

# Quantum Mechanics on a Planck Lattice and Black Hole Physics

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**Abstract.** We study uncertainty relations as formulated in a crystal-like universe, whose lattice spacing is of order of Planck length. For Planck energies, the uncertainty relation for position and momenta has a lower bound equal to zero. Connections of this result with double special relativity, and with 't Hooft's deterministic quantization proposal, are briefly pointed out. We then apply our formulae to micro black holes, and we derive a new mass-temperature relation for Schwarzschild (micro) black holes. In contrast to standard results based on Heisenberg and stringy uncertainty relations, we obtain both a finite Hawking's temperature and a zero rest-mass remnant at the end of the micro black hole evaporation.

## 1. Introduction

The idea of a discrete structure of the space-time can be traced back to the mid 50's when John Wheeler presented his *space-time foam* concept [1]. Recent advances in physics indicate that the notion of space-time as a continuum is indeed likely to be superseded at the Planck scale, where gravitation and strong-electro-weak interactions become comparable in strength. In fact, discrete structures of space-time appear, as a rule, in many quantum-gravity models. Loop quantum gravity, non-commutative geometry or cosmic cellular automata may serve as examples. Perhaps the simplest toy-model for planck physics is a *discrete lattice*.

Recently one of us (HK) proposed a universe with a peculiar “foam” structure, that can be modeled by a simple discrete space-time Planck Lattice, whose lattice constant is of order Planck length. This discrete, crystal-like universe was termed “world crystal”. There, the geometry of Einstein and Einstein-Cartan spaces can be considered as being a manifestation of the defect structure of the crystal [2]. Curvature is due to rotational defects, torsion due to translational defects. On the coarse-grained scale, the memory of the crystalline structure is smeared out, leading thus to ordinary Einstein's and Einstein-Cartan's theory.

In this paper we study the generalized uncertainty principle [3, 4, 5] (GUP) associated with quantum mechanics formulated on the world crystal, and derive physical consequences for the corresponding micro black holes. As micro black holes could be potentially created at LHC

already in the TeV range (as certain theories with extra-dimensions indicate), our results could contribute into the ongoing discussion about observable signatures of micro black holes.

## 2. Differential calculus on a lattice

On a 1D lattice, the nodes are at  $x_n = n\epsilon$ , with  $n \in \mathcal{Z}$ . One can then define two fundamental derivatives, so called forward and backward derivatives, namely

$$(\nabla f)(x) = \frac{1}{\epsilon} [f(x + \epsilon) - f(x)] \quad \text{and} \quad (\bar{\nabla} f)(x) = \frac{1}{\epsilon} [f(x) - f(x - \epsilon)] . \quad (1)$$

The corresponding integration is performed as a summation:

$$\int dx f(x) \equiv \epsilon \sum_x f(x) , \quad (2)$$

where  $x$  runs over all  $x_n$ . For periodic functions on the lattice or for functions vanishing at the boundary of the world crystal, the lattice derivatives can be subjected to the lattice version of the integration by parts rule:

$$\sum_x f(x) \nabla g(x) = - \sum_x g(x) \bar{\nabla} f(x) , \quad (3)$$

$$\sum_x f(x) \bar{\nabla} g(x) = - \sum_x g(x) \nabla f(x) . \quad (4)$$

One can also define the lattice Laplacian as

$$\nabla \bar{\nabla} f(x) = \bar{\nabla} \nabla f(x) = \frac{1}{\epsilon^2} [f(x + \epsilon) - 2f(x) + f(x - \epsilon)] , \quad (5)$$

which reduces in the continuum limit to an ordinary Laplace operator  $\partial_x^2$ . The above calculus can be naturally extended to any number  $D$  of dimensions.

