Eur. Phys. J. C (2024) 84:1330
https://doi.org/10.1140/epjc/s10052-024-13660-2

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

The Chini integrability condition in second order Lovelock gravity

Mohammed O. E. Ismail[1,a], **Sunil D. Maharaj**[1,b], **Byron P. Brassel**[1,2,c]

[1] Astrophysics Research Centre, School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Private Bag X54001, Durban 4000, South Africa
[2] Department of Mathematics, Faculty of Applied Sciences, Durban University of Technology, P O Box 1334, Durban 4000, South Africa

Received: 6 September 2024 / Accepted: 27 November 2024

Abstract We analyse neutral and charged matter distributions in second order Lovelock gravity, also known as Einstein–Gauss–Bonnet gravity, in arbitrary dimensions for a static, spherically symmetric spacetime. We first transform the charged condition of pressure isotropy, an Abel differential equation of the second kind, into canonical form. We then determine a systematic approach to integrate the condition of pressure isotropy by showing that the canonical form is a Chini differential equation. The Chini invariant, which allows the master differential equation to be separable, is identified. This enables us to find three new general solutions, in implicit form, to the condition of pressure isotropy. We also show that previously obtained exact specific solutions arise as special cases in our general class of models. The Chini invariant does not arise in general relativity; it is a distinguishing feature of Einstein–Gauss–Bonnet gravity.

1 Introduction

General relativity (GR), introduced by Einstein in 1915, revolutionised the way we think about gravity. It replaced Newtonian gravity and has been validated on numerous occasions via observations including the perihelion shift of the planet Mercury, gravitational lensing and the detection of gravitational waves. However, there are several physical phenomena it cannot explain such as the late-time acceleration of the universe and dark matter [1]. To address the limitations of general relativity, we use the Lovelock theory of gravity [2,3]. Lovelock theory is a natural extension of general relativity, and contains a series of curvature corrections. The Lovelock action is derived by considering the Lagrangian action to be in the form of a polynomial. It is important to note that the Bianchi identities are still satisfied in the framework of Lovelock gravity, ensuring consistency within the theory. More information on Lovelock gravity can be found in the papers by Padmanabhan and Kothawala [4] and Garraffo and Giribet [5]. The geometrical properties of Einstein–Gauss–Bonnet (EGB) gravity are contained in treatments by Ishihara [6], Wiltshire [7], Deruelle and Dolezel [8] and Brassel [9]. In second order Lovelock gravity, or EGB gravity, we modify the Einstein–Hilbert action of GR to include terms that are quadratic in the Riemann tensor, Ricci tensor and Ricci scalar. This leads to new and modified field equations. Interestingly, this modified action of EGB gravity appears in the low energy limit of heterotic string theory [10]. Another noteworthy feature of EGB gravity is that the resulting equations of motion are second order quasilinear differential equations, which implies a theory free of Ostrogradsky ghosts. Various exact solutions to the EGB field equations have been found. Some interesting solutions to the EGB field equations can be found in treatments by Naicker et al. [11,12]; they extend earlier solutions such as the model of Hansraj and Mkhize [13] in six dimensions to N dimensions. Other classes of exact solutions are contained in the works by Hansraj et al. [14], Maharaj et al. [15], Chilambwe et al. [16], Das et al. [17], Rej et al. [18], and Bhatti et al. [19], amongst others. The papers mentioned above have been studied in spherically symmetric distributions with pressure. Interestingly charged dust models in EGB gravity exist as shown by Hansraj [20] and Naicker et al. [21].

EGB gravity is being applied to a number of astrophysical applications. Motivated by string theory considerations, Bogadi et al. [22] found compactified structures with a colour-flavour-locked equation of state. Stars with quark and strange equations of state have been considered by Rej et al. [18], Maurya et al. [23], Malaver et al. [24], Hansraj et al. [25] and Tangphati et al. [26,27]. The generalized case of

[a] e-mail: 219012653@stu.ukzn.ac.za
[b] e-mail: maharaj@ukzn.ac.za (corresponding author)
[c] e-mail: ByronB@dut.ac.za

polytropic matter distributions was considered by Kaisavelu et al. [28]. The matter distributions and gravitational potentials found in these works describe self-gravitating and compact astronomical objects with the vacuum Boulware–Deser exterior. Further, the EGB field equations for an extended Vaidya-like source of matter were solved for the mass function and matter variables in [29]. Theoretical results obtained are consistent with stellar objects such as the millisecond pulsar PSR J1614-2230, and LMC X-4, a high-mass X-ray binary (HMXB) [23]. The framework of minimal geometric decoupling has been used to study the strength of the Gauss–Bonnet coupling constant in EGB gravity. The decoupling analysis shows consistency with observations related to compact objects [23] and the gravitational wave events GW 170817 and GW 190814 [30], as shown in the treatments by [31,32]. It is therefore important to understand the structure of the EGB field equations, generate exact solutions and relate them to observational results.

In this paper, we generate the EGB field equations for a static, spherically symmetric and charged spacetime, assuming the matter distribution to be that of a perfect fluid. The inclusion of charge introduces more complexity through the electromagnetic field which is described by Maxwell's equations. The resulting field equations are a system of four, highly nonlinear differential equations. In order to simplify the Einstein–Gauss–Bonnet–Maxwell (EGBM) field equations, we impose a restriction in the form of the isotropic pressure condition. If we assume tangential pressure to be equal to radial pressure, then the EGBM field equations reduce to an Abel differential equation of the second kind, one of the most difficult types of differential equations to solve. There are no known general solutions to this type of differential equation. Naicker et al. [11,12] showed that the condition of pressure isotropy can be transformed into canonical form. We relate the canonical form of the condition of pressure isotropy to the Chini differential equation [33]. The Chini differential equation is a first order, nonlinear differential equation that cannot be integrated in general. It can be transformed into a separable differential equation, provided a certain integrability condition is met. The resulting separable differential equation can then, in principle, be integrated to give a solution to the Chini differential equation. We believe that our study is the first treatment utilising the Chini differential equation in any gravitational theory. Certain known solutions in EGB gravity arise as special cases in our general treatment.

2 Second order Lovelock gravity

Second order Lovelock gravity, also known as EGB gravity, is a generalization of general relativity. The action of EGB gravity modifies the Einstein–Hilbert action of general relativity to include terms which are quadratic in the Ricci scalar, Ricci tensor and Riemann tensor. The EGB field equations are derived by varying the action, leading to the form

$$G_{ab} - \frac{\alpha}{2} H_{ab} + \Lambda g_{ab} = \kappa_N \mathcal{T}_{ab}. \tag{1}$$

In Eq. (1), **G** is the Einstein tensor, α is the Gauss–Bonnet coupling constant, Λ is the cosmological constant, and **H** is the Lovelock tensor. This has the explicit form

$$\begin{aligned} H_{ab} &= g_{ab} L_{GB} - 4 R R_{ab} + 8 R_{ac} R^{c}{}_{b} \\ &\quad + 8 R^{cd} R_{acbd} - 4 R_{a}{}^{cde} R_{bcde}, \end{aligned} \tag{2}$$

where L_{GB} is the Gauss–Bonnet term given by

$$L_{GB} = R^2 + R_{abcd} R^{abcd} - 4 R_{cd} R^{cd}. \tag{3}$$

The gravitational coupling constant, κ_N, is given in terms of the spacetime dimension N by

$$\kappa_N = \frac{2\,(N-2)\,\pi^{\frac{N-1}{2}}}{(N-3)\,\widetilde{\Gamma}\left(\frac{N-1}{2}\right)}. \tag{4}$$

where the function $\widetilde{\Gamma}(z)$ is the gamma function defined by

$$\widetilde{\Gamma}(z) = \int_0^\infty t^{z-1} e^{-t}\, dt, \tag{5}$$

which is related to the generalized factorial function with non-integer inputs.

