# Improved analytic solution of Kerr-(Newman) black hole superradiance

**Qi-Xuan Xu** [1], **Yin-Da Guo** [2]

[1] CENTRA, Departamento de Física, Instituto Superior Técnico – IST, Universidade de Lisboa – UL, Avenida Rovisco Pais 1, 1049-001 Lisboa, Portugal
[2] Key Laboratory of Particle Physics and Particle Irradiation (MOE), Institute of Frontier and Interdisciplinary Science, Shandong University, Qingdao, Shandong 266237, China

E-mail: qixuan.xu@tecnico.ulisboa.pt

**Abstract.** We revisit an improved analytic calculation of the superradiant instability rates of Kerr–(Newman) black holes for (charged) massive scalars in the limit of $M\mu \ll 1$, achieving good agreement with numerical results. We also point out earlier analyses that lack full rigor for the first time.

Black holes (BH), traditionally regarded as simple end states of gravitational collapse, have recently emerged as unique laboratories for testing fundamental physics [1]. Their extreme gravitational environments allow for nontrivial interactions with bosonic fields [2]. In particular, it has been shown that a massive scalar field can become unstable around a rotating BH [3–8]. The superradiant instability was shown to be maximized when the Compton wavelength of the field is comparable to the BH's gravitational radius, enabling the field to extract rotational energy from the BH and form long-lived *bosonic clouds* [9, 10]. This mechanism offers a compelling window into new physics beyond the Standard Model [11–24]. In particular, the presence (or absence) of rapidly spinning BHs in specific mass ranges can impose strong observational constraints on the properties of ultralight fields [25–28]. Moreover, BH superradiance has been extensively investigated within gravitational theories beyond General Relativity [29–37].

The phenomenological study of BH superradiance depends on the accurate determination of the bound state's eigenfrequency. For Kerr BHs, in the limit where the Compton wavelength of the scalar field is much larger than the BH's gravitational radius, the asymptotic-matching method yields an analytic approximation for the complex eigenfrequency $\omega$ first obtained by Detweiler [6]. In addition, the numerical continued fraction method was first proposed by Leaver for massless scalars [38] and it was later developed for massive scalars in Refs. [7, 39]. Furuhashi and Nambu subsequently extended this approximation and developed a shooting method for the case where both BH and the scalar field can carry electric charge [40]. Nonetheless, the approximation is not in good agreement with existing numerical calculations in all cases. This discrepancy has recently been resolved in Refs. [41, 42] by including the higher-order contribution to the analytic approximation. The improved analytical solution has been directly applied in various phenomenological studies [27, 28, 43, 44]. In this work, we revisit the asymptotic matching method and point out aspects of previous work that lack sufficient rigor. Without loss of generality, we consider a complex scalar field $\Phi$ with mass $\mu$ and electric charge $q$ that perturbs a Kerr-Newman BH with mass $M$, angular momentum $J = aM$ and charge $Q$. Our analysis also applies to Kerr BH superradiance, by simply setting $Q = 0$. We adopt units in which $\sqrt{1/4\pi\epsilon_0} = G = \hbar = c = 1$ throughout the paper.

In this work, we study a free test scalar field in the stationary Kerr–Newman background. Starting from the Klein–Gordon equation and separating variables in Boyer–Lindquist coordinates, one obtains [45–47]: $\Phi(t, r, \theta, \varphi) = R_{lm}(r)S_{lm}(\theta)e^{im\varphi}e^{-i\omega t}$. This leads to an angular equation for $S_{lm}$ solved by spheroidal harmonics [48], and a radial equation for $R_{lm}$ whose solution must vanish at infinity. Following Ref. [42], we introduce the power-counting parameter $\alpha$, which can be $M\mu$ or any other quantity with the same scaling for the expansion. The scaling of other parameters are $\mu \sim \Re(\omega) \sim q$ and $M \sim Q \sim a \sim r_+ \sim r_-$. In the large $r$ limit ($r \gg M$), the radial equation can be simplified as

$$\frac{d^2}{dr^2}(rR) + \Big[-\kappa^2 + \frac{2\kappa\lambda}{r} - \frac{l'(l'+1)}{r^2}\Big](rR) = 0, \tag{1}$$

where $\kappa = \sqrt{\mu^2 - \omega^2}$,

$$\lambda = \frac{M\mu^2 - qQ\mu}{\kappa}, \tag{2}$$

and

$$l'(l'+1) = l(l+1) - 8r_g^2\mu^2 + Q^2(\mu^2 - q^2) + 8qQr_g\mu + \mathcal{O}(\alpha^4). \tag{3}$$

