Anomalies and the Standard Model of Particle Physics

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For my parents

Declaration

This dissertation is based on original research done by the author while he was a graduate student at the Department of Applied Mathematics and Theoretical Physics, University of Cambridge, between October 2016 and August 2020. The material in Chapters 2 and 5 is based on the work done by the author under the supervision of David Tong and has been partly published in References [97, 96], while Chapters 3 and 4 are based on research done in collaboration with Joe Davighi, part of which is published in References [53, 52].

Except for part of Chapter 3 that has been previously submitted for a degree of doctor of philosophy by Joe Davighi at the University of Cambridge, no other part of this work has been submitted, or is being concurrently submitted, for a degree or other qualification at the University of Cambridge or any other university or similar institution.

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This dissertation aims to study quantum anomalies and some other aspects of the Standard Model of Particle physics. In any quantum gauge field theory, anomalies place a very restrictive condition on the matter content and the dynamics. The former is due to the cancellation of gauge anomalies while 't Hooft anomaly matching constraints produce the latter. As the Standard Model, which is our most fundamental and most accurate description of particle physics, is constructed as a gauge field theory, it is also subject to these anomalies. Here we explore subtleties in anomalies that could arise from the Standard Model and also use them to provide a consistency check as we explore its phase diagram.

We start by reexamining local anomaly cancellation in the Standard Model. It has long been known that the requirement that all gauge anomalies and the mixed gauge-gravitational anomaly cancel lead to the quantisation of hypercharge and essentially give the unique hypercharge assignment to the fermion content of the theory. However, if we take the view that hypercharge must be quantised from the outset, then it is enough to prove that the fermions have the Nature-assigned hypercharges using the cancellation of gauge anomalies alone. This remarkable result is made more astounding by the fact that Fermat's Last Theorem plays a crucial role in completing the proof.

We then move on to search for subtler global anomalies in the Standard Model and beyond from the modern viewpoint of cobordism theory, where a global anomaly can be computed as a homomorphism from a bordism group of manifolds equipped with appropriate spin and gauge bundle structure to a circle group. Since the gauge interaction depends on the gauge group G only through its Lie algebra, there are many possibilities for the gauge group of a gauge theory as long as the global structure is consistent with the matter content. In the Standard Model, the options for the gauge group G are $U(1)\times SU(2)\times SU(3)$, $U(2)\times SU(3)$, $SU(2)\times U(3)$, or $U(2)\times SU(3)/\mathbb{Z}_3$. We compute the fifth spin-bordism group of manifolds equipped with these G-bundle structures $\Omega_5^{\mathrm{Spin}}(BG)$ and show that it is at most $\mathbb{Z}_2$. Therefore, the global anomaly that can appear in the Standard Model is a mod 2 anomaly which can be identified with the well-known Witten anomaly in the gauge group $SU(2)$. We repeat the bordism group calculation for some beyond the Standard Model gauge groups and obtain a similar result: there is a mod 2 anomaly whenever there is an $SU(2)$ factor in the gauge group.

A curious fact from these bordism calculations is that the bordism group is trivial when $U(2)$ appears in lieu of $SU(2)$. Driven by this curiosity, we investigate further and find that there is an interplay between the local and the global anomaly. The condition for the gauge

anomaly cancellation on the $SU(2)$-representations of the fermions coupled to a gauge theory is the same whether the gauge group is $SU(2)$ or $U(2)$. However, the condition comes from the cancellation of the global Witten anomaly in the former case while it arises from the mixed anomaly cancellation between the $U(1)$ sector and $SU(2)$ sector in the latter case. We investigate further to see whether we can give the same interpretation to the new $SU(2)$ anomaly of Wang, Wen, and Witten when we place a $U(2)$ gauge theory on a non-spin manifold. We find that even though the requirement that the mixed gauge and the mixed gauge-gravitational anomalies cancel automatically cancel the new $SU(2)$ anomaly, it cannot be thought of as arising from the local anomalies. The reason is essentially because the transformation that induces the new $SU(2)$ anomaly involves a non-trivial diffeomorphism on the underlying manifold. Mathematically, we can compute the bordism group and still see a factor of $\mathbb{Z}_2$ associated with this new global $SU(2)$ anomaly.

