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Non-Minimal Einstein–Dirac-Axion Theory: Spinorization of the Early Universe Induced by Curvature

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Abstract: A new non-minimal version of the Einstein–Dirac-axion theory is established. This version of the non-minimal theory describing the interaction of gravitational, spinor, and axion fields is of the second order in derivatives in the context of the Effective Field Theory and is of the first order in the spinor particle number density. The model Lagrangian contains four parameters of non-minimal coupling and includes, in addition to the Riemann tensor, Ricci tensor, and Ricci scalar, as well as left-dual and right-dual curvature tensors. The pseudoscalar field appears in the Lagrangian in terms of trigonometric functions providing the discrete symmetry associated with axions, which is supported. The coupled system of extended master equations for the gravitational, spinor, and axion fields is derived; the structure of new non-minimal sources that appear in these master equations is discussed. Application of the established theory to the isotropic homogeneous cosmological model is considered; new exact solutions are presented for a few model sets of guiding non-minimal parameters. A special solution is presented, which describes an exponential growth of the spinor number density; this solution shows that spinor particles (massive fermions and massless neutrinos) can be born in the early Universe due to the non-minimal interaction with the spacetime curvature.

Keywords: alternative theories of gravity; Einstein–Dirac theory; axion; spinor

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1. Introduction

For more than fifty years, the problem of non-minimal coupling of fields and matter to the spacetime curvature has attracted the serious attention of scientists who study the cosmic sector of fundamental interactions. These investigations are considered to be an important part of the modern trend, which is indicated as Modified Theories of Gravity [1,2]. Based on the mathematical aspects, one can say that two paradigms exist in this trend. The first paradigm is connected with the Effective Field Theory (EFT) (see, e.g., [3–5]), in the framework of which the theories can be classified according to the maximal order of derivatives, which participate in the construction of the Lagrangian. In fact, the first version of the theory of non-minimal coupling of the scalar field φ to the spacetime curvature, which can be indicated as the theory of the second order in derivatives, was presented in [6]. Indeed, in this work, the term φR was introduced into the Lagrangian, where R is the Ricci scalar. The models of non-minimal derivative coupling of the scalar field to curvature [7–9], which contain the convolution of the Ricci tensor with two gradients of the scalar field, are of the fourth order in the context of the EFT. All the works concerning the non-minimal Einstein–Maxwell theory are of the fourth and higher orders in derivatives. Indeed, the author of the work [10] has introduced the one-parameter term $\lambda R^{mnpq} F_{mn} F_{pq}$

into the Lagrangian so that the Riemann tensor R^{mnpq} contains two derivatives of the spacetime metric, and each of the two Maxwell tensors F_{mn} includes one derivative of the electromagnetic potential four-vector A_j. The famous paper [11]), which presented the one-loop Quantum Electrodynamic version of the theory of the photon coupling to curvature, fixed the idea that the true non-minimal theory of photon–graviton coupling, linear in curvature, has to be the three-parameter one. If the pseudoscalar field ϕ can be used as an element of the Lagrangian, the new true scalar term $\phi F^*_{mn} F^{mn}$ appears [12], and this term plays an important role in the mathematical formalism of the axion theory [13–16]. The theory of non-minimal axion–photon coupling, linear in curvature, has been established in [17]. Then, this idea has been extended as follows: the term $\phi F^*_{mn} F^{mn}$ in the Lagrangian has been replaced by the term $\phi R^*_{mnpq} R^{mnpq}$, which is quadratic in curvature and thus is of the fourth order in derivatives. Subsequently, ϕ was replaced by the odd function $U(\phi)$ (see, e.g., [18,19]). Clearly, if $\phi = \phi_0$ is a constant parameter, i.e., this quantity looses the status of pseudoscalar, the term $\phi_0 F^*_{mn} F^{mn}$ in electrodynamics converts to complete divergence. Similarly, when $\phi = \phi_0$, the term $\phi_0 R^*_{mnpq} R^{mnpq}$, which can be indicated as the Chern–Simons pseudoscalar or the Pontryagin density (see, e.g., [20]), also converts to complete divergence and thus does not give contribution to the evolutionary equations for the gravity field.

The second paradigm in the non-minimal approach is based on the non-linear representation of the coupling terms; the most known models in this trend are indicated as $f(R)$, $f(G)$, $f(R, F^2)$, $f(R, T_{\mu\nu} T^{\mu\nu})$, etc., theories (see, e.g., [1,2,21–23]). In these theories, f is considered as an arbitrary function of the Ricci scalar R, of the Gauss–Bonnet scalar G, of the square of the Maxwell tensor $F^2 = F_{mn} F^{mn}$, of the square of the matter stress–energy tensor $T_{\mu\nu}$, etc. This approach does not give us a possibility to initially fix the maximal order of derivatives, which are used in the Lagrangian construction, but opens the window for non-linear modeling in the Modified Theories of Gravity. Both approaches have their own advantages and lead to many interesting results in application to the non-minimal theories of the electromagnetic, scalar, pseudoscalar, and Yang–Mills fields coupling to gravity (see, e.g., [24–37]).

Among the models of non-minimal coupling, the spinor field theory occupies a special place. Such a specific role is due to two factors. First, it is well known that the Einstein–Dirac theory operates with spinor tensors that do not contain derivatives: they have the structure $\bar{\psi}\gamma^m \cdots \gamma^j \psi$, where γ^m denotes the Dirac matrices, ψ is the spinor field, and $\bar{\psi}$ is its Dirac conjugated quantity. In this sense, the simplest non-minimal scalar term $R\bar{\psi}\psi$, which has been used in the work [38], is linear in curvature, is linear in the spinor particle number density $\mathcal{N} = \bar{\psi}\psi$, and is the term of the second order in derivatives in the context of EFT. One can find an evident analogy between the terms φR and $R\bar{\psi}\psi$ introduced in [6] and [38], respectively. Clearly, if we will follow the idea of non-linear extensions of the Einstein–Dirac theory, we can consider the Lagrangian with the term $f(R, \mathcal{N})$ as an analog of the $f(R, F^2)$ theory.

The second factor that distinguishes the non-minimal Einstein–Dirac theory from theories of non-spinor fields is connected with the idea of the use of the left- and right-dual Riemann (pseudo)tensors ${}^*R_{mnpq}$ and R^*_{mnpq} as independent quantities of the Lagrangian decomposition along with the Riemann tensors, Ricci tensors, and Ricci scalar. Indeed, when one deals with the Einstein–Maxwell theory, the dual Maxwell (pseudo)tensor F^*_{mn} and dual Riemann tensors cannot be used separately for the construction of the pure scalar Lagrangian. Convolutions of two mentioned pseudotensors, which are constructed on the basis of the Levi–Civita pseudotensor, give the linear combination of the known terms and thus cannot be used as independent elements. A new situation arises in the Einstein–Dirac theory, since in this theory, there exist spinor pseudotensors of the form $\bar{\psi}\gamma^m \cdots \gamma^j \gamma^5 \psi$, convolutions of which with the left- and right-dual Riemann tensors give new independent true scalars that are suitable for the extension of the Lagrangian. When we work with the pseudoscalar field ϕ, new non-minimal scalar terms of the type $\sin\phi R^{*}_{mnpq}\bar{\psi}\gamma^{m}\gamma^{n}\gamma^{p}\gamma^{q}\psi$ appear as elements of the Lagrangian modeling, and they are also of the second order in derivative and of the first order in the spinor particle number density $\mathcal{N}$. From the physical point of view, the non-minimal coupling of the spinor field to the spacetime curvature can be interesting as an explanation of the abundance of fermions born in the early Universe, when the spacetime curvature was much bigger than now. In other words, the early Universe spinorization (the last term means the anomalous growth of the fermion number density, see, e.g., [39–41]) can be connected with the curvature-induced effects in the spinor systems. Another way of explaining the Universe spinorization can be connected with non-linear interactions of the spinor field with gravitation. The corresponding models are non-linear in the spinor particle number density $\mathcal{N}$; they were discussed, e.g., in the works [42–44].

Keeping in mind the mentioned details of the history of the non-minimal field theory, we established here a new complete non-minimal version of the Einstein–Dirac-axion theory, which is linear in curvature, is of the second order in derivatives, is linear in the number density of spinor particles, and is non-linear in the axion field. The paper is organized as follows. In Section 2, we construct the Lagrangian of the non-minimal Einstein–Dirac-axion theory and derive the extended master equations for the spinor, pseudoscalar, and gravitational fields. In Sections 3–5, we apply the established extended theory to the homogeneous isotropic cosmological model, describe the evolution of the spinor field invariants, and analyze the behavior of the Hubble function and of the scale factor. To be more precise, in Section 3, we reduce the master equations of the spinor, axion, and gravity fields, taking into account the symmetry of the Friedmann–Lemaître–Robertson–Walker (FLRW) spacetime platform; in Section 4, we deal with the exactly integrable model, in which the axion field is frozen in one of the minima of the axion field potential and thus is in the equilibrium state; in Section 5, the axion field is fixed in the unstable state related to one of the maxima of the axion potential. In Section 6, we discuss physical consequences of the non-minimal graviton–spinor–axion interactions.

2. The Formalism

2.1. Action Functional of the Non-Minimal Einstein–Dirac-Axion Theory

The presented version of the non-minimal Einstein–Dirac-axion theory is based on three assumptions.

First, we work in the context of the Effective Field Theory and use the model up to the second order in derivatives.

Second, we consider models in which the master equations for the spinor field remain of the first order in derivatives.

Third, we consider the scheme of model formulation linear in the spinor particle number density.

The total action functional is considered to be of the form

$$\mathcal{S} = \int d^4x\sqrt{-g}\left\{-\frac{1}{2\kappa}(R+2\Lambda)+L_{(\mathrm{NM})}+\frac{1}{2}\Psi_0^2[g^{mn}\nabla_m\phi\nabla_n\phi - V(\phi)]+\right.$$
$$\left.+\left[\frac{i}{2}\left(\bar{\psi}\gamma^k D_k\psi - D_k\bar{\psi}\gamma^k\right)-m\bar{\psi}\psi\right]\right\}. \qquad (1)$$

The elements of the canonic Lagrangian of the minimal Einstein-axion model are well documented. The term g is the determinant of the spacetime metric g_{mn}; R is the Ricci scalar; Λ is the cosmological constant; $\kappa = 8\pi G$ includes the Newtonian coupling constant

G ($c = 1$). The pseudoscalar dimensionless function ϕ relates to the axion field, and the parameter Ψ_0 is equal to the inverse parameter of the axion–photon coupling. The axion field with mass m_A is assumed to be described by the periodic potential

$$V(\phi) = 2m_A^2(1 - \cos\phi)\,. \tag{2}$$

The last term in the integral (1) describes the canonic Lagrangian of the spinor field ψ; the term $L_{(\mathrm{NM})}$ accumulates the non-minimal interaction terms; these elements of the total Lagrangian will be described in the next two subsections.

