tag{2.55}$$

Then D is a.s. determined by h.

Proof of Lemma 4.32. Let $p \in (0,1)$ be as in Theorem 4.33. Lemma 4.29 implies that D_h is a ξ-additive local metric for h. Lemma 4.25 along with the translation invariance of the law of h, modulo additive constant, implies that there exists $C > 0$ (depending only on the choice of subsequence) such that for each $z \in \mathbb{C}$ and each $r > 0$,

$$\mathbb{P}\left[D(\partial B_{r/2}(z), \partial B_r(z)) \geq C^{-1/2}\mathfrak{c}_r e^{\xi h_r(z)}\right] \geq \frac{1-p}{2} \quad \text{and}$$

$$\mathbb{P}\left[\sup_{u,v\in\partial B_r(z)} D_h\left(u,v; B_{2r}(z)\setminus\overline{B_{r/2}(z)}\right) \leq C^{1/2}\mathfrak{c}_r e^{\xi h_r(z)}\right] \geq \frac{1-p}{2}.$$

This implies that (2.55) holds for two conditionally independent samples from the conditional law of D_h given h. Hence the criteria of Theorem 4.33 are satisfied, so D_h is a.s. determined by h. The last statement follows from Lemma 4.3. □

Proof of Theorem 4.2. Step 1: Defining a D_{h} for a whole-plane GFF plus a bounded continuous function. Let h be a whole-plane GFF normalized so that $h_1(0) = 0$. Lemma 4.17 implies that for any sequence of ε's tending to zero, there is a subsequence $\varepsilon_n \to 0$ along which $(h, D_h^{\varepsilon_n}) \to (h, D_h)$ in law. By Lemma 4.32, D_h is a.s. determined by h and $D_h^{\varepsilon_n} \to D_h$ in probability. Hence every deterministic subsequence of the ε_n's admits a further deterministic subsequence ε_{n_k} along which $D_h^{\varepsilon_{n_k}} \to D_h$ a.s. By Lemma 4.24, it is a.s. the case that for every bounded continuous function $f : \mathbb{C} \to \mathbb{R}$ simultaneously, we have $D_{h+f}^{\varepsilon_{n_k}} \to e^{\xi f} \cdot D_h$. We define $D_{h+f} := e^{\xi f} \cdot D_h$. Then D_{h+f} is a.s. determined by $h+f$ and $D_{h+f}^{\varepsilon_n}$ converges in probability to D_{h+f}.

This gives us a measurable function $\mathsf{h} \mapsto D_{\mathsf{h}}$ from distributions to continuous metrics on $\mathbb{C}$ which is a.s. defined whenever h is a whole-plane GFF plus a bounded continuous function: in particular, D_{h} is the a.s. limit of $D_{\mathsf{h}}^{\varepsilon_{n_k}}$. With this definition of D, Axiom I holds with h constrained to be a whole-plane GFF plus a bounded continuous function since we know that the limiting metric in the setting of Lemma 4.17 is a length metric. By the preceding paragraph, Axiom III holds for this definition of D and with f constrained to be bounded. It is immediate from the definition of LFPP that also Axiom IV holds. By Lemma 4.25, also Axiom V holds.

Step 2: locality for a whole-plane GFF plus a bounded continuous function. Axiom II in the case of a whole-plane GFF is immediate from Lemma 4.29 now that we know that D_h is a.s. determined by h. We now prove Axiom II in the case when h is a whole-plane GFF plus a bounded continuous function. Indeed, let $V \subset \mathbb{C}$ be open and let $O \subset O' \subset V$ be open and bounded with $\overline{O} \subset O'$ and $\overline{O}' \subset V$. Let $u, v \in O$ be deterministic. We will show that

$$D_{\mathsf{h}}(u,v)\mathbb{1}_{\{D_{\mathsf{h}}(u,v) < D_{\mathsf{h}}(u,\partial O')\}} \in \sigma\left(\mathsf{h}|_V\right). \tag{2.56}$$

Since $(u,v) \mapsto D_{\mathsf{h}}(u,v)$ is a.s. continuous, (2.56) implies that in fact $\mathsf{h}|_V$ a.s. determines the random function $O \ni (u,v) \mapsto D_{\mathsf{h}}(u,v)\mathbb{1}_{\{D_{\mathsf{h}}(u,v) < D_{\mathsf{h}}(u,\partial O')\}}$. Since $\overline{O}$ is a compact subset of O', O can be covered by finitely many sets of the form $\{v \in O : D_{\mathsf{h}}(u,v) < D_{\mathsf{h}}(u,\partial O')\}$ for points $u \in O$. By the definition of the internal metric $D_{\mathsf{h}}(\cdot,\cdot;O)$, this shows that $\mathsf{h}|_V$ a.s. determines $D_{\mathsf{h}}(\cdot,\cdot;O)$. Letting O increase to all of V then shows that $\mathsf{h}|_V$ a.s. determines $D_{\mathsf{h}}(\cdot,\cdot;V)$.

To prove (2.56), note that if we define the localized LFPP metric $\widehat{D}_{\mathsf{h}}^{\varepsilon_n}$ as in (2.17), then by Lemma 4.13 we have $\mathfrak{a}_{\varepsilon_n}^{-1}\widehat{D}_{\mathsf{h}}^{\varepsilon_n}(u,v) \to D_{\mathsf{h}}(u,v)$ and $\mathfrak{a}_{\varepsilon_n}^{-1}\widehat{D}_{\mathsf{h}}^{\varepsilon_n}(u,\partial O') \to D_{\mathsf{h}}(u,\partial O')$ in probability. Therefore,

$$\mathfrak{a}_{\varepsilon_n}^{-1}\widehat{D}_{\mathsf{h}}^{\varepsilon_n}(u,v)\mathbb{1}_{\left\{\widehat{D}_{\mathsf{h}}^{\varepsilon_n}(u,v) < \widehat{D}_{\mathsf{h}}^{\varepsilon_n}(u,\partial O')\right\}} \to D_{\mathsf{h}}(u,v)\mathbb{1}_{\{D_{\mathsf{h}}(u,v) < D_{\mathsf{h}}(u,\partial O')\}}, \quad \text{in probability.} \tag{2.57}$$

By (2.18) and since $\overline{O}' \subset V$, the random variable on the left side of (2.57) is a.s. determined by $\mathsf{h}|_V$ for large enough $n \in \mathbb{N}$. Thus (2.56) holds.

Step 3: extending to unbounded continuous function. We will now extend the definition of D to the case of a whole-plane GFF plus an unbounded continuous function and check that the axioms remain true. To this end, let h be a whole-plane GFF and let f be a possibly random unbounded continuous function. If $V \subset \mathbb{C}$ is open and bounded and ϕ is a smooth compactly supported bump function which is identically equal to 1 on V, then ϕf is bounded so we can define the metric $D_{h+f}^V := D_{h+\phi f}(\cdot,\cdot;V)$. By Axiom II in the case of a whole-plane GFF plus a bounded continuous function, this metric is a.s. determined by $(h+\phi f)|_V = (h+f)|_V$, in a manner which does not depend on ϕ. We now define the D_{h+f}-length of any continuous path P in $\mathbb{C}$ to be the D_{h+f}^V-length of P, where $V \subset \mathbb{C}$ is a bounded open set which contains P The definition does not depend on the choice of V. We define $D_{h+f}(z,w)$ for $z,w \in \mathbb{C}$ to be the infimum of the D_{h+f}-lengths of continuous paths from z to w. Then D_{h+f} is a length metric on $\mathbb{C}$ which is a.s. determined by D_{h+f} and which satisfies $D_{h+f}(\cdot,\cdot;V) = D_{h+f}^V$ for each bounded open set $V \subset \mathbb{C}$.

With the above definition, it is immediate from the case of a whole-plane GFF plus a bounded continuous function that the axioms in the definition of a weak γ-LQG metric are satisfied to the mapping $\mathsf{h} \mapsto D_{\mathsf{h}}$, which is a.s. defined whenever h is a whole-plane GFF plus a continuous function. □

4.3 Proofs of quantitative properties of weak LQG metrics

In this section we will prove the estimates stated in Section 4.1.2. Actually, in many cases we will prove a priori stronger estimates which are required to be uniform across different Euclidean scales. With what we know now, these estimates are not implied by the estimates stated in Section 4.1.2 since we are working with a weak γ-LQG metric so we have tightness across scales instead of exact scale invariance. However, *a posteriori*, once it is proven that a weak γ-LQG metric satisfies the coordinate change formula (1.6) (which will be done in [59], building on the results in the present chapter), the estimates in this section are equivalent to the estimates in Section 4.1.2. Throughout this section, D denotes a weak LQG metric and h denotes a whole-plane GFF normalized so that

$h_1(0) = 0$.

4.3.1 Estimate for the distance between sets

The goal of this subsection is to prove the following more precise version of Theorem 4.9 which is required to be uniform across scales. For the statement, we recall the scaling constants $\mathfrak{c}_r$ for $r > 0$ from Axiom V.

Proposition 4.34. *Let $U \subset \mathbb{C}$ be an open set (possibly all of $\mathbb{C}$) and let $K_1, K_2 \subset U$ be connected, disjoint compact sets which are not singletons. For each $\mathbb{r} > 0$, it holds with superpolynomially high probability as $A \to \infty$, at a rate which is uniform in the choice of $\mathbb{r}$, that*

$$A^{-1}\mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)} \leq D_h(\mathbb{r}K_1, \mathbb{r}K_2; \mathbb{r}U) \leq A\mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)}. \tag{3.58}$$

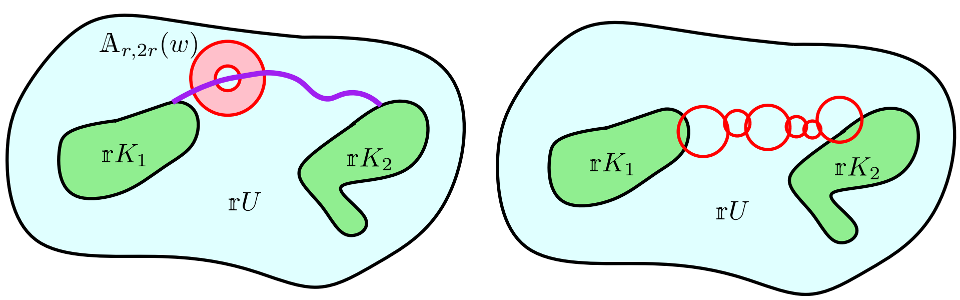

Figure 4.2 – **Left:** To prove the lower bound in Proposition 4.34, we cover $\mathbb{r}U$ by balls $B_{r/2}(w)$ such that the D_h-distance across the annulus $\mathbb{A}_{r,2r}(w)$ is bounded below. Each path from $\mathbb{r}K_1$ to $\mathbb{r}(K_2 \cup \partial U)$ must cross at least one of these annuli (one such path is shown in purple). **Right:** To prove the upper bound in Proposition 4.34, we cover $\mathbb{r}U$ by balls $B_{r/2}(w)$ for which the D_h-diameter of the circle $\partial B_r(w)$ is bounded above, then string together a path of such circles from K_1 to K_2.

We now explain the idea of the proof of Proposition 4.34; see Figure 4.2 for an illustration. Using Axiom V and a general "local independence" lemma for the GFF (see Lemma 4.36 below), we can, with extremely high probability, cover $\mathbb{r}U$ by small Euclidean balls $B_{r/2}(w)$ such that $r \in [\varepsilon^2\mathbb{r}, \varepsilon\mathbb{r}]$ and the D_h-distance across the annulus $\mathbb{A}_{r,2r}(w)$ is bounded below by a constant times $\mathfrak{c}_r e^{\xi h_r(w)}$. Any path from $\mathbb{r}K_1$ to $\mathbb{r}K_2$ must cross at least one of these annuli. This leads to a lower bound for

$D_h(\mathbb{r}K_1, \mathbb{r}K_2; \mathbb{r}U)$ in terms of

$$\inf_{r \in [\varepsilon^2 \mathbb{r}, \varepsilon \mathbb{r}]} \mathfrak{c}_r \quad \text{and} \quad \inf_{r \in [\varepsilon^2 \mathbb{r}, \varepsilon \mathbb{r}]} \inf_{w \in \mathbb{r}U} e^{\xi h_r(w)}. \tag{3.59}$$

The first infimum in (3.59) can be bounded below by a positive power of ε times $\mathfrak{c}_{\mathbb{r}}$ by (1.5). By being a little more careful about how we choose the balls $B_{r/2}(w)$, the second term in (3.59) can be reduced to an infimum over finitely many values of r and w, which can then be bounded below by a positive power of ε times $e^{\xi h_{\mathbb{r}}(0)}$ using the Gaussian tail bound and a union bound (see Lemma 4.37). Choosing ε to be an appropriate power of A then concludes the proof.

The upper bound in (3.58) is proven similarly, but in this case we instead cover U by balls $B_{r/2}(w)$ for which the D_h-diameter of the circle $\partial B_r(w)$ is bounded above by a constant times $\mathfrak{c}_r e^{\xi h_r(w)}$, then "string together" a collection of such circles to get a path from $\mathbb{r}K_1$ to $\mathbb{r}K_2$ whose D_h-length is bounded above. The hypothesis that K_1 and K_2 are connected and are not singletons allows us to force some of the circles in this path to intersect K_1 and K_2.

We now explain how to cover U by Euclidean balls with the desired properties. For $C > 1$, $z \in \mathbb{C}$, and $r > 0$, let $E_r(z; C)$ be the event that

$$\sup_{u,v \in \partial B_r(z)} D_h\left(u, v; \mathbb{A}_{r/2,2r}(z)\right) \leq C \mathfrak{c}_r e^{\xi h_r(0)} \quad \text{and} \quad D_h\left(\partial B_r(z), \partial B_{2r}(z)\right) \geq C^{-1} \mathfrak{c}_r e^{\xi h_r(0)}. \tag{3.60}$$

Lemma 4.35. *For each $\nu > 0$ and each $M > 0$, there exists $C = C(\nu, M) > 1$ such that for each $\mathbb{r} > 0$, it holds with probability at least $1 - O_\varepsilon(\varepsilon^M)$ as $\varepsilon \to 0$, at a rate which is uniform in $\mathbb{r}$, that the following is true. For each $z \in B_{\mathbb{r}\varepsilon^{-M}}(0)$, there exists $w \in B_{\mathbb{r}\varepsilon^{-M}}(0) \cap \left(\frac{\varepsilon^{1+\nu}\mathbb{r}}{4}\mathbb{Z}^2\right)$ and $r \in [\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}] \cap \{2^{-k}\mathbb{r}\}_{k \in \mathbb{N}}$ such that $E_r(w; C)$ occurs and $z \in B_{\mathbb{r}\varepsilon^{1+\nu}/2}(w)$.*

We will prove Lemma 4.35 using the following result from [57], which in turn follows from the near-independence of the GFF across disjoint concentric annuli. See in particular [57, Lemma 3.1].

Lemma 4.36. *Fix $0 < s_1 < s_2 < 1$. Let $\{r_k\}_{k \in \mathbb{N}}$ be decreasing with the r_i's positive and s.t. $r_{k+1}/r_k \leq s_1$ for each $k \in \mathbb{N}$ and let $\{E_{r_k}\}_{k \in \mathbb{N}}$ be events s.t. $E_{r_k} \in \sigma\left((h - h_{r_k}(0))|_{\mathbb{A}_{s_1 r_k, s_2 r_k}(0)}\right)$ for each $k \in \mathbb{N}$. For $K \in \mathbb{N}$, let $N(K)$ be the number of $k \in [1, K]_{\mathbb{Z}}$ for which E_{r_k} occurs.*

For each $a > 0$ and each $b \in (0, 1)$, there exists $p = p(a, b, s_1, s_2) \in (0, 1)$ and $c = c(a, b, s_1, s_2) > 0$

such that if

$$\mathbb{P}\left[E_{r_k}\right] \geq p, \quad \forall k \in \mathbb{N}, \tag{3.61}$$

then

$$\mathbb{P}\left[N(K) < bK\right] \leq ce^{-aK}, \quad \forall K \in \mathbb{N}. \tag{3.62}$$

Proof of Lemma 4.35. By Axioms IV and V (also see (1.7)), for each $p \in (0,1)$ there exists $C > 1$ such that for every $z \in \mathbb{C}$ and $r > 0$, $\mathbb{P}\left[E_r(z;C)\right] \geq p$. By the locality of D_h and Axiom III, the event $E_r(z;C)$ is determined by $(h - h_{3r}(z))|_{\mathbb{A}_{r/2,2r}(z)}$. We can therefore apply Lemma 4.36 to a logarithmic (in ε) number of values of $r \in [\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}] \cap \{2^{-k}\mathbb{r}\}_{k\in\mathbb{N}}$ to find that for any choice of $\nu > 1$ and $\widetilde{M} > 0$, there is a large enough $C = C(\nu, \widetilde{M}) > 1$ such that the following is true. For each $z \in \mathbb{C}$ it holds with probability at least $1 - O_\varepsilon(\varepsilon^{\widetilde{M}})$ that $E_r(z;C)$ occurs for at least one value of $r \in [\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}] \cap \{2^{-k}\mathbb{r}\}_{k\in\mathbb{N}}$. We now conclude the proof by choosing $\widetilde{M}$ to be sufficiently large, in a manner depending only on ν, M, and taking a union bound over all $z \in B_{\mathbb{r}\varepsilon^{-M}}(0) \cap \left(\frac{\varepsilon^{1+\nu}\mathbb{r}}{4}\mathbb{Z}^2\right)$. □

The occurrence of the event $E_r(z;C)$ allows us to bound distances in terms of circle averages and the scaling coefficients $\mathfrak{c}_r$. The $\mathfrak{c}_r$'s can be bounded using (1.5). To bound the circle averages, we will need the following lemma.

Lemma 4.37. *For each $\nu > 0$, each $q > 2+2\nu$, each $R > 0$, and each $\mathbb{r} > 0$, it holds with probability $1 - O_\varepsilon\left(\varepsilon^{\frac{q^2}{2(1+\sqrt{\nu})^2} - 2 - 2\nu}\right)$, at a rate depending only on q and R (not on $\mathbb{r}$) that*

$$\sup\left\{|h_r(w) - h_{\mathbb{r}}(0)| : w \in B_{R\mathbb{r}}(0) \cap \left(\frac{\varepsilon^{1+\nu}\mathbb{r}}{4}\mathbb{Z}^2\right), \, r \in [\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}]\right\} \leq q \log \varepsilon^{-1}. \tag{3.63}$$

Proof. Fix $s \in (0,q)$ to be chosen momentarily. For each $w \in B_{R\mathbb{r}}(0)$, the random variable $t \mapsto h_{e^{-t}\varepsilon\mathbb{r}}(w) - h_{\varepsilon\mathbb{r}}(w)$ is a standard linear Brownian motion [44, Section 3]. We can therefore apply the Gaussian tail bound to find that

$$\mathbb{P}\left[\sup_{r\in[\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}]} |h_r(w) - h_{\varepsilon\mathbb{r}}(w)| \leq s \log \varepsilon^{-1}\right] \geq 1 - O_\varepsilon\left(\varepsilon^{s^2/(2\nu)}\right). \tag{3.64}$$

The random variables $h_{\varepsilon\mathbb{r}}(w) - h_{\mathbb{r}}(0)$ for $w \in B_{R\mathbb{r}}(0)$ are centered Gaussian with variance $\log \varepsilon^{-1} +$

$O_\varepsilon(1)$. Applying the Gaussian tail bound again therefore gives

$$\mathbb{P}\left[|h_{\varepsilon\mathbbm{r}}(w) - h_{\mathbbm{r}}(0)| \leq (q-s)\log\varepsilon^{-1}\right] \geq 1 - O_\varepsilon\left(\varepsilon^{(q-s)^2/2}\right). \tag{3.65}$$

Combining (3.64) and (3.65) applied with $s = q\sqrt{\nu}/(1+\sqrt{\nu})$ shows that for $w \in B_{R\mathbbm{r}}(0)$,

$$\mathbb{P}\left[\sup_{r\in[\varepsilon^{1+\nu}\mathbbm{r},\varepsilon\mathbbm{r}]} |h_r(w) - h_{\mathbbm{r}}(0)| \leq q\log\varepsilon^{-1}\right] \geq 1 - O_\varepsilon\left(\varepsilon^{\frac{q^2}{2(1+\sqrt{\nu})^2}}\right). \tag{3.66}$$

We now conclude by means of a union bound over $O_\varepsilon(\varepsilon^{-2-2\nu})$ values of $w \in B_{R\mathbbm{r}}(0) \cap \left(\frac{\varepsilon^{1+\nu}\mathbbm{r}}{4}\mathbb{Z}^2\right)$. $\square$

Proof of Proposition 4.34. Throughout the proof, all $O(\cdot)$ and $o(\cdot)$ errors are required to be uniform in the choice of $\mathbbm{r}$. We also impose the requirement that U is bounded — we will explain at the very end of the proof how to get rid of this requirement.

Set $\nu = 1$, say, and fix a large $M > 1$, which we will eventually send to ∞. Let $C = C(1, M) > 1$ be chosen as in Lemma 4.35 and for $\varepsilon \in (0,1)$ and $\mathbbm{r} > 0$, let $F_{\mathbbm{r}}^\varepsilon$ be the event of Lemma 4.35 for this choice of ν, M, C, so that $\mathbb{P}[F_{\mathbbm{r}}^\varepsilon] = 1 - O_\varepsilon(\varepsilon^M)$. We will eventually take $\varepsilon = A^{-b/\sqrt{M}}$ for a small constant $b > 0$, so ε^M will be a large negative power of A (i.e., the power goes to ∞ as $M \to \infty$) but $\varepsilon^{\sqrt{M}}$ will be a fixed negative power of A (which does not go to ∞ when $M \to \infty$).

By Lemma 4.37 (applied with $\nu = 1$ and $q = 2\sqrt{2}\sqrt{4+M}$), it holds with probability $1 - O_\varepsilon(\varepsilon^M)$ that

$$\sup\left\{|h_r(w) - h_{\mathbbm{r}}(0)| : w \in B_{\mathbbm{r}}(\mathbbm{r}U) \cap \left(\frac{\varepsilon^2\mathbbm{r}}{4}\mathbb{Z}^2\right),\ r \in [\varepsilon^2\mathbbm{r}, \varepsilon\mathbbm{r}]\right\} \leq 2\sqrt{2}\sqrt{4+M}\log\varepsilon^{-1}. \tag{3.67}$$

Henceforth assume that $F_{\mathbbm{r}}^\varepsilon$ occurs and (3.67) holds, which happens with probability $1 - O_\varepsilon(\varepsilon^M)$. We will now prove lower and upper bounds for $D_h(\mathbbm{r}K_1, \mathbbm{r}K_2; \mathbbm{r}U)$ in terms of ε.

Step 1: lower bound. By the definition of $F_{\mathbbm{r}}^\varepsilon$, if ε is sufficiently small, depending on K_1, K_2, U, then each path from $\mathbbm{r}K_1$ to $\mathbbm{r}(K_2 \cup \partial U)$ must cross from $\partial B_r(w)$ to $\partial B_{2r}(w)$ for some $w \in$

$B_{\varepsilon\mathbbm{r}}(\mathbbm{r}U) \cap \left(\frac{\varepsilon^2\mathbbm{r}}{4}\mathbb{Z}^2\right)$ and $r \in [\varepsilon^2\mathbbm{r}, \varepsilon\mathbbm{r}] \cap \{2^{-k}\mathbbm{r}\}_{k\in\mathbb{N}}$ for which $E_r(w;C)$ occurs. Therefore,

$$\begin{aligned} D_h(\mathbbm{r}K_1, \mathbbm{r}K_2) &\geq \inf\left\{C^{-1}\mathfrak{c}_r e^{\xi h_r(w)} : w \in B_{\varepsilon\mathbbm{r}}(\mathbbm{r}U) \cap \left(\frac{\varepsilon^2\mathbbm{r}}{4}\mathbb{Z}^2\right),\ r \in [\varepsilon^2\mathbbm{r}, \varepsilon\mathbbm{r}]\right\} \quad \text{(by (3.60))} \\ &\geq C^{-1}\varepsilon^{\xi 2\sqrt{2}\sqrt{4+M}} e^{\xi h_{\mathbbm{r}}(0)} \inf\left\{\mathfrak{c}_r : r \in [\varepsilon^2\mathbbm{r}, \varepsilon\mathbbm{r}]\right\} \quad \text{(by (3.67))} \\ &\geq \Lambda^{-1}\varepsilon^{\xi 2\sqrt{2}\sqrt{4+M}+2\Lambda+o_\varepsilon(1)}\mathfrak{c}_{\mathbbm{r}} e^{\xi h_{\mathbbm{r}}(0)} \quad \text{(by (1.5))}. \end{aligned} \tag{3.68}$$

Step 2: upper bound. It is easily seen from the definition of $F_{\mathbbm{r}}^\varepsilon$ (see Lemma 4.38 below) that if ε is sufficiently small (depending only on K_1, K_2, and U) then the union of the circles $\partial B_r(w)$ for $w \in B_{\varepsilon\mathbbm{r}}(\mathbbm{r}U) \cap \left(\frac{\varepsilon^2\mathbbm{r}}{4}\mathbb{Z}^2\right)$ and $r \in [\varepsilon^2\mathbbm{r}, \varepsilon\mathbbm{r}] \cap \{2^{-k}\mathbbm{r}\}_{k\in\mathbb{N}}$ such that $E_r(w;C)$ occurs contains a path from $\mathbbm{r}K_1$ to $\mathbbm{r}K_2$ which is contained in $\mathbbm{r}U$. The total number of such circles is at most $\varepsilon^{-4-o_\varepsilon(1)}$, so by the triangle inequality and by (3.60),

$$\begin{aligned} D_h(\mathbbm{r}K_1, \mathbbm{r}K_2; \mathbbm{r}U) &\leq \varepsilon^{-4-o_\varepsilon(1)} \sup\left\{C\mathfrak{c}_r e^{\xi h_r(w)} : w \in B_{\varepsilon\mathbbm{r}}(\mathbbm{r}U) \cap \left(\frac{\varepsilon^2\mathbbm{r}}{4}\mathbb{Z}^2\right),\ r \in [\varepsilon^2\mathbbm{r}, \varepsilon\mathbbm{r}]\right\} \\ &\leq \varepsilon^{-4-\xi 2\sqrt{2}\sqrt{4+M}-o_\varepsilon(1)} e^{\xi h_{\mathbbm{r}}(0)} \sup\left\{\mathfrak{c}_r : r \in [\varepsilon^2\mathbbm{r}, \varepsilon\mathbbm{r}]\right\} \quad \text{(by (3.67))} \\ &\leq \Lambda\varepsilon^{-4-\xi 2\sqrt{2}\sqrt{4+M}-2\Lambda-o_\varepsilon(1)}\mathfrak{c}_{\mathbbm{r}} e^{\xi h_{\mathbbm{r}}(0)} \quad \text{(by (1.5))}. \end{aligned} \tag{3.69}$$

Step 3: choosing ε. The bounds (3.68) and (3.69) hold with probability $1 - O_\varepsilon(\varepsilon^M)$. Given $A > 0$, we now choose $\varepsilon = A^{-b/\sqrt{M}}$, where $b > 0$ is a small constant (depending only on ξ, Λ) chosen so that the right side of (3.68) is at least $A^{-1}\mathfrak{c}_{\mathbbm{r}} e^{\xi h_{\mathbbm{r}}(0)}$ and the right side of (3.69) is at most $A\mathfrak{c}_{\mathbbm{r}} e^{\xi h_{\mathbbm{r}}(0)}$. Then (3.68) and (3.69) imply that

$$\mathbb{P}\left[D_h(\mathbbm{r}K_1, \mathbbm{r}K_2) \geq A^{-1}\mathfrak{c}_{\mathbbm{r}} e^{\xi h_{\mathbbm{r}}(0)},\ D_h(\mathbbm{r}K_1, \mathbbm{r}K_2; \mathbbm{r}U) \leq \mathfrak{c}_{\mathbbm{r}} e^{\xi h_{\mathbbm{r}}(0)}\right] \geq 1 - O_A(A^{-b\sqrt{M}}). \tag{3.70}$$

If U' is a possibly unbounded open subset with $U \subset U'$, then $D_h(\mathbbm{r}K_1, \mathbbm{r}K_2) \leq D_h(\mathbbm{r}K_1, \mathbbm{r}K_2; \mathbbm{r}U') \leq D_h(K_1, K_2; \mathbbm{r}U)$. Since M can be made arbitrarily large, we now obtain (3.58) (with U possibly unbounded) from (3.70). □

The following lemma was used in the proof of the upper bound of Proposition 4.34.

Lemma 4.38. *Assume that we are in the setting of Proposition 4.34, with U bounded. Define the* *event $F_{\mathbb{r}}^{\varepsilon}$ as in the proof of Proposition 4.34. For small enough $\varepsilon > 0$ (depending on K_1, K_2, U), on $F_{\mathbb{r}}^{\varepsilon}$, the union of the circles $\partial B_r(w)$ for $w \in B_{\varepsilon \mathbb{r}}(\mathbb{r}U) \cap \left(\frac{\varepsilon^2 \mathbb{r}}{4}\mathbb{Z}^2\right)$ and $r \in [\varepsilon^2 \mathbb{r}, \varepsilon \mathbb{r}] \cap \{2^{-k}\mathbb{r}\}_{k\in\mathbb{N}}$ such that $E_r(w;C)$ occurs contains a path from $\mathbb{r}K_1$ to $\mathbb{r}K_2$ which is contained in $\mathbb{r}U$.*

Proof. Throughout the proof we assume that $F_{\mathbb{r}}^{\varepsilon}$ occurs. By the definition of $F_{\mathbb{r}}^{\varepsilon}$ and since U is connected, if ε is chosen so be sufficiently small then the union of the balls $B_r(w)$ for w, r as in the lemma statement contains a path from $\mathbb{r}K_1$ to K_2 which is contained in U. Let $\mathcal{B}$ be a sub-collection of these balls which is minimal in the sense that $\bigcup_{B\in\mathcal{B}} B$ contains a path from $\mathbb{r}K_1$ to $\mathbb{r}K_2$ in $\mathbb{r}U$ and no proper sub-collection of the balls in $\mathcal{B}$ has this property. Choose a path P from $\mathbb{r}K_1$ to $\mathbb{r}K_2$ in $(\mathbb{r}U) \cap \bigcup_{B\in\mathcal{B}} B$.

We first observe that $\bigcup_{B\in\mathcal{B}} B$ is connected. Indeed, if this set had two proper disjoint open subsets, then each would have to intersect P (by minimality) which would contradict the connectedness of P. Furthermore, by minimality, no ball in $\mathcal{B}$ is properly contained in another ball in $\mathcal{B}$.

We claim that $\bigcup_{B\in\mathcal{B}} \partial B$ is connected. Indeed, if this were not the case then we could partition $\mathcal{B} = \mathcal{B}_1 \sqcup \mathcal{B}_2$ such that $\mathcal{B}_1$ and $\mathcal{B}_2$ are non-empty and $\bigcup_{B\in\mathcal{B}_1} \partial B$ and $\bigcup_{B\in\mathcal{B}_2} \partial B$ are disjoint. By the minimality of $\mathcal{B}$, it cannot be the case that any ball in $\mathcal{B}_2$ is contained in $\bigcup_{B\in\mathcal{B}_1} B$. Furthermore, since $\bigcup_{B\in\mathcal{B}_1} \partial B$ and $\bigcup_{B\in\mathcal{B}_2} \partial B$ are disjoint, it cannot be the case that any ball in $\mathcal{B}_2$ intersects both $\bigcup_{B\in\mathcal{B}_1} B$ and $\mathbb{C} \setminus \bigcup_{B\in\mathcal{B}_1} B$ (otherwise, such a ball would have to intersect the boundary of some ball in $\mathcal{B}_1$). Therefore, $\bigcup_{B\in\mathcal{B}_1} B$ and $\bigcup_{B\in\mathcal{B}_2} \partial B$ are disjoint. Since no element of $\mathcal{B}_1$ can be contained in $\bigcup_{B\in\mathcal{B}_2} B$, we get that $\bigcup_{B\in\mathcal{B}_1} B$ and $\bigcup_{B\in\mathcal{B}_2} B$ are disjoint. This contradicts the connectedness of $\bigcup_{B\in\mathcal{B}} B$, and therefore gives our claim.

Since P is a path from $\mathbb{r}K_1$ to $\mathbb{r}K_2$ and each of $\mathbb{r}K_1$ and $\mathbb{r}K_2$ is connected and not a single point, if $\varepsilon < \frac{1}{2}(\operatorname{diam}(K_1) \wedge \operatorname{diam}(K_2))$, then the boundaries of the balls in $\mathcal{B}$ which contain the starting and endpoint points of P must intersect K_1 and K_2, respectively. Hence for such an ε, $\bigcup_{B\in\mathcal{B}} \partial B$ contains a path from $\mathbb{r}K_1$ to $\mathbb{r}K_2$, as required. □

4.3.2 Asymptotics of the scaling constants

The goal of this section is to prove Theorem 4.5. We will accomplish this by comparing D_h-distances to a variant of the Liouville first passage percolation (LFPP) which we now define.

For $\varepsilon \in (0,1)$ and $U \subset \mathbb{C}$, we view $U \cap (\varepsilon \mathbb{Z}^2)$ as a graph with adjacency defined by

$$z, w \in U \cap (\varepsilon \mathbb{Z}^2) \text{ are connected by an edge if and only if } |z-w| \in \{\varepsilon, \sqrt{2}\varepsilon\}. \tag{3.71}$$

Note that this differs from the standard nearest-neighbor graph structure in that we also include the diagonal edges. We define the *discretized ε-LFPP metric with parameter ξ* on U by

$$\widetilde{D}_h^\varepsilon(z,w;U) := \min_{\pi : z \to w} \sum_{j=0}^{|\pi|} e^{\xi h_\varepsilon(\pi(j))}, \quad \forall z, w \in U \cap (\varepsilon \mathbb{Z}^2), \tag{3.72}$$

where the minimum is over all paths $\pi : [0, |\pi|]_{\mathbb{Z}} \to U \cap (\varepsilon \mathbb{Z}^2)$ from z to w in $U \cap (\varepsilon \mathbb{Z}^2)$ (the tilde is to distinguish this from the variant of LFPP defined in (1.2)).

Recall that $\mathbb{S} = (0,1)^2$ denotes the open Euclidean unit square. Below, we will show, using Proposition 4.34 and a union bound over a polynomial number of $\delta\mathbb{r} \times \delta\mathbb{r}$ squares contained in $\mathbb{r}\mathbb{S}$, that with high probability,

$$\mathfrak{c}_{\mathbb{r}} = \delta^{o_\delta(1)} \mathfrak{c}_{\delta\mathbb{r}} \times \left(\widetilde{D}_h^{\delta\mathbb{r}} \text{ distance between two sides of } \mathbb{r}\mathbb{S} \right). \tag{3.73}$$

The reason why discretized LFPP comes up in this estimate is the circle average term $e^{\xi h_{\mathbb{r}}(0)}$ in Proposition 4.34. We know that the $\widetilde{D}_h^{\delta\mathbb{r}}$ distance across the square $\mathbb{r}\mathbb{S}$ is of order $\delta^{-\xi Q + o_\delta(1)}$, uniformly in $\mathbb{r}$, by the results of [28] (see Lemma 4.39 just below). Hence (3.73) leads to $\mathfrak{c}_{\delta\mathbb{r}} = \delta^{\xi Q + o_\delta(1)} \mathfrak{c}_{\mathbb{r}}$, as required.

For a square $S \subset \mathbb{C}$, we write $\partial_{\mathrm{L}}^\varepsilon S$ and $\partial_{\mathrm{R}}^\varepsilon S$ for the set of leftmost (resp. rightmost) vertices of $S \cap (\varepsilon \mathbb{Z}^2)$.

Lemma 4.39. *Fix $\zeta \in (0,1)$. For $\mathbb{r} > 0$, it holds with probability tending to 1 as $\delta \to 0$, uniformly in the choice of $\mathbb{r}$, that*

$$\widetilde{D}_h^{\delta\mathbb{r}} \left(\partial_{\mathrm{L}}^{\delta\mathbb{r}}(\mathbb{r}\mathbb{S}), \partial_{\mathrm{R}}^{\delta\mathbb{r}}(\mathbb{r}\mathbb{S}); \mathbb{r}\mathbb{S} \right) \in \left[\delta^{-\xi Q + \zeta} e^{\xi h_{\mathbb{r}}(0)}, \delta^{-\xi Q - \zeta} e^{\xi h_{\mathbb{r}}(0)} \right]. \tag{3.74}$$

Proof. We first reduce to the case when $\mathbb{r} = 1$. Indeed, by the scale and translation invariance of the law of h, modulo additive constant, we have $h(\mathbb{r}\cdot) - h_{\mathbb{r}}(0) \stackrel{d}{=} h$. Moreover, from the definition (3.72) it is easily seen that

$$\widetilde{D}^{\delta}_{h(\mathbb{r}\cdot)-h_{\mathbb{r}}(0)}(\cdot,\cdot;\mathbb{S}) = e^{-\xi h_{\mathbb{r}}(0)}\widetilde{D}^{\delta\mathbb{r}}_{h}(\cdot,\cdot;\mathbb{r}\mathbb{S}). \tag{3.75}$$

Hence $e^{-\xi h_{\mathbb{r}}(0)}\widetilde{D}^{\delta\mathbb{r}}_{h}(\cdot,\cdot;\mathbb{r}\mathbb{S}) \overset{d}{=} \widetilde{D}^{\delta}_{h}(\cdot,\cdot;\mathbb{S})$, so we only need to prove the lemma when $\mathbb{r}=1$, i.e., we need to show that with probability tending to 1 as $\delta \to 0$, we have

$$\widetilde{D}^{\delta}_{h}\left(\partial^{\delta}_{\mathrm{L}}\mathbb{S},\partial^{\delta}_{\mathrm{R}}\mathbb{S};\mathbb{S}\right) = \delta^{-\xi Q+o_\delta(1)}. \tag{3.76}$$

This follows from the LFPP distance exponent computation in [28]. To be more precise, [28, Theorem 1.5] shows that for continuum LFPP defined using the circle average process of the GFF, as in (1.2), the δ-LFPP distance between the left and right boundaries of $\mathbb{S}$ is of order $\delta^{1-\xi Q+o_\delta(1)}$ with probability tending to 1 as $\delta \to 0$. Combining this with [28, Lemma 3.7] shows that the same is true for continuum LFPP defined using the white-noise approximation $\{\widehat{h}_\delta\}_{\delta>0}$, as defined in [28, Equation (3.1)], in place of the circle average process. The same argument as in the proof of [28, Proposition 3.16] then shows that (3.76) holds if we replace the circle average by the white-noise approximation in the definition of $\widetilde{D}^{\delta}_{h}$ (here we note that the definition of discretized LFPP in [28, Equation (3.32)] has an extra factor of δ as compared to (3.72), which is why we get $\delta^{-\xi Q+o_\delta(1)}$ instead of $\delta^{1-\xi Q+o_\delta(1)}$). The desired formula (3.76) now follows by combining this with the uniform comparison of h_δ and $\widehat{h}_\delta$ from [28, Lemma 3.7]. □

For the proof of Theorem 4.5 (and at several later places in this section) we will use the following terminology.

Definition 4.40 (Distance around an annulus)**.** For a set $A \subset \mathbb{C}$ with the topology of a an annulus, we define the D_h*-distance around* A to be the infimum of the D_h-lengths of the paths in A which disconnect the inner and outer boundaries of A.

Proof of Theorem 4.5. Step 1: estimates for D_h*.* For $z \in \varepsilon\mathbb{Z}^2$, we write S^{ε}_{z} for the square of side length ε centered at z and $B_\varepsilon(S^{\varepsilon}_{z})$ for the ε-neighborhood of this square. Fix $\zeta \in (0,1)$. By Proposition 4.34 and a union bound over all $z \in (\mathbb{r}\mathbb{S}) \cap (\delta\mathbb{r}\mathbb{Z}^2)$, it holds with superpolynomially high probability as $\delta \to 0$ that (in the terminology of Definition 4.40)

$$\left(D_h\text{-distance around } B_{\delta\mathbb{r}}(S_z^{\delta\mathbb{r}}) \setminus S_z^{\delta\mathbb{r}}\right) \leq \delta^{-\zeta} \mathfrak{c}_{\delta\mathbb{r}} e^{\xi h_{\delta\mathbb{r}}(z)}, \quad \forall z \in (\mathbb{r}\mathbb{S}) \cap (\delta\mathbb{r}\mathbb{Z}^2). \tag{3.77}$$

Similarly, it holds with superpolynomially high probability as $\delta \to 0$ that

$$D_h\left(S_z^{\delta\mathbb{r}}, \partial B_{\delta\mathbb{r}}(S_z^{\delta\mathbb{r}})\right) \geq \delta^{\zeta} \mathfrak{c}_{\delta\mathbb{r}} e^{\xi h_{\delta\mathbb{r}}(z)}, \quad \forall z \in (\mathbb{r}\mathbb{S}) \cap (\delta\mathbb{r}\mathbb{Z}^2). \tag{3.78}$$

Henceforth assume that (3.77) and (3.78) both hold.

Step 2: lower bound for $\mathfrak{c}_{\delta\mathbb{r}}/\mathfrak{c}_{\mathbb{r}}$. Let $\pi : [0, |\pi|]_{\mathbb{Z}} \to (\mathbb{r}\mathbb{S}) \cap (\delta\mathbb{r}\mathbb{Z}^2)$ be a path in $(\mathbb{r}\mathbb{S}) \cap (\delta\mathbb{r}\mathbb{Z}^2)$ (with the graph structure defined by (3.71)) from $\partial_{\mathrm{L}}^{\delta\mathbb{r}}(\mathbb{r}\mathbb{S})$ to $\partial_{\mathrm{R}}^{\delta\mathbb{r}}(\mathbb{r}\mathbb{S})$ for which the sum in (3.72) equals $\widetilde{D}_h^{\delta\mathbb{r}}\left(\partial_{\mathrm{L}}^{\delta\mathbb{r}}(\mathbb{r}\mathbb{S}), \partial_{\mathrm{R}}^{\delta\mathbb{r}}(\mathbb{r}\mathbb{S}); \mathbb{r}\mathbb{S}\right)$. For each $j \in [0, |\pi|]_{\mathbb{Z}}$, let P_j be a path in $B_{\delta\mathbb{r}}(S_{\pi(j)}^{\delta\mathbb{r}}) \setminus S_{\pi(j)}^{\delta\mathbb{r}}$ which disconnects the inner and outer boundaries of $B_{\delta\mathbb{r}}(S_{\pi(j)}^{\delta\mathbb{r}}) \setminus S_{\pi(j)}^{\delta\mathbb{r}}$ and whose D_h-length is at most $2\delta^{-\zeta}\mathfrak{c}_{\delta\mathbb{r}} e^{\xi h_{\delta\mathbb{r}}(z)}$. Such a path exists by (3.77).

We have $P_j \cap P_{j-1} \neq \emptyset$ for each $j \in [0, |\pi|]_{\mathbb{Z}}$, so the union of the P_j's is connected and contains a path between the left and right boundaries of $\mathbb{r}\mathbb{S}$. Therefore, the triangle inequality implies that

$$\begin{aligned} D_h\left(\mathbb{r}\partial_{\mathrm{L}}\mathbb{S}, \mathbb{r}\partial_{\mathrm{R}}\mathbb{S}\right) &\leq \sum_{j=0}^{|\pi|} (D_h\text{-length of } P_j) \leq 2\delta^{-\zeta}\mathfrak{c}_{\delta\mathbb{r}} \sum_{j=0}^{|\pi|} e^{\xi h_{\delta\mathbb{r}}(0)} \\ &= 2\delta^{-\zeta}\mathfrak{c}_{\delta\mathbb{r}} \widetilde{D}_h^{\delta\mathbb{r}}\left(\partial_{\mathrm{L}}^{\delta\mathbb{r}}(\mathbb{r}\mathbb{S}), \partial_{\mathrm{R}}^{\delta\mathbb{r}}(\mathbb{r}\mathbb{S}); \mathbb{r}\mathbb{S}\right). \end{aligned} \tag{3.79}$$

By Axiom V, the left side of (3.79) is at least $\delta^{\zeta}\mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)}$ with probability tending to 1 as $\delta \to 0$, uniformly in $\mathbb{r}$. By Lemma 4.39, the right side of (3.79) is at most $\delta^{-\xi Q - 2\zeta}\mathfrak{c}_{\delta\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)}$ with probability tending to 1 as $\delta \to 0$, uniformly in $\mathbb{r}$. Combining these relations and sending $\zeta \to 0$ shows that $\mathfrak{c}_{\mathbb{r}} \leq \delta^{-\xi Q - o_\delta(1)}\mathfrak{c}_{\delta\mathbb{r}}$, as desired.

Step 3: upper bound for $\mathfrak{c}_{\delta\mathbb{r}}/\mathfrak{c}_{\mathbb{r}}$. Let $P : [0, |P|] \to \mathbb{S}$ be a path between the left and right boundaries of $\mathbb{r}\mathbb{S}$ with D_h-length at most $2D_h\left(\mathbb{r}\partial_{\mathrm{L}}\mathbb{S}, \mathbb{r}\partial_{\mathrm{R}}\mathbb{S}; \mathbb{r}\mathbb{S}\right)$. We will use P to construct a path in $(\mathbb{r}\mathbb{S}) \cap (\delta\mathbb{r}\mathbb{Z}^2)$ from $\partial_{\mathrm{L}}^{\delta\mathbb{r}}(\mathbb{r}\mathbb{S})$ to $\partial_{\mathrm{R}}^{\delta\mathbb{r}}(\mathbb{r}\mathbb{S})$ for which the sum in (3.72) can be bounded above.