Finally, we turn our attention towards the Standard Model itself, leaving anomalies as a tool we occasionally use to provide a consistency check on the IR dynamics. We apply the philosophy that one can get information and intuition on a theory by studying a collection of theories in the parameter space to the Standard Model. In these variations of the Standard Model, we deviate the Yukawa couplings from the actual values so that they are insensitive to the generations of fermions. We then vary the relative strength between the strong nuclear force and the weak nuclear force and see what happens. The results are surprising. No phase transition seems to be present when there is only one generation of fermions. More remarkably, the leptons seem to smoothly mutate into quarks as we slowly dial the relative strength between the weak and the strong gauge group. When more than one generations of fermions are present, however, the global symmetry group on either end of the phase diagram is not a subgroup of the other, and a first-order phase transition is expected to occur.

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Chapter 1

Introduction

Quantum Field Theory is a basis for much of modern physics. The most fundamental and accurate theory of particle physics to date, the Standard Model, is formulated as a quantum field theory. Many-body phenomena in condensed matter physics can often be phrased in this way, too. However, apart from free field theories, it is almost impossible to solve a quantum field theory exactly. In a few cases, it is possible to use perturbation theory to extract relevant quantities as a power series in small parameters of the theory, as done in quantum electrodynamics to predict the magnetic dipole moment of the electron to high precision. In most other cases where the interaction is strong, there is no small parameter anywhere in sight, leading to the break down of perturbation theory. In such *strongly coupled* quantum field theories, one has to rely on other methods to probe the dynamics and understand the behaviour of the systems.

One such powerful tool goes by the name of *anomalies*. From a modern point of view, an anomaly in its purest form is an obstruction to gauging a global symmetry. However, the subject is so rich and multi-faceted that a simple description like this does not do enough justice to it. There are also a lot of subtleties involved and connections to other areas of physics where quantum field theory is applicable, such as condensed matter physics. Examples include global anomalies in d dimensions and symmetry-protect topological (SPT) phases in $d+1$ dimensions. Most importantly, its robustness under RG flow makes it a suitable means for probing low energy dynamics of a system once its UV description, which is often much more straightforward to describe due to its weakly interacting nature, is given.

In this introduction, we will explore anomalies from different points of view through some representative examples and how to use them to explore the deep IR dynamics of strongly coupled quantum field theories. In the end, I will give a brief overview of how my work fits into the picture.

1.1 What are anomalies?

Consider a quantum field theory given by the action $S[\phi]$ where ϕ are a collection of fields. We say that G is a classical symmetry of the system if $S[g \cdot \phi] = -S[\phi]$ for all $g \in G$. If G is a Lie group, then Noether's Theorem tells us that there exists a conserved current for each generator t^a of G, where $a = 1, 2, \ldots, \operatorname{rank} G$. More precisely, there exist a set of current vector fields $j^{a\mu}$ such that

$$\partial_\mu j^{a\mu} = 0, \tag{1.1}$$

whenever the equations of motion are satisfied. Since G is a symmetry, the action is invariant under an infinitesimal transformation $g = (\mathbf{1} + \mathrm{i}\varepsilon^a t^a)$ (no sum) for a constant ε^a . Therefore, if ε^a are taken to be spacetime-dependent $\varepsilon^a = \varepsilon^a(x)$, the variation of the action is

$$S[g \cdot \phi] - S[\phi] = \int \mathrm{d}^4x \, j^{a\mu} \partial_\mu \varepsilon^a(x) = -\int \mathrm{d}^4x \, \partial_\mu j^{a\mu} \, \varepsilon^a(x), \tag{1.2}$$

But when the equations of motion are satisfied, the action is stationary under any variation of the field. Hence the current is conserved as claimed.