2.2. Auxiliary Mathematical Details Connected with the Spinor Field

2.2.1. Dirac Matrices and Tetrad Vectors

We use the standard duet: the spinor field ψ and its Dirac conjugated quantity $\bar{\psi}$. Tensors in the Dirac theory are constructed using the scheme $\bar{\psi}\gamma^j \cdots \gamma^n \psi$, and the corresponding pseudotensors have the form $\bar{\psi}\gamma^j \cdots \gamma^m \gamma^n \gamma^5 \psi$. The scalar $S = \bar{\psi}\psi$ is standardly interpreted as a spinor particle number density $\mathcal{N}$. The basic pseudoscalar P is defined as $P = i\bar{\psi}\gamma^5\psi$. The matrices γ^j in the Riemann spacetime with the metric g_{pq} are connected with the constant Dirac matrices $\gamma^{(a)}$, defined in the Minkowski spacetime with the metric $\eta_{(a)(b)}$, via the tetrad vectors $X^j_{(a)}$: $\gamma^k = X^k_{(a)}\gamma^{(a)}$. The following relationships are used in this context.

First, we deal with the normalization of the tetrad four-vectors:

$$g_{mn} X^m_{(a)} X^n_{(b)} = \eta_{(a)(b)}\,, \quad \eta^{(a)(b)} X^m_{(a)} X^n_{(b)} = g^{mn}\,. \tag{3}$$

If we consider, formally, the tetrad four-vector as an element of the Jacobi matrix $X^j_{(a)} = \frac{\partial x^j}{\partial x^{(a)}}$, which describes the local coordinate transformation from the Minkowski spacetime to the Riemann one, we can see that $|\det(X^j_{(a)})| = \frac{1}{\sqrt{-g}}$. Below, we assume that $\det(X^j_{(a)})$ is positive.

Second, the Dirac matrices satisfy the fundamental anti-commutation relations

$$\gamma^{(a)}\gamma^{(b)} + \gamma^{(b)}\gamma^{(a)} = 2E\eta^{(a)(b)} \Longleftrightarrow \gamma^m\gamma^n + \gamma^n\gamma^m = 2Eg^{mn}\,, \tag{4}$$

where E is the four-dimensional unit matrix.

2.2.2. Definition and Properties of the Matrix γ^5

Special attention should be attracted to the definition of matrix γ^5. We follow the basic definition of the Levi–Civita tensor ϵ_{mnpq} via the absolutely anti-symmetric symbol E_{mnpq}:

$$\epsilon_{mnpq} = \sqrt{-g} E_{mnpq}\,, \quad E_{0123} = -1\,. \tag{5}$$

Taking into account the standard definition of the determinant, we obtain

$$\epsilon_{mnpq} X^m_{(a)} X^n_{(b)} X^p_{(c)} X^q_{(d)} = \sqrt{-g}\, \det(X^j_{(a)}) \epsilon_{(a)(b)(c)(d)} = \epsilon_{(a)(b)(c)(d)}\,, \tag{6}$$

keeping in mind that in the Minkowski spacetime, $\epsilon_{(a)(b)(c)(d)} \equiv E_{(a)(b)(c)(d)}$, with $E_{(0)(1)(2)(3)} = -1$. Using (6), we can introduce in the covariant way the link between the Dirac matrices γ^5 and $\gamma^{(5)}$ as follows:

$$\gamma^5 \equiv -\frac{1}{4!}\epsilon_{mnpq}\gamma^m\gamma^n\gamma^p\gamma^q = -\frac{1}{4!}\epsilon_{mnpq} X^m_{(a)} X^n_{(b)} X^p_{(c)} X^q_{(d)} \gamma^{(a)}\gamma^{(b)}\gamma^{(c)}\gamma^{(d)} = \tag{7}$$

$$= -\frac{1}{4!}\epsilon_{(a)(b)(c)(d)}\gamma^{(a)}\gamma^{(b)}\gamma^{(c)}\gamma^{(d)} = \gamma^{(0)}\gamma^{(1)}\gamma^{(2)}\gamma^{(3)} \equiv \gamma^{(5)}.$$

In other words, the matrix γ^5 does not depend on the metric and, in addition to the unit matrix E, is a constant matrix. The multiplier i in the definition of the pseudoscalar P allows the matrix $i\gamma^5$ to be free of the imaginary unit.

2.2.3. The Fock–Ivanenko Connection Coefficients

The covariant derivatives of the spinor fields

$$D_k\psi = \partial_k\psi - \mathcal{Q}_k\psi\,, \quad D_k\bar{\psi} = \partial_k\bar{\psi} + \bar{\psi}\mathcal{Q}_k \tag{8}$$

are constructed using the Fock–Ivanenko connection matrices $\mathcal{Q}_k$ [45]

$$\mathcal{Q}_k = \frac{1}{4}g_{mn}X^{(a)}_s\gamma^s\gamma^n\nabla_k X^m_{(a)}\,. \tag{9}$$

2.3. Non-Minimal Extension of the Action Functional

2.3.1. Geometric and Spinor Elements of the Lagrangian Decomposition

We classify the terms appearing in the extended non-minimal Lagrangian in the context of the second-order model of the EFT. For the Riemann tensor $R^j{}_{kmn}$, its convolutions $R_{kn} = R^m_{kmn}$ and $R = R^m_m$, as well as the left- and right-dual tensors, are defined as

$$^*R_{jkmn} = \frac{1}{2}\epsilon_{jkpq}R^{pq}{}_{mn}\,, \quad R^*_{jkmn} = \frac{1}{2}R_{jk}{}^{pq}\epsilon_{pqmn}\,, \tag{10}$$

which are just of the second order in derivatives, so we cannot use derivatives of other quantities in the corresponding decomposition. In order to introduce the odd and even contributions of the axion field ϕ, we define two new matrices

$$\mathcal{A} = E\sin\phi + i\nu_*\gamma^5\cos\phi\,, \quad \mathcal{B} = E\cos\phi - i\nu_*\gamma^5\sin\phi\,, \tag{11}$$

which are characterized by one real parameter ν_*. They possess the evident properties

$$\frac{d}{d\phi}\mathcal{A} = \mathcal{B}\,, \quad \frac{d}{d\phi}\mathcal{B} = -\mathcal{A}\,, \quad \mathcal{A}^2 + \mathcal{B}^2 = (1+\nu_*^2)E\,. \tag{12}$$

The following new scalars containing both spinor and axion fields are useful for further calculations defined as

$$\bar{\psi}i\gamma^5\mathcal{A}\psi = \nu_* S\cos\phi + P\sin\phi\,, \quad \bar{\psi}\mathcal{B}\psi = S\cos\phi - \nu_* P\sin\phi\,. \tag{13}$$

Both scalars are periodic, i.e., do not vary under the discrete transformation $\phi \to \phi + 2\pi k$.

2.3.2. Non-Minimal Contributions to the Total Lagrangian

We present the scalar linear in R and linear in the spinor particle density as the product

$$\beta_1 R\bar{\psi}\mathcal{B}\psi = \beta_1 R(S\cos\phi - \nu_* P\sin\phi)\,. \tag{14}$$

Since the Ricci tensor is symmetric, we see that $R_{mn}(\bar{\psi}\gamma^m\gamma^n\psi) = R_{mn}(\bar{\psi}g^{mn}\psi) = RS$, i.e., there are no independent terms with the Ricci tensor linear in the spinor particle number density. We present the scalar containing the curvature tensor in the following form:

$$\beta_2 R_{jkmn}(\bar{\psi}\gamma^j\gamma^k\gamma^m\gamma^n\mathcal{B}\psi)\,. \tag{15}$$

Similarly, we present the scalars linear in the left- and right-dual Riemann tensors as follows:

$$i\beta_* \left({}^*R_{jkmn} + R^*_{jkmn} \right) \left(\bar{\psi}\gamma^j\gamma^k\gamma^m\gamma^n\mathcal{A}\psi \right). \tag{16}$$

This term is chosen in the symmetric form, since we assume that the Lagrangian is symmetric with respect to the left and right duality. Thus, the non-minimal terms in the Lagrangian can be written as follows:

$$L_{(\mathrm{NM})} = \beta_1 R(\bar{\psi}\mathcal{B}\psi) + \beta_2 R_{jkmn}\left(\bar{\psi}\gamma^j\gamma^k\gamma^m\gamma^n\mathcal{B}\psi \right) + i\beta_* \left({}^*R_{jkmn} + R^*_{jkmn} \right) \left(\bar{\psi}\gamma^j\gamma^k\gamma^m\gamma^n\mathcal{A}\psi \right). \tag{17}$$

This element of the Lagrangian decomposition is the four-parameter one: the guiding parameters are β_1, β_2, β_*, and ν_*.

2.4. Master Equations

The variation procedure with respect to the spinor field ψ, its Dirac conjugate quantity $\bar{\psi}$, axion field ϕ, and the metric g^{pq} gives us the coupled system of the extended master equations.

2.4.1. Master Equations for the Spinor Field

Variation in the action functional (1) with respect to $\bar{\psi}$ and ψ gives the extended Dirac equations, which have, formally speaking, the standard structure

$$i\gamma^n D_n\psi = M\psi\,, \quad iD_n\bar{\psi}\gamma^n = -\bar{\psi}M\,. \tag{18}$$

However, the matrix M is now of a much more complicated form

$$M = mE + M_{(\mathrm{NM})}\,, \tag{19}$$

$$-M_{(\mathrm{NM})} = \beta_1 R\mathcal{B} + \beta_2 R_{jkmn}\gamma^j\gamma^k\gamma^m\gamma^n\mathcal{B} + i\beta_*\left({}^*R_{jkmn} + R^*_{jkmn} \right)\gamma^j\gamma^k\gamma^m\gamma^n\mathcal{A}\,. \tag{20}$$

The matrix of the effective mass M, in addition to the seed mass term mE, contains the term with non-minimally induced mass $M_{(\mathrm{NM})}$, which depends on the spacetime curvature and on the axion field. In other words, the extension of the Dirac equations is connected with the introduction of a new effective spinor field mass.