To this end, let $\tau_0 = 0$ and let $z_0 \in (\mathbb{r}\mathbb{S}) \cap (\delta\mathbb{r}\mathbb{Z}^2)$ be chosen so that $P(0) \in S_{z_0}^{\delta\mathbb{r}}$. Inductively, suppose $j \in \mathbb{N}$, a time $\tau_{j-1} \in [0, |P|]$, and a point $z_{j-1} \in (\mathbb{r}\mathbb{S}) \cap (\delta\mathbb{r}\mathbb{Z}^2)$ have been defined in such a way that $P(\tau_{j-1}) \in S_{z_{j-1}}^{\delta\mathbb{r}}$. Let τ_j be the first time after τ_{j-1} at which P exits $B_{\delta\mathbb{r}}(S_{z_{j-1}}^{\delta\mathbb{r}})$, if such a time exists, and otherwise set $\tau_j = |P|$. Let $z_j \in (\mathbb{r}\mathbb{S}) \cap (\delta\mathbb{r}\mathbb{Z}^2)$ be chosen so that $P(\tau_j) \in S_{z_j}^{\delta\mathbb{r}}$. Let J be the smallest $j \in \mathbb{N}$ for which $\tau_j = |P|$, and note that $P(|P|) \in S_{z_j}^{\delta\mathbb{r}}$.

Successive squares $S_{z_{j-1}}^{\delta\mathbb{r}}$ and $S_{z_j}^{\delta\mathbb{r}}$ necessarily share a vertex. Hence z_{j-1} and z_j lie at $(\mathbb{r}\mathbb{S})\cap(\delta\mathbb{r}\mathbb{Z}^2)$-graph distance 1 from one another, so $\pi(j) := z_j$ for $j \in [0, J]_{\mathbb{Z}}$ is a path from $\partial_{\mathrm{L}}^{\delta\mathbb{r}}(\mathbb{r}\mathbb{S})$ to $\partial_{\mathrm{R}}^{\delta\mathbb{r}}(\mathbb{r}\mathbb{S})$ in $(\mathbb{r}\mathbb{S}) \cap (\delta\mathbb{r}\mathbb{Z}^2)$.

We will now bound $\sum_{j=0}^J e^{\xi h_{\delta\mathbb{r}}(\pi(j))}$. For each $j \in [1, J]_{\mathbb{Z}}$, the path P crosses between the inner and outer boundaries of $B_{\delta\mathbb{r}}(S_{z_{j-1}}^{\delta\mathbb{r}}) \setminus S_{z_{j-1}}^{\delta\mathbb{r}}$ between time τ_{j-1} and time τ_j. By (3.78), for each $j \in [1, J]_{\mathbb{Z}}$,

$$D_h\left(P(\tau_{j-1}), P(\tau_j)\right) \geq \delta^{\zeta} \mathfrak{c}_{\delta\mathbb{r}} e^{\xi h_{\delta\mathbb{r}}(\pi(j))}. \tag{3.80}$$

Using (3.80) and the definition of P, we therefore have

$$\begin{aligned}\sum_{j=0}^J e^{\xi h_{\delta\mathbb{r}}(\pi(j))} &\leq \delta^{-\zeta} \mathfrak{c}_{\delta\mathbb{r}}^{-1} \sum_{j=0}^J D_h\left(P(\tau_{j-1}), P(\tau_j)\right) \\ &\leq \delta^{-\zeta} \mathfrak{c}_{\delta\mathbb{r}}^{-1} D_h\left(\mathbb{r}\partial_{\mathrm{L}}\mathbb{S}, \mathbb{r}\partial_{\mathrm{R}}\mathbb{S}\right). \end{aligned} \tag{3.81}$$

By Axiom V, the right side of (3.81) is at most $\delta^{-2\zeta}\mathfrak{c}_{\delta\mathbb{r}}^{-1}\mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)}$ with probability tending to 1 as $\delta \to 0$, uniformly in $\mathbb{r}$. By Lemma 4.39, the left side of (3.79) is at least $\delta^{-\xi Q - \zeta} e^{\xi h_{\mathbb{r}}(0)}$ with probability tending to 1 as $\delta \to 0$, uniformly in $\mathbb{r}$. Combining these relations and sending $\zeta \to 0$ shows that $\mathfrak{c}_{\delta\mathbb{r}}^{-1}\mathfrak{c}_{\mathbb{r}} \geq \delta^{-\xi Q - o_\delta(1)}$. □

Theorem 4.5 has the following useful corollary.

Lemma 4.41. *Let h be a whole-plane GFF normalized so that $h_1(0) = 0$. Almost surely, for every compact set $K \subset \mathbb{C}$ we have $\lim_{r\to\infty} D_h(K, \partial B_r(0)) = \infty$. In particular, every closed, D_h-bounded subset of $\mathbb{C}$ is compact.*

Proof. By tightness across scales (Axiom V), there exists $a > 0$ such that for each $r > 0$, $\mathbb{P}\left[D_h(B_r(0), B_{2r}(0)) \geq a\mathfrak{c}_r e^{\xi h_r(0)}\right] \geq 1/2$. By the locality of D_h (Axiom II) and since the sigma-algebra $\sigma\left(\bigcap_{r>0} h|_{\mathbb{C}\setminus B_r(0)}\right)$ is trivial, a.s. there are infinitely many $k \in \mathbb{N}$ for which we have

$D_h(B_{2^k}(0), B_{2^{k+1}}(0)) \geq a\mathfrak{c}_{2^k} e^{\xi h_{2^k}(0)}$. By Theorem 4.5, $\mathfrak{c}_r = r^{\xi Q + o_r(1)}$. Since $t \mapsto h_{e^t}(0)$ is a standard linear Brownian motion [44, Section 3.1], we get that a.s. $\lim_{r\to\infty} \mathfrak{c}_r e^{\xi h_r(0)} = \infty$. Hence a.s. $\limsup_{k\to\infty} D_h(B_{2^k}(0), B_{2^{k+1}}(0)) = \infty$. Since D_h is a length metric, for any $r \geq 2^{k+1}$ and any compact set $K \subset B_{2^k}(0)$, we have $D_h(K, \partial B_r(0)) \geq D_h(B_{2^k}(0), B_{2^{k+1}}(0))$. We thus obtain the first assertion of the lemma. The first assertion (applied with K equal to a single point, say) implies that any D_h-bounded subset of $\mathbb{C}$ must be contained in a Euclidean-bounded subset of $\mathbb{C}$, which must be compact since D_h induces the Euclidean topology on $\mathbb{C}$. □

4.3.3 Moment bound for diameters

In this section we will prove the following more quantitative version of the moment bound from Theorem 4.8, which is required to be uniform across scales.

Proposition 4.42. *Let $U \subset \mathbb{C}$ be open and let $K \subset U$ be a compact connected set with more than one point. For each $p \in (-\infty, 4d_\gamma/\gamma^2)$, there exists $C_p > 0$ which depends on U and K but not on $\mathbbm{r}$ such that for each $\mathbbm{r} > 0$,*

$$\mathbb{E}\left[\left(\mathfrak{c}_{\mathbbm{r}}^{-1} e^{-\xi h_{\mathbbm{r}}(0)} \sup_{z,w\in \mathbbm{r}K} D_h(z,w;\mathbbm{r}U)\right)^p\right] \leq C_p. \tag{3.82}$$

We will deduce Proposition 4.42 from the following variant, which allows us to bound internal D_h-distances all the way up to the boundary of a square. Recall that $\mathbb{S} := (0,1)^2$.

Proposition 4.43. *For each $p \in (-\infty, 4d_\gamma/\gamma^2)$, there is a constant $C_p > 0$ such that for each $\mathbbm{r} > 0$,*

$$\mathbb{E}\left[\left(\mathfrak{c}_{\mathbbm{r}}^{-1} e^{-\xi h_{\mathbbm{r}}(0)} \sup_{z,w\in \mathbbm{r}\mathbb{S}} D_h\left(z,w;\mathbbm{r}\mathbb{S}\right)\right)^p\right] \leq C_p. \tag{3.83}$$

Proof of Proposition 4.42, assuming Proposition 4.43. For $p < 0$, the bound (3.82) follows from the lower bound of Proposition 4.34. Now assume $p \in (0, 4d_\gamma/\gamma^2)$. We can cover K by finitely many Euclidean squares $S_1, \ldots, S_n$ which are contained in U, chosen in a manner depending only on K and U. For $k = 1, \ldots, n$, let u_k be the bottom left corner of S_k and let ρ_k be its side length. Proposition 4.43 together with Axiom IV shows that there is a constant $\widetilde{C}_p > 0$ depending only on

p such that for each $k = 1, \ldots, n$,

$$\mathbb{E}\left[\left(\mathfrak{c}_{\mathbb{r}\rho_k}^{-1} e^{-\xi h_{\mathbb{r}\rho_k}(\mathbb{r}u_k)} \sup_{z,w\in\mathbb{r}S_k} D_h\left(z,w;\mathbb{r}S_k\right)\right)^p\right] \le \widetilde{C}_p. \tag{3.84}$$

We apply the Gaussian tail bound to bound each of the Gaussian random variables $h_{\mathbb{r}\rho_k}(\mathbb{r}u_k) - h_{\mathbb{r}}(0)$ (which have constant order variance) and Theorem 4.5 to compare $\mathfrak{c}_{\mathbb{r}\rho_k}$ to $\mathfrak{c}_{\mathbb{r}}$ up to a constant-order multiplicative error. This allows us to deduce (3.82) from (3.84). □

To prove Proposition 4.43, we first use the upper bound in Proposition 4.34 and a union bound to build paths between the two shorter sides of each $2^{-n}\mathbb{r} \times 2^{-n-1}\mathbb{r}$ or $2^{-n-1}\mathbb{r} \times 2^{-n}\mathbb{r}$ rectangle with corners in $2^{-n-1}\mathbb{r}\mathbb{Z}^2$ which is contained in $\mathbb{S}$. We then string together such paths at all scales (in the manner illustrated in Figure 4.3) to get a bound for the internal D_h-diameter of $\mathbb{r}\mathbb{S}$. The following lemma is needed to control the circle average terms which appear when we apply Proposition 4.34.

Lemma 4.44. *Fix $R > 0$ and $q > 2$. For $C > 1$ and $\mathbb{r} > 0$, it holds with probability $1 - C^{-q-\sqrt{q^2-4}+o_C(1)}$ as $C \to \infty$, at a rate which is uniform in $\mathbb{r}$, that*

$$\sup\left\{|h_{2^{-n}\mathbb{r}}(w) - h_{\mathbb{r}}(0)| : w \in B_{R\mathbb{r}}(0) \cap \left(2^{-n-1}\mathbb{r}\mathbb{Z}^2\right)\right\} \le \log(C2^{qn}), \quad \forall n \in \mathbb{N}_0. \tag{3.85}$$

When we apply Lemma 4.44, we will take q to be a little bit less than $Q = 2/\gamma + \gamma/2$. The fact that $Q + \sqrt{Q^2-4} = 4/\gamma$ is the reason why γ (instead of just ξ) appears in our moment bounds.

Proof of Lemma 4.44. To lighten notation, define the event

$$E_{\mathbb{r}}^n := \left\{\sup\left\{|h_{2^{-n}\mathbb{r}}(w) - h_{\mathbb{r}}(0)| : w \in B_{R\mathbb{r}}(0) \cap \left(2^{-n-1}\mathbb{r}\mathbb{Z}^2\right)\right\} \le \log(C2^{qn})\right\}. \tag{3.86}$$

We want a lower bound for the probability that $E_{\mathbb{r}}^n$ occurs for every $n \in \mathbb{N}_0$ simultaneously.

Fix $\zeta > 0$ (which we will eventually send to 0) and a partition $\zeta = \alpha_0 < \cdots < \alpha_N = 1/\zeta$ of $[\zeta, 1/\zeta]$ with $\max_{k=1,\ldots,N}(\alpha_k - \alpha_{k-1}) \le \zeta$. We will separately bound the probability of $E_{\mathbb{r}}^n$ for $2^n \in [C^{\alpha_{k-1}}, C^{\alpha_k}]$ for $k = 1, \ldots, N$, for $2^n \ge C^{1/\zeta}$, and for $2^n \le C^{\zeta}$.

By Lemma 4.37 applied with $\varepsilon = 2^{-n}$, $\nu = 0$, and $q + 1/\alpha_k$ in place of q, we find that for each

$k = 1, \dots, N$ and each $n \in \mathbb{N}_0$ with $2^n \in [C^{\alpha_{k-1}}, C^{\alpha_k}]$,

$$\mathbb{P}\left[(E_{\mathbb{r}}^n)^c\right] \le \mathbb{P}\left[\sup\left\{|h_{2^{-n}\mathbb{r}}(w) - h_{\mathbb{r}}(0)| : w \in B_{R\mathbb{r}}(0) \cap \left(2^{-n-1}\mathbb{r}\mathbb{Z}^2\right)\right\} > \left(q + \frac{1}{\alpha_k}\right)\log(2^n)\right]$$
$$\le 2^{-n\left(\frac{(q+1/\alpha_k)^2}{2}-2\right)} \le C^{-\alpha_{k-1}\left(\frac{(q+1/\alpha_k)^2}{2}-2\right)} \le C^{2\alpha_k - \frac{(q\alpha_k+1)^2}{2\alpha_k} + o_\zeta(1)} \tag{3.87}$$

with the rate of the $o_\zeta(1)$ depending only on q. Note that in the last inequality, we have done some trivial algebraic manipulations then used that $\alpha_k - \alpha_{k-1} \le \zeta$ (which is what produces the $o_\zeta(1)$). By a union bound over logarithmically many (in C) values of $n \in \mathbb{N}_0$ with $2^n \in [C^{\alpha_{k-1}}, C^{\alpha_k}]$, we get

$$\mathbb{P}\left[E_{\mathbb{r}}^n,\ \forall n \in \mathbb{N}_0 \text{ with } C^{\alpha_{k-1}} \le 2^n \le C^{\alpha_k}\right] \ge 1 - C^{2\alpha_k - \frac{(q\alpha_k+1)^2}{2\alpha_k} + o_\zeta(1) + o_C(1)}. \tag{3.88}$$

For $n \in \mathbb{N}_0$ with $2^n \ge C^{1/\zeta}$, Lemma 4.37 applied with $\varepsilon = 2^{-n}$, $\nu = 0$, and $q + \zeta$ in place of q gives

$$\mathbb{P}\left[(E_{\mathbb{r}}^n)^c\right] \le 2^{-n\left((q+\zeta)^2/2-2\right)}.$$

Summing this estimate over all such n shows that

$$\mathbb{P}\left[E_{\mathbb{r}}^n,\ \forall n \in \mathbb{N} \text{ with } 2^n \ge C^{1/\zeta}\right] \ge 1 - C^{-\frac{(q+\zeta)^2-4}{2\zeta} + o_C(1)}. \tag{3.89}$$

Finally, if $n \in \mathbb{N}_0$ and $2^n \le C^\zeta$, then the Gaussian tail bound and a union bound, applied as in the proof of Lemma 4.37, shows that $\mathbb{P}[(E_{\mathbb{r}}^n)^c] \le C^{2\zeta - (q\zeta+1)^2/(2\zeta) + o_C(1)}$ (in fact, if 2^n is of constant order, this probability will decay superpolynomially in C due to the Gaussian tail bound). By a union bound over a logarithmic number (in C) of such values of n we get

$$\mathbb{P}\left[E_{\mathbb{r}}^n,\ \forall n \in \mathbb{N} \text{ with } 2^n \le C^\zeta\right] \ge 1 - C^{2\zeta - \frac{(q\zeta+1)^2}{2\zeta} + o_C(1)}. \tag{3.90}$$

The quantity $2\alpha - (q\alpha+1)^2/(2\alpha)$ is maximized over all $\alpha > 0$ when $\alpha = (q^2-4)^{-1/2}$, in which case it equals $-(q + \sqrt{q^2-4})$. Consequently, by combining the estimates (3.88), (3.89), and (3.90), we get that if ζ is chosen sufficiently small relative to q, then

$$\mathbb{P}\left[E_{\mathbb{r}}^n,\ \forall n \in \mathbb{N}_0\right] \ge 1 - C^{-q-\sqrt{q^2-4} + o_\zeta(1) + o_C(1)}. \tag{3.91}$$

Sending $\zeta \to 0$ now concludes the proof. $\square$

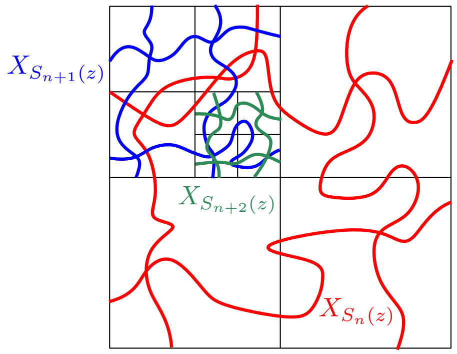

Figure 4.3 – Three of the sets $X_{S_n(z)}$ for dyadic squares containing z used in the proof of Proposition 4.43. As $n \to \infty$, the D_h-diameter of $S_n(z)$ shrinks to zero (by the continuity of $(z,w) \mapsto D_h(z,w)$), so the distance from z to $X_{S_N(z)}$ is bounded above by the sum over all $n \geq N$ of the D_h-lengths of the four paths which comprise $X_{S_n(z)}$.

Proof of Proposition 4.43. For $p < 0$, the bound (3.83) follows from the lower bound of Proposition 4.34. We will bound the positive moments up to order $4d_\gamma/\gamma^2$.

Step 1: constructing short paths across rectangles. Fix $q \in (2, Q)$ which we will eventually send to Q. By Lemma 4.44 it holds with probability $1 - C^{-q-\sqrt{q^2-4}+o_C(1)}$ that

$$\sup\left\{|h_{2^{-n}\mathbb{r}}(w) - h_{\mathbb{r}}(0)| : w \in \mathbb{r}\mathbb{S} \cap \left(2^{-n-1}\mathbb{r}\mathbb{Z}^2\right)\right\} \leq \log(C2^{qn}), \quad \forall n \in \mathbb{N}_0. \tag{3.92}$$

Now fix $\zeta \in (0, Q-q)$, which we will eventually send to zero. For $n \in \mathbb{N}_0$, let $\mathcal{R}_{\mathbb{r}}^n$ be the set of open $2^{-n}\mathbb{r} \times 2^{-n-1}\mathbb{r}$ or $2^{-n-1}\mathbb{r} \times 2^{-n}\mathbb{r}$ rectangles $R \subset \mathbb{r}\mathbb{S}$ with corners in $2^{-n-1}\mathbb{r}\mathbb{Z}^2$. For $R \in \mathcal{R}_{\mathbb{r}}^n$ let w_R be the bottom-left corner of R.

Let

$$N_C := \lfloor \log_2 C^\zeta \rfloor. \tag{3.93}$$

By the upper bound of Proposition 4.34 (applied with $2^{-n}\mathbb{r}$ in place of $\mathbb{r}$ and with $A = 2^{\zeta\xi n}$), Axiom IV, and a union bound over all $R \in \mathcal{R}_{\mathbb{r}}^n$ and all $n \geq N_C$, we get that except on an event of probability decaying faster than any negative power of C (the rate of decay depends on ζ), the following is true. For each $n \geq N_C$ and each $R \in \mathcal{R}_{\mathbb{r}}^n$, the distance between the two shorter sides of R w.r.t. the internal metric $D_h(\cdot,\cdot;R)$ is at most $2^{\zeta\xi n}\mathfrak{c}_{2^{-n}\mathbb{r}}e^{\xi h_{2^{-n}\mathbb{r}}(w_R)}$.

Combining this with (3.67) shows that with probability $1 - C^{-q-\sqrt{q^2-4}+o_C(1)}$, it holds for each $n \geq N_C$ and each $R \in \mathcal{R}_{\mathbb{r}}^n$ that there is a path P_R in R between the two shorter sides of R with D_h-length at most $C^{\xi} 2^{(q+\zeta)\xi n} \mathfrak{c}_{2^{-n}\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)}$. By applying Theorem 4.5 to bound $\mathfrak{c}_{2^{-n}\mathbb{r}}$, we get that in fact

$$(D_h\text{-length of } P_R) \leq C^{\xi} 2^{-(Q-q-\zeta)\xi n + o_n(n)} \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)}. \tag{3.94}$$

Henceforth assume that such paths P_R exist. We will establish an upper bound for the D_h-diameter of $\mathbb{r}\mathbb{S}$.

Step 2: stringing together paths in rectangles. For each square $S \subset \mathbb{r}\mathbb{S}$ with side length $2^{-n}\mathbb{r}$ and corners in $2^{-n}\mathbb{r}\mathbb{S}$, there are exactly four rectangles in $\mathcal{R}_{\mathbb{r}}^n$ which are contained in S. If $n \geq N_C$, let X_S be the #-shaped region which is the union of the paths P_R for these four rectangles, as illustrated in Figure 4.3. If S' is one of the four dyadic children of S, then $X_S \cap X_{S'} \neq \emptyset$. Since the four paths which comprise X_S have D_h-length at most $C^{\xi} 2^{-(Q-q-\zeta)\xi n + o_n(n)} e^{\xi h_{\mathbb{r}}(0)} \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)}$, this means that each point of X_S can be joined to $X_{S'}$ by a path in S of D_h-length at most $C^{\xi} 2^{-(Q-q-\zeta)\xi n + o_n(n)} \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)}$.

Since the metric D_h is a continuous function on $\mathbb{C} \times \mathbb{C}$, if $z \in \mathbb{r}\mathbb{S}$ and we let $S_n(z)$ for $n \in \mathbb{N}_0$ be the square of side length $2^{-n}\mathbb{r}$ with corners in $2^{-n}\mathbb{r}\mathbb{Z}^2$ which contains z, so that $S_0(z) = \mathbb{S}$, then the D_h-diameter of $S_n(z)$ tends to zero as $n \to \infty$. Consequently,

$$\sup_{w \in S_{N_C}(z)} D_h(z, w; \mathbb{r}\mathbb{S}) \leq C^{\xi} \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)} \sum_{n=N_C}^{\infty} 2^{-(Q-q-\zeta)\xi n + o_n(n)} \leq O_C(C^{\xi}) \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)}.$$

Since this holds for every $z \in \mathbb{r}\mathbb{S}$, we get that with probability at least $1 - C^{-q-\sqrt{q^2-4} o_C(1)}$, for each $n \geq N_C$, each $2^{-n}\mathbb{r} \times 2^{-n}\mathbb{r}$ square $S \subset \mathbb{r}\mathbb{S}$ with corners in $2^{-n}\mathbb{r}\mathbb{Z}^2$ has $D_h(\cdot, \cdot; \mathbb{r}\mathbb{S})$-diameter at most $O_C(C^{\xi}) \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)}$.

Step 3: conclusion. Since $2^{N_C} \leq C^{\zeta}$, we can use the triangle inequality to get that if the event at the end of the preceding step occurs, then the $D_h(\cdot, \cdot; \mathbb{r}\mathbb{S})$-diameter of $\mathbb{r}\mathbb{S}$ is at most $O_C(C^{\xi+\zeta}) \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)}$. Setting $\widetilde{C} := C^{\xi+\zeta}$, then sending $\zeta \to 0$, shows that

$$\mathbb{P}\left[\mathfrak{c}_{\mathbb{r}}^{-1} e^{-\xi h_{\mathbb{r}}(0)} \sup_{z, w \in \mathbb{r}\mathbb{S}} D_h(z, w; \mathbb{r}\mathbb{S}) > \widetilde{C} \right] \leq \widetilde{C}^{-\xi^{-1}(q + \sqrt{q^2-4}) + o_{\widetilde{C}}(1)}.$$

By sending $q \to Q$ and noting that $Q + \sqrt{Q^2 - 4} = 4/\gamma$, we get

$$\mathbb{P}\left[\mathfrak{c}_{\mathbb{r}}^{-1} e^{-\xi h_{\mathbb{r}}(0)} \sup_{z,w \in \mathbb{r}\mathbb{S}} D_h(z, w; \mathbb{r}\mathbb{S}) > \widetilde{C}\right] \leq \widetilde{C}^{-\frac{4}{\gamma\xi} + o_{\widetilde{C}}(1)} = \widetilde{C}^{-\frac{4d_\gamma}{\gamma^2} + o_{\widetilde{C}}(1)}.$$

For $p \in (0, 4d_\gamma/\gamma^2)$, we can multiply this last estimate by $\widetilde{C}^{p-1}$ and integrate to get the desired pth moment bound (3.83). □

4.3.4 Pointwise distance bounds

In this subsection we will prove the following more quantitative versions of Theorems 4.10 and 4.11, which are required to be uniform across scales. Recall that h is a whole-plane GFF normalized so that $h_1(0) = 0$.

Proposition 4.45 (Distance from a point to a circle)**.** *Let $\alpha \in \mathbb{R}$ and let $h^\alpha := h - \alpha \log|\cdot|$. If $\alpha \in (-\infty, Q)$, then for each $p \in (-\infty, \frac{2d_\gamma}{\gamma}(Q - \alpha))$, there exists $C_p > 0$ such that for each $\mathbb{r} > 0$,*

$$\mathbb{E}\left[\left(\mathfrak{c}_{\mathbb{r}}^{-1} \mathbb{r}^{\alpha\xi} e^{-\xi h_{\mathbb{r}}(0)} D_{h^\alpha}\left(0, \partial B_{\mathbb{r}}(0)\right)\right)^p\right] \leq C_p. \tag{3.95}$$

If $\alpha > Q$, then a.s. $D_{h^\alpha}(0, z) = \infty$ for every $z \in \mathbb{C} \setminus \{0\}$.

Proposition 4.46 (Distance between two points)**.** *Let $\alpha, \beta \in \mathbb{R}$, let $z, w \in \mathbb{C}$ be distinct, and let $h^{\alpha,\beta} := h - \alpha \log|\cdot - z| - \beta \log|\cdot - w|$. Set $\mathbb{r} := |z - w|/2$. If $\alpha, \beta \in (-\infty, Q)$, then for each $p \in \left(-\infty, \frac{2d_\gamma}{\gamma}(Q - \max\{\alpha, \beta\})\right)$, there exists $C_p > 0$ such that for each choice of z, w as above,*

$$\mathbb{E}\left[\left(\mathfrak{c}_{\mathbb{r}}^{-1} \mathbb{r}^{\alpha\xi} e^{-\xi h_{\mathbb{r}}(z)} D_{h^\alpha}\left(z, w; B_{8\mathbb{r}}(z)\right)\right)^p\right] \leq C_p. \tag{3.96}$$

If either $\alpha > Q$ or $\beta > Q$, then a.s. $D_{h^{\alpha,\beta}}(z, w) = \infty$.

Propositions 4.45 and 4.46 are immediate consequences of the following sharper distance estimates and a calculation for the standard linear Brownian motion $t \mapsto h_{\mathbb{r}e^{-t}}(0) - h_{\mathbb{r}}(0)$.

Proposition 4.47. *Assume that we are in the setting of Proposition 4.45. If $\alpha \in (-\infty, Q)$, then there is a deterministic function $\psi : [0, \infty) \to [0, \infty)$ which is bounded in every neighborhood of 0* *and satisfies* $\lim_{t\to\infty}\psi(t)/t = 0$*, depending only on* α *and the choice of metric* D,[3] *such that the following is true. For each* $\mathbbm{r} > 0$*, it holds with superpolynomially high probability as* $C\to\infty$*, at a rate which is uniform in the choice of* $\mathbbm{r}$*, that*

$$C^{-1}\frac{\mathfrak{c}_{\mathbbm{r}}}{\mathbbm{r}^{\alpha\xi}}\int_0^\infty e^{\xi h_{\mathbbm{r}e^{-t}}(0)-\xi(Q-\alpha)t-\psi(t)}\,dt \le D_{h^\alpha}\left(0,\partial B_{\mathbbm{r}}(0)\right) \le C\frac{\mathfrak{c}_{\mathbbm{r}}}{\mathbbm{r}^{\alpha\xi}}\int_0^\infty e^{\xi h_{\mathbbm{r}e^{-t}}(0)-\xi(Q-\alpha)t+\psi(t)}\,dt \tag{3.97}$$

and the D_{h^α}*-distance around the annulus* $B_{\mathbbm{r}}(0)\setminus B_{\mathbbm{r}/e}(0)$ *(Definition 4.40) is at most the right side of* (3.97)*. If* $\alpha > Q$*, then a.s.* $D_{h^\alpha}(0,z)=\infty$ *for every* $z\in\mathbb{C}\setminus\{0\}$*.*

Proposition 4.48. *Assume that we are in the setting of Proposition 4.46. If* $\alpha,\beta\in(-\infty,Q)$*, then there is a deterministic function* $\psi:[0,\infty)\to[0,\infty)$ *which is bounded in every neighborhood of 0 and satisfies* $\lim_{t\to\infty}\psi(t)/t=0$*, depending only on* α *and the choice of metric* D*, such that the following is true. With superpolynomially high probability as* $C\to\infty$*, at a rate which is uniform in the choice of* z *and* w*,*

$$D_{h^{\alpha,\beta}}(z,w) \ge C^{-1}\frac{\mathfrak{c}_{\mathbbm{r}}}{\mathbbm{r}^{\alpha\xi}}\int_0^\infty\left(e^{\xi h_{\mathbbm{r}e^{-t}}(z)-\xi(Q-\alpha)t-\psi(t)}+e^{\xi h_{\mathbbm{r}e^{-t}}(w)-\xi(Q-\beta)t-\psi(t)}\right)dt \tag{3.98}$$

and

$$D_{h^{\alpha,\beta}}(z,w;B_{8\mathbbm{r}}(z)) \le C\frac{\mathfrak{c}_{\mathbbm{r}}}{\mathbbm{r}^{\alpha\xi}}\int_0^\infty\left(e^{\xi h_{\mathbbm{r}e^{-t}}(z)-\xi(Q-\alpha)t+\psi(t)}+e^{\xi h_{\mathbbm{r}e^{-t}}(w)-\xi(Q-\beta)t+\psi(t)}\right)dt. \tag{3.99}$$

If either $\alpha>Q$ *or* $\beta>Q$*, then a.s.* $D_{h^{\alpha,\beta}}(z,w)=\infty$*.*

Remark 4.49. It will be shown in [59] that every weak LQG metric is a strong LQG metric, so in particular it satisfies Axiom V with $\mathfrak{c}_r = r^{\xi Q}$. Once this is established, our proof shows that Propositions 4.47 and 4.48 hold with $\psi(t)=0$.

Proof of Proposition 4.45, assuming Proposition 4.47. For $t\ge 0$, let $B_t := h_{\mathbbm{r}e^{-t}}(0)-h_{\mathbbm{r}}(0)$. Then B is a standard linear Brownian motion [44, Section 3.1]. By Proposition 4.47, for each $\zeta\in(0,1)$, it

[3] At this point we do not know that the weak LQG metric $D: h\mapsto D_h$ is unique. When we say that something is allowed to depend on the choice of D, we mean that it is allowed to depend on which particular weak LQG metric we are looking at.

holds with superpolynomially high probability as $C \to \infty$, uniformly over the choice of $\mathbbm{r}$, that

$$C^{-\zeta} \int_0^\infty e^{\xi B_t - (Q-\alpha)\xi t - \zeta t}\, dt \le \mathfrak{c}_{\mathbbm{r}}^{-1} \mathbbm{r}^{\alpha\xi} e^{-\xi h_{\mathbbm{r}}(0)} D_{h^\alpha}\left(0, \partial B_{\mathbbm{r}}(0)\right) \le C^{\zeta} \int_0^\infty e^{\xi B_t - (Q-\alpha)\xi t + \zeta t}\, dt. \quad (3.100)$$

To prove the proposition, we will use an exact formula for the laws of the integrals appearing in (3.100). To write down such a formula, let $\widetilde{B}_s := \xi B_{s/\xi^2}$. Then $\widetilde{B}$ is a standard linear Brownian motion and $B_t = \xi^{-1} \widetilde{B}_{\xi^2 t}$. Making the change of variables $t = s/\xi^2$ gives

$$\int_0^\infty e^{\xi B_t - (Q-\alpha)\xi t + \zeta t}\, dt = \frac{1}{\xi^2} \int_0^\infty e^{\widetilde{B}_s - (Q-\alpha)s/\xi + \zeta s/\xi^2}\, ds. \quad (3.101)$$

It is shown in [40] (see also [111, Example 3.3] with $c = (Q-\alpha)/\xi - \zeta/\xi^2$) that

$$\mathbb{P}\left[\int_0^\infty e^{\widetilde{B}_s - (Q-\alpha)s/\xi + \zeta s/\xi^2}\, ds \in dx\right] = b x^{-2(Q-\alpha)/\xi + 2\zeta/\xi^2 - 1} e^{-2/x}, \quad \forall x \ge 0, \quad (3.102)$$

where b is a normalizing constant depending only on Q, α, ξ. Combining the upper bound in (3.100) with (3.101) and the upper tail asymptotics of the density (3.102), then sending $\zeta \to 0$, shows that

$$\mathbb{P}\left[\mathfrak{c}_{\mathbbm{r}}^{-1} \mathbbm{r}^{\alpha\xi} e^{-\xi h_{\mathbbm{r}}(0)} D_{h^\alpha}\left(0, \partial B_{\mathbbm{r}}(0)\right) > C\right] \le C^{-2(Q-\alpha)/\xi - o_C(1)}, \quad (3.103)$$

uniformly in $\mathbbm{r}$. Recall that $\xi = \gamma/d_\gamma$. Multiplying both sides of (3.103) by pC^{p-1} and integrating gives the desired bound for positive moments from (3.95). We similarly obtain the desired bound for negative moments using the lower bound in (3.100) and the exponential lower tail of the density (3.102). □

Proof of Proposition 4.46, assuming Proposition 4.48. The bound for positive moments in (3.96) is obtained in essentially the same way as the analogous bound in Proposition 4.45. We apply the upper bound in Proposition 4.48 and use the exact formula (3.102) to bound the integral of each of the two summands appearing on the right side of (3.99), then multiply the resulting tail estimate by pC^{p-1} and integrate. We use that $h_{\mathbbm{r}}(z) - h_{\mathbbm{r}}(w)$ is Gaussian with constant-order variance to get an estimate which depends only on $h_{\mathbbm{r}}(z)$, not $h_{\mathbbm{r}}(w)$. The bound for negative moments in (3.96) can similarly be extracted from the lower bound in Proposition 4.48, or can be deduced from

Proposition 4.45 and the fact that a path from z to w must cross $\partial B_{\mathbb{r}}(z)$. □

It remains only to prove Propositions 4.47 and 4.48. We will prove Proposition 4.47 by applying Proposition 4.34 to bound the distances across and around concentric annuli surrounding 0 with dyadic radii, then summing over all of these annuli (see Figure 4.4 for an illustration). We will then deduce Proposition 4.48 from Proposition 4.47 by considering two overlapping Euclidean disks centered at z and w, respectively. For this purpose the statement concerning the D_h-distance around $B_{\mathbb{r}}(0) \setminus B_{\mathbb{r}/e}(0)$ is essential to link up paths in these two disks.

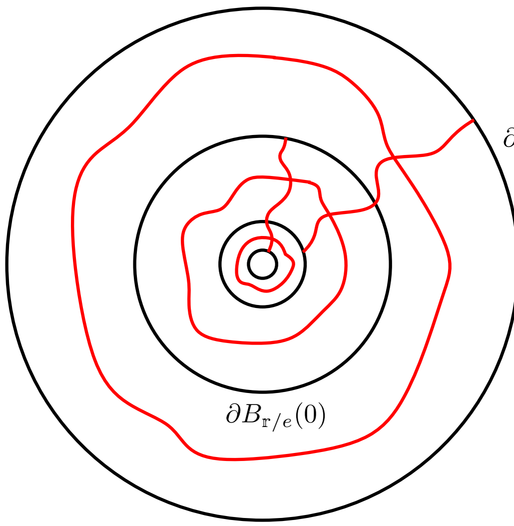

Figure 4.4 – To prove Proposition 4.47, we use Proposition 4.34 to show that with high probability, the following bounds hold simultaneously for each $k \in \mathbb{N}_0$: a lower bound for the D_h-distance across the annulus $B_{\mathbb{r}e^{-k}}(0) \setminus B_{\mathbb{r}e^{-k-1}}(0)$; an upper bound for the D_h-distance around this annulus; and a lower bound for the D_h-distance across the larger annulus $B_{\mathbb{r}e^{-k}}(0) \setminus B_{\mathbb{r}e^{-k-2}}(0)$. Summing the lower bounds for the distances across these annuli leads to the lower bound in (3.97). The paths involved in the upper bounds are shown in red in the figure. Concatenating all of these paths gives a path from 0 to $\partial B_{\mathbb{r}}(0)$, which leads to the upper bound in (3.97).

Proof of Proposition 4.47. See Figure 4.2 for an illustration. The proof is divided into four steps.

1. We apply Proposition 4.34 in the annuli $\mathbb{A}_{\mathbb{r}e^{-k-1},\mathbb{r}e^{-k}}$ for $k \in \mathbb{N}_0$ to prove upper and lower bounds for $D_h(0, \partial B_{\mathbb{r}}(0))$ in terms of sums over such annuli.

2. Using Brownian motion estimates, we convert from sums over annuli to integrals of quantities of the form $e^{\xi h_{\mathbb{r}e^{-t}}(z) - \xi(Q-\alpha)t + o_t(t)}$.

3. We show that the contribution of the small error terms in our estimates coming from sums/integrals at superpolynomially small scales is negligible.

4. We put the above pieces together to conclude the proof.

Step 1: applying Proposition 4.34 at exponential scales. We will apply Proposition 4.34 and take a union bound over exponential scales. In this step we allow any value of $\alpha \in \mathbb{R}$.

Fix a small parameter $\zeta \in (0,1)$, which we will eventually send to zero. By Proposition 4.34 and Axiom III (to deal with the addition of $-\alpha \log|\cdot|$) and a union bound over all $k \in [0, C^{1/\zeta}]_{\mathbb{Z}}$, we find that with superpolynomially high probability as $C \to \infty$, the following is true for each $k \in [0, C^{1/\zeta}]_{\mathbb{Z}}$.

1. The D_{h^α}-distance from $\partial B_{\mathbb{r}e^{-k-1}}(0)$ to $\partial B_{\mathbb{r}e^{-k}}(0)$ is at least $C^{-1}\mathfrak{c}_{\mathbb{r}e^{-k}}\mathbb{r}^{-\xi\alpha}\exp\left(\xi h_{\mathbb{r}e^{-k}}(0) + \xi\alpha k\right)$.

2. There is a path from $\partial B_{\mathbb{r}e^{-k-2}}(0)$ to $\partial B_{\mathbb{r}e^{-k}}(0)$ which has D_{h^α}-length at most $C\mathfrak{c}_{\mathbb{r}e^{-k}}\mathbb{r}^{-\xi\alpha}\exp\left(\xi h_{\mathbb{r}e^{-k}}(0) + \xi\alpha k\right)$. Moreover, there is also a path in $B_{\mathbb{r}e^{-k}}(0) \setminus \overline{B_{\mathbb{r}e^{-k-1}}(0)}$ which disconnects $\partial B_{\mathbb{r}e^{-k-1}}(0)$ from $\partial B_{\mathbb{r}e^{-k}}(0)$ and which has D_{h^α}-length at most $C\mathfrak{c}_{\mathbb{r}e^{-k}}\mathbb{r}^{-\xi\alpha}\exp\left(\xi h_{\mathbb{r}e^{-k}}(0) + \xi\alpha k\right)$.

To deal with the scales for which $k \geq C^{1/\zeta}$, we apply Proposition 4.34 with k^ζ in place of C and take a union bound over all such values of k to find that superpolynomially high probability as $C \to \infty$, the above two conditions hold for each $k \in [0, C^{1/\zeta}]_{\mathbb{Z}}$, and furthermore the following condition holds for each integer $k \geq C^{1/\zeta}$.

2′. There is a path from $\partial B_{\mathbb{r}e^{-k-2}}(0)$ to $\partial B_{\mathbb{r}e^{-k}}(0)$ which has D_{h^α}-length at most $k^\zeta\mathfrak{c}_{\mathbb{r}e^{-k}}\mathbb{r}^{-\xi\alpha}\exp\left(\xi h_{\mathbb{r}e^{-k}}(0) + \xi\alpha k\right)$. Moreover, there is also a path in $B_{\mathbb{r}e^{-k}}(0) \setminus \overline{B_{\mathbb{r}e^{-k-1}}(0)}$ which disconnects $\partial B_{\mathbb{r}e^{-k-1}}(0)$ from $\partial B_{\mathbb{r}e^{-k}}(0)$ and which has D_{h^α}-length at most $k^\zeta\mathfrak{c}_{\mathbb{r}e^{-k}}\mathbb{r}^{-\xi\alpha}\exp\left(\xi h_{\mathbb{r}e^{-k}}(0) + \xi\alpha k\right)$.

Henceforth assume that conditions 1 and 2 hold for each $k \in [0, C^{1/\zeta}]_{\mathbb{Z}}$ and condition 2′ holds for each integer $k \geq C^{1/\zeta}$, which happens with superpolynomially high probability as $C \to \infty$.

Any path from 0 to $\partial B_{\mathbb{r}}(0)$ must cross each of the annuli $B_{\mathbb{r}e^{-k}}(0) \setminus B_{\mathbb{r}e^{-k-1}}(0)$ for $k \in [0, C^{1/\zeta}]_{\mathbb{Z}}$. Furthermore, the union of $\{0\}$ and the paths from conditions 2 and $2'$ for all $k \in \mathbb{N}_0$ contains a path from 0 to $\partial B_{\mathbb{r}}(0)$. By Theorem 4.5, there is a deterministic function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(k) = o_k(k)$, depending only on the choice of metric D, such that

$$e^{-\xi Qk - \phi(k)} \mathfrak{c}_{\mathbb{r}} \leq \mathfrak{c}_{\mathbb{r}e^{-k}} \leq e^{-\xi Qk + \phi(k)} \mathfrak{c}_{\mathbb{r}}, \quad \forall \mathbb{r} > 0. \tag{3.104}$$

Summing the bounds from conditions 1 and 2 over all $k \in [0, C^{1/\zeta}]_{\mathbb{Z}}$ and the bounds from condition $2'$ over all integers $k \geq C^{1/\zeta}$ and plugging in (3.104) shows that with superpolynomially high probability as $C \to \infty$,

$$\begin{aligned} C^{-1} \frac{\mathfrak{c}_{\mathbb{r}}}{\mathbb{r}^{\alpha\xi}} \sum_{k=0}^{\lfloor C^{1/\zeta} \rfloor} e^{\xi h_{\mathbb{r}e^{-k}}(0) - \xi(Q-\alpha)k - \phi(k)} &\leq D_{h^\alpha}(0, \partial B_{\mathbb{r}}(0)) \\ &\leq C \frac{\mathfrak{c}_{\mathbb{r}}}{\mathbb{r}^{\alpha\xi}} \sum_{k=0}^{\lfloor C^{1/\zeta} \rfloor} e^{\xi h_{\mathbb{r}e^{-k}}(0) - \xi(Q-\alpha)k + \phi(k)} + \frac{\mathfrak{c}_{\mathbb{r}}}{\mathbb{r}^{\alpha\xi}} \sum_{k=\lfloor C^{1/\zeta} \rfloor + 1}^{\infty} k^{\zeta} e^{\xi h_{\mathbb{r}e^{-k}}(0) - \xi(Q-\alpha)k + \phi(k)}. \end{aligned} \tag{3.105}$$

Furthermore, by condition 2 for $k = 0$ the D_{h^α}-distance around $B_{\mathbb{r}}(0) \setminus B_{\mathbb{r}/e}(0)$ is at most the right side of (3.105).