## 3. Position and momentum operator on a 1D lattice

To build Quantum Mechanics on a 1D lattice we define a *scalar product* as

$$\langle \psi_1 | \psi_2 \rangle = \epsilon \sum_x \psi_1^*(x) \psi_2(x) . \quad (6)$$

It follows from Eq. (3) that

$$\langle f | \nabla g \rangle = - \langle \bar{\nabla} f | g \rangle , \quad (7)$$

so that  $(i\nabla)^\dagger = i\bar{\nabla}$ , and neither  $i\nabla$  nor  $i\bar{\nabla}$  are hermitian operators. The lattice Laplacian (5), however, is hermitian under the scalar product (6).

The position operator  $\hat{X}_\epsilon$  acting on wave functions of  $x$  is defined by a multiplication with  $x$

$$(\hat{X}_\epsilon f)(x) = x f(x) . \quad (8)$$

To ensure hermiticity of the lattice momentum operator  $\hat{P}_\epsilon$ , we relate it to the symmetric lattice derivative. Using (1) we have

$$(\hat{P}_\epsilon f)(x) = \frac{\hbar}{2i} [(\nabla f)(x) + (\bar{\nabla} f)(x)] = \frac{\hbar}{2i\epsilon} [f(x + \epsilon) - f(x - \epsilon)] . \quad (9)$$

For small  $\epsilon$ , this reduces to the ordinary momentum operator  $\hat{p} \equiv -i\hbar\partial_x$ :

$$\hat{P}_\epsilon = \hat{p} + \mathcal{O}(\epsilon^2). \quad (10)$$

Thus, the “canonical” commutator between  $\hat{X}_\epsilon$  and  $\hat{P}_\epsilon$  on the lattice reads

$$([\hat{X}_\epsilon, \hat{P}_\epsilon]f)(x) = \frac{i\hbar}{2}[f(x+\epsilon) + f(x-\epsilon)] \equiv i\hbar(\hat{I}_\epsilon f)(x). \quad (11)$$

The operators  $\hat{X}_\epsilon$ ,  $\hat{P}_\epsilon$ , and  $\hat{I}_\epsilon$  are hermitian under the scalar product (6). They form a  $E(2)$  Lie algebra, which reduces to the standard Weyl-Heisenberg algebra in the limit  $\epsilon \rightarrow 0$ :  $\hat{X}_\epsilon \rightarrow \hat{x}$ ,  $\hat{P}_\epsilon \rightarrow \hat{p}$ ,  $\hat{I}_\epsilon \rightarrow \hat{1}$ . Ordinary QM is thus obtained from lattice QM by a contraction of the  $E(2)$  algebra via the limit  $\epsilon \rightarrow 0$ .

Reminding that  $\exp(\epsilon\partial_x)f(x) = f(x+\epsilon)$ , we can rewrite the commutation relation (11) as

$$([\hat{X}_\epsilon, \hat{P}_\epsilon]f)(x) = i\hbar \cos(\epsilon\hat{p}/\hbar)f(x). \quad (12)$$

The latter allows to identify the lattice unit operator  $\hat{I}_\epsilon$  with  $\cos(\epsilon\hat{p}/\hbar)$ .

#### 4. Uncertainty relations on lattice

Let us define the uncertainty of an observable  $A$  in a state  $\psi$  as the standard deviation, i.e.,

$$(\Delta A)_\psi \equiv [\langle\psi|(\hat{A} - \langle\psi|\hat{A}|\psi\rangle)^2|\psi\rangle]^{1/2}. \quad (13)$$

Via the Schwarz inequality we get the Uncertainty Relation on a lattice in the form

$$\Delta X_\epsilon \Delta P_\epsilon \geq \frac{1}{2} |\langle\psi|[\hat{X}_\epsilon, \hat{P}_\epsilon]|\psi\rangle| = \frac{\hbar}{2} |\langle\psi|\hat{I}_\epsilon|\psi\rangle| = \frac{\hbar}{2} |\langle\psi|\cos(\epsilon\hat{p}/\hbar)|\psi\rangle|. \quad (14)$$

We denote  $\langle\psi|\cdots|\psi\rangle$  as  $\langle\cdots\rangle_\psi$  and study two critical energy regimes of (14):