2.4.2. Master Equations for the Axion Field

Variation with respect to the pseudoscalar field ϕ gives the axion field equation

$$g^{mn}\nabla_m\nabla_n\phi + m_A^2 \sin\phi = \frac{1}{\Psi_0^2}\mathcal{J} \tag{21}$$

with the curvature-induced pseudoscalar source

$$\mathcal{J} = -\beta_1 R(\bar{\psi}\mathcal{A}\psi) - \beta_2 R_{jkmn}\left(\bar{\psi}\gamma^j\gamma^k\gamma^m\gamma^n\mathcal{A}\psi \right) + i\beta_*\left({}^*R_{jkmn} + R^*_{jkmn} \right)\left(\bar{\psi}\gamma^j\gamma^k\gamma^m\gamma^n\mathcal{B}\psi \right), \tag{22}$$

which depends on the spinor and axion fields.

2.4.3. Master Equations for the Gravity Field

Variation in the action functional (1) with respect to the metric gives the equation which can be formally written in the Einstein-like form

$$R_{pq} - \frac{1}{2}g_{pq}R - \Lambda g_{pq} = \kappa T^{(\mathrm{D})}_{pq} + \kappa T^{(\mathrm{A})}_{pq} + \kappa T^{(\mathrm{NM})}_{pq}\,. \tag{23}$$

The first term in the right-hand side of this equation

$$T^{(\mathrm{D})}_{pq} = -g_{pq}\left[\frac{i}{2}[\bar{\psi}\gamma^k D_k\psi - D_k\bar{\psi}\gamma^k\psi] - m\bar{\psi}\psi\right] + \frac{i}{4}\left[\bar{\psi}\gamma_p D_q\psi + \bar{\psi}\gamma_q D_p\psi - (D_p\bar{\psi})\gamma_q\psi - (D_p\bar{\psi})\gamma_q\psi\right] \quad (24)$$

describes the canonic stress–energy tensor of the spinor field. The second term in the right-hand side of (23)

$$T^{(\mathrm{A})}_{pq} = \Psi_0^2\left\{\nabla_p\phi\nabla_q\phi - \frac{1}{2}g_{pq}[g^{mn}\nabla_m\phi\nabla_n\phi - V(\phi)]\right\} \quad (25)$$

relates to the canonic stress–energy tensor of the axion field.

We divide the non-minimal contributions to the total stress–energy tensor into three parts, which have in front three non-minimal parameters β_1, β_2, and β_*, respectively:

$$T^{(\mathrm{NM})}_{pq} = \beta_1 T^{(\mathrm{NM1})}_{pq} + \beta_2 T^{(\mathrm{NM2})}_{pq} + i\beta_* T^{(\mathrm{NM}*)}_{pq}\,, \quad (26)$$

where the following definitions have been used:

$$T^{(\mathrm{NM1})}_{pq} = 2\left[R_{pq} - \frac{1}{2}Rg_{pq} + g_{pq}\nabla_m\nabla^m - \nabla_{(p}\nabla_{q)}\right](\bar{\psi}\mathcal{B}\psi)\,, \quad (27)$$

$$T^{(\mathrm{NM2})}_{pq} = 2\bar{\psi}\left[\gamma_{(p}R_{q)kmn} - \frac{1}{2}g_{pq}R_{jkmn}\gamma^j\right]\gamma^k\gamma^m\gamma^n\mathcal{B}\psi + 4\bar{\psi}\left[2\gamma_{(p}R_{q)m}\gamma^m - R_{pq}\right]\mathcal{B}\psi +$$
$$2\nabla_p\nabla_q\bar{\psi}\mathcal{B}\psi - 4g_{pq}\nabla_s\nabla_n\bar{\psi}\gamma^s\gamma^n\mathcal{B}\psi - 2(\nabla_s\nabla_q - \nabla_q\nabla_s)\bar{\psi}\gamma_p\gamma^s\mathcal{B}\psi - 2(\nabla_s\nabla_p - \nabla_p\nabla_s)\bar{\psi}\gamma_q\gamma^s\mathcal{B}\psi\,, \quad (28)$$

$$T^{(\mathrm{NM}*)}_{pq} = -2\bar{\psi}\gamma^*_{(p}R_{q)kmn}\gamma^k\gamma^m\gamma^n\mathcal{A}\psi - 4\nabla_s\nabla_m\bar{\psi}\gamma_l\gamma_k\gamma^s\gamma_{(p}\epsilon_{q)}^{\cdot mlk}\mathcal{A}\psi - \quad (29)$$
$$-4\epsilon^{lkm\cdot}_{(p}\nabla_{q)}\nabla_m\bar{\psi}\gamma_l\gamma_k\mathcal{A}\psi - 2\bar{\psi}\gamma^k\gamma^m\gamma^n R^*_{kmn(p}\gamma_{q)}\mathcal{A}\psi -$$
$$-4\epsilon^{mlk\cdot}_{(p}\nabla_{q)}\nabla_m\bar{\psi}\gamma_l\gamma_k\mathcal{A}\psi - 4\nabla_s\nabla_m\epsilon^{mlk\cdot}_{(p}\bar{\psi}\gamma_{q)}\gamma^s\gamma_l\gamma_k\mathcal{A}\psi\,.$$

The decomposition (26) of the non-minimal part of the total stress–energy tensor is convenient, in particular, for the interpretation of the obtained results; for instance, below, we will show that the parameter β_* is responsible for the description of the growth of the spinor particle number density, and thus, the term $T^{(\mathrm{NM}*)}_{pq}$ provides the energy support of the spinorization of the early Universe.

We used three groups of auxiliary relationships in the variation procedure. The first group is connected with the variation in the following geometric objects:

$$\delta\Gamma^j_{sn} = \frac{1}{2}\left(g_{sp}g_{nq}\nabla^j - \delta^j_p g_{sq}\nabla_n - \delta^j_p g_{nq}\nabla_s\right)\delta g^{pq}\,, \quad (30)$$

$$\delta R^j{}_{smn} = \frac{1}{2}\left[g_{s(p}g_{q)n}\nabla_m\nabla^j - g_{s(p}g_{q)m}\nabla_n\nabla^j + \delta^j_{(p}g_{q)s}(\nabla_n\nabla_m - \nabla_m\nabla_n) + \right. \quad (31)$$
$$\left. + \delta^j_{(p}g_{q)m}\nabla_n\nabla_s - \delta^j_{(p}g_{q)n}\nabla_m\nabla_s\right]\delta g^{pq}\,,$$

$$\delta R_{sn} = \frac{1}{2}\left[g_{s(p}g_{q)n}\nabla_m\nabla^m - g_{s(p}g_{q)m}\nabla_n\nabla^m + \delta^m_{(p}g_{q)s}(\nabla_n\nabla_m - \nabla_m\nabla_n) + \right.$$
$$\left. + g_{pq}\nabla_n\nabla_s - \delta^m_{(p}g_{q)n}\nabla_m\nabla_s\right]\delta g^{pq}, \quad (32)$$

$$\delta R = \left(R_{pq} + g_{pq}\nabla_m\nabla^m - \nabla_{(p}\nabla_{q)}\right)\delta g^{pq}\,. \quad (33)$$

The second group of auxiliary relationships is connected with the variation in the tetrad vectors, which depend on the metric due to the following normalization conditions (4):

$$\delta X^j_{(a)} = \frac{1}{4}\Big[X_{p(a)}\delta^j_q + X_{q(a)}\delta^j_p\Big]\delta g^{pq}\,, \quad \delta X_{j(a)} = -\frac{1}{4}\Big[X_{p(a)}g_{jq} + X_{q(a)}g_{jp}\Big]\delta g^{pq}\,. \tag{34}$$

The third group gives us variations of the Dirac matrices

$$\delta\gamma^{(a)} = 0\,, \quad \delta\gamma^{(5)} = 0 = \delta\gamma^5\,, \tag{35}$$

$$\delta\gamma^k = \gamma^{(a)}\delta X^k_{(a)} = \frac{1}{4}\delta g^{pq}\Big(\gamma_p\delta^k_q + \gamma_q\delta^k_p\Big)\,, \quad \delta\gamma_k = -\frac{1}{4}\delta g^{pq}\Big(\gamma_p g_{kq} + \gamma_q g_{kp}\Big)\,. \tag{36}$$

Finally, we presented the complete set of coupled master equations for the spinor field (see (18)–(20)), for the axion field (see (21) and (22), and for the gravitational field (see (23)–(29)).

