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Coframe geometry, gravity and electromagnetism

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Abstract. The extensions of GR for description of fermions on a curved space, for supergravity, and for the loop quantum gravity ordinary use a set of 16 independent variables instead of 10 components of metric. These variables can be assembled in a coframe field, i.e., a set of four linearly independent 1-forms. In this presentation we review a geometrical structure based on the coframe field. We construct a complete class of the coframe connections which are linear in the first order derivatives of the coframe field on an n dimensional manifolds with and without a metric. The subclasses of the torsion-free, metric-compatible and flat connections are derived. We also study the behavior of the geometrical structures under local transformations of the coframe. The remarkable fact is an existence of a subclass of connections which are invariant when the infinitesimal transformations satisfy the Maxwell-like system of equations.

1. Introduction. Why do we have to go beyond Riemannian geometry?

General relativity (GR) is, probably, the best of the known theories of gravity. From mathematical and aesthetic points of view, it can be used as a standard of what a physical theory has to be. Einstein's theory is in a very good agreement with all up-to-date observation data. The central idea of Einstein's GR is that the physical properties of the gravitational field are in one-to-one correspondence with the geometry of the base manifold. The standard GR is based on a (pseudo) Riemannian geometry with a unique metric tensor and a unique Levi-Civita connection constructed from this tensor.

After classical works of Weyl, Cartan and others, it is known that Riemannian construction is not a unique possible geometry. A most general geometric framework involves independent metric and independent connection. Gravity field models based on such general geometry (metric-affine gravity) was studied intensively, see [1]—[17] and the references given therein. Probably a main problem of these constructions is a huge number of geometrical fields which do not find their physical co-partners.

In this paper we study a much more economical construction based on a unique geometrical object — coframe field. Absolute (teleparallel) frame/coframe variables (repèr, vierbein, ...) were introduced in physics by Einstein in 1928 with an aim of a unification of gravitational and electromagnetic fields (for classical references, see [18]). The physical models for gravity based on the coframe variable are well studied, see [19]—[43]. In some aspects such models are even preferable to the standard GR. In particular, they involve a meaningful definition of the gravitational energy, which is in a proper correspondence with the Noether procedure [33], [36]. Moreover some problems inside and beyond Einstein's gravity, such as: i) Hamiltonian formulation of GR[44],[45]; (ii) positive energy proofs [46]; (iii) fermions on a curved manifold

[47],[48]; (iv) supergravity [49]; (v) loop quantum gravity [50], require a richer set of 16 independent variables of the coframe.

In this paper, we present a geometric structure that can be constructed from the vierbein (frame/coframe) variable.

The organization of the paper is as follows:

In the first section, we construct a geometrical structure based on a coframe variable as unique building block. In an addition to the coframe volume element and metric, we present a most general coframe connection. The Levi-Civita and flat connections are special cases of it. The torsion and nonmetricity tensors of the general coframe connection are calculated. We identify the subclasses of symmetric (torsion-free) connections and of metric-compatible connections. The unique symmetric metric-compatible connection is of Levi-Civita. In section 2, we study the transformation properties of the coframe field and identify a subclass of connections which are invariant under restricted coframe transformations. In section 3, we study the invariant conditions on the coframe transformations. In particular, we present a quite remarkable fact that invariance conditions are approximated by a Maxwell-type system. It gives a possibility to model the electromagnetic field by the elements of the transformation group. Exact spherical symmetric solution for our model is presented. It is bounded near the Schwarzschild radius. Further off, the solution is close to the Coulomb field.

In the last section, some proposals of possible developments of a geometrical coframe construction and its applications to gravity and electromagnetism are presented.

2. Coframe geometry

In this section, we define a geometry based on a coframe field. It is instead of the of the standard Riemannian geometry based on a metric tensor field.

2.1. Coframe manifold. Definitions and notations

Coframe field. Let an $(n+1)$-dimensional differential manifold M endowed with a smooth nondegenerate coframe field ϑ^α be given. The coframe comes together with its dual — the frame field e_α. In an arbitrary chart of local coordinates $\{x^i\}$, these fields are expressed as

$$\vartheta^\alpha = \vartheta^\alpha{}_i dx^i\,, \qquad e_\alpha = e_\alpha{}^i\partial_i\,, \tag{2.1}$$

i.e., by two nondegenerate matrices $\vartheta^\alpha{}_i$ and $e_\alpha{}^i$ which are the inverse to each-other. In other words, we are dealing with a set of n^2 independent smooth functions on M. The coframe indices change in the same range $\alpha, \beta, \ldots = 0, \ldots, n$ as the coordinate indices $i, j, \ldots$. However they must be strictly distinguished. In particular, the indices in $\vartheta^\alpha{}_i$ or $e_\alpha{}^i$ cannot be contracted.

Coframe transformation. For most physical models based on the coframe field, this field is defined only up to global transformations. It is natural to consider a wider class of coframe fields related by local pointwise transformations

$$\vartheta^\alpha \to L^\alpha{}_\beta(x)\vartheta^\beta\,, \qquad e_\alpha \to L_\alpha{}^\beta(x) e_\beta\,. \tag{2.2}$$

Here $L^\alpha{}_\beta(x)$ and $L_\alpha{}^\beta(x)$ are inverse to each-other at arbitrary point x. Denote the group of matrices $L^\alpha{}_\beta(x)$ by G. Note two specially important cases: (i) G is a group of global transformations with a constant matrix $L^\alpha{}_\beta$; (ii) G is a group of arbitrary local transformations such that the entries of $L^\alpha{}_\beta$ are arbitrary functions of a point.

Consequently we involve an additional element of the coframe structure — *the coframe transformations group*

$$G = \Big\{ L^\alpha{}_\beta(x) \in GL(n+1, \mathbb{R});\ \text{for every}\ x \in M \Big\}\,. \tag{2.3}$$

On this stage, we only require the matrices $L^\alpha{}_\beta(x)$ to be invertible at an arbitrary point $x \in M$. The successive specializations of the coframe transformation matrix will be considered in sequel.

Coframe field volume element. We assume the coframe field to be non-degenerate at an arbitrary point $x \in M$. Consequently, a special $n+1$-form, *the coframe field volume element*, is defined and nonzero. Define

$$\mathrm{vol}(\vartheta^\alpha) = \frac{1}{n!}\,\varepsilon_{\alpha_o\cdots\alpha_n}\vartheta^{\alpha_o}\wedge\cdots\wedge\vartheta^{\alpha_n}\,, \tag{2.4}$$

where $\varepsilon_{\alpha_o\cdots\alpha_n}$ is the Levi-Civita permutation symbol normalized by $\varepsilon_{01\cdots n} = 1$. Treating the coframe volume element as one of the basic elements of the coframe geometric structure, we obtain the following invariance condition.

Volume element invariance condition: Volume element $\mathrm{vol}(\vartheta^\alpha)$ is assumed to be invariant under pointwise transformations of the coframe field

$$\mathrm{vol}\,(\vartheta^\alpha) = \mathrm{vol}\left(L^\alpha{}_\beta\vartheta^\beta\right)\,. \tag{2.5}$$

This condition is satisfied by matrices with unit determinant. Consequently, the coframe transformation group (2.3) is restricted to

$$G = \left\{L^\alpha{}_\beta(x) \in SL(n+1,\mathbb{R});\ \text{for every}\ x \in M\right\}. \tag{2.6}$$

Metric tensor. For a meaningful physical field model, it is necessary to have a metric structure on M. Moreover, the metric tensor has to be of the Lorentzian signature. In a coordinate basis, a generic metric tensor is written as

$$g = g_{ij}dx^i \otimes dx^j\,, \tag{2.7}$$

where the components g_{ij} and g_{ab} are smooth functions of a point $x \in M$.