- I) The long-wavelengths regime, where  $\langle\hat{p}\rangle_\psi \rightarrow 0$ . In this regime the GUP (14) can be written for Planck lattices as (cf. (10))

$$\Delta X_\epsilon \Delta P_\epsilon \geq \frac{\hbar}{2} \left(1 - \frac{\epsilon^2}{2\hbar^2} (\Delta P_\epsilon)^2\right). \quad (15)$$

- II) The regime near the boundary of the first Brillouin zone, where the averaged momentum takes its maximum value  $\langle\hat{p}\rangle_\psi \rightarrow \pi\hbar/2\epsilon$ . Here, using again (10), we can write

$$\Delta X_\epsilon \Delta P_\epsilon \geq \frac{\hbar}{2} \left(\frac{\pi}{2} - \frac{\epsilon}{\hbar} \langle\hat{P}_\epsilon\rangle_\psi\right). \quad (16)$$

We see that, as the momentum reaches the boundary of the Brillouin zone, the right-hand side of (16) vanishes, so that lattice quantum mechanics at short wavelengths *can* exhibit classical behavior — there is no irreducible lower bound for uncertainties of two complementary observables! This result is analogous to the one found, on a different ground, by Magueijo and Smolin in *deformed special relativity* [6]. So we see that the world crystal physics can become “classical” for energies close to the Brillouin zone, i.e., for Planck energies. This is also a recurrent theme in ’t Hooft’s “deterministic” quantum mechanics[7], where a deterministic theory at the Planck scale, supplemented with a dissipation mechanism, can give rise to a genuine quantum mechanical behavior at larger scales.

### 5. World-crystal micro black hole physics

Following a strategy outlined in Ref. [8] we can now use (15) and (16) to obtain the mass-temperature relation for the world crystal (micro) black holes. Heisenberg's microscope argument implies that the smallest resolvable spatial detail  $\delta x$  goes roughly as the wavelength of the probing photon. If  $E$  is the (average) energy of photons used in the microscope, then standard Heisenberg principle dictates

$$\delta x = \frac{\hbar c}{2E}. \quad (17)$$

From (15), the lattice version of the Heisenberg formula (17) reads

$$\delta X_\epsilon \simeq \frac{\hbar c}{2E_\epsilon} \left[ 1 - \frac{\epsilon^2}{2\hbar^2 c^2} (E_\epsilon)^2 \right]. \quad (18)$$

This links the (average) wavelength of a photon to  $E_\epsilon$ . The position uncertainty  $\delta X_\epsilon$  of unpolarized photons of Hawking radiation just outside the event horizon is of the order of the Schwarzschild radius  $R_S$ . This follows directly from the geometry, reminding that the average wavelength of the Hawking radiation is of the order of the geometrical size of the hole. So, we can write

$$\delta X_\epsilon \simeq 2\mu R_S = 2\mu \ell_p m, \quad (19)$$

with  $R_S = \ell_p m$ , where  $m = M/M_p$  is the black hole mass in Planck units ( $M_p = \mathcal{E}_p/c^2$ ,  $\mathcal{E}_p = \hbar c/2\ell_p$ ), and  $\mu$  is a free parameter to be specified. According to the equipartition law, the average energy  $E_\epsilon$  of unpolarized photons is  $E_\epsilon \simeq k_B T$ . For a lattice spacing  $\epsilon = a\ell_p$  the constant  $\mu$  can be fixed by the condition that in the continuum limit  $\epsilon \rightarrow 0$  the formula (18) should predict the standard semiclassical Hawking temperature  $T = \hbar c/(4\pi k_B R_S)$ . This gives  $\mu = \pi$ . Again using Planck units for the temperature  $\mathcal{E}_p = k_B T_p/2$ ,  $\Theta = T/T_p$ , we may finally rewrite (18) as

$$2m = \frac{1}{2\pi\Theta} - \zeta^2 2\pi\Theta, \quad (20)$$

where we define the deformation parameter  $\zeta = a/2\sqrt{2}\pi$ .