3. Cosmological Application

3.1. Geometrical Aspects of the Model

As an application, we consider the spatially isotropic homogeneous spacetime platform with the Friedmann–Lemaître–Robertson–Walker-type metric

$$ds^2 = dt^2 - a^2(t)[dx^{1^2} + dx^{2^2} + dx^{3^2}] \tag{37}$$

with the scale factor $a(t)$. As usual, we introduce the Hubble function $H(t) \equiv \frac{\dot{a}}{a}$ (here and below, the dot symbolizes the derivative with respect to time). For the metric (37), the tetrad vectors take the simple form

$$X^i_{(0)} = \delta^i_0\,, \quad X^i_{(1)} = \delta^i_1\frac{1}{a(t)}\,, \quad X^i_{(2)} = \delta^i_2\frac{1}{a(t)}\,, \quad X^i_{(3)} = \delta^i_3\frac{1}{a(t)}\,, \tag{38}$$

and the spinor connection coefficients $\mathcal{Q}_k$ are

$$\mathcal{Q}_0 = 0\,, \quad \mathcal{Q}_1 = \frac{1}{2}\dot{a}\gamma^{(1)}\gamma^{(0)}\,, \quad \mathcal{Q}_2 = \frac{1}{2}\dot{a}\gamma^{(2)}\gamma^{(0)}\,, \quad \mathcal{Q}_3 = \frac{1}{2}\dot{a}\gamma^{(3)}\gamma^{(0)}\,. \tag{39}$$

As a direct consequence of (39), we obtain the auxiliary formula

$$\gamma^k\mathcal{Q}_k = -\frac{3}{2}H\gamma^0 = -\mathcal{Q}_k\gamma^k\,. \tag{40}$$

For convenience, we list the non-vanishing components of the Riemann tensor, Ricci tensor, and Ricci scalar:

$$R^{01}{}_{01} = R^{02}{}_{02} = R^{03}{}_{03} = -\frac{\ddot{a}}{a}\,, \quad R^{12}{}_{12} = R^{13}{}_{13} = R^{23}{}_{23} = -\left(\frac{\dot{a}}{a}\right)^2\,,$$

$$R^0_0 = -3\frac{\ddot{a}}{a}\,, \quad R^1_1 = R^2_2 = R^3_3 = -\left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2\right]\,, \quad R = -6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] = -6\Big(\dot{H} + 2H^2\Big)\,. \tag{41}$$

The non-vanishing components of the left- and right-dual Riemann tensors are

$${}^*R^{01}{}_{23} = {}^*R^{02}{}_{31} = {}^*R^{03}{}_{12} = -\frac{\dot{a}^2}{a}\,, \quad R^{*12}{}_{03} = R^{*23}{}_{01} = R^{*31}{}_{02} = \frac{\ddot{a}}{a^2}\,. \tag{42}$$

Using (41), (42), and the anti-commutation relations (4), one can calculate the terms from the non-minimal part of the Lagrangian and represent them in the form

$$R_{jkmn}\left(\bar{\psi}\gamma^j\gamma^k\gamma^m\gamma^n\mathcal{B}\psi\right) = 12(\bar{\psi}\mathcal{B}\psi)\left[\frac{\ddot{a}}{a}+\left(\frac{\dot{a}}{a}\right)^2\right] = -2R\,(\bar{\psi}\mathcal{B}\psi)\,, \tag{43}$$

$$\left({}^*R_{jkmn}+R^*_{jkmn}\right)\left(\bar{\psi}\gamma^j\gamma^k\gamma^m\gamma^n\mathcal{A}\psi\right) = 24\left(\bar{\psi}\gamma^5\mathcal{A}\psi\right)\left[\left(\frac{\dot{a}}{a}\right)^2+\frac{\ddot{a}}{a}\right] = -4R\,\left(\bar{\psi}\gamma^5\mathcal{A}\psi\right). \tag{44}$$

The matrix M, which describes the non-minimal effective mass of the spinor particle, can now be presented as follows:

$$M = mE - R\left(E\chi_1\cos\phi + i\gamma^5\chi_2\sin\phi\right), \tag{45}$$

where two auxiliary coefficients are introduced as follows:

$$\chi_1 = (\beta_1 - 2\beta_2 - 4\nu_*\beta_*)\,, \quad \chi_2 = [\nu_*(2\beta_2-\beta_1) - 4\beta_*]\,. \tag{46}$$

The scalar of the effective mass has the form

$$\mathcal{M} \equiv \frac{(\bar{\psi}M\psi)}{(\bar{\psi}\psi)} = m - R\left(\chi_1\cos\phi + \frac{P}{S}\chi_2\sin\phi\right). \tag{47}$$

Calculation of the source term $\mathcal{J}$ (see (22)) in the extended equation for the axion field (21) gives

$$\mathcal{J} = -R(\chi_1 S\sin\phi - \chi_2 P\cos\phi)\,. \tag{48}$$

3.2. Reduced Dirac Equations and Evolution of the Spinor Scalars

The high symmetry of the FLRW spacetime allows us to reduce the Dirac Equation (18) to the following simple pair of the ordinary differential equations of the first order:

$$i\gamma^0\frac{d}{dt}\left[a^{\frac{3}{2}}(t)\psi\right] = M\left[a^{\frac{3}{2}}(t)\psi\right], \quad i\frac{d}{dt}\left[a^{\frac{3}{2}}(t)\bar{\psi}\right]\gamma^0 = -\left[a^{\frac{3}{2}}(t)\bar{\psi}\right]M\,. \tag{49}$$

This symmetry opens a possibility to establish a closed system of evolutionary equations for three quantities—one scalar function $S(t)$ and two pseudoscalars $P(t)$ and $\Omega(t)$—which are defined as follows:

$$S = \bar{\psi}E\psi\,, \quad P = \bar{\psi}i\gamma^5\psi\,, \quad \Omega = \bar{\psi}\gamma^5\gamma^0\psi\,. \tag{50}$$

Indeed, using the reduced Dirac Equation (49) and the presentation (45) of the effective mass matrix, we obtain

$$\dot{S} + 3HS = i\bar{\psi}(M\gamma^0 - \gamma^0 M)\psi = 2\chi_2 R\,\Omega\sin\phi\,, \tag{51}$$

$$\dot{P} + 3HP = \bar{\psi}\left(\gamma^5\gamma^0 M - M\gamma^0\gamma^5\right)\psi = 2\Omega(m - \chi_1 R\cos\phi)\,, \tag{52}$$

$$\dot{\Omega} + 3H\Omega = -i\bar{\psi}(M\gamma^5 + \gamma^5 M)\psi = 2P(\chi_1 R\cos\phi - m) + 2\chi_2 R\,S\sin\phi\,. \tag{53}$$

The idea to use the evolutionary equation for S and P for the description of the spinor system has already been proposed earlier by the authors of the works [42–44]. Here, we have to emphasize that in the non-minimal model, similarly to the model studied in [46], the evolutionary system for S, P, and Ω again happened to be the closed one. In other words, for the non-minimal model based on the FLRW platform, there is no necessity to extend the basis of S, P, and Ω of the spinor scalars/pseudoscalars.

If we introduce the functions

$$X(t) = S(t)\left(\frac{a(t)}{a(t_0)}\right)^3, \quad Y(t) = P(t)\left(\frac{a(t)}{a(t_0)}\right)^3, \quad Z(t) = \Omega(t)\left(\frac{a(t)}{a(t_0)}\right)^3, \tag{54}$$

and rewrite the evolutionary Equations (51)–(53) in the form

$$\dot{X} = 2\chi_2 R\, Z \sin\phi\,, \quad \dot{Y} = 2Z(m - \chi_1 R\cos\phi)\,, \quad \dot{Z} = 2Y(\chi_1 R\cos\phi - m) + 2\chi_2 R\, X\sin\phi\,, \tag{55}$$

we obtain immediately that the following integral of this system of equations exists as

$$Y^2 + Z^2 - X^2 = K\,, \tag{56}$$

where K is constant. In other words, one can extract $Z(t)$ from (56) and put it into the first and second equations of the set (55), thus reducing the system to the pair of equations for X and Y only. In this sense, just the quantities S and P predetermine the character of the spinor system evolution.

3.3. Reduced Equation of the Axion Dynamics

The evolutionary equation of the pseudoscalar (axion) field (21) supplemented by (22) now takes the form

$$\ddot{\phi} + 3H\dot{\phi} + \left[m_A^2 + \frac{R}{\Psi_0^2}\chi_1 S\right]\sin\phi = \frac{R}{\Psi_0^2}\chi_2 P\, \cos\phi\,. \tag{57}$$

One can state that there exists an effective non-minimally induced axion mass M_A given by the formula

$$M_A = \sqrt{m_A^2 + \frac{R}{\Psi_0^2}\chi_1 S}\,. \tag{58}$$

It is important to note that the axion field can be frozen in one of the minima of the periodic potential (2) (it is possible when $\phi = 2\pi k$) if and only if $\chi_2 P = 0$. In other words, the equilibrium in the axionic system (see, e.g., [47]) is admissible, when $\chi_2 = 0$, or the state is characterized by the requirement $P(t) = 0$.

3.4. Key Equation for the Gravitational Dynamics

It is well known that when one deals with the symmetric FLRW spacetime platform, the set of the gravity field equations can be reduced to one equation. For our case, one has to calculate the covariant derivatives in (23)–(29) and to choose the equation with the indices $p = 0$ and $q = 0$. This equation takes the form

$$\begin{aligned} 3H^2 - \Lambda = \kappa m S + \kappa\Psi_0^2\left[\dot{\phi}^2 + 2m_A^2(1-\cos\phi)\right] + 6\beta_1\left(H^2 + H\frac{d}{dt}\right)(S\cos\phi - \nu_* P\sin\phi) - \\ -12\beta_2\left(\dot{H} + 2H^2\right)(S\cos\phi - \nu_* P\sin\phi) - 6\beta_2\frac{d^2}{dt^2}(S\cos\phi - \nu_* P\sin\phi) - \\ -12\beta_2 H\frac{d}{dt}(S\cos\phi - \nu_* P\sin\phi) - 12\beta_*\left(\dot{H} + 2H^2\right)(\nu_* S\cos\phi + P\sin\phi) \\ +48\beta_*\left(H^2 - H\frac{d}{dt}\right)(\nu_* S\cos\phi + P\sin\phi)\,. \end{aligned} \tag{59}$$

Short Resume

Thus, we obtained the key system, which contains four evolutionary equations for four unknown functions. The first key equation is the gravity field Equation (59); it includes

ϕ, S, P, and their derivatives up to the second order, as well as H and $\dot{H}$. The second key equation is the axion field Equation (57); one can extract $\ddot{\phi}$ from this equation and put it into (59). The third and fourth key equations are the equations for the spinor quantities S and P (see (51), (52), and the integral (56), which excludes the function Ω from consideration). One can extract $\dot{S}$ and $\dot{P}$ from these equations and put them into (59). Integration of the key system of evolutionary equations in the general case is possible by the numerical methods only; we hope to fulfill such analysis in future. But below, we consider two particular submodels, which are exactly integrable.