Step 2: from summation to integration. We now want to convert from sums to integrals in (3.105). Since $t \mapsto h_{\mathbb{r}e^{-t}}(0) - h_{\mathbb{r}}(0)$ is a standard linear Brownian motion [44, Section 3.1], the Gaussian tail bound and the union bound show that with superpolynomially high probability as $C \to \infty$,

$$\sup_{t \in [k, k+1]} |h_{\mathbb{r}e^{-t}}(0) - h_{\mathbb{r}e^{-k}}(0)| \leq \frac{1}{\xi} \log C, \quad \forall k \in \left[0, C^{1/\zeta}\right]_{\mathbb{Z}}. \tag{3.106}$$

Let $\psi(t) := \phi(\lfloor t \rfloor)$, where ϕ is as in (3.104). Then $\psi(t) = o_t(t)$ and if (3.106) holds, then for each $k \in [0, C^{1/\zeta}]_{\mathbb{Z}}$,

$$\begin{aligned} &e^{\xi h_{\mathbb{r}e^{-k}}(0) - \xi(Q-\alpha)k - \phi(k)} \geq C^{-1} \int_k^{k+1} e^{\xi h_{\mathbb{r}e^{-t}}(0) - \xi(Q-\alpha)t - \psi(t)} \, dt \quad \text{and} \\ &e^{\xi h_{\mathbb{r}e^{-k}}(0) - \xi(Q-\alpha)k + \phi(k)} \leq C \int_k^{k+1} e^{\xi h_{\mathbb{r}e^{-t}}(0) - \xi(Q-\alpha)t + \psi(t)} \, dt. \end{aligned} \tag{3.107}$$

By summing (3.107) over all $k \in [0, C^{1/\zeta}]_{\mathbb{Z}}$, we obtain

$$\sum_{k=0}^{\lfloor C^{1/\zeta} \rfloor} e^{\xi h_{\mathbb{r}e^{-k}}(0) - \xi(Q-\alpha)k - \phi(k)} \geq C^{-1} \int_0^{\lfloor C^{1/\zeta} \rfloor + 1} e^{\xi h_{\mathbb{r}e^{-t}}(0) - \xi(Q-\alpha)t - \psi(t)}\, dt \quad \text{and}$$
$$\sum_{k=0}^{\lfloor C^{1/\zeta} \rfloor} e^{\xi h_{\mathbb{r}e^{-k}}(0) - \xi(Q-\alpha)k + \phi(k)} \leq C \int_0^{\lfloor C^{1/\zeta} \rfloor + 1} e^{\xi h_{\mathbb{r}e^{-t}}(0) - \xi(Q-\alpha)t + \psi(t)}\, dt. \tag{3.108}$$

Step 3: bounding the sum of the small scales. To deduce our desired bounds from (3.105) and (3.108), we now need an upper bound for $\int_{\lfloor C^{1/\zeta} \rfloor}^{\infty} e^{\xi h_{\mathbb{r}e^{-t}}(0) - \xi(Q-\alpha)t + \psi(t)}\, dt$ and an upper bound for the second sum on the right side of (3.105). This is the only step where we need to assume that $\alpha < Q$.

Since $t \mapsto h_{\mathbb{r}e^{-t}}(0) - h_{\mathbb{r}}(0)$ is a standard linear Brownian motion and for $q \in (0,1]$, $x \mapsto x^q$ is concave, hence subadditive, if $q \in (0,1]$ is chosen small enough that $\xi q(Q-\alpha) - \xi^2 q^2/2 > 0$, then

$$\begin{aligned}\mathbb{E}\left[\left(\int_{\lfloor C^{1/\zeta} \rfloor}^{\infty} e^{\xi h_{\mathbb{r}e^{-t}}(0) - \xi(Q-\alpha)t + \psi(t)}\, dt\right)^q\right] &\preceq e^{q h_{\mathbb{r}}(0)} \int_{\lfloor C^{1/\zeta} \rfloor}^{\infty} \exp\left(-\left(\xi q(Q-\alpha) - \frac{\xi^2 q^2}{2}\right) t + o_t(t)\right) dt \\ &\preceq e^{q h_{\mathbb{r}}(0)} \exp\left(-\frac{1}{2}\left(\xi q(Q-\alpha) - \frac{\xi^2 q^2}{2}\right) C^{1/\zeta}\right),\end{aligned}$$

where here the $o_t(t)$ and the implicit constants in $\preceq$ do not depend on C or $\mathbb{r}$. Therefore, the Chebyshev inequality shows that

$$\mathbb{P}\left[\int_{\lfloor C^{1/\zeta} \rfloor}^{\infty} e^{\xi h_{\mathbb{r}e^{-t}}(0) - \xi(Q-\alpha)t + \psi(t)}\, dt > e^{\xi h_{\mathbb{r}}(0) - C^{1/(2\zeta)}}\right] \tag{3.109}$$

decays faster than any negative power of C. On the other hand, it is easily seen from the Gaussian tail bound that

$$\mathbb{P}\left[\int_0^{\lfloor C^{1/\zeta} \rfloor} e^{\xi h_{\mathbb{r}e^{-t}}(0) - \xi(Q-\alpha)t + \psi(t)}\, dt < e^{\xi h_{\mathbb{r}}(0) - C^{1/(2\zeta)}}\right] \tag{3.110}$$

decays faster than any negative power of C. Hence with superpolynomially high probability as $C \to \infty$,

$$\int_0^{\infty} e^{\xi h_{\mathbb{r}e^{-t}}(0) - \xi(Q-\alpha)t + \psi(t)}\, dt \leq 2 \int_0^{\lfloor C^{1/\zeta} \rfloor} e^{\xi h_{\mathbb{r}e^{-t}}(0) - \xi(Q-\alpha)t + \psi(t)}\, dt. \tag{3.111}$$

Similarly, we get that with superpolynomially high probability as $C \to \infty$,

$$\sum_{k=\lfloor C^{1/\zeta} \rfloor+1}^{\infty} k^{\zeta} e^{\xi h_{\mathbbm{r}e^{-k}}(0) - \xi(Q-\alpha)k - \phi(k)} \leq \int_0^{\infty} e^{\xi h_{\mathbbm{r}e^{-t}}(0) - \xi(Q-\alpha)t - \psi(t)}\, dt. \tag{3.112}$$

Step 4: conclusion. By applying (3.108), (3.111), and (3.112) to bound the left and right sides of (3.105), we get that if $\alpha < Q$, then with superpolynomially high probability, the bounds (3.97) as well as the bound stated just below (3.97) (here we use the sentence just below (3.105)) all hold with $2C^2$, say, in place of C. Since we are claiming that these bounds hold with superpolynomially high probability as $C \to \infty$, this is sufficient.

Finally, we consider the case when $\alpha > Q$. Since $h_{\mathbbm{r}e^{-t}}(0) - h_{\mathbbm{r}}(0)$ evolves as a standard linear Brownian motion, for each $\beta \in (0, \alpha - Q)$ it is a.s. the case that the summand $e^{\xi h_{\mathbbm{r}e^{-k}}(0) - \xi(Q-\alpha)k - \phi(k)}$ in the lower bound in (3.105) is bounded below by $e^{\beta k}$ for large enough k. (How large is random). Since (3.105) holds with superpolynomially high probability as $C \to \infty$, the Borel-Cantelli lemma combined with the preceding sentence shows that a.s. for large enough (random) $C > 1$, we have $D_{h^\alpha}(0, \partial B_{\mathbbm{r}}(0)) \geq C^{-1} e^{\beta \lfloor C^{1/\zeta} \rfloor}$, which tends to ∞ as $C \to \infty$. This shows that a.s. $D_{h^\alpha}(0, \partial B_{\mathbbm{r}}(0)) = \infty$. Since this holds a.s. for each rational $\mathbbm{r} > 0$, it follows that a.s. $D_{h^\alpha}(0, z) = \infty$ for every $z \in \mathbb{C} \setminus \{0\}$. □

Proof of Proposition 4.48. We first observe that by Axiom IV, Proposition 4.47 still holds with 0 replaced by any $z \in \mathbb{C}$, with the rate of convergence as $C \to \infty$ uniform in z and $\mathbbm{r}$. Applying the lower bound of Proposition 4.47 with each of z and w in place of 0 immediately gives (3.98) since any path from z to w must contain disjoint sub-paths from z to $\partial B_{\mathbbm{r}/2}(z)$ and from w to $B_{\mathbbm{r}/2}(w)$. Moreover, by comparing the local behavior of $D_{h^{\alpha,\beta}}$ near z and near w to D_{h^α} and D_{h^β}, respectively, we get that a.s. $D_{h^{\alpha,\beta}}(z, w) = \infty$ if either $\alpha > Q$ or $\beta > Q$.

It remains to prove (3.99). Assume $\alpha < Q$. We first apply Proposition 4.47 with $8\mathbbm{r}$ in place of $\mathbbm{r}$ to find that with superpolynomially high probability as $C \to \infty$, there is a path $P_{z,1}$ from z to $\partial B_{8\mathbbm{r}}(z)$ and a path $P_{z,2}$ in $B_{\mathbbm{r}}(z) \setminus B_{8\mathbbm{r}/e}(z)$ which disconnects $\partial B_{8\mathbbm{r}/e}(z)$ from $\partial B_{8\mathbbm{r}}(z)$ which each have D_h-length at most

$$\int_{-\log 8}^{\infty} e^{\xi h_{\mathbbm{r}e^{-t}}(z) - \xi(Q-\alpha)t + \psi(t)}\, dt;$$

and the same is true with w in place of z. Since $w \in B_{8\mathbbm{r}/e}(z)$, the union of the paths $P_{z,1}, P_{z,2}$, and $P_{w,1}$ contains a path from z to w in $B_{8\mathbbm{r}}(z)$. This gives (3.99) but with $-\log 8$ instead of 0 in the lower bound of integration for the integral on the right.

To get the estimate with the desired lower bound of integration, we use that $t \mapsto h_{\mathbbm{r}e^{-t}}(z) - h_{\mathbbm{r}}(z)$ is a standard two-sided linear Brownian motion. In particular, two applications of the Gaussian tail bound show that with superpolynomially high probability as $C \to \infty$,

$$\sup_{t \in [-\log 8, 0]} h_{\mathbbm{r}e^{-t}}(z) \leq \inf_{t \in [0, \log 2]} h_{\mathbbm{r}e^{-t}}(z) + \log C.$$

Therefore, with superpolynomially high probability as $C \to \infty$,

$$\begin{aligned} \int_{-\log 8}^{\infty} e^{\xi h_{\mathbbm{r}e^{-t}}(z) - \xi(Q-\alpha)t + \psi(t)} \, dt \leq & \int_0^{\infty} e^{\xi h_{\mathbbm{r}e^{-t}}(z) - \xi(Q-\alpha)t + \psi(t)} \, dt \\ & + C^{\xi} \int_0^{\log 2} e^{\xi h_{\mathbbm{r}e^{-t}}(z) - \xi(Q-\alpha)t + \psi(t)} \, dt. \end{aligned}$$

Combining this with the analogous estimate with w in place of z and the aforementioned analog of (3.99) with $-\log 8$ instead of 0 in the lower bound of integration gives (3.99). □

Although it is not needed for the proofs of Propositions 4.47 and 4.48, we record the following generalization of Proposition 4.42 which tells us in particular that D_{h^α} induces the Euclidean topology on $\mathbb{C}$ when $Q > 2$ and $\alpha < Q$ (which is a stronger statement than just that $D_{h^\alpha}(0, z) < \infty$ for every $z \in \mathbb{C}$).

Proposition 4.50. *Let h, α, h^α, and D_{h^α} be as in Proposition 4.47. If $Q = 2/\gamma + \gamma/2 > 2$ and $\alpha \in (-\infty, Q)$, then for each $-\infty < p < \min\{\frac{4d_\gamma}{\gamma^2}, \frac{2d_\gamma}{\gamma}(Q - \alpha)\}$, there exists $C_{\alpha,p} > 0$ such that for each $\mathbbm{r} > 0$,*

$$\mathbb{E}\left[\left(e^{-\xi h_{\mathbbm{r}}(0)} \mathfrak{c}_{\mathbbm{r}}^{-1} \mathbbm{r}^{\alpha\xi} \sup_{z,w \in B_{\mathbbm{r}}(0)} D_{h^\alpha}(z, w)\right)^p\right] \leq C_{\alpha,p}. \tag{3.113}$$

In particular, a.s. D_{h^α} induces the Euclidean topology on $\mathbb{C}$.

We note that the range of moments $-\infty < p < \min\{\frac{4d_\gamma}{\gamma^2}, \frac{2d_\gamma}{\gamma}(Q - \alpha)\}$ for the D_{h^α}-diameter of $\mathbb{D}$ appearing in Proposition 4.50 is the same as the range of moments for the μ_{h^α}-mass of $\mathbb{D}$, but scaled by d_γ; see, e.g., [54, Lemma A.3]. This is natural from the perspective that d_γ is the scaling exponent relating γ-LQG distances and areas.

Proof of Proposition 4.50. On $B_{\mathbbm{r}}(0) \setminus B_{\mathbbm{r}/2}(0)$, we have that $-\alpha \log|\cdot|$ is bounded above and below by $-\alpha \log \mathbbm{r}$ times constants depending only on α. Therefore, the existence of negative moments is immediate from Axiom III and Proposition 4.42 applied with $U = \mathbb{D} \setminus B_{1/2}(0)$.

To get the desired positive moments, for $k \in \mathbb{N}_0$ let A_k be the annulus $B_{\mathbbm{r}e^{-k}}(0) \setminus B_{\mathbbm{r}e^{-k-1}}(0)$. The random variable $h_{\mathbbm{r}e^{-k}}(0) - h_{\mathbbm{r}}(0)$ is Gaussian with variance k, so for $p > 0$,

$$\mathbb{E}\left[e^{p\xi\left(h_{\mathbbm{r}e^{-k}}(0) - h_{\mathbbm{r}}(0)\right)}\right] = e^{p^2\xi^2 k/2}, \quad \forall p > 0. \tag{3.114}$$

By Proposition 4.42 (applied with $K = A_0$, $U = \mathbb{C}$, and $\mathbbm{r}e^{-k}$ in place of $\mathbbm{r}$),

$$\mathbb{E}\left[\left(\mathfrak{c}_{\mathbbm{r}e^{-k}}^{-1} e^{-\xi h_{\mathbbm{r}e^{-k}}(0)} e^{-\alpha\xi k} \mathbbm{r}^{\alpha k} \sup_{z,w \in A_k} D_{h^\alpha}(z,w)\right)^p\right] \preceq 1, \quad \forall p < \frac{4d_\gamma}{\gamma^2}. \tag{3.115}$$

By (3.114) and (3.115) and since $(h - h_{\mathbbm{r}e^{-k}}(0))|_{A_k}$ is independent from $h_{\mathbbm{r}e^{-k}}(0) - h_{\mathbbm{r}}(0)$, we find that for $p \in (0, 4d_\gamma/\gamma^2)$,

$$\begin{aligned}
&\mathbb{E}\left[\left(e^{-\xi h_{\mathbbm{r}}(0)} \mathfrak{c}_{\mathbbm{r}}^{-1} \mathbbm{r}^{\alpha\xi} \sup_{z,w \in A_k} D_{h^\alpha}(z,w)\right)^p\right] \\
&\quad = \left(\frac{\mathfrak{c}_{\mathbbm{r}e^{-k}}}{\mathfrak{c}_{\mathbbm{r}}}\right)^p e^{p\alpha\xi k} \mathbb{E}\left[e^{p\xi\left(h_{\mathbbm{r}e^{-k}}(0) - h_{\mathbbm{r}}(0)\right)}\right] \mathbb{E}\left[\left(\mathfrak{c}_{\mathbbm{r}e^{-k}}^{-1} e^{-\xi h_{\mathbbm{r}e^{-k}}(0)} e^{-\alpha\xi k} \mathbbm{r}^{\alpha k} \sup_{z,w \in A_k} D_{h^\alpha}(z,w)\right)^p\right] \\
&\quad \le \exp\left(-\left(\xi p(Q-\alpha) - \frac{p^2\xi^2}{2}\right) k + o_k(k)\right),
\end{aligned} \tag{3.116}$$

at a rate depending only on α, p. Note that in the last line we used Theorem 4.5 to bound $\mathfrak{c}_{\mathbbm{r}e^{-k}}/\mathfrak{c}_{\mathbbm{r}}$.

The quantity inside the exponential on the right side of (3.116) is negative provided $p < \min\{\frac{4d_\gamma}{\gamma^2}, \frac{2d_\gamma}{\gamma}(Q-\alpha)\}$ (recall that $\xi = \gamma/d_\gamma$). For $0 < p < \min\{1, \frac{2d_\gamma}{\gamma}(Q-\alpha)\}$, the function $x \mapsto x^p$

is concave, hence subadditive, so summing (3.116) over all $k \in \mathbb{N}_0$ gives

$$\begin{aligned}\mathbb{E}\left[\left(e^{-\xi h_{\mathbb{r}}(0)}\mathfrak{c}_{\mathbb{r}}^{-1}\mathbb{r}^{\alpha\xi}\sup_{z,w\in B_{\mathbb{r}}(0)} D_{h^\alpha}(z,w)\right)^p\right] &\le \sum_{k=0}^\infty \mathbb{E}\left[\left(e^{-\xi h_{\mathbb{r}}(0)}\mathfrak{c}_{\mathbb{r}}^{-1}\mathbb{r}^{\alpha\xi}\sup_{z,w\in A_k} D_{h^\alpha}(z,w)\right)^p\right]\\ &\preceq \sum_{k=0}^\infty \exp\left(-\left(\xi p(Q-\alpha)-\frac{p^2\xi^2}{2}\right)k + o_k(k)\right)\\ &\preceq 1. \end{aligned} \tag{3.117}$$

This gives (3.113) in the case when $0 < p < \min\{1, \frac{2d_\gamma}{\gamma}(Q-\alpha)\}$. In the case when $1 \le p < \min\{\frac{4d_\gamma}{\gamma^2}, \frac{2d_\gamma}{\gamma}(Q-\alpha)\}$, (3.113) follows from a similar calculation with the triangle inequality for the L^p norm used in place of sub-additivity.

Finally, we know that the restriction of D_{h^α} to $\mathbb{C}\setminus\{0\}$ induces the Euclidean topology (see the discussion just above Theorem 4.10), so to check that that D_{h^α} induces the Euclidean topology, we need to show that a.s. $\sup_{z,w\in B_{e^{-k}}(0)} D_{h^\alpha}(z,w) \to 0$ as $k\to\infty$. This follows from the bound (3.117) applied with $\mathbb{r}=1$ and the Borel-Cantelli lemma. □

4.3.5 Hölder continuity

We will prove the following more quantitative version of Theorem 4.7 which is required to be uniform across scales.

Proposition 4.51. *Fix a compact set $K \subset \mathbb{C}$ and exponents $\chi \in (0, \xi(Q-2))$ and $\chi' > \xi(Q+2)$. For each $\mathbb{r} > 0$, it holds with polynomially high probability as $\varepsilon \to 0$, at a rate which is uniform in $\mathbb{r}$, that*

$$\left|\frac{u-v}{\mathbb{r}}\right|^{\chi'} \le \mathfrak{c}_{\mathbb{r}}^{-1} e^{-\xi h_{\mathbb{r}}(0)} D_h(u,v) \le \left|\frac{u-v}{\mathbb{r}}\right|^{\chi}, \quad \forall u,v\in \mathbb{r}K \text{ with } |u-v|\le \varepsilon\mathbb{r}. \tag{3.118}$$

We will actually prove a slightly stronger version of the upper bound for D_h in Proposition 4.51, which bounds internal distances relative to a small neighborhood of u instead of just distances along paths in all of $\mathbb{C}$; see Lemma 4.53 just below. This stronger version is used in [59].

For the proof of Proposition 4.51, we assume that $Q > 2$ and we fix a compact set $K \subset \mathbb{C}$. The basic idea of the proof of the upper bound in (3.118) is to apply Proposition 4.42 to Euclidean balls of radius ε and take a union bound over many such Euclidean balls which cover K. The basic idea for the proof of the lower bound in (3.118) is to apply the lower bound in Proposition 4.34 to lower bound the D_h-distance across Euclidean annuli of the form $B_{2\varepsilon}(z) \setminus B_\varepsilon(z)$, then take a union bound over many such annuli whose inner balls cover K. We first prove an upper bound for D_h-distances in terms of Euclidean distances. For this purpose we will use the following consequence of Propositions 4.42 and 4.43.

Lemma 4.52. *For each $s \in (0, \xi Q)$, each $\mathbb{r} > 0$, and each $z \in \mathbb{r}K$,*

$$\mathbb{P}\left[\sup_{u,v \in B_{\varepsilon\mathbb{r}}(z)} D_h\left(u,v; B_{2\varepsilon\mathbb{r}}(z)\right) \le \varepsilon^s \mathfrak{c}_\mathbb{r} e^{\xi h_\mathbb{r}(0)}\right] \ge 1 - \varepsilon^{\frac{(\xi Q - s)^2}{2\xi^2} + o_\varepsilon(1)}, \quad \text{as } \varepsilon \to 0, \tag{3.119}$$

uniformly over the choices of $\mathbb{r}$ and $z \in \mathbb{r}K$. Furthermore, if we let $S^{\varepsilon\mathbb{r}}(z)$ be the square of side length $\varepsilon\mathbb{r}$ centered at z, then for $\mathbb{r} > 0$ and $z \in \mathbb{r}K$, the D_h-internal diameter of $S^{\varepsilon\mathbb{r}}(z)$ satisfies

$$\mathbb{P}\left[\sup_{u,v \in S^{\varepsilon\mathbb{r}}(z)} D_h\left(u,v; S^{\varepsilon\mathbb{r}}(z)\right) \le \varepsilon^s \mathfrak{c}_\mathbb{r} e^{\xi h_\mathbb{r}(0)}\right] \ge 1 - \varepsilon^{\frac{(\xi Q - s)^2}{2\xi^2} + o_\varepsilon(1)}, \quad \text{as } \varepsilon \to 0, \tag{3.120}$$

uniformly over the choices of $\mathbb{r}$ and $z \in \mathbb{r}K$.

Proof. We know that $h_{2\varepsilon\mathbb{r}}(z) - h_\mathbb{r}(z)$ is centered Gaussian of variance $\log \varepsilon^{-1} - \log 2$ and is independent from $(h - h_{2\varepsilon\mathbb{r}}(z))|_{B_{2\varepsilon\mathbb{r}}(z)}$. By Axioms II and III, $h_{2\varepsilon\mathbb{r}}(z) - h_\mathbb{r}(z)$ is also independent from the internal metric

$$D_{h - h_{2\varepsilon\mathbb{r}}(z)}\left(u,v; B_{2\varepsilon\mathbb{r}}(z)\right) = e^{-\xi h_{2\varepsilon\mathbb{r}}(z)} D_h\left(u,v; B_{2\varepsilon\mathbb{r}}(z)\right).$$

Consequently, we can apply Theorem 4.5 and Proposition 4.42 (with $\varepsilon\mathbb{r}$ in place of $\mathbb{r}$) together with the formula $\mathbb{E}[e^X] = e^{\mathrm{Var}(X)/2}$ for a Gaussian random variable X to get that for $p \in (0, 4/(\gamma\xi))$,

$$\begin{aligned} &\mathbb{E}\left[\left(\mathfrak{c}_\mathbb{r}^{-1} e^{-\xi h_\mathbb{r}(0)} \sup_{u,v \in B_{\varepsilon\mathbb{r}}(z)} D_h\left(u,v; B_{2\varepsilon\mathbb{r}}(z)\right)\right)^p\right] \\ &\quad = \left(\frac{\mathfrak{c}_{\varepsilon\mathbb{r}}}{\mathfrak{c}_\mathbb{r}}\right)^p \mathbb{E}\left[e^{\xi p (h_{\varepsilon\mathbb{r}}(z) - h_\mathbb{r}(z)}\right] \mathbb{E}\left[\left(\mathfrak{c}_{\varepsilon\mathbb{r}}^{-1} e^{-\xi h_{\varepsilon\mathbb{r}}(z)} \sup_{u,v \in B_{\varepsilon\mathbb{r}}(z)} D_h\left(u,v; B_{2\varepsilon\mathbb{r}}(z)\right)\right)^p\right] \\ &\quad \le \varepsilon^{\xi Q p - \xi^2 p^2/2 + o_\varepsilon(1)}, \end{aligned} \tag{3.121}$$

with the $o_\varepsilon(1)$ uniform over all $\mathbb{r} > 0$ and $z \in \mathbb{C}$.

By (3.121) and the Chebyshev inequality,

$$\mathbb{P}\left[\sup_{u,v\in B_{\varepsilon\mathbbm{r}}(z)} D_h\left(u,v;B_{2\varepsilon\mathbbm{r}}(z)\right) > \varepsilon^s \mathfrak{c}_{\mathbbm{r}} e^{\xi h_{\mathbbm{r}}(z)}\right] \leq \varepsilon^{p\xi Q - \frac{p^2\xi^2}{2} - ps + o_\varepsilon(1)}. \tag{3.122}$$

The exponent on the right side is maximized for $p = (\xi Q - s)/\xi^2$, which is always at most $4/(\xi\gamma)$ for $s > 0$ (since $\gamma < 2$) and is positive provided $s < \xi Q$. Making this choice of p gives (3.119) but with $h_{\mathbbm{r}}(z)$ in place of $h_{\mathbbm{r}}(0)$. The random variables $h_{\mathbbm{r}}(z) - h_{\mathbbm{r}}(0)$ for $z \in \mathbbm{r}K$ are Gaussian with variance bounded above by a constant depending only on K. Consequently, we can apply the Gaussian tail bound to get (3.119) in general.

The bound (3.120) is proven similarly but with Proposition 4.43 used in place of Proposition 4.42. □

We can now prove a slightly sharper version of the upper bound of Proposition 4.51.

Lemma 4.53. *For each $\chi \in (0, \xi(Q-2))$ and each $\mathbbm{r} > 0$, it holds with polynomially high probability as $\varepsilon \to 0$, at a rate which is uniform in $\mathbbm{r}$, that*

$$\mathfrak{c}_{\mathbbm{r}}^{-1} e^{-\xi h_{\mathbbm{r}}(0)} D_h\left(u,v;B_{2|u-v|}(u)\right) \leq \left|\frac{u-v}{\mathbbm{r}}\right|^\chi, \quad \forall u,v \in \mathbbm{r}K \text{ with } |u-v| \leq \varepsilon\mathbbm{r}. \tag{3.123}$$

Furthermore, it also holds with polynomially high probability as $\varepsilon \to 0$, at a rate which is uniform in $\mathbbm{r}$, that for each $k \in \mathbb{N}_0$ and each $2^{-k}\varepsilon\mathbbm{r} \times 2^{-k}\varepsilon\mathbbm{r}$ square S with corners in $2^{-k}\varepsilon\mathbbm{r}\mathbb{Z}^2$ which intersects $\mathbbm{r}K$, we have

$$\mathfrak{c}_{\mathbbm{r}}^{-1} e^{-\xi h_{\mathbbm{r}}(0)} \sup_{u,v\in S} D_h(u,v;S) \leq (2^{-k}\varepsilon)^\chi. \tag{3.124}$$

Proof. The bound (3.123) follows from (3.119), applied with $s = \chi$ and with $2^{-k}\varepsilon$ for $k \in \mathbb{N}_0$ in place of ε, together with a union bound over all $z \in B_{\varepsilon\mathbbm{r}}(K) \cap (2^{-k-2}\varepsilon\mathbbm{r}\mathbb{Z}^2)$ and then over all $k \in \mathbb{N}_0$. The bound (3.124) similarly follows from (3.120). □

To prove the Hölder continuity of the Euclidean metric w.r.t. D_h, we first need the following estimate which plays a role analogous to Lemma 4.52.

Lemma 4.54. *For each $s > \xi Q$, each $\mathbb{r} > 0$, and each $z \in \mathbb{r}K$,*

$$\mathbb{P}\left[D_h\left(B_{\varepsilon\mathbb{r}}(z), \partial B_{2\varepsilon\mathbb{r}}(z)\right) \geq \varepsilon^s \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)}\right] \geq 1 - \varepsilon^{\frac{(s-\xi Q)^2}{2\xi^2} + o_\varepsilon(1)}, \quad \text{as } \varepsilon \to 0, \tag{3.125}$$

uniformly over the choices of $\mathbb{r}$ and $z \in \mathbb{r}K$.

Proof. The proof is similar to that of Lemma 4.52 but we use Proposition 4.34 instead of Proposition 4.42. Proposition 4.34 implies that $\mathfrak{c}_{\varepsilon\mathbb{r}}^{-1} e^{-\xi h_{\varepsilon\mathbb{r}}(z)} D_h\left(B_{\varepsilon\mathbb{r}}(z), \partial B_{2\varepsilon\mathbb{r}}(z)\right)$ has finite moments of all negative orders which are bounded above uniformly over all $z \in \mathbb{C}$ and $\mathbb{r} > 0$. By the same calculation as in (3.121), for each $p > 0$ we have

$$\mathbb{E}\left[\left(\mathfrak{c}_{\mathbb{r}}^{-1} e^{-\xi h_{\mathbb{r}}(z)} D_h\left(B_{\varepsilon\mathbb{r}}(z), \partial B_{2\varepsilon\mathbb{r}}(z)\right)\right)^{-p}\right] = \varepsilon^{-\xi Q p - \xi^2 p^2/2 + o_\varepsilon(1)}, \tag{3.126}$$

uniformly over all $z \in \mathbb{C}$ and $\mathbb{r} > 0$. Applying the Chebyshev inequality and setting $p = (s - \xi Q)/\xi^2$ gives (3.125) with $h_{\mathbb{r}}(z)$ in place of $h_{\mathbb{r}}(0)$. For $z \in \mathbb{r}K$, we can replace $h_{\mathbb{r}}(z)$ with $h_{\mathbb{r}}(0)$ via exactly the same argument as in the proof of Lemma 4.52. □

Lemma 4.55. *For each $\chi' > \xi(Q+2)$ and each $\mathbb{r} > 0$, it holds with polynomially high probability as $\varepsilon \to 0$, at a rate which is uniform in $\mathbb{r}$, that*

$$\mathfrak{c}_{\mathbb{r}}^{-1} e^{-\xi h_{\mathbb{r}}(0)} D_h(u, v) \geq \left|\frac{u - v}{\mathbb{r}}\right|^{\chi'}, \quad \forall u, v \in K \text{ with } |u - v| \leq \varepsilon. \tag{3.127}$$

Proof. This follows from (3.119), applied with $s = \chi'$ and with $2^{-k}\varepsilon$ for $k \in \mathbb{N}_0$ in place of ε, together with a union bound over all $z \in B_{\varepsilon\mathbb{r}}(K) \cap (2^{-k-2}\varepsilon\mathbb{r}\mathbb{Z}^2)$ and then over all $k \in \mathbb{N}_0$. □

Proof of Proposition 4.51. Combine Lemmas 4.53 and 4.55. □

To conclude the proof of Theorem 4.7, we need to check that the Hölder exponents $\xi(Q-2)$ and $(\xi(Q+2))^{-1}$ are optimal.

Lemma 4.56. *Let $V \subset \mathbb{C}$ be an open set. Almost surely, the identity map from V, equipped with the Euclidean metric, to $(V, D_h|_V)$ is* not *Hölder continuous with any exponent greater than $\xi(Q-2)$.*

Furthermore, the inverse of this map is not *Hölder continuous with any exponent greater than* $\xi^{-1}(Q+2)^{-1}$.

Proof. The idea of the proof is to use Proposition 4.47 to study D_h-distances as we approach an α-thick point of h for α close to 2 or to -2. To produce such a thick point, we will sample a point from the α-LQG measure induced by the zero-boundary part of $h|_V$. By Axiom III, we can assume without loss of generality that h is normalized so that $h_1(0) = 0$. We can also assume without loss of generality that V is bounded with smooth boundary. Let h^V be the zero-boundary part of $h|_V$, so that $h - h^V$ is harmonic on V.

Let $\alpha \in (-2,2)$ which we will eventually send to either -2 or 2, and let $\mu_{h^V}^\alpha$ be the α-LQG measure induced by h^V. Also let $\mathbb{z}$ be sampled uniformly from μ_h^α, normalized to be a probability measure. Let $\widetilde{\mathbb{P}}$ be the law of $(h,\mathbb{z})$ weighted by the total mass $\mu_{h^V}^\alpha(V)$, so that under $\widetilde{\mathbb{P}}$, h is sampled from its marginal law weighted by $\mu_{h^V}^\alpha(V)$ and conditional on h, $\mathbb{z}$ is sampled from $\mu_{h^V}^\alpha$, normalized to be a probability measure. By a well-known property of the α-LQG measure (see, e.g., [43, Lemma A.10]), a sample $(h,\mathbb{z})$ from the law $\widetilde{\mathbb{P}}$ can be equivalently be produced by first sampling $\widetilde{h}$ from the unweighted marginal law of h, then independently sampling $\mathbb{z}$ uniformly from Lebesgue measure on $\mathbb{S}'$ and setting $h = \widetilde{h} - \alpha \log|\cdot - \mathbb{z}| + \mathfrak{g}_{\mathbb{z}}$, where $\mathfrak{g}_{\mathbb{z}} : V \to \mathbb{R}$ is a deterministic continuous function.

By Proposition 4.47 (applied with the field $\widetilde{h} - \alpha\log|\cdot - \mathbb{z}|$ in place of h^α), the fact that $\mathfrak{g}_{\mathbb{z}}$ is a.s. bounded in a neighborhood of $\mathbb{z}$ (by continuity), and the Borel-Cantelli lemma, we find that a.s.

$$D_h(\mathbb{z}, \partial B_r(\mathbb{z})) = r^{o_r(1)} \frac{c_r}{r^{\alpha\xi}} \int_0^\infty e^{\xi \widetilde{h}_{re^{-t}}(\mathbb{z}) - \xi(Q-\alpha)t o_t(t)}\, dt, \tag{3.128}$$

where here the $o_t(t)$ is deterministic and tends to 0 as $t \to \infty$ (it comes from the error $\psi(t)$ in Proposition 4.47) and the $o_r(1)$ denotes a random variable which tends to 0 a.s. as $r \to 0$. The description in the preceding paragraph shows that conditional on $\mathbb{z}$, the process $t \mapsto \widetilde{h}_{re^{-t}}(\mathbb{z}) - \widetilde{h}_r(\mathbb{z})$ evolves as a standard linear Brownian motion. Consequently, the Gaussian tail bound shows that with probability tending to 1 as $r \to 0$,

$$\int_0^\infty e^{\xi \widetilde{h}_{re^{-t}}(\mathbb{z}) - \xi(Q-\alpha)t + o_t(t)}\, dt = r^{o_r(1)} e^{\xi \widetilde{h}_r(\mathbb{z})} = r^{o_r(1)}. \tag{3.129}$$

By plugging (3.129) into (3.128) and using the fact that $\mathfrak{c}_r = r^{\xi Q + o_r(1)}$ (Theorem 4.5), it therefore follows that with probability tending to 1 as $r \to 0$,

$$D_h(\mathbb{z}, \partial B_r(\mathbb{z})) = r^{\xi(Q-\alpha)+o_r(1)}.$$

Since α can be made arbitrarily close to 2, this shows the desired lack of Hölder continuity for identity map $(V, |\cdot|) \to (V, D_h)$. Since α can be made arbitrarily close to -2, we also get the desired lack of Hölder continuity for the inverse map $(V, D_h) \to (V, |\cdot|)$. $\square$

4.4 Constraints on the behavior of D_h-geodesics

Let D be a weak γ-LQG metric. By Lemma 4.41, for a whole-plane GFF h, the metric space $(\mathbb{C}, D_h)$ is a boundedly compact length space (i.e., closed bounded subsets are compact) so there is a D_h-geodesic — i.e., a path of minimal D_h-length — between any two points of $\mathbb{C}$ [17, Corollary 2.5.20]. In this section we will apply the main results of this chapter to prove two estimates which constrain the behavior of D_h-geodesics. The first of these estimates, Proposition 4.57, tells us that paths which stay in a small Euclidean neighborhood of a straight line or an arc of the boundary of a circle have large D_h-lengths. In particular, D_h-geodesics are unlikely to stay in such a neighborhood. The second estimate, Proposition 4.59, says that a D_h-geodesic cannot spend a long time near the boundary of a D_h-metric ball.

4.4.1 Lower bound for D_h-distances in a narrow tube

Proposition 4.57. *Let $L \subset \mathbb{C}$ be a compact set which is either a line segment, an arc of a circle, or a whole circle and fix $b > 0$. For each $\mathbb{r} > 0$ and each $p > 0$, it holds with probability at least $1 - \varepsilon^{p^2/(2\xi^2)+o_\varepsilon(1)}$ that*

$$\inf\{D_h(u, v; B_{\varepsilon\mathbb{r}}(\mathbb{r}L)) : u, v \in B_{\varepsilon\mathbb{r}}(\mathbb{r}L), |u-v| \geq b\mathbb{r}\} \geq \varepsilon^{p+\xi Q-1-\xi^2/2} \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)}, \tag{4.130}$$

where the rate of the $o_\varepsilon(1)$ depends on L, b, p but not on $\mathbb{r}$.

By [4, Theorem 1.9], for each $\gamma \in (0, 2)$ we have $\xi Q \leq 1$ and hence $\xi Q - 1 - \xi^2/2 < 0$. Therefore, the power of ε on the right side of (4.130) is negative for small enough p. Hence, Proposition 4.57 implies that when ε is small and $u,v \in B_{\varepsilon\mathbb{r}}(\mathbb{r}L)$ with $|u-v| \geq b\mathbb{r}$, it holds with high probability that $D_h(u,v;B_{\varepsilon\mathbb{r}}(\mathbb{r}L))$ is much larger than $D_h(u,v)$. In particular, a D_h-geodesic from u to v cannot stay in $B_{\varepsilon\mathbb{r}}(L)$.

Proof of Proposition 4.57. Step 1: bounding distances in terms of circle averages. View L as a path $[0,|L|] \to \mathbb{C}$ parametrized by Euclidean unit speed. For $k \in [0,|L|/(6\varepsilon)]_{\mathbb{Z}}$, let $z_k^\varepsilon := \mathbb{r}L(6k\varepsilon)$. Then the balls $B_{3\varepsilon\mathbb{r}}(z_k^\varepsilon)$ are disjoint and the balls $B_{7\varepsilon\mathbb{r}}(z_k^\varepsilon)$ cover $B_{\varepsilon\mathbb{r}}(\mathbb{r}L)$.

Fix $\zeta \in (0,1)$, which we will eventually send to zero. By Proposition 4.34 and a union bound, it holds with superpolynomially high probability as $\varepsilon \to 0$ that

$$D_h\left(B_{2\varepsilon\mathbb{r}}(z_k^\varepsilon), B_{3\varepsilon\mathbb{r}}(z_k^\varepsilon)\right) \geq \varepsilon^\zeta \mathfrak{c}_{\varepsilon\mathbb{r}} e^{\xi h_{\varepsilon\mathbb{r}}(z_k^\varepsilon)}, \quad \forall k \in [0,|L|/(6\varepsilon)]_{\mathbb{Z}}. \tag{4.131}$$

Henceforth assume that (4.131) holds. The idea of the proof is that a path in $B_{\varepsilon\mathbb{r}}(\mathbb{r}L)$ has to cross between the inner and outer boundaries of a large number of the annuli $B_{3\varepsilon\mathbb{r}}(z_k^\varepsilon) \setminus B_{2\varepsilon\mathbb{r}}(z_k^\varepsilon)$. Thus (4.131) reduces our problem to proving a lower bound for the sum of the quantities $\varepsilon^\zeta \mathfrak{c}_{\varepsilon\mathbb{r}} e^{\xi h_{\varepsilon\mathbb{r}}(z_k^\varepsilon)}$ for these annuli, which in turn can be proven using Theorem 4.5 and basic estimates for the circle average process.

Step 2: lower-bounding lengths of paths in $B_{\varepsilon\mathbb{r}}(\mathbb{r}L)$ in terms of circle averages. There is a constant $c > 0$ depending only on b and L such that for small enough $\varepsilon > 0$ (depending only on b and L), the following is true. If $u,v \in B_{\varepsilon\mathbb{r}}(\mathbb{r}L)$ satisfy $|u-v| \geq b\mathbb{r}$, there are integers $0 \leq k_1' < k_2' \leq |L|/(6\varepsilon)$ such that $k_2' - k_1' \geq c\varepsilon^{-1}$, $u \in B_{7\varepsilon\mathbb{r}}(z_{k_1'}^\varepsilon)$, and $v \in B_{7\varepsilon\mathbb{r}}(z_{k_2'}^\varepsilon)$. Each path from u to v in $B_{\varepsilon\mathbb{r}}(\mathbb{r}L)$ must enter $B_{2\varepsilon\mathbb{r}}(z_k^\varepsilon)$ for each $k \in [k_1'+2, k_2'-2]_{\mathbb{Z}}$, and hence must cross the annulus $\mathbb{A}_{2\varepsilon\mathbb{r},3\varepsilon\mathbb{r}}(z_k^\varepsilon)$ for each such k. Combining this with (4.131) shows that

$$D_h\left(u,v;B_{\varepsilon\mathbb{r}}(\mathbb{r}L)\right) \geq \varepsilon^\zeta \mathfrak{c}_{\varepsilon\mathbb{r}} \sum_{k=k_1'+2}^{k_2'-2} e^{\xi h_{\varepsilon\mathbb{r}}(z_k^\varepsilon)}. \tag{4.132}$$

Step 3: proof conditional on a circle average estimate. We claim that for any fixed $k_1, k_2 \in$

$[0,|L|/(6\varepsilon)]_{\mathbb{Z}}$ with $k_2 - k_1 \geq (c/2)\varepsilon^{-1}$ and any $p > 0$,

$$\mathbb{P}\left[\sum_{k=k_1}^{k_2} e^{\xi h_{\varepsilon\mathbbm{r}}(z_k^\varepsilon)} \geq \varepsilon^{p-1-\xi^2/2} e^{\xi h_{\varepsilon\mathbbm{r}}(0)}\right] \geq 1 - \varepsilon^{\frac{p^2}{2\xi^2}+o_\varepsilon(1)} \tag{4.133}$$

where the rate of the $o_\varepsilon(1)$ depends on L, b, p but not on $\mathbbm{r}$ or the particular choice of k_1, k_2. We will prove (4.133) just below using standard Gaussian estimates.

Let us first conclude the proof assuming (4.133). We can find a constant-order number of pairs $k_1, k_2 \in [0,|L|/(6\varepsilon)]_{\mathbb{Z}}$ with $k_2 - k_1 \geq (c/2)\varepsilon^{-1}$ such that for small enough ε (depending only on L and b), each interval $[k_1'+2, k_2'-2] \subset [0,|L|/(6\varepsilon)]_{\mathbb{Z}}$ with $|k_2' - k_1'| \geq c\varepsilon^{-1}$ contains one of the intervals $[k_1, k_2]$.

By applying (4.133) (with $p - 2\zeta$ in place of p) to each such pair k_1, k_2, then taking a union bound, we get that with probability at least $1 - \varepsilon^{\frac{(p-2\zeta)^2}{2\xi^2}+o_\varepsilon(1)}$, the sum on the right side of (4.132) is bounded below by $\varepsilon^{p-1-\xi^2/2-2\zeta} e^{\xi h_{\varepsilon\mathbbm{r}}(0)}$ simultaneously for every possible choice of k_1', k_2'. By (4.132), with probability at least $1 - \varepsilon^{\frac{(p-2\zeta)^2}{2\xi^2}+o_\varepsilon(1)}$ it holds simultaneously for each $u, v \in B_{\varepsilon\mathbbm{r}}(\mathbbm{r}L)$ satisfying $|u - v| \geq b\mathbbm{r}$ that

$$D_h\left(u, v; B_{\varepsilon\mathbbm{r}}(\mathbbm{r}L)\right) \geq \varepsilon^{p-1-\xi^2/2-\zeta} \mathfrak{c}_{\varepsilon\mathbbm{r}} e^{\xi h_{\mathbbm{r}}(0)} \geq \varepsilon^{p+\xi Q-1-\xi^2/2-\zeta+o_\varepsilon(1)} \mathfrak{c}_{\mathbbm{r}} e^{\xi h_{\mathbbm{r}}(0)} \tag{4.134}$$

where in the second inequality we use Theorem 4.5. Sending $\zeta \to 0$ now gives (4.130).

Step 4: proof of the circle average estimate. The rest of the proof is devoted to proving the inequality (4.133). To lighten notation, write $X_k := h_{\varepsilon\mathbbm{r}}(z_k^\varepsilon) - h_{\mathbbm{r}}(0)$. By the calculations in [44, Section 3.1] (and the scale invariance of the law of h, modulo additive constant), the X_k's are jointly centered Gaussian with variances satisfying

$$\mathrm{Var}(X_k) = \log \varepsilon^{-1} + O(1), \tag{4.135}$$

where here $O(1)$ denotes a quantity which is bounded above and below by constants depending only on L, b (not on $\varepsilon, \mathbbm{r}, j, k$). Since $z_k^\varepsilon = \mathbbm{r}L(6k\varepsilon)$ and L is parametrized by Euclidean unit speed, we also have the following covariance formula for $j \neq k$:

$$\operatorname{Cov}(X_j, X_k) = \log\left(\frac{\mathbb{r}}{|z_j^\varepsilon - z_k^\varepsilon|}\right) + O(1) = \log\left(\frac{1}{\varepsilon|k-j|}\right) + O(1). \tag{4.136}$$

Recall the formula $\mathbb{E}[e^X] = e^{\operatorname{Var}(X)/2}$ for a centered Gaussian random variable X. Applying this to the X_k's and recalling (4.135) and the fact that $k_2 - k_1 \asymp \varepsilon^{-1}$ gives

$$\mathbb{E}\left[\sum_{k=k_1}^{k_2} e^{\xi X_k}\right] \asymp \varepsilon^{-1-\xi^2/2}, \tag{4.137}$$

with the implicit constant depending only on L, b. From (4.135) and (4.136) we obtain $\operatorname{Var}(X_j + X_k) = \log\left(\varepsilon^{-4}|k-j|^{-2}\right) + O(1)$ for $j \neq k$. Hence

$$\begin{aligned}\mathbb{E}\left[\left(\sum_{k=k_1}^{k_2} e^{\xi X_k}\right)^2\right] &= \sum_{k=k_1}^{k_2} \mathbb{E}\left[e^{2\xi X_k}\right] + 2\sum_{k=k_1}^{k_2}\sum_{j=k+1}^{k_2} \mathbb{E}\left[e^{\xi(X_j+X_k)}\right] \\ &\preceq \varepsilon^{-1-2\xi^2} + 2\varepsilon^{-2\xi^2}\sum_{k=k_1}^{k_2}\sum_{j=k+1}^{k_2} |j-k|^{-\xi^2} \\ &\preceq \varepsilon^{-1-2\xi^2} + \varepsilon^{-2-\xi^2} \preceq \varepsilon^{-2-\xi^2}\end{aligned} \tag{4.138}$$

with the implicit constants depending only on L, b, where in the last inequality we use that $\xi < 2/d_2 < 1$, so $1 + 2\xi^2 < 2 + \xi^2$.