The result coming from ordinary Heisenberg uncertainty principle, to which, in the continuum limit  $\epsilon \rightarrow 0$ , Eq.(18) reduces, is

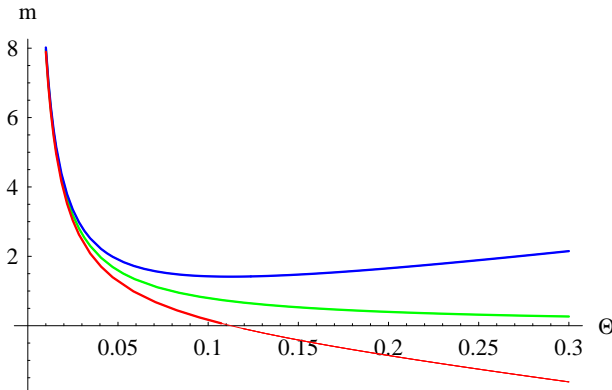
$$m = \frac{1}{4\pi\Theta}, \quad (21)$$

which is the dimensionless version of Hawking's formula for macroscopic black holes. The result implied by the stringy uncertainty relation [4] is

$$2m = \frac{1}{2\pi\Theta} + \zeta^2 2\pi\Theta. \quad (22)$$

The phenomenological consequences of these three relations are quite different. In Fig. 1 we compare the three results, including also the curve for the ordinary Hawking relation (21). Considering  $m$  and  $\Theta$  as functions of time, we can follow the evolution of a micro black hole from the curves plotted there.

For the stringy GUP, we have a maximum temperature  $\Theta_{\max} = 1/(2\pi\zeta)$ , and a minimum rest mass  $m_{\min} = \zeta$ . The end of the evaporation process is reached after a finite time, the



**Figure 1.** Diagrams for the three *mass-temperature* relations: ours (red), Hawking's (green), and stringy GUP result (blue), with  $\zeta = \sqrt{2}$ , as an example. As a consequence of the lattice uncertainty principle the evaporation ends at a *finite* temperature with a *zero* rest-mass remnant.

final temperature is finite, and there is a remnant of a finite rest mass. Should we have used the standard Heisenberg uncertainty principle we would have found the usual Hawking formula. This predicts that the evaporation process ends, after a finite time, with a zero mass and a worrisome infinite temperature. In contrast to these results, our lattice GUP predicts a finite final temperature  $\Theta_{\max} = 1/(2\pi\zeta)$ , with a *zero-mass remnant*. Our result (20) coming from the lattice GUP formula (15) avoids at once the difficulties of an infinite final temperature and the existence of finite-mass black hole remnants in the universe. The analysis with the GUP short wave limit (16) fully confirms the results previously obtained.

## 6. Entropy

From the first law of black hole thermodynamics [9] we know that the differential of the thermodynamical entropy of a Schwarzschild black hole reads

$$dS = \frac{dE}{T_H}, \quad (23)$$

where  $dE$  is the amount of energy swallowed by a black hole with Hawking temperature  $T_H$ . In Eq. (23) the increase in the internal energy is equal to the added heat because a black hole makes no mechanical work when its entropy/surface changes (its expanding surface does not exert any pressure). Rewriting Eq. (23) with the dimensionless variables  $m$  and  $\Theta$  we get  $dS = k_B dm/(2\Theta)$ . Inserting here formula (20) we find

$$dS = \frac{k_B}{2} \frac{dm}{\Theta} = -\frac{k_B}{4} \left( \frac{1}{2\pi\Theta^3} + \frac{2\pi\zeta^2}{\Theta} \right) d\Theta. \quad (24)$$