Here, we incorporate the term $\mathcal{O}(\alpha^2)/r^2$ by defining $l' = l + \epsilon$ where $\epsilon \sim \mathcal{O}(\alpha^2)$. Ref. [41] shows that $\epsilon$ plays the role of a regulator and cannot be simply dropped. It can be calculated from the definition of $l'$ in Eq. (3),

$$\epsilon = \frac{-8r_g^2\mu^2 + Q^2\mu^2 + 8r_g qQ\mu - q^2Q^2}{2l+1} + \mathcal{O}(\alpha^4). \tag{4}$$

Up to an arbitrary normalization, the solution of Eq. (1) can be written in terms of the confluent hypergeometric function,

$$R(r) = e^{-\kappa r}(2\kappa r)^{l'} U(l' + 1 - \lambda, 2\,l' + 2; 2\,\kappa\, r). \tag{5}$$

Strictly speaking, dropping the term $\mathcal{O}(l)/r^3$ in the region $r \gg M$ is not fully justified. Given that $(M\kappa)^2 \sim \mathcal{O}(\alpha^4)$, the condition under which this term can be neglected relative to $\kappa^2$ is $r/M \gg l^2\alpha^{-4/3}$. Although one can always take sufficiently large $r/M$ so that the $\mathcal{O}(l)/r^3$ term becomes negligible, this choice eliminates the overlapping region with the near-horizon solution, as will be clarified later. For a similar reason, the $\mathcal{O}(\alpha^2)/r^2$ and $\mathcal{O}(l)/r^4$ terms should also be retained, and these terms were neglected in Ref. [6]. Nevertheless, in order to keep the analytical solution in Eq. (1) solvable and compact, we adopt the standard approximation and neglect the $\mathcal{O}(l)/r^3$ and $\mathcal{O}(l)/r^4$ terms.

In order to facilitate further analysis for the radial equation in small $r$ limit, it is more convenient to rewrite the radial equation in terms of $z = (r - r_+)/2b$,

$$z(z+1)\frac{d}{dz}\left[z(z+1)\frac{dR}{dz}\right] + U(z)R = 0, \tag{6}$$

where $U(z)$ is a polynomial of $z$,

$$\begin{aligned} U(z) = p^2 + z\left[\frac{4Mr_+\omega}{b}\left(r_+\omega - \frac{am}{2r_+} - \frac{Q^2\omega}{2M}\right) - (\Lambda_{lm} + r_+^2\mu^2 + a^2\omega^2) + \frac{qQ}{b}(am + r_+qQ - a^2\omega - 3r_+^2\omega)\right] \\ + z^2(a^2\omega^2 - \Lambda_{lm} + 2\mu^2a^2 - 3\mu^2r_+^2 + 6r_+^2\omega^2 + 2Q^2\mu^2 + q^2Q^2 - 6r_+qQ\omega) \\ + 4z^3b\left[M\mu^2 + 2r_+(\omega^2 - \mu^2) - qQ\omega\right] + 4z^4b^2(\omega^2 - \mu^2), \end{aligned} \tag{7}$$

in which $p = (r_+^2 + a^2)(\omega - \omega_c)/2b$ and $\omega_c = (ma + qQr_+)/(r_+^2 + a^2)$. In the small $r$ limit ($r \ll \max(l/\omega, l/\mu)$), Refs. [6, 40] approximate $U(z)$ as $p^2 - l(l+1)z(z+1)$. This approximation implicitly assumes $b \sim \mathcal{O}(r_g)$. However, for a fast-rotating BH, the value of $b$ is very small and this assumption is not satisfied. Thus, it is crucial to retain the terms proportional to $1/b$ in the coefficient of $z$ in Eq. (7) [41, 42]. In order to make the expression more compact, we set $U(z) = p^2 - l'(l'+1)z(z+1) + zd$, where $d$ is defined as

$$d = (4r_g\mu - 2qQ)p - 2(4r_g - r_+)r_g\mu^2 - q^2Q^2 + 2\mu qQ(4r_g - r_+) + \mathcal{O}(\alpha^3). \tag{8}$$

Changing the variable from $z$ back to $r$, the solution is the confluent hypergeometric function,

$$R(r) = \frac{(r-r_-)^{\sqrt{d-p^2}}}{(r-r_+)^{ip}}\, {}_2F_1\Big(-l'-ip+\sqrt{d-p^2}, l'+1-ip+\sqrt{d-p^2}; 1-2ip; -\frac{r-r_+}{2b}\Big). \tag{9}$$

up to an arbitrary normalization.