By (4.137), (4.138), and the Payley-Zygmund inequality, we find that there is a constant $a = a(L) > 0$ such that

$$\mathbb{P}\left[\sum_{k=k_1}^{k_2} e^{\xi X_k} \geq a\varepsilon^{-1-\xi^2/2}\right] \geq a. \tag{4.139}$$

To improve the lower bound for this probability, we will apply the following elementary Gaussian concentration bound (see, e.g., [32, Lemma 2.1]):

Lemma 4.58. *For any $a > 0$, there exists $C = C(a) > 0$ such that the following is true. Let $\mathbf{X} = (X_1, \dots, X_n)$ be a centered Gaussian vector taking values in $\mathbb{R}^n$ and let $\sigma^2 := \max_{1 \leq j \leq n} \operatorname{Var}(X_j)$.*

If $B \subset \mathbb{R}^n$ such that $\mathbb{P}[X \in B] \geq a$, then for any $\lambda \geq C\sigma$,

$$\mathbb{P}\left[\inf_{\mathbf{x}\in B} |\mathbf{X}-\mathbf{x}|_\infty > \lambda\right] \leq e^{-\frac{(\lambda - C\sigma)^2}{2\sigma^2}}, \tag{4.140}$$

where $|\cdot|_\infty$ is the L^∞ norm on $\mathbb{R}^n$.

We now apply Lemma 4.58 with a as in (4.139), with $\sigma^2 = \log \varepsilon^{-1} + O(1)$ (recall (4.135)), with

$$B = \left\{ (x_{k_1}, \dots, x_{k_2}) \in \mathbb{R}^{k_1+k_2+1} : \sum_{k=k_1}^{k_2} e^{\xi x_k} \geq a \varepsilon^{-1-\xi^2/2} \right\}, \tag{4.141}$$

and with $\lambda = \frac{p}{\xi}\log \varepsilon^{-1}$. This shows that with probability $1-\varepsilon^{p^2/(2\xi^2)+o_\varepsilon(1)}$, there exists $(x_{k_1}, \dots, x_{k_2}) \in B$ such that $\max_{k\in[k_1,k_2]_\mathbb{Z}} |X_k - x_k| \leq \frac{p}{\xi}\log\varepsilon^{-1}$. If this is the case, then

$$\sum_{k=k_1}^{k_2} e^{\xi X_k} \geq \varepsilon^p \sum_{k=k_1}^{k_2} e^{\xi x_k} \geq a \varepsilon^{p-1-\xi^2/2}. \tag{4.142}$$

Since $X_k = h_{\varepsilon \mathbb{r}}(z_k^\varepsilon) - h_\mathbb{r}(0)$, this implies (4.133). □

4.4.2 D_h-geodesics cannot trace the boundaries of D_h-metric balls

For $s > 0$ and $z \in \mathbb{C}$, we write $\mathcal{B}_s(z; D_h)$ for the D_h-metric ball of radius s centered at z. The following proposition prevents a D_h-geodesic from spending a long time near the boundary of a D_h-metric ball.

Proposition 4.59. *For each $M > 0$ and each $\mathbb{r} > 0$, it holds with superpolynomially high probability as $\varepsilon \to 0$, at a rate which is uniform in the choice of $\mathbb{r}$, that the following is true. For each $s > 0$ for which $\mathcal{B}_s(0; D_h) \subset B_{\varepsilon^{-M}\mathbb{r}}(0)$ and each D_h-geodesic P from 0 to a point outside of $\mathcal{B}_s(0; D_h)$,*

$$\operatorname{area}\left(B_{\varepsilon\mathbb{r}}(P) \cap B_{\varepsilon\mathbb{r}}\left(\partial \mathcal{B}_s(0; D_h)\right)\right) \leq \varepsilon^{2-1/M}\mathbb{r}^2, \tag{4.143}$$

where area *denotes 2-dimensional Lebesgue measure.*

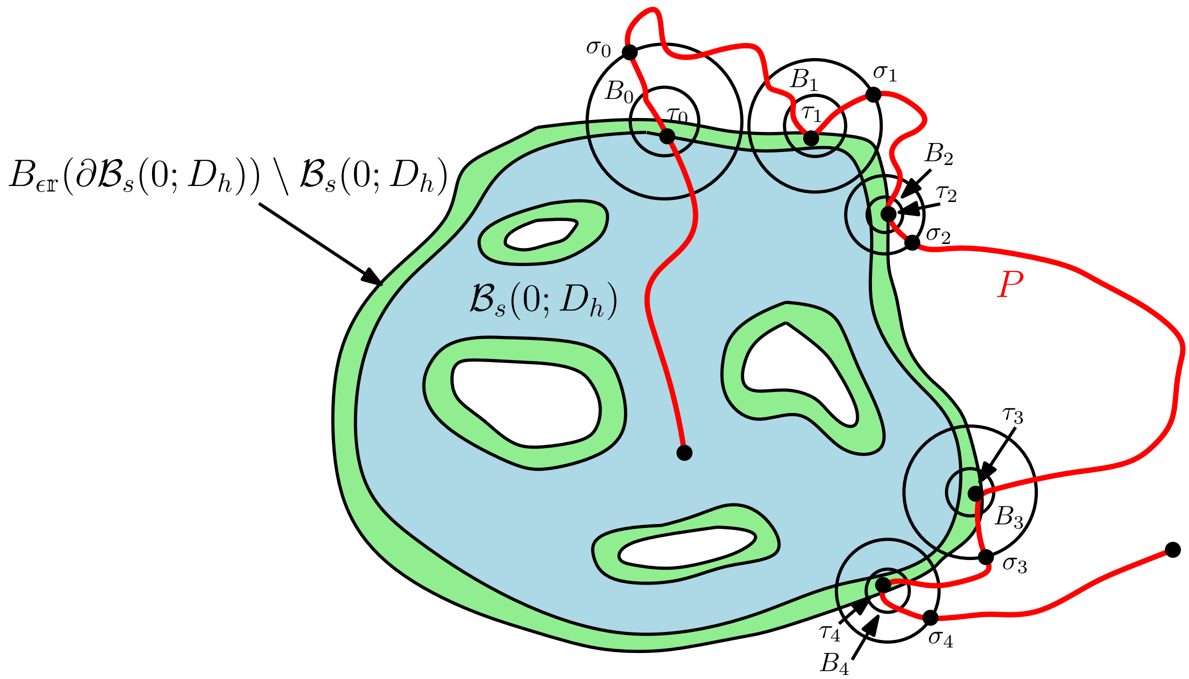

Figure 4.5 – Illustration of the proof of Proposition 4.59. By considering successive times at which P enters $B_{\varepsilon\mathbb{r}}(\mathcal{B}_s(0;D_h))$, we can find $K \in \mathbb{N}$ and a collection of K C-good Euclidean balls $B_0,\dots,B_K$ with radii in $[2\varepsilon\mathbb{r}, \varepsilon^{1-\zeta}\mathbb{r}]$ with the following properties: (a) each B_k intersects $\partial\mathcal{B}_s(0;D_h)$; (b) the D_h-geodesic P crosses the annuli $(2B_k)\setminus B_k$ for $k \in [0,K-1]_{\mathbb{Z}}$ in numerical order; and (c) the balls of radii $4\varepsilon^{1-\zeta}\mathbb{r}$ with the same centers as the B_k's cover $P \cap B_{\varepsilon\mathbb{r}}(\mathcal{B}_s(0;D_h))$. This last property implies that area $(B_{\varepsilon\mathbb{r}}(P) \cap B_{\varepsilon\mathbb{r}}(\partial\mathcal{B}_s(0;D_h)) \le \text{const} \times \varepsilon^{2-2\zeta}\mathbb{r}^2 K$, so we are left to bound K. To this end, we show using the definition (4.144) of a C-good ball and the fact that P is a D_h-geodesic that $D_h(\partial B_k, \partial(2B_k))$ increases exponentially in k. Due to Lemma 4.61, this implies that $K \le \varepsilon^{-1/(2M)}$.

For $C>1$, $z\in\mathbb{C}$, and $r>0$, we say that the Euclidean ball $B_r(z)$ is C-*good* if

$$\sup_{u,v\in\partial B_r(z)} D_h\left(u,v;\mathbb{A}_{r/2,2r}(z)\right) \le C D_h\left(\partial B_r(z), \partial B_{2r}(z)\right). \tag{4.144}$$

To prove Proposition 4.59, we will consider C-good balls which intersect $\partial\mathcal{B}_s(0;D_h)$ and which are hit by a given D_h-geodesic started from 0. See Figure 4.5 for an illustration and outline of the proof.

Lemma 4.60. *For each $\zeta\in(0,1)$ and each $M>0$, there exists $C=C(\zeta,M)>1$ such that for each $\mathbb{r}>0$, it holds with probability at least $1-O_\varepsilon(\varepsilon^M)$, at a rate which is uniform in $\mathbb{r}$, that the Euclidean ball $B_{\varepsilon^{-M}\mathbb{r}}(0)$ can be covered by C-good balls with radii in $[2\varepsilon\mathbb{r}, \varepsilon^{1-\zeta}\mathbb{r}]$.*

Proof. This is an immediate consequence of Lemma 4.35 applied with $\varepsilon^{1-\zeta}$ in place of ε and any choice of $\nu\in(0,\frac{1}{1-\zeta}-1)$. ∎

We will also need the following easy consequence of the distance bounds from Section 4.3.

Lemma 4.61. *For each $M > 0$, there exists $A = A(M) > 0$ such that for each $\mathbb{r} > 0$, the following holds with probability $1 - O_\varepsilon(\varepsilon^M)$ as $\varepsilon \to 0$, at a rate which is uniform in $\mathbb{r}$. For each $z, w \in B_{\varepsilon^{-M}\mathbb{r}}(0)$ with $|z - w| \geq \varepsilon\mathbb{r}$,*

$$D_h(z, w) \geq \varepsilon^A \sup_{u,v \in B_{\varepsilon^{-M}\mathbb{r}}(0)} D_h(u, v). \tag{4.145}$$

Proof. We will prove a lower bound for the left side of (4.145) (see (4.149)) and an upper bound for the right side of (4.145) (see (4.151)), then compare them.

By Proposition 4.34 and a union bound, it holds with superpolynomially high probability as $\varepsilon \to 0$ that

$$D_h(\partial B_{\varepsilon\mathbb{r}/4}(x), \partial B_{\varepsilon\mathbb{r}/2}(x)) \geq \varepsilon \mathfrak{c}_{\varepsilon\mathbb{r}} e^{\xi h_{\varepsilon\mathbb{r}}(x)}, \quad \forall x \in B_{\varepsilon^{-M}\mathbb{r}}(0) \cap \left(\frac{\varepsilon\mathbb{r}}{8}\mathbb{Z}^2\right). \tag{4.146}$$

The circle averages $h_{\varepsilon\mathbb{r}}(x) - h_\mathbb{r}(0)$ for $x \in B_{\varepsilon^{-M}\mathbb{r}}(0)$ are Gaussian with variance at most $(M+1)\log\varepsilon^{-1}$. By the Gaussian tail bound and a union bound, if we choose $A_0 = A_0(M)$ to be sufficiently large, then it holds with probability $1 - O_\varepsilon(\varepsilon^M)$ that

$$|h_{\varepsilon\mathbb{r}}(x) - h_\mathbb{r}(0)| \leq A_0 \log\varepsilon^{-1} \quad \forall x \in B_{\varepsilon^{-M}\mathbb{r}}(0) \cap \left(\frac{\varepsilon\mathbb{r}}{8}\mathbb{Z}^2\right). \tag{4.147}$$

By Theorem 4.5,

$$\mathfrak{c}_{\varepsilon\mathbb{r}} = \varepsilon^{\xi Q + o_\varepsilon(1)} \mathfrak{c}_\mathbb{r}. \tag{4.148}$$

If $z, w \in B_{\varepsilon^{-M}\mathbb{r}}(0)$ with $|z - w| \geq \varepsilon\mathbb{r}$, then any path from z to w must cross between the inner and outer boundaries of an annulus of the form $B_{\varepsilon\mathbb{r}/2}(x) \setminus B_{\varepsilon\mathbb{r}/4}(x)$ for some $x \in B_{\varepsilon^{-M}\mathbb{r}}(0) \cap \left(\frac{\varepsilon\mathbb{r}}{8}\mathbb{Z}^2\right)$. Combining this last observation with (4.146) shows that with superpolynomially high probability as $\varepsilon \to 0$, $D_h(z, w)$ is at least the right side of (4.146) for each such z, w. We then apply (4.147) and (4.148) to lower-bound the right side of (4.146). This shows that with probability $1 - O_\varepsilon(\varepsilon^M)$,

$$D_h(z, w) \geq \varepsilon^{\xi A_0 + \xi Q + 1 + o_\varepsilon(1)} \mathfrak{c}_\mathbb{r} e^{\xi h_\mathbb{r}(0)}, \quad \forall z, w \in B_{\varepsilon^{-M}\mathbb{r}}(0) \text{ with } |z - w| \geq \varepsilon\mathbb{r}. \tag{4.149}$$

By Proposition 4.42,

$$\mathbb{E}\left[\mathfrak{c}_{\varepsilon^{-M}\mathbb{r}}^{-1} e^{-\xi h_{\varepsilon^{-M}\mathbb{r}}(0)} \sup_{u,v\in B_{\varepsilon^{-M}\mathbb{r}}(0)} D_h(u,v)\right] \preceq 1, \tag{4.150}$$

with the implicit constant uniform over all $\mathbb{r} > 0$ and $\varepsilon \in (0,1)$. By Theorem 4.5, $\mathfrak{c}_{\varepsilon^{-M}\mathbb{r}} = \varepsilon^{-\xi QM + o_\varepsilon(1)}\mathfrak{c}_{\mathbb{r}}$. By the Gaussian tail bound, we can find $A_1 = A_1(M) > 0$ such that with probability $1 - O_\varepsilon(\varepsilon^M)$, we have $|h_{\varepsilon^{-M}\mathbb{r}}(0) - h_{\mathbb{r}}(0)| \leq A_0 \log \varepsilon^{-1}$. Combining these estimates with (4.150) and Markov's inequality shows that with probability $1 - O_\varepsilon(\varepsilon^M)$,

$$\sup_{u,v\in B_{\varepsilon^{-M}\mathbb{r}}(0)} D_h(u,v) \leq \varepsilon^{-\xi A_1 - \xi QM - M + o_\varepsilon(1)} \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)}. \tag{4.151}$$

Combining (4.149) and (4.151) gives (4.145) for any choice of $A > \xi A_1 + \xi QM + M + \xi A_0 + \xi Q + 1$. □

Proof of Proposition 4.59. Step 1: defining a regularity event. For $\widetilde{M} > 0$, $\zeta \in (0,1)$, $C > 1$, and $A > 1$, let $G_{\mathbb{r}}^\varepsilon = G_{\mathbb{r}}^\varepsilon(\widetilde{M}, \zeta, C, A)$ be the event that the following is true.

1. The ball $B_{\varepsilon^{-\widetilde{M}}\mathbb{r}}(0)$ can be covered by C-good Euclidean balls with radii in $[2\varepsilon\mathbb{r}, \varepsilon^{1-\zeta}\mathbb{r}]$.
2. For each $z, w \in B_{\varepsilon^{-\widetilde{M}}\mathbb{r}}(0)$ with $|z-w| \geq \varepsilon\mathbb{r}$,

$$D_h(z,w) \geq \varepsilon^A \sup_{u,v\in B_{\varepsilon^{-\widetilde{M}}\mathbb{r}}(0)} D_h(u,v). \tag{4.152}$$

By Lemmas 4.60 and 4.61, for any $\widetilde{M} > 0$ and $\zeta \in (0,1)$ we can find $C, A > 1$ for which

$$\mathbb{P}[G_{\mathbb{r}}^\varepsilon] \geq 1 - O_\varepsilon(\varepsilon^{\widetilde{M}}), \quad \text{uniformly over all } \mathbb{r} > 0. \tag{4.153}$$

Henceforth assume that $G_{\mathbb{r}}^\varepsilon$ occurs for such a choice of C, A and that $\widetilde{M} > M$.

Step 2: reducing to a bound for the number of excursions of a geodesic. Let $s > 0$ such that $\mathcal{B}_s(0; D_h) \subset B_{\varepsilon^{-M}\mathbb{r}}(0)$ and let P be a D_h-geodesic from 0 to a point outside of $\mathcal{B}_s(0; D_h)$. Let $\tau_0 = s$ and inductively for $k \in \mathbb{N}$ let τ_k be the first time t after the exit time of P from $B_{4\varepsilon^{1-\zeta}\mathbb{r}}(P(\tau_{k-1}))$ for which $P(t) \in B_{\varepsilon\mathbb{r}}(\partial\mathcal{B}_s)$, or $\tau_k = \infty$ if no such time exists. Let K be the smallest $k \in \mathbb{N}$ for which $\tau_k = \infty$.

We claim that there exists a constant $c > 0$ depending on C, A such that $K \le c \log \varepsilon^{-1}$ on $G_{\mathbb{r}}^{\varepsilon}$. If this is the case, then $P \cap B_{\varepsilon\mathbb{r}}(\partial\mathcal{B}_s)$ can be covered by at most $c \log \varepsilon^{-1}$ Euclidean balls of radius $4\varepsilon^{1-\zeta}\mathbb{r}$. This means that $\operatorname{area}(B_{\varepsilon\mathbb{r}}(P) \cap B_{\varepsilon\mathbb{r}}(\partial\mathcal{B}_s(0; D_h)) \le 4\pi\varepsilon^{2-2\zeta+o_\varepsilon(1)}\mathbb{r}^2$. Choosing $\zeta < 1/(2M)$ and sending $\widetilde{M} \to \infty$ then concludes the proof. Hence we only need to prove a logarithmic upper bound for K assuming that $G_{\mathbb{r}}^{\varepsilon}$ occurs.

Step 3: bounding excursions using C-good balls. For $k \in [0, K-1]_{\mathbb{Z}}$, we can find a C-good Euclidean ball B_k with radius in $[\varepsilon\mathbb{r}, \varepsilon^{1-\zeta}\mathbb{r}]$ which contains $P(\tau_k)$. Write $2B_k$ for the Euclidean ball with the same center as B_k and twice the radius of B_k. Let σ_k be the first time after τ_k at which P exits $2B_k$. The time σ_k is smaller than the exit time of P from $B_{4\varepsilon^{1-\zeta}\mathbb{r}}(P(\tau_k))$. Consequently, the definition of the τ_k's shows that $\sigma_k \in [\tau_k, \tau_{k+1}]$ for each $k \in [0, K]_{\mathbb{Z}}$.

Since P is a D_h-geodesic and P crosses the annulus $(2B_k) \setminus B_k$ between times τ_k and σ_k,

$$\sigma_k - \tau_k \ge D_h(\partial B_k, \partial(2B_k)). \tag{4.154}$$

We now argue that

$$\tau_k \le s + C D_h(\partial B_k, \partial(2B_k)). \tag{4.155}$$

Indeed, since B_k intersects $B_{\varepsilon\mathbb{r}}(\partial\mathcal{B}_s(0; D_h))$ and has radius at least $2\varepsilon\mathbb{r}$, it follows that B_k intersects $\partial\mathcal{B}_s(0; D_h)$. Let $z \in \partial\mathcal{B}_s(0; D_h)$ and let $t \in [\tau_k, \sigma_k]$ such that $P(t) \in \partial B_k$ (such a t exists by the definition of σ_k). By the definition of a C-good ball, the D_h-diameter of ∂B_k is at most $C D_h(\partial B_k, \partial(2B_k))$. Hence

$$\tau_k \le t \le D_h(0, z) + D_h(z, P(t)) \le s + C D_h(\partial B_k, \partial(2B_k)),$$

which is (4.155).

By (4.154) and (4.155) and the fact that the intervals $[\tau_k, \sigma_k] \subset [s, \infty)$ are disjoint, we get

$$\sum_{j=0}^{k-1} (\sigma_j - \tau_j) \le \tau_k - s \le C(\sigma_k - \tau_k).$$

This holds for each $k \in [0, K-1]_{\mathbb{Z}}$, from which we infer that

$$\sigma_{K-1} - \tau_{K-1} \geq C^{-1}(1 + C^{-1})^K (\sigma_0 - \tau_0). \tag{4.156}$$

By the definition of σ_0, we have $|P(\sigma_0) - P(\tau_0)| = \varepsilon\mathbb{r}$. Moreover, since $P(\tau_{K-1}) \in B_{\varepsilon\mathbb{r}}(B_s(0; D_h))$, $B_s(0; D_h) \subset B_{\varepsilon^{-M}\mathbb{r}}(0)$, and $\widetilde{M} > M$, we have $P(\sigma_{K-1}), P(\tau_{K-1}) \in B_{\varepsilon^{-\widetilde{M}}\mathbb{r}}(0)$. By (4.152) in the definition of $G_{\mathbb{r}}^{\varepsilon}$, it follows that

$$\sigma_0 - \tau_0 \geq \varepsilon^A (\sigma_{K-1} - \tau_{K-1}). \tag{4.157}$$

Combining this with (4.156) shows that $C^{-1}(1 + C^{-1})^K \leq \varepsilon^{-A}$, so $K \leq \frac{A}{\log(1+C^{-1})} \log \varepsilon^{-1} + O_\varepsilon(1)$, as required. $\square$

Chapter 5: Volume of metric balls in Liouville quantum gravity

This chapter is a joint work [5] with Morris Ang and Xin Sun.

5.1 Introduction

In this chapter, we study the volume of metric balls in Liouville quantum gravity (LQG). For $\gamma \in (0,2)$, it has been known since the early work of Kahane [67] and Molchan [87] that the LQG volume of Euclidean balls has finite moments exactly for $p \in (-\infty, 4/\gamma^2)$. Here, we prove that the LQG volume of LQG metric balls admits all finite moments. This answers a question of Gwynne and Miller and generalizes a result obtained by Le Gall for the Brownian map, namely, the $\gamma = \sqrt{8/3}$ case. We use this moment bound to show that on a compact set the volume of metric balls of size r is given by $r^{d_\gamma + o_r(1)}$, where d_γ is the dimension of the LQG metric space. Using similar techniques, we prove analogous results for the first exit time of Liouville Brownian motion from a metric ball. Gwynne, Miller and Sheffield [60] proved that the metric measure space structure of γ-LQG a.s. determines its conformal structure when $\gamma = \sqrt{8/3}$; their argument and our estimate yield the result for all $\gamma \in (0,2)$.

Let us now give a precise formulation of our results. The main result of this chapter is the following theorem concerning the volume of metric balls.

Theorem 5.1. *Fix $\gamma \in (0,2)$ and let h be a whole-plane GFF normalized to have average zero on the unit circle. Let $\mathcal{B}_s(z; D_h)$ be the D_h-ball of radius s centered at z. Then*

$$\mathbb{E}\left[\mu_h(\mathcal{B}_1(0; D_h))^p\right] < \infty \quad \textit{for all } p \in \mathbb{R}. \tag{1.1}$$

Moreover, for any compact set $K \subset \mathbb{C}$ and $\varepsilon > 0$, we have almost surely that

$$\sup_{s\in(0,1)} \sup_{z\in K} \frac{\mu_h(\mathcal{B}_s(z; D_h))}{s^{d_\gamma - \varepsilon}} < \infty \quad \textit{and} \quad \inf_{s\in(0,1)} \inf_{z\in K} \frac{\mu_h(\mathcal{B}_s(z; D_h))}{s^{d_\gamma + \varepsilon}} > 0. \tag{1.2}$$

Consequently, the Minkowski dimension of γ-LQG is d_γ almost surely.

This result is in stark contrast to the LQG volume of a deterministic bounded open set, which only has finite moments for $p \in (-\infty, 4/\gamma^2)$. Roughly speaking, $\mu_h(\mathcal{B}_1(0; D_h))$ has finite positive moments because the metric ball $\mathcal{B}_1(0; D_h)$ in some sense *avoids* regions where h (and thus μ_h) is large. Our arguments also show (1.1) when we replace h by $h + \alpha \log|\cdot|^{-1}$ for $\alpha < \frac{\gamma}{2} + \frac{2}{\gamma}$ (see Propositions 5.8 and 5.26).

Similar arguments allow us to prove an analogous result for the first exit time of the Liouville Brownian motion (LBM) from metric balls. Classically, Brownian motion is well defined on smooth manifolds and on some random fractals. Formally, LBM is Brownian motion associated to the metric tensor "$e^{\gamma h}(dx^2 + dy^2)$", and can be rigorously constructed via regularization and renormalization [11, 49]. It is a time-change of an ordinary Brownian motion independent of h. For a set $X \subset \mathbb{C}$ and $z \in \mathbb{C}$, denote by $\tau_h(z; X)$ the first exit time of the Liouville Brownian motion started at z from the set X. When X is a deterministic bounded open set, $\tau_h(z; X)$ has finite moments for $p \in (-\infty, 4/\gamma^2)$. Here, we study the case where X is given by a metric ball.

Theorem 5.2. *Fix $\gamma \in (0, 2)$ and let h be a whole-plane GFF normalized to have average zero on the unit circle. Then*

$$\mathbb{E}\left[\tau_h(0; \mathcal{B}_1(0; D_h))^p\right] < \infty \quad \text{for all } p \in \mathbb{R}.$$

Moreover, for any compact set $K \subset \mathbb{C}$ and $\varepsilon > 0$, we have at a rate uniform in $z \in K$ that

$$\lim_{s \to 0} \mathbb{P}[\tau_h(z; \mathcal{B}_s(z; D_h)) \in (s^{d_\gamma + \varepsilon}, s^{d_\gamma - \varepsilon})] = 1.$$

As an application of Theorem 5.1, we can extend results of [60] to the case of general $\gamma \in (0, 2)$. The following theorem resolves another question of [59].

Theorem 5.3. *Let $\gamma \in (0, 2)$ and h be a whole-plane GFF h normalized to have average zero on the unit circle. Then the field h up to rotation and scaling of the complex plane is almost surely determined by (i.e. measurable with respect to) the random pointed metric measure space $(\mathbb{C}, 0, D_h, \mu_h)$.*

We emphasize that the input is $(\mathbb{C}, 0, D_h, \mu_h)$ as a *pointed metric measure space*, so in particular we forget the exact parametrization in the complex plane of D_h and μ_h. More precisely we view it as an element in the space of pointed metric measure spaces endowed with the local Gromov-Hausdorff-Prokhorov topology (local here refers to metric balls about the point). For the special case $\gamma = \sqrt{8/3}$, [60] proves an analogous theorem for the quantum disk (see also [82]). Their results depend on the correspondence between the Brownian map and $\sqrt{8/3}$-LQG [81–83, 86], and rely on the estimates obtained by Le Gall [74] for the Brownian map. Theorem 5.1 provides the estimates needed to generalize the results of [60] to *all* $\gamma \in (0,2)$, yielding Theorem 5.3 and a statement of the convergence of the simple random walk on a Poisson-Voronoi tessellation of γ-LQG to Brownian motion (viewed as curves modulo time-parametrization) in the quenched sense; see Section 5.5.3.

Chapter outline. In Section 5.2, we discuss preliminary material about LQG. We prove the finiteness of moments statement of Theorem 5.1 in Sections 5.3 and 5.4, which bound the positive and negative moments of the unit LQG ball volume respectively. In Section 5.5.1, we complete the proof of Theorem 5.1. Section 5.5.2 addresses Theorem 5.2. Finally Section 5.5.3 discusses Theorem 5.3. In the appendix, we recollect some ingredients of the proof by Le Gall for the Brownian map case as a comparison.

5.2 Background and preliminaries

5.2.1 Notation

We write $\mathbb{N} = \{1, 2, 3 \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $x \in \mathbb{R}$, $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the floor and ceiling functions evaluated at x. We write $|E|$ for the cardinality of a finite set E. If f is a function from a set X to $\mathbb{R}^n$ for some $n \geq 1$, we denote the supremum norm of f by $\|f\|_X := \sup_{x \in X} |f(x)|$.

In our arguments, it is natural to consider both Euclidean balls and metric balls. We use the notation $B_r(z)$ to denote the *Euclidean* ball of radius r centered at z, and $\mathcal{B}_r(z; D_h)$ to denote the *metric* ball of radius r centered at z (i.e. the ball with respect to the metric D_h). We also distinguish the unit disk $\mathbb{D} := B_1(0)$. We denote by $\overline{X}$ the closure of a set X. For any $r > 0$ and $z \in \mathbb{C}$, let $A_r(z)$ stand for the annulus $B_r(z) \setminus \overline{B_{r/2}(z)}$. Furthermore, for $0 < s < r$, we set $A_{s,r}(z) := B_r(z) \setminus \overline{B_s(z)}$.

The LQG metric D_h is almost surely a length metric, i.e. $D_h(z, w)$ is the infimum of the

D_h-lengths of continuous paths between z, w. For an open set $U \subset \mathbb{C}$, the internal metric D_h^U on U is given by the infimum of the D_h-lengths of continuous paths in U.

We write $\fint_C f$ for the average of f over the circle C. For a GFF h, we write $h_r(z)$ for the average of h on the circle $\partial B_r(z)$.

We write $X \sim \mathcal{N}(m, \sigma^2)$ to express that the random variable X is distributed according to a Gaussian probability measure with mean m and variance σ^2.

We say that an event E_ε, depending on ε, occurs with superpolynomially high probability if for every fixed $p > 0$, for all ε small enough, $\mathbb{P}[E_\varepsilon] \geq 1 - \varepsilon^p$. We similarly define events which occur with superpolynomially high probability as a parameter tends to ∞.

5.2.2 The whole-plane Gaussian free field

We give here a brief introduction to the whole-plane GFF. For more details see [84].

Let H be the Hilbert space closure of smooth compactly supported functions f on $\mathbb{C}$, equipped with the Dirichlet inner product

$$(f, g)_\nabla = (2\pi)^{-1} \int_{\mathbb{C}} \nabla f(z) \cdot \nabla g(z) \, dz.$$

Let $\{f_n\}$ be any orthonormal basis of H, and consider the equivalence relation on the space of distributions given by $T_1 \sim T_2$ when $T_1 - T_2$ is a constant. The *whole-plane GFF modulo additive constant* h is a random equivalence class of distributions, a representative of which is given by $\sum \alpha_n f_n$ where $\{\alpha_n\}$ is a sequence of i.i.d. $\mathcal{N}(0,1)$ random variables. The law of h does not depend on the choice of $\{f_n\}$.

For any complex affine transformation of the complex plane A, it is easy to verify that $(f \circ A, g \circ A)_\nabla = (f, g)_\nabla$. Consequently, h has a law that is invariant under affine transformations: for each $r, z \in \mathbb{C}$ we have $h \stackrel{d}{=} h(r \cdot + z)$.

Write $\widetilde{H} \subset H$ for the subspace of functions f with $\int_{\mathbb{C}} f = 0$. Although we cannot define $\langle h, f \rangle$ for general $f \in H$, the distributional pairing makes sense for $f \in \widetilde{H}$ (the choice of additive constant does not matter). Explicitly, for $f \in \widetilde{H}$ the pairing $\langle h, f \rangle$ is a centered Gaussian with variance

$$\mathrm{Var}(\langle h, f \rangle) = \iint_{\mathbb{C}^2} f(w) f(z) \log |w - z|^{-1} \, dw dz. \tag{2.3}$$

It is easy to check that (2.3) in fact *defines* the whole-plane GFF modulo additive constant.

We will often fix the additive constant of h, i.e. choose an equivalence class representative. This can be done by specifying the value of $\langle h, f \rangle$ for some $f \in H$ with $\int_{\mathbb{C}} f \neq 0$, or the average of h on a circle (see [44, Section 3] for details on the circle averages of h). Recalling that $h_r(z)$ means the circle average of h on $\partial B_r(z)$, we will typically work with a whole-plane GFF h normalized so $h_1(0) = 0$ (this is a distribution *not* modulo additive constant).

Let $\mathcal{H}_1 \subset H$ (resp. $\mathcal{H}_2 \subset H$) be the Hilbert space completion of compactly supported functions which are constant (resp. have mean zero) on $\partial B_r(0)$ for all $r > 0$. It is easy to verify the orthogonal decomposition $H = \mathcal{H}_1 \oplus \mathcal{H}_2$. This allows us to write the whole-plane GFF h with $h_1(0) = 0$ as the sum of independent fields h^1 and h^2; these are respectively the projections of h to $\mathcal{H}_1$ and $\mathcal{H}_2$. Moreover, we can explicitly describe the law of h^1: Writing $X_t = h_{e^{-t}}(0)$, the processes $(X_t)_{t \geq 0}$ and $(X_{-t})_{t \geq 0}$ are independent Brownian motions started at zero. The strong Markov property tells us that for any stopping time T of $(X_t)_{t \geq 0}$, the random process $(X_{s+T} - X_T)_{s \geq 0}$ is independent from X_T. Also, by the scale invariance of the whole-plane GFF, the law of h^2 is scale invariant. These observations (with the independence of h^1, h^2) give us the following.

Lemma 5.4. *Let h be a whole-plane GFF with $h_1(0) = 0$, and let $T \geq 0$ be a stopping time of the circle average process $(h_{e^{-t}}(0))_{t \geq 0}$. Then we have, as fields on $\mathbb{D}$,*

$$h(e^{-T} \cdot)|_{\mathbb{D}} - h_{e^{-T}}(0) \stackrel{d}{=} h|_{\mathbb{D}}.$$

Moreover, $h(e^{-T} \cdot)|_{\mathbb{D}} - h_{e^{-T}}(0)$ is independent of $h_{e^{-T}}(0)$.

We note that there exist variants of the GFF on bounded domains $D \subset \mathbb{C}$, such as the zero boundary GFF and the Neumann GFF; we do not go into further detail, but remark that their LQG measures (Section 1.2) are well defined.

Finally, we present a version of the Markov property for the whole-plane GFF, taken from [62, Lemma 2.2]. It essentially follows from the orthogonal decomposition $H = \mathcal{H}_{\mathbb{D}} \oplus \mathcal{H}_{\text{harm}}$ where $\mathcal{H}_{\mathbb{D}}$ (resp. $\mathcal{H}_{\text{harm}}$) is the Hilbert space completion of functions which are compactly supported (resp. harmonic) in $\mathbb{D}$.

Lemma 5.5 (Markov property of GFF)**.** *Let h be a whole-plane GFF normalized so $h_1(0) = 0$. For each open set $U \subset \mathbb{C}$ with harmonically non-trivial boundary and $U \cap \partial\mathbb{D} = \emptyset$, we have the decomposition*

$$h = \mathfrak{h} + \mathring{h}$$

where $\mathfrak{h}$ is a random distribution which is harmonic on U, and $\mathring{h}$ is independent from $\mathfrak{h}$ and has the law of a zero-boundary GFF on U (in particular, $\mathring{h}|_{U^c} \equiv 0$).

5.2.3 LQG volume of Euclidean balls

Tails estimates for the LQG volume of Euclidean balls are quite well understood. It has been known since the work of Kahane [67] and Molchan [87] that it admits finite moments for $p \in (-\infty, 4/\gamma^2)$. This result contrasts a very different behavior between the right tails and the left tails.

Negative moments The finiteness of all negative moments goes back to Molchan [87]; moreover it is more generally true that for any base measure of the GMC, the total GMC mass has negative moments of all order [48]. Duplantier and Sheffield obtained the following more explicit tail behavior [44, Lemma 4.5]: writing μ_h for the LQG measure corresponding to a zero boundary GFF h on $\mathbb{D}$, they showed that if $U \subset\subset \mathbb{D}$ is an open set, then there exists $C, c > 0$ such that for all $s > 0$,

$$\mathbb{P}\left[\mu_h(U) \leq e^{-s}\right] \leq Ce^{-cs^2}. \tag{2.4}$$

We note that this result is sharp in the sense that

$$\mathbb{P}\left[\mu_h(U) \leq e^{-s}\right] \geq ce^{-Cs^2}.$$

by a simple application of the Cameron-Martin formula. When h is replaced by $h - \fint_U h dz$, a sharper tail estimate is obtained in [71].

Positive moments Recently, Rhodes and Vargas [92] obtained a precise asymptotic result about the upper tails of GMC when $\gamma \in (0,2)$. They obtained a power law and identified the constant. This result has been generalized to a more general family of Gaussian fields in [115], and extended to the critical case $\gamma = 2$ in [114].

As already mentioned, the LQG volume of Euclidean balls has finite p moments for $p < 4/\gamma^2$. This can be easily seen for integer moments $k < 4/\gamma^2$, which we review below. (This will also serve as a preparation to some of our arguments.) Indeed, due to the logarithmic correlations of the field, the problem is essentially equivalent to the finiteness of

$$u_k := \int_{\mathbb{D}^k} \frac{dz_1, \dots dz_k}{\prod_{i<j} |z_i - z_j|^{\gamma^2}}.$$

By introducing

$$u_k(r) := \int_{r\mathbb{D}^k} \frac{dz_1, \dots dz_k}{\prod_{i<j} |z_i - z_j|^{\gamma^2}} \quad \text{and} \quad v_k(r) = \int_{\mathbb{D}^k} \frac{1_{r/2 \le \max_{i<j} |z_i - z_j| \le r}}{\prod_{i<j} |z_i - z_j|^{\gamma^2}} dz_1 \dots dz_k, \tag{2.5}$$

we note that when $u_k < \infty$ then $u_k(r) = r^{2k - \gamma^2 \frac{k(k-1)}{2}} u_k$. Furthermore, the v_k's provide the following inductive inequality, obtained by splitting the points $\{z_1, \dots, z_k\}$ into two well-separated clusters (see Lemma 5.47 in the Appendix for details):

$$v_k(r) \le C_k r^{-2} \sum_{i=1}^{k-1} r^{-\gamma^2 i(k-i)} u_i(4r) u_{k-i}(4r) \le C_k r^{k\gamma Q - \frac{1}{2}\gamma^2 k^2 - 2} \sum_{i=1}^{k-1} u_i u_{k-i}.$$

Finally, we note that

$$k\gamma Q - \frac{1}{2}\gamma^2 k^2 - 2 = k(2 + \frac{\gamma^2}{2}) - \frac{1}{2}\gamma^2 k^2 - 2 = 2(k-1) - \frac{\gamma^2}{2} k(k-1) > 0 \quad \text{if} \quad 1 < k < 4/\gamma^2,$$

and the conclusion follows from $u_k = \sum_{p \ge -1} v_k(2^{-p})$ and an induction on k.

Our later arguments in Section 5.3.1 follow a similar structure to the above, but also have to account for the random geometry of the metric ball $\mathcal{B}_1(0; D_h)$.

5.2.4 LQG metric

Recall that for $\gamma \in (0,2)$, the γ-LQG metric is the unique metric D_h determined by a field h (a whole-plane GFF plus a possibly random bounded continuous function) which induces the Euclidean topology and satisfies the following.

I. **Length space.** $(\mathbb{C}, D_h)$ is almost surely a length space. That is, the D_h-distance between any two points in $\mathbb{C}$ is the infimum of the D_h-lengths of continuous paths between the two points.

II. **Locality.** Let $U \subset \mathbb{C}$ be a deterministic open set. Then the internal metric D_h^U is almost surely determined by $h|_U$.

III. **Weyl scaling.** Recall ξ in (3.4). For each continuous function $f : \mathbb{C} \to \mathbb{R}$, define

$$(e^{\xi f} \cdot D_h)(z, w) := \inf_{P: z \to w} \int_0^{\mathrm{len}(P; D_h)} e^{\xi f(P(t))} dt, \quad \text{for all } z, w \in \mathbb{C}, \tag{2.6}$$

where we take the infimum over all continuous paths from z to w parametrized by D_h-length. Then almost surely $e^{\xi f} \cdot D_h = D_{h+f}$ for every continuous $f : \mathbb{C} \to \mathbb{R}$.

IV. **Coordinate change for translation and scaling.** Recall Q in (2.3). For fixed deterministic $z \in \mathbb{C}$ and $r > 0$ we have almost surely

$$D_h(ru + z, rv + z) = D_{h(r\cdot + z) + Q \log r}(u, v) \quad \text{for all } u, v \in \mathbb{C}.$$

To be precise, D_h is unique up to a global multiplicative constant, which can be fixed in some way, e.g. requiring the median of $D_h(0,1)$ to be 1 for h a whole-plane GFF normalized so $h_1(0) = 0$. We emphasize that the metric D_h depends on the parameter $\gamma \in (0,2)$; to follow previous works and avoid clutter we will omit γ in the notation.

Basic estimates for distances The main quantitative input we need when working with the LQG metric is the following estimate relating the D_h-distance between compact sets to circle averages of h.

Proposition 5.6 (Concentration of side-to-side crossing distance [39, Proposition 3.1])**.** *Let $U \subset \mathbb{C}$ be an open set (possibly $U = \mathbb{C}$) and let $K_1, K_2 \subset U$ be disjoint connected compact sets which are not singletons. Then for $r > 0$, it holds with superpolynomially high probability as $A \to \infty$ (at a rate uniform in r) that*

$$A^{-1} r^{\xi Q} e^{\xi h_r(0)} \le D_h^{rU}(rK_1, rK_2) \le A r^{\xi Q} e^{\xi h_r(0)}.$$

This formulation is slightly different from that of [39, Proposition 3.1], but by [39, Remark 3.16] they are equivalent. Note that by taking $r = 1$, this includes the superpolynomial tails of side-to-side crossing distances.

Euclidean balls within LQG balls The next lemma is an important input in the proof of the finiteness of the negative moments.

Proposition 5.7 (LQG balls contain Euclidean balls of comparable diameter [60, Proposition 4.5])**.** *Fix $\zeta \in (0,1)$ and compact $K \subset \mathbb{C}$. Let h be a whole-plane GFF normalized so $h_1(0) = 0$. With superpolynomially high probability as $\delta \to 0$, each D_h-metric ball $B \subset K$ with* $\operatorname{diam}(B) \le \delta$ *contains a Euclidean ball of radius at least* $\operatorname{diam}(B)^{1+\zeta}$*.*

Proof. [60, Proposition 4.5] gives this result with K replaced by $\mathbb{D}$ and with the specific choice $\gamma = \sqrt{8/3}$. To get the result for K, we simply note that the law of the whole-plane GFF (viewed modulo additive constant) is scale-invariant, and that the set of all D_h-metric balls (viewed as subsets of $\mathbb{C}$) does not depend on the choice of additive constant. To generalize to $\gamma \in (0,2)$, we remark that the proof of [60, Proposition 4.5] uses only the following few inputs for the $\sqrt{8/3}$ LQG metric, which we ascertain hold for general γ:

- The scaling relation [60, Lemma 2.3]. In our setting, this is Axiom III (Weyl scaling), plus the following easy consequence of Weyl scaling: for h a whole-plane GFF plus a bounded continuous function and $f : \mathbb{C} \to \mathbb{R}$ a (possibly random) bounded continuous function, almost surely

$$\exp\left(\xi \inf_{\mathbb{C}} f\right) D_h(z,w) \le D_{h+f}(z,w) \le \exp\left(\xi \sup_{\mathbb{C}} f\right) D_h(z,w) \quad \text{for all } z, w \in \mathbb{C}.$$

- With probability tending to 1 as $C \to \infty$, the D_h-distance from $S = [0,1]^2$ to $\partial B_{1/2}(S)$ is at least $1/C$ (here, $B_{1/2}(S)$ is the Euclidean 1/2-neighborhood of S). This follows immediately from Proposition 5.6.

- Fix $n \geq 1$. With probability tending to 1 as $C \to \infty$, each Euclidean ball of radius $e^{-Cn^{2/3}}$ which intersects $[0,1]^2$ has D_h-diameter at most $e^{-n^{2/3}}$. This follows from the fact that D_h is a.s. bi-Hölder with respect to the Euclidean metric [39, Theorem 1.7], and that $e^{-Cn^{2/3}} \to 0$ as $C \to \infty$.

□

We point out that this is possible to obtain a more quantative version of this Proposition, with essentially the same arguments as in [60], which can then be used to obtain more precise lower tail estimates for the volume of LQG metric balls.

5.3 Positive moments

The main result of this section is the following.