By integrating  $dS$  we obtain  $S = S(\Theta)$ . Just as formula (20), the relation (24) can be trusted only for  $\Theta \ll \Theta_{\max} = 1/2\pi\zeta$ . Thus, when integrating (24), we should do this only up to a cutoff  $\tilde{\Theta}_{\max} \ll \Theta_{\max}$ . The additive constant in  $S$  can then be fixed by requiring that  $S = 0$  when  $\Theta \rightarrow \tilde{\Theta}_{\max}$ . This is equivalent to what is usually done when calculating the Hawking temperature for a Schwarzschild black hole. There, one fixes the additive constant in the entropy integral to be zero for  $m = 0$ , so that  $S(m = 0) = S(\Theta \rightarrow \infty) = 0$  (the minimum mass attainable in the standard Hawking effect is  $m = 0$ ). The integral yields

$$S(\Theta) = \frac{k_B}{16\pi} \left( \frac{1}{\Theta^2} - \frac{1}{\tilde{\Theta}_{\max}^2} + 8\pi^2\zeta^2 \log \frac{\tilde{\Theta}_{\max}}{\Theta} \right). \quad (25)$$

The entropy is always positive, and  $S \rightarrow 0$  for  $\Theta \rightarrow \tilde{\Theta}_{\max}$ .

## 7. Heat Capacity

With the entropy formulae (24) and (25) at hand we can now compute the heat capacity of a (micro) black hole in the world-crystal. This will give us important insights on the final stage of the evaporation process. Again, we shall obtain formulae valid only for  $\Theta \ll \Theta_{\max} = 1/(2\pi\zeta)$ . The heat capacity  $C$  of a black hole is defined via the relation  $dQ = dE = CdT$ . The pressure exerted on the environment by the expanding black hole surface is zero. Hence we do not need to specify which  $C$  is meant. Therefore we have  $C = T \left( \frac{dS}{dT} \right) = \Theta \left( \frac{dS}{d\Theta} \right)$ , which yields

$$C = -\frac{\pi k_B}{2} \left[ \zeta^2 + \frac{1}{(2\pi\Theta)^2} \right]. \quad (26)$$

From this clearly follows that  $C$  is always negative. Most condensed-matter systems have  $C > 0$ . However, because of instabilities induced by gravity this is generally not the case in astrophysics [10], especially in black hole physics. A Schwarzschild black hole has  $C < 0$  which indicates that the black hole becomes hotter by radiating. The result (26) implies that this scenario holds also for micro black holes in the world-crystal. In case of stringy GUP, we have to use Eq. (22) as the mass-temperature formula. The expression for the heat capacity then reads

$$C = \frac{\pi k_B}{2} \left[ \zeta^2 - \frac{1}{(2\pi\Theta)^2} \right]. \quad (27)$$

Since also here  $0 < \Theta < \Theta_{\max} = 1/2\pi\zeta$ , black holes have negative specific heat also according to the stringy GUP. However, stringy GUP displays a striking difference with respect to lattice GUP. In fact, since in principle we can trust Eq. (22) also when  $\Theta \simeq \Theta_{\max} = 1/2\pi\zeta$ , then from (27) we have, in such limit,  $C = 0$ . This means that for the stringy GUP the specific heat vanishes at the end point of the evaporation process in a finite time, so that the black hole at the end of its evolution cannot exchange energy with the surrounding space. In other words, the black hole stops to interact thermodynamically with the environment. The final stage of the Hawking evaporation, according to the stringy GUP scenario, contains a Planck-size remnant with a maximal temperature  $\Theta = \Theta_{\max}$ , but thermodynamically inert. The remnant behaves like an elementary particle — there are no internal degrees of freedom to excite in order to produce a heat absorption or emission.

## 8. Connection with double special relativity

A different application relates to the idea of double (or doubly, or deformed) special relativity (DSR) (see, e.g. Ref. [6]). The general idea is that if the Planck length is a truly universal quantity, then it should look the same to any inertial observer. This demands a modification (deformation) of the Lorentz transformations, to accommodate an invariant length scale. In Ref. [6] the nonlinearity of the deformed Lorentz transformations lead the authors to novel commutators between space-time coordinates and momenta, depending on the energy

$$[\hat{x}^i, \hat{p}_j] = i\hbar \left( 1 - \frac{E}{\mathcal{E}_p} \right) \delta_j^i, \quad (28)$$