On a coframe manifold, a metric tensor is not an independent quantity. Instead, we are looking for a metric explicitly constructed from a given coframe field, $g = g(\vartheta^\alpha)$. We assume the metric tensor to be quadratic in the coframe field components and independent of its derivatives. Moreover, it should be reducible at a point to the Lorentzian metric $\eta_{\alpha\beta} = \mathrm{diag}(-1,1,\cdots,1)$. With these restrictions, we come to a definition of the *coframe field metric tensor*

$$g = \eta_{\alpha\beta}\vartheta^\alpha\otimes\vartheta^\beta\,, \qquad g_{ij} = \eta_{\alpha\beta}\vartheta^\alpha{}_i\vartheta^\beta{}_j\,. \tag{2.8}$$

Note that the equations (2.8) often appear as a definition of a (non unique) orthonormal basis of reference for a given metric. Another interpretation treats (2.8) as an expression of a given metric in a special orthonormal basis of reference, as in (2.7). In our approach, (2.8) has a principle different meaning. It is a definition of the metric tensor field via the coframe field. Certainly the form of the metric $\eta_{\alpha\beta}$ in the tangential vector space T_xM is an additional axiom of our construction. With an aim to define an invariant coframe geometric structure we require:

Metric tensor invariance condition: Metric tensor is assumed to be invariant under pointwise transformations of the coframe field, i.e.,

$$g\,(\vartheta^\alpha) = g\left(L^\alpha{}_\beta\vartheta^\beta\right)\,. \tag{2.9}$$

This condition is satisfied by pseudo-orthonormal matrices,

$$\eta_{\mu\nu}L^\mu{}_\alpha L^\nu_\beta = \eta_{\alpha\beta}. \tag{2.10}$$

Consequently, the invariance of the coframe metric restricts the coframe transformation group to

$$G=\Big\{L^\alpha{}_\beta(x)\in O(1,n,\mathbb{R});\ \text{for every}\ x\in M\Big\}. \tag{2.11}$$

Volume&metric invariance condition: In order to have simultaneously a metric and a volume element structures both constructed from the coframe field, we have to assume a successive restriction of the coframe transformation group:

$$G=\Big\{L^\alpha{}_\beta(x)\in SO(1,n,\mathbb{R});\ \text{for every}\ x\in M\Big\}. \tag{2.12}$$

Topological restrictions. A global smooth coframe field may be defined only on a parallelizable manifold, i.e., on a topological manifold of a zero second Whitney class. This topological restriction is equivalent to existence of a spinorial structure on M. We restrict ourselves to a local consideration, thus the global definiteness problems will be neglected. Moreover, we assume the coframe field to be smooth and nonsingular only in a "weak" sense. Namely, the components $\vartheta^\alpha{}_i$ and $e_\alpha{}^i$ are required to be differentiable and linearly independent at almost all points of M, except of a zero measure set. So, in general, the coframe field can degenerate at singular points, on singular lines (strings), or even on singular submanifolds (p-branes). This assumption leaves a room for the standard singular solutions of the physics field equations such as the Coulomb field, the Schwarzschild metric, the Kerr metric etc..

2.2. Coframe connections

A differential manifold endowed with a coframe field is a rather poor structure. In particular, we can not determine if two vectors attached at distance points are parallel to each-other or not. In order to have a meaningful geometry and, consequently, meaningful geometrical field models, we have to consider a reacher structure. In this section we define a coframe manifold with a linear coframe connection. The connection 1-form $\mathit{\Gamma}_a{}^b$ will not be an independent variable, as in the Cartan geometry or in MAG [5]. Alternatively, in our construction, the connection will be explicitly constructed from the coframe field and its first order derivatives. Thus we are dealing with a category of *coframe manifolds with a linear coframe connection:*

$$\Big\{M\,,\vartheta^\alpha,G\,,\Gamma_a{}^b(\vartheta^\alpha)\Big\}. \tag{2.13}$$

We start with a coframe manifold without an addition metric structure. Metric contributions to the connection will be considered in sequel.

Affine connection. Recall the main properties of a generic linear affine connection on an $(n+1)$ dimensional differential manifold. Relative to a local coordinate chart x^i, a connection is represented by a set of $(n+1)^3$ independent functions $\Gamma^k{}_{ij}(x)$ — *the coefficients of the connection.* The only condition these functions have to satisfy is to transform, under a change of coordinates $x^i\mapsto y^i(x)$, by an inhomogeneous linear rule:

$$\Gamma^i{}_{jk}\mapsto\left(\Gamma^l{}_{mn}\frac{\partial y^m}{\partial x^j}\frac{\partial y^n}{\partial x^k}+\frac{\partial^2 y^l}{\partial x^j\partial x^k}\right)\frac{\partial x^i}{\partial y^l}. \tag{2.14}$$

When instead of the coordinate basis $\{dx^i,\partial/\partial x^i\}$, an arbitrary reference basis $\{\theta^a\,,f_b\}$ is involved, the coefficients of the connection are arranged in a $GL(n,\mathbb{R})$-valued *connection 1-form*, which is defined as [51]

$$\mathit{\Gamma}_a{}^b=f_a{}^k\left(\theta^b{}_i\Gamma^i{}_{jk}-\theta^b{}_{k,j}\right)dx^j. \tag{2.15}$$

To return to the holonomic coordinate basis, we can simply use the identities $\theta^a{}_i = \delta^a_i$ and $f_a{}^i = \delta^i_a$. Consequently, in a coordinate basis, the connection 1-form is

$$\mathit{\Gamma}_j{}^i = \Gamma^i{}_{jk}\, dx^k\,. \tag{2.16}$$

Due to (2.14), this quantity transforms under the coordinate transformations as

$$\mathit{\Gamma}_j{}^i \to \left[\Gamma_m{}^l \frac{\partial y^m}{\partial x^j} + d\left(\frac{\partial y^l}{\partial x^j}\right)\right]\frac{\partial x^i}{\partial y^l}\,. \tag{2.17}$$

Alternatively, the connection 1-form (2.15) is invariant under smooth transformations of coordinates. The inhomogeneous linear behavior is shifted here to the transformations of $\mathit{\Gamma}_a{}^b$ under a linear local map of the reference basis ($\theta^a \to A^a{}_b\theta^b, f_a \to A_a{}^b f_b$)

$$\mathit{\Gamma}_a{}^b \mapsto \left(\Gamma_c{}^d A_a{}^c + dA_a{}^d\right) A^b{}_d\,. \tag{2.18}$$

On a manifold with a given coframe field ϑ^α, the connection 1-form (2.15), can also be referred to this field. We denote this quantity by $\mathit{\Gamma}_\alpha{}^\beta$. Due to (2.15) it reads:

$$\mathit{\Gamma}_\alpha{}^\beta = e_\alpha{}^k\left(\vartheta^\beta{}_i\Gamma^i{}_{jk} - \vartheta^\beta{}_{k,j}\right) dx^j\,. \tag{2.19}$$

This quantity can be treated as an expression of a generic connection (2.15) in a special basis. An essential difference between two very similar equations (2.15) and (2.19) is vizualized when we use in both formulas the connection constructed explicitly from the derivatives of the coframe field itself $\Gamma^i{}_{jk}(\vartheta^\alpha)$.