Proposition 5.8. *Let h be a whole-plane GFF such that $h_1(0) = 0$. Then, $\mu_h(\mathcal{B}_1(0; D_h))$ has finite kth moments for all $k \geq 1$. Furthermore, this result still holds if we add to the field h an α-*log *singularity at the origin for $\alpha < Q$, i.e. replace h with $h + \alpha \log|\cdot|^{-1}$.*

In the following paragraphs, we present heuristic arguments and an outline of the proof. Recall the definition of the annulus $A_1 = B_1(0)\backslash\overline{B_{1/2}(0)}$. The key difficulty to prove this result is in arguing that $\mathbb{E}[\mu_h(\mathcal{B}_1(0; D_h) \cap A_1)^k] < \infty$. So we want to prove

$$\mathbb{E}\left[\int_{(A_1)^k} \prod_{i=1}^{k} 1_{D_h(0,z_i)<1} \mu_h(dz_1)\dots\mu_h(dz_k)\right] < \infty, \tag{3.7}$$

and the starting point is to rewrite it via a Cameron-Martin shift, as

$$\int_{(A_1)^k} \exp(\gamma^2 \sum_{i<j} \mathrm{Cov}(h(z_i), h(z_j)))\mathbb{P}\left[D_{h+\gamma\sum_j \mathrm{Cov}(h(z_j),h(\cdot))}(0, z_i) < 1, \forall i\right] dz_1 \dots dz_k < \infty. \tag{3.8}$$

A first heuristic We present a heuristic explaining why $\mathbb{E}\left[\mu_h(\mathcal{B}_1(0;D_h)\cap A_1)^k\right] < \infty$. As remarked above and since h is log-correlated, the left-hand side of (3.7) is bounded from above by

$$\int_{A_1^k} \frac{P_{z_1,\dots,z_k}}{\prod_{i<j}|z_i - z_j|^{\gamma^2}} dz_1 \dots dz_k \tag{3.9}$$

where

$$P_{z_1,\dots,z_k} = \mathbb{P}[D_{h+\gamma\sum_j \mathrm{Cov}(h(z_j),h(\cdot))}(z_i, \partial B_{1/2}(z_i)) < 1 \text{ for all } i].$$

The volume of Euclidean balls have infinite kth moments when k is large due to the contribution of clusters at mutual distance r (collection of points in the domain whose pairwise distance are between cr and Cr). Indeed, for such clusters $\{z_1,\dots,z_k\}$, the singularities contributes as $\prod_{i<j}|z_i - z_j|^{-\gamma^2} \approx r^{-\binom{k}{2}\gamma^2}$, on a macroscopic domain, we have r^{-2} possibilities for placing this cluster and the volume associated is r^{2k}. The total contribution is then $r^{-2+2k-\binom{k}{2}\gamma^2}$ and the sum over dyadic r is finite if and only if $k < 4/\gamma^2$. Now, we explain how this is counterbalanced by the $P_{z_1,\dots,z_k}$ term when $k \geq 4/\gamma^2$. By the annulus crossing distance bound from Proposition 5.6, for any $z \in K = \{z_1,\dots,z_k\}$, the following lower bound holds

$$D_{h+\gamma\sum_{i\leq k}\log|\cdot - z_i|^{-1}}(z, \partial B_{1/2}(z)) \gtrsim r^{\xi Q} e^{\xi h_r(z)} r^{-\xi k\gamma}.$$

Indeed, one can use an annulus centered at z, separating z from $\partial B_{1/2}(z)$ and at distance r of z, whose width is of the same order. Then, we see that the circle average of the log-singularity gives the $r^{-\xi k\gamma}$ term. So, by the condition defining $P_{z_1,\dots,z_k}$, on the associated event, for $z \in \{z_1,\dots,z_k\}$,

$$1 \gtrsim r^{\xi Q} e^{\xi h_r(z)} r^{-\xi k\gamma}.$$

By a Gaussian tail estimate, introducing the term $c_k = k\gamma - Q \geq \frac{4}{\gamma^2}\gamma - Q = 2/\gamma - \gamma/2 > 0$, we have

$$P_{z_1,\dots,z_k} \lesssim \mathbb{P}\left[h_r(z) \leq -c_k \log r^{-1}\right] \approx r^{\frac{1}{2}c_k^2}.$$

An elementary computation, namely $-2 + 2k - \binom{k}{2}\gamma^2 + \frac{1}{2}c_k^2 = \frac{1}{2}Q^2 - 2$, gives then that for such a cluster, the scale r contribution to (3.9) is $r^{\frac{1}{2}Q^2 - 2}$, which is summable for all k since $Q = \frac{\gamma}{2} + \frac{2}{\gamma} > 2$ for $\gamma \in (0,2)$ and this is essentially the reason of the finiteness of all moment.

Outline of the proof To turn this argument into a proof requires us to take care of all configurations of clusters $K = \{z_1, \ldots, z_k\}$. Similarly to the one presented in Section 5.2.3, our proof works by induction on k. We will partition $K = \{z_1, \ldots, z_k\}$ into two clusters I and J such that the pairwise distance of points between I and J is $\geq r$, since both $\prod_{i<j} |z_i - z_j|^{\gamma^2}$ and $P_{z_1,\ldots,z_k}$ have a nice hierarchical clusters structure (see (3.15) for the exact splitting procedure partitioning $K = I \cup J$ and the definition of r). Indeed, for such a cluster, we can bound from above

$$\prod_{i<j} |z_i - z_j|^{-\gamma^2} \lesssim r^{-|I||J|\gamma^2} \prod_I |z_a - z_b|^{-\gamma^2} \prod_J |z_a - z_b|^{-\gamma^2}. \tag{3.10}$$

Now, we discuss $P_{z_1,\ldots,z_k}$. The aforementioned annuli crossing distance bounds from Proposition 5.6 imply that for all $z \in K$, $\varepsilon \in (0, 1/2)$,

$$h_\varepsilon(z) + \gamma \sum_{z_a \in K} \fint_{\partial B_\varepsilon(z)} \log |\cdot - z_a|^{-1} + x \leq Q \log \varepsilon^{-1}, \tag{3.11}$$

for $x = 0$. From now, denote by $\widehat{P}^x_{z_1,\ldots,z_k}$ the circle average variant of $P_{z_1,\ldots,z_k}$ associated with (3.11): this is the probability that (3.11) holds for every $z \in K = \{z_1, \ldots, z_k\}$ and $\varepsilon \in (0, 1/2)$, with this extra parameter $x \in \mathbb{R}$, which is necessary to consider when deriving an inductive inequality. Note that when I and J are at distance of order r and the diameters of both I and J are smaller than $O(r)$, for $\varepsilon \in (0, r)$, then $\forall z, z_a \in K$ and $\forall z_i \in I, z_j \in J$,

$$\fint_{\partial B_r(z)} \log |\cdot - z_a|^{-1} \approx \log r^{-1} \quad \text{and} \quad \fint_{\partial B_\varepsilon(z_i)} \log |\cdot - z_j|^{-1} \approx \log r^{-1}.$$

Therefore, we can rewrite the condition (3.11) for $z \in I$ as follows

$$\begin{aligned}(h_\varepsilon(z) - h_r(z)) + \left(\gamma \sum_{z_i \in I} \fint_{\partial B_\varepsilon(z)} \log |\cdot - z_i|^{-1} + |J|\gamma \log r^{-1}\right) - k\gamma \log r^{-1} \\ + \left(x + h_r(z) + k\gamma \log r^{-1} - Q \log r^{-1}\right) \leq Q \log(\varepsilon/r)^{-1}.\end{aligned}$$

Hence, after simplification, for $z \in I$, we have

$$(h_\varepsilon(z) - h_r(z)) + \gamma \sum_{z_i \in I} \fint_{\partial B_\varepsilon(z)} \log |\cdot/r - z_i/r|^{-1} + \left(x + h_r(z) + c_k \log r^{-1}\right) \leq Q \log(\varepsilon/r)^{-1}$$

which is a variant of (3.11), and a similar condition holds for $z \in J$. Furthermore, note that the processes $((h_\varepsilon(z) - h_r(z))_{\varepsilon \in (0,r)})_{z \in I}$ and $((h_\varepsilon(z) - h_r(z))_{\varepsilon \in (0,r)})_{z \in J}$ are approximately independent and $h_r(z) \approx h_r(w)$ for all $z, w \in K$, which we then denote by X_r (this can thought as their common approximate value; to be rigorous, by monotonicity, one can take their maximum). From this, and the fact that circle average processes evolve as correlated Brownian motions, it is natural to expect

$$\widehat{P}_K^0 \lesssim \mathbb{E}\left[1_{X_r + c_k \log r^{-1} \le 0} \widehat{P}_{I/r}^{x + X_r + c_k \log r^{-1}} \widehat{P}_{J/r}^{x + X_r + c_k \log r^{-1}}\right], \tag{3.12}$$

which is the hierarchical structure we were looking for. Altogether, (3.10) and (3.12) allow to inductively bound from above the term

$$\int_{A_1^k} \frac{P^x_{z_1,\ldots,z_k}}{\prod_{i<j} |z_i - z_j|^{\gamma^2}} dz_1 \ldots dz_k,$$

by a quantitative estimate in term of x. This provides not only $\mathbb{E}[\mu_h(\mathcal{B}_1(0; D_h) \cap A_1)^k] < \infty$ but also a quantitative estimate which allows to get $\mathbb{E}[\mu_h(\mathcal{B}_1(0; D_h) \cap A_s^k] < s^{\alpha_k}$ for some $\alpha_k > 0$ and all $s \in (0,1)$, via a standard scaling/decoupling argument. An application of Hölder's inequality shows $\mathbb{E}[\mu_h(\mathcal{B}_1(0; D_h) \cap \mathbb{D})^k] < \infty$ and similar techniques concludes that $\mathbb{E}[\mu_h(\mathcal{B}_1(0; D_h) \cap \mathbb{C} \setminus \mathbb{D})^k] < \infty$, yielding the proof of Proposition 5.8.

In our implementation of these ideas, because we have to carry the Euclidean domains associated with the clusters I, J and K, we use $\star$-scale invariant fields. The short-range correlation of the fine field gives independence between well-separated clusters, and invariance properties of the $\star$-scale invariant field simplifies our multiscale analysis.

In Section 5.3.1, we prove a quantitative variant of (3.8) where the field h is replaced by a $\star$-scale invariant field plus some constant, and the probability in the integrand is replaced by the probability of *coarse-field distance approximations* being less than 1. In Section 5.3.2, we use these estimates to first bound $\mathbb{E}[\mu_h(\mathcal{B}_1(0; D_h) \cap A_1)^k]$, by using a truncated moment estimate, then extend our arguments to all annuli to deduce the finiteness of the kth moment $M_k := \mathbb{E}[\mu_h(\mathcal{B}_1(0; D_h)^k]$ for all $k \ge 1$. By keeping track of the k dependence, it turns out that it is possible to bound M_k by Ck^{ck^2} for some constants C, c depending only on γ. To simplify the presentation of our arguments, we omit these precise estimates.

5.3.1 Inductive estimate for the $\star$-scale invariant field

We derive a key estimate for the positive moments (Proposition 5.15), which is like a quantitative version of (3.8) where we add a constant to the field. We will use $\star$-scale invariant fields, which satisfy properties convenient for multiscale analysis. Relevant references are [3, 38, 66].

Proposition 5.9 ($\star$-scale decomposition of h). *The whole plane GFF h normalized so $h_1(0) = 0$ can be written as*

$$h = g + \phi = g + \phi_1 + \phi_2 + \dots$$

where the fields $g, \phi_1, \phi_2, \dots$ satisfy the following properties:

1. *g and the ϕ_n's are continuous centered Gaussian fields.*
2. *The law of ϕ_n is invariant under Euclidean isometries.*
3. *ϕ_n has finite range dependence with range of dependence e^{-n}, i.e. the restrictions of ϕ_n to regions with pairwise distance at least e^{-n} are mutually independent.*
4. *$(\phi_n(z))_{z\in\mathbb{R}^2}$ has the law of $\left(\phi_1(ze^{n-1})\right)_{z\in\mathbb{R}^2}$.*
5. *The ϕ_n's are mutually independent fields.*
6. *The covariance kernel of ϕ is $C_{0,\infty}(z,z') = -\log|z-z'| + q(z-z')$ for some smooth function q.*
7. *We have $\mathbb{E}[\phi_n(z)^2] = 1$ for all n, z.*

The convergence of this infinite sum is with respect to the weak topology on $\mathcal{S}'(\mathbb{R}^2)$.

Proof. Lemma 5.49 gives the coupling $h = g + \phi$ with g continuous. The fields ϕ_n are defined in Appendix 5.6.2, and are shown to satisfy these properties there. $\square$

Define also the field $\phi_{a,b}$ from scales a to b via

$$\phi_{a,b} := \begin{cases} \phi_{a+1} + \dots + \phi_b & \text{if } a < b \\ 0 & \text{if } a \geq b \end{cases} \tag{3.13}$$

so that $\phi = \phi_{0,\infty}$ and set, for $z, z' \in \mathbb{C}$,

$$C_{a,b}(z,z') := \mathbb{E}\left(\phi_{a,b}(z)\phi_{a,b}(z')\right). \tag{3.14}$$

We will construct a hierarchical representation of a set of points $K = \{z_1, \ldots, z_k\} \subset \mathbb{C}$. Roughly speaking, starting with K, we will iteratively split each cluster into smaller clusters that are well separated. We formalize the splitting procedure below.

Splitting procedure Define for any finite set S of points in the plane (with $|S| \geq 2$) the *separation distance* $s(S)$ to be the largest $t \geq 0$ for which we can partition $S = I \cup J$ such that $d(I,J) \geq t$, i.e.

$$s(S) := \max_{S=I\cup J, |I|,|J|\geq 1} d(I,J). \tag{3.15}$$

Define $I_S, J_S \subset S$ to be any partition of S with $d(I,J) = s(S)$. Note that if $\operatorname{diam} S$ denote the diameter of the set S, we have the following inequality

$$\frac{\operatorname{diam} S}{|S|} \leq s(S) \leq \operatorname{diam} S. \tag{3.16}$$

For the edge case where $|S| = 1$ define $s(S) = 0$.

Lemma 5.10. *For $|S| \geq 2$, we have $s(I_S), s(J_S) \leq s(S)$.*

Proof. It suffices to prove the lemma for S such that all pairwise distances in S are distinct, then continuity yields the result for general S. Suppose for the sake of contradiction that $s(J) > s(S)$, then there is a partition $J = J_1 \cup J_2$ satisfying $d(J_1, J_2) > s(S)$. Since distances are pairwise distinct, we must have $d(I, J_i) = s(S)$ and $d(I, J_{3-i}) > s(S)$ for some i. Then $d(I \cup J_i, J_{3-i}) = \min(d(I, J_{3-i}), d(J_i, J_{3-i})) > s(S)$. This contradicts the definition of $s(S)$. □

Hierarchical structure of $K = \{z_1, \ldots, z_k\}$ and definition of $T_K^a(\{\phi\})$ By iterating the splitting procedure above, we can decompose a set $K = \{z_1, \ldots, z_k\} \subset \mathbb{C}$ into a binary tree of clusters. This decomposition into hierarchical clusters is unique for Lebesgue typical points $\{z_1, \ldots, z_k\}$. Two vertices in this tree are separated by at least the separation distance of their first common ancestor. See Figure 5.1 for an illustration.

A *labeled (binary) tree* is a rooted binary tree with k leaves. For each $K = \{z_1, \dots, z_k\} \subset \mathbb{C}$, collection of fields $\{\phi\} = (\phi_n)_{n \geq 0}$, and nonnegative integer $a \leq \lceil \log s(K)^{-1} \rceil$ we will define a labeled binary tree denoted by $T_K^a(\{\phi\})$. Each internal vertex of this tree is labeled with a quadruple (S, m, ψ, η) with $S \subset K$ and $|S| \geq 2$, an integer m, and $\psi, \eta \in \mathbb{R}$, whereas each leaf is labeled with just a singleton $\{z\} \subset K$. The truncated labels (S, m) depend only on the recursive splitting procedure described above: S is one of the clusters associated with this hierarchical cluster decomposition, and $m = \lceil \log s(S)^{-1} \rceil$. The variable a represents an initial scale.

For such a labeled tree T we write $T + (\psi_0, \eta_0)$ to be the tree obtained by replacing each internal vertex label (S, m, ψ, η) with $(S, m, \psi + \psi_0, \eta + \eta_0)$. We also write $\mathrm{Left}(S)$ to denote the leftmost point of S, viz. $\arg\min_{z \in S} \Re(z)$, where $\Re(z)$ denotes the real part of the complex number z.

We explain how the remaining parts (ψ, η) of the labels are obtained. For $(K, \{\phi\}, a)$ as above, we proceed as follows to complete the definition of the labeled tree $T_K^a(\{\phi\})$. For $k := |K| = 1$, we simply set $T_K^a(\{\phi\})$ to be the tree with one vertex, labeled with the singleton K. For $k > 1$, setting $m := \lceil \log s(K)^{-1} \rceil \geq a$, the root vertex of $T_K^a(\{\phi\})$ is labeled $(K, m, \phi_{a,m}(\mathrm{Left}(K)), (m-a)k\gamma)$, and its two child subtrees are given by $T_{I_K}^m(\{\phi\}) + (\phi_{a,m}(\mathrm{Left}(K)), (m-a)k\gamma)$ and $T_{J_K}^m(\{\phi\}) + (\phi_{a,m}(\mathrm{Left}(K)), (m-a)k\gamma)$. Essentially, after making the split $K = I \cup J$, we add up the contribution of the coarse field $\phi_{a,m}$ and the contribution of the γ-log singularities to get the scale m field approximation for the clusters I and J.

We note that the tree structure of $T_K^a(\{\phi\})$ is deterministic, and for each internal vertex with label (S, m, ψ, η), only $\psi = \psi(\{\phi\})$ is random; the other components are deterministic. Roughly speaking, S is a cluster in our hierarchical decomposition, m is the scale of the cluster (i.e. $s(S) \approx e^{-m}$), ψ (resp. η) approximates a radius e^{-m} circle average of the field $\phi_{a,m}$ (resp. $\gamma \sum_{z \in K} \log |z - \cdot|^{-1} - \gamma k a$) at the cluster.

Remark 5.11. *For the labeled tree $T_K^a(\{\phi\})$, at each internal vertex the field approximation ψ can be explicitly described in terms of the fields $\{\phi\}$ as follows. Let $(S_i, m_i, \psi_i, \eta_i)$ for $i = 1, \dots, n$ be a*

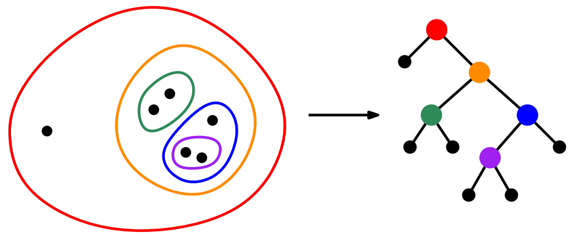

Figure 5.1 – **Left:** The set of points K is iteratively divided into smaller and smaller clusters. **Right:** From this clustering algorithm we obtain a hierarchical binary tree $T_K^a(\{\phi\})$ (labels not shown), where internal vertices correspond to clusters $S \subset K$ and leaves correspond to points $z \in K$.

path from the root $(S_1, m_1, \psi_1, \eta_1)$*. Then, writing* $m_0 = a$*, we have*

$$\psi_n = \sum_{i=1}^{n} \phi_{m_{i-1},m_i}(\mathrm{Left}(S_i)). \tag{3.17}$$

The γ*-singularity approximation* η *can likewise be stated non-recursively, as*

$$\eta_n = \gamma \sum_{i=1}^{n} (m_i - m_{i-1})|S_i|. \tag{3.18}$$

Remark 5.12. *The choice* $\mathrm{Left}(S_i)$ *is arbitrary; any other deterministic choice of point in* S_i *works. Replacing* $\phi_{m_{i-1},m_i}(\mathrm{Left}(S_i))$ *with the average* $|S_i|^{-1} \sum_{z \in S_i} \phi_{m_{i-1},m_i}(z)$ *would also work without affecting our proofs much.*

Definitions of key observables In this paragraph, we provide analogous definitions of the quantities appearing in (3.8). The first one corresponds to a variant of $\mathbb{P}[D_{h+\gamma\sum_j \mathrm{Cov}(h(z_j),h(\cdot))}(0, z_i) < 1$ for all $i]$, with an extra parameter x. For $x \in \mathbb{R}$, let $P_K^{a,x}$ be the probability that the tree with random labels $T_K^a(\{\phi\})$ satisfies

$$\psi + \eta + x \le Q(m - a) \quad \text{for each internal vertex labeled } (S, m, \psi, \eta). \tag{3.19}$$

Note that this probability is taken over the randomness of the fields $\{\phi\}$, and that this definition yields for $|K| = 1$ that $P_K^{a,x} = 1$. Let us comment a bit on this definition and its relation with the conditions $D_{h+\gamma\sum_j \mathrm{Cov}(h(z_j),h(\cdot))}(0, z_i) < 1$. These distances being less than one implies upper bounds for annuli crossing distances for annuli separating the origin from the singularities. The ψ

term corresponds to field average over these annuli, η is an approximation for the γ-singularities and the Q term stands for the scaling of the metric. Altogether, roughly speaking, $P_K^{0,x}$ is the probability that for the field $\phi_{0,\infty} + \sum_{z\in K} \gamma \log|z - \cdot|^{-1} + x$, for all clusters S of K the field-average approximation of annulus-crossing distances near S is less than 1.

The following observable stands for a variant of the integral in (3.8). Writing $K = \{z_1, \dots, z_k\}$ and $dz_K = dz_1 \dots dz_k$, we define

$$u_k^n(x) := \int_{B_n(0)^k} \frac{P_K^{0,x}}{\prod_{i<j} |z_i - z_j|^{\gamma^2}} 1_{s(K)\le e} dz_K. \tag{3.20}$$

In Proposition 5.15, we show that $u_k^n(x) < \infty$, and bound it in terms of x. Note that the statement $u_k^n(x) < \infty$ is comparable to (3.8) by the fact that $\exp(\gamma \operatorname{Cov}(h(z_i), h(z_j))) \asymp |z_i - z_j|^{-\gamma^2}$.

The next lemma establishes basic properties of $P_K^{a,x}$. To state it, we first define

$$c_k := k\gamma - Q. \tag{3.21}$$

Lemma 5.13. *The $P_K^{a,x}$'s satisfy the following properties:*

1. *Monotonicity: $P_K^{0,x}$ is decreasing in x.*
2. *Markov decomposition: for the partition $I_K \cup J_K = K$ with separation distance satisfying $e^{-m} \le s(K) < e^{-m+1}$ we have*

$$P_K^{0,x} = \mathbb{E}\left[1_{X_r + x + c_k \log r^{-1} \le 0} P_{I_K}^{\log r^{-1}, X_r + x + c_k \log r^{-1}} P_{J_K}^{\log r^{-1}, X_r + x + c_k \log r^{-1}}\right],$$

   where $r = e^{-m}$ and $X_r = \phi_{0,m}(\mathrm{Left}(K))$ is a centered Gaussian with variance $\log r^{-1}$.
3. *Scaling: $P_{rz_1,\dots rz_k}^{\log r^{-1},x} = P_{z_1,\dots z_k}^{0,x}$ for any $r = e^{-m}$ with $m \in \mathbb{Z}$.*
4. *Invariance by translation: $P_{z_1+w,\dots,z_k+w}^{0,x} = P_{z_1,\dots,z_k}^{0,x}$.*

The first condition corresponds to a shift of the field. The second condition is an identity with three terms in the right-hand side: the term X_r represents the coarse field, the indicator says that the "coarse field approximation of quantum distances" at Euclidean scale r are less than 1, and the product of the two other terms represent a Markovian decomposition conditional on the coarse field. Properties 3 and 4 are clear from the translation invariance and scaling properties of ϕ_n.

Proof. The monotonicity Property 1 is clear from the definition.

Property 2 follows from the inductive definition of $P_K^{0,x}$, by looking at the first split $K = I \cup J$. Indeed, recall $X_r = \phi_{0,m}(\mathrm{Left}(K))$. The event $\{X_r + x + c_k \log r^{-1} \leq 0\}$ corresponds to inequality (3.19) for the root vertex $(K, m, \phi_{0,m}(\mathrm{Left}(K)), mk\gamma)$.

Then, if the set K is decomposed as $K = I \cup J$, note that the trees $T_I^m(\{\phi\})$ and $T_J^m(\{\phi\})$ are independent. Indeed, $d(I, J) \geq e^{-m}$, so the restrictions of the field ϕ_m (and each finer field) to I and J are independent. Therefore, since $(\phi_{0,m}(\mathrm{Left}(K)), T_I^m(\{\phi\}), T_J^m(\{\phi\}))$ are independent, conditionally on $\phi_{0,m}(\mathrm{Left}(K))$, the trees $T_I^m(\{\phi\}) + (\phi_{0,m}(\mathrm{Left}(K)), mk\gamma)$ and $T_J^m(\{\phi\}) + (\phi_{0,m}(\mathrm{Left}(K)), mk\gamma)$ are independent. Thus, all conditions in the definition of $P_K^{0,x}$ associated to the child subtrees are conditionally independent. To conclude, we just have to explain that this is indeed the term $P_I^{m, X_r + x + c_k m}$ which appears. For a non-root vertex (S, b, ψ, η) of $T_K^{0,x}$ belonging to the genealogy of I, the condition (3.19) can be rewritten,

$$\psi + \eta + x = (X_r + \psi') + (mk\gamma + \eta') + x \leq Qb = Q(b - m) + Qm,$$

hence $\psi' + \eta' + (X_r + x + c_k m) \leq Q(a - m)$, which is exactly the condition we were looking for at the vertex (S, b, ψ', η') in the tree $T_I^m(\{\phi\})$.

The scaling Property 3 follows from the scaling property of the ϕ_m and the observation that $s(rK) = rs(K)$ (and hence $\lceil \log s(rK)^{-1} \rceil = \log r^{-1} + \lceil \log s(K)^{-1} \rceil$).

The invariance by translation Property 4 follows from the translation invariance of the fields ϕ_m. $\square$

Using these properties, we derive the following inductive inequality.

Lemma 5.14. *For each $n, k > 0$, there exists a constant $C_{n,k}$ such that the following inductive* *inequality holds, for all $x \in \mathbb{R}$, where $X_r \sim \mathcal{N}(0, \log r^{-1})$.*

$$u_k^n(x) \leq C_{n,k} \sum_{i=1}^{k-1} \sum_{r=e^{-m}, m \geq 0} r^{k\gamma Q - \frac{1}{2}\gamma^2 k^2 - 2} \times$$
$$\mathbb{E}\left[1_{X_r + x + c_k \log r^{-1} \leq 0} u_i^{6k}(X_r + x + c_k \log r^{-1}) u_{k-i}^{6k}(X_r + x + c_k \log r^{-1})\right].$$

We now turn to the proof of the inductive relation. The argument is close to that of Lemma 5.47, the difference being that we have to take care of the decoupling of $P_K^{0,x}$.

Proof. We first introduce some notation. In what follows we will be integrating over k-tuples of points $z_1, \dots, z_k$; write K for this collection of points and $dz_K = dz_1 \dots dz_k$. Write $f(K) := \prod |z - z'|^{-\gamma^2/2}$ where the product is taken over all pairs $z, z' \in K$ with $z \neq z'$.

We first split the integral in the definition (3.20) of $u_k^n(x)$ as

$$u_k^n(x) = \sum_{r=e^{-m}, m \geq 0} v_k^n(x, r)$$

where for $r \in (0, 1]$, $v_k^n(x, r)$ is defined by

$$v_k^n(x, r) := \int_{B_n(0)^k} P_K^{0,x} f(K) 1_{r \leq s(K) \leq er} dz_K. \tag{3.22}$$

Notice that $s(K) \leq er$ implies $\operatorname{diam} K \leq ekr$, so any K contributing to the integral in (3.22) is contained in a ball of radius $6kr$ centered in $r\mathbb{Z}^2 \cap B(0, n)$. Taking a sum over the $O(n^2 r^{-2})$ such balls and by translation invariance, we get the bound

$$v_k^n(x, r) \leq O(n^2 r^{-2}) \int_{B_{6kr}(0)^k} P_K^{0,x} f(K) 1_{r \leq s(K) \leq er} dz_K.$$

Write $K = I_K \cup J_K$ for the partition described before Lemma 5.10. For $z \in I_K$ and $z' \in J_K$ we have $|z - z'|^{-\gamma^2} \leq s(K)^{-\gamma^2} \leq r^{-\gamma^2}$, and $s(I_K), s(J_K) \leq s(K) \leq er$ by Lemma 5.10, so

$$v_k^n(x, r) \leq O(n^2 r^{-2}) \int_{B_{6kr}(0)^{6k}} r^{-\gamma^2 |I_K||J_K|} P_K^{0,x} f(I_K) 1_{s(I_K) \leq er} f(J_K) 1_{s(J_K) \leq er} \, dz_K.$$

The Markov property decomposition 2 Lemma 5.13 allows us to split $P_K^{0,x}$ into an expectation over a product of terms, yielding an upper bound of $v_k^n(x,r)$ as an integral of terms which 'split' into z_{I_K} and z_{J_K} parts. This expression is in terms of the partition $I_K \cup J_K = K$; we can upper bound it by summing over *all* $I, J \subset K$. To be precise, for each $i = 1, \dots, k-1$ we sum over all pairs $I, J \subset K$ with $|I| = i$ and $|J| = k - i$. Absorbing combinatorial terms like $\binom{k}{i}$ and the prefactor n^2 into the constant $C_{n,k}$, we get

$$v_k^n(x,r) \le C_{n,k} r^{-2} \sum_{i=1}^{k-1} r^{-\gamma^2 i(k-i)} \mathbb{E}_{X_r}\left[\int_{B_{6kr}(0)^i} \frac{P_{z_1,\dots,z_i}^{\log r^{-1}, X_r + x + c_k \log r^{-1}}}{\prod_{a<b} |z_a - z_b|^{\gamma^2}} 1_{s(z_1,\dots,z_i) \le er} dz_1 \dots dz_i \right.$$
$$\left. \left(\int_{B_{6kr}(0)^{k-i}} \frac{P_{w_1,\dots,w_{k-i}}^{\log r^{-1}, X_r + x + c_k \log r^{-1}}}{\prod_{a<b} |w_a - w_b|^{\gamma^2}} 1_{s(w_1,\dots,w_{k-i}) \le er} dw_1 \dots dw_{k-i} \right) 1_{X_r + x + c_k \log r^{-1} \le 0} \right].$$

We analyze the first integral (we can deal with the second one along the same lines). Changing the domain of integration from $B_{6kr}(0)^i$ to $B_{6k}(0)^i$, we get

$$\int_{B_{6kr}(0)^i} \frac{P_{z_1,\dots,z_i}^{\log r^{-1}, X_r + x + c_k \log r^{-1}}}{\prod_{a<b} |z_a - z_b|^{\gamma^2}} 1_{s(z_1,\dots,z_i) \le er} dz_1 \dots dz_i$$
$$= r^{2i - \gamma^2 \binom{i}{2}} \int_{B_{6k}(0)^i} \frac{P_{rz_1,\dots,rz_i}^{\log r^{-1}, X_r + x + c_k \log r^{-1}}}{\prod_{a<b} |z_a - z_b|^{\gamma^2}} 1_{s(z_1,\dots,z_i) \le e} dz_1 \dots dz_i,$$

and then applying the scaling property 3 of P, the integral on the right hand side is equal to

$$\int_{B_{6k}(0)^i} \frac{P_{z_1,\dots,z_i}^{0, X_r + x + c_k \log r^{-1}}}{\prod_{a<b} |z_a - z_b|^{\gamma^2}} 1_{s(z_1,\dots,z_i) \le e} dz_1 \dots dz_i = u_i^{6k}(X_r + x + c_k \log r^{-1}).$$

By gathering the previous bounds and identities, and noting that the power of r is

$$r^{-2 - \gamma^2 i(k-i) + 2k - \gamma^2 \binom{i}{2} - \gamma^2 \binom{k-i}{2}} = r^{\gamma k Q - \frac{1}{2}\gamma^2 k^2 - 2},$$

and this completes the proof of the inductive inequality. □

Using the inductive relation and the base case, we derive the following proposition, which provides a bound on the quantity (3.20) introduced at the beginning of the section.

Proposition 5.15. *Recall that $c_k = k\gamma - Q$. For $x \in \mathbb{R}$ we have*

$$u_k^n(x) \le C_{n,k} e^{-c_k x} \quad \text{when } k \ge 4/\gamma^2,$$

and

$$u_k^n(x) \le C_{n,k} \quad \text{when } k < 4/\gamma^2,$$

where $C_{n,k}$ is a constant depending only on n, k.

Proof. We first address the case where $k < 4/\gamma^2$. In this setting, by the trivial bound $P_K^{0,x} \le 1$ we have

$$u_k^n(x) \le \int_{B_n(0)^k} \prod_{i<j} |z_i - z_j|^{-\gamma^2} \, dz_1 \dots dz_k,$$

and the right-hand side is finite by the discussion in Section 5.2.3.

Now consider $k \ge 4/\gamma^2$. We proceed inductively, assuming that the statement of the proposition has been shown for all $k' < k$. Lemma 5.14 gives us the bound

$$u_k^n(x) \le C_{n,k} \sum_{i=1}^{k-1} \sum_{r=e^{-m}, m\ge 0} r^{k\gamma Q - \frac{1}{2}\gamma^2 k^2 - 2} \times \\ \mathbb{E}\left[1_{X_r + x + c_k \log r^{-1} \le 0} u_i^{6k}(X_r + x + c_k \log r^{-1}) u_{k-i}^{6k}(X_r + x + c_k \log r^{-1}) \right], \quad (3.23)$$

where $X_r \sim \mathcal{N}(0, \log r^{-1})$. We bound each term $u_i^{6k} u_{k-i}^{6k}$ using the inductive hypothesis. We need to split into cases based on which bound of the statement of the proposition is applicable (i.e. based on the sizes of $i, k-i$), but the different cases are almost identical, so we present the first case in detail and simply record the computation for the remaining cases.

Case 1: $i, k - i \ge 4/\gamma^2$. By the inductive hypothesis we can bound the ith term of (3.23) by a constant times

$$\sum_{r=e^{-m}, m\ge 0} r^{k\gamma Q - \frac{1}{2}\gamma^2 k^2 - 2} \mathbb{E}\left[e^{-(c_i + c_{k-i})(X_r + x + c_k \log r^{-1})} 1_{X_r + x + c_k \log r^{-1} \le 0} \right] \\ = \sum_{r=e^{-m}, m\ge 0} r^{k\gamma Q - \frac{1}{2}\gamma^2 k^2 - 2 + c_k(c_k - Q)} e^{(Q - c_k)x} \mathbb{E}\left[e^{-(c_k - Q)X_r} 1_{X_r + x + c_k \log r^{-1} \le 0} \right], \quad (3.24)$$

where we have used the identity $c_i + c_{k-i} = c_k - Q$. For each r we can write the expectation in the equation (3.24) by a Cameron-Martin shift as

$$\mathbb{E}[e^{-(c_k-Q)X_r}]\mathbb{P}[X_r + x + c_k \log r^{-1} - (c_k - Q)\operatorname{Var}(X_r) \le 0] \\ = r^{-\frac{1}{2}(c_k-Q)^2}\mathbb{P}[X_r \le -(Q\log r^{-1} + x)]. \quad (3.25)$$

We claim that

$$\mathbb{P}[X_r \le -(Q\log r^{-1} + x)] \le r^{\frac{1}{2}Q^2}e^{-Qx}. \quad (3.26)$$

Indeed, in the case where $Q\log r^{-1} + x \ge 0$, we have by a standard Gaussian tail bound that

$$\mathbb{P}[X_r \le -(Q\log r^{-1} + x)] \le e^{-\frac{(Q\log r^{-1}+x)^2}{2\log r^{-1}}} = r^{\frac{1}{2}Q^2}e^{-Qx}e^{-\frac{x^2}{2\log r^{-1}}} \le r^{\frac{1}{2}Q^2}e^{-Qx},$$

and in the cases where $Q\log r^{-1} + x < 0$ we have

$$\mathbb{P}[X_r \le -(Q\log r^{-1} + x)] \le 1 \le e^{-Q(Q\log r^{-1}+x)} = r^{Q^2}e^{-Qx} \le r^{\frac{1}{2}Q^2}e^{-Qx}.$$

Finally, we combine (3.24), (3.25) and (3.26) to upper bound the ith term of (3.23). This upper bound is a sum over r of terms of the form $r^{\text{power}}e^{-c_k x}$ where the power is

$$k\gamma Q - \frac{1}{2}\gamma^2k^2 - 2 + c_k(c_k - Q) - \frac{1}{2}(c_k - Q)^2 + \frac{1}{2}Q^2 = \frac{1}{2}Q^2 - 2 > 0.$$

So we can bound the ith term of (3.23) by a constant times

$$\sum_{r=e^{-m}, m\ge 0} r^{\frac{Q^2}{2}-2}e^{-c_k x} = O(e^{-c_k x}).$$

Case 2: $i \ge 4/\gamma^2$ and $k - i < 4/\gamma^2$. By the inductive hypothesis we can bound the ith term of (3.23) by a constant times

$$\begin{aligned}
&\sum_{r=e^{-m},m\geq 0} r^{k\gamma Q-\frac{1}{2}\gamma^2k^2-2}\mathbb{E}\left[e^{-c_i(X_r+x+c_k\log r^{-1})}1_{X_r+x+c_k\log r^{-1}\leq 0}\right]\\
&=\sum_{r=e^{-m},m\geq 0} r^{k\gamma Q-\frac{1}{2}\gamma^2k^2-2+c_ic_k}e^{-c_ix}\mathbb{E}\left[e^{-c_iX_r}1_{X_r+x+c_k\log r^{-1}\leq 0}\right]\\
&=\sum_{r=e^{-m},m\geq 0} r^{k\gamma Q-\frac{1}{2}\gamma^2k^2-2+c_ic_k-\frac{1}{2}c_i^2}e^{-c_ix}\mathbb{P}\left[X_r\leq -((c_k-c_i)\log r^{-1}+x)\right]\\
&\leq\sum_{r=e^{-m},m\geq 0} r^{k\gamma Q-\frac{1}{2}\gamma^2k^2-2+c_ic_k-\frac{1}{2}c_i^2+\frac{1}{2}(c_k-c_i)^2}e^{-c_kx}\\
&=\sum_{r=e^{-m},m\geq 0} r^{\frac{1}{2}Q^2-2}e^{-c_kx}=O(e^{-c_kx}).
\end{aligned}$$

Note that by symmetry Case 2 also settles the case where $i<4/\gamma^2$ and $k-i\geq 4/\gamma^2$.

Case 3: $i,k-i<4/\gamma^2$**.** By the inductive hypothesis we can bound the ith term of (3.23) by a constant times

$$\begin{aligned}
\sum_{r=e^{-m},m\geq 0} r^{k\gamma Q-\frac{1}{2}\gamma^2k^2-2}\mathbb{P}\left[X_r\leq -(c_k\log r^{-1}+x)\right]&\leq \sum_{r=e^{-m},m\geq 0} r^{k\gamma Q-\frac{1}{2}\gamma^2k^2-2+\frac{1}{2}c_k^2}e^{-c_kx}\\
&=\sum_{r=e^{-m},m\geq 0} r^{\frac{1}{2}Q^2-2}e^{-c_kx}=O(e^{-c_kx}).
\end{aligned}$$

This completes the proof. □

The proof of Proposition 5.15 depends on the exponent $\frac{1}{2}Q^2-2=\frac{1}{2}(\frac{2}{\gamma}-\frac{\gamma}{2})^2$ being positive. If we make a slight perturbation to our definitions, so long as the resulting exponent is still positive, we get a variant of Proposition 5.15. In particular, for $\delta>0$, we define $P_K^{a,x,\delta}$ similarly to $P_K^{a,x}$ by replacing the inequality (3.19) with $\psi+\eta+x\leq(Q+\delta)(m-a)$, and define $u_k^{n,\delta}$ analogously to (3.20) with $P_K^{0,x,\delta}$. We record the following result as a corollary since the proof follows the same steps as in the proof of Proposition 5.15.

Corollary 5.16. *For $k\geq 1$ and $n\geq 1$, for δ small enough, there exist constants $C_{n,k,\delta}$ and $c_{k,\delta}$ such that,*

$$u_k^{n,\delta}(x)\leq C_{n,k,\delta}e^{-c_{k,\delta}x}\quad \textit{for all } x\in\mathbb{R} \textit{ when } k\geq 4/\gamma^2,$$

and

$$u_k^{n,\delta}(x) \le C_{n,k,\delta} \qquad \textit{for all } x \in \mathbb{R} \textit{ when } k < 4/\gamma^2.$$

Furthermore, $\lim_{\delta \to 0} c_{k,\delta} = k\gamma - Q$ *for fixed* k.

Remark 5.17. *Alternatively, one could modify the definition of* $u_k^n(x)$ *in* (3.20) *to have a different denominator* $|z_i - z_j|^{\gamma^2+\delta}$. *Namely, by setting*

$$\hat{u}_k^{n,\delta}(x) := \int_{B_n(0)^k} \frac{P_K^{0,x}}{\prod_{i<j} |z_i - z_j|^{\gamma^2+\delta}} 1_{s(K) \le e} dz_K,$$

the statement of Corollary 5.16 applies to $\hat{u}_k^{n,\delta}(x)$ *instead of* $u_k^{n,\delta}(x)$.

5.3.2 Moment bounds for the whole-plane GFF

In this section, we use our previous estimate (Proposition 5.15 or its variant Corollary 5.16) to obtain the moment bounds for a whole-plane GFF h such normalized such that $h_1(0) = 0$ and therefore prove Proposition 5.8. Additionally, in this section we write C or $C_{k,\delta}$ to represent large constants depending only on k and δ, and may not necessarily represent the same constant in different contexts or equations.

Proxy estimate for whole-plane GFF

Recall the notation $A_{s,r} := B_r(0) \setminus \overline{B_s(0)}$ for $0 < s < r$. We introduce the following proxy

$$P_h^{r,d} := \{z \in \mathbb{C} : D_h(z, \partial B_{r/4}(z)) \le d\}. \tag{3.27}$$

The set $P_h^{r,d}$ contains points whose "local distances" are small. We work with $P_h^{r,d}$ because the event $z \in P_h^{r,d}$ depends only on the field $h|_{B_{r/4}(z)}$, and is thus more tractable than the event $z \in \mathcal{B}_1(0; D_h)$ (which depends on the field in a more "global" way). Moreover we have $\mathcal{B}_1(0; D_h) \cap A_r \subset P_h^{r,1} \cap A_r$, so to bound from above $\mu_h(\mathcal{B}_1(0; D_h))$ it suffices to bound from above the volume of the proxy set. We emphasize that $P_h^{r,d}$ is different from the quantity $P_K^{a,x}$ introduced in (3.19): the former is associated with a field h and is considered on the full plane without restriction; the latter is associated with $\star$-scale invariant fields, and the capital letter K refers to a finite number of points where the condition is localized.

Proposition 5.18. *Let h be a whole-plane GFF such that $h_1(0) = 0$. For $k \geq 4/\gamma^2$, $\delta \in (0, 1/2)$, there exists a constant $C_{k,\delta}$ such that for all $x \in \mathbb{R}$,*

$$\mathbb{E}\left[\mu_h\left(B_{10}(0) \cap P_h^{1,e^{-\xi x}}\right)^k\right] \leq C_{k,\delta} e^{-c_{k,\delta} x},$$

where we recall that $c_k = k\gamma - Q$ and $c_{k,\delta} \to c_k$ as $\delta \to 0$.

In fact, for $x > 0$ it is possible, by using tail estimates for side-to-side distances, to show that the decay is Gaussian in x. We do not need this result so we omit it.

Proof. In order to keep the key ideas of the proof transparent, we postpone the proofs of some intermediate elementary lemmas to the end of this section. Consider the collection of balls

$$\mathfrak{B} = \left\{B_{e^{-\ell}}(z) \,:\, \ell \in \mathbb{N}_0, z \in e^{-\ell-2}\mathbb{Z}^2, B_{e^{-\ell}}(z) \cap B_{10}(0) \neq \emptyset\right\}. \tag{3.28}$$

We will work with three events in the proof: $E_{\delta,M}$ is a global regularity event, $F_{K,\delta,M}$ is an approximation of the event $\{K \subset P_h^{1,e^{-\xi x}}\}$ which replaces the conditions on the metric by conditions on the field, and $F'_{K,\delta,M}$ is a variant of $F_{K,\delta,M}$ where γ-log singularities are added to the field at the points $z \in K$ (this is related to $P_K^{0,x}$). Here, M is a parameter that is sent to $+\infty$ and δ is a small positive parameter. The integer k is fixed throughout the proof, so the events are allowed to depend on k and we omit it in the notation.

Step 1: truncating over a global regularity event E. The event $E_{\delta,M}$ is given by the following criteria:

1. For all $\ell \geq 0$, the annulus crossing distance of $B \backslash 0.99B$ is at least $M^{-\xi} e^{-\xi \ell^{\frac{1}{2}+\delta}} e^{-\xi Q \ell} e^{\xi \fint_{\partial B} h}$ for all $B \in \mathfrak{B}$ with radius $e^{-\ell}$.
2. For all integers $\ell > \ell' \geq 0$, for all $B \in \mathfrak{B}$ of radius $e^{-\ell-2}$, we have $e^{-\ell} \sup_{6kB} |\nabla \phi_{\ell',\ell}| \leq \ell^{\frac{1}{2}+\delta} + \log M$.
3. For all $\ell \geq 0$ and all $B \in \mathfrak{B}$ of radius $e^{-\ell-2}$, $\fint_{\partial B} \phi_{\ell,\infty} \leq \ell^{\frac{1}{2}+\delta} + \log M$.

4. $\|\phi - h\|_{\mathbb{D}} = \|g\|_{\mathbb{D}} \le \log M$.

As we see later in Lemma 5.22, for fixed δ the event $E_{\delta,M}$ occurs with superpolynomially high probability in M as $M \to \infty$. Therefore, when looking at moments of $\mu_h(\mathcal{B}_1(0; D_h) \cap \mathbb{D})$, one can restrict to moments truncated on $E_{\delta,M}$.