The total energy momentum tensor $\mathcal{T}$ is of the form

$$\mathcal{T}_{ab} = T_{ab} + E_{ab}. \tag{6}$$

In the above, T_{ab} is the energy momentum tensor for a perfect fluid

$$T_{ab} = (\rho + p) u_a u_b + p g_{ab}, \tag{7}$$

where ρ is the energy density, p is the isotropic pressure, **u** is the fluid N-velocity and **g** is the metric tensor. The electromagnetic tensor **E** is a function of the metric tensor, Faraday tensor **F** and the surface area of the unit $(N-2)$-sphere $\mathcal{A}_{N-2}$. It takes the form

$$E_{ab} = \frac{1}{\mathcal{A}_{N-2}} \left(F_{ac} F_b{}^c - \frac{1}{4} g_{ab} F_{cd} F^{cd} \right). \tag{8}$$

The electromagnetic tensor is trace-free only in four dimensions [34]. The surface area of the unit $(N-2)$-sphere has the form

$$\mathcal{A}_{N-2} = \frac{2\pi^{\frac{N-1}{2}}}{\widetilde{\Gamma}\left(\frac{N-1}{2}\right)}, \tag{9}$$

and the Faraday tensor **F** can be written as

$$F_{ab} = A_{b;a} - A_{a;b}. \tag{10}$$

The N-vector **A** is the electromagnetic potential which, for a static spherically symmetric spacetime, can be chosen to be of the form

$$A_a = (\Omega(r), 0, 0, \ldots, 0)\,, \tag{11}$$

where $\Omega(r)$ is a scalar potential. Maxwell's equations can be written as

$$F_{ab;c} + F_{bc;a} + F_{ca;b} = 0, \tag{12a}$$
$$F^{ab}{}_{;b} = \mathcal{A}_{N-2} J^a, \tag{12b}$$

where

$$J^a = \sigma u^a, \tag{13}$$

is the current vector and σ is the proper charge density.

The EGB field equations (1), along with Maxwell's equations (12) describe the evolution of a charged gravitational fluid in EGB gravity. The line element for a static, spherically symmetric spacetime in N dimensions is

$$ds^2 = -e^{2\nu}dt^2 + e^{2\lambda}dr^2 + r^2 d\Omega^2_{N-2}, \tag{14}$$

where

$$d\Omega^2_{N-2} = \sum_{i=1}^{N-2}\left[\left(\prod_{j=1}^{i-1}\sin^2(\theta_j)\right)(d\theta_i)^2\right],$$

is the metric of the unit $(N-2)$-sphere. We have a radial dependence for the static potentials $\nu = \nu(r)$ and $\lambda = \lambda(r)$. We obtain the following system of nonlinear differential equations in static spherical geometry

$$\begin{aligned}\kappa_N\left(\rho + \frac{E^2}{2\mathcal{A}_{N-2}}\right) &= \frac{N-2}{r^4 e^{4\lambda}}\Bigg[r^3 e^{2\lambda}\lambda' + \frac{(N-3)r^2 e^{4\lambda}}{2} \\ &\quad - \frac{(N-3)r^2 e^{2\lambda}}{2} + \hat{\alpha}\left(e^{2\lambda}-1\right) \\ &\quad \times\left(2r\lambda' + \frac{(N-5)\left(e^{2\lambda}-1\right)}{2}\right)\Bigg] - \Lambda,\end{aligned} \tag{15a}$$

$$\begin{aligned}\kappa_N\left(p - \frac{E^2}{2\mathcal{A}_{N-2}}\right) &= \frac{N-2}{r^4 e^{4\lambda}}\Bigg[r^3 e^{2\lambda}\nu' + \frac{(N-3)r^2 e^{2\lambda}}{2} \\ &\quad - \frac{(N-3)r^2 e^{4\lambda}}{2} + \hat{\alpha}\left(e^{2\lambda}-1\right) \\ &\quad \times\left(2r\nu' - \frac{(N-5)\left(e^{2\lambda}-1\right)}{2}\right)\Bigg] + \Lambda,\end{aligned} \tag{15b}$$

$$\begin{aligned}\kappa_N\left(p + \frac{E^2}{2\mathcal{A}_{N-2}}\right) &= \frac{1}{r^2 e^{2\lambda}}\Bigg[\frac{(N-3)(N-4)}{2} \\ &\quad + r^2\nu'' + r^2(\nu')^2 - r^2\nu'\lambda' \\ &\quad + (N-3)r(\nu'-\lambda')\Bigg] \\ &\quad + \hat{\alpha}\Bigg[\frac{2}{r^2 e^{2\lambda}}\left(\nu'' + (\nu')^2 - \nu'\lambda'\right) \\ &\quad + \frac{2}{r^2 e^{4\lambda}}\left[3\nu'\lambda' - \nu'' - (\nu')^2\right] \\ &\quad + \frac{2(N-5)}{r^3 e^{4\lambda}}\left(e^{2\lambda}-1\right)(\nu'-\lambda') \\ &\quad - \frac{(N-5)(N-6)(e^{2\lambda}-1)^2}{2r^4 e^{4\lambda}}\Bigg] \\ &\quad - \frac{(N-3)(N-4)}{2r^2} + \Lambda,\end{aligned} \tag{15c}$$

$$\sigma = \frac{e^{-\lambda}\left(r^{N-2}E\right)'}{r^{N-2}\mathcal{A}_{N-2}}, \tag{15d}$$

describing the gravitational dynamics of a charged static fluid in EGB gravity. In the above, we have set

$$\hat{\alpha} = \alpha(N-3)(N-4). \tag{16}$$

Simplification of the field equations arises if we utilise the coordinate transformation of Durgapal and Bannerji [35],

$$e^{2\nu(r)} = y^2(x), \quad e^{-2\lambda(r)} = Z(x), \quad x = r^2,$$

to obtain the following system

$$\begin{aligned}\kappa_N\left(\rho + \frac{E^2}{2\mathcal{A}_{N-2}}\right) &= (N-2)\Bigg[\frac{(N-3)(1-Z) - 2x\dot{Z}}{2x} \\ &\quad + \frac{\hat{\alpha}(1-Z)}{2x^2}\big(-4x\dot{Z} \\ &\quad + (N-5)(1-Z)\big)\Bigg] - \Lambda,\end{aligned} \tag{17a}$$

$$\begin{aligned}\kappa_N\left(p - \frac{E^2}{2\mathcal{A}_{N-2}}\right) &= (N-2)\Bigg[\frac{2Z\dot{y}}{y} + \frac{(N-3)(Z-1)}{2x} \\ &\quad + \hat{\alpha}(1-Z)\Bigg(\frac{4Z\dot{y}}{xy} \\ &\quad - \frac{(N-5)(1-Z)}{2x^2}\Bigg)\Bigg] + \Lambda,\end{aligned} \tag{17b}$$

$$\begin{aligned}\kappa_N\left(p + \frac{E^2}{2\mathcal{A}_{N-2}}\right) &= \frac{2}{y}\left[2xZ\ddot{y} + x\dot{Z}\dot{y} + (N-2)\dot{y}Z\right] \\ &\quad + (N-3)\left[\dot{Z} + \frac{(N-4)(Z-1)}{2x}\right] \\ &\quad + \hat{\alpha}\Bigg[\frac{4(N-4)Z(1-Z)\dot{y}}{xy} \\ &\quad + \frac{8Z(1-Z)\ddot{y}}{y} + \frac{4\dot{Z}\dot{y}(1-3Z)}{y}\end{aligned}$$

$$+\frac{2(N-5)\dot{Z}(1-Z)}{x} - \frac{(N-5)(N-6)(1-Z)^2}{2x^2}\Bigg] + \Lambda, \tag{17c}$$

$$\sigma^2 = \frac{Z\left[2x^{\frac{N-1}{2}}\dot{E} + (N-2)x^{\frac{N-3}{2}}E\right]^2}{(\mathcal{A}_{N-2})^2 x^{(N-2)}}. \tag{17d}$$

By equating (17b) and (17c), we obtain the charged condition of isotropic pressure

$$\Big[2x^3\dot{y} + 4\hat{\alpha}x^2\dot{y} - 12\hat{\alpha}x^2\dot{y}Z + (N-3)x^2y + 2\hat{\alpha}(N-5)xy - 2\hat{\alpha}(N-5)xyZ\Big]\dot{Z} + 2\hat{\alpha}\Big[4x\dot{y} - 4x^2\ddot{y} + (N-5)y\Big]Z^2 + \Big[4x^3\ddot{y} + 8\hat{\alpha}x^2\ddot{y} - 8\hat{\alpha}x\dot{y} - (N-3)xy - 4\hat{\alpha}(N-5)y\Big]Z + \Big[(N-3)x + 2\hat{\alpha}(N-5) - \left(\frac{N-2}{N-3}\right)x^2E^2\Big]y = 0. \tag{18}$$

When $\alpha = 0$, we regain N-dimensional general relativity. To solve the field equations (17), we need to integrate (18).