where  $E$  is the energy scale of the particle to which the deformed Lorentz boost is to be applied, while  $\mathcal{E}_p$  is the Planck energy. This suggests that they have an energy-dependent Planck “constant”  $\hbar(E) = \hbar(1 - E/\mathcal{E}_p)$ . Their model also implies that  $\hbar(E) \rightarrow 0$  for  $E \rightarrow \mathcal{E}_p$ . For energies much below that Planck regime, the usual Heisenberg commutators are recovered, but when  $E \simeq \mathcal{E}_p$  one has  $\hbar(\mathcal{E}_p) \simeq 0$ . So the Planck energy is not only an invariant in this model, but the world looks also apparently classical at the Planck scale, similarly as in ’t Hooft’s proposal.

The connections of the DSR model with our proposal are at this point self evident. Our GUP (14) implies that, at the boundary of the Brillouin zone, when  $\langle \hat{p} \rangle_\psi \rightarrow \hbar\pi/2\epsilon$ , i.e. for Planck energies  $E_\epsilon \simeq (2\sqrt{2}/a)\mathcal{E}_p$ , the fundamental commutator vanishes  $[\hat{X}_\epsilon, \hat{P}_\epsilon] \simeq 0$ , and since  $\Delta X_\epsilon \Delta P_\epsilon \geq 0$ , lattice quantum mechanics at short wavelengths allows for classical behavior, that is uncertainties of two complementary observables *can* be *simultaneously* zero. However, if we express the fundamental commutator (12) of our model in terms of energy, using the exact dispersion relation for photons inferred from the lattice Laplacian (5), we find (for  $\epsilon = a\ell_p$ )

$$[\hat{X}_\epsilon, \hat{P}_\epsilon] f(x) = i\hbar \left( 1 - \frac{a^2 \hat{E}^2}{8 \mathcal{E}_p^2} \right) f(x). \quad (29)$$

This means that the deforming term in our model is quadratic in the energy, instead of being linear in the energy as the DSR model (28).

## 9. Discussion and Summary

It should be noted that the present lattice generalization of the uncertainty principle is not an approximate description, but it is an exact formula necessarily implied by our model of lattice space time. The great majority of the GUP research has always borrowed the deformed commutator  $[\hat{x}, \hat{p}] = i\hbar(1 + \kappa\hat{p}^2)$  either from string theory [4], or from heuristic arguments about black holes [5]. To be precise, even in string theory the formula expressing the GUP is not derived from the basic features of the model, but instead it appears as an economic description of the high-energy gedanken experiments of string scatterings. In contrast to this, we have *derived* all the results from a simple lattice model of space-time, and from the analytic structure of the basic commutator (12).

We have calculated the uncertainties on a crystal-like universe whose lattice spacing is of the order of Planck length — the so-called world crystal. When the energies lie near the border of the Brillouin zone, i.e., for Planck energies, the uncertainty relations for position and momenta pose a zero lower bound. Hence the world crystal universe can become “deterministic” at Planck energies. In this high-energy regime, our lattice commutator resembles the double special relativity result of Magueijo and Smolin. The scenario in which the universe at Planck energies is deterministic and classical, rather than being dominated by quantum fluctuations, is also a starting point in ’t Hooft’s “deterministic” quantum mechanics.

Using this new generalized uncertainty relation we derived a new mass-temperature relation for Schwarzschild (micro) black holes. In contrast to standard results based on Heisenberg or stringy uncertainty relations, our mass-temperature formula predicts both finite Hawking’s temperature and a zero rest-mass remnant at the end of the evaporation process. Especially the absence of remnants is a welcome bonus which allows to avoid such conceptual difficulties as entropy/information problem or why we do not experimentally observe the remnants that must have been prodigiously produced in the early stage of the universe.

Finally, the connections between DSR and our generalized uncertainty relation have been illustrated. The fundamental commutator in DSR as well as in our lattice GUP goes to zero at Planck energy. The world should therefore be manifestly “classical” in the Planck regime — a feature very different from the common believe.

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