Linear coframe connections. We restrict ourselves to the quasi-linear $\Gamma^i{}_{jk}(\vartheta^\alpha)$, i.e., we consider a connection constructed as a linear combination of the first order derivatives of the coframe field. The coefficients in this linear expression may depend on the frame/coframe components. In other words, we are looking for a coframe analog of an ordinary Levi-Civita connection.

Let us assist ourselves with a similar construction from the Riemannian geometry. In this case, we are looking for a most general connection that can be constructed from the metric tensor components. Consider a general linear combination of the first order derivatives of the metric tensor:

$$g^{mi}(\alpha_1 g_{mj,k} + \alpha_2 g_{mk,j} + \alpha_3 g_{jk,m})\,. \tag{2.20}$$

Although this expression has the same index content as $\Gamma^i{}_{jk}$, it is a connection only for some special values of the parameters $\alpha_1, \alpha_2.\alpha_3$. Indeed, any two connections are differ by a tensor. Thus an arbitrary connection can be expressed as a certain special connection plus a tensor

$$\Gamma^i{}_{jk} = \overset{*}{\Gamma}{}^i{}_{jk} + K^i{}_{jk}\,. \tag{2.21}$$

Use for $\overset{*}{\Gamma}{}^i{}_{jk}$ the Levi-Civita connection

$$\overset{*}{\Gamma}{}^i{}_{jk} = \frac{1}{2}g^{im}(g_{mj,k} + g_{mk,j} - g_{jk,m})\,. \tag{2.22}$$

However in Riemannian geometry, does not exist a tensor constructed from the first order derivatives of the metric. Therefore $K^i{}_{jk} = 0$, thus the Levi-Civita connection is a unique connection that can be constructed from the first order derivatives of the metric tensor. It is evidently symmetric and metric compatible.

In an analogy to this construction, we will look for a most general coframe connection of the form

$$\Gamma^i{}_{jk}(\vartheta^\alpha) = \overset{\circ}{\Gamma}{}^i{}_{jk}(\vartheta^\alpha) + K^i{}_{jk}(\vartheta^\alpha)\,. \tag{2.23}$$

Here $\overset{\circ}{\Gamma}{}^i{}_{jk}$ is a certain special connection, while $K^i{}_{jk}$ is a tensor. To start with, we need a certain analog of the Levi-Civita connection, i.e., a special connection constructed from the coframe field. Note that alternatively to the metric geometry, non-trivial tensor $K^i{}_{jk}(\vartheta^\alpha)$ do exist.

The flat Weitzenböck connection. On a bare differentiable manifold M, without any additional structure, the notion of parallelism of two vectors attached to distance points depends on a curve joint the points. Oppositely, on a coframe manifold $\{M\,,\vartheta^\alpha\}$, a certain type of the parallelism of distance vectors may be defined in an absolute (curve independent) sense [53]. Namely, two vectors $u(x_1)$ and $v(x_2)$ may be declared parallel to each other, if, being referred to the local elements of the coframe field $u(x_1) = u_\alpha(x_1)\vartheta^\alpha(x_1)$ and $v(x_2) = v_\alpha(x_2)\vartheta^\alpha(x_2)$, they have the proportional components $u_\alpha(x_1) = Cv_\alpha(x_2)$. This definition is independent on the coordinates used on the manifold and on the nonholonomic frame of reference. It do depends on the coframe field. Since, by local transformations, the coframes at distance points change differently, only rigid linear coframe transformations preserve such type of a parallelism.

This geometric picture may be reformulated in term of a special connection. The elements of the coframe field attached to distinct points have to be assumed parallel to each other. It means that a special connection $\overset{\circ}{\Gamma}{}^i{}_{jk}$ exists such that the corresponding covariant derivative of the coframe field components is zero:

$$\vartheta^\alpha{}_{j;k} = \vartheta^\alpha{}_{j,k} - \overset{\circ}{\Gamma}{}^i{}_{jk}\vartheta^\alpha{}_i = 0\,. \tag{2.24}$$

Multiplying by $e_\alpha{}^i$, we have an explicit expression

$$\overset{\circ}{\Gamma}{}^i{}_{jk} = e_\alpha{}^i\vartheta^\alpha{}_{k,j}\,. \tag{2.25}$$

Under a smooth transform of coordinates, this expression is transformed in accordance with the inhomogeneous linear rule (2.14). Consequently, (2.25) indeed gives the coefficients of a special connection which is referred to as the *Weitzenböck flat connection*. This connection is unique for a class of coframes related by rigid linear transformations.

In an arbitrary nonholonomic reference basis (θ^a, f_a), we have correspondingly a unique Weitzenböck's connection 1-form which is constructed by (2.15) from (2.25)

$$\varGamma_a{}^b = f_a{}^k\left(\theta^b{}_i\,\overset{\circ}{\Gamma}{}^i{}_{jk} - \theta^b{}_{k,j}\right)dx^j\,. \tag{2.26}$$

Substituting the coframe field ϑ^α instead of the nonholonomic basis θ^a we have

$$\overset{\circ}{\varGamma}{}_\alpha{}^\beta = \left(-\vartheta^\beta{}_{i,j} + \vartheta^\beta{}_k e_\alpha{}^k\vartheta^\alpha{}_{i,j}\right)e_\alpha{}^i\,dx^j = 0\,. \tag{2.27}$$

Thus the Weitzenböck connection 1-form is zero, when it is referred to the coframe field $(\vartheta^\alpha, e_\alpha)$ itself. Certainly, this property is only a basis related fact. It yields, however, vanishing of the curvature of the Weitzenböck connection, which is a basis independent property.

General coframe connections. Recall that we are looking for a general coframe connection constructed from the first order derivatives of the coframe field components. In the Riemannian geometry, the analogous construction yields an unique connection of Levi-Civita. In the coframe geometry, however, the situation is different [37].

Proposition 3: The general linear connection constructed from the first order derivatives of the coframe field is given by a 3-parametric family:

$$\Gamma^i{}_{jk} = \overset{\circ}{\Gamma}{}^i{}_{jk} + \alpha_1 C^i{}_{jk} + \alpha_2 C_j \delta^i_k + \alpha_3 C_k \delta^i_j \,, \tag{2.28}$$

where

$$C^i{}_{jk} = \frac{1}{2} e_\alpha{}^i \left(\vartheta^\alpha{}_{k,j} - \vartheta^\alpha{}_{j,k}\right) \,, \qquad C_i = C^m{}_{mi} \,. \tag{2.29}$$

By (2.15), the connection 1-form corresponded to the coefficients (2.28), being referred to a nonholonomic basis, takes the form

$$\mathit{\Gamma}_a{}^b = f_a{}^k \left(-\theta^b{}_{k,m} + \theta^b{}_l \overset{\circ}{\Gamma}{}^l{}_{mk} + K^l{}_{mk}\theta^b{}_l\right) dx^m \,. \tag{2.30}$$