Equation (18) is an Abel differential equation of the second kind in the variable Z if y is specified. Particular solutions to (18) have been found in recent treatments by Hansraj and Mkhize [13] and Naicker et al. [11,12] by specifying a form for y. However, a systematic approach leads to a simpler form of (18). Naicker et al. [11] showed that with the transformation

$$w = (Z - \mathfrak{D}(x))\,W, \tag{19}$$

where

$$\mathfrak{D}(x) = \frac{[2\hat{\alpha}(N-5) + x(N-3)]y + 2x(x+2\hat{\alpha})\dot{y}}{x[6x\dot{y} + (N-5)y]}. \tag{20}$$

and

$$W = \exp\left(-\int \frac{\left(4x\dot{y} - 4x^2\ddot{y} + (N-5)\,y\right)}{x\left[6x\dot{y} + (N-5)\,y\right]}\,dx\right), \tag{21}$$

the condition of pressure isotropy (18) can be written in canonical form as

$$\dot{w} = F_0 w^{-1} + F_1, \tag{22}$$

where F_1 and F_0 are given by

$$F_1 = \frac{x\left[y - 2\left(x + 2\hat{\alpha}\right)\dot{y}\right]\left[(N-3)\,\dot{y} - 2x\ddot{y}\right]W}{\hat{\alpha}\left[6x\dot{y} + (N-5)\,y\right]^2}, \tag{23a}$$

$$F_0 = -2x^2\left(y - 2\left(x + 2\hat{\alpha}\right)\dot{y}\right)\Big[y\left((N-3)\dot{y}\right. \tag{23b}$$

$$\left.+2\left(x(N-3) + 2\hat{\alpha}\,(N-5)\right)\ddot{y}\right) + 2\dot{y}\left[\left(x - 4\hat{\alpha}\right)\dot{y} + 2x\left(x + 2\hat{\alpha}\right)\ddot{y}\right]\Big] \tag{23c}$$

$$\times \frac{W^2}{2\hat{\alpha}^2\left[6x\dot{y} + (N-5)\,y\right]^3} - \frac{(N-2)yxE^2W^2}{2\hat{\alpha}(N-3)\left[6x\dot{y} + (N-5)\,y\right]}. \tag{23d}$$

Note that although the functions F_0 and F_1 have specific forms, in the following analysis we will treat F_0 and F_1 as arbitrary functions of x. The reduced equation (22), in terms of w is clearly simpler than (18), in terms of Z.

3 The Chini invariant

Particular solutions to (22) were found by Naicker et al. [11,12] by placing restrictions on F_1 and F_0. However new classes of exact solutions are possible if we relate (22) to the Chini differential equation. The Chini differential equation was first studied by Chini [33]. Since then, a number of studies have focused on this equation, investigating its mathematical properties. The global existence of solutions to the Chini differential equation and its generalizations have been studied by Redheffer [36]. It has also been studied recently by Chamberland and Gasull [37]. They considered the number of limit cycles of a T-periodic Chini differential equation. In a more recent analysis by Pinto [38], the Chini method was used to analyse related Riccati differential equations. More information about the Chini differential equation can be found in the book by Kamke [39]. As far as we are aware, the Chini differential equation has not been studied in a realistic physical application in gravity. Here we will show that it arises in the evolution of a gravitational fluid in modified theories of gravity. As far as we are aware, this is the first application of the Chini differential equation in a gravitational theory. This highlights the importance of the Chini differential equation in gravitational dynamics.

The Chini differential equation in canonical form is presented in Appendix A. It is a first order nonlinear differential equation. On inspection of the condition of pressure isotropy (22), we observe that it is a Chini differential equation (A1) with $n = -1$, $f(x) = F_0$, $g(x) = 0$, and $h(x) = F_1$. The quantity

$$\beta_0 = h^3\left[\dot{f}h - f\dot{h}\right]^{-1},$$

must be constant. (In the context of differential equations the quantity β_0 is called an invariant: it leads to separability of the differential equation.) Hence the Chini invariant is therefore given by

$$\beta_0 = F_1\left[\frac{d}{dx}\left(\frac{F_0}{F_1}\right)\right]^{-1}, \tag{24}$$

in terms of the variables used in this paper. It is critical to realise that only if β_0 is constant, can the Chini differential equation be transformed into a separable and hence integrable differential equation. We may interpret (24) as an integrability condition for the existence of solutions. Note that our result holds for both $E = 0$ and $E \neq 0$.

We summarize our result as follows:

Theorem 1 *The gravitational dynamics in EGB gravity, for a neutral fluid, and in EGBM gravity, for a charged fluid, are connected to the Chini differential equation with the Chini invariant* β_0.

Corollary 1 *The condition of pressure isotropy in EGB gravity is a separable differential equation if the Chini invariant* β_0 *is constant.*

Corollary 2 *The Chini invariant does not arise in general relativity.*

This is the first demonstration of the Chini differential equation in a gravitational theory. Hence from the above we observe that *the Chini differential equation distinguishes EGB gravity from standard general relativity*.

Some general comments for the Chini equation in relation to general relativity and EGB gravity are relevant at this point. In the case of general relativity, $\hat{\alpha} = 0$ and the charged pressure isotropy condition (18) becomes

$$\begin{aligned}&\left[2x^2\dot{y} + (N-3)xy\right]\dot{Z}\\&\quad+\left[4x^2\ddot{y} - (N-3)y\right]Z\\&\quad+\left[(N-3)y - \left(\frac{N-2}{N-3}\right)xyE^2\right] = 0.\end{aligned} \tag{25}$$

Note that the charged condition of pressure isotropy (25) is linear in y (if Z is specified) *and linear* in Z (if y is specified). In EGB gravity the charged condition of pressure isotropy (18) holds with $\hat{\alpha} \neq 0$. This is a *nonlinear* differential equation in Z (if y is specified). In fact the resulting equation is an Abel differential equation of the second kind. We have shown that the Abel equation (18) can be transformed into the Chini differential equation

$$\frac{F_0}{F_1^2}\dot{\Phi}\Phi = \Phi + 1 - \frac{1}{\beta_0}\Phi^2, \tag{26}$$

where β_0 is the Chini invariant given by (24) and

$$w = \frac{F_0}{F_1}\Phi. \tag{27}$$

The solutions of (26) in EGB gravity will be necessarily different in general, and the gravitational potential Z will differ from the forms obtained in general relativity for a chosen form of the potential y. As an example, if $y = \sqrt{x}$ in EGB gravity, then the potential Z is given implicitly in general as shown by Naicker et al. [12] for a charged gravitating fluid. The Chini differential equation does not arise in general relativity as the underlying condition of pressure isotropy equation in linear in Z. The nonlinearity of the potential Z in EGB gravity leads to the Chini invariant, and this distinguishes the EGB theory from general relativity.

The transformation which makes (22) separable is (27). Applying this transformation leads to (26) which is a separable differential equation in $\Phi(x)$. Equivalently, we can write (26) as

$$\int \frac{-\beta_0\Phi}{\Phi^2 - \beta_0\Phi - \beta_0}\,d\Phi = \int \frac{{F_1}^2}{F_0}\,dx. \tag{28}$$

This equation can be integrated for different values of β_0.

Three classes of solutions are possible. We present the families of solutions below.

3.1 Case A: $\beta_0 \in (-\infty, -4) \cup (0, \infty)$

Integrating Eq. (28) for these values of β_0, we obtain the general solution for Φ as

$$\begin{aligned}\int \frac{{F_1}^2}{F_0}\,dx + C &= -\frac{\beta_0}{2}\ln|\Phi^2 - \beta_0\Phi - \beta_0|\\&\quad-\frac{\beta_0^2}{2\sqrt{\beta_0^2 + 4\beta_0}}\\&\quad\times\ln\left|\frac{2\Phi - \beta_0 - \sqrt{\beta_0^2 + 4\beta_0}}{2\Phi - \beta_0 + \sqrt{\beta_0^2 + 4\beta_0}}\right|,\end{aligned} \tag{29}$$

where C is a constant of integration. Then using (27) and (19), we can write the solution in terms of the gravitational potential Z as

$$\int \frac{{F_1}^2}{F_0}\,dx + C$$

$$
\begin{aligned}
&= -\frac{\beta_0}{2}\ln\left|\left[\frac{F_1}{F_0}\left(Z-\mathfrak{D}(x)\right)W\right]^2 \right.\\
&\quad \left. -\beta_0\frac{F_1}{F_0}\left(Z-\mathfrak{D}(x)\right)W-\beta_0\right|\\
&\quad -\frac{\beta_0^2}{2\sqrt{\beta_0^2+4\beta_0}}\ln\left|2\frac{F_1}{F_0}\left(Z-\mathfrak{D}(x)\right)W\right.\\
&\quad -\beta_0-\sqrt{\beta_0^2+4\beta_0}\left[2\frac{F_1}{F_0}\left(Z-\mathfrak{D}(x)\right)W\right.\\
&\quad \left.\left.-\beta_0+\sqrt{\beta_0^2+4\beta_0}\right]^{-1}\right|, \qquad (30)
\end{aligned}
$$

where $\mathfrak{D}(x)$ is given by (20). In general, this is an implicit solution for the potential Z, and W is specified by (21).