4. First Exactly Integrable Submodel

According to Equation (57), the axion equilibrium state $\phi = 2\pi k$ with integer k is admissible when $\chi_2 = 0$, i.e., when the constants of the non-minimal coupling are linked by the relations

$$\chi_2 = \nu_*(2\beta_2 - \beta_1) - 4\beta_* = 0 \Rightarrow \chi_1 = -4\beta_* \left(\frac{1+{\nu_*}^2}{\nu_*} \right). \tag{60}$$

4.1. Evolution of the Spinor Scalar S and Pseudoscalars P, Ω

For the axion field in the state of equilibrium, Equation (55) can be transformed into

$$\dot{X} = 0\,, \quad \dot{Y} = 2Z(m - R\chi_1)\,, \quad \dot{Z} = -2Y(m - R\chi_1)\,. \tag{61}$$

Clearly, $X(t) = const = S(t_0)$, and the number density of the spinor particles $\mathcal{N} = S(t) = S(t_0)\left(\frac{a(t_0)}{a(t)}\right)^3$ decreases when the Universe expands. Analyzing the equations for Y and Z, we can formally introduce the new variable

$$\tau(t) = 2\int_{t_0}^{t} d\xi [m - R(\xi)\chi_1]\,, \quad \tau(t_0) = 0\,, \tag{62}$$

and find the exact solutions

$$Y(\tau) = Y(0)\cos\tau + Y'(0)\sin\tau\,, \quad Z(\tau) = Y'(0)\cos\tau - Y(0)\sin\tau\,, \tag{63}$$

$$Y^2(\tau) + Z^2(\tau) = Y^2(0) + {Y'}^2(0) \Rightarrow K = Y^2(0) + {Y'}^2(0) - X^2(0)\,. \tag{64}$$

This means that the functions $Y(\tau)$ and $Z(\tau)$ are bounded, and the pseudoscalars $P(t)$ and $\Omega(t)$ are also decreasing in the expanding Universe. Mention should be made that in this submodel, the effective mass of the spinor field

$$\mathcal{M} = \frac{\bar{\psi}M\psi}{\bar{\psi}\psi} = m - \chi_1 R(t) \tag{65}$$

is directly connected with the auxiliary time $\tau(t) = 2\int_{t_0}^{t} d\xi \mathcal{M}(\xi)$.

4.2. Evolution of the Hubble Function

In the framework of this submodel, the pseudoscalar P does not enter the key equation of the gravitational field (59), and this key equation can be written as follows:

$$H^2 - \frac{\Lambda}{3} = S(t_0)\frac{a^3(t_0)}{a^3(t)}\left[\frac{1}{3}\kappa m + 2H^2\left(28\beta_*\nu_* - 11\beta_2 + 8\frac{\beta_*}{\nu_*}\right) + 2\dot{H}(\beta_2 - 2\beta_*\nu_*)\right]. \tag{66}$$

For the analysis of this equation, we use the new variable $x = \frac{a(t)}{a(t_0)}$ and the dimensionless function $\frac{H(x)}{H_\infty}$, where $H_\infty = \sqrt{\frac{\Lambda}{3}}$, and the rule of differentiation is $\frac{d}{dt} = xH(x)\frac{d}{dx}$. In these terms, the key Equation (66) takes the form

$$\Gamma_1 x \frac{d\mathcal{U}}{dx} + (\Gamma_2 - x^3)\mathcal{U} + \Gamma_3 = 0\,, \tag{67}$$

where the following definitions are used:

$$\mathcal{U} = \frac{H^2}{H^2_\infty} - 1\,, \quad \Gamma_1 = S(t_0)(\beta_2 - 2\beta_* \nu_*)\,, \quad \Gamma_2 = 2S(t_0)\left(28\beta_*\nu_* - 11\beta_2 + 8\frac{\beta_*}{\nu_*}\right), \quad \Gamma_3 = \frac{\kappa m S(t_0)}{3H^2_\infty} + \Gamma_2\,. \tag{68}$$

Clearly, the solutions to the Equation (67) essentially depend on the value and sign of the parameter Γ_1, and below, we consider the cases $\Gamma_1 = 0$, $\Gamma_1 > 0$, and $\Gamma_1 < 0$ separately.

4.2.1. The Case $\Gamma_1 > 0$

When $\Gamma_1 \neq 0$, we deal with the linear differential equation of the first order, and the corresponding solution is

$$\mathcal{U} = \left[\frac{H^2(1)}{H^2_\infty} - 1\right] x^{-\frac{\Gamma_2}{\Gamma_1}} e^{\frac{(x^3-1)}{3\Gamma_1}} - \frac{\Gamma_3}{\Gamma_1} x^{-\frac{\Gamma_2}{\Gamma_1}} e^{\frac{x^3}{3\Gamma_1}} \int_1^x d\xi \xi^{\frac{\Gamma_2}{\Gamma_1}-1} e^{-\frac{\xi^3}{3\Gamma_1}}\,. \tag{69}$$

When $\Gamma_1 > 0$, the behavior of the Hubble function is predetermined by the properties of the function $e^{\frac{(x^3-1)}{3\Gamma_1}}$, which grows infinitely at $x \to \infty$. A typical behavior of $H(x)$ in this case can be visualized by the solution with $\Gamma_3 = 0$, i.e., with $\Gamma_2 = -\frac{\kappa m S(t_0)}{3H^2_\infty} < 0$. We now obtain

$$H(x) = \pm H_\infty \sqrt{1 + \left[\frac{H^2(1)}{H^2_\infty} - 1\right] x^{\frac{|\Gamma_2|}{|\Gamma_1|}} e^{\frac{(x^3-1)}{3|\Gamma_1|}}}\,,$$

$$\frac{2HH'(x)}{H^2_\infty} = \frac{1}{|\Gamma_1|}\left[\frac{H^2(1)}{H^2_\infty} - 1\right] x^{\frac{|\Gamma_2|}{|\Gamma_1|}-1} e^{\frac{(x^3-1)}{3|\Gamma_1|}} \left(x^3 + |\Gamma_2|\right). \tag{70}$$

Clearly, when $|H(1)| > H_\infty$, the function $H(x)$ (see (70)) has no extrema and grows monotonically from $H(1)$ to infinity. The integral in the right-hand side of the equality

$$H_\infty(t - t_0) = \int_1^{\frac{a(t)}{a(t_0)}} \frac{dx}{x\sqrt{1 + \left[\frac{H^2(1)}{H^2_\infty} - 1\right] x^{\frac{|\Gamma_2|}{|\Gamma_1|}} e^{\frac{(x^3-1)}{3|\Gamma_1|}}}} \tag{71}$$

converges when the upper limit tends to infinity. This means that the scale factor $a(t)$ reaches the infinite value during the finite time interval, and we deal with the Big Rip according to the classification presented in [48].

When $|H(1)| < H_\infty$, the Hubble function monotonically decreases, takes a zero value, and then becomes an imaginary quantity. This scenario is not physically motivated.

When $|H(1)| = H_\infty$, we deal with the de Sitter solution, since now $H(t) = H_\infty = \sqrt{\frac{\Lambda}{3}}$, and $a(t) = a(t_0)e^{H_\infty(t-t_0)}$. In this case, the auxiliary function $\tau(t)$, which enters the solution $P(\tau)$ as the argument, takes the form

$$\tau(t) = \frac{2(t-t_0)}{\nu_*}\left[m\nu_* - 48(1+\nu^2_*)H^2_\infty\right]. \tag{72}$$

REMARK: This concerns the behavior of the axion field perturbations.

Let us imagine that, due to fluctuations, the equilibrium value of the axion field acquires a small perturbation $\phi = 2\pi k + \varphi$ at the moment $t = t_1$ ($\varphi(t_1) = 0$, $\varphi'(t_1) \neq 0$). In terms of x, Equation (57), being modified, can be written as

$$x^2\varphi'' + 2x\varphi'\left[1 - \frac{R}{12H^2}\right] + \left[\frac{m_A^2}{H^2} + \frac{\chi_1 S(t_0)R}{\Psi_0^2 x^3 H^2}\right]\varphi = 0\,, \tag{73}$$

where the term $\frac{R(x)}{H^2}$ calculated on the base of the solution (70) is of the form

$$\frac{R(x)}{H^2} = -12\left[1 + \frac{1}{4}x\frac{(H^2)'}{H^2}\right] = -12\left\{1 + \frac{\frac{1}{4|\Gamma_1|}\left[\frac{H^2(1)}{H_\infty^2} - 1\right]x^{\frac{|\Gamma_2|}{|\Gamma_1|}}\, e^{\frac{(x^3-1)}{3|\Gamma_1|}}\left(x^3 + |\Gamma_2|\right)}{1 + \left[\frac{H^2(1)}{H_\infty^2} - 1\right]x^{\frac{|\Gamma_2|}{|\Gamma_1|}}\, e^{\frac{(x^3-1)}{3|\Gamma_1|}}}\right\}. \tag{74}$$

When $x \to \infty$, this term tends to $\frac{R(x)}{H^2} \to -\frac{3x^3}{|\Gamma_1|}$, and the asymptotic version of the Equation (73) transforms into

$$\varphi'' + \frac{x^2}{2|\Gamma_1|}\varphi' - \frac{3\chi_1 S(t_0)}{\Psi_0^2|\Gamma_1|x^2}\varphi = 0\,. \tag{75}$$

In the asymptotic regime $\varphi'(x) \propto e^{-\frac{x^3}{6|\Gamma_1|}}$, the perturbations remain bounded. In other words, even in the case of the Big Rip scenario, the equilibrium state of the axion field remains stable.

4.2.2. The Case $\Gamma_1 < 0$

In order to illustrate the behavior of the Hubble function in this case, we consider the particular submodel with $\Gamma_2 = 3\Gamma_1$ and obtain that $\Gamma_3 = \frac{\kappa m S(t_0)}{3H_\infty^2} - 3|\Gamma_1|$, and

$$H(x) = H_\infty\sqrt{1 + \left[\frac{H^2(1)}{H_\infty^2} - 1 - \Gamma_3\right]x^{-3}\, e^{-\frac{(x^3-1)}{3|\Gamma_1|}} + \Gamma_3 x^{-3}}\,. \tag{76}$$

We add the information about the derivative

$$\frac{2HH'(x)}{H_\infty^2} = -3x^{-4}\left\{\Gamma_3 + \left[\frac{H^2(1)}{H_\infty^2} - 1 - \Gamma_3\right]\left(1 + \frac{x^3}{3|\Gamma_1|}\right)e^{-\frac{(x^3-1)}{3|\Gamma_1|}}\right\}, \tag{77}$$

and can state the following:

1. When $\frac{H^2(1)}{H_\infty^2} - 1 > \Gamma_3 > 0$, and thus, $H(1) > H_\infty$, and $H'(x) < 0$, the Hubble function (76) has no extrema; it decreases monotonically and tends to H_∞ when $x \to \infty$.
2. When $\frac{H^2(1)}{H_\infty^2} < 1 + \Gamma_3 < 1$, and thus, $H_\infty > H(1)$, and $H'(x) > 0$, the Hubble function (76) has no extrema; it grows monotonically and tends to H_∞ when $x \to \infty$.
3. For other requirements for $\frac{H^2(1)}{H_\infty^2}$, Γ_3, and $|\Gamma_1|$, the extrema on the lines $H(x)$ are admissible, but in any cases, $H(x)$ tends to H_∞ asymptotically.