G is a symmetry of the quantum theory if both the exponentiated action and the path integral measure inside the path integral

$$Z = \int [\mathrm{d}\phi] \, \exp(-S[\phi]) \tag{1.3}$$

is invariant under G. The conservation laws $\partial_\mu j^{a\mu} = 0$ must now be interpreted as an operator equation; the correlation function of $\partial_\mu j^{a\mu}(x)$ with any number of operators $\mathscr{O}_i(x_i)$ at other points $x_i \neq x$ must vanish:

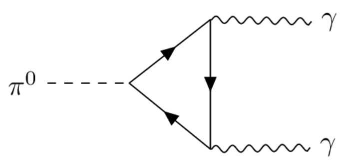
$$\left\langle \partial_\mu j^{a\mu}(x) \prod_i \mathscr{O}_i(x_i) \right\rangle = 0. \tag{1.4}$$

This is known as *Ward's identity*, which is the quantum version of Noether's theorem.

Promoting a system with symmetry group G to a gauge theory with gauge group G is a two-step process: (a) one first makes the symmetry transformation local and couple a classical background gauge field A to the theory so that the current remains conserved, and (b) one then make the field A dynamical by integrating over the space of gauge field A modulo gauge transformations. The first step can also be done without difficulties through minimal coupling. However, there is a possibility that the effective action is not invariant under gauge transformations in the presence of the background field. This proves to be an obstruction to carry out procedure (b) successfully. This obstruction is what we called an *anomaly*. If, however, G is already a gauge group with a dynamical gauge field, the existence of an anomaly signifies the breakdown of gauge invariance which renders the theory inconsistent. The former situation is commonly known as an *'t Hooft anomaly* while the latter goes by the name of a *gauge anomaly*.

1.1.1 Perturbative anomaly

Chiral anomaly



It is perhaps most transparent to derive the gauge and 't Hooft anomalies starting from the first known example of anomalies, which has a slightly different interpretation from what is stated earlier: it is the failure of a classical symmetry to remain the symmetry of the system at the quantum level. This happens in quantum electrodynamics coupled to a massless Dirac fermion. At the classical level there is a $U(1)$ global symmetry, called the chiral $U(1)$ symmetry, which rotates the phases of the left-handed and right-handed components of the fermion by the same amount but in the opposite direction. However, the chiral $U(1)$ symmetry does not remain a symmetry in the quantum theory. Through calculation of the pion decay $\pi^0 \to \gamma\gamma$ at one-loop in perturbation theory from the triangle diagrams of the form shown above, Bell and Jackiw discovered that the current associated with the chiral symmetry is not conserved [34], a result independently found by Adler through his investigation on Ward identities of axial vectors in quantum electrodynamics [3]. Subsequently, Adler and Bardeen performed a careful analysis to all order in perturbation theory and showed that the violation of chiral symmetry is one-loop exact [4]. It has since been known simply as the *anomaly*. If one needs to be more specific, one can refer to the anomaly as the *Adler–Bell–Jackiw* (ABJ) anomaly after its discoverers, or the *chiral* anomaly after the explicitly broken symmetry. One must not confuse the anomaly, where the chiral symmetry is explicitly broken by quantum fluctuations, with the spontaneous breakdown of global symmetry, where the symmetry is a genuine symmetry of the path integral, but does not leave the vacuum invariant, leading to degenerate vacua.

The system of a massless Dirac fermion Ψ coupled to a dynamical $U(1)$ gauge field A has the action

$$S_\Psi = \int \mathrm{d}^4x\, \overline{\Psi} \mathrm{i} \not{D} \Psi, \tag{1.5}$$

where the Dirac operator is defined by $\mathrm{i}\not{D} = \mathrm{i}\gamma^\mu(\partial_\mu - \mathrm{i}A_\mu)$. It is easy to see that, apart from enjoying the $U(1)$ gauge invariance, the action is invariant under another $U(1)$ symmetry, denoted by $U(1)_A$:

$$U(1)_A : \Psi \mapsto \mathrm{e}^{\mathrm{i}\beta\gamma_5}\Psi, \tag{1.6}$$

with a 2π-periodic parameter α. Since it rotates the two chiral components of the Dirac fermion in opposite directions, it is usually called the chiral or axial symmetry. By Noether's theorem there exists an associated conserved Noether current, denoted by j_A^μ.