3.2 Case B: $\beta_0 \in (-4, 0)$

For this range of values for β_0, integrating Eq. (28) gives us the following solution

$$
\begin{aligned}
\int \frac{{F_1}^2}{F_0}\,dx + C &= -\frac{\beta_0}{2}\ln\left|\Phi^2-\beta_0\Phi-\beta_0\right|\\
&\quad -\frac{\beta_0^2}{\sqrt{|\beta_0^2+4\beta_0|}}\\
&\quad \times\arctan\left(\frac{2\Phi-\beta_0}{\sqrt{|\beta_0^2+4\beta_0|}}\right). \qquad (31)
\end{aligned}
$$

With the help of (27) and (19), we obtain

$$
\begin{aligned}
&\int \frac{{F_1}^2}{F_0}\,dx + C\\
&= -\frac{\beta_0}{2}\ln\left|\left[\frac{F_1}{F_0}\left(Z-\mathfrak{D}(x)\right)W\right]^2\right.\\
&\quad \left.-\beta_0\left[\frac{F_1}{F_0}\left(Z-\mathfrak{D}(x)\right)W-1\right]\right|\\
&\quad -\frac{\beta_0^2}{2\sqrt{|\beta_0^2+4\beta_0|}}\arctan\left(\left(2\frac{F_1}{F_0}\left(Z-\mathfrak{D}(x)\right)W\right.\right.\\
&\quad \left.\left.-\beta_0\right)\frac{1}{\sqrt{|\beta_0^2+4\beta_0|}}\right), \qquad (32)
\end{aligned}
$$

where we have that $\mathfrak{D}(x)$ is given by (20). Again, in general, this is an implicit solution involving the potential Z, and W is given by (21).

3.3 Case C: $\beta_0 = -4$

Integrating Eq. (28), we now aquire the general solution

$$
\int \frac{{F_1}^2}{F_0}\,dx + C = 4\ln\left(\Phi+2\right)+\frac{8}{\Phi+2}. \qquad (33)
$$

Similarly, using (27) and (19) we can write

$$
\begin{aligned}
&\int \frac{{F_1}^2}{F_0}\,dx + C\\
&= 4\ln\left(\frac{F_1}{F_0}\left(Z-\mathfrak{D}(x)\right)W+2\right)\\
&\quad +8\left[\frac{F_1}{F_0}\left(Z-\mathfrak{D}(x)\right)W+2\right]^{-1}, \qquad (34)
\end{aligned}
$$

where $\mathfrak{D}(x)$ is given by (20). This gives a solution in implicit form for the potential Z, and W is defined in (21).

The three classes of solutions that arise depend on the value of β_0. It is important to observe that the above three general solutions have been found without specifying y and E and they are valid for both charged and neutral fluids in EGB gravity. In general the function $\Phi(x)$ is given implicitly. Note that the condition (A5) must be satisfied for these classes of solutions to exist. The three families of exact solutions that we have found, utilizing the Chini equation, are *new* solutions of the condition of pressure isotropy in EGB gravity.

We can summarize the results as follows:

Theorem 2 *The Chini differential equation in EGB gravity can be integrated so that three classes of exact solutions are possible in implicit form for the invariant β_0. The potential Z is given by (30), (32) and (34).*

It is important to note that our results, and the above three classes of exact solutions given by (29), (31) and (33), hold for *any* Abel differential equation of the second kind. These results will also provide exact solutions to Abel differential equations of the first kind as both types of equations are related. The Chini form of the condition of pressure isotropy leads to exact solutions which are given in implicit form. Explicit forms for the gravitational potentials also arise in particular cases. We demonstrate this property in the next two sections for neutral matter and charged matter respectively.

4 Charged fluids

In general, the integrability condition (24) is a differential equation with the dependent variable $y(x)$. In the presence of charge, we obtain

$$
\begin{aligned}
E^2 = \Big[&-2x^2\left(y-2\left(x+2\hat{\alpha}\right)\dot{y}\right)\Big[y\left((N-3)\dot{y}\right.\\
&+2\left(x(N-3)+2\hat{\alpha}\left(N-5\right)\right)\ddot{y}\Big)
\end{aligned}
$$

$$+2\dot{y}\left[\left(x-4\hat{\alpha}\right)\dot{y}+2x\left(x+2\hat{\alpha}\right)\ddot{y}\right]\Big]$$
$$\times\frac{W^2}{2\hat{\alpha}^2\left[6x\dot{y}+(N-5)\,y\right]^3}-\frac{1}{\beta_0}F_1\int F_1\,dx\Big]$$
$$\times\frac{2\hat{\alpha}(N-3)\left[6x\dot{y}+(N-5)\,y\right]}{(N-2)yW^2}, \qquad (35)$$

where W is given by (21) and $\beta_0 \neq 0$ is a constant. Effectively, (35) becomes a definition for the charge E^2, which will be determined once y is specified. Hence the solution will be given by one of the three general cases given above depending on what is chosen for the value of the Chini invariant β_0. In order to find an exact solution to the charged condition of isotropic pressure, an Abel differential equation of the second kind, we need to evaluate three integrals. Firstly to determine W, we must integrate

$$\int\frac{\left(4x\dot{y}-4x^2\ddot{y}+(N-5)\,y\right)}{x\left[6x\dot{y}+(N-5)\,y\right]}\,dx,$$

and then to determine the form for the electromagnetic field intensity E^2 we must evaluate the integral given by

$$\int\frac{x\left[y-2\left(x+2\hat{\alpha}\right)\dot{y}\right]\left[(N-3)\,\dot{y}-2x\ddot{y}\right]W}{\hat{\alpha}\left[6x\dot{y}+(N-5)\,y\right]^2}\,dx.$$

Lastly to generate the solution we need to integrate

$$\int\frac{{F_1}^2}{F_0}\,dx.$$

Clearly integration is possible only in limited cases depending on the choice of the potential y.

We present two cases in which the integrations are possible and exact solutions can be found.

4.1 Specific potential: $y = x^{-4}$

We demonstrate the above algorithm by determining a new solution to the isotropic pressure condition for the specific potential

$$y = x^{-4}. \qquad (36)$$

For the electromagnetic field intensity, we substitute

$$\beta_0 = -4, \qquad (37)$$

into (34) and set the integration constant $C = 0$. This leads to

$$E^2=\frac{(N-3)}{x\hat{\alpha}(N-2)(N-101)}\Big[9x^2(N-101)+16x\hat{\alpha}(N-263)-4608\hat{\alpha}^2\Big]. \qquad (38)$$

Therefore we need to integrate

$$w\dot{w} = F_1 w + F_0,$$

with

$$F_1=-\frac{4(N+7)x^{-\frac{2(N-65)}{N-29}}(9x+16\hat{\alpha})}{\hat{\alpha}(N-29)^2}, \qquad (39)$$

and

$$F_0=\frac{(N+7)^2x^{-\frac{3(N-77)}{N-29}}}{2\hat{\alpha}^2(N-101)(N-29)^3}\Big[-9x^2(N-101)-16x\hat{\alpha}(N-173)+2048\hat{\alpha}^2\Big]. \qquad (40)$$

With these choices, we see that the solution falls in *Case C* of the general solutions. Substituting into (34), we obtain the exact solution

$$\left[\frac{4(N-101)\mathcal{F}(x)}{\mathcal{G}(x)}+2\right]^4\times\exp\left(\frac{8\mathcal{G}(x)}{4(N-101)\mathcal{F}(x)+2\mathcal{G}(x}\right)=\frac{C_1x^{\frac{4(N-101)}{N-29}}}{\left[x(N-101)-128\hat{\alpha}\right]^4}. \qquad (41)$$

where

$$\mathcal{F}(x)=\left[(N-11)x+2\hat{\alpha}[(N-13)-(N-29)Z]\right],$$

and

$$\mathcal{G}(x)=(N+7)\left[128\hat{\alpha}-(N-101)x\right].$$

This is a *new* implicit solution of the charged condition of pressure isotropy.