When this quantity is referred to the coframe field itself, it can be expressed by the exterior derivative of the coframe:

$$\mathit{\Gamma}_\alpha{}^\beta = \left(\alpha_1 C^\beta{}_{\gamma\alpha} + \alpha_2 C_\gamma \delta^\beta_\alpha + \alpha_3 C_\alpha \delta^\beta_\gamma\right) \vartheta^\gamma \,. \tag{2.31}$$

Metric-coframe connection. Consider a manifold endowed with the coframe metric tensor (2.8). Again, we are looking for a most general coframe connection that can be constructed from the first order derivatives of the coframe field. We will refer to it as the *metric-coframe connection.* Thus we are deal with a category of *coframe manifolds with a coframe metric and a linear coframe connection:*

$$\Big\{M\,, \vartheta^\alpha, G\,, g(\vartheta^a), \Gamma_a{}^b(\vartheta^\alpha)\Big\} \,. \tag{2.32}$$

Now the connection expression will involve some additional terms which depend on the metric tensor (2.8). To describe all possible combinations of the metric tensor components and frame/coframe components it is useful to pull down all the indices. Define:

$$\Gamma_{ijk} = g_{im}\Gamma^m{}_{jk} \,, \qquad C_{ijk} = g_{im} C^m{}_{jk} \,. \tag{2.33}$$

Proposition 4: The most general metric-coframe connection constructed from the first order derivatives of the coframe field is represented by a 6-parametric family:

$$\Gamma_{ijk} = \overset{\circ}{\Gamma}{}_{ijk} + \alpha_1 C_{ijk} + \alpha_2 g_{ik} C_j + \alpha_3 g_{ij} C_k + \beta_1 g_{jk} C_i + \beta_2 C_{jki} + \beta_3 C_{kij} \,. \tag{2.34}$$

The expression (2.34) can be rewritten in a

$$\Gamma^i{}_{jk} = \overset{\circ}{\Gamma}{}^i{}_{jk} + \alpha_1 C^i{}_{jk} + \alpha_2 \delta^i_k C_j + \alpha_3 \delta^i_j C_k + \beta_1 g^{il} g_{jk} C_l + \beta_2 g^{il} C_{jkl} + \beta_3 g^{il} C_{klj} \,. \tag{2.35}$$

Here we can identify: (i) The terms with the coefficient α_i that do not depend on the metric; (ii) The terms with the coefficient β_i that can be constructed only by use of the metric tensor.

With respect to a nonholonomic basis (f_a, θ^a), the coefficients of a connection (2.35) correspond to a connection 1-form (2.15)

$$\mathit{\Gamma}_a{}^b = \overset{\circ}{\mathit{\Gamma}}{}_a{}^b + K^i{}_{jk} f_a{}^k \theta^b{}_i dx^j \,. \tag{2.36}$$

When (2.36) is referred to the coframe field itself, it can be expressed by the exterior derivative of the coframe. We have

$$\Gamma_\alpha{}^\beta = \left(\alpha_1 C^\beta{}_{\gamma\alpha} + \alpha_2 C_\gamma \delta^\beta_\alpha + \alpha_3 C_\alpha \delta^\beta_\gamma + \beta_1 C^\beta \eta_{\alpha\gamma} + \beta_2 C_{\gamma\alpha\nu}\eta^{\beta\nu} + \beta_3 C_{\alpha\nu\gamma}\eta^{\beta\nu}\right)\vartheta^\gamma . \qquad (2.37)$$

2.3. Torsion of the coframe connection

Torsion tensor and torsion 2-form. Definitions. Consider a connection 1-form $\Gamma_b{}^a$ referred to an arbitrary basis (θ^a, f_a). For a tensor valued p-form of a representation type $\rho\left(A_a{}^b\right)$, the *covariant exterior derivative* operator $D : \Omega^p(\mathcal{M}) \to \Omega^{p+1}(\mathcal{M})$ is defined as [24], [5]

$$D = d + \Gamma_b{}^a \rho\left(A_a{}^b\right) \wedge . \qquad (2.38)$$

In particular, the covariant exterior derivative of a scalar-valued form ϕ is $D\phi = d\phi$. For a vector-valued form ϕ^a, it is given by $D\phi^a = d\phi^a + \Gamma_b{}^a \wedge \phi^b$, etc.

For a connection 1-form $\Gamma_a{}^b$ written with respect to a nonholonomic basis, the *torsion 2-form* $\mathcal{T}^a$ is defined as

$$\mathcal{T}^a = D\theta^a = d\theta^a + \Gamma_b{}^a \wedge \theta^b . \qquad (2.39)$$

Substituting (2.15) into (2.39), we observe that the coframe derivative term $d\vartheta^a$ cancels out. Hence,

$$\mathcal{T}^a = \Gamma^i{}_{jk}\theta^a{}_i dx^j \wedge dx^k = \Gamma^i{}_{[jk]}\theta^a{}_i dx^j \wedge dx^k . \qquad (2.40)$$

In a coordinate coframe, this expression is simplified to

$$\mathcal{T}^i = \Gamma^i{}_{[jk]} dx^j \wedge dx^k . \qquad (2.41)$$

Consequently, the torsion 2-form $\mathcal{T}^a$ is completely determined by an antisymmetric combination of the coefficients of the connection. Observe that such combination is a tensor. Thus, the torsion 2-form is completely equivalent to a $(1, 2)$-rank *torsion tensor* which is defined as

$$T^i{}_{jk} = 2\Gamma^i{}_{[jk]} . \qquad (2.42)$$

In a holonomic and a nonholonomic bases, the torsion 2-form is expressed respectively as

$$\mathcal{T}^i = \frac{1}{2}T^i{}_{jk} dx^j \wedge dx^k , \qquad \mathcal{T}^a = \frac{1}{2}T^i{}_{jk}\theta^a{}_i dx^j \wedge dx^k . \qquad (2.43)$$

It is useful to define also a quantity

$$\mathcal{T}^\alpha = \frac{1}{2}T^i{}_{jk}\vartheta^\alpha{}_i dx^j \wedge dx^k . \qquad (2.44)$$

With respect to the coframe field, the torsion 2-form of the Weitzenböck connection (2.44) reads

$$\overset{\circ}{\mathcal{T}}{}^\alpha = d\vartheta^\alpha . \qquad (2.45)$$

Torsion of the metric-coframe connection. For the metric-coframe connection (2.34), the covariant components $T_{ijk} = 2g_{im}\Gamma^m{}_{[jk]}$ of the torsion tensor take the form

$$T_{ijk} = 2(1 + \alpha_1)C_{ijk} + (\alpha_2 - \alpha_3)(g_{ik}C_j - g_{ij}C_k) + (\beta_2 + \beta_3)(C_{jki} + C_{kij}) . \qquad (2.46)$$

The corresponded torsion 2-form is expressed in the coordinate basis as

$$T^i = \Big[(1+\alpha_1)C^i{}_{jk} + (\alpha_2-\alpha_3)C_j\delta^i_k + (\beta_2+\beta_3)g^{im}C_{jkm}\Big]dx^j\wedge dx^k\,. \tag{2.47}$$

Torsion-free metric-coframe connection. Let us look for which values of the parameters the torsion of the metric-coframe connection is identically zero. The corresponded connection is called *the symmetric or torsion-free connection.* It is clear from (2.46) that the metric-coframe connection is symmetric if

$$\alpha_1=-1\,,\qquad \alpha_2=\alpha_3\,,\qquad \beta_2=-\beta_3\,. \tag{2.48}$$

The necessity of this condition can be derived from the irreducible decomposition [37].