By using Property 4 of $E_{\delta,M}$ and the definition of μ_ϕ as a Gaussian multiplicative chaos (see Section 1.2), we get

$$\mathbb{E}\left[\mathbb{1}_{E_{\delta,M}} \mu_h \left(B_{10}(0) \cap P_h^{1,e^{-\xi x}}\right)^k\right] \le C_k M^{\gamma k} \mathbb{E}\left[\mathbb{1}_{E_{\delta,M}} \mu_\phi \left(B_{10}(0) \cap P_h^{1,e^{-\xi x}}\right)^k\right]$$

and

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_{E_{\delta,M}} \mu_\phi \left(B_{10}(0) \cap P_h^{1,e^{-\xi x}}\right)^k\right] \\
&\qquad = \mathbb{E}\left[\int_{B_{10}(0)^k} \mathbb{1}_{E_{\delta,M}} \mathbb{1}\{z_i \in P_h^{1,e^{-\xi x}} \text{ for all } i\} \mu_\phi(dz_1) \dots \mu_\phi(dz_k)\right] \\
&\qquad\qquad \le \mathbb{E}\left[\int_{B_{10}(0)^k} \mathbb{1}_{F_{K,\delta,M}} \mu_\phi(dz_1) \dots \mu_\phi(dz_k)\right],
\end{aligned}$$

where the event $F_{K,\delta,M}$ is defined in the following lemma. In the first inequality above, the constant C_k appears from the difference of definition between Gaussian multiplicative chaos measures and the Liouville quantum gravity measure; the former one is defined by renormalizing by a pointwise expectation whereas the latter one by $\varepsilon^{\frac{\gamma^2}{2}}$.

Lemma 5.19. *For $k \ge 2$, there exists a constant C so that for any k-tuple of points $K = \{z_1, \dots, z_k\} \subset \mathbb{D}$ we have the inclusion of events*

$$E_{\delta,M} \cap \{z_i \in P_h^{1,e^{-\xi x}} \text{ for all } i = 1, \dots, k\} \subset F_{K,\delta,M}$$

where $F_{K,\delta,M}$ is the event that for all vertices (S, m, ψ, η) of $T_K^0(\{\phi\})$ we have

$$\psi + x < Qm + Cm^{\frac{1}{2}+\delta} + C \log M. \tag{3.29}$$

Essentially, Lemma 5.19 holds because $K \subset P_h^{1,e^{-\xi x}}$ implies that distances near each cluster are small. Then for each cluster, Property 1 of $E_{\delta,M}$ lets us convert bounds on distances to bounds on circle averages of h, Property 2 lets us replace the coarse field circle average with the coarse field evaluated at any nearby point, and Properties 3 and 4 allow us to neglect the fine field and the random continuous function $h-\phi$; this gives (3.29).

Step 2: shifting LQG mass as γ-singularties. We then use the following lemma to replace the terms $\mu_\phi(dz_i)$'s by dz_i and γ-singularities.

Lemma 5.20. *If f is a bounded nonnegative measurable function, and $C_{a,b}$ are the covariances of $\phi_{a,b}$ (defined as in* (3.14)*), we have*

$$\mathbb{E}\left[\int_{B_{10}(0)^k} f(\phi, z_1, \dots, z_k, \phi_1, \dots, \phi_\ell, \dots)\mu_\phi(dz_1)\dots\mu_\phi(dz_k)\right]$$
$$\le \int_{B_{10}(0)^k} \mathbb{E}[f(\phi+\gamma\sum_{i\le k} C_{0,\infty}(\cdot, z_i), z_1, \dots, z_k, \phi_1+\gamma\sum_{i\le k} C_{0,1}(\cdot, z_i), \dots, \phi_\ell+\gamma\sum_{i\le k} C_{\ell-1,\ell}(\cdot, z_i), \dots)]$$
$$\times \exp\left(\frac{\gamma^2}{2}\sum_{i\neq j} C_{0,\infty}(z_i, z_j)\right) dz_1, \dots dz_k.$$

We apply Lemma 5.20 with $f = 1_{F_{K,\delta,M}}$ and we get

$$\mathbb{E}\int_{B_{10}(0)^k} \mathbb{1}_{F_{K,\delta,M}}\mu_\phi(dz_1)\dots\mu_\phi(dz_k) \le \int_{B_{10}(0)^k} \mathbb{P}[F'_{K,\delta,M}]\exp(\frac{\gamma^2}{2}\sum_{i\neq j} C_{0,\infty}(z_i, z_j))dz_1\dots dz_k,$$

where $F'_{K,\delta,M}$ is the event that in the labeled tree $T_K^0(\{\phi\})$, for any path from the root $(S_1, m_1, \psi_1, \eta_1)$ to $(S_n, m_n, \psi_n, \eta_n)$, we have

$$\psi_n + \gamma\sum_{i=1}^n\sum_{z\in K} C_{m_{i-1},m_i}(z, \mathrm{Left}(S_i)) + x \le Qm_n + Cm_n^{\frac{1}{2}+\delta} + C\log M. \tag{3.30}$$

Note that by Lemma 5.21 below, (3.30) implies that for each vertex $(S_n, m_n, \psi_n, \eta_n)$ we have

$$\psi_n + \eta_n + x \le (Q+\delta)m_n + C\log M + 2C. \tag{3.31}$$

(The term $2C$ comes from Lemma 5.21 and the bound $Cm_n^{\frac{1}{2}+\delta} \le \delta m_n + C$, using that $\delta\in(0,1/2)$.) Now, the probability that (3.31) occurs for each vertex is precisely $P_K^{0,x-C\log M-2C,\delta}$, defined in just before the Corollary 5.16, so we conclude that $\mathbb{P}[F'_{K,\delta,M}] \leq P_K^{0,x-C\log M-2C,\delta}$.

Lemma 5.21. *For $k \geq 2$, there exists C_k such that for $K \in B_{10}(0)^k$, for any path from the root $(S_1, m_1, \psi_1, \eta_1)$ to $(S_n, m_n, \psi_n, \eta_n)$ in the labeled tree $T_K^0(\{\phi\})$ we have, writing $m_0 = 0$,*

$$\left|\eta_n - \gamma \sum_{i=1}^n \sum_{z \in K} C_{m_{i-1},m_i}(z, \mathrm{Left}(S_i))\right| = \left|\gamma \sum_{i=1}^n (m_i - m_{i-1})|S_i| - \gamma \sum_{i=1}^n \sum_{z \in K} C_{m_{i-1},m_i}(z, \mathrm{Left}(S_i))\right| < C.$$

By Proposition 5.9, for $K \subset B_{10}(0)$ we have $\exp(\frac{\gamma^2}{2}\sum_{i\neq j} C_{0,\infty}(z_i, z_j)) \leq C\prod_{i<j}|z_i - z_j|^{-\gamma^2}$. Combining all of the above bounds yields

$$\mathbb{E}\left[\mathbb{1}_{E_{\delta,M}}\mu_h\left(B_{10}(0) \cap P_h^{1,e^{-\xi x}}\right)^k\right] \leq C_k M^{\gamma k} \int_{B_{10}(0)^k} \frac{P_{z_1,\ldots,z_k}^{0,x-C\log M-2C,\delta}}{\prod |z_i - z_j|^{\gamma^2}} dz_1 \ldots dz_k.$$

Finally, by Corollary 5.16 we conclude that for all $x \in \mathbb{R}$ we have

$$\mathbb{E}\left[\mathbb{1}_{E_{\delta,M}}\mu_h\left(B_{10}(0) \cap P_h^{1,e^{-\xi x}}\right)^k\right] \leq C_{k,\delta} M^C e^{-c_{k,\delta} x}. \tag{3.32}$$

Step 3: concluding the proof. By Markov's inequality, we get,

$$\begin{aligned}\mathbb{P}[\mu_h(B_{10}(0) \cap P_h^{1,e^{-\xi x}}) \geq t] &\leq \mathbb{P}[E_{\delta,M}^c] + \mathbb{P}[E_{\delta,M}, \mu_h(B_{10}(0) \cap P_h^{1,e^{-\xi x}}) \geq t] \\ &\leq \mathbb{P}[E_{\delta,M}^c] + t^{-k}\mathbb{E}[1_{E_{\delta,M}}\mu_h(B_{10}(0) \cap P_h^{1,e^{-\xi x}})^k]. \end{aligned} \tag{3.33}$$

The second term is bounded by (3.32). To control the first term, we use the following lemma.

Lemma 5.22. *For fixed $\delta \in (0, 1/2)$, the regularity event $E_{\delta,M}$ occurs with superpolynomially high probability as $M \to \infty$.*

Combining these bounds, namely starting from (3.33), using (3.32) and the previous lemma, we get, for all δ, k, p, a constant $C_{\delta,k,p}$ such that for all $x \in \mathbb{R}$ and for all $M, t > 0$,

$$\mathbb{P}[\mu_h(B_{10}(0) \cap P_h^{1,e^{-\xi x}}) \geq t] \leq C_{k,\delta,p}\left(M^{-p} + t^{-k}M^C e^{-c_{k,\delta}x}\right).$$

By taking $M = t^{k/(p+C)} e^{c_{k,\delta} x/(p+C)}$, we get

$$\mathbb{P}[\mu_h(B_{10}(0) \cap P_h^{1,e^{-\xi x}}) \geq t] \leq C_{k,\delta,p} t^{-\frac{p}{p+C}k} e^{-\frac{p}{p+C} c_{k,\delta} x}$$

so by choosing p large and integrating the tail estimate to obtain moment bounds, we obtain

$$\mathbb{E}[\mu_h(B_{10}(0) \cap P_h^{1,e^{-\xi x}})^{k-\delta}] \leq C e^{-(c_{k,\delta}-\delta)x}.$$

Then, by (3.32) and the Cauchy-Schwartz inequality, we get

$$\mathbb{E}\left[\mu_h\left(B_{10}(0) \cap P_h^{1,e^{-\xi x}}\right)^k\right] \leq C_{k,\delta} M^C e^{-c_{k,\delta} x} + \mathbb{P}[E_{\delta,M}^c]^{1/2} \mathbb{E}\left[\mu_h\left(B_{10}(0) \cap P_h^{1,e^{-\xi x}}\right)^{2k}\right]^{1/2}$$

and we conclude the proof of Proposition 5.18 by taking $M = e^{\varepsilon|x|}$ for some small $\varepsilon > 0$ (indeed, for this choice of M we have $\mathbb{P}[E_{\delta,M}^c] \lesssim e^{-a|x|}$ for any $a > 0$, and our earlier bound says that the $2k$th moment is at most exponential in x). □

Annuli contributions and α-singularities.

Here, we use the proxy estimate to study moments of metric balls when the field has singularities. The link is made with the following deterministic remark. Recall that $A_{r/2} := B_{r/2}(0) \backslash \overline{B_{r/4}(0)}$. If $z \in \mathcal{B}_1(0; D_h) \cap A_{r/2}$ then $D_h(0, \partial B_{r/4}(0)) \leq 1$ and $z \in P_h^{r,1-D_h(0,\partial B_{r/4}(0))}$ (recall (3.27) for the definition of $P_h^{r,d}$).

In the following lemma, we will study the LQG volume of the intersection of the unit metric ball with the unit Euclidean disk. To do so, we study first the contribution of small annuli to the volume and then use a Hölder inequality to conclude.

Lemma 5.23. *Let h be a whole-plane GFF such that $h_1(0) = 0$. Then for $\alpha < Q$,*

$$\mathbb{E}\left[\mu_{h+\alpha\log|\cdot|^{-1}}(\mathcal{B}_1(0; D_{h+\alpha\log|\cdot|^{-1}}) \cap \mathbb{D})^k)\right] < \infty.$$

Proof. Note that $\mathcal{B}_1(0; D_h) \cap A_{r/2} \subset P_h^{r,1} \cap A_{r/2}$ and that the latter one is measurable with respect to the field $h|_{B_r(0)}$. We use a decoupling/scaling argument as follows. We write,

$$\begin{aligned}\mu_h(\mathcal{B}_1(0;D_h)\cap A_{r/2}) &\le 1_{D_h(0,\partial B_{r/4}(0))\le 1}\mu_h(P_h^{r,1}\cap A_{r/2})\\ &= 1_{e^{\xi h_r(0)}D_{h-h_r(0)}(0,\partial B_{r/4}(0))\le 1}e^{\gamma h_r(0)}\mu_{h-h_r(0)}\left(A_{r/2}\cap P_{h-h_r(0)}^{r,e^{-\xi h_r(0)}}\right),\end{aligned}$$

and set $\widetilde{h} := h(r\cdot) - h_r(0)$. By Lemma 5.4 we have the equality in law $\widetilde{h}|_{\mathbb{D}} \overset{(d)}{=} h|_{\mathbb{D}}$, and also $\widetilde{h}|_{\mathbb{D}}$ is independent of $h_r(0)$. Using the scaling of the metric and of the measure, we get

$$\begin{aligned}&\mathbb{E}\left[\mu_h(\mathcal{B}_1(0;D_h)\cap A_{r/2})^k\right]\\ &\le \mathbb{E}\left[1_{e^{\xi h_r(0)}D_{h-h_r(0)}(0,\partial B_{r/4}(0))\le 1}e^{\gamma k h_r(0)}\mu_{h-h_r(0)}\left(A_{r/2}\cap P_{h-h_r(0)}^{r,e^{-\xi h_r(0)}}\right)^k\right]\\ &\le r^{k\gamma Q}\mathbb{E}\left[1_{e^{\xi h_r(0)}r^{\xi Q}D_{\tilde{h}}(0,\partial B_{1/4}(0))\le 1}e^{\gamma k h_r(0)}\mu_{\tilde{h}}\left(A_{1/2}\cap P_{\tilde{h}}^{1,r^{-\xi Q}e^{-\xi h_r(0)}}\right)^k\right]. \qquad (3.34)\end{aligned}$$

We split the expectation with $1_{D_{\tilde{h}}(0,\partial B_{1/4})\le r^\delta}$ and $1_{D_{\tilde{h}}(0,\partial B_{1/4})\ge r^\delta}$. Note first that for $p>1$, by Proposition 5.18 and a moment computation for the exponential of a Gaussian variable with variance constant times $\log r^{-1}$,

$$\mathbb{E}\left[e^{\gamma p k h_r(0)}\mu_{\tilde{h}}\left(A_{1/2}\cap P_{\tilde{h}}^{1,r^{-\xi Q}e^{-\xi h_r(0)}}\right)^{kp}\right]\le C r^{\text{power}},$$

for some power whose value does not matter. Indeed, because of the superpolynomial decay of the event $\{D_{\tilde{h}}(0,\partial B_{1/4})\le r^\delta\}$ coming from Proposition 5.6, the quantity

$$\begin{aligned}&\mathbb{E}\left[1_{D_{\tilde{h}}(0,\partial B_{1/4})\le r^\delta}e^{\gamma k h_r(0)}\mu_{\tilde{h}}\left(A_{1/2}\cap P_{\tilde{h}}^{1,r^{-\xi Q}e^{-\xi h_r(0)}}\right)^k\right]\\ &\qquad\le \mathbb{P}[D_{\tilde{h}}(0,\partial B_{1/4})\le r^\delta]^{1/q}\,\mathbb{E}\left[e^{\gamma p k h_r(0)}\mu_{\tilde{h}}\left(A_{1/2}\cap P_{\tilde{h}}^{1,r^{-\xi Q}e^{-\xi h_r(0)}}\right)^{kp}\right]^{1/p}\end{aligned}$$

decays superpolynomially fast in r, by using Hölder's inequality with $\frac{1}{p}+\frac{1}{q}=1$.

From now on, we truncate on the event $\{D_{\tilde{h}}(0,\partial B_{1/4})\ge r^\delta\}$ and we want to bound from above

$$r^{k\gamma Q}\mathbb{E}\left[1_{e^{\xi h_r(0)}r^{\xi Q+\delta}\le 1}e^{\gamma k h_r(0)}\mu_{\tilde{h}}\left(A_{1/2}\cap P_{\tilde{h}}^{1,r^{-\xi Q}e^{-\xi h_r(0)}}\right)^k\right].$$

By Proposition 5.18, since $A_{1/2}\subset B_{10}(0)$ and $h_r(0)$ is independent of $\widetilde{h}|_{\mathbb{D}}$, by writing $c_{k,\delta}=$

$k\gamma - Q + \alpha_\delta$ for some small α_δ, we get

$$
\begin{aligned}
& r^{k\gamma Q}\mathbb{E}\left[1_{e^{\xi h_r(0)}r^{\xi Q+\delta}\leq 1}e^{\gamma k h_r(0)}\mu_{\tilde{h}}\left(A_{1/2}\cap P_{\tilde{h}}^{1,r^{-\xi Q}e^{-\xi h_r(0)}}\right)^k\right] \\
& \leq C_k r^{k\gamma Q} r^{-c_{k,\delta}Q}\mathbb{E}\left[1_{e^{\xi h_r(0)}r^{\xi Q+\delta}\leq 1}e^{\gamma k h_r(0)}e^{-c_{k,\delta}h_r(0)}\right] \\
& = C_k r^{Q^2 - Q\alpha_\delta}\mathbb{E}\left[1_{e^{\xi h_r(0)}r^{\xi Q+\delta}\leq 1}e^{(Q-\alpha_\delta)h_r(0)}\right].
\end{aligned}
$$

Furthermore, since

$$\mathbb{E}\left[1_{e^{\xi h_r(0)}r^{\xi Q+\delta}\leq 1}e^{(Q-\alpha_\delta)h_r(0)}\right] \leq \mathbb{E}\left[e^{(Q-\alpha_\delta)h_r(0)}\right],$$

by a Gaussian computation we get

$$\mathbb{E}\left[\mu_h(\mathcal{B}_1(0;D_h)\cap A_{r/2})^k\right] \leq C_k r^{\frac{1}{2}Q^2+\beta_\delta},$$

for some arbitrarily small β_δ.

Furthermore, note that when one replaces h by $h + \alpha\log|\cdot|^{-1}$ for $\alpha < Q$, we get

$$\mathbb{E}\left[\mu_{h+\alpha\log|\cdot|^{-1}}(\mathcal{B}_1(0;D_{h+\alpha\log|\cdot|^{-1}})\cap A_{r/2})^k\right] \leq C_k r^{\frac{1}{2}(Q-\alpha)^2+\beta_\delta}. \tag{3.35}$$

Indeed, on $A_{r/2}$, $\alpha\log|\cdot|^{-1}$ is of order $-\log r + O(1)$ so the volume term contributes an additional $r^{-k\gamma\alpha}$. Furthermore, by monotonicity, we can replace the intersection of the unit $D_{h+\alpha\log|\cdot|^{-1}}$-metric ball with $A_{r/2}$ by an order $r^{\xi\alpha}$ D_h-metric ball intersected with $A_{r/2}$. Then, instead of using $\mathcal{B}_1(0;D_h)\cap A_{r/2}\subset P_h^{r,1}\cap A_{r/2}$ at the beginning of the proof, we use $\mathcal{B}_{r^{\alpha\xi}}(0;D_h)\cap A_{r/2}\subset P_h^{r,r^{\alpha\xi}}\cap A_{r/2}$. Then we note that the term $r^{\xi Q}$ in (3.34) is replaced by $r^{\xi(Q-\alpha)}$. Therefore, (3.35) follows by replacing Q with $Q-\alpha$.

We can conclude as follows. Set $V_r^{\gamma,\alpha} := \mu_{h+\alpha\log|\cdot|^{-1}}(\mathcal{B}_1(0;D_{h+\alpha\log|\cdot|^{-1}}\cap A_r))$. By monotone convergence,

$$\mathbb{E}\left[\mu_{h+\alpha\log|\cdot|^{-1}}(\mathcal{B}_1(0;D_{h+\alpha\log|\cdot|^{-1}})\cap\mathbb{D})^k\right] = \lim_{n\to\infty}\mathbb{E}\left[\left(\sum_{i=0}^{n}V_{2^{-i}}^{\gamma,\alpha}\right)^k\right].$$

We introduce some deterministic $\Lambda > 1$ to be chosen. By Hölder's inequality we get

$$\left(\sum_{i=0}^{n} V_{2^{-i}}^{\gamma,\alpha}\right)^k = \left(\sum_{i=0}^{n} \Lambda^i V_{2^{-i}}^{\gamma,\alpha} \Lambda^{-i}\right)^k \leq \left(\sum_{i=0}^{n} \Lambda^{ki} (V_{2^{-i}}^{\gamma,\alpha})^k\right) \left(\sum_{i=0}^{n} \Lambda^{-i\frac{k}{k-1}}\right)^{k-1}.$$

Taking expectations, and using the bound (3.35), we get, uniformly in n,

$$\mathbb{E}\left[\left(\sum_{i=0}^{n} V_{2^{-i}}^{\gamma,\alpha}\right)^k\right] \leq \left(\frac{1}{1-\Lambda^{-\frac{k}{k-1}}}\right)^{k-1} \sum_{i=0}^{\infty} \Lambda^{ki} 2^{-i(\frac{1}{2}(Q-\alpha)^2+\beta_\delta)}.$$

Taking Λ close enough to one such that $\Lambda^k 2^{-\frac{1}{2}(Q-\alpha)^2+\beta_\delta} < 1$, this series is absolutely convergent, as desired. □

Lemma 5.24 (Large annuli)**.** *Let h be a whole-plane GFF such that $h_1(0) = 0$. Then, for $\alpha < Q$, $\mathbb{E}\left[\mu_{h+\alpha \log|\cdot|^{-1}}(\mathcal{B}_1(0; D_{h+\alpha\log|\cdot|^{-1}}) \cap \mathbb{C} \setminus \mathbb{D})^k\right] < \infty$.*

Proof. The proof uses the proxy estimate and a decomposition over annuli with a scaling argument. This is similar to Lemma 5.23. We point out here only the main differences with the proof of this lemma.

Write $D_h(0, \partial B_{R/4}(0)) =: R^{\xi Q} e^{\xi h_{R/4}(0)} X_R$. Since $\mathcal{B}_1(0; D_h) \cap A_R \subset P_h^{R,1} \cap A_R$

$$\begin{aligned}\mathbb{E}[\mu_h(\mathcal{B}_1(0; D_h) \cap A_R)^k] &\leq \mathbb{E}[1_{D_h(0,\partial B_{R/4}(0)) \leq 1} \mu_h(P_h^{R,1} \cap A_R)^k] \\ &= \mathbb{E}[1_{R^{\xi Q} e^{\xi h_{R/4}(0)} X_R \leq 1} e^{k\gamma h_{R/4}(0)} \mu_{h-h_{R/4}(0)}(P_{h-h_{R/4}(0)}^{R,e^{-\xi h_{R/4}(0)}} \cap A_R)^k]\end{aligned}$$

We truncate again with $1_{X_R \leq R^{-\delta}}$ and $1_{X_R \geq R^{-\delta}}$. Because of the superpolynomial decay of $\mathbb{P}(X_R \leq R^{-\delta})$, the term associated with the former truncation is negligible compared to the other one. Furthermore, since we will have some room at the level of exponent, we will simply assume that $\delta = 0$ for the remaining steps. By using that $h - h_{R/4}(0))|_{A_{R/4,2R}(0)}$ is independent of $h_{R/4}(0)$ and that the proxy $P_h^{R,x} \cap A_r$ is measurable with respect to $h|_{A_{R/4,2R}}$, we get by scaling,

$$\begin{aligned}&\mathbb{E}(1_{R^Q e^{h_{R/4}(0)} \leq 1} \mu_{h-h_{R/4}(0)}(P_{h-h_{R/4}(0)}^{R,e^{-\xi h_{R/4}(0)}} \cap A_R)^k) \\ &\qquad = R^{k\gamma Q} \mathbb{E}(1_{R^Q e^{h_{R/4}(0)} \leq 1} e^{k\gamma h_{R/4}(0)} \mu_{\tilde{h}}(P_{\tilde{h}}^{1,e^{-\xi h_{R/4}(0)} R^{-\xi Q}} \cap A_1)^k)\end{aligned}$$

At this stage we use the estimate from Proposition 5.18. Therefore, we compute

$$R^{k\gamma Q}\mathbb{E}(1_{h_{R/4}(0)\le -Q\log R}e^{k\gamma h_{R/4}(0)}e^{-c_k(h_{R/4}(0)+Q\log R)}) = R^{k\gamma Q}e^{-c_k Q\log R}\mathbb{E}\left(1_{h_{R/4}(0)\le -Q\log R}e^{Qh_{R/4}(0)}\right)$$

and by using the Cameron-Martin formula we get

$$\begin{aligned} R^{k\gamma Q}e^{-c_k Q\log R}\mathbb{E}\left(1_{h_{R/4}(0)\le -Q\log R}e^{Qh_{R/4}(0)}\right) & \\ &\approx R^{Q^2}R^{\frac{Q^2}{2}}\mathbb{E}\left(1_{h_{R/4}(0)\le -Q\log R}e^{Qh_{R/4}(0)-\frac{1}{2}Q^2\log R/4}\right) \\ &\approx R^{\frac{3}{2}Q^2}\mathbb{P}\left(h_{R/4}(0)\le -2Q\log R\right)\approx R^{-\frac{Q^2}{2}}. \end{aligned}$$

where $A_R \approx B_R$ if $A_R/B_R = R^{o(1)}$. So this gives

$$\mathbb{E}[\mu_h(\mathcal{B}_1(0;D_h)\cap A_R)^k] \le R^{-\frac{Q^2}{2}+o(1)}$$

The rest of the proof, namely taking into account all the annuli contributions and using Hölder inequality, is the same as the one of Lemma 5.23. □

Proof of Proposition 5.8. Let h be a whole-plane GFF such that $h_1(0) = 0$ and fix $\alpha < Q$. The proof follows easily by writing

$$\mu_{h+\alpha\log|\cdot|^{-1}}(\mathcal{B}_1(0;D_{h+\alpha\log|\cdot|^{-1}})) = \mu_{h+\alpha\log|\cdot|^{-1}}(\mathcal{B}_1(0;D_{h+\alpha\log|\cdot|^{-1}})\cap\mathbb{D}) + \mu_{h+\alpha\log|\cdot|^{-1}}(\mathcal{B}_1(0;D_{h+\alpha\log|\cdot|^{-1}})\cap\mathbb{C}\setminus\mathbb{D})$$

and using the inequality $(x+y)^k \le 2^{k-1}(x^k+y^k)$ together with Lemma 5.23 and Lemma 5.24. □

Lemma 5.25 (Upper bound for small metric balls)**.** *For $\varepsilon \in (0,1)$, $k \ge 1$, there exists a constant $C_{k,\varepsilon}$ such that for all $s \in (0,1)$,*

$$\mathbb{E}[\mu_h(\mathcal{B}_s(0;D_h))^k] \le C_{k,\varepsilon}s^{kd_\gamma-\varepsilon}$$

Proof. The proof is very similar to the one of Lemma 5.23, therefore we omit the details and just provide the differences. By replacing 1 by s in the proof, we get $\mathbb{E}[\mu_h(\mathcal{B}_s(0;D_h)\cap A_r)^k] \leq C_k s^{kd_\gamma - c_\gamma} r^{\frac{Q^2}{2}}$ where $c_\gamma = \frac{d_\gamma}{\gamma}Q$. By using Hölder's inequality, we get $\mathbb{E}[\mu_h(\mathcal{B}_s(0;D_h)\cap A_r)^k] \leq C_{kp}^{1/p} s^{kd_\gamma - \frac{c_\gamma}{p}} r^{\frac{Q^2}{2p}}$. We then take p such that $c_\gamma/p < \varepsilon$ and the rest of the proof follows the same line as those of Lemma 5.23. □

Proofs of the intermediate lemmas for Proposition 5.18

We recall here the definition of the event $E_{\delta,M}$ (recall the definition of $\mathfrak{B}$ in (3.28)). It is given by the following criteria:

1. For all $\ell \geq 0$, the annulus crossing distance of $B\backslash 0.99B$ is at least $M^{-\xi}e^{-\xi\ell^{\frac{1}{2}+\delta}}e^{-\xi Q\ell}e^{\xi \fint_{\partial B} h}$ for all $B \in \mathfrak{B}$ with radius $e^{-\ell}$,
2. for all integers $\ell > \ell' \geq 0$, for all $B \in \mathfrak{B}$ of radius $e^{-\ell-2}$, we have $e^{-\ell}\sup_{6kB}|\nabla\phi_{\ell',\ell}| \leq \ell^{\frac{1}{2}+\delta} + \log M$,
3. for all $\ell \geq 0$ and for all $B \in \mathfrak{B}$ of radius $e^{-\ell-2}$, $\fint_{\partial B}\phi_{\ell,\infty} \leq \ell^{\frac{1}{2}+\delta} + \log M$,
4. and $\|\phi - h\|_{\mathbb{D}} = \|g\|_{\mathbb{D}} \leq \log M$.

Proof of Lemma 5.19. We prove here that for any k-tuple of points $K = \{z_1,\ldots,z_k\} \subset \mathbb{D}$ we have

$$\begin{aligned} &E_{\delta,M} \cap \{z_i \in P_h^{1,e^{-\xi x}} \text{ for all } i = 1,\ldots,k\} \\ &\qquad \subset \{\psi + x \leq Qm + 8k^2(m^{\frac{1}{2}+\delta} + \log M) \text{ for each vertex } (S,m,\psi,\eta) \text{ of } T_K^0(\{\phi\})\}. \end{aligned}$$

Fix K and consider any vertex (S,m,ψ,η) of $T_K^0(\{\phi\})$. Recall first that by (3.17),

$$\psi = \psi_n = \sum_{i=1}^n \phi_{m_{i-1},m_i}(\mathrm{Left}(S_i)), \tag{3.36}$$

where we write (S_i,m_i,ψ_i,η_i) for the path from the root (S_1,m_1,ψ_1,η_1) to $(S_n,m_n,\psi_n,\eta_n) = (S,m,\psi,\eta)$. The proof is to compare a circle average around $z \in S$ (which can be bounded since $z \in P_h^{1,e^{-\xi x}}$) with the right-hand side above. Pick any point $z \in S$. Since $z \in P_h^{1,e^{-\xi x}}$,

$$D_h(z,\partial B_{e^{-m-1}}(z)) \leq D_h(z,\partial B_{1/4}(z)) \leq e^{-\xi x},$$

and we can find a ball $B \in \mathfrak{B}$, centered at a point in $e^{-m-4}\mathbb{Z}^2$ with radius e^{-m-2} whose boundary separates z from $\partial B_{e^{-m-1}}(z)$. Hence the annulus crossing distance of $B\backslash 0.99B$ is at most $e^{-\xi x}$. By Property 1, we have,

$$M^{-\xi} e^{-\xi(m+2)^{\frac{1}{2}+\delta}} e^{-\xi Q(m+2)} e^{\xi \fint_{\partial B} h} \leq e^{-\xi x},$$

or equivalently

$$\fint_{\partial B} h + x \leq Q(m+2) + (m+2)^{\frac{1}{2}+\delta} + \log M. \tag{3.37}$$

Now we lower bound $\fint_{\partial B} h$ in term of (3.36) by using properties 2, 3 and 4 of $E_{\delta,M}$.

- By Property 4 we have

$$\fint_{\partial B} h \geq \sum_{i=1}^{n} \fint_{\partial B} \phi_{m_{i-1},m_i} + \fint_{\partial B} \phi_{m,\infty} - \log M.$$

- For each i, notice that $z \in S_i$, and so $d(z, \mathrm{Left}(S_i)) \leq eke^{-m_i}$ by (3.16). Consequently, by Property 2 we have for each $i = 1, \ldots, n$

$$\fint_{\partial B} \phi_{m_{i-1},m_i} \geq \phi_{m_{i-1},m_i}(\mathrm{Left}(S_i)) - 4km_i^{\frac{1}{2}+\delta} - 4k \log M.$$

- By Property 3 we have

$$\fint_{\partial B} \phi_{m,\infty} \geq -m^{\frac{1}{2}+\delta} - \log M.$$

Combining these yields (see Remark 5.11)

$$\fint_{\partial B} h \geq \sum_{i=1}^{n} \phi_{m_{i-1},m_i}(\mathrm{Left}(S_i)) - 6k^2 m^{\frac{1}{2}+\delta} - 6k^2 \log M = \psi - 6k^2 m^{\frac{1}{2}+\delta} - 6k^2 \log M.$$

Together with (3.37), this gives $\psi + x \leq Qm + 8k^2(m^{\frac{1}{2}+\delta} + \log M)$ and concludes the proof. □

Proof of Lemma 5.20. This is an application of the Cameron-Martin theorem. We outline here the main idea, assuming for notational simplicity that the function f depends only on $\phi, z_1, \ldots, z_k$. The argument works the same way for f depending also on $(\phi_n)_{n\geq 0}$.

Assume first that f is continuous. Fix $k \geq 2$, $\delta > 0$ and set $C_\delta := \{(z_1, \ldots, z_k) \in B_{10}(0)^k :$

$\min_{i<j}|z_i - z_j| \geq \delta\}$. Then, by using Fatou's lemma and the Cameron-Martin formula, we have

$$\begin{aligned}
&\mathbb{E}\left[\int_{B_{10}(0)^k \cap C_\delta} f(\phi, z_1, \ldots, z_k)\mu_\phi(dz_1)\ldots\mu_\phi(dz_k)\right] \\
&\leq \liminf_{\varepsilon\to 0} \mathbb{E}\left[\int_{B_{10}(0)^k\cap C_\delta} f(\phi, z_1,\ldots,z_k)\frac{e^{\gamma\phi_\varepsilon(z_1)}}{\mathbb{E}[e^{\gamma\phi_\varepsilon(z_1)}]}\cdots\frac{e^{\gamma\phi_\varepsilon(z_k)}}{\mathbb{E}[e^{\gamma\phi_\varepsilon(z_k)}]}dz_1\ldots dz_k\right] \\
&= \liminf_{\varepsilon\to 0}\int_{B_{10}(0)^k\cap C_\delta}\frac{dz_1\ldots dz_k}{e^{-\frac{\gamma^2}{2}\sum_{i\neq j}\mathrm{Cov}(\phi_\varepsilon(z_i),\phi_\varepsilon(z_j))}}\mathbb{E}\left[e^{\gamma\sum_{i\leq k}\phi_\varepsilon(z_i)-\frac{\gamma^2}{2}\mathrm{Var}(\sum_{i\leq k}\phi_\varepsilon(z_i))}f(\phi,z_1,\ldots,z_k)\right] \\
&= \liminf_{\varepsilon\to 0}\int_{B_{10}(0)^k\cap C_\delta}\frac{dz_1\ldots dz_k}{e^{-\frac{\gamma^2}{2}\sum_{i\neq j}\mathrm{Cov}(\phi_\varepsilon(z_i),\phi_\varepsilon(z_j))}}\mathbb{E}\left[f(\phi+\gamma\sum_{i\leq k}\mathrm{Cov}(\phi(\cdot),\phi_\varepsilon(z_i)),z_1,\ldots,z_k)\right] \\
&= \int_{B_{10}(0)^k\cap C_\delta}\frac{dz_1\ldots dz_k}{e^{-\frac{\gamma^2}{2}\sum_{i\neq j}\mathrm{Cov}(\phi(z_i),\phi(z_j))}}\mathbb{E}\left[f(\phi+\gamma\sum_{i\leq k}\mathrm{Cov}(\phi(\cdot),\phi(z_i)),z_1,\ldots,z_k)\right].
\end{aligned}$$

using dominated convergence theorem in the last equality (the term $\sum_{i\neq j}\mathrm{Cov}(\phi(z_i),\phi(z_j))$ is uniformly bounded for $(z_1,\ldots,z_n)\in C_\delta$). The Cameron-Martin formula is used by writing

$$\gamma\sum_{i\leq k}\phi_\varepsilon(z_i) = \langle\phi,\gamma\sum_{i\leq k}\rho_{\varepsilon,z_i}\rangle$$

where ρ_{ε,z_i} denote the uniform probability measure on the circle $\partial B_\varepsilon(z_i)$. Note that the above inequality was only shown for continuous f, but we can approximate general bounded nonnegative measurable f by a sequence of continuous f_n which converge pointwise to f, and apply the dominated convergence theorem. Thus the above inequality holds for general f.

Finally, letting δ going to zero and using the monotone convergence theorem, we get

$$\begin{aligned}
&\mathbb{E}\left[\int_{B_{10}(0)^k} f(\phi,z_1,\ldots,z_k)\mu_\phi(dz_1)\ldots\mu_\phi(dz_k)\right] \\
&\qquad\leq \int_{B_{10}(0)^k} e^{\frac{\gamma^2}{2}\sum_{i\neq j}\mathrm{Cov}(\phi(z_i),\phi(z_j))}\mathbb{E}\left[f(\phi+\gamma\sum_{i\leq k}\mathrm{Cov}(\phi(\cdot),\phi(z_i)),z_1,\ldots,z_k)\right]dz_1\ldots dz_k.
\end{aligned}$$

This concludes the proof. □

Proof of Lemma 5.21. It suffices to show that for some constant C, for each $z\in K$ and each

$i = 1, \dots, n$, writing $w = \mathrm{Left}(S_i)$ we have

$$\left|C_{m_{i-1},m_i}(z,w) - (m_i - m_{i-1})1_{z\in S_i}\right| < C.$$

If $z \notin S_i$, then by definition $d(z,w) \geq d(z,S_i) \geq e^{-m_{i-1}}$. This is larger than the range of dependence of ϕ_{m_{i-1},m_i}, so $C_{m_{i-1},m_i}(z,w) = 0$ as desired.

Now suppose $z \in S_i$. By (3.16), we know that S_i is contained in a ball of radius $6ke^{-m_i}$; by translation invariance we may assume this ball is centered at the origin. On $B_{6k}(0) \times B_{6k}(0)$, the correlation of $\phi_{0,\infty}$ is $C_{0,\infty}(\cdot,\cdot) = \log|\cdot - \cdot|^{-1} + q(\cdot - \cdot)$ for some bounded continuous q. Thus, by scale invariance, we can write

$$\begin{aligned} C_{m_{i-1},m_i}(z,w) &= C_{0,m_i-m_{i-1}}(e^{m_{i-1}}z, e^{m_{i-1}}w) \\ &= \log|e^{m_{i-1}}(z-w)|^{-1} - C_{m_i-m_{i-1},\infty}(e^{m_{i-1}}z, e^{m_{i-1}}w) + O(1). \end{aligned}$$

But again by scale invariance we have

$$C_{m_i-m_{i-1},\infty}(e^{m_{i-1}}z, e^{m_{i-1}}w) = C_{0,\infty}(e^{m_i}z, e^{m_i}w) = \log|e^{m_i}(z-w)|^{-1} + O(1).$$

Comparing these two equations we conclude that $C_{m_{i-1},m_i}(z,w) = m_i - m_{i-1} + O(1)$, as needed. □

Finally we check the bound on the regularity event E.

Proof of Lemma 5.22. We prove here the estimate of the occurence of the event $E_{\delta,M}$.

For all integers $\ell > \ell' \geq 0$, for all $B \in \mathfrak{B}$ of radius $e^{-\ell-2}$, the probability that $e^{-\ell}\sup_{6kB}|\nabla\phi_{\ell',\ell}| > \ell^{\frac{1}{2}+\delta} + \log M$ is $\leq Ce^{-c(\log M)^2}e^{-c\ell^{1+2\delta}}$ by Lemma 5.48. Therefore, the probability that Condition 2 does not hold is $\leq Ce^{-c(\log M)^2}\sum_{\ell\geq 0}\ell e^{2\ell}e^{-c\ell^{1+2\delta}}$.

For Condition 3, for a $B \in \mathfrak{B}$ of size $e^{-\ell-2}$, by scaling $⨍_{\partial B}\phi_{\ell,\infty}$ is distributed as $⨍_{\partial B_0}\phi_{0,\infty}$ where B_0 is of size e^{-2} and this is a centered Gaussian variable with bounded variance. Therefore, the probability it is at least $\ell^{\frac{1}{2}+\delta} + \log M$ is less than $Ce^{-c(\ell^{\frac{1}{2}+\delta}+\log M)^2} \leq Ce^{-c\ell^{1+2\delta}}e^{-c(\log M)^2}$. For each ℓ, there are $O(e^{2\ell})$ balls of size $e^{-\ell-2}$ in $\mathfrak{B}$, hence the probability that Condition 3 does not hold is less than $Ce^{-c(\log M)^2}\sum_{\ell\geq 0}e^{2\ell}e^{-c\ell^{1+2\delta}}$.

For Condition 4, since $\phi - h$ is continuous by Proposition 5.9, and applying Fernique's theorem, the probability that $\|\phi - h\|_{\mathbb{D}} \leq \log M$ occurs is $\geq 1 - Ce^{-c(\log M)^2}$. For Condition 1, we use Proposition 5.6 and again a union bound. $\square$

5.4 Negative moments

In this section, we prove the following lower bound on the LQG volume of the unit metric ball.

Proposition 5.26 (Negative moments of LQG ball volume)**.** *Let h be a whole-plane GFF normalized so $h_1(0) = 0$. Then*

$$\mathbb{E}\left[\mu_h(\mathcal{B}_1(0; D_h))^{-p}\right] < \infty \text{ for all } p \geq 0.$$

This result also holds if we instead consider the LQG measure and metric associated with the field $\widetilde{h} = h - \alpha \log|\cdot|$ for $\alpha < Q$.

In Section 5.4.1, we prove the finiteness of negative moments of $\mu_h(\mathcal{B}_1(0; D_h^{\mathbb{D}}))$, the unit ball with respect to the $\mathbb{D}$-internal metric $D_h^{\mathbb{D}}$. This immediately implies Proposition 5.26 since $\mathcal{B}_1(0; D_h^{\mathbb{D}}) \subset \mathcal{B}_1(0; D_h)$. In Section 5.4.2 we bootstrap our results to obtain lower bounds on $\mu_h(\mathcal{B}_s(0; D_h))$ for $s \in (0, 1)$; these lower bounds will be useful in our applications in Section 5.5.

5.4.1 Lower tail of the unit metric ball volume

The goal of this section is the following result.

Proposition 5.27 (Superpolynomial decay of internal metric ball volume lower tail)**.** *Let h be a whole-plane GFF normalized so $h_1(0) = 0$. Let $D_h^{\mathbb{D}} : \mathbb{D} \times \mathbb{D} \to \mathbb{R}$ be the internal metric in $\mathbb{D}$ induced by D_h, and $\mathcal{B}_1(0; D_h^{\mathbb{D}}) \subset \mathbb{D}$ the $D_h^{\mathbb{D}}$-metric ball. Then for any $p > 0$, for all sufficiently large $C > 0$ we have*

$$\mathbb{P}\left[\mu_h(\mathcal{B}_1(0; D_h^{\mathbb{D}})) \geq C^{-1}\right] \geq 1 - C^{-p}.$$

This result also holds if we instead consider the LQG measure and metric associated with the field $\widetilde{h} = h - \alpha \log|\cdot|$ for $\alpha < Q$.

Let $N > 1$ be a parameter which we keep fixed as $C \to \infty$ (taking N large yields p large in

Proposition 5.27) and define

$$k_0 = \left\lfloor \frac{1}{N} \log C \right\rfloor, \qquad k_1 = \lfloor N \log C \rfloor.$$

Let P be a $D_{\widetilde{h}}$-geodesic from 0 to $\partial B_{e^{-k_0}}(0)$. See Figure 5.2 (left) for the setup.

Proof sketch of Proposition 5.27. The proof follows several steps. Each step below holds with high probability.

- We find an annulus $B_{e^{-k+1}}(0) \backslash \overline{B_{e^{-k}}(0)}$ with $k > k_0$ not too large, such that the annulus-crossing length of P is not too small. This is possible because the $D_{\widetilde{h}}$-length of P between $\partial B_{e^{-k_1}}(0)$ and $\partial B_{e^{-k_0}}(0)$ is at least $C^{-\beta}$ for some fixed $\beta > 0$. We conclude that the circle average $\widetilde{h}_{e^{-k}}(0)$ is not small ($\widetilde{h}_{e^{-k}} \gtrsim -\log C$).
- We find a D_h-metric ball which is "tangent" to $\partial B_{e^{-k}}(0)$ and $\partial B_{e^{-k-1}}(0)$. Then, by Proposition 5.7, this metric ball (and hence $\mathcal{B}_1(0; D_{\widetilde{h}}^{\mathbb{D}})$) contains a Euclidean ball B with Euclidean radius not too small (say $e^{-(1+\zeta)k}$ for small $\zeta > 0$). Since $\widetilde{h}_{e^{-k}}(0)$ is not small, neither is the average of $\widetilde{h}$ on ∂B (i.e. $⨍_{\partial B} \widetilde{h} \gtrsim -\log C$).
- Finally, we have a good lower bound on $\mu_{\widetilde{h}}(B)$ in terms of the average of $\widetilde{h}$ on ∂B, so we find that B has not-too-small LQG volume. Since B lies in $\mathcal{B}_1(0; D_{\widetilde{h}}^{\mathbb{D}})$, we obtain a lower bound $\mu_{\widetilde{h}}(\mathcal{B}_1(0; D_{\widetilde{h}}^{\mathbb{D}})) \gtrsim C^{-\text{power}}$. This last exponent does not depend on N, so we may take $N \to \infty$ to conclude the proof of Proposition 5.27.

We now turn to the details of the proof. Let L_k be the $D_{\widetilde{h}}$-length of the subpath of P from 0 until the first time one hits $\partial B_{e^{-k}}(0)$. We emphasize that L_k is *not* the $D_{\widetilde{h}}$ distance from 0 to $\partial B_{e^{-k}}(0)$.