In path integral formalism, the chiral anomaly arises because the path integral measure is not invariant under the chiral symmetry even though the exponentiated action is, as shown by Fujikawa in [68, 69]. The first step we need to take is to carefully define the path integral measure. Let $\{\phi_n\}$ be an orthonormal basis for the Dirac operator:

$$\mathrm{i}\not{D}\phi_n = \lambda_n \phi_n, \qquad \int \mathrm{d}^4x\, \phi_m^\dagger \phi_n = \delta_{mn}, \tag{1.7}$$

Note that the eigenvalues λ_n are real because the Dirac operator $\mathrm{i}\not{D}$ is Hermitian. If the Dirac fermion Ψ is expanded in terms of the orthonormal basis as

$$\Psi = \sum_n a_n \phi_n, \qquad \overline{\Psi} = \sum \bar{b}_n \bar{\phi}_n, \tag{1.8}$$

where $a_n, \bar{b}_n$ are Grasmann variables, then the fermionic path integral measure can be defined (at least formally) by

$$[\mathrm{d}\overline{\Psi}][\mathrm{d}\Psi] = \prod_n \mathrm{d}\bar{b}_n \mathrm{d}a_n, \tag{1.9}$$

and the path integral becomes

$$\int [\mathrm{d}\overline{\Psi}][\mathrm{d}\Psi]\mathrm{e}^{-S_\Psi} = \prod_n \mathrm{d}\bar{b}_n \mathrm{d}a_n \mathrm{e}^{\sum_n a_n \bar{b}_n \lambda_n} = \prod_n \lambda_n. \tag{1.10}$$

The formal infinite product $\prod_n \lambda_n$ can be thought of as the determinant of the Dirac operator. Of course one still needs to regulate it to obtain finite results.

Under the infinitesimal chiral transformation $\Psi \mapsto (1 + \mathrm{i}\alpha(x)\gamma_5)\Psi$, the Grassmann variables $a_n, \bar{b}_n$ transform as

$$\begin{aligned} a_n \mapsto a'_n &= \sum_m \left(\int \mathrm{d}^4x\, \phi_n^\dagger (1 + \mathrm{i}\gamma_5\alpha)\phi_m \right) a_m \\ \bar{b}_n \mapsto \bar{b}'_n &= \sum_m \bar{b}_m \left(\int \mathrm{d}^4x\, \phi_m^\dagger (1 + \mathrm{i}\gamma_5\alpha)\phi_n \right). \end{aligned} \tag{1.11}$$

From this we can deduce that the path integral measure transforms as

$$\prod_n \mathrm{d}\bar{b}'_n \mathrm{d}c'_n = J \prod_n \mathrm{d}\bar{b}_n \mathrm{d}c_n, \tag{1.12}$$

where the Jacobian J is given by

$$\log J = (-2\mathrm{i}) \lim_{N\to\infty} \sum_{n=1}^{N} \int \mathrm{d}^4x\, \phi_n^\dagger \alpha \gamma_5 \phi_n \tag{1.13}$$

To regulate this, we suppress contributions from high frequency modes. Construct a cut-off smooth function $f : [0,\infty) \to [0,1]$ such that $f(s) = 1$ for $s < 1-\varepsilon/2$ and vanishes for $s > 1+\varepsilon/2$ where ε is a small positive number. Then the regulated version of $\log J$ can be written as

$$\log J = (-2\mathrm{i}) \lim_{M\to\infty} \sum_{n=1}^{\infty} \int \mathrm{d}^4x\, \phi_n^\dagger \beta \gamma_5 f\left(\tfrac{\lambda_n^2}{M^2}\right) \phi_n \tag{1.14}$$

$$= (-2\mathrm{i}) \lim_{M\to\infty} \sum_{n=1}^{\infty} \int \mathrm{d}^4x\, \phi_n^\dagger \beta \gamma_5 f\left(\tfrac{-\not{D}^2}{M^2}\right) \phi_n, \tag{1.15}$$

This can be succinctly rewritten as

$$\log J = (-2\mathrm{i}) \lim_{M\to\infty} \mathrm{Tr}\left[\alpha \gamma_5 f\left(-\frac{\not{D}^2}{M^2}\right)\right]. \tag{1.16}$$