4.2 Specific potential: $y = \sqrt{x}$

We can show that known solutions are also contained in the Chini class. A specific solution to the charged condition of isotropic pressure was found by Naicker et al. [12] for the gravitational potential

$$y = \sqrt{x}, \qquad (42)$$

and the electromagnetic field intensity

$$E^2=\frac{2A(N-3)^2(N-4)}{x^2(N-2)^2}. \qquad (43)$$

The solution has the form

$$4C_1\left(2+\Theta\right)x^2=\Big[2\left(2+\Theta\right)\Phi-\mathcal{K}_1(x)\Big]^{1+\frac{1}{\sqrt{9+4\Theta}}}\times\Big[2\left(2+\Theta\right)\Phi-\mathcal{K}_2(x)\Big]^{1+\frac{1}{\sqrt{9+4\Theta}}}, \qquad (44)$$

where

$$\mathcal{K}_1(x) = 1 + \sqrt{9 + \frac{4A(N-3)(N-4)}{\hat{\alpha}}},$$

and

$$\mathcal{K}_2(x) = 1 - \sqrt{9 + \frac{4A(N-3)(N-4)}{\hat{\alpha}}},$$

with $\Theta = \frac{A(N-3)(N-4)}{\hat{\alpha}}$. In terms of the potential Z, the solution was found to be of the form

$$\begin{aligned}&\left[2(N-2)\left(Z-\Gamma(x)\right)-\mathcal{K}_1(x)\right]^{1+\frac{1}{\sqrt{9+4\Theta}}}\\&\quad\times\left[2(N-2)\left(Z-\Gamma(x)\right)-\mathcal{K}_2(x)\right]^{1+\frac{1}{\sqrt{9+4\Theta}}}\\&\quad=4C_1\left(2+\Theta\right)x^2,\end{aligned} \tag{45}$$

where

$$\Gamma(x) = \frac{(N-2)x + 2\hat{\alpha}(N-4)}{2\hat{\alpha}(N-2)}.$$

This solution is contained in the Chini general solutions. If we substitute

$$\beta_0 = \frac{\hat{\alpha}}{2\hat{\alpha} + A(N-3)(N-4)}, \tag{46}$$

and set the integration constant $C = 0$ in (34), we regain the electromagnetic field intensity (43) chosen by Naicker et al. [12]. We notice that $0 < \beta_0 < 1$, therefore the solution falls under *Case A* of the Chini general class. Substituting into (29), we obtain the potential given by Eq. (44). This particular solution was obtained by Naicker et al. [12] using an ad hoc approach. We have shown that it is part of a wider class of solutions associated with the Chini invariants of *Class A* given above.

5 Neutral fluids

In the case of neutral fluids, the integrability condition (24) has to be solved with $E^2 = 0$. This becomes a nonlinear differential equation in the dependent variable y. Therefore neutral fluids are more difficult to analyse using the Chini invariant. We present two cases as examples where integration is possible for specific choices of the potential y.

5.1 Specific potential: $y = ax + b$

An exact solution to the linear case of the uncharged condition of pressure isotropy was found by Chilambwe et al. [16] in five dimensions. This particular solution is given in terms of rational functions for the potential Z. We seek to find a solution for Z in an arbitrary number of dimensions N using the Chini differential equation. If we choose the form

$$y = ax + b, \tag{47}$$

for the gravitational potential y, we find that (22) becomes

$$\begin{aligned}w\dot{w} &= \frac{a\left[b(N-5)+ax(N+1)\right]^{-\frac{2N}{N+1}}\left[b-a(x+4\hat{\alpha})\right]}{\hat{\alpha}(N-4)}w\\&\quad+\frac{a}{\hat{\alpha}^2}\left[b(N-5)+ax(N+1)\right]^{\frac{4-3(N-1)}{N+1}}\\&\quad\times\left[-b+a(x+4\hat{\alpha})\right]\left[b(N-3)\right.\\&\quad\left.-a(x(N-1)+8\hat{\alpha})\right].\end{aligned} \tag{48}$$

Computing the Chini invariant for (48), we obtain the following

$$\beta_0 = \frac{(N-3)^2}{2(N-1)}, \tag{49}$$

which is clearly independent of x. Furthermore we observe that β_0 is always positive for $N \geq 4$ which implies that the solution of (48) falls under *Case A* in the Chini class of the above three general solutions. Substituting into (29), we obtain the following solution for (48) containing Φ:

$$\begin{aligned}&C_1\left[b(N-5)+ax(N+1)\right]^{-\frac{(N-3)^2}{2(N+1)}}\\&\quad\times\left[b(N-3)+ax(N-1)-8a\hat{\alpha}\right]^{\frac{(N-3)^2}{2(N-1)}}\\&\quad=\left(\Phi^2-\beta_0\Phi-\beta_0\right)^{-\frac{\beta_0}{2}}\\&\quad\times\left(\frac{2\Phi-\beta_0-\sqrt{\beta_0^2+4\beta_0}}{2\Phi-\beta_0+\sqrt{\beta_0^2+4\beta_0}}\right)^{-\frac{\beta_0^2}{2\sqrt{\beta_0^2+4\beta_0}}},\end{aligned} \tag{50}$$

where C_1 is a constant of integration.

We can then use transformations (27) and (19) to obtain a solution with the gravitational potential Z:

$$\begin{aligned}&C_1\left(\left[b(N-5)+ax(N+1)\right]^{-\frac{(N-3)^2}{2(N+1)}}\right)\\&\quad\times\left(\left[b(N-3)+ax(N-1)-8a\hat{\alpha}\right]^{\frac{(N-3)^2}{2(N-1)}}\right)\\&\quad=\left[\left(\mathfrak{C}(x)\left(\mathfrak{B}(x)-Z\right)\right)^2+\beta_0\mathfrak{C}(x)\left(Z-\mathfrak{B}(x)\right)-\beta_0\right]^{-\frac{\beta_0}{2}}\\&\quad\times\left[\frac{-2\mathfrak{C}(x)\big(Z-\mathfrak{B}(x)\big)}{-2\mathfrak{C}(x)\big(Z-\mathfrak{B}(x)\big)-\beta_0+\sqrt{\beta_0^2+4\beta_0}}\right.\end{aligned}$$

$$+\frac{\beta_0+\sqrt{\beta_0^2+4\beta_0}}{2\mathfrak{C}(x)\big(Z-\mathfrak{B}(x)\big)+\beta_0-\sqrt{\beta_0^2+4\beta_0}}\Bigg]^{\frac{-\beta_0^2}{2\sqrt{\beta_0^2+4\beta_0}}}, \tag{51}$$

where we have the following:

$$\mathfrak{C}(x)=\frac{\hat{\alpha}(N-3)\,[b(N-5)+ax(N+1)]}{x\left[b(N-3)+a\left(x(N-1)-8\hat{\alpha}\right)\right]},$$
$$\mathfrak{B}(x)=\frac{\mathcal{H}(x)+\mathcal{B}(x)}{2\hat{\alpha}\,[ax(N+1)+b(N-5)]},$$
$$\mathcal{H}(x)=ax\left[x(N-1)+2\hat{\alpha}(N-3)\right],$$
$$\mathcal{B}(x)=b\left[(N-3)x+2\hat{\alpha}(N-5)\right].$$

This is a *new* solution to the condition of isotropic pressure in N dimensions. This extends the particular Chilambwe et al. [16] model to N dimensions. In general, our class of solutions has an implicit form for the potential Z. The Chilambwe et al. [16] solution was found by inspection; here we have shown that it is a part of a general class of solutions associated with the Chini invariant in *Case A*.