Lemma 5.28 (Length bounds along P)**.** *There exist positive constants $c = c(\gamma, \alpha)$ and $\beta = \beta(\gamma, \alpha)$ independent of N such that for sufficiently large C, with probability $1 - O(C^{-cN})$ the following all hold:*

$$L_{k_0} > C^{-\beta}, \tag{4.38}$$

$$L_{k_1} < C^{-\beta-1}, \tag{4.39}$$

$$L_{k-1} - L_k < C \exp\left(-k\xi(Q-\alpha) + \xi h_{e^{-k}}(0)\right) \qquad \text{for all } k \in [k_0+1, k_1]. \tag{4.40}$$

Proof. We focus first on (4.38). Using Proposition 5.6 to bound the annulus crossing distance of $B_{e^{-k_0}}(0)\backslash\overline{B_{e^{-k_0-1}}(0)}$, we see that with superpolynomially high probability as $C \to \infty$ we have

$$L_{k_0} \geq C^{-1}\left(e^{-k_0}\right)^{\xi(Q-\alpha)} \exp(\xi h_{e^{-k_0}}(0)). \tag{4.41}$$

Note that since $\mathrm{Var}(h_{e^{-k_0}}(0)) = k_0 \leq N^{-1}\log C$, we have

$$\mathbb{P}\left[\xi h_{e^{-k_0}}(0) < -\log C\right] \leq \exp\left(-\frac{(\log C)^2}{2\xi^2 N^{-1}\log C}\right) = C^{-cN}$$

for $c = 1/(2\xi^2)$. Notice that when we have both (4.41) and $\{\xi h_{e^{-k_0}} \geq -\log C\}$, then

$$L_{k_0} \geq C^{-1} \cdot C^{-\xi(Q-\alpha)/N} \cdot C^{-1} \geq C^{-\beta}$$

for the choice $\beta = 2 + \xi(Q-\alpha)$. Thus (4.38) holds with probability $1 - O(C^{-cN})$.

To prove the upper bound (4.40), we glue paths to bound $L_{k-1} - L_k$. By Proposition 5.6 and a union bound, with superpolynomially high probability as $C \to \infty$ the following event E_C holds:

- For each $k \in [k_0+1, k_1]$, there exists a path from $\partial B_{e^{-k+1}}(0)$ to $\partial B_{e^{-k-1}}(0)$ and paths in the annuli $B_{e^{-k}}(0)\backslash\overline{B_{e^{-k-1}}(0)}$ and $B_{e^{-k+2}}(0)\backslash\overline{B_{e^{-k+1}}(0)}$ which separate the circular boundaries of the annuli, and such that each of these path has $D_{\widetilde{h}}$-length at most $\frac{1}{3}C\exp\left(-k\xi(Q-\alpha) + \xi h_{e^{-k}}(0)\right)$.

Since the segment on P measured by $L_{k-1} - L_k$ is the restriction of a geodesic which crosses a larger annulus, by triangular equality, (4.40) holds on E_C.

Finally, we check that for our choice of β, the inequality (4.39) holds with probability $1 - C^{-cN}$ (possibly by choosing a smaller value of $c > 0$). By the triangle inequality, L_{k_1} is bounded from above by the sum of the $D_{\widetilde{h}}$-distance from the origin to $\partial B_{e^{-k_1+1}}(0)$ plus the $D_{\widetilde{h}}$-length of any circuit in the annulus $B_{e^{-k_1+1}}(0)\backslash\overline{B_{e^{-k_1}}(0)}$. Hence, using the circuit bound on E_C, we have

$$L_{k_1} \leq D_{\widetilde{h}}(0, \partial B_{e^{-k_1+1}}(0)) + Ce^{-k_1\xi(Q-\alpha)}e^{\xi h_{e^{-k_1}}(0)}.$$

By scaling of the metric, $D_{\widetilde{h}}(0,\partial B_{e^{-k_1+1}}(0))$ is bounded from above by $e^{\xi h_{e^{-k_1+1}}(0)}e^{(-k_1+1)\xi(Q-\alpha)}Y$ where Y is distributed as $D_{\widetilde{h}}(0,\partial B_1(0))$. Now, since $k_1=\lfloor N\log C\rfloor$ and $h_{e^{-k_1}}(0)$ has variance $N\log C$, by a Gaussian tail estimate we get

$$\mathbb{P}\left[h_{e^{-k_1}}(0)>\frac{1}{4}k_1(Q-\alpha)\right]\leq C^{-cN}.$$

Furthermore, since Y has some finite small moments for $\alpha<Q$ (by [39, Theorem 1.10]), the Markov's inequality provides

$$\mathbb{P}\left[Ye^{-\frac{1}{4}k_1\xi(Q-\alpha)}>1\right]\leq C^{-cN}.$$

Altogether, we obtain (4.39) with probability $1-O(C^{-cN})$. □

As an immediate consequence of the above lemma, we can find a scale $k\in(k_0,k_1]$ such that $\mathcal{B}_1(0;D^{\mathbb{D}}_{\widetilde{h}})$ intersects $\partial B_{e^{-k}}(0)$, and the field average at scale k is large. We introduce here a small parameter $\zeta>0$ which does not depend on C, whose value we fix at the end.

Lemma 5.29 (Existence of large field average near $\mathcal{B}_1(0;D^{\mathbb{D}}_{\widetilde{h}})$)**.** *Consider c and β as in Lemma 5.28. With probability $1-O(C^{-cN})$, there exists $k\in[k_0,k_1]$ such that $D_{\widetilde{h}}(0,\partial B_{e^{-k}}(0))<1$ and*

$$-k(Q-\alpha)+h_{e^{-k}}(0)\geq-\xi^{-1}(\beta+2)\log C; \tag{4.42}$$

moreover, there exists a Euclidean ball $B_r(z)$ with $r=e^{-k(1+\zeta)}$ and $z\in r\mathbb{Z}^2$ such that $B_r(z)\subset B_{e^{-k}}(0)\backslash\overline{B_{e^{-k-1}}(0)}$ and $B_r(z)\subset\mathcal{B}_1(0;D^{\mathbb{D}}_{\widetilde{h}})$.

Proof. To prove (4.42), we first claim that when the event of Lemma 5.28 holds, there exists $k\in[k_0+1,k_1]$ such that $L_k<1$ and $L_{k-1}-L_k\geq C^{-\beta-1}$. Let $k_\star$ be the smallest $k\in(k_0,k_1]$ such that $L_{k_\star}<C^{-\beta}$, then

$$\sum_{k=k_\star}^{k_1}L_{k-1}-L_k=L_{k_\star-1}-L_{k_1}\geq C^{-\beta}-C^{-\beta-1}.$$

Since the LHS is a sum over at most $N\log C$ terms, we indeed find some index $k\in[k_\star,k_1]$ such that

$$L_{k-1}-L_k\geq\frac{C^{-\beta}-C^{-\beta-1}}{N\log C}>C^{-\beta-1}.$$

For this choice of k, we have $D_{\widetilde{h}}(0, \partial B_{e^{-k}}(0)) \le L_k \le L_{k_\star} < C^{-\beta} < 1$, and by (4.40) we have (4.42) also.

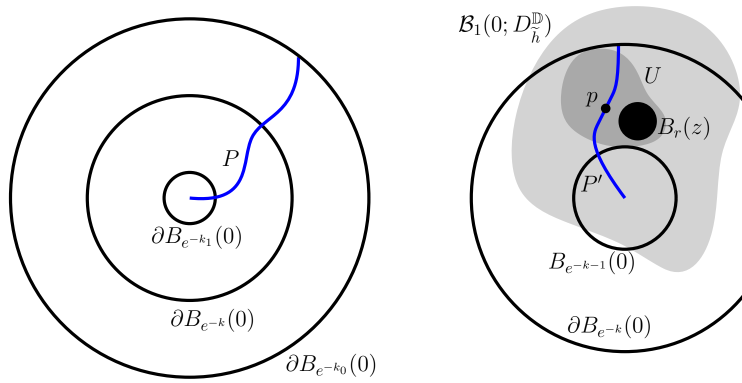

Figure 5.2 – **Left:** Setup of Lemma 5.28. Given C that we eventually sent to ∞, we take the circles with radii $e^{-k_0} \approx C^{-1/N}$ and $e^{-k_1} = C^{-N}$, and draw all circles with radii e^{-k} with $k_0 \le k \le k_1$. In Lemma 5.29 we follow the geodesic P from the outer circle to the inner until we find an annulus on which the geodesic segment is long. **Right:** Illustration of the second assertion of Lemma 5.29. We find a D_h-metric ball $U \subset \mathcal{B}_1(0; D^{\mathbb{D}}_{\widetilde{h}})$ such that U is "tangent" to $\partial B_{e^{-k}}$ and $\partial B_{e^{-k-1}}$, then apply Proposition 5.7 to find a Euclidean ball $B_r(z) \subset U$.

Now we turn to the second assertion of the lemma; see Figure 5.2 (right). Let P' be a $D_{\widetilde{h}}$-geodesic from 0 to $\partial B_{e^{-k}}(0)$. By the continuity of $D_{\widetilde{h}}$, we can find a point $p \in P'$ in the annulus $B_{e^{-k}}(0) \backslash \overline{B_{e^{-k-1}}(0)}$ such that $D_{h+(k+1)\alpha}(p, \partial B_{e^{-k}}(0)) = D_{h+(k+1)\alpha}(p, \partial B_{e^{-k-1}}(0))$; let U be the $D_{h+(k+1)\alpha}$-ball with this radius centered at p.

We claim that $U \subset \mathcal{B}_1(0; D^{\mathbb{D}}_{\widetilde{h}})$. We assume that $\alpha \ge 0$ (the other case is similar). Since $(k+1)\alpha \ge \alpha \log |\cdot|^{-1} \ge k\alpha$ on $B_{e^{-k}}(0) \backslash \overline{B_{e^{-k-1}}(0)}$, we have for all $w \in U$ that

$$D^{\mathbb{D}}_{\widetilde{h}}(p, w) \le e^{\xi\alpha} D^{\mathbb{D}}_{h+\alpha k}(p, w) \le e^{\xi\alpha} D^{\mathbb{D}}_{h+\alpha k}(p, \partial B_{e^{-k}}(0)) \le e^{\xi\alpha} D^{\mathbb{D}}_{\widetilde{h}}(p, \partial B_{e^{-k}}(0)),$$

and consequently

$$D^{\mathbb{D}}_{\widetilde{h}}(0, w) \le D^{\mathbb{D}}_{\widetilde{h}}(0, p) + D^{\mathbb{D}}_{\widetilde{h}}(p, w) \le D^{\mathbb{D}}_{\widetilde{h}}(0, p) + e^{\xi\alpha} D^{\mathbb{D}}_{\widetilde{h}}(p, \partial B_{e^{-k}}(0)) \le e^{\xi\alpha} D^{\mathbb{D}}_{\widetilde{h}}(0, \partial B_{e^{-k}}(0));$$

this last inequality follows from the fact that p lies on P' so $D^{\mathbb{D}}_{\widetilde{h}}(0, p) + D^{\mathbb{D}}_{\widetilde{h}}(p, \partial B_{e^{-k}}(0)) = D^{\mathbb{D}}_{\widetilde{h}}(0, \partial B_{e^{-k}}(0))$. Since $D^{\mathbb{D}}_{\widetilde{h}}(0, \partial B_{e^{-k}}(0)) \le L_{k_\star} < C^{-\beta}$, we conclude that $D^{\mathbb{D}}_{\widetilde{h}}(0, w) < e^{\xi\alpha} C^{-\beta} \le 1$, and hence $U \subset \mathcal{B}_1(0; D^{\mathbb{D}}_{\widetilde{h}})$.

Since U is a $D_{h+(k+1)\alpha}$ metric ball, it is also a D_h metric ball. Furthermore, since $\operatorname{diam}(U) \in (\frac{1}{2}e^{-k}, 2e^{-k})$, Proposition 5.7 gives us a Euclidean ball of radius $e^{-k(1+\zeta/2)}$ in U, and hence a Euclidean ball $B_r(z) \subset U$ with $z \in r\mathbb{Z}^2$. Since U lies in $B_{e^{-k}}(0) \backslash \overline{B_{e^{-k-1}}(0)}$ and in $\mathcal{B}_1(0; D^{\mathbb{D}}_{\widetilde{h}})$, so does $B_r(z)$, so we have shown Lemma 5.29. □

Finally, we need a regularity event to say that the $\mu_{\widetilde{h}}$-volumes of Euclidean balls are close to their field average approximations, and that the field does not fluctuate too much on each scale. The bounds in the following lemma are standard in the literature. We introduce a large parameter $q > 0$ that does not depend on C, and fix its value at the end.

Lemma 5.30 (Regularity of field averages and ball volumes)**.** *Fix $\zeta \in (0,1)$ and $q > 0$. Then for all sufficiently large $C > C_0(q, \zeta, N)$, with probability $1 - C^{-\zeta(\frac{q^2}{2N} - 2N - 1)}$ the following is true. For each $k \in [k_0, k_1]$, writing $r = e^{-k(1+\zeta)}$, for all $z \in r\mathbb{Z}^2$ such that $B_r(z) \subset B_{e^{-k}}(0) \backslash \overline{B_{e^{-k-1}}(0)}$ we have*

$$|h_r(z) - h_{e^{-k}}(0)| < kq\zeta \tag{4.43}$$

and

$$\mu_{\widetilde{h}}(B_r(z)) \geq C^{-1} r^{\gamma Q} \exp(\gamma \widetilde{h}_r(z)). \tag{4.44}$$

Proof. By standard GFF estimates, we have $\operatorname{Cov}(h_r(z), h_{e^{-k}}(0)) = k + O(1)$, $\operatorname{Var} h_r(z) = -\log r + O(1) = k(1+\zeta) + O(1)$ and $\operatorname{Var} h_{e^{-k}}(0) = k + O(1)$. Consequently,

$$\operatorname{Var}(h_r(z) - h_{e^{-k}}(0)) = \zeta k + O(1),$$

and hence by the Gaussian tail bound,

$$\mathbb{P}[|h_r(z) - h_{e^{-k}}(0)| < kq\zeta] \geq 1 - O(e^{-\frac{q^2 \zeta k}{2}}).$$

Taking a union bound over all $O(e^{2k\zeta})$ points in $r\mathbb{Z}^2 \cap B_{e^{-k}}(0)$, then summing over all $k \in [k_0, k_1]$, we see that the probability (4.43) holds for all k and all suitable z is at least

$$1 - O\left(\sum_{k=k_0}^{k_1} e^{2k\zeta} e^{-q^2\zeta k/2}\right) \geq 1 - O\left(N \log C \cdot e^{2k_1\zeta} e^{-q^2 \zeta k_0/2}\right) \geq 1 - C^{-\zeta(\frac{q^2}{2N} - 2N - 1)}.$$

Now, we establish that for each fixed choice of k, z, the inequality (4.44) holds with superpolynomially high probability as $C \to \infty$ (then we are done by a union bound over a collection of polynomially many k, z); since $-\alpha \log|\cdot| - \alpha k$ is bounded on the annulus, it suffices to show (4.44) with $\widetilde{h}$ replaced by $h + \alpha k$ (or equivalently by h, since both sides of the equation (4.44) scale the same way under adding a constant to the field). By the Markov property of the GFF (Lemma 5.5) we can decompose $h = \mathfrak{h} + \widehat{h}$, where $\mathfrak{h}$ is a distribution which is harmonic in $B_{2r}(z)$, and $\widehat{h}$ is a zero boundary GFF in the domain $B_{2r}(z)$; moreover $\mathfrak{h}$ and $\widehat{h}$ are independent. We can then write

$$
\begin{aligned}
\mu_h(B_r(z)) &\geq e^{\gamma \inf_{B_r(z)} \mathfrak{h}} \mu_{\widehat{h}}(B_r(z)) \\
&= (2r)^{\gamma Q} e^{\gamma h_r(z)} e^{-\gamma \widehat{h}_r(z)} e^{\gamma \inf_{B_r(z)} \mathfrak{h} - \gamma \mathfrak{h}(z)} \mu_g(B_{\frac{1}{2}}(0)),
\end{aligned}
$$

where $g := \widehat{h}(2r \cdot + z)$ has the law of a zero boundary GFF on $\mathbb{D}$. (This follows from an affine change of coordinates mapping $B_{2r}(z) \mapsto \mathbb{D}$; then by the coordinate change formula $\mu_{\widehat{h}}(B_r(z)) = (2r)^{\gamma Q} \mu_g(B_{\frac{1}{2}}(0))$.)

Since $\widehat{h}_r(z)$ is a mean zero Gaussian with fixed variance, and by the quantum volume lower bound (2.4), we have $e^{-\gamma \widehat{h}_r(z)} \geq C^{-1/3}$ and $\mu_g(B_{\frac{1}{2}}(0)) \geq C^{-1/3}$ with superpolynomially high probability in C. Combining these bounds with the above estimate, with superpolynomially high probability in C we have

$$
\mu_h(B_r(z)) \geq (2r)^{\gamma Q} C^{-2/3} e^{\gamma \inf_{B_r(z)} \mathfrak{h} - \gamma \mathfrak{h}(z)}.
$$

Hence we are done once we check that with superpolynomially high probability in C,

$$
e^{\gamma \inf_{B_r(z)} \mathfrak{h} - \gamma \mathfrak{h}(z)} \geq C^{-1/3}. \tag{4.45}
$$

Since $h = \mathfrak{h} + \widehat{h}$ and $\mathfrak{h}, \widehat{h}$ are independent, for $x, x' \in B_r(z)$ we have

$$
\operatorname{Var}\left(\mathfrak{h}(x) - \mathfrak{h}(x')\right) \leq \operatorname{Var}\left(h_r(x) - h_r(x')\right) = O(1).
$$

Moreover, by the scale and translation invariance of the GFF modulo additive constant and the fact that $\mathfrak{h}$ is continuous in $B_{\frac{3}{2}r}(z)$, we know that $\mathfrak{h}(z) - \inf_{B_r(z)} \mathfrak{h} > -\infty$ and has a law independent of

r, z, so by the Borell-TIS inequality we see that for some absolute constants m, c, we have

$$\mathbb{P}\left[\mathfrak{h}(z) - \inf_{B_r(z)} \mathfrak{h} > u + m\right] \leq e^{-cu^2} \quad \text{for all } u > 0.$$

This immediately implies (4.45). Thus, for each fixed choice of k, z, the inequality (4.44) holds with superpolynomially high probability as $C \to \infty$. Taking a union bound, we obtain (4.44). $\square$

Proof of Proposition 5.27. Let c, β be as in Lemma 5.28. We will work with parameters N, ζ, q, and choose their values at the end. Assume that the events of Lemmas 5.29 and 5.30 hold; this occurs with probability at least $1 - C^{-cN} - C^{-\zeta(\frac{q^2}{2N} - 2N - 1)}$. Let k, r, and $B_r(z)$ be as in Lemma 5.29.

We now lower bound the quantum volume of $B_r(z)$. By (4.42) and (4.43), we see that

$$\begin{aligned} r^{\gamma Q} \exp\left(\gamma \widetilde{h}_r(z)\right) &\geq \exp\left(-\gamma k Q(1+\zeta) + \gamma h_r(z) + \gamma \alpha k\right) \\ &\geq \exp(-\gamma \zeta k(Q+q) - \gamma k(Q-\alpha) + \gamma h_{e^{-k}}(0)) \\ &\geq \exp(-\gamma \zeta k(Q+q)) C^{-\frac{\gamma}{\xi}(\beta+2)} \\ &\geq C^{-\gamma \zeta N(Q+q)} C^{-\frac{\gamma}{\xi}(\beta+2)}. \end{aligned}$$

The last inequality follows from $k \leq k_1 = \lfloor N \log C \rfloor$. Choose $q = N^3$ and $\zeta = N^{-4}$. Then by the above inequality, (4.44), and $B_r(z) \subset \mathcal{B}_1(0; D_{\widetilde{h}}^{\mathbb{D}})$, we see that for a constant $\beta' = \beta'(\gamma) > 0$ we have

$$\mu_{\widetilde{h}}(\mathcal{B}_1(0; D_{\widetilde{h}}^{\mathbb{D}})) \geq \mu_{\widetilde{h}}(B_r(z)) \geq C^{-\beta'}.$$

Since this occurs with probability $1 - C^{-cN} - C^{-\zeta(\frac{q^2}{2N} - 2N - 1)} = 1 - O(C^{-cN})$, and N can be made arbitrarily large, we have proved Proposition 5.27. $\square$

5.4.2 Lower tail of small metric balls

Using Proposition 5.27 and the scaling properties of the LQG metric and measure, we can easily prove a similar result for metric balls centered at the origin of all radii $s \in (0, 1)$. We emphasize that in the following proposition, we are considering the D_h-metric balls, rather than $D_h^{\mathbb{D}}$-metric balls.

Lemma 5.31. *Let h be a whole-plane GFF normalized so $h_1(0) = 0$. For any $p > 0$, there exists*

C_p *such that for all* $C > C_p$ *and* $s \in (0,1)$*, we have*

$$\mathbb{P}\left[\mu_h(\mathcal{B}_s(0; D_h)) \geq C^{-1}s^{d_\gamma}\right] \geq 1 - C^{-p}.$$

Proof. The process $t \mapsto h_{e^{-t}}(0)$ for $t \geq 0$ evolves as standard Brownian motion started at 0. Fix $s \in (0,1)$ and let $T > 0$ be the first time $t > 0$ that $-Qt + h_{e^{-t}}(0) = \xi^{-1}\log s$. Notice that

$$\begin{aligned} h(e^{-T}\cdot) + Q\log e^{-T} &= \big(h(e^{-T}\cdot) - h_{e^{-T}}(0)\big) - QT + h_{e^{-T}}(0) \\ &= \big(h(e^{-T}\cdot) - h_{e^{-T}}(0)\big) + \xi^{-1}\log s. \end{aligned}$$

By Lemma 5.4, conditioned on T, we have $(h(e^{-T}\cdot) + Q\log e^{-T})\big|_{\mathbb{D}} \stackrel{d}{=} (\widehat{h} + \xi^{-1}\log s)\big|_{\mathbb{D}}$ where $\widehat{h}$ is a whole-plane GFF normalized to have mean zero on $\partial\mathbb{D}$. Couple these fields to agree. By the Weyl scaling relations and the change of coordinates formula for quantum volume and distances, and the locality property of the internal metric (Axiom II), we have the internal metric relation

$$D_h^{e^{-T}\mathbb{D}}(e^{-T}z, e^{-T}w) = D^{\mathbb{D}}_{\widehat{h}+\xi^{-1}\log s}(z,w) = sD^{\mathbb{D}}_{\widehat{h}}(z,w)$$

and the volume measure relation

$$\mu_h(e^{-T}\cdot) = \mu_{\widehat{h}+\xi^{-1}\log s}(\cdot) = s^{d_\gamma}\mu_{\widehat{h}}(\cdot).$$

Thus we can relate the quantum volume of the internal metric balls $\mathcal{B}_s(0; D_h^{e^{-T}\mathbb{D}}) \subset e^{-T}\mathbb{D}$ and $\mathcal{B}_1(0; D^{\mathbb{D}}_{\widehat{h}})$:

$$\mu_h\left(\mathcal{B}_s(0; D_h^{e^{-T}\mathbb{D}})\right) = s^{d_\gamma}\mu_{\widehat{h}}(\mathcal{B}_1(0; D^{\mathbb{D}}_{\widehat{h}+\xi^{-1}\log s})),$$

and consequently we have

$$\left\{\mu_h(\mathcal{B}_s(0; D_h^{e^{-T}\mathbb{D}})) \geq C^{-1}s^{d_\gamma}\right\} = \left\{\mu_{\widehat{h}}(\mathcal{B}_1(0; D^{\mathbb{D}}_{\widehat{h}})) \geq C^{-1}\right\}.$$

Since $\mu_h(\mathcal{B}_s(0; D_h)) \geq \mu_h(\mathcal{B}_s(0; D_h^{e^{-T}\mathbb{D}}))$, our claim follows from Proposition 5.27. $\square$

5.5 Applications and other results

5.5.1 Uniform volume estimates and Minkowski dimension

In this section, we prove the remaining assertions of Theorem 5.1. Namely, the Minkowski dimension of a bounded open set S is almost surely equal to d_γ and for any compact set $K \subset \mathbb{C}$ and $\varepsilon > 0$, we have, almost surely

$$\sup_{s\in(0,1)} \sup_{z\in K} \frac{\mu_h(\mathcal{B}_s(z;D_h))}{s^{d_\gamma-\varepsilon}} < \infty \quad \text{and} \quad \inf_{s\in(0,1)} \inf_{z\in K} \frac{\mu_h(\mathcal{B}_s(z;D_h))}{s^{d_\gamma+\varepsilon}} > 0.$$

Since the whole-plane GFF modulo additive constants has a translation invariant law, we can deduce a version of Lemma 5.31 for metric balls centered at $z \neq 0$.

Proposition 5.32 (Uniform lower tail for $\mu_h(\mathcal{B}_s(z;D_h))$)**.** *Let h be a whole-plane GFF normalized so $h_1(0) = 0$, and $K \subset \mathbb{C}$ be any compact set. For any $p > 0$, there exists $C_{p,K} > 0$ such that*

$$\sup_{s\in(0,1),z\in K} \mathbb{P}\left[\mu_h(\mathcal{B}_s(z;D_h)) \geq C^{-1}s^{d_\gamma}\right] \geq 1 - C^{-p} \quad \textit{for each } C > C_{p,K}.$$

Proof. Fix $z \in K$. We can write $h = \widehat{h} + X$ where $\widehat{h}$ is a whole-plane GFF normalized so $\widehat{h}_1(z) = 0$, and $X = h_1(z)$ is a random real number. On the event $\{|X| \leq \gamma^{-1}\log C\}$ we have $C^{-1} \leq e^{\gamma X} \leq C$, so

$$\begin{aligned}
\{\mu_h(\mathcal{B}_s(z;D_h)) < C^{-3}s^{d_\gamma}\} &= \{e^{\gamma X}\mu_{\widehat{h}}(\mathcal{B}_{e^{-\xi X}s}(z;D_{\widehat{h}})) < C^{-3}s^{d_\gamma}\} \\
&\subset \{C^{-1}\mu_{\widehat{h}}(\mathcal{B}_{C^{-1/d_\gamma}s}(z;D_{\widehat{h}})) < C^{-3}s^{d_\gamma}\} \cup \{|X| > \gamma^{-1}\log C\} \\
&= \{\mu_{\widehat{h}}(\mathcal{B}_{C^{-1/d_\gamma}s}(z;D_{\widehat{h}})) < C^{-1}(C^{-1/d_\gamma}s)^{d_\gamma}\} \cup \{|X| > \gamma^{-1}\log C\}.
\end{aligned}$$

In the last line, the first event is superpolynomially rare in C by Lemma 5.31, and the second because X is a centered Gaussian. Note that $\operatorname{Var} X = \operatorname{Var} h_1(z)$ is uniformly bounded for all $z \in K$, so the decay of the second event is uniform for $z \in K$. This completes the proof. □

Similarly, we can bootstrap Lemma 5.25 to a statement uniform for D_h-balls centered in a compact set.

Proposition 5.33 (Uniform upper tail for $\mu_h(\mathcal{B}_s(z;D_h))$). *Let h be a whole-plane GFF normalized so $h_1(0)=0$. For any compact set $K\subset\mathbb{C}$, $p>0$, $\varepsilon\in(0,1)$, there exists a constant $C_{p,\varepsilon,K}>0$ such that*

$$\sup_{s\in(0,1),z\in K}\mathbb{P}\left[\mu_h(\mathcal{B}_s(z;D_h))\le Cs^{d_\gamma-\varepsilon}\right]\ge 1-C^{-p}\quad \textit{for each } C>C_{p,\varepsilon,K}.$$

Proof. We note that Lemma 5.25 implies an upper bound version of Lemma 5.31 (with an exponent of $d_\gamma-\varepsilon$ instead of d_γ), and we deduce Proposition 5.33 in the same way that we obtain Proposition 5.32 from Lemma 5.31. □

Before moving to the proof of the almost sure uniform estimate, we first prove volume bounds on a countable collection of metric balls.

Lemma 5.34. *For any $\varepsilon>0$ and bounded open set $2\mathbb{D}$, the following is true almost surely. For all sufficiently large m, for all $z\in 2^{-m}\mathbb{Z}^2\cap 2\mathbb{D}$, and for all dyadic $s=2^{-k}\in(0,1]$ we have*

$$s^{d_\gamma-\varepsilon}2^{\varepsilon m}>\mu_h(\mathcal{B}_s(z;D_h))>s^{d_\gamma+\varepsilon}2^{-\varepsilon m}.$$

Proof. The proof is a straightforward application of Propositions 5.33 and 5.32 and the Borel-Cantelli lemma. We prove the lower bound; the upper bound follows the same argument.

Pick any large $p>0$, and let $C_{p,2\mathbb{D}}$ be the constant from Proposition 5.32. Consider any m such that $2^{\varepsilon m}>C_{p,2\mathbb{D}}$, then for any $z\in 2\mathbb{D}$ we have

$$\mathbb{P}\left[\mu_h(\mathcal{B}_s(z;D_h))>s^{d_\gamma+\varepsilon}2^{-\varepsilon m}\text{ for all dyadic } s\in(0,1]\right]>1-2^{-\varepsilon pm}\sum_{\text{dyadic } s}s^{\varepsilon p}.$$

Taking a union bound over all the $O(2^{2m})$ points in $2^{-m}\mathbb{Z}^2\cap 2\mathbb{D}$ yields

$$\mathbb{P}\left[\mu_h(\mathcal{B}_r(z;D_h))>s^{d_\gamma+\varepsilon}2^{-\varepsilon m}\text{ for all dyadic } s\in(0,1]\text{ and } z\in 2^{-m}\mathbb{Z}^2\cap 2\mathbb{D}\right]$$
$$>1-O(2^{-(\varepsilon p-2)m})\sum_{\text{dyadic } s}s^{\varepsilon p}.$$

For p large enough we have $\varepsilon p-2>0$, so by the Borel-Cantelli lemma, a.s. at most finitely many of the above events fail, i.e. the lower bound of Lemma 5.34 holds. The upper bound follows the same argument. □

With this lemma and the bi-Hölder continuity of D_h with respect to Euclidean distance, we can prove the second part of Theorem 5.1.

Proof of Theorem 5.1 part 2. We first prove that a.s. for some random $r \in (0,1)$, we have

$$\inf_{s\in(0,r]} \inf_{z\in\mathbb{D}} \frac{\mu_h(\mathcal{B}_s(z;D_h))}{s^{d_\gamma+\zeta}} > 0. \tag{5.46}$$

We use the bi-Hölder continuity of D_h with respect to Euclidean distance (see e.g. [39, Theorem 1.7]) and the Borel-Cantelli lemma to obtain the following. There exist deterministic constants $\chi, \chi' > 0$ and random constant c, C such that, almost surely,

$$c|u-v|^{\chi'} \leq D_h(u,v) \leq C|u-v|^{\chi} \quad \text{for all } u,v \in 2\mathbb{D}.$$

Moreover, Proposition 5.7 and Borell-Cantelli yield that a.s. every metric ball B contained in $2\mathbb{D}$ and having sufficiently small Euclidean diameter contains a Euclidean ball of radius at least $\operatorname{diam}(B)^2$.

Consequently, for all sufficiently small s and any $z \in \mathbb{D}$, we have

$$\frac{s}{2} \leq C \operatorname{diam}(\mathcal{B}_{s/2}(z;D_h))^{\chi},$$

and since any two points in $\mathcal{B}_{s/2}(w;D_h)$ have D_h-distance at most s, the bi-Hölder lower bound gives

$$c \operatorname{diam}(\mathcal{B}_{s/2}(z;D_h))^{\chi'} \leq s.$$

Since the ball $\mathcal{B}_{s/2}(z;D_h)$ has a small diameter, it a.s. contains a Euclidean ball of radius at least $\operatorname{diam}(\mathcal{B}_{s/2}(z;D_h))^2 \geq (s/2C)^{2/\chi}$ hence contains a point $w \in 2^{-m}\mathbb{Z}^2$ with $m = \lceil -\frac{2}{\chi}\log_2(s/2C)\rceil < -\frac{3}{\chi}\log_2(s/2C)$.

Thus, for a random constant c', for sufficiently small s, applying Lemma 5.34 to m as above and dyadic $s_1 \in (\frac{s}{4}, \frac{s}{2}]$, we have

$$\mu_h(\mathcal{B}_{s/2}(w; D_h)) \geq \mu_h(\mathcal{B}_{s_1}(w; D_h)) \geq s_1^{d_\gamma+\varepsilon} \cdot 2^{-\varepsilon m} \geq \left(\frac{s}{4}\right)^{d_\gamma+\varepsilon} \left(\frac{s}{2C}\right)^{\frac{3\varepsilon}{\chi}} = c' s^{d_\gamma+\varepsilon+\frac{3\varepsilon}{\chi}}.$$

Since $w \in \mathcal{B}_{s/2}(z; D_h)$, by the triangle inequality we have $\mathcal{B}_{s/2}(w; D_h) \subset \mathcal{B}_s(z; D_h)$, so

$$\mu_h(\mathcal{B}_s(z; D_h)) > c' s^{d_\gamma+3\varepsilon+\frac{3\varepsilon}{\chi}}.$$

Almost surely, this holds for all sufficiently small $s > 0$ and all $z \in \mathbb{D}$. Choosing $\varepsilon > 0$ so that $\varepsilon + \frac{3\varepsilon}{\chi} < \zeta$, we obtain (5.46).

The supremum analog of (5.46) follows almost exactly the same proof, except that instead of finding a "dyadic" metric ball inside each radius s metric ball, we find a dyadic metric ball $\widetilde{\mathcal{B}}$ (with dyadic radius $s_1 \in [2s, 4s)$) around each metric ball $\mathcal{B}$, then apply Lemma 5.34 to upper bound $\mu_h(\widetilde{\mathcal{B}})$ (and hence $\mu_h(\mathcal{B})$).

Now, we extend (5.46) to a supremum/infimum over all $s \in (0, 1]$. For any $s \in (r, 1]$ and $z \in \mathbb{D}$, we have

$$\frac{\mu_h(\mathcal{B}_s(z; D_h))}{s^{d_\gamma+\zeta}} \geq \mu_h(\mathcal{B}_s(z; D_h)) \geq r^{d_\gamma+\zeta} \frac{\mu_h(\mathcal{B}_r(z; D_h))}{r^{d_\gamma+\zeta}},$$

and noting that a.s. for sufficiently large R we have $D_h(\mathbb{D}, \partial B_R(0)) > 1$,

$$\frac{\mu_h(\mathcal{B}_s(z; D_h))}{s^{d_\gamma-\zeta}} \leq r^{-d_\gamma+\zeta} \mu_h(B_R(0)) < \infty.$$

This concludes the proof of the uniform volume estimates. □

Finally, we prove the statement from Theorem 5.1 about the Minkowski dimension of a set.

Proof of Theorem 5.1, part 3. Consider any bounded measurable set S containing an open set and fix $\delta \in (0, 1)$. Let N_ε^S be the minimal number of LQG metric balls with radius ε needed to cover the set S and denote by $\mathcal{C}_\varepsilon$ the set of centers associated to such a covering. Then, since

$$\mu_h(S) \leq \sum_{z \in \mathcal{C}_\varepsilon} \mu_h(\mathcal{B}_\varepsilon(z; D_h)) \leq N_\varepsilon^S \max_{z \in \mathcal{C}_\varepsilon} \mu_h(\mathcal{B}_\varepsilon(z; D_h)),$$

the uniform volume estimate and the fact that $\mu_h(S) > 0$ a.s. imply that for every $\delta > 0$, we have the a.s. lower bound $\liminf_{\varepsilon \to 0} \frac{\log N_\varepsilon^S}{\log \varepsilon^{-1}} \geq d_\gamma - \delta$. Now, denote by M_ε^S the maximal number of pairwise disjoint LQG metric balls with radius ε whose union is included in S. Denote by $\mathcal{D}_\varepsilon$ the set of centers associated to such a collection of metric balls. Note that $M_\varepsilon^S \geq N_{2\varepsilon}^S$. Therefore,

$$\mu_h(S) \geq \sum_{z \in \mathcal{D}_\varepsilon} \mu_h(\mathcal{B}_\varepsilon(z; D_h)) \geq M_\varepsilon^S \min_{z \in \mathcal{D}_\varepsilon} \mu_h(\mathcal{B}_\varepsilon(z; D_h)) \geq N_{2\varepsilon}^S \min_{z \in \mathcal{D}_\varepsilon} \mu_h(\mathcal{B}_\varepsilon(z; D_h))$$

from which we get the a.s. upper bound $\limsup_{\varepsilon \to 0} \frac{\log N_\varepsilon^S}{\log \varepsilon^{-1}} \leq d_\gamma + \delta$ by the uniform volume estimate and the fact that $\mu_h(S) < \infty$ almost surely. Letting $\delta \to 0$ completes the proof. □

5.5.2 Estimates for Liouville Brownian motion metric ball exit times

Liouville Brownian motion is, roughly speaking, Brownian motion associated to the LQG metric tensor "$e^{\gamma h}(dx^2 + dy^2)$", and was rigorously constructed independently in the works [49] and [11]. These papers consider fields different from our field h (a whole-plane GFF normalized so $h_1(0) = 0$), but their results are applicable in our setting. This can be verified either directly or by local absolute continuity arguments.

Liouville Brownian motion was defined in [11, 49] by applying an h-dependent time-change to standard planar Brownian motion. Letting B_t be standard planar Brownian motion from the origin sampled independently from h, we can define Liouville Brownian motion as $X_t = B_{F^{-1}(t)}$ for $t \geq 0$, where F is a random time-change defined h-almost surely. The function $F(t)$ should be understood as the quantum time elapsed at Euclidean time t, and has the following explicit description. Defining the approximation

$$F^\varepsilon(t) = \int_0^t \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(B_s)} ds, \tag{5.47}$$

and writing T_R for the Euclidean time that B_t exits the ball $B_R(0)$, the sequence $F^\varepsilon|_{[0,T_R]}$ converges almost surely as $\varepsilon \to 0$ to $F|_{[0,T_R]}$ in the uniform metric [11, Theorem 1.2].

For a set $X \subset \mathbb{C}$ and $z \in \mathbb{C}$, denote by $\tau_h(z; X)$ the first exit time of the Liouville Brownian motion started at z from the set X. We discuss now the results of [49] on the moments of $\tau_h(z; B_1(z))$ and of $F(t)$, i.e. the moments of the elapsed quantum time at some Euclidean time. These results are analogous to the moments of the LQG volume of a Euclidean ball (Section 5.2.3).

Proposition 5.35 (Moments of quantum time [49, Theorem 2.10, Corollary 2.12, Corollary 2.13])**.** *For all $q \in (-\infty, 4/\gamma^2)$, $t > 0$, the following holds,*

$$\mathbb{E}[\tau_h(0; B_1(0))^q] + \mathbb{E}[F(t)^q] < \infty.$$

Heuristically, the nonexistence of large moments is due to the Brownian motion hitting regions of small Euclidean size but large quantum size. On the other hand, the random set $\mathcal{B}_1(0; D_h)$ in some sense avoids such regions.

In this section we prove the finiteness of *all* moments of the LBM first exit time of $\mathcal{B}_1(0; D_h)$, which we abbreviate as τ, and discuss the moments of $\tau_h(0; \mathcal{B}_s(0; D_h))$ for small $s \in (0, 1)$.

Upper bound for LBM exit time of metric balls

Theorem 5.36 (Positive moments for quantum exit time of metric ball)**.** *Let h be a whole-plane GFF normalized so $h_1(0) = 0$, and consider Liouville Brownian motion associated to h. Let τ be the first exit time of the Liouville Brownian motion started at the origin from the ball $\mathcal{B}_1(0; D_h)$, i.e.*

$$\tau = \inf\{t \geq 0 \,:\, X_t \notin \mathcal{B}_1(0; D_h)\}.$$

Then

$$\mathbb{E}[\tau^k] < \infty \quad \text{for all } k \geq 0.$$

Proof sketch: In computing $\mathbb{E}[\tau^k]$, by first averaging out the randomness of $(B_t)_{t \geq 0}$, we obtain an expectation in h of an integral over k-tuples of points in $\mathcal{B}_1(0; D_h)$; this is similar to the integral in Step 1 of the proof of Proposition 5.18, but with additional log-singularities between these points. Because the arguments of Proposition 5.18 had some room in the exponents, the log-singularities pose no issue for us, and we can carry out the same arguments from Section 5.3. We will be succinct when adapting these arguments.

Let τ_n be the quantum time LBM spends in the annulus $A_{2^n} := B_{2^n}(0) \backslash \overline{B_{2^{n-1}}(0)}$ before exiting $\mathcal{B}_1(0; D_h)$. As in [49, (B.2)], we have the following representation of $\mathbb{E}[\tau_n^k]$ for k a positive integer, which follows from taking an expectation over the standard Brownian motion $(B_t)_{t\geq 0}$ used to define $(X_t)_{t\geq 0}$ (see (5.47)),

$$\mathbb{E}[\tau_n^k] = \mathbb{E}\left[\int_{(A_{2^n})^k} f(z_1,\dots,z_k,h)\mathbb{1}\{z_1,\dots,z_k \in \mathcal{B}_1(0;D_h)\}\mu_h(dz_1)\dots\mu_h(dz_k)\right], \tag{5.48}$$

and where, writing $t_0 = 0$ and $z_0 = 0$ for notational convenience, f is given by

$$f(z_1,\dots,z_k,h) := \int_{0\leq t_1\leq\dots\leq t_k<\infty} \frac{k!}{(2\pi)^{k/2}\prod_{i=1}^k (t_i - t_{i-1})} \exp\left(-\sum_{i=1}^k \frac{|z_i - z_{i-1}|^2}{2(t_i - t_{i-1})}\right) \tag{5.49}$$
$$\times\, \mathbb{P}\left[B|_{[0,t_k]} \text{ stays in } \mathcal{B}_1(0;D_h) \mid h, B_{t_i} = z_i \text{ for } i = 1,\dots,k\right] dt_1\dots dt_k.$$

The function $f(z_1,\dots,z_k)$ is an integral of the Brownian motion transition density at times $t_1,\dots,t_k$ times the conditional probability that the Brownian motion does not escape $\mathcal{B}_1(0;D_h)$. We will need the following bound on f, whose proof is postponed to the end of the section.

Lemma 5.37. *There exists a constant $C > 0$ such that for all sufficiently large $R > 0$, on the event $\{\mathcal{B}_1(0;D_h) \subset B_R(0)\}$ we have*

$$f(z_1,\dots,z_k,h) \leq C(\log R)^k g(z_1,\dots,z_k) \quad \textit{for all } z_1,\dots,z_k \in R\mathbb{D},$$

where, recalling $z_0 = 0$,

$$g(z_1,\dots,z_k) = \prod_{i=1}^k \max(-\log|z_i - z_{i-1}|, 1).$$

Proof of Theorem 5.36. Our strategy is to fix some large $R > 0$ then truncate on the event $E'_R := \{\mathcal{B}_1(0;D_h) \subset B_R(0)\}$. Subsequently, we show an analog of Proposition 5.18, and use it to bound $\mathbb{E}[\tau_n^k \mathbb{1}_{E'_R}]$ for all n. Combining these, we obtain a bound on $\mathbb{E}[\tau^k \mathbb{1}_{E'_R}]$. Finally, we verify that $\mathbb{P}[E'_R]$ decays sufficiently quickly in R, and we are done.

Step 1: Proving an analog of Proposition 5.18. Recall the definition $P_h^{r,d} = \{z \in \mathbb{C} : D_h(z, \partial B_{r/4}(z)) \leq d\}$ in (3.27). The argument of Proposition 5.18 bounded

$$\mathbb{E}\left[\int_{(A_1)^k} \mathbb{1}\{z_1,\dots,z_k \in A_1 \cap P_h^{1,e^{-\xi x}}\}\mu_h(dz_1)\dots\mu_h(dz_k)\right]$$

by using a Cameron-Martin shift (placing γ-log singularities at each z_i and replacing $\prod \mu_h(dz_i)$ by $\prod_{i<j}|z_i - z_j|^{-\gamma^2} \prod dz_i$), then using Proposition 5.15 to bound the integral. Recalling Remark 5.17, Proposition 5.18 can be proved even if the exponent γ^2 is made slightly larger. Any such exponent increase will upper bound the log-singularities of g, hence we have the following analog of Proposition 5.18:

$$\mathbb{E}\left[\int_{(A_1)^k} g(z_1, \ldots, z_k)\mathbb{1}\{z_1, \ldots, z_k \in A_1 \cap P_h^{1,e^{-\xi x}}\}\mu_h(dz_1)\ldots\mu_h(dz_k)\right] \lesssim e^{-c_{k,\delta}x}.$$

Step 2: Bounding $\mathbb{E}[\tau_n^k \mathbb{1}_{E'_R}]$ *for each* n*.* We start with $n = 0$. Using Lemma 5.37 and (5.48) (and noting that $\mathcal{B}_1(0; D_h) \cap A_1 \subset A_1 \cap P_h^{1,1}$), we obtain that $\mathbb{E}[\tau_0^k \mathbb{1}_{E'_R}]$ is bounded from above by

$$(\log R)^k\, \mathbb{E}\left[\int_{(A_1)^k} g(z_1, \ldots, z_k)\mathbb{1}\{z_1, \ldots, z_k \in \mathcal{B}_1(0; D_h)\}\mu_h(dz_1)\ldots\mu_h(dz_k)\right] \lesssim (\log R)^k\,,$$

where the last inequality follows from Step 1. Likewise, building off of Step 1, similar arguments as in Lemmas 5.23 and 5.24 yield

$$\mathbb{E}\left[\tau_n^k \mathbb{1}_{E'_R}\right] \lesssim \begin{cases} (\log R)^k\, 2^{-\frac{Q^2}{2}|n|} 2^{\alpha_\delta |n|} & \text{if } n < 0, \\ (\log R)^k\, 2^{-\frac{Q^2}{2}n} & \text{if } n > 0. \end{cases}$$

for some arbitrarily small $\alpha_\delta > 0$.