After inserting a complete set of plane wave solutions and a lengthy calculation, we obtain

$$\log J = \frac{\mathrm{i}}{16\pi^2} \int \mathrm{d}^4x\, \alpha(x)\, \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}. \tag{1.17}$$

and the fermion path integral measure turns out to be non-invariant under a local $U(1)_A$ transformation:

$$[\mathrm{d}\bar{\Psi}][\mathrm{d}\Psi] \mapsto [\mathrm{d}\bar{\Psi}][\mathrm{d}\Psi] \exp\left(\frac{\mathrm{i}}{16\pi^2} \int \mathrm{d}^4x\, \alpha(x)\, \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}\right). \tag{1.18}$$

Hence, through a modification of the Ward identity, $\partial_\mu j_A^\mu$ does not vanish. Instead, it satisfies

$$\partial_\mu j_A^\mu = -\frac{\mathrm{i}}{16\pi^2} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}, \tag{1.19}$$

where we must take this equation as an operator statement. Since the gauge field is dynamical and must be integrated over in the full path integral, we cannot turn it off and the right-hand side of (1.19) is generically non-vanishing. Therefore, $U(1)_A$ should not be considered a symmetry of the quantum theory.

A similar anomaly also arises when the gauge group G is non-abelian:

$$\partial_\mu j_A^\mu = -\frac{\mathrm{i}}{16\pi^2}\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\,F_{\mu\nu}F_{\rho\sigma}. \tag{1.20}$$

This result can be derived in exact the same way. However, there is an alternative approach that shows the relationship between anomalies and topology of the gauge field in a more transparent way. Note that to any mode ϕ_n of the Dirac operator $\mathrm{i}\not{D}$ with eigenvalue λ_n, there is a corresponding mode $\phi_{-n} := \gamma_5\phi_n$ with the opposite eigenvalue $-\lambda_n$. If λ_n is non-zero, the two modes are orthogonal and the summand $\int \mathrm{d}^4x\,\phi_n^\dagger\gamma_5\phi_n$ in (1.14) vanishes. We are therefore concerned only with the zero modes of the Dirac operator. Since γ_5 anticommutes with the Dirac operator, one can choose the zero mode to be either left-handed or right-handed:

$$\mathrm{i}\not{D}\phi_{-,n} = 0, \qquad \gamma_5\phi_{-,n} = -\phi_{-,n}, \tag{1.21}$$

$$\mathrm{i}\not{D}\phi_{+,n} = 0, \qquad \gamma_5\phi_{+,n} = +\phi_{+,n}. \tag{1.22}$$

Then the Jacobian in (1.14) becomes

$$J = \exp\left(2\mathrm{i}\,(n_+ - n_-)\right), \tag{1.23}$$

where $n_+ - n_-$ is difference between the numbers of the right-handed and the left-handed zero modes called the (analytical) index of the Dirac operator $\mathrm{i}\not{D}$. The Atiyah–Singer index theorem states that the analytical index of the Dirac operator can be calculated in terms of a topological quantity built from the field strength:

$$\mathrm{ind}(\mathrm{i}\not{D}) = -\frac{1}{32\pi^2}\int \mathrm{d}^4x\,\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\,F_{\mu\nu}F_{\rho\sigma}, \tag{1.24}$$

from which we obtain the nonconservation of the axial current as stated in (1.20).

Gauge anomaly

From the ABJ anomaly we can deduce the purest type of anomalies, which occurs in a chiral gauge theory where the gauge field couples to the left-handed component of the fermions differently from the right-handed ones. This can be done by considering the previous example of a massless Dirac fermion with gauge field $a_\mu/2$ coupled to the axial $U(1)_A$ in addition to the gauge field A_μ which we set to be $-a_\mu/2$. The action now becomes

$$S = \int \mathrm{d}^4x\left[\overline{\Psi}\mathrm{i}(\not{\partial} - \mathrm{i}\not{a})\frac{1}{2}(1-\gamma_5)\Psi + \overline{\Psi}\mathrm{i}\not{\partial}\frac{1}{2}(1+\gamma_5)\Psi\right] \tag{1.25}$$

with the desired effect that the gauge field a_μ only couples to the left-handed component $\psi_L = \frac{1}{2}(1-\gamma_5)\Psi$ of the Dirac fermion Ψ. The corresponding gauge transformation is given by

$$a_\mu \mapsto a'_\mu = a_\mu + \partial_\mu \alpha(x), \quad \psi_L \mapsto \psi'_L = \mathrm{e}^{\mathrm{i}\alpha(x)}\psi_L. \tag{1.26}$$