Thus on a manifold of the dimension $D\geq 3$ there exists a 3-parametric family of the symmetric (torsion-free) connections:

$$\Gamma^i{}_{jk} = \overset{\circ}{\Gamma}{}^i{}_{jk} - C^i{}_{jk} + \alpha_2\left(\delta^i_kC_j+\delta^i_jC_k\right) + \beta_1 g_{jk}g^{im}C_m + \beta_2 g^{im}\left(C_{jkm}-C_{kmj}\right)\,. \tag{2.49}$$

2.4. Nonmetricity of the metric-coframe connection

Nonmetricity tensor and nonmetricity 2-form. Definition. When Cartan's manifold is endowed with a metric tensor, the connection generates an additional tensor field called *the nonmetricity tensor.* It is expressed as a covariant derivative of the metric tensor components. For a metric given in a local system of coordinates as $g=g_{ij}dx^i\otimes dx^j$, the nonmetricity tensor is defined as

$$Q_{kij} = -\nabla_k g_{ij} = -g_{ij,k} + \Gamma^m{}_{ik}g_{mj} + \Gamma^m{}_{jk}g_{im}\,, \tag{2.50}$$

or,

$$Q_{kij} = -g_{ij,k} + \Gamma_{jik} + \Gamma_{ijk}\,. \tag{2.51}$$

Evidently, this tensor is symmetric in the last pair of indices $Q_{kij}=Q_{kji}$. Hence, on a D dimensional manifold, the nonmetricity tensor has $D(D^2+D)/2$ independent components.

For the exterior form representation, it is useful to define *the nonmetricity 1-form.* In a coordinate basis, it is given by

$$Q_{ij} = Q_{kij}dx^k = -dg_{ij} + \Gamma_{ij} + \Gamma_{ji}\,. \tag{2.52}$$

In an arbitrary reference basis $(f_a\,,\theta^a)$, the metric tensor is expressed as $g=g_{ab}\theta^a\otimes\theta^b$. Correspondingly, the nonmetricity 1-form reads

$$Q_{ab} = -dg_{ab} + \Gamma_{ab} + \Gamma_{ba}\,. \tag{2.53}$$

With respect to the coframe field ϑ^α, the components of the metric are constants $\eta_{\alpha\beta}$, thus the nonmetricity is merely the symmetric combination of the connection 1-form components

$$Q_{\alpha\beta} = \Gamma_{\alpha\beta} + \Gamma_{\beta\alpha}\,. \tag{2.54}$$

Note, that this expression is not a usual tensorial quantity. In fact, it is an expression of a tensor-valued 1-form of nonmetricity with respect to a special class of bases. Its relation to a proper tensorial valued 1-form (2.52) is, however, very simple. By a substitution of (2.19) into (2.54) we have

$$Q_{ij} = Q_{\alpha\beta}\vartheta^\alpha{}_i\vartheta^\beta{}_j\,. \tag{2.55}$$

The following generalization of the Levi-Civita theorem from the Riemannian geometry provides a decomposition of an arbitrary affine connection [54].

Proposition 5: Let a metric g on a manifold M be fixed and two tensors T_{ijk} and Q_{ijk} with the symmetries

$$T_{ijk} = -T_{ikj}\,, \qquad Q_{kij} = Q_{kji}\,. \tag{2.56}$$

be given. A unique connection Γ_{ijk} exists on M such that T_{ijk} is its torsion and Q_{ijk} is its nonmetricity. Explicitly,

$$\Gamma_{ijk} \;=\; \overset{*}{\Gamma}_{ijk} - \frac{1}{2}\big(Q_{ijk} - Q_{jki} - Q_{kij}\big) + \frac{1}{2}\big(T_{ijk} + T_{jki} - T_{kij}\big)\,, \tag{2.57}$$

where

$$\overset{*}{\Gamma}_{ijk} = \frac{1}{2}\big(g_{ij,k} + g_{ik,j} - g_{jk,i}\big) \tag{2.58}$$

are the components of the Levi-Civita connection.

Nonmetricity of the metric-coframe connection. We calculate now the nonmetricity tensor of the metric-coframe connection (2.34)

$$\begin{aligned} Q_{kij} \;=\; & \Big(-g_{ij,k} + \overset{\circ}{\Gamma}_{ijk} + \overset{\circ}{\Gamma}_{jik}\Big) + (\alpha_1 - \beta_2)(C_{ijk} - C_{jki}) + \\ & (\alpha_2 + \beta_1)(g_{ik}C_j + g_{jk}C_i) + 2\alpha_3 g_{ij} C_k\,. \end{aligned} \tag{2.59}$$

The first parenthesis represent the nonmetricity tensor of the Weitzenböck connection. This expression vanishes identically, i.e., the Weitzenböck connection is metric-compatible. Indeed, we have

$$g_{ij,k} = \eta_{\alpha\beta}(\vartheta^{\alpha}{}_{i,k}\vartheta^{\beta}_{j} + \vartheta^{\alpha}{}_{i}\vartheta^{\beta}_{j,k}) = \overset{\circ}{\Gamma}_{ijk} + \overset{\circ}{\Gamma}_{jik}\,. \tag{2.60}$$

Consequently, (2.59) is simplified to

$$Q_{kij} = (\alpha_1 - \beta_2)(C_{ijk} + C_{jki}) + (\alpha_2 + \beta_1)(g_{ik}C_j + g_{jk}C_i) + 2\alpha_3 g_{ij} C_k\,. \tag{2.61}$$

Metric compatible metric-coframe connection. Let us look for which values of the coefficients the connection is *metric-compatible*, i.e., has an identically zero non-metricity tensor. Recall that both quantities, the metric tensor and the connection, are constructed from the same building block — the coframe field ϑ^{α}. It is clear from (2.61) that the metric-coframe connection is metric-compatible if

$$\alpha_1 = \beta_2\,, \qquad \alpha_2 = -\beta_1\,, \qquad \alpha_3 = 0\,. \tag{2.62}$$

The necessity of this condition can be derived from the irreducible decomposition of the nonmetricity tensor [37].

Metric compatible and torsion-free metric-coframe connection. Let us look now for a general coframe connection of a zero torsion and zero non-metricity, i.e., for a symmetric metric compatible connection constructed from the coframe field. The system of conditions (2.48) and (2.62) has a unique solution

$$\alpha_1 = \beta_2 = -\beta_3 = -1\,, \qquad \beta_1 = \alpha_2 = \alpha_3 = 0\,. \tag{2.63}$$

Consequently, a metric-compatible symmetric connection is unique. This is in a correspondence to the original Levi-Civita theorem, and the unique connection is of Levi-Civita. Moreover, substituting (2.63) into (2.34) we can express now the standard Levi-Civita connection $\overset{*}{\Gamma}{}^{i}{}_{jk}$ via the flat connection of Weitzenböck $\overset{\circ}{\Gamma}{}^{i}{}_{jk}$

$$\overset{*}{\Gamma}_{ijk} \;=\; \overset{\circ}{\Gamma}_{i(jk)} + C_{kij} - C_{jki}\,. \tag{2.64}$$