Step 3: Bounding the upper tail of τ*.* By Hölder's inequality (see end of proof of Lemma 5.23), the above bounds on $\mathbb{E}\left[\tau_n^k \mathbb{1}_{E'_R}\right]$ yield

$$\mathbb{E}\left[\tau^k \mathbb{1}_{E'_R}\right] \lesssim (\log R)^k\,.$$

By Lemma 5.38 (see end of section) we also have for some fixed $a > 0$ that

$$\mathbb{P}[(E'_R)^c] \leq R^{-a}$$

Combining these assertions, we have

$$\mathbb{P}\left[\tau > t\right] \lesssim \mathbb{P}\left[(E'_R)^c\right] + \mathbb{E}\left[\tau^k \mathbb{1}_{E'_R}\right] t^{-k} \lesssim R^{-a} + (\log R)^k t^{-k}.$$

Taking R equal to some large power of t, we conclude that for all $p < k$ we have $\mathbb{E}[\tau^p] < \infty$. Taking $k \to \infty$, we obtain Theorem 5.36. □

Proof of Lemma 5.37. We instead prove the stronger statement

$$f(z_1, \ldots, z_k, h) \le C \prod_{i=1}^{k} \left(\log R - \log|z_i - z_{i-1}|\right) \quad \text{for all } z_1, \ldots, z_k \in A_1.$$

We split the integral (5.49) into two parts (integrating over $t_k < R^2$ and $t_k \ge R^2$ respectively), and bound each part separately.

There exists $p > 0$ such that the following is true: Let $t \ge 1/k$ and consider a Brownian bridge of duration t with endpoints B_0, B_t specified in $\mathbb{D}$. Then this Brownian bridge stays in $\mathbb{D}$ with probability at most e^{-pt}. If $t_k \ge R^2$, then there exists some $i \in \{1, \ldots, k\}$ such that $t_i - t_{i-1} \ge t_k/k \ge R^2/k$, and so $B|_{[t_{i-1}, t_i]}$ conditioned on $B_{t_{i-1}} = z_{i-1}$ and $B_{t_i} = z_i$ stays in $R\mathbb{D}$ with probability at most e^{-pt_k/kR^2}. This allows us to upper bound the integral (5.49) on the restricted domain with $t_k \ge R^2$:

$$\begin{aligned}
&\int_{0 \le t_1 \le \cdots \le t_k < \infty} \frac{k! \quad dt_1 \ldots dt_k}{(2\pi)^{k/2} \prod_{i=1}^{k} (t_i - t_{i-1})} \exp\left(- \sum_{i=1}^{k} \frac{|z_i - z_{i-1}|^2}{2(t_i - t_{i-1})} - \frac{p}{kR^2}(t_i - t_{i-1}) \right) \\
=& \frac{k!}{(2\pi)^{k/2}} \prod_{i=1}^{k} \int_0^\infty \frac{1}{t} \exp\left(- \frac{|z_i - z_{i-1}|^2}{2t} - \frac{p}{kR^2} t \right) dt = O\left(\prod_{i=1}^{k} \left(\log R - \log|z_i - z_{i-1}|\right) \right),
\end{aligned}$$

by using the bound $\int_0^\infty e^{-t/x} e^{-1/t} \frac{dt}{t} \le \int_0^1 e^{-1/t} \frac{dt}{t} + \int_x^\infty e^{-t/x} \frac{dt}{t} + \int_1^x \frac{dt}{t} \le C + \log x$ for $x \ge 1$ and a change of variable.

Now we upper bound the integral (5.49) on the restricted domain $0 \le t_1 \le \cdots \le t_k < R^2$:

$$\int_{0\le t_1\le\cdots\le t_k<R^2} \frac{k!}{(2\pi)^{k/2}\prod_{i=1}^k (t_i - t_{i-1})} \exp\left(-\sum_{i=1}^k \frac{|z_i - z_{i-1}|^2}{2(t_i - t_{i-1})}\right) dt_1 \dots dt_k$$
$$\le \frac{k!}{(2\pi)^{k/2}} \prod_{i=1}^k \int_0^{R^2} \frac{1}{t} \exp\left(-\frac{|z_i - z_{i-1}|^2}{2t}\right) dt = O\left(\prod_{i=1}^k (\log R - \log|z_i - z_{i-1}|)\right),$$

where the final inequality follows from $\int_0^{R^2} e^{-a/2t}\frac{dt}{t} = \int_0^1 e^{-1/2u}\frac{du}{u} + \int_1^{R^2 a^{-2}} e^{-1/2u}\frac{du}{u} \le C + \log R^2 a^{-2}$. Combining these two upper bounds, we are done. $\square$

Lemma 5.38 (Polynomial tail for Euclidean diameter of $\mathcal{B}_1(0; D_h)$)**.** *Let h be a whole-plane GFF with $h_1(0) = 0$. Then for all $a \in (0, Q^2/2)$, for all sufficiently large R we have*

$$\mathbb{P}\left[\mathcal{B}_1(0; D_h) \subset B_R(0)\right] \ge 1 - R^{-a}.$$

Proof. Fix $\varepsilon > 0$ small. By Proposition 5.6 we have with superpolynomially high probability as $R \to \infty$ that

$$D_h(0, \partial B_R(0)) \ge D_h(\partial B_{R/2}(0), \partial B_R(0)) \ge R^{\xi(Q-\varepsilon)} e^{\xi h_R(0)}.$$

By a standard Gaussian tail bound we also have

$$\mathbb{P}[h_R(0) > -(Q-\varepsilon)\log R] \le \exp\left(-\frac{(Q-\varepsilon)^2 \log R}{2}\right) = R^{-(Q-\varepsilon)^2/2}.$$

Altogether, we see that with probability $1 - O(R^{-(Q-\varepsilon)^2/2})$ we have $D_h(0, \partial B_R(0)) > 1$, as desired. $\square$

Lower bound for LBM exit time of metric balls

Theorem 5.39. *Recall that τ is the first exit time of the Liouville Brownian motion $(X_t)_{t\ge 0}$ from the LQG metric ball $\mathcal{B}_1(0; D_h)$. For all $k \ge 1$, we have*

$$\mathbb{E}[\tau^{-k}] < \infty.$$

We now sketch the proof. We restrict to a regularity event on which annulus-crossing distances and the quantum time taken to cross an annulus are well approximated by field averages. We can find a collection of annuli separating 0 from X_τ. Gluing circuit and crossing paths associated to the annuli, we obtain a path from 0 to X_τ. Since the D_h-length of these is bounded from above by a circle average approximation, the condition $D_h(0, X_\tau) = 1$ gives a lower bound for a certain sum of (exponentials of) circle averages terms. Raising the exponent by a factor of d_γ by Jensen's inequality, we get a lower bound for a circle average approximation of the quantum time spent across these annuli. Thus τ is unlikely to be very small.

Consider standard Brownian motion $(B_t)_{t\geq 0}$ started at the origin, and recall that Liouville Brownian motion is given by a random time-change: $X_t = B_{F^{-t}(t)}$, where the quantum clock F is formally given by $F(t) = \int_0^t e^{\gamma h(B_s)} ds$ (see (5.47)). Consider an annulus $A_{r/e,r}(z)$ with $0 \notin A_{r/e,r}(z)$. Define $\tau_r(z)$ to be the quantum passage time of the annulus. That is, for the case where the annulus encircles the origin, writing t_1 for the first time B_t hits $\partial B_r(z)$, and t_0 for the last time before t_1 that B_t hits $\partial B_{r/e}(z)$, we set $\tau_r(z) = F(t_1) - F(t_0)$, and define it analogously in the case that the annulus does not encircle the origin.

We need the following input, which can be seen as a variant of [49, Proposition 2.12] combined with the scaling relation [49, Equation (2.25)] and which can be obtained by using the same techniques.

Proposition 5.40. *For any compact set $K \subset \mathbb{C}$, there exists a random variable $X \geq 0$ having all negative moments such that the following is true. For fixed $r \in (0, 1)$ and $z \in K$ such that $0 \notin A_{r/e,r}(z)$, the quantum passage time $\tau_r(z)$ is stochastically dominated by $r^{\gamma Q} e^{\gamma h_r(z)} X$.*

As an immediate consequence of the $r = 1$ case of this proposition, we have the following.

Corollary 5.41. *The event $\{X_\tau \notin \mathbb{D}$ and $\tau < C^{-1}\}$ is superpolynomially unlikely as $C \to \infty$.*

Similarly to Section 5.4.1, we set

$$k_1 = \lfloor N \log C \rfloor.$$

Lemma 5.42. *There exist γ-dependent constants $\chi, c > 0$ so that the following holds. Consider the event E_C that each ball $B_{e^{-k_1}}(z)$ included in $2\mathbb{D}$ has quantum diameter at most $2e^{-\chi k_1}$. Then, E_C occurs with probability at least $1 - e^{-cN}$.*

Proof. This is an application of the Hölder estimate [39, Proposition 3.18] which implies that there exist positive constants χ, α such that, as $\varepsilon \to 0$, with probability at least $1 - \varepsilon^\alpha$,

$$D_h(u, v) \le |u - v|^\chi, \quad \forall u, v \in 2\mathbb{D} \text{ with } |u - v| \le \varepsilon.$$

Therefore, taking $\varepsilon = e^{-k_1}$, for z such that $B_{e^{-k_1}}(z) \subset 2\mathbb{D}$, for all $w \in B_{e^{-k_1}}(z)$, $D_h(z, w) \le e^{-\chi k_1}$ and the quantum diameter of that ball is bounded from above by twice this upper bound. □

We consider the grid $\mathbb{Z}_C := \frac{1}{100} e^{-k_1} \mathbb{Z}^2$.

Lemma 5.43. *Consider the event F_C that for every point $z \in \mathbb{Z}_C \cap 2\mathbb{D}$, for all $k \in [0, k_1]$, the following conditions hold. There is a circuit of D_h-length at most $e^{-k\xi Q} e^{\xi h_{e^{-k}}(z)} C$ in the annulus $A_{e^{-k-1}, e^{-k}}(z)$, the crossing length $D_h(\partial B_{e^{-k-1}}(z), \partial B_{e^{-k+1}}(z))$ is at most $e^{-k\xi Q} e^{\xi h_{e^{-k}}(z)} C$, $\tau_{e^{-k}}(z) \ge e^{-k\gamma Q} e^{\gamma h_{e^{-k}}(z)} C^{-1}$ and, finally, $|h_{e^{-k}}(z) - h_{e^{-k+1}}(z)| \le \xi^{-1} \log C$. Then, F_C occurs with superpolynomially high probability as $C \to \infty$.*

Proof. This follows from Proposition 5.40 and Proposition 5.6 together with a union bound. □

Proof of Theorem 5.39. We will show that $P[\tau > C^{-1}]$ occurs with superpolynomially high probability. By Corollary 5.41 and Lemmas 5.42 and 5.43, we see that the probability of $\{\tau < C^{-1}$ and $X_\tau \notin \mathbb{D}\} \cup E_C^c \cup F_C^c$ is at most C^{-cN} for some fixed c.

Now restrict to the event $\{X_\tau \in \mathbb{D}\} \cap E_C \cap F_C$; we show that for some constant α not depending on C, N we have $\tau > C^{-\alpha}$ for sufficiently large C, then we are done since N is arbitrary. On this event the distances $D_h(0, \partial B_{e^{-k_1}}(0))$ and $D_h(X_\tau, \partial B_{e^{-k_1}}(X_\tau))$ are small, so we have $D_h(\partial B_{e^{-k_1}}(0), \partial B_{e^{-k_1}}(X_\tau)) \ge \frac{1}{2}$. Let $w \in \mathbb{Z}_C$ be the closest point to X_τ, and grow the annuli centered at 0 and w until they first hit; let $k_* \in [0, k_1]$ satisfy $2e^{-k_*} \le |w| < 2e^{-k_*+1}$. By Lemma 5.43 we get

$$\tau \ge \sum_{k \in [k_*, k_1]} \tau_{e^{-k}}(0) + \tau_{e^{-k}}(w) \ge C^{-1} \sum_{k \in [k_*, k_1]} e^{-k\gamma Q} e^{\gamma h_{e^{-k}}(0)} + e^{-k\gamma Q} e^{\gamma h_{e^{-k}}(w)}$$

and, by taking an additional annulus crossing and circuit, using the circle average regularity between two annuli,

$$\frac{1}{2} \le D_h(\partial B_{e^{-k_1}}(0), \partial B_{e^{-k_1}}(X_\tau)) \le 10C^2 \sum_{k\in[k_*,k_1]} e^{-k\xi Q} e^{\xi h_{e^{-k}}(0)} + e^{-k\xi Q} e^{\xi h_{e^{-k}}(w)}.$$

Therefore, by raising the inequality above to the power d_γ and using Jensen's inequality for the right-hand side, as well as the lower bound for τ, we get

$$\frac{1}{2^{d_\gamma}} \le (10C^2)^{d_\gamma} k_1^{d_\gamma - 1} \sum_{k\in[k_*,k_1]} e^{-k\gamma Q} e^{\gamma h_{e^{-k}}(0)} + e^{-k\gamma Q} e^{\gamma h_{e^{-k}}(w)} \le (10C^2)^{d_\gamma} k_1^{d_\gamma - 1} C\tau.$$

hence $\tau \ge C^{-\alpha}$ for some fixed power α and C large enough. Since N is arbitrary (α does not depend on N), we conclude the proof of Theorem 5.39. □

Scaling relations for small balls Finally we explain the behavior of small ball exit times. Recall that $\tau_h(z; \mathcal{B}_s(z; D_h))$ is the first time that Liouville Brownian motion started at z exits the ball $\mathcal{B}_s(z; D_h)$.

Theorem 5.44. *Let h be a whole-plane GFF normalized so $h_1(0) = 0$, and let $K \subset \mathbb{C}$ be any compact set. For any $\varepsilon \in (0,1)$, there exists a constant $C_{p,\varepsilon,K}$ so that for $C > C_{p,\varepsilon,K}$, for all $s \in (0,1)$ and $z \in K$ we have*

$$\mathbb{P}[\tau_h(z; \mathcal{B}_s(z; D_h)) \le C s^{d_\gamma - \varepsilon}] \ge 1 - C^p, \tag{5.50}$$

and

$$\mathbb{P}[\tau_h(z; \mathcal{B}_s(z; D_h)) \ge C^{-1} s^{d_\gamma}] \ge 1 - C^p. \tag{5.51}$$

Proof. We first discuss the proofs of (5.50) and (5.51) for the specific case $z = 0$. For the $z = 0$ upper bound, recall that we proved $\mathbb{E}[\tau_h(0; \mathcal{B}_1(0; D_h))^k] < \infty$ for all $k > 0$ in Theorem 5.36 by adapting the proof of Proposition 5.8 . An extension of these arguments like in Lemma 5.25 yields $\mathbb{E}[\tau_h(0; \mathcal{B}_s(0; D_h))^k] \lesssim s^{kd_\gamma - \varepsilon}$ with implicit constant depending only on k, ε, and hence by Markov's inequality, for all $s \in (0,1)$ and sufficiently large C that

$$\mathbb{P}[\tau_h(0; \mathcal{B}_s(0; D_h)) \le C s^{d_\gamma - \varepsilon}] \ge 1 - C^p. \tag{5.52}$$

For the $z = 0$ lower bound, Theorem 5.39 gives $\mathbb{E}[\tau_h(0; \mathcal{B}_1(0; D_h))^{-k}] < \infty$ for all $k > 0$, and applying the rescaling argument of Lemma 5.31 then yields for all $s \in (0, 1)$ and sufficiently large C that

$$\mathbb{P}[\tau_h(0; \mathcal{B}_s(0; D_h)) \geq C^{-1} s^{d_\gamma}] \geq 1 - C^p. \tag{5.53}$$

Finally, the arguments of Proposition 5.32 allow us to extend (5.52) and (5.53) to (5.50) and (5.51). □

5.5.3 Recovering the conformal structure from the metric measure space structure of γ-LQG

The Brownian map is constructed as a random metric measure space (see [75, 76]) and has been proved to be the Gromov-Hausdorff limit of uniform triangulations and $2p$-angulations in [73–75, 78]. The Brownian map was later endowed with a canonical conformal structure (i.e. an embedding into a flat domain, defined up to conformal automorphism of the domain) via identification with $\sqrt{8/3}$-LQG [81–83, 86] but this construction was non-explicit. The work of [60] gives an explicit way to recover the conformal structure of a Brownian map from its metric measure space structure, and their proof mostly carries over directly to the general setting $\gamma \in (0, 2)$, except for certain Brownian map metric ball volume estimates of Le Gall [74]. The missing ingredient for general γ was exactly the uniform volume estimates (1.2)(cf. [60, Lemma 4.9]).

As an immediate consequence of (1.2) and the arguments of [60] (see discussion before [60, Remark 1.3]), we obtain the following generalization of [60, Theorem 1.1] to all $\gamma \in (0, 2)$. Let h be a whole-plane GFF normalized so $h_1(0) = 0$, and write $\mathcal{B}^\bullet_R(0; D_h)$ for the *filled* D_h-ball centered at 0 with radius R (i.e. the union of $\mathcal{B}_R(0; D_h)$ and all μ_h-finite complementary regions). Let $\mathcal{P}^\lambda$ be a sample from the intensity λ Poisson point process associated to μ_h. We can obtain a D_h-Voronoi tessellation of $\mathbb{C}$ into cells $\{H^\lambda_z\}_{z \in P^\lambda}$ by defining $H^\lambda_z = \{w \in \mathbb{C} : D_h(w, z) \leq D_h(w, z') \,\forall\, z' \in P^\lambda\}$. We define a graph structure on $\mathcal{P}^\lambda$ by saying that $z, z' \in P^\lambda$ are adjacent if their Voronoi cells $H^\lambda_z, H^\lambda_{z'}$ intersect along their boundaries, and define ∂P^λ to be the vertices corresponding to Voronoi cells intersecting the boundary. Let Y^λ be a simple random walk on $\mathcal{P}^\lambda$ started from the point whose Voronoi cell contains 0, extend Y^λ from the integers to $[0, \infty)$ by interpolating along D_h-geodesics, and finally stop Y^λ when it hits ∂P^λ.

Theorem 5.45 (Generalization of [60, Theorem 1.1])**.** *As $\lambda \to \infty$, the conditional law of Y^λ given $(\mathbb{C}, 0, D_h, \mu_h)$ converges in probability as $\lambda \to 0$ to standard Brownian motion in $\mathbb{C}$ started at 0 and stopped when it hits $\partial \mathcal{B}^\bullet_R(0; D_h)$ (viewed as curves modulo time parametrization).*

Here, the metric on curves modulo time parametrization is given as follows. For curves $\eta_j : [0, T_j] \to \mathbb{C}$ $(j = 1, 2)$, we set

$$d(\eta_1, \eta_2) = \inf_{\phi} \sup_{t \in [0,T_1]} |\eta_1(t) - \eta_2(\phi(t))|$$

where the infimum is over increasing homeomorphisms $\phi : [0, T_1] \to [0, T_2]$. We remark that the convergence in Theorem 5.45 holds uniformly for the random walk and Brownian motion started in a compact set, and moreover holds for a range of quantum surfaces such as quantum spheres, quantum cones, quantum wedges, and quantum disks; see [60, Theorem 3.3]. Consequently, the Tutte embedding of the Poisson-Voronoi tessellation of the quantum disk converges to the quantum disk as $\lambda \to \infty$ (see the proof of [60, Theorem 1.2]).

Proof. Since we have the estimates (1.2), the general $\gamma \in (0, 2)$ version of [60, Theorem 3.3] holds. In particular, Theorem 5.45 holds if we replace the field h with that of a 0-quantum cone. By comparing h to the field of a 0-quantum cone and using local absolute continuity arguments, we obtain Theorem 5.45. □

Notice that the construction of Y^λ involves only the pointed metric measure space structure of $(\mathbb{C}, 0, D_h, \mu_h)$, so Theorem 5.45 roughly tells us that we can recover the conformal structure of $(\mathbb{C}, 0, D_h, \mu_h)$ from its metric measure space structure. The following variant of [60, Theorem 1.2] makes this observation explicit, resolving a question of [59].

Theorem 5.46 (Pointed metric measure space $(\mathbb{C}, 0, D_h, \mu_h)$ determines conformal structure)**.** *Let h be a whole-plane GFF normalized so $h_1(0) = 0$. Almost surely, given the pointed metric measure space $(\mathbb{C}, 0, D_h, \mu_h)$, we can recover its conformal embedding into $\mathbb{C}$ and hence recover h (both modulo rotation and scaling).*

Proof. To simplify the notation, suppose the two-pointed metric measure space $(\mathbb{C}, 0, 1, D_h, \mu_h)$ is given, then we show we can recover exactly the embedding of μ_h in $\mathbb{C}$ (otherwise, one can arbitrarily pick any other point from the pointed metric measure space and use that in place of 1, and only recover the embedded measure modulo rotation and scaling). Since μ_h (with its embedding in $\mathbb{C}$) determines h [14] and hence D_h, it suffices to recover μ_h.

Consider R large so $1 \in \mathcal{B}^\bullet_R(0; D_h)$. In the same way that [60, Theorem 1.1] is used to prove [60, Theorem 1.2], we can use Theorem 5.45 to obtain an embedding of the two-pointed metric measure space $(\mathcal{B}^\bullet_R(0; D_h), 0, 1, D_h, \mu_h)$ into the unit disk $\mathbb{D}$ with the correct conformal structure and sending 0 to 0 and 1 to a point in $(0, 1)$.

This is done by taking a λ-intensity Poisson-Voronoi tessellation of $(\mathcal{B}^\bullet_R(0; D_h), 0, 1, D_h, \mu_h)$, and embedding its adjacency graph P^λ in $\mathbb{D}$ via the *Tutte embedding* Φ^λ: let $x_0, \ldots, x_n$ be the vertices in ∂P^λ in counterclockwise order with x_0 arbitrarily chosen, and let z_0 (resp. z_1) be the vertex corresponding to the Poisson-Voronoi cell containing 0 (resp. 1). Define the map $\widetilde{\Phi}^\lambda : P^\lambda \to \overline{\mathbb{D}}$ via $\widetilde{\Phi}^\lambda(z_0) = 0, \widetilde{\Phi}(x_0) = 1$ and $\widetilde{\Phi}^\lambda(x_j) = e^{2\pi i p_j}$ where p_j is the probability that Y^λ hits ∂P^λ at one of the points $x_0, \ldots, x_j$, and extend $\widetilde{\Phi}$ to the rest of P^λ so it is discrete harmonic. Finally, define $\Phi^\lambda(z) = e^{i\theta}\widetilde{\Phi}^\lambda(z)$ where $\theta \in [0, 2\pi)$ is chosen so $\Phi(z_1) \in \mathbb{R}$. Taking $\lambda \to \infty$, the Φ^λ-pushforward of the counting measure on the vertices of the embedded graph normalized by λ^{-1} converges weakly in probability to the desired conformally embedded measure. See [60, Section 3.3] for details.

Rescale this embedding (and forget the metric) to obtain an equivalent two-pointed measure space $(c_R\mathbb{D}, 0, 1, \mu^R)$ with the LQG measure and conformal structure. That is, there exists a conformal map $\varphi^R : \mathcal{B}^\bullet_R(0; D_h) \to c_R\mathbb{D}$ such that $\varphi^R(0) = 0, \varphi^R(1) = 1$, and the pushforward $(\varphi^R)^*\mu_h$ equals μ^R. We emphasize that since we are only given $(\mathbb{C}, 0, 1, D_h, \mu_h)$ as a two-pointed metric measure space, we know neither the embedding $\mathcal{B}^\bullet_R(0; D_h) \subset \mathbb{C}$ nor the conformal map φ^R, but we do know c_R and μ^R.

Now, by a simple estimate on the distortion of conformal maps [85, Lemma 2.4] (stated for the cylinder $\mathbb{R} \times [0, 2\pi]$ but applicable to our setting via the map $z \mapsto e^{-z}$), we see that for any compact $K \subset \mathbb{C}$ we have $\lim_{R\to\infty} \sup_{z\in K} |\varphi^R(z) - z| = 0$ and $\lim_{R\to\infty} \sup_{z\in K} |(\varphi^R)^{-1}(z) - z| = 0$. Thus, for any fixed rectangle A, the measure of the symmetric difference $\mu_h \left(A \triangle (\varphi^R)^{-1}(A)\right)$ converges to zero as $R \to \infty$; this implies $\lim_{R\to\infty} |\mu^R(A) - \mu_h(A)| = 0$. Since μ^R is a function of the two-pointed metric measure space $(\mathbb{C}, 0, 1, D_h, \mu_h)$, we conclude that $\mu_h(A)$ is also. Therefore the two-pointed metric measure space $(\mathbb{C}, 0, 1, D_h, \mu_h)$ determines μ_h and hence h. $\square$

5.6 Appendix

5.6.1 Proof of the inductive relation for small moments

Lemma 5.47. *Recall $v_k(r)$ and $u_k(r)$ from* (2.5)*. The following relation holds.*

$$v_k(r) \le Cr^{-2} \sum_{i=1}^{k-1} \binom{k}{i} (4k)^{\gamma^2 i(k-i)} r^{-\gamma^2 i(k-i)} u_i(4r) u_{k-i}(4r). \tag{6.54}$$

Proof. Set $f_k(z_1, \dots z_k) := \prod_{i<j} |z_i - z_j|^{-\gamma^2}$. Note that when $\max_{i<j} |z_i - z_j| \le r$, the k points are included in $B(z_1, r)$ which itself is included in a ball of radius $4r$ centered at at point of $r\mathbb{Z}^2 \cap \mathbb{D}$. Since f_k is a function of the pairwise distance, which is translation invariant, we get

$$\begin{aligned} v_k(r) &= \int_{\mathbb{D}^k} \frac{1_{r/2 \le \max_{i<j} |z_i - z_j| \le r}}{\prod_{i<j} |z_i - z_j|^{\gamma^2}} dz_1 \dots dz_k \\ &\le Cr^{-2} \int_{4r\mathbb{D}^k} 1_{r/2 \le \max_{i<j} |z_i - z_j|} f_k(z_1, \dots, z_k) dz_1 \dots dz_k \end{aligned}$$

Then, take two points at distance $r/2$ in $4r\mathbb{D}$, say z and w among $\{z_1, \dots, z_k\}$. We cut $k+1$ orthogonal sections of same width to the segment $[z, w]$. At least one should be empty and this separates two clusters of points, $I = \{z_{p_1}, \dots, z_{p_i}\}$ and $J = \{z_{q_1}, \dots, z_{q_{k-i}}\}$ for some $1 \le i \le k-1$. All points between the two clusters I and J are separated by $|z-w|/(k+1) \ge r/4k$. We decouple $f_k(z_1, \dots, z_k)$ for two clusters I and J of size i and $k-i$ by $f_k(z_1, \dots, z_k) \le (4k)^{\gamma^2 i(k-i)} r^{-\gamma^2 i(k-i)} f_i(I) f_{k-i}(J)$. In particular, splitting over the possibles cases we get

$$v_k(r) \le Cr^{-2} \sum_{i=1}^{k-1} \sum_{I} (4k)^{\gamma^2 i(k-i)} r^{-\gamma^2 i(k-i)} \int_{4r\mathbb{D}^k} f_i(I) f_{k-i}(J) dz_1 \dots dz_k,$$

where for each i, I ranges over all subsets of $\{z_1 \dots, z_k\}$ with i elements. This gives

$$v_k(r) \le Cr^{-2} \sum_{i=1}^{k-1} \binom{k}{i} (4k)^{\gamma^2 i(k-i)} r^{-\gamma^2 i(k-i)} u_i(4r) u_{k-i}(4r).$$

and completes the proof. □

5.6.2 Whole-plane GFF and $\star$-scale invariant field

In this section we recall some properties of $\star$-scale invariant fields and explain that the whole-plane GFF modulo constants can be seen as a $\star$-scale invariant field.

We will denote by $\mathcal{S}(\mathbb{C})$ the space of space of Schwartz functions and by $L^2(\mathbb{C})$ the space of square integrable functions, on $\mathbb{C}$. For $f, g \in L^2(\mathbb{C})$, let $\langle f, g \rangle$ stands for the $L^2(\mathbb{C})$ inner product. Furthermore, $*$ denotes the convolution operator.

$\star$-scale invariant field ϕ We introduce here the field $\phi = \sum_{k \geq 1} \phi_k$ we work with in Section 5.3.1. The notation and definition are close to the one in [38, Section 2.1] and we refer the reader to this Section for more details.

Consider k, a smooth, radially symmetric and nonnegative bump function supported in $B_{1/(2e)}(0)$, such that k is normalized in $L^2(\mathbb{C})$. We set $c = k * k$ which has therefore compact support included in $B_{1/e}(0)$ and satisfies $c(0) = 1$. We consider a space-time white noise $\xi(dx, dt)$ on $\mathbb{C} \times [0, \infty)$ and define the random Schwartz distribution

$$\phi(x) := \int_0^1 \int_{\mathbb{C}} k\left(\frac{x-y}{t}\right) t^{-3/2} \xi(dy, dt).$$

The covariance kernel of ϕ is given by $\mathbb{E}(\phi(x)\phi(x')) = \int_0^1 c(\frac{x-x'}{t})\frac{dt}{t}$. We decompose $\phi = \sum_{k \geq 1} \phi_k$ where $\phi_k(x) := \int_{e^{-k}}^{e^{-(k-1)}} \int_{\mathbb{C}} k\left(\frac{x-y}{t}\right) t^{-3/2} \xi(dy, dt)$ and whose covariance kernel is given by $C_k(x, x') := \int_{e^{-k}}^{e^{-(k-1)}} c(\frac{x-x'}{t})\frac{dt}{t}$. Note that $C_k(x, x') = C_1(e^{(k-1)}x, e^{(k-1)}x')$ and that if $|x - x'| \geq e^{-1}$, $C_1(x, x') = 0$ hence ϕ_k has finite range dependence with range of dependence e^{-k}. Note also that the pointwise variance of $\phi_{0,n} := \sum_{1 \leq k \leq n} \phi_k$ is equal to n.

Lemma 5.48. *There exists $C, c > 0$ such that for all $k \geq 0$, $x > 0$, $\mathbb{P}(e^{-k} \|\nabla \phi_{0,k}\|_{e^{-k}S} \geq x) \leq Ce^{-cx^2}$, where S denotes the unit square $[0,1] \times [0,1]$.*

Proof. This is essentially the argument as in the proof of Lemma 10.1 in [38] which we recall. By Fernique's theorem, $\mathbb{P}(\|\nabla \phi_1\|_S \geq x) \leq Ce^{-cx^2}$. Therefore, by scaling, $\mathbb{P}(e^{-\ell} \|\nabla \phi_\ell\|_{e^{-\ell}S} \geq x) \leq Ce^{-cx^2}$ for $\ell \geq 1$. By setting $X_\ell := e^{-\ell} \|\nabla \phi_\ell\|_{e^{-\ell}S}$, by the triangle inequality and since $e^{-k}S \subset e^{-\ell}S$ for $\ell \leq k$, $e^{-k} \|\nabla \phi_{0,k}\|_{e^{-k}S} \leq \sum_{0 \leq \ell \leq k} e^{-(k-\ell)} X_\ell$. By inspecting the Laplace functional, and using that the X_ℓ's are independent and identically distributed, we conclude the proof of the Lemma. $\square$

Whole-plane GFF We explain here why $\int_0^\infty k(\frac{x-y}{t})t^{-3/2}\xi(dy,dt)$ is a whole-plane GFF modulo constants. Set $\phi_\varepsilon(x) = \int_\varepsilon^{\varepsilon^{-1}}\int_{\mathbb{C}} k\left(\frac{x-y}{t}\right)t^{-3/2}\xi(dy,dt)$ and take $f \in \mathcal{S}(\mathbb{C})$ such that $\int_{\mathbb{C}} f dx = 0$. Writing $C_\varepsilon(x) := \int_\varepsilon^{\varepsilon^{-1}} c\left(\frac{x}{t}\right)\frac{dt}{t} = \int_\varepsilon^{\varepsilon^{-1}} c_t(x)\frac{dt}{t}$ with $c_t(\cdot) = c(\cdot/t)$, we have

$$\mathbb{E}\left(\langle\phi_\varepsilon, f\rangle^2\right) = \int_{\mathbb{C}\times\mathbb{C}} f(x)C_\varepsilon(x-y)f(y)dxdy = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\hat{C}_\varepsilon(\xi)|\hat{f}(\xi)|^2 d\xi$$

where our convention for the Fourier transform is $\hat{g}(\xi) := \int_{\mathbb{C}} g(x)e^{-i\xi\cdot x}$.

We compute the Fourier transform $\hat{C}_\varepsilon(\xi) = \int_\varepsilon^{\varepsilon^{-1}}\hat{c}_t(\xi)\frac{dt}{t} = \int_\varepsilon^{\varepsilon^{-1}} t\hat{c}(t\xi)dt$ and since $c = k*k$, $\hat{c} = \hat{k}^2$, then $\hat{C}_\varepsilon(\xi) = \int_\varepsilon^{\varepsilon^{-1}} t\hat{k}(t\xi)^2dt = \|\xi\|^{-2}\int_{\varepsilon\|\xi\|}^{\varepsilon^{-1}\|\xi\|} u\hat{k}(u)^2du$. By monotone convergence, we get

$$\begin{aligned}\mathbb{E}\left(\langle\phi_\varepsilon, f\rangle^2\right) &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\|\xi\|^{-2}\int_{\varepsilon\|\xi\|}^{\varepsilon^{-1}\|\xi\|} u\hat{k}(u)^2du|\hat{f}(\xi)|^2d\xi \\ &\underset{\varepsilon\to 0}{\to}\left(\int_0^\infty u\hat{k}(u)^2du\right)\times\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\|\xi\|^{-2}|\hat{f}(\xi)|^2d\xi.\end{aligned}$$

Since $\hat{k}$ is radially symmetric and k is normalized in L^2, by Plancherel's theorem $\int_0^\infty u\hat{k}(u)^2du = 2\pi$. Furthermore, by setting $g(x) = \int_{\mathbb{C}}\log|x-y|f(y)dy$ we get $\Delta g = 2\pi f$ and in Fourier modes, $-\|\xi\|^2\hat{g}(\xi) = 2\pi\hat{f}(\xi)$ hence, by Plancherel's theorem,

$$\begin{aligned}\int_{\mathbb{C}^2} f(x)(-\log|x-y|)f(y)dxdy = -\int_{\mathbb{C}} f(x)g(x)dx &= \frac{-1}{(2\pi)^2}\int_{\mathbb{R}^2}\hat{f}(\xi)\hat{g}(\xi)d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}^2}\|\xi\|^{-2}|\hat{f}(\xi)|^2d\xi.\end{aligned}$$

Note that this term is finite because under the assumption $\int_{\mathbb{C}} f dx = 0$, we have $\hat{f}(0) = 0$ so the above singularity at the origin is compensated by the first term in the development of $\hat{f}$. Altogether, we get

$$\mathbb{E}\left(\langle\phi_\varepsilon, f\rangle^2\right)\underset{\varepsilon\to 0}{\to}\int_{\mathbb{C}^2} f(x)(-\log|x-y|)f(y)dxdy$$

Hence the convergence of the characteristic functionals: $\mathbb{E}(e^{i\langle\phi_\varepsilon,f\rangle}) = e^{-\frac{1}{2}\mathbb{E}\left(\langle\phi_\varepsilon,f\rangle^2\right)}\underset{\varepsilon\to 0}{\to} e^{-\frac{1}{2}\mathbb{E}\left(\langle h,f\rangle^2\right)}$.

The following lemma will be useful when working with the whole plane GFF *not* modulo additive constant.

Lemma 5.49. *There exists a coupling of the whole-plane GFF h normalized such that $h_1(0) = 0$ and the $\star$-scale invariant field ϕ such that the difference $h - \phi$ is a continuous field.*

Proof. Recall the notation $\phi_{k,\ell} = \int_{e^{-\ell}}^{e^{-k}} k(\frac{x-y}{t}) t^{-3/2} \xi(dy, dt)$. We know $\phi_{-\infty,\infty}$ is a whole-plane GFF modulo constant. The fine field $\phi = \phi_{0,\infty}$ is a well-defined Schwartz distribution. Also, the gradient field $\nabla \phi_{-\infty,0}$ is a well-defined continuous Gaussian vector (this can be checked by inspecting the covariance kernel and applying the Kolmogorov continuity theorem). Thus, $\phi_{-\infty,0}$ is well defined modulo additive constant, so $\phi_L :=$ "$\phi_{-\infty,0} - \fint_{\partial B_1(0)} \phi_{-\infty,0}$" is a well-defined continuous Gaussian field, independent of ϕ. By setting $g := \phi_L - \fint_{\partial B_1(0)} \phi$, we get that $h := \phi + g$ is a whole-plane GFF normalized such that $h_1(0) = 0$. □

5.6.3 Volume of small balls in the Brownian map

We do not use any material in this section in our proofs, but include it to facilitate an easier comparison between our argument in Section 5.3 and the analogous result for the Brownian map case. Le Gall obtained the following uniform estimate on the volume of small balls in the Brownian map. For $\beta \in (0, 1)$, there exists a random $K_\beta > 0$ such that for every $r > 0$, the volume of any ball of radius r in the Brownian map is bounded from above by $K_\beta r^{4-\beta}$. Our proof of the finiteness of LQG ball volume positive moments (Section 5.3) shares some similarities with his only at a very high level; no explicit formulas are available in our framework, and the techniques are very different. We discuss some of the arguments used in the Brownian map setting and we refer the reader to [72–74, 77] for details. This estimate was used in the proofs of the uniqueness of the Brownian map [75, 78].

Tree of Brownian paths A binary marked tree is a pair $\theta = (\tau, (h_v)_{v\in\tau})$ where τ is a binary plane tree and where for $v \in \tau$, h_v is the length of the branch associated to v. We denote by $\Lambda_k(d\theta)$ the uniform measure on the set of binary marked trees with k leaves (uniform measure over binary plane trees and Lebesgue measures for the length of the branches). $I(\theta)$ and $L(\theta)$ will denote respectively the internal nodes and leaves of θ. One can define a Brownian motion on such a tree: the process is a standard Brownian motion over a branch, and after an intersection, the two processes evolve independently conditioning on the value at the node. We will denote by P_x^θ this process, started from the root of the tree with initial value x. Similarly, instead of using a Brownian motion, one can consider a 9-dimensional Bessel process and we will denote it by Q_x^θ.

Similarly, for trees given by a contour function $(h(s))_{s\leq\sigma}$ with lifetime σ, one can associate the so-called Brownian snake given by the process $(W_s)_{s\leq\sigma}$ of Brownian type path (for each s, W_s is a Brownian type path with lifetime $h(s)$, its last value is denoted by $\widehat{W}_s$ and corresponds to the Brownian label above the point of the tree corresponding to s). We can add another level of randomness by taking h given by a Brownian type excursion: $\mathbb{N}_0$ is the measure associated to the unconditioned lifetime Itô excursion, $\overline{\mathbb{N}}_0$ is also associated to the unconditioned lifetime Ito excursion but the Brownian labels are conditioned to stay positive.

Explicit formulas The following explicit formula (see [72], Proposition IV.2), relates the objects of the previous paragraph. For $p \geq 1$, $x \in \mathbb{R}$ and F a symmetric nonnegative measurable function on W^p, where W denotes the space of finite continuous paths,

$$\mathbb{N}_x\left[\int_{(0,\sigma)^p} F(W_{s_1},\dots,W_{s_p})ds_1\dots ds_p\right] = 2^{p-1}p!\int \Lambda_p(d\theta)P_x^\theta\left[F((w^{(a)})_{a\in L(\theta)})\right]. \tag{6.55}$$

Here, w is the tree-indexed Brownian motion with law P_x^θ and $w^{(a)}$ the restriction of w to the path joining a to the root, and $\mathbb{N}_x$ is the measure $\mathbb{N}_0$ where each Brownian snake has its labels incremented by x. This formula involves combining the branching structure of certain discrete trees with spatial displacements. It relies on nice Markovian properties, in particular on specific properties of the Itô measure. The proof of the uniform volume bound for metric ball is based on an explicit formula obtained in [77] for the finite-dimensional marginal distributions of the Brownian tree under $\overline{\mathbb{N}}_0$,

$$\overline{\mathbb{N}}_0\left[\int_{(0,\sigma)^p} F(W_{s_1},\dots,W_{s_p})ds_1\dots ds_p\right] = 2^{p-1}p!\int \Lambda_p(d\theta)Q_0^\theta\left[F((\overline{w}^{(a)})_{a\in L(\theta)}\prod_{b\in I(\theta)}\overline{V}_b^4\prod_{c\in L(\theta)}\overline{V}_c^{-4}\right]. \tag{6.56}$$

Here, we write $\overline{w}$ and $\overline{w}^{(a)}$ for the nine-dimensional Bessel process counterparts of w and $w^{(a)}$, and $\overline{V}_v$ for the value of the Bessel process at the vertex v. Because of the conditioning of $\overline{\mathbb{N}}_0$, the spatial displacements are given by nine-dimensional Bessel processes rather than linear Brownian motions. To derive such a formula, in [77] the authors generalize (6.55) to functionals including the range of labels and lifetime σ and then use results on absolute continuity relations between Bessel processes, which are consequences of the Girsanov theorem (note that integrals over time of Brownian motions are integral over branches of trees of Brownian motion).

Positive moment estimates In the proof of the upper bound on small ball volumes of the Brownian map in [74], a key estimate is to show that, for $k \geq 1$, $c_k < \infty$ where

$$
\begin{aligned}
c_k &:= \overline{\mathbb{N}}_0 \left[\left(\int_0^\sigma 1_{\{\widehat{W}_s \leq 1\}} ds \right)^k \right] \\
&= 2^{k-1} k! \int Q_0^\theta \left[\big(\prod_{a \in I(\theta)} \overline{V}_a^4 \big) \big(\prod_{b \in L(\theta)} \overline{V}_b^{-4} 1_{\overline{V}_b \leq 1} \big) \right] \Lambda_k(d\theta) =: 2^{k-1} k! \tilde{d}_k. \quad (6.57)
\end{aligned}
$$

Note that the second inequality follows by (6.56) with $F(W_{s_1}, \ldots, W_{s_k}) = 1_{\widehat{W}_{s_1} \leq 1}, \ldots, 1_{\widehat{W}_{s_k} \leq 1}$. The proof works by induction, introducing an additional parameter to take care of the value of the label at the splitting node in the branching structure, by setting

$$
\tilde{d}_k(r) := \int Q_r^\theta \left[\left(\prod_{a \in I(\theta)} \overline{V}_a^4 \right) \left(\prod_{b \in L(\theta)} \overline{V}_b^{-4} 1_{\overline{V}_b \leq 1} \right) \right] \Lambda_k(d\theta).
$$

In this framework, the base case and inductive relation are quite straightforward because of the exact underlying branching structure. Let R denote a 9-dimensional Bessel process that starts from r. The base case corresponds to

$$
\tilde{d}_1(r) = \mathbb{E} \left[\int_0^\infty R_t^{-4} 1_{\{R_t \leq 1\}} dt \right] = c \int_{\mathbb{R}^9} |r - z|^{-7} |z|^{-4} 1_{\{|z| \leq 1\}} dz \quad (6.58)
$$

and the inductive relation states

$$
\tilde{d}_\ell(r) = \mathbb{E} \left[\int_0^\infty R_t^4 \left(\sum_{j=1}^{\ell-1} \tilde{d}_j(R_t) \tilde{d}_{\ell-j}(R_t) \right) \right]. \quad (6.59)
$$

Now, one can easily derive the bounds $\tilde{d}_1(r) \leq Mr^{-2} \wedge r^{-7}$ and for $j \geq 2$ $\tilde{d}_j(r) \leq M_j 1 \wedge r^{-7}$. We underline that the exact branching structure of the framework is expressed through the equality (6.59).

Comparison Let us compare our proof of the finiteness of positive moments with the one in the Brownian map setting. In our setup, no nice branching structure for distances is known. Furthermore, by working with a given embedding or a restriction to a specific domain, we have to carry in the analysis information about the Euclidean domain, including an additional layer of difficulty.

In the case of the Brownian map, when one considers the "volume" associated with (6.57) thanks to the explicit formulas (6.55) and (6.56), one ends up with branching Bessel processes on uniform trees. In our framework, analogous observables of "distances" are not well understood so far. Instead, circle averages processes are tractable. They evolve as correlated Brownian motions. These are a good proxy for the metric because of the superconcentration of side-to-side crossing distances. Furthermore, when one weights the distribution with singularities (after a Cameron-Martin argument), these Brownian motions are shifted by drifts. (Note that the passage from (6.55) to (6.56) uses Girsanov.)

Similarities can be seen as the level of induction where the value of the Bessel process at the first node is comparable with the value of the circle average of the field at the first branching in our hierarchical decomposition. So Lemma 5.14 is similar to (6.59) and Proposition 5.15 to (6.57).

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