Using the Jacobian (1.17), the path integral after ψ_R is decoupled transforms under this gauge transformation as

$$\int [\mathrm{d}\bar{\psi}_L][\mathrm{d}\psi_L]\mathrm{e}^{-S_{\psi_L}} \mapsto \int [\mathrm{d}\bar{\psi}_L][\mathrm{d}\psi_L]\exp\left(\frac{-\mathrm{i}}{32\pi^2}\int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma}\beta(x) f_{\mu\nu}f_{\rho\sigma}\right)\mathrm{e}^{-S_{\psi_L}}, \tag{1.27}$$

implying that the gauge current is not conserved and the theory violates gauge invariance. This is unacceptable. Violation of classical gauge invariance at the quantum level like this is referred to as a *gauge anomaly* and must be cancelled at all cost for the theory to be consistent.

Even though a $U(1)$ chiral gauge theory with one left-handed (or right-handed) Weyl fermion has a gauge anomaly, we can construct a $U(1)$ chiral gauge theory with more complicated matter content such that the total anomaly vanishes. First note that we can always pair a left-handed and a right-handed Weyl fermions of the same charge q and give them a mass through the Dirac mass term. Since there is no gauge anomaly associated with a Dirac fermion, the gauge anomaly for $U(1)$ must depends only on the *index* ℓ_q of the representation q which is the difference between the number of left-handed Weyl fermions and the right-handed fermions of the same representation q. It is straightforward to show that for an arbitrary fermion content with index ℓ_q for charge q, the nonconservation equation becomes

$$\partial_\mu j^\mu = -\frac{\mathrm{i}\mathscr{A}}{32\pi^2}\varepsilon^{\mu\nu\rho\sigma} f_{\mu\nu}f_{\rho\sigma}, \tag{1.28}$$

where the *anomaly coefficient* $\mathscr{A}$ is given by

$$\mathscr{A} = \sum_{q\in\mathbb{Z}} \ell_q q^3. \tag{1.29}$$

The indices $\ell(q)$ must satisfy $\mathscr{A} = 0$, known as the cubic anomaly cancellation condition. ncWrite this in two steps: with the basic rep and general rep.

Chiral gauge anomalies also arise in a non-abelian gauge theory. If the fermion matter content consists of a left-handed Weyl fermions in the representation $\mathbf{r}$ of the gauge group G, the nonconservation of the gauge current is given by

$$\partial_\mu j^{a\mu} = -\frac{\mathrm{i}}{32\pi^2} d^{abc}(\mathbf{r})\varepsilon^{\mu\nu\rho\sigma} f^b_{\mu\nu} f^c_{\rho\sigma}, \tag{1.30}$$

where the totally symmetric rank 3 tensor $d^{abc}(\mathbf{r})$ of the representation $\mathbf{r}$ defined by the trace

$$d^{abc}(\mathbf{r}) := \frac{1}{2}\,\mathrm{tr}_{\mathbf{r}}(t^a\{t^b,t^c\}) \tag{1.31}$$

over the representation $\mathbf{r}$. The tensor $d^{abc}(\mathbf{r})$ in a general representation $\mathbf{r}$ is related to the tensor $d^{abc} := d^{abc}(\mathbf{F})$ in the fundamental representation $\mathbf{F}$ through $d^{abc}(\mathbf{r}) = A(\mathbf{r})d^{abc}$, where the coefficient of proportionality $A(\mathbf{r})$ is simply called the *anomaly* of the representation $\mathbf{r}$.

When the theory has an arbitrary fermion content, with index $\ell(\mathbf{r})$ for each representation $\mathbf{r}$, the tensor $d^{abc}(\mathbf{r})$ in the current nonconservation (1.30) is replaced by $\mathscr{A}d^{abc}$ where the anomaly coefficient $\mathscr{A}$ is given

$$\mathscr{A} = \sum_{\mathbf{r}} \ell(\mathbf{r})A(\mathbf{r}). \tag{1.32}$$

The result for the gauge anomaly in $U(1)$ given by the formula (1.29) can be obtained from this by realising that the generator of the charge q representation is just q.