3. Gauge transformations

Local transformations of the coframe field. The geometrical structure considered above is well defined for a fixed coframe field e_α. Moreover, it is invariant under rigid coframe transformations. The gauge paradigm suggests now to look for a localization of such transformations:

$$\vartheta^\alpha \mapsto L^\alpha{}_\beta\, \vartheta^\beta\,, \qquad e_\alpha \mapsto L_\alpha{}^\beta\, e_\beta\,, \tag{3.1}$$

or, in the components,

$$\vartheta^\alpha{}_i \mapsto L^\alpha{}_\beta\, \vartheta^\beta{}_i\,, \qquad e_\alpha{}^i \mapsto L_\alpha{}^\beta\, e_\beta{}^i\,. \tag{3.2}$$

Here the matrix $L^\alpha{}_\beta$ and its inverse $L_\alpha{}^\beta$ are functions of a point $x \in M$. We require the volume element (2.4) and the metric tensor (2.8) both to be invariant under the pointwise transformations (3.1). Consequently, $L^\alpha{}_\beta$ is assumed to be a pseudo-orthonormal matrix whit enters are smooth functions of a point. We will also use an infinitesimal version of the transformation (3.2) with $L^\alpha{}_\beta = \delta^\alpha_\beta + X^\alpha{}_\beta$. In the components, it takes the form

$$\vartheta^\alpha{}_i \mapsto \vartheta^\alpha{}_i + X^\alpha{}_\beta\, \vartheta^\beta{}_i\,, \qquad e_\alpha{}^i \mapsto e_\alpha{}^i - X^\beta{}_\alpha\, e_\beta{}^i\,. \tag{3.3}$$

As the elements of the algebra $so(1,n)$, the matrix $X_{\alpha\beta} = \eta_{\alpha\mu}X^\mu{}_\beta$ is antisymmetric. We define a corresponded antisymmetric tensor

$$F_{ij} = \vartheta^\alpha{}_i \vartheta^\beta{}_j X_{\alpha\beta}\,. \tag{3.4}$$

We have already postulated the invariance of the volume element and of the metric tensor under the coframe transformations. It is natural to involve now an additional invariance requirement concerning the affine connection.

Connection invariance condition. Affine coframe connection is assumed to be invariant under pointwise transformations of the coframe field

$$\Delta\Gamma^i{}_{jk} := \Gamma^i{}_{jk}\left(L^\alpha{}_\beta \vartheta^\beta\right) - \Gamma^i{}_{jk}\left(\vartheta^\alpha\right) = 0\,. \tag{3.5}$$

Since the coframe connection is constructed from the first order derivatives of the coframe field, (3.5) is a first order PDE for the elements of the group G and for the components of the coframe field.

Weitzenböck connection transformation. Since the Weitzenböck connection is a basis tool of our construction, it is useful to calculate the change of this quantity under the coframe transformations (3.1). We have

$$\Delta \overset{\circ}{\Gamma}{}^i{}_{jk} = e_\alpha{}^i \vartheta^\beta{}_k Y^\alpha{}_{\beta j}\,, \qquad \text{where} \qquad Y^\alpha{}_{\beta j} = L^\alpha{}_\gamma L^\gamma{}_{\beta,j}\,. \tag{3.6}$$

All matrices involved here are nonsingular, consequently the Weitzenböck connection is preserved only under the rigid transformations of the coframe field with $L^\gamma{}_{\beta,j} = 0$.

Let us rewrite (3.6) in alternative forms. Since the metric tensor is invariant under the transformations (3.1) we have

$$\Delta \overset{\circ}{\Gamma}{}_{ijk} = \Delta\left(g_{im} \overset{\circ}{\Gamma}{}^m{}_{jk}\right) = g_{im} \Delta \overset{\circ}{\Gamma}{}^m{}_{jk}\,. \tag{3.7}$$

Consequently

$$\Delta \overset{\circ}{\Gamma}{}_{ijk} = \vartheta^\alpha{}_i \vartheta^\beta{}_k Y_{\alpha\beta j}\,, \qquad \text{where} \qquad Y_{\alpha\beta j} = \eta_{\alpha\mu} Y^\mu{}_{\beta j}\,. \tag{3.8}$$

In the infinitesimal approximation, (3.6) takes the form

$$\Delta \overset{\circ}{\Gamma}{}^i{}_{jk} = e_\alpha{}^i \vartheta^\beta{}_k X^\alpha{}_{\beta,j} \,. \tag{3.9}$$

while (3.7) with $X_{\alpha\beta} = \eta_{\alpha\mu} X^\mu{}_\beta$ reads

$$\Delta \overset{\circ}{\Gamma}{}_{ijk} = \vartheta^\alpha{}_i \vartheta^\beta{}_k X_{\alpha\beta,j} \,. \tag{3.10}$$

Note that since $X_{\alpha\beta}$ is antisymmetric, we have in this approximation

$$\Delta \overset{\circ}{\Gamma}{}_{ijk} = -\Delta \overset{\circ}{\Gamma}{}_{kji} \,. \tag{3.11}$$

We will also consider an additional physical meaningful approximation when the derivatives of the coframe is considered to be small relative to the derivatives of the transformation matrix. In this case, (3.9) and (3.10) read

$$\Delta \overset{\circ}{\Gamma}{}^i{}_{jk} = F^i{}_{k,j} \,, \qquad \text{where} \qquad F^i{}_k = e_\alpha{}^i \vartheta^\beta{}_k X^\alpha{}_\beta \,, \tag{3.12}$$

and

$$\Delta \overset{\circ}{\Gamma}{}_{ijk} = F_{ik,j} \,, \qquad \text{where} \qquad F_{ij} = \vartheta^\alpha{}_i \vartheta^\beta{}_j X_{\alpha\beta} \,. \tag{3.13}$$

Transformations preserved the geometric structure. Since the coframe field appears in the coframe geometrical structure only implicitly, (3.1) is a type of a gauge transformation. Invariance of the metric tensor and of the volume element restricts $L^\alpha{}_\beta$ to a pseudo-orthonormal matrix $G = SO(1,n)$. Let us ask now, under what conditions the general coframe connection (2.34) is invariant under the coframe transformations (3.1). First we rewrite (2.34) via the Levi-Civita connection. Using (2.63) we have

$$\overset{\circ}{\Gamma}{}_{ijk} = \overset{*}{\Gamma}{}_{ijk} + C_{ijk} - C_{kij} + C_{jki} \,. \tag{3.14}$$

Thus (2.34) takes the form

$$\Gamma_{ijk} = \overset{*}{\Gamma}{}_{ijk} + (\alpha_1+1)C_{ijk} + \alpha_2 g_{ik}C_j + \alpha_3 g_{ij}C_k + \beta_1 g_{jk}C_i + (\beta_2+1)C_{jki} + (\beta_3-1)C_{kij} \,. \tag{3.15}$$

Since the Levi-Civita connection $\overset{*}{\Gamma}{}_{ijk}$ is invariant under the transformations (3.1), the equation $\Delta\Gamma_{ijk} = 0$ takes the form

$$(\alpha_1+1)\Delta C_{ijk} + \alpha_2 g_{ik}\Delta C_j + \alpha_3 g_{ij}\Delta C_k + \beta_1 g_{jk}\Delta C_i + (\beta_2+1)\Delta C_{jki} + (\beta_3-1)\Delta C_{kij} = 0 \,. \tag{3.16}$$

Hence in order to have an invariant coframe connection, we have to look for possible solutions of equation (3.16).