5.2 Specific potential: $y=\sqrt{x}$

The choice

$$y=\sqrt{x}, \tag{52}$$

for the gravitational potential was considered by Naicker et al. [11]. It simplifies the condition of pressure isotropy to the form

$$w\dot{w}=-\frac{1}{(N-2)\,x^2}w-\frac{2}{(N-2)^2\,x^3}.$$

Substituting F_0 and F_1 in (24), we obtain

$$\beta_0=\frac{1}{2}. \tag{53}$$

Clearly β_0 is a constant, and the solution must fall under *Case A* of the general cases. Substituting β_0, F_0 and F_1 into (29), we aquire

$$\ln\left|\sqrt{\Phi^2-\frac{1}{2}\Phi-\frac{1}{2}}\left(\frac{2(\Phi-1)}{2\Phi+1}\right)^{\frac{1}{6}}\right|=\int\frac{1}{x}\,dx+C. \tag{54}$$

Integrating and using properties of the logarithm the expression (54) can be written as

$$(2\Phi+1)^2(\Phi-1)^4=C_1x^6. \tag{55}$$

The solution in terms of the potential Z is obtained using the transformations (27) and (19), and is of the form

$$Z=\frac{1}{N-2}\Bigg[\left(-1\pm2\sqrt{C_1}x^3+\mathcal{N}(x)\right)^{-\frac{1}{3}}+\left(-1\pm2\sqrt{C_1}x^3+\mathcal{N}(x)\right)^{\frac{1}{3}}+1\Bigg]+\frac{(N-2)\,x+2\alpha\,(N-3)\,(N-4)^2}{2\alpha\,(N-2)\,(N-3)\,(N-4)}, \tag{56}$$

where

$$\mathcal{N}(x)=\sqrt{\pm4\sqrt{C_1}x^3+4C_1x^6}.$$

This result is equivalent to the solution found by Naicker et al. [11]; the Chini approach also leads to the same class of models. It is important to remember that the Chini *Class A* solution is a general class of exact solutions, and this case corresponding to $y=\sqrt{x}$ is only a special case.

There exists cases in which the potential y has a simple form for neutral and charged relativistic fluids. These produce expressions for the potential Z in implicit form in general. We have identified certain cases in which y leads to constant Chini invariants and consequently, a resulting separable differential equation for the charged condition of pressure isotropy. In Table 1, we list the potentials y, the Chini invariant β_0 and indicate special cases found earlier.

6 Physical features

We consider two physical aspects related to the exact solutions generated in this paper: graphical plots and junction conditions. In this paper, we have determined solutions for the interior of a static, spherically symmetric spacetime in the presence of charge. We note that the solution (45) is implicit in nature and so attempting to analyse its physical behaviour is nontrivial. However, it is possible to visually represent the charge arising from expression (43), as can be seen in Fig. 1, where we have chosen the parameter $A=1$. The electric charge decreases away from the centre as a general trend. However, the electric charge increases with increasing dimension N and is positive and well behaved throughout the distribution.

Since the neutral solution (56), arising from the choice $y=\sqrt{x}$, is explicit, we are able to study its physical features.

In the following figures, we have chosen $\alpha=2$, $\Lambda=1$ and $C_1=2$. In Figs. 1 and 2, we present the overall behaviour of the energy density and pressure for the neutral solution. It can be seen that the behaviour of the curves is smooth and well defined, in both cases. We also note that both the energy density and pressure are greater as the dimension N increases. However, the overall trend in the case of the pressure is that it decreases, for all dimensions, further away from

Table 1 Summary of Chini invariants and associated potentials

Case	Chini invariant β_0	Chini class	Potential y	Fluid	Comment
I	-4	C	x^{-4}	Charged	New solution
II	$\frac{\alpha}{2\alpha+A}$	A	$\sqrt{x}$	Charged	Special case found by Naicker et al. [12] and Hansraj and Mkhize [13]
III	$\frac{(N-3)^2}{2(N-1)}$	A	$ax+b$	Neutral	New solution, special case found by Chilambwe et al. [16]
IV	$\frac{1}{2}$	A	$\sqrt{x}$	Neutral	Special case found by Naicker et al. [11] and Hansraj and Mkhize [13]

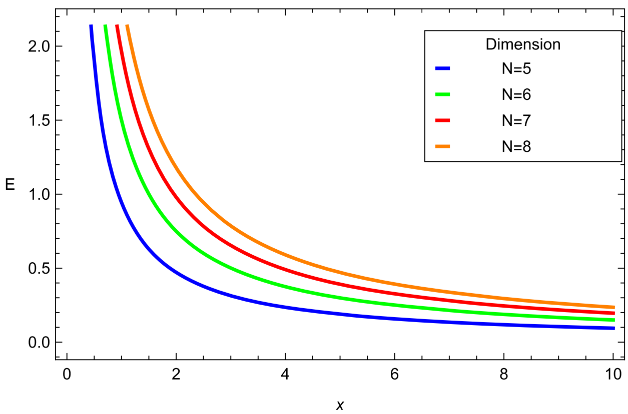

Fig. 1 The electromagnetic field E plotted against x

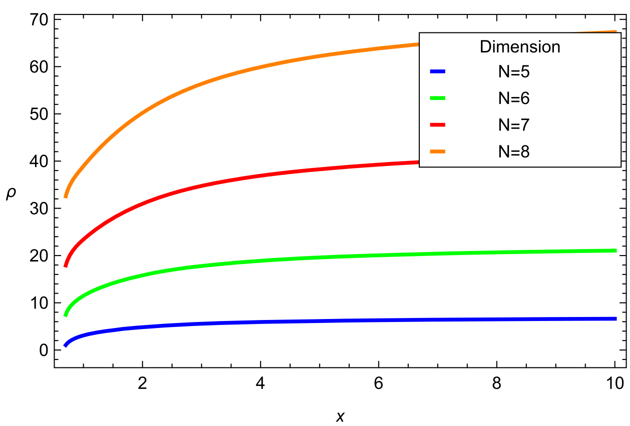

Fig. 2 The energy density ρ plotted against x

the centre of the distribution. Note that this general behaviour of the pressure in higher dimensions is similar to the profiles obtained by Paul [40] for dense neutron stars (Fig. 3).

We now further consider the junction conditions for the matching of the interior charged distribution of matter to an external vacuum spacetime. In order to match these solutions to an exterior metric, certain junction conditions at the boundary must be satisfied. In general relativity, the Israel–

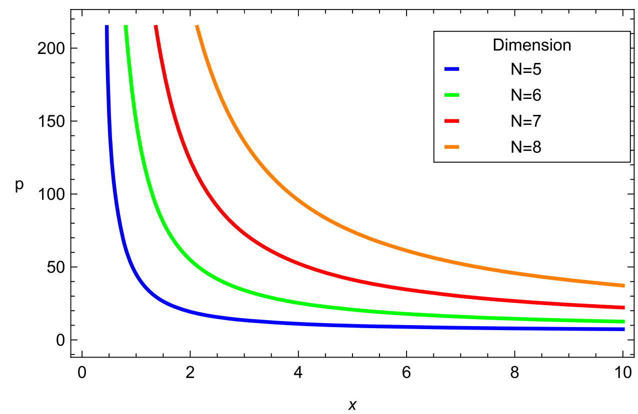

Fig. 3 The pressure p plotted against x

Darmois junction conditions

$$(\mathrm{d}s_-^2)_\Sigma = (\mathrm{d}s_+^2)_\Sigma = \mathrm{d}s_\Sigma^2, \tag{57a}$$

$$K_{ab}^- = K_{ab}^+ = K_{ab}|_\Sigma, \tag{57b}$$

must be satisfied in order to match two spacetime manifolds $\mathcal{M}^\pm$ across a comoving boundary surface Σ. In the above, $\mathrm{d}s^2$ is the line element and $\mathbf{K}$ is the extrinsic curvature. The junction conditions of EGB gravity were determined by Davis [41] to be of the form

$$(\mathrm{d}s_-^2)_\Sigma = (\mathrm{d}s_+^2)_\Sigma = \mathrm{d}s_\Sigma^2, \tag{58a}$$

$$[K_{ab} - Kh_{ab}]^\pm + 2\alpha[3J_{ab} - Jh_{ab} + 2\hat{P}_{abcd}K^{bc}]^\pm = 0, \tag{58b}$$

where $\mathbf{P}$ is the divergence-free part of the Riemann tensor, given by

$$\hat{P}_{abcd} = \hat{R}_{abcd} + 2\hat{R}_{b[c}h_{d]a} - 2\hat{R}_{a[c}h_{d]b} + \hat{R}h_{a[c}h_{d]b}. \tag{59}$$

The caret "^" symbol indicates the tensors associated with the induced metric **h**. The quantity **J** is the tensor given by

$$J_{ab} = \frac{1}{3}\big(2KK_{ab}K^{c}{}_{b} + K_{cd}K^{cd}K_{ab} - 2K_{ac}K^{cd}K_{db} - K^2K_{ab}\big), \tag{60}$$

with J as its trace. In an important advance in matching in EGB gravity, Brassel et al. [42] showed that if the Israel-Darmois conditions hold, then the EGB junction conditions (58) are satisfied. This result makes it possible to model a relativistic star in EGB gravity.