The last statement can be advocated as follows. Let us require that $H'(x_E) = 0$ and rewrite this condition as

$$\Gamma_3 e^z = \left(1 + \Gamma_3 - \frac{H^2(1)}{H_\infty^2}\right)\left(z + 1 + \frac{1}{3|\Gamma_1|}\right), \quad z = \frac{x_E^3 - 1}{3|\Gamma_1|}\,. \tag{78}$$

Depending on the values of the mentioned parameters, the graph of the exponential function may intersect the graph of the linear function once or not at all. For instance, the point of extremum $x_E = (1 + 3|\Gamma_1|)^{\frac{1}{3}}$ corresponds to the case $\Gamma_3 = \frac{(1+6|\Gamma_1|)(H^2(1) - H_\infty^2)}{[1+3|\Gamma_1|(2-e)]H_\infty^2}$, while for $\Gamma_3 = 0$ at $H^2(1) \neq H^2_\infty$, we obtain the value $x_E = -(3|\Gamma_1|)^{\frac{1}{3}} < 0$, which does not belong to the admissible interval $(1, \infty)$. In other words, there are four variants of the Hubble function behavior: when $H(1) > H_\infty$, the monotonic decreasing or passing through the minimum is possible; when $H(1) < H_\infty$, the monotonic growing or passing through the maximum is possible.

The scale factor $a(t)$ can now be obtained from the integral

$$H_\infty(t - t_0) = \int_1^{\frac{a(t)}{a(t_0)}} \frac{dx}{x\sqrt{1 + \left[\frac{H^2(1)}{H^2_\infty} - 1 - \Gamma_3\right] x^{-3}\, e^{-\frac{(x^3-1)}{3|\Gamma_1|}} + \Gamma_3 x^{-3}}} \,. \tag{79}$$

Since $e^{-\frac{(x^3-1)}{3|\Gamma_1|}}$ is the rapidly decaying function, the typical behavior of $a(t)$ can be illustrated by the submodel with $\frac{H^2(1)}{H^2_\infty} = 1+\Gamma_3$, $\Gamma_3 > -1$, for which the scale factor is equal to

$$\frac{a(t)}{a(t_0)} = \left\{\cosh\left[\frac{3}{2}H_\infty(t - t_0)\right] + \sqrt{1+\Gamma_3}\, \sinh\left[\frac{3}{2}H_\infty(t - t_0)\right]\right\}^{\frac{2}{3}} . \tag{80}$$

When $t \to \infty$, the scale factor behaves as $a(t) \to a(t_0)\left[\frac{1}{2}(1+\sqrt{1+\Gamma_3})\right]^{\frac{2}{3}} e^{H_\infty(t-t_0)}$. The Hubble function is presented by the formula

$$H(t) = H_\infty \left\{ \frac{\tanh\left[\frac{3}{2}H_\infty(t - t_0)\right] + \sqrt{1+\Gamma_3}}{1 + \sqrt{1+\Gamma_3}\, \tanh\left[\frac{3}{2}H_\infty(t - t_0)\right]} \right\}. \tag{81}$$

The acceleration parameter

$$-q(t) = \frac{\ddot{a}}{aH^2} = \frac{3}{2}\left\{ \frac{\sqrt{1+\Gamma_3}\tanh\left[\frac{3}{2}H_\infty(t - t_0)\right] + 1}{\sqrt{1+\Gamma_3} + \tanh\left[\frac{3}{2}H_\infty(t - t_0)\right]} \right\}^2 - \frac{1}{2} \tag{82}$$

takes the initial value $-q(t_0) = \frac{2-\Gamma_3}{2(1+\Gamma_3)}$ and tends to one asymptotically, where $-q(\infty) = 1$. Since the function (82) is monotonic, this means that if $\Gamma_3 > 2$, the initial and final values of the acceleration parameter have opposite signs, and thus, there exists the time moment t_T when $-q$ takes zero value as follows:

$$t_T = t_0 + \frac{2}{3H_\infty}\text{Artanh}\left[\frac{\sqrt{1+\Gamma_3} - \sqrt{3}}{\sqrt{3}\sqrt{1+\Gamma_3} - 1}\right] . \tag{83}$$

This is the point of transition from the decelerated expansion of the Universe to the accelerated one.

4.2.3. The Case $\Gamma_1 = 0$

In this submodel, the key Equation (67) converts into the algebraic equation

$$\frac{H^2(x)}{H^2_\infty} - 1 = \frac{\Gamma_3}{x^3 - \Gamma_2} \,. \tag{84}$$

In order to eliminate the Big Rip scenario, we assume that the parameter Γ_2 has to satisfy the requirement $\Gamma_2 < 1$, providing that at $x \geq 1$, the denominator does not take zero value. The Hubble function

$$H(x) = \pm H_{\infty}\sqrt{\frac{x^3 - \Gamma_2 + \Gamma_3}{x^3 - \Gamma_2}} \tag{85}$$

does not take the zero value on the admissible interval $x > 1$ if $\Gamma_3 - \Gamma_2 = \frac{\kappa m S(t_0)}{3H_{\infty}^2} < 1$. In this case, it monotonically decreases and tends asymptotically to H_{∞}. The acceleration parameter

$$-q(x) = 1 + x\frac{H'(x)}{H} = 1 - \frac{3\Gamma_3 x^3}{2(x^3 - \Gamma_2)(x^3 - \Gamma_2 + \Gamma_3)} \tag{86}$$

tends to one asymptotically, where $-q(\infty) = 1$. The initial value $-q(1)$ is negative when

$$\frac{2 - \rho^2}{5 + 2\rho^2} < \Gamma_2 < 1\,, \quad \rho = \sqrt{\frac{\kappa m S(t_0)}{3H_{\infty}^2}}\,, \tag{87}$$

and thus, the function $-q(x)$ takes zero value at $x = x_{\rm T} > 1$, where

$$x_{\rm T} = \left[\frac{5\Gamma_2 + \rho^2}{4} + \frac{1}{4}\sqrt{(\rho^2 + 13\Gamma_2)^2 - 144\Gamma_2^2}\right]^{\frac{1}{3}}. \tag{88}$$

If, in addition, Γ_2 is negative, the function $-q(x)$ reaches the maximum at $x = x_{\rm E} = \left(\rho^2|\Gamma_2|\right)^{\frac{1}{6}}$; when $0 < \Gamma_2 < 1$, the function $-q(x)$ tends to one monotonically.

The scale factor $a(t)$ can now be obtained from the equation

$$3H_{\infty}(t - t_0) = \int_1^{\left(\frac{a(t)}{a(t_0)}\right)^3} \frac{dz}{z}\sqrt{\frac{z - \Gamma_2}{z + \rho^2}}\,. \tag{89}$$

Integration of this equation reveals two different cases: $\Gamma_2 < 0$ and $0 < \Gamma_2 < 1$.

When $\Gamma_2 < 0$, i.e., $\beta_2 > \frac{4\beta_*}{11\nu_*}(2 + 7\nu_*^2)$, using (for the sake of convenience) the parameter $\alpha = \frac{1}{\rho}\sqrt{|\Gamma_2|}$, we obtain the implicit dependence $x(t)$ as follows:

$$e^{3H_{\infty}(t-t_0)} = \left[\left(\frac{\left|\sqrt{\frac{x^3+|\Gamma_2|}{\alpha^2x^3+|\Gamma_2|}} - 1\right|}{\sqrt{\frac{x^3+|\Gamma_2|}{\alpha^2x^3+|\Gamma_2|}} + 1}\right)\left(\frac{\sqrt{\frac{1+|\Gamma_2|}{\alpha^2+|\Gamma_2|}} + 1}{\left|\sqrt{\frac{1+|\Gamma_2|}{\alpha^2+|\Gamma_2|}} - 1\right|}\right)\right]^{\alpha} \times$$
$$\times\left(\frac{\alpha\sqrt{\frac{x^3+|\Gamma_2|}{\alpha^2x^3+|\Gamma_2|}} + 1}{\left|\alpha\sqrt{\frac{x^3+|\Gamma_2|}{\alpha^2x^3+|\Gamma_2|}} - 1\right|}\right)\left(\frac{\left|\alpha\sqrt{\frac{1+|\Gamma_2|}{\alpha^2+|\Gamma_2|}} - 1\right|}{\alpha\sqrt{\frac{1+|\Gamma_2|}{\alpha^2+|\Gamma_2|}} + 1}\right). \tag{90}$$

Clearly, when $x \to \infty$, we obtain

$$e^{3H_{\infty}(t-t_0)} \approx \frac{4\alpha^2x^3}{|\Gamma_2|(\alpha+1)^2}\left(\frac{\sqrt{\frac{1+|\Gamma_2|}{\alpha^2+|\Gamma_2|}} + 1}{\left|\sqrt{\frac{1+|\Gamma_2|}{\alpha^2+|\Gamma_2|}} - 1\right|}\right)^{\alpha}\left(\frac{\left|\alpha\sqrt{\frac{1+|\Gamma_2|}{\alpha^2+|\Gamma_2|}} - 1\right|}{\alpha\sqrt{\frac{1+|\Gamma_2|}{\alpha^2+|\Gamma_2|}} + 1}\right). \tag{91}$$

In other words, we obtain the de Sitter asymptotic regime $a(t) \propto e^{H_{\infty}t}$.