The solution in Eq. (5) is valid when $r \gg M$. The solution in Eq. (9) requires $r \ll \max(l/\omega, l/\mu)$. In the small $r$ limit ($r \ll 1/\kappa \sim M\alpha^{-2}$), function (5) becomes,

$$\frac{(2\kappa)^{l'}\Gamma(-2l'-1)}{\Gamma(-l'-\lambda)}r^{l'} + \frac{(2\kappa)^{-l'-1}\Gamma(2l'+1)}{\Gamma(l'+1-\lambda)}r^{-l'-1}, \tag{10}$$

and the large $r$ limit ($r \gg M\max(p,l)$) of function (9), which takes the form,

$$\frac{(2b)^{-l'-ip+\sqrt{d-p^2}}\Gamma(2l'+1)\Gamma(1-2ip)}{\Gamma(l'+1-ip-\sqrt{d-p^2})\Gamma(l'+1-ip+\sqrt{d-p^2})}r^{l'} + \frac{(2b)^{l'+1-ip+\sqrt{d-p^2}}\Gamma(-2l'-1)\Gamma(1-2ip)}{\Gamma(-l'-ip-\sqrt{d-p^2})\Gamma(-l'-ip+\sqrt{d-p^2})}r^{-l'-1}. \tag{11}$$

These two solutions (5) and (9) have an overlapping region $M\max(p,l) \ll r \ll \max(l/\omega, l/\mu)$ when $\alpha \ll 1$. In deriving Eq. (1), the $\mathcal{O}(l)/r^3$ and $\mathcal{O}(l)/r^4$ terms can be neglected only if $r/M \gg l^2\alpha^{-4/3}$ and $r/M \gg l^2\alpha^{-1}$, respectively. However, these conditions lie outside the overlapping region, which is why we think these two terms should be retained. For terms with higher powers of $r$, the conditions under which they can be neglected do overlap with the overlapping region. Therefore, these higher-power terms can be safely dropped.

The ratio of the coefficients of $r^{l'}$ and $r^{-l'-1}$ must be the same for the two solutions (10) and (11). It can be solved perturbatively by noting that the second term in Eq. (10) must be suppressed at small $r$, which implies that $l'+1-\lambda$ is very close to zero or to a negative integer,

$$l'+1-\lambda = -n-\delta\lambda, \tag{12}$$

where $|\delta\lambda| \ll 1$ and $n$ is zero or a positive integer. Defining $\bar{n} = n+l+1$, the above relation can be re-expressed as $\lambda = \bar{n}+\epsilon+\delta\lambda$. Since $|\delta\lambda| \ll 1$, $\delta\lambda$ can be solved perturbatively with expressions (10) and (11). Following similar matching steps as described in Ref. [6, 41], one could obtain $\delta\lambda$ after some algebra,

$$\delta\lambda = \left(\frac{d}{2\epsilon} - \frac{\epsilon}{2} - ip\right)\frac{(4\kappa b)^{2l'+1}\,\Gamma(n+2l'+2)\Gamma_{pd}}{n!\,[\Gamma(2l'+1)\Gamma(2l'+2)]^2}, \tag{13}$$

where $\Gamma_{pd}$ is defined as

$$\Gamma_{pd} = \frac{\left|\Gamma(l'+1+ip+\sqrt{d-p^2})\Gamma(l'+1+ip-\sqrt{d-p^2})\right|^2\Gamma(1+2\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-ip-\sqrt{d-p^2}-\epsilon)\Gamma(1+ip+\sqrt{d-p^2}+\epsilon)\Gamma(1-ip+\sqrt{d-p^2}-\epsilon)\Gamma(1+ip-\sqrt{d-p^2}+\epsilon)}. \tag{14}$$

The eigenfrequency $\omega$ can be expressed in terms of $\delta\lambda$ with Eqs. (2) and (12). Defining $\omega = \omega_0 + \omega_1\delta\lambda$ in Eq. (2) and expanding it to the linear term of $\delta\lambda$, one arrives at

$$\lambda = \frac{(r_g\mu - qQ)\mu}{\sqrt{\mu^2-\omega_0^2}} + \frac{(r_g\mu-qQ)\mu^2\omega_1}{(\mu^2-\omega_0^2)^{3/2}}\delta\lambda + \mathcal{O}\left(\delta\lambda^2\right).$$