We will match the interior solution (41) found in this paper to the exterior Boulware–Deser–Wiltshire vacuum solution, given by

$$\mathrm{d}s^2 = -F(r)\mathrm{d}t^2 + \frac{1}{F(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{N-2}, \tag{61}$$

where

$$F(r) = 1 + \frac{r^2}{2\hat{\alpha}}\left[1 - \left(1 + \frac{4\hat{\alpha}}{(N-3)}\left(\frac{2M}{r^{N-1}} - \frac{Q^2}{(N-3)r^{2N-4}}\right)\right)^{\frac{1}{2}}\right]. \tag{62}$$

In the above, the quantities Q and M represent the charge and gravitational mass of the hypersphere respectively.

The junction conditions require that the line elements of the interior and exterior must be equal at the boundary. The interior potentials at the boundary $r = \mathcal{R}$ are given by

$$\xi_1 = y(\mathcal{R}^2), \tag{63a}$$
$$\xi_2 = Z(\mathcal{R}^2). \tag{63b}$$

The matching of the potentials at $r = \mathcal{R}$ gives the conditions

$$\xi_1^2 = 1 + \frac{\mathcal{R}^2}{2\hat{\alpha}}\left[1 - \left(1 + \frac{4\hat{\alpha}}{(N-3)}\left(\frac{2M}{\mathcal{R}^{N-1}} - \frac{Q^2}{(N-3)\mathcal{R}^{2N-4}}\right)\right)^{\frac{1}{2}}\right], \tag{64a}$$
$$\xi_2 = \frac{1}{4\mathcal{R}^2}\xi_1^2. \tag{64b}$$

The gravitational mass M can be written as

$$M = M_E + M_{GB}, \tag{65}$$

where M_E is the mass contribution of general relativity, given by

$$M_E = \frac{N-3}{2}\left[\mathcal{R}^{N-3}(1-\xi_2) - \frac{\Lambda r^{N-1}}{(N-1)(N-2)} + \frac{Q^2}{(N-3)^2\mathcal{R}^{N-3}}\right], \tag{66}$$

and M_{GB} is the mass contribution of EGB, given by

$$M_{GB} = \frac{1}{2}\left[\hat{\alpha}(N-3)\mathcal{R}^{N-5}(1-\xi_2)^2\right]. \tag{67}$$

The gravitational mass M is affected by dimension N. We also require that the radial pressure vanishes at the boundary. From (17), we have

$$(N-2)\left[\frac{2\xi_2\xi_3}{\xi_1} + \frac{(N-3)(\xi_2-1)}{2\mathcal{R}^2} + \hat{\alpha}(1-\xi_2)\left(\frac{4\xi_2\xi_3}{\mathcal{R}^2\xi_1} - \frac{(N-5)(1-\xi_2)}{2\mathcal{R}^4}\right)\right] + \frac{N-2}{2(N-3)}\eta_1^2 + \Lambda = 0, \tag{68}$$

where $\xi_1'(\mathcal{R}) = \xi_3$ and $\eta_1 = E(\mathcal{R}^2)$. The charge density at the boundary is given by

$$\sigma = \frac{\sqrt{\xi_2}\left[\mathcal{R}^{N-2}\eta_2 + (N-2)\mathcal{R}^{N-3}\eta_1\right]}{\mathcal{A}_{N-2}\mathcal{R}^{N-2}}, \tag{69}$$

where $\eta_2 = \eta_1'(\mathcal{R})$. The total charge inside of the hypersphere of radius $\mathcal{R}$ as measured by a distant observer is given by

$$Q = \int_0^{\mathcal{R}} [r^2E'(r^2) + (N-2)rE(r^2)]\mathrm{d}r, \tag{70}$$

which, in principle, can be determined when the electric field E is specified. Equation (70), along with the system (64) and (68) give restrictions that must be satisfied for the matching of the two solutions at $r = \mathcal{R}$. This is an algebraic system of six unknowns ($\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \mathcal{R}$) in four equations. Therefore a real solution to the system can be determined when two of the unknown parameters are specified. Therefore the model is closed and complete once particular forms for the potential y, the potential Z, and the electric field E are specified. The potential Z may arise implicitly, with a complicated analytic form, as in the treatment of Naicker et al. [12]. In that case we may have to obtain the determining parameters using numerical techniques; it is important to note that this is always possible in principle.

7 Pure EGB reduction

We comment on the possible extension to the interesting class of theories called pure EGB gravity. The pure Lovelock theory of gravity is a subclass of the full theory where there is essentially only one Euler density term within the gravitational action, that is, the highest order term of the theory. In general relativity (or first order Lovelock gravity), this amounts to the omission of the cosmological constant, leaving only the standard field equations. Significant work has been done in the higher order pure Lovelock gravity theory, for example see [43–46]. With regards to the pure EGB theory, the only surviving term in the action is the Gauss–Bonnet term, i.e.

$$S = \int d^N x \sqrt{-g}\,(\alpha L_{GB}) + S_{\text{matter}}, \tag{71}$$

where α is the Gauss–Bonnet coupling constant. The cosmological constant term and first order term (the Ricci scalar) fall away leaving only second order term in the above action. There is therefore no Einstein limit. Thus, upon variation of the reduced action (71) with respect to the metric, the pure EGB field equations can be given by

$$-\frac{\alpha}{2} H_{ab} = \kappa_N T_{ab}, \tag{72}$$

where the second order Lovelock tensor is defined as before by (2). In the vacuum scenario, the pure EGB equations reduce further to

$$H_{ab} = 0,$$

which is of a similar form to the Einstein vacuum field equations; however the Einstein tensor, which is the highest order Euler density in the first order theory, is replaced by the second order Lovelock tensor in the EGB scenario.

The charged pressure isotropy condition in pure EGB gravity then takes the form

$$\begin{aligned} &4\hat{\alpha} x\,[2x(1-3Z)\dot{y} + (N-5)(1-Z)y]\,\dot{Z} \\ &\quad -2(N-4)(N-5)\hat{\alpha} y(1-Z)^2 \\ &\quad -8\hat{\alpha} x(\dot{y}-2x)(1-Z)Z + 4x^2E^2 y = 0. \end{aligned} \tag{73}$$

Observe that Eq. (73) is still an Abel differential equation of the second kind in Z, for a specified y. Thus, despite being a subclass of the EGB theory, there is no real simplification in the governing equation in pure EGB gravity. The same notions are true in the five dimensional subcase which implies that the pure EGB scenario is more restrictive than the more general class. We note that in the absence of charge, the coupling constant $\hat{\alpha}$ can be divided out entirely, which is altogether unsurprising; in Einstein gravity, the coupling constant is set to one, and so is not present in any explicit sense. Due to the complicated nature of the above Eq. (73), it will be analysed in a subsequent treatise, as pure EGB gravity has several interesting features.

8 Discussion

We have studied the EGB field equations in a static, spherically symmetric spacetime, assuming the matter distribution to be that of a charged perfect fluid. We then generated the charged condition of pressure isotropy from the system of nonlinear differential equations. This leads to an Abel differential equation of the second kind, a particularly difficult type of ordinary differential equation which has no known general solutions. We transformed this equation into a simpler canonical form. We then showed that the canonical form of the charged condition of pressure isotropy is a Chini differential equation. The associated Chini invariant β_0 is an integrability condition for the existence of solutions. We showed that three classes of exact solutions, in implicit form, exist. Some known solutions are contained in our three classes of solutions. It is important to note that this is the *first* appearance of the Chini differential equation in a gravity theory. Also the Chini differential equation is a distinguishing feature of EGB gravity; it does *not* arise in general relativity. The new solutions found in this paper may be used to model a charged star in EGB gravity. This was used by Naicker et al. [12], with the potential $y = \sqrt{x}$, to illustrate the existence of charged objects. Other forms of y may be used to model compact objects in EGB gravity. In this regard, the recently found junction conditions of Brassel et al. [42] should be used at the matching surface. We also commented on pure EGB gravity and indicated that the approach of this paper can be extended as the governing equation remains an Abel differential equation of the second kind. This will be pursued in future research.