Trivial solutions of the invariance equation. Consider first two trivial solutions of (3.16) which turn out to be non-dynamical.

(i) *Arbitrary transformations — Levi-Civita connection.*
The equation (3.16) is evidently satisfied when all the numerical coefficients mutually equal to zero. It is easy to check that these six relations are equivalent to (2.63). Thus the corresponded connection is of Levi-Civita. In this case, the elements of the matrix $L^\alpha{}_\beta$ are arbitrary functions of a point. Thus we come to a trivial fact that the Levi-Civita connection is a unique coframe connection which is invariant under arbitrary local $SO(1,n)$ transformations of the coframe field.

(ii) *Rigid transformations.*
Another trivial solution of the system (3.16) emerges when we require $\Delta C_{ijk} = 0$. All permutations and traces of this tensor are also equal to zero so (3.16) is trivially valid. Due to (3.8), it means that the matrix of transformations is independent on a point. In this case, an arbitrary coframe connection, in particular the Weitzenböck connection, remains unchanged. Thus we come to another trivial fact that the coframe connection is invariant under rigid transformations of the coframe field.

Dynamical solution. We will look now for nontrivial solutions of the system (3.16). Three traces of this system yield the equations $\Delta C_i = 0$. Thus we obtain the first condition

$$\Delta C_i = 0\,. \tag{3.17}$$

The system (3.16) remains now in the form

$$(\alpha_1 + 1)\Delta C_{ijk} + (\beta_2 + 1)\Delta C_{jki} + (\beta_3 - 1)\Delta C_{kij} = 0\,. \tag{3.18}$$

Applying the complete antisymetrization in three indices we derive the second equation

$$\Delta C_{[ijk]} = 0\,. \tag{3.19}$$

The equation (3.18) remains now in the form

$$(\beta_2 - \alpha_1)\Delta C_{jki} + (\beta_3 - \alpha_1 - 2)\Delta C_{jki} = 0\,. \tag{3.20}$$

We have to restrict now the coefficients, otherwise we obtain $\Delta C_{ijk} = 0$, i.e., only the rigid transformations. Consequently we require

$$\beta_2 = \alpha_1\,, \qquad \beta_3 = \alpha_1 + 2\,. \tag{3.21}$$

Thus we have proved

Proposition 6: The coframe connection

$$\Gamma_{ijk} = \overset{*}{\Gamma}_{ijk} + (\alpha_1 + 1)C_{[ijk]} + \alpha_2 g_{ik} C_j + \alpha_3 g_{ij} C_k + \beta_1 g_{jk} C_i\,. \tag{3.22}$$

is invariant under the coframe transformations satisfied the equations

$$\Delta C_i = 0\,. \qquad \Delta C_{[ijk]} = 0\,. \tag{3.23}$$

Observe that this family includes the Levi-Civita connection, which is invariant under a wider group of arbitrary transformations of the coframe field.

The torsion tensor of the connection (3.22) is expressed as

$$T_{ijk} = (\alpha_1 + 1)C_{[ijk]} + (\alpha_2 - \alpha_3)(g_{ik} C_j - g_{ij} C_k)\,. \tag{3.24}$$

Thus a torsion-free subfamily of (3.22) is given by

$$\Gamma_{ijk} = \overset{*}{\Gamma}_{i(jk)} + \alpha_2 (g_{ik} C_j + g_{ij} C_k) + \beta_1 g_{jk} C_i\,. \tag{3.25}$$

The nonmetricity tensor of the connection (3.22) reads

$$Q_{kij} = (\alpha_2 + \beta_1)(g_{ik} C_j + g_{jk} C_i) + 2\alpha_3 g_{jj} C_k\,. \tag{3.26}$$

Thus a metric compatible subfamily of (3.22) is given by

$$\Gamma_{ijk} = \overset{*}{\Gamma}_{i(jk)} + (\alpha_1+1)C_{[ijk]} + \alpha_2(g_{ik}C_j - g_{jk}C_i)\,. \tag{3.27}$$

From (3.24) and (3.26) we derive an interesting conclusions:

$$\Delta Q_{kij} = 0 \quad \Longleftrightarrow \quad \Delta C_i = 0\,. \tag{3.28}$$

and, together with this relation,

$$\Delta T_{ijk} = 0 \quad \Longleftrightarrow \quad \Delta C_{[ijk]} = 0\,. \tag{3.29}$$

Thus the relations (3.23) obtain a geometric meaning, they correspond to invariance of the torsion and nonmetricity tensors under coframe transformations.

4. Maxwell-type system

Let us examine now what physical meaning can be given to the invariance conditions [42]

$$\Delta C_{[ijk]} = 0\,, \qquad \Delta C_i = 0\,. \tag{4.1}$$

Denote $K_{ijk} = \Delta C_{ijk}$. Thus (4.1) takes the form

$$K_{[ijk]} = 0\,, \qquad K^m{}_{im} = 0\,. \tag{4.2}$$

The tensor K_{ijk} depends on the derivatives of the Lorentz parameters $X_{\alpha\beta}$ and on the components of the coframe field

$$K_{ijk} = \frac{1}{2}\,\vartheta^\alpha{}_k\Big(X_{\alpha\beta,j}\vartheta^\beta{}_i - X_{\alpha\beta,i}\vartheta^\beta{}_j\Big)\,. \tag{4.3}$$

Thus, in fact, we have in (4.2), two first order partial differential equations for the entries of an antisymmetric matrix $X_{\alpha\beta}$. We can translate this matrix into an antisymmetric tensor F_{ij}

$$F_{ij} = X_{\mu\nu}\vartheta^\mu{}_i\vartheta^\nu{}_j\,, \qquad X_{\mu\nu} = F_{ij}e_\mu{}^i e_\nu{}^j\,. \tag{4.4}$$

Substituting into (4.3), we derive

$$\begin{aligned} K_{ijk} &= F_{k[i,j]} - \frac{1}{2}X_{\alpha\beta}\Big[(\vartheta^\alpha{}_k\vartheta^\beta{}_i)_{,j} - (\vartheta^\alpha{}_k\vartheta^\beta{}_j)_{,i}\Big] \\ &= F_{k[i,j]} - F_{km}C^m{}_{ij} - \frac{1}{2}\Big(F_{mi}\,\overset{\circ}{\Gamma}{}^m{}_{kj} - F_{mj}\,\overset{\circ}{\Gamma}{}^m{}_{ki}\Big)\,. \end{aligned} \tag{4.5}$$

Consequently, the first equation from (4.2) takes the form

$$F_{[ij,k]} = \frac{2}{3}(C^m{}_{ij}F_{km} + C^m{}_{jk}F_{im} + C^m{}_{ki}F_{jm})\,, \tag{4.6}$$

while the second equation from (4.2) is rewritten as

$$F^i{}_{j,i} = -2F^i{}_mC^m{}_{ij} + F_{kj}g^{ki}{}_{,i} + F_{mj}g^{ki}\,\overset{\circ}{\Gamma}{}^m{}_{ki} - F_{mi}g^{ki}\,\overset{\circ}{\Gamma}{}^m{}_{kj}\,. \tag{4.7}$$

Observe first a significant approximation to (4.6—4.7). If the right hand sides in both equations are neglected, the equations take the form of the ordinary Maxwell equations for the electromagnetic field in vacuum —

$$F_{[ij,k]} = 0\,, \qquad F^i{}_{j,i} = 0\,. \tag{4.8}$$

In the coframe models, the gravity is modeled by a variable coframe field, i.e., by nonzero values of the quantities $\overset{\circ}{\Gamma}{}_{ij}{}^{k}$. Consequently, the right hand sides of (4.6—4.7) can be viewed as curved space additions, i.e., as the gravitational corrections to the electromagnetic field equations. In the flat spacetime, when a suitable coordinate system is chosen, these corrections are identically equal to zero. Consequently, in the flat spacetime, the invariance conditions (4.2) take the form of the vacuum Maxwell system.