On the other hand, since $\lambda = \bar{n}+\delta\lambda$ from Eq. (12), one can combine this result with Eq. (15) to solve for $\omega_0$ and $\omega_1$ perturbatively, retaining terms up to $\mathcal{O}(\alpha^2)$,

$$\frac{\omega_0}{\mu} = 1 - \frac{1}{2}\left(\frac{r_g\mu - qQ}{\bar{n}}\right)^2 + \mathcal{O}(\alpha^4), \quad \frac{\omega_1}{\mu} = \frac{(r_g\mu-qQ)^2}{\bar{n}^3} + \mathcal{O}(\alpha^4). \tag{15}$$

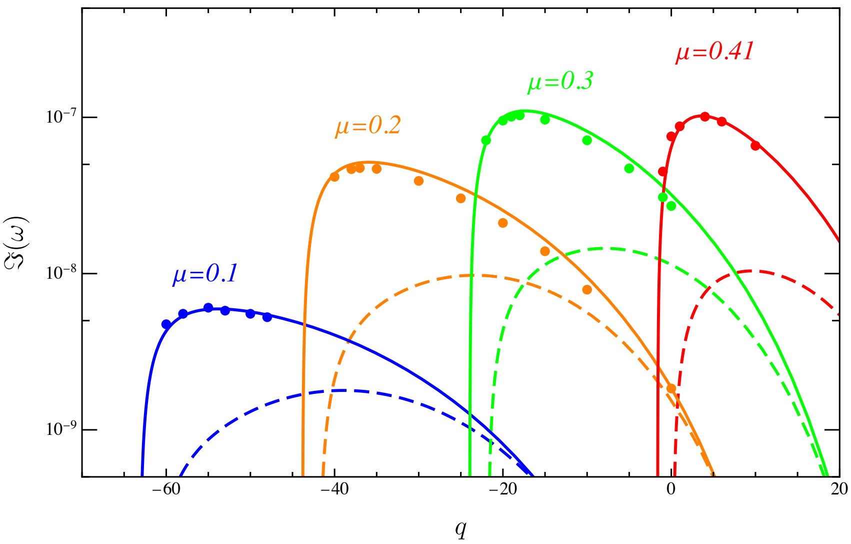


Figure 1: Comparison of the numerical result and the analytic approximations for $n = 0$, $l = m = 1$, $a = 0.98$, and $Q = 0.01$, with $M$ chosen to be 1 for compacity. $\Im(\omega)$ is plotted as a function of the scalar field charge $q$. The dashed (solid) curves are the Detweiler (Improved) approximations and the scattered dots are numerical results taken from Ref. [40].

Note that $\delta\lambda$ is not purely imaginary. Here, however, we only focus on the imaginary part of the eigenfrequency, which is $\Im(\omega) = \omega_1 \Im(\delta\lambda)$. In addition, the real part of the eigenfrequency $\Re(\omega)$ can be obtained using the method in Ref. [49] more precisely. In order to prevent the inclusion of higher-order terms in the $\delta\lambda$ formula, the $\omega$ in $p$ and $\kappa$ should be replaced with $\omega_0$.

Figure 1 shows the comparison of the improved analytic solution to the Detweiler approximations and numerical results from Ref. [40]. The improved approximation shows significantly better agreement with the numerical results. In particular, the average percentage errors of improved solution for the points in Fig. 1 are 1%, 12%, 10% and 7% for $M\mu = 0.1,\ 0.2,\ 0.3$ and 0.41, respectively. However, the large-$r$ treatment neglects terms of order $\mathcal{O}(l)/r^3$ and $\mathcal{O}(l)/r^4$, whose neglect requires $r/M \gg l^2\alpha^{-4/3}$ and $r/M \gg l^2\alpha^{-1}$, respectively. These conditions lie entirely outside the overlapping region, so meeting them would eliminate the overlapping region altogether. This issue will be addressed in future work.

**Acknowledgements**
We are grateful to Richard Brito for comments on the final draft of this manuscript. We are indebted to Hong Zhang for very valuable discussions and suggestions. QX acknowledges financial support provided by FCT – Fundação para a Ciência e a Tecnologia, I.P., under the Project "GravNewFields" funded through the ERC-Portugal program. Y-D. Guo acknowledges financial supports provided by the National Natural Science Foundation of China (Grants Nos. 12075136 and 124B2098) and the Natural Science Foundation of Shandong Province (Grant No. ZR2020MA094).

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Journal of Physics: Conference Series **3177** (2026) 012050 doi:10.1088/1742-6596/3177/1/012050

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