Data Availability Statement This manuscript has no associated data. [Author's' comment: Data sharing not applicable to this article as no datasets were generated or analysed during the current study.]

Code Availability Statement This manuscript has no associated code/software. [Author's comment: Code/Software sharing not applicable to this article as no code/software was generated or analysed during the current study.]

Declarations

Conflict of interest We confirm no conflict of interest exist.

Funded by SCOAP[3].

Appendix A: The Chini differential equation

We provide details on the form of the Chini differential equation, and indicate how the Chini invariant leads to separable differential equations. We achieve this by writing the Chini differential equation in canonical form.

The Chini differential equation takes the form

$$\dot{Y} = f(x)Y^n + g(x)Y + h(x), \tag{A1}$$

where f, g and h are arbitrary functions of x, and $n \in \mathbb{R}$, $n \neq 0, 1$. This is a generalisation of the Abel differential equation. We notice that if $n = 0$ or $n = 1$, Eq. (A1) becomes a simple linear differential equation. In general we cannot integrate Eq. (A1). However a special case arises for which (A1) is separable. To demonstrate this, we define a new variable $\Phi(x)$. We introduce the following transformation

$$Y(x) = \left(\frac{h}{f}\right)^{\frac{1}{n}} \Phi(x), \tag{A2}$$

where $\Phi(x)$ is an arbitrary function of x.

Then Eq. (A1) becomes

$$nh^{\frac{-n+1}{n}} f^{-\frac{1}{n}} \dot{\Phi} - n(\Phi^n + 1) - h^{\frac{-2n+1}{n}} f^{\frac{-n-1}{n}} [(nfg + \dot{f})h - f\dot{h}]\Phi = 0. \tag{A3}$$

If we let

$$\beta = f^{-n-1}h^{-2n+1}\left[(nfg + \dot{f})h - f\dot{h}\right]^n,$$

then Eq. (A3) becomes

$$nh^{\frac{-n+1}{n}} f^{-\frac{1}{n}} \dot{\Phi} = n(\Phi^n + 1) + \beta^{\frac{1}{n}}\Phi. \tag{A4}$$

Notice that if we take

$$\beta = \beta_0, \tag{A5}$$

where β_0 is a constant, then (A4) is a separable differential equation and can be written as

$$\int \frac{d\Phi}{n(\Phi^n + 1) + \beta_0^{\frac{1}{n}}\Phi} = \int \frac{dx}{nh^{\frac{-n+1}{n}} f^{-\frac{1}{n}}}. \tag{A6}$$

This integration can be completed to give a solution for Φ, which in turn, using (A2), will give a solution to the Chini equation (A1). Note that the constant β_0 in (A5) is known as the Chini invariant and hence is an integrability condition of (A1). Note that the variables in (A1) do not separate in general. The advantage of the Chini invariant is that it leads to a separable equation.

References

1. M.S. Turner, Phys. Scr. **2000**, 210 (2000)
2. D. Lovelock, J. Math. Phys. **498**, 12 (1971)
3. D. Lovelock, J. Math. Phys. **13**, 874 (1972)
4. T. Padmanabhan, D. Kothawala, Phys. Rep. **531**, 115 (2013)
5. C. Garrafo, G. Giribet, Mod. Phys. Lett. A **23**, 1801 (2008)
6. H. Ishihara, Phys. Lett. B **179**, 217 (1986)
7. D.L. Wiltshire, Phys. Lett. B **169**, 36 (1986)
8. N. Deruelle, T. Dolezel, Phys. Rev. D **62**, 103502 (2000)
9. B.P. Brassel, Gen. Relativ. Gravit. **56**, 49 (2024)
10. D. Gross, Nucl. Phys. Proc. Suppl. **74**, 426 (1999)
11. S. Naicker, S.D. Maharaj, B.P. Brassel, Universe **9**, 47 (2023)
12. S. Naicker, S.D. Maharaj, B.P. Brassel, Eur. Phys. J. C **83**, 343 (2023)
13. S. Hansraj, N. Mkhize, Phys. Rev. D **102**, 084028 (2020)
14. S. Hansraj, B. Chilambwe, S.D. Maharaj, Eur. Phys. J. C **75**, 277 (2015)
15. S.D. Maharaj, B. Chilambwe, S. Hansraj, Phys. Rev. D **91**, 084049 (2015)
16. B. Chilambwe, S. Hansraj, S.D. Maharaj, Int. J. Mod. Phys. D **24**, 1550051 (2015)
17. B. Das, S. Dey, S. Das, B.C. Paul, Eur. Phys. J. C **82**, 519 (2022)
18. P. Rej, A. Errehymy, M. Daoud, Eur. Phys. J. C **83**, 392 (2023)
19. M.Z. Bhatti, B. Almutairi, A.S. Khan, Eur. Phys. J. C **83**, 724 (2023)
20. S. Hansraj, Eur. Phys. J. C **82**, 218 (2023)
21. S. Naicker, S.D. Maharaj, B.P. Brassel, Gen. Relativ. Gravit. **55**, 116 (2023)
22. R.S. Bogadi, M. Govender, S. Moyo, Eur. Phys. J. Plus **138**, 426 (2023)
23. S.K. Maurya, K.N. Singh, M. Govender, S. Hansraj, Eur. Phys. J. C **82**, 49 (2022)
24. M. Malaver, R. Iyer, I. Khan, Phys. Sci. Int. J. **26**, 1 (2022)
25. S. Hansraj, M. Govender, L. Moodley, K.N. Singh, Phys. Rev. D **105**, 044030 (2022)
26. T. Tangphati, A. Pradhan, A. Errehymy, A. Banerjee, Phys. Lett. B **819**, 136423 (2021)
27. T. Tangphati, A. Pradhan, A. Errehymy, A. Banerjee, Ann. Phys. **430**, 168498 (2021)
28. A. Kaisavelu, M. Govender, S. Hansraj, D. Krupanandan, Eur. Phys. J. Plus **136**, 1029 (2021)
29. B.P. Brassel, S.D. Maharaj, Eur. Phys. J. C **80**, 971 (2020)
30. W. Lu, P. Benjamin, C. Bonnerot, Mon. Not. R. Astron. Soc. **500**, 1817 (2021)
31. S.K. Maurya, K.N. Singh, M. Govender, S. Ray, Mon. Not. R. Astron. Soc. **519**, 4303 (2023)
32. S.K. Maurya, A. Pradhan, F. Tello-Ortiz, A. Banerjee, R. Nag, Eur. Phys. J. C **81**, 848 (2021)
33. M. Chini, Rendiconti Instituto Lombardo **57**, 506 (1924)
34. S.C. Gumede, S.D. Maharaj, K.S. Govinder, Eur. Phys. J. C **84**, 862 (2024)
35. M.C. Durgapal, R. Bannerji, Phys. Rev. D **27**, 328 (1983)
36. R. Redheffer, Aequ. Math. **58**, 60 (1999)
37. M. Chamberland, A. Gasull, Qual. Theory Dyn. Syst. **9**, 29 (2010)

38. A.S. Pinto, P.N. da Silva, A.L.C. dos Santos, Int. J. Innov. Educ. Res. **10**, 4 (2022)
39. E. Kamke, *Differetialgleichungen lösungsmethoden und lösungen* (Springer, Berlin, 1977)
40. B.C. Paul, Class. Quantum Gravity **18**, 2637 (2001)
41. S.C. Davis, Phys. Rev. D **67**, 024030 (2003)
42. B.P. Brassel, S.D. Maharaj, R. Goswami, Class. Quantum Gravity **40**, 125004 (2023)
43. R.-G. Cai, N. Ohta, Phys. Rev. D **74**, 064001 (2006)
44. N. Dadhich, R. Durka, N. Merino, O. Miskovic, Phys. Rev. D **93**, 064009 (2016)
45. P. Concha, E. Rodríguez, Phys. Lett. B **774**, 616 (2017)
46. J.M. Toledo, V.B. Bezerra, Eur. Phys. J. C **79**, 117 (2019)