On a curved manifold, the standard Maxwell equations are formulated in a covariant form. Let us show that our system (4.6—4.7) is already covariant. We rewrite (4.5) as

$$K_{ijk} = \frac{1}{2}(F_{ki,j} - F_{km}\,\overset{\circ}{\Gamma}{}^{m}{}_{ij} - F_{mi}\,\overset{\circ}{\Gamma}{}^{m}{}_{kj}) - \frac{1}{2}(\,i \longleftrightarrow j\,)\,. \tag{4.9}$$

Consequently,

$$K_{ijk} = F_{k[i;j]}\,, \tag{4.10}$$

where the covariant derivative (denoted by the semicolon) is taken relative to the Weitzenböck connection. Consequently, the system (4.6—4.7) takes the covariant form

$$F_{[ij;k]} = 0\,, \qquad F^{i}{}_{j;i} = 0\,. \tag{4.11}$$

These equations are literally the same as the electromagnetic sector field equations of the Maxwell-Einstein system. The crucial difference is encoded in the type of the covariant derivative. In the Maxwell-Einstein system, the covariant derivative is taken relative to the Levi-Civita connection, while, in our case, the corresponding connection is of Weitzenböck. Observe that, due to our approach, the Weitzenböck connection is rather natural in (4.11). Indeed, since the electromagnetic-type field describes the local change of the coframe field, it should itself be referred only to the global changes of the coframe. As we have shown, such global transformations correspond precisely to the teleparallel geometry with the Weitzenböck connections.

Let us show how these modifications alternate the electromagnetic field solutions. On Schwarzschild space-time the spherically symmetric solution of (4.11) is given [42] by

$$F_0{}^{\hat{i}} = x^{\hat{i}}\frac{Q}{\rho^3}\,\frac{1-m/2\rho}{(1+m/2\rho)^3}\,. \tag{4.12}$$

The Coulomb-type force acted on a test charge q (of a small mass) takes the form

$$\mathcal{F} = \frac{Qq}{\rho^2}\,\frac{1-m/2\rho}{(1+m/2\rho)^3}\,. \tag{4.13}$$

Here the isotropic coordinates are used. The ordinary Cartesian radius r is related to the isotropic radius ρ as

$$r = \rho\left(1+\frac{m}{2\rho}\right)^2\,. \tag{4.14}$$

Hence

$$\rho = \frac{r-m+\sqrt{r^2-2mr}}{2} \approx r\left(1-\frac{m}{r}\right)\,. \tag{4.15}$$

Observe that the isotropic coordinates are defined only for $r > 2m$. Consequently, the modernized Coulomb force takes the form

$$F = \frac{Qq}{r^2}\left(1-\frac{m^2}{4\rho^2}\right)\,. \tag{4.16}$$

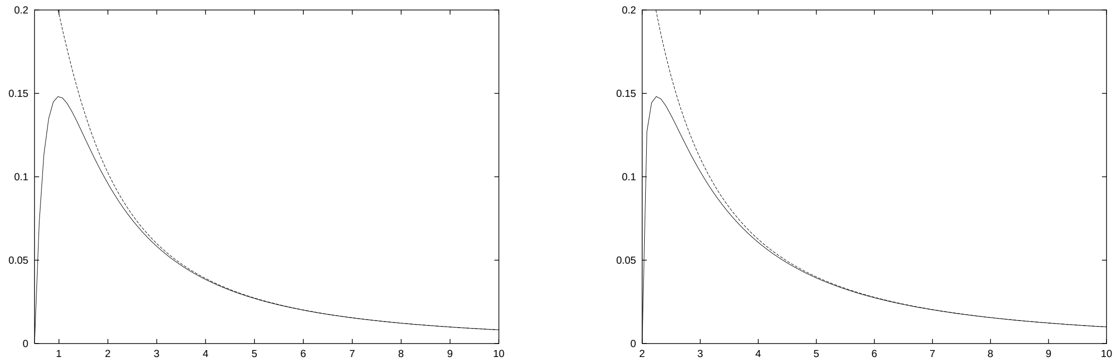

Figure 1. The graphs represent the dependence of the force F on ρ/m and r/m correspondingly relative to the Coulomb force (the top lines). In both cases F is given in the units of Qq/m^2.

Finally, in the Cartesian coordinates, the mass-correction terms take the form

$$\begin{aligned} F &= Qq\,\frac{r-2m+\sqrt{r^2-2mr}}{4(r+\sqrt{r^2-2mr})^3} \\ &= \frac{Qq}{r^2}\left(1-\frac{m^2}{4r^2}-\frac{m^3}{2r^3}+\cdots\right). \end{aligned} \tag{4.17}$$

We depict the dependence of the force F on the distances ρ and r on Fig. 2. The graphs start from $\rho = m/2$ and $r = 2m$ which are the minimal possible value. The deviation from the Coulomb values appears only for small distances $\rho, r \sim m$. The maximal value of the force between two charged particles predicts as

$$F \approx 0.15\,Qq/m^2\,. \tag{4.18}$$

For two electrons, it gives $F \approx 0.76 \cdot 10^{86} N$.

5. Conclusion

GR is a well-posed classical field theory for 10 independent variables — the components of the metric tensor. Although, this theory is completely satisfactory in the pure gravity sector, its possible extensions to other physics phenomena is rather problematic. In particular, the description of fermions on a curved space and the supergravity constructions require a richer set of 16 independent variables. These variables can be assembled in a coframe field, i.e., a local set of four linearly independent 1-forms. Moreover, in supergravity, it is necessary to involve a special flat connection constructed from the derivatives of the coframe field. These facts justify the study of the field models based on a coframe variable alone.

The classical field construction of the coframe gravity is based on a Yang-Mills-type Lagrangian which is a linear combination of quadratic terms with dimensionless coefficients. Such model turns to be satisfactory in the gravity sector and has the viable Schwarzschild solutions even being alternative to the standard GR. Moreover, the coframe model treating of the gravity energy makes it even preferable than the ordinary GR where the gravity energy cannot be defined at all. A principle problem that the coframe gravity construction does not have any connection to a specific geometry even being constructed from the geometrical meaningful objects. A geometrization of the coframe gravity is an aim of this presentation.

We construct a general family of coframe connections which involves as the special cases the Levi-Civita connection of GR and the flat Weitzenböck connection. Every specific connection generates a geometry of a specific type. We identify the subclasses of metric-compatible and torsion-free connections. Moreover we study the local linear transformations of the coframe fields and identify a class of connections which are invariant under restricted coframe transformations. Quite remarkable that the restriction conditions are necessary approximated by a Maxwell-type system of equations.

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