When $0 < \Gamma_2 < 1$, integration in (89) yields

$$\exp\left\{3H_\infty(t-t_0)+2\alpha\left[\arctan\sqrt{\frac{x^3-\Gamma_2}{\alpha^2x^3+\Gamma_2}}-\arctan\sqrt{\frac{1-\Gamma_2}{\alpha^2+\Gamma_2}}\right]\right\} =$$

$$= \left(\frac{\alpha\sqrt{\frac{x^3-\Gamma_2}{\alpha^2x^3+\Gamma_2}}+1}{1-\alpha\sqrt{\frac{x^3-\Gamma_2}{\alpha^2x^3+\Gamma_2}}}\right)\left(\frac{1-\alpha\sqrt{\frac{1-\Gamma_2}{\alpha^2+\Gamma_2}}}{\alpha\sqrt{\frac{1-\Gamma_2}{\alpha^2+\Gamma_2}}+1}\right). \tag{92}$$

In the asymptotic limit, we obtain

$$\exp\left\{3H_\infty(t-t_0)+2\alpha\left[\arctan\left|\frac{1}{\alpha}\right|-\arctan\sqrt{\frac{1-\Gamma_2}{\alpha^2+\Gamma_2}}\right]\right\} \approx \left(\frac{4\alpha^2x^3}{\Gamma_2}\right)\left(\frac{1-\alpha\sqrt{\frac{1-\Gamma_2}{\alpha^2+\Gamma_2}}}{\alpha\sqrt{\frac{1-\Gamma_2}{\alpha^2+\Gamma_2}}+1}\right), \tag{93}$$

i.e., we obtain again the de Sitter regime of the Universe expansion $a(t) \propto e^{H_\infty t}$.

5. Second Exactly Integrable Model

5.1. Evolution of the Spinor Particle Number Density

Let us suppose that the axion field was found to be in an unstable state provided by one of the maxima of the axion field potential (2). The value $\phi = \frac{\pi}{2}+2\pi k$ can be an exact solution to Equation (57) when the axion effective mass is equal to zero as $M_A = 0$ (see (58))—in particular, when $m_A = 0$ and $\chi_1 = 0$. The last condition gives

$$\beta_1 = 2\beta_2 + 4\nu_*\beta_*\,, \quad \chi_2 = -4\beta_*(\nu_*^2+1)\,. \tag{94}$$

If, in addition, we assume that $m = 0$, the Equation (55) converts into the system

$$\dot{X} = 2\chi_2 R\, Z\,, \quad \dot{Y} = 0\,, \quad \dot{Z} = 2\,\chi_2 R\, X\,. \tag{95}$$

We now see that $Y(t) = Y(t_0) = const$, and thus, $P(t) = P(t_0)\left(\frac{a(t_0)}{a(t)}\right)^3$. Again, we introduce the new time function

$$\tilde{\tau}(t) = 2\chi_2\int_{t_0}^{t} d\xi R(\xi)\,, \quad \tilde{\tau}(t_0) = 0\,, \tag{96}$$

and immediately solve the remaining equations

$$\frac{dX}{d\tilde{\tau}} = Z\,, \quad \frac{dZ}{d\tilde{\tau}} = X\,, \tag{97}$$

obtaining the solutions in terms of hyperbolic functions as

$$X(\tilde{\tau}) = X(0)\cosh\tilde{\tau} + X'(0)\sinh\tilde{\tau}\,, \quad Z(\tilde{\tau}) = X(0)\sinh\tilde{\tau} + X'(0)\cosh\tilde{\tau}\,. \tag{98}$$

Clearly, we have two consequences of (98):

$$X^2(\tilde{\tau}) - Z^2(\tilde{\tau}) = X^2(0) - Z^2(0)\,, \quad K = Y^2(t_0) - X^2(0) + Z^2(0)\,. \tag{99}$$

Since we are interested in studying the evolution of the spinor particle number density $\mathcal{N}$, we can say that

$$\mathcal{N}(t) = S(t) = \left(\frac{a(t_0)}{a(t)}\right)^3\left\{S(t_0)\cosh\tilde{\tau}(t) + \frac{\left[\dot{S}(t_0)+3H(t_0)S(t_0)\right]}{\chi_2 R(t_0)}\sinh\tilde{\tau}(t)\right\}. \tag{100}$$

The presence of the hyperbolic functions in (100) is the symptom of exponential growth of the fermion number density, but now we have to precisely determine the behavior of the argument $\tilde{\tau}$, which is given by the integral of the Ricci scalar (96); thus, we have to solve the gravity field Equation (59) and calculate the Ricci scalar $R(t)$ based on this solution.

5.2. Gravity Field Evolution

The Hubble function can be found from the reduced Equation (59), which takes now the form

$$x\tilde{\Gamma}_1 \frac{d}{dx}\tilde{\mathcal{U}} + \tilde{\mathcal{U}}\left(\tilde{\Gamma}_2 - x^3\right) + \tilde{\Gamma}_2 = 0\,, \tag{101}$$

where the following definitions are introduced:

$$\tilde{\mathcal{U}} = \frac{H^2}{H^2_{\infty}} - 1\,, \quad \tilde{\Gamma}_1 = -P(t_0)(\beta_2 \nu_* + 2\beta_*)\,, \quad \tilde{\Gamma}_2 = P(t_0)(22\nu_*\beta_2 + 16\nu^2_*\beta_* + 56\beta_*)\,. \tag{102}$$

Equation (101) has a structure similar to (67); however, we have to emphasize two differences. First, now $m = 0$, and thus, $\tilde{\Gamma}_3$ (the analog of Γ_3 in the previous submodel) coincides with $\tilde{\Gamma}_2$, and $\rho = 0$. Second, in the previous submodel, the value $S(t_0) = \mathcal{N}(t_0)$ was assumed to be positive; now, there are no physical arguments to fix the sign of the quantity $P(t_0)$. Again, we consider three cases: $\tilde{\Gamma}_1 > 0$, $\tilde{\Gamma}_1 < 0$, and $\tilde{\Gamma}_1 = 0$.

5.2.1. The Case $\tilde{\Gamma}_1 > 0$

The parameter $\tilde{\Gamma}_1$ is positive when $P(t_0)(\beta_2\nu_*{+}2\beta_*) < 0$. The solution to the Equation (101) is of the form

$$\mathcal{U} = \left[\frac{H^2(1)}{H^2_{\infty}} - 1\right] x^{-\frac{\tilde{\Gamma}_2}{\tilde{\Gamma}_1}}\, e^{\frac{(x^3-1)}{3\tilde{\Gamma}_1}} - \frac{\tilde{\Gamma}_2}{\tilde{\Gamma}_1}\, x^{-\frac{\tilde{\Gamma}_2}{\tilde{\Gamma}_1}}\, e^{\frac{x^3}{3\tilde{\Gamma}_1}} \int_1^x d\xi \xi^{\frac{\tilde{\Gamma}_2}{\tilde{\Gamma}_1}-1}\, e^{-\frac{\xi^3}{3\tilde{\Gamma}_1}}\,. \tag{103}$$

As in the previous case, the rapidly increasing exponent $e^{\frac{(x^3-1)}{3|\tilde{\Gamma}_1|}}$ predetermines the behavior of the Hubble function, and again, we can illustrate three possible cases of Universe expansion if we assume that $\tilde{\Gamma}_2 = 0$. Then, we see from the formula

$$H(x) = \pm H_{\infty}\sqrt{1{+}\left[\frac{H^2(1)}{H^2_{\infty}} - 1\right] e^{\frac{(x^3-1)}{3\tilde{\Gamma}_1}}} \tag{104}$$

that, when $H(1) > H_{\infty}$, we deal with the Big Rip; when $H(1) < H_{\infty}$, the model describes the Big Crunch. When $H(1) = H_{\infty}$, the Hubble function is constant, and the Universe expands according to the de Sitter law.

REMARK: This concerns solutions with the constant Hubble function.

As we have fixed above, the models with the constant Hubble function $H(t) = H_{\infty}$ appear for some special sets of the guiding parameters (see, e.g., (104) for $H(1) = H_{\infty}$). Such a case is convenient for the interpretation of the obtained solutions, since the Ricci scalar is now also constant $R(t) = -12H^2_{\infty}$, and the dimensionless argument of the hyperbolic functions in (100) is

$$\tilde{\tau}(t) = 96\beta_*(1+\nu^2_*)H^2_{\infty}(t-t_0)\,. \tag{105}$$

Clearly, we deal with an exponential growth of the spinor particle number according to the law

$$\mathcal{N} \propto e^{3H_{\infty}(t-t_0)\left[32\beta_*(1+\nu^2_*)H_{\infty}-1\right]}\,, \tag{106}$$

if $32\beta_*(1+\nu_*^2) > \sqrt{\frac{3}{\Lambda}}$. Since the rate of growth depends on the parameter β_*, we can state that such behavior is caused by the non-minimal coupling of the spinor field to the dual curvature tensors.

5.2.2. The Case $\tilde{\Gamma}_1 < 0$

Formally speaking, solutions to the Equation (101) have to be similar to the ones associated with Equation (67). However, the coefficients, which enter the first mentioned equation, essentially differ from the ones in the second equation so that the analysis presented above should be modified. We do not intend to repeat the corresponding analysis, and we consider the new exactly integrable model with parameters

$$\tilde{\Gamma}_2 = 6\tilde{\Gamma}_1 , \quad \frac{H^2(1)}{H^2_{\infty}} = 1 + 18|\tilde{\Gamma}_1|\left(|\tilde{\Gamma}_1| - \frac{1}{3}\right). \tag{107}$$

The Hubble function, given by the formula

$$H(x) = H_{\infty}\sqrt{\left(1-3|\tilde{\Gamma}_1|x^{-3}\right)^2 + 9|\tilde{\Gamma}_1|^2 x^{-6}}\,, \tag{108}$$

tends to H_{∞} at $x \to \infty$. In the behavior of the one-parameter function $h(x, |\tilde{\Gamma}_1|) = \frac{H}{H_{\infty}}$, there are three interesting details.

When $0 < |\tilde{\Gamma}_1| < \frac{1}{6}$, the function h starts with the value $h(1) < 1$ and tends to one monotonically.

When $\frac{1}{6} < |\tilde{\Gamma}_1| < \frac{1}{3}$, the function h starts with the value $h(1) < 1$, passes the minimum $h_{\min} = \frac{1}{\sqrt{2}}$ at $x_{\min} = (6|\tilde{\Gamma}_1|)^{\frac{1}{3}}$, and then tends to one monotonically.

When $|\tilde{\Gamma}_1| > \frac{1}{3}$, the function h starts with the value $h(1) > 1$, passes the minimum $h_{\min} = \frac{1}{\sqrt{2}}$ at $x_{\min} = (6|\tilde{\Gamma}_1|)^{\frac{1}{3}}$, and tends to one monotonically.

In addition, one can see that, when $|\tilde{\Gamma}_1| = \frac{1}{6}$, the minimum appears at the starting point $x = 1$; when $|\tilde{\Gamma}_1| = \frac{1}{3}$, the starting value of the Hubble function coincides with the final one, $H(1) = H_{\infty}$.

The acceleration parameters is now described by the formula

$$-q(x) = \frac{x^6 + 3|\tilde{\Gamma}_1|x^3 - 36|\tilde{\Gamma}_1|^2}{x^6 - 6|\tilde{\Gamma}_1|x^3 + 18|\tilde{\Gamma}_1|^2}\,. \tag{109}$$

The function $-q(x)$ can change the sign at $x = x_{\mathrm{T}} = \left(\frac{3}{2}|\tilde{\Gamma}_1|(\sqrt{17}-1)\right)^{\frac{1}{3}} > 1$, if $|\tilde{\Gamma}_1| > \frac{\sqrt{17}+1}{24}$. These are the conditions for Universe transition from the decelerated to accelerated expansion regime.

The scale factor $a(t)$ satisfies the equation

$$H_{\infty}(t - t_0) = \int_1^{\frac{a(t)}{a(t_0)}} \frac{dx}{x\sqrt{1 - 6|\tilde{\Gamma}_1|x^{-3} + 18|\tilde{\Gamma}_1|^2 x^{-6}}} \tag{110}$$

and the integration yields

$$\frac{a(t)}{a(t_0)} = \left\{3|\tilde{\Gamma}_1| + (1-3|\tilde{\Gamma}_1|)\cosh\left[3H_{\infty}(t-t_0)\right] + \sqrt{(1-3|\tilde{\Gamma}_1|)^2 + 9|\tilde{\Gamma}_1|^2}\sinh\left[3H_{\infty}(t-t_0)\right]\right\}^{\frac{1}{3}}, \tag{111}$$

$$\frac{H(t)}{H_{\infty}} = \frac{(1-3|\tilde{\Gamma}_1|)\sinh\left[3H_{\infty}(t-t_0)\right] + \sqrt{(1-3|\tilde{\Gamma}_1|)^2 + 9|\tilde{\Gamma}_1|^2}\cosh\left[3H_{\infty}(t-t_0)\right]}{3|\tilde{\Gamma}_1| + (1-3|\tilde{\Gamma}_1|)\cosh\left[3H_{\infty}(t-t_0)\right] + \sqrt{(1-3|\tilde{\Gamma}_1|)^2 + 9|\tilde{\Gamma}_1|^2}\sinh\left[3H_{\infty}(t-t_0)\right]}\,. \tag{112}$$

5.2.3. The Case $\tilde{\Gamma}_1 = 0$

We can obtain the main results for this case from the formulas of the previous model with $\Gamma_1 = 0$ if we put $\Gamma_3 = \Gamma_2$, $m = 0$, or $\rho = 0$ and replace Γ_2 with $\tilde{\Gamma}_2 < 1$. In particular, we obtain $H(x) = H_\infty(1 - \tilde{\Gamma}_2 x^{-3})^{-\frac{1}{2}}$. Only Formulas (90) and (92) require reformulation, since we have to assume that $\alpha \to \infty$. Now, we obtain from (89)

$$\exp\left\{3H_\infty(t-t_0) + 2\left(\sqrt{1-\tilde{\Gamma}_2 x^{-3}} - \sqrt{1-\tilde{\Gamma}_2}\right)\right\} = \frac{\left(\sqrt{1-\tilde{\Gamma}_2 x^{-3}}+1\right)\left|\sqrt{1-\tilde{\Gamma}_2}-1\right|}{\left|\sqrt{1-\tilde{\Gamma}_2 x^{-3}}-1\right|\left(\sqrt{1-\tilde{\Gamma}_2}+1\right)}. \tag{113}$$

When $x \to \infty$, we can replace $\left|\sqrt{1-\tilde{\Gamma}_2 x^{-3}}-1\right|$ by $\frac{1}{2}|\tilde{\Gamma}_2|x^{-3}$, thus recovering the asymptotic de Sitter law $a(t) \propto e^{H_\infty t}$.

6. Discussion

We discuss the new version of the non-minimal Einstein–Dirac-axion theory of the second order in derivatives and of the first order in the spinor particle number density $\mathcal{N} = \bar{\psi}\psi$. What are the interesting details of this theory?

The non-minimal terms in the Lagrangian of this theory are formulated as tensorial products of the Riemann tensor R_{mnpq}, of its left-dual, ${}^*R_{mnpq}$, and right-dual, R^*_{mnpq}, tensors, as well as of the Ricci tensor R_{mn} and Ricci scalar R on the one hand and of the spinor–axionic tensors and pseudotensors on the other hand. All the derivatives in the non-minimal terms originated from the curvature tensor and its convolutions; the spinor–axion tensors and pseudotensors do not contain derivatives. In this sense, the set of the non-minimal terms listed in (17) is complete: there are no new independent geometrical objects up to the second order in derivatives in the context of the Effective Field Theory.

In order to construct the complete Lagrangian of the presented theory, we used the so-called spinor–axion tensors and pseudotensors. One can explain this idea as follows. The standard spinor tensors $\bar{\psi}\gamma^m \cdots \gamma^j\psi$, being multiplied by $\cos\phi$, the even function of the pseudoscalar field, remain true tensors. If we use the odd function $\sin\phi$ as the multiplier, we obtain the spinor–axion pseudotensor. Similarly, when we work with the spinor pseudotensor $\bar{\psi}\gamma^m \cdots \gamma^j\gamma^5\psi$, we can obtain a spinor–axion pseudotensor by multiplying it by $\cos\phi$ or a spinor–axion tensor if we replace $\cos\phi$ with $\sin\phi$. That is why we introduced two families of the one-parameter matrices $\mathcal{A}$ and $\mathcal{B}$, given by (11), and used them in the procedure of the Lagrangian construction.

Why do we restrict ourselves to the frameworks of the linear theory with respect to the spinor particle number density $\mathcal{N} = \bar{\psi}\psi$? If we consider, e.g., the theory of the second order, we should include into the Lagrangian a lot of terms, which have the forms $R_{mnpq}(\bar{\psi}\gamma^m\gamma^n\psi)(\bar{\psi}\gamma^p\gamma^q\psi)$, $R_{mnpq}(\bar{\psi}\gamma^m\psi)(\bar{\psi}\gamma^n\gamma^p\gamma^q\psi)$, $R_{mn}(\bar{\psi}\gamma^m\psi)(\bar{\psi}\gamma^n\psi)$, …, etc. Clearly, such a model would become non-effective because of the large number of phenomenological parameters. Let us recall that the constructed theory contains only four non-minimal parameters: β_1, β_2, β_*, and ν_*.

We would like to attract attention to the fact that the extended equations of the spinor field (18) have the structure of the canonic Dirac's equation, but instead of the seed mass m multiplied by the unit matrix E, we obtain an effective mass matrix M (see (19) and (20)), which depends on the curvature tensor and on the axion field. Similarly, the extended equation for the axion field (57) contains the effective axion mass M_A (see (58)), which depends on the scalars R and $\mathcal{N}$ instead of the seed mass m_A. These effective masses, associated with the spacetime curvature, predetermine the behavior of the spinor particles (massive fermions and massless neutrinos) and the dynamics of the axions in the early Universe.

Application of the formulated non-minimal theory to the isotropic homogeneous cosmological model revealed an interesting detail. If we introduce one spinor scalar $S = \bar{\psi}\psi$ and two spinor pseudoscalars $P = i\bar{\psi}\gamma^5\psi$ and $\Omega = \bar{\psi}\gamma^0\gamma^5\psi$, the evolutionary equations for these quantities form the closed system of differential equations of the first order (51)–(53). Moreover, this system admits the explicit first integral (56), and thus, only two quantities, $S(t)$ and $P(t)$, characterize the evolution of the gravitational field (see, e.g., (59)) and enter the source term $\mathcal{J}$ (see (48)), which predetermines the evolution of the axion field.

We analyzed in detail two exactly integrable submodels of evolution of the non-minimally coupled spinor–axion system in the gravity field of the FLRW type. In the first submodel, the pseudoscalar field was assumed to be frozen in one of the minima of the potential of the axion field, $\phi = 2\pi k$, i.e., the axions are considered to be in the equilibrium state [47]. Analysis of the exact solutions describing the Hubble function $H(t)$, scale factor $a(t)$, and acceleration parameter $-q(t)$ shows that, depending on the values of the non-minimal guiding parameters β_1, β_2, β_*, and ν_*, the Universe can be characterized by the Big Rip, Big Crunch, and Pseudo Rip scenarios. It is important to note that the equilibrium state of the axion system remains stable, even if the catastrophic Big Rip scenario is realized. The spinor particle number density decreases in this case monotonically as $\mathcal{N}(t) = \mathcal{N}(t_0)\left(\frac{a(t_0)}{a(t)}\right)^3$.

If the pseudoscalar field is fixed in one of the maxima of the axion field potential, i.e., $\phi = \frac{\pi}{2}+2\pi k$, the law of evolution of the gravity field is similar to the previous one; however, the exact solution for the spinor scalar S (see (100)) is expressed in the hyperbolic functions. This means that the spinor particle number density $\mathcal{N}$ grows exponentially, i.e., the so-called spinorization of the early Universe takes place. One can see from the formula (106) that it is possible when $32\beta_*(1+\nu_*^2) > \sqrt{\frac{3}{\Lambda}}$. It is important to mention the following detail of analysis: since the rate of growth is predetermined by the value of the parameter β_*, one can state that such behavior is caused by the non-minimal coupling of the spinor field to the left/right-dual curvature tensors.

Finally, when $\chi_1 = \chi_2 = 0$ and $H = 0$, the master equation for the axion field (57) can be reduced to the sine-Gordon equation $\frac{d^2\phi}{d\zeta^2} + \sin\phi = 0$ with $\zeta = m_A t$, which admits the kink-type solution of the form $\phi(\zeta) = 4\arctan e^{\rho\zeta}$ with some parameter ρ. Kink and anti-kink solutions to the classical sine-Gordon equation link basic states of the axion field associated with extrema of the periodic axion field potential (2). It is not clear whether the kink-type solutions to the extended Equation (57) exist in the general case, but this problem is very interesting for further investigations.

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