It is worth pointing out that the tensor $d^{abc}(\mathbf{r})$ vanishes when the representation $\mathbf{r}$ is either real or pseudoreal. These are representations that are equivalent to its complex conjugate; the generators $t^a_{\mathbf{r}}$ of a real or pseudoreal representation $\mathbf{r}$ satisfy

$$(\mathrm{i}t^a_{\mathbf{r}})^* = S t^a_{\mathbf{r}} S^{-1}. \tag{1.33}$$

Plugging this condition into the definition of $d^{abc}(\mathbf{r})$ we obtain $d^{abc}(\mathbf{r}) = -d^{abc}(\mathbf{r})$. Simple Lie groups that have only real or pseudoreal representations are $SO(2n+1)$ with $n \geq 1$ (including $SU(2)$ which has the same Lie algrebra as $SO(3)$), $SO(4n)$ with $n \geq 2$, $\mathrm{Sp}(n)$ with $n \geq 3$, and the exceptional groups G_2, F_4, E_7, E_8 [100, 101]. Additionally, the groups $SO(4n+2)$ with $n \geq 2$ and E_6 have $d^{abc}(\mathbf{r}) = 0$ for all representations $\mathbf{r}$ even though they admit complex representations [76]. Hence, the non-abelian gauge anomaly arises only when the gauge group G is a special unitary group $SU(n)$, $n > 2$ which are actually of most relevance to particle physics.

’t Hooft anomaly

Consider a chiral gauge theory with a gauge group G whose gauge field is denoted by a. If we take a to be not dynamical, it does not have quantum fluctuations and the theory can be interpreted as having a global symmetry group G with a classical background gauge field a. The theory now make sense mathematically even when there is an anomaly in G because the anomaly vanishes when the background gauge field is turned off, which is now allowed because it is not integrated over in the path integral and is a free parameter that we can set by hand. This type of anomalies goes by the name of an *'t Hooft anomaly*. It does not render the theory inconsistent but should be interpreted as an obstruction to gauging a global symmetry. Since it is invariant under the RG flow, it is a powerful tool used to probe the dynamics of strongly coupled quantum field theories, as will be described in Section 1.2.

Mixed anomaly

Oftentimes, the symmetry group of our theory can be written as a product $G = G_1 \times G_2$ (or more precisely, when its Lie algebra $\mathfrak{g}$ is a direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2$). If the fermions are in the representation $\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2)$ of the symmetry group, then one can derive more constraints on the rank-3 symmetric tensor that determines the anomaly.

Let indices from the start of the Greek alphabet label generators of G_1 and indices from the start of the Latin alphabet label generators of G_2. Then a generator of G can either be of the form $t^\alpha_{\mathbf{r}_1} \otimes \mathbf{1}_{\mathbf{r}_2}$ or $\mathbf{1}_{\mathbf{r}_1} \otimes t^a_{\mathbf{r}_2}$. One can then show that

$$d_G^{\alpha\beta\gamma}(\mathbf{r}) = \dim(\mathbf{r}_2)\, d_{G_1}^{\alpha\beta\gamma}(\mathbf{r}_1), \tag{1.34}$$

$$d_G^{abc}(\mathbf{r}) = \dim(\mathbf{r}_1)\, d_{G_2}^{abc}(\mathbf{r}_2), \tag{1.35}$$

$$d_G^{\alpha bc}(\mathbf{r}) = \mathrm{tr}_{\mathbf{r}_1} t^\alpha\, C(\mathbf{r}_2) \frac{1}{2} \delta^{bc}, \tag{1.36}$$

$$d_G^{a\beta\gamma}(\mathbf{r}) = \mathrm{tr}_{\mathbf{r}_2} t^a\, C(\mathbf{r}_1) \frac{1}{2} \delta^{\beta\gamma}, \tag{1.37}$$

where the *Dynkin index* $C(\mathbf{r})$ of a representation $\mathbf{r}$ is defined by

$$\mathrm{tr}_{\mathbf{r}}\, t^a t^b = C(\mathbf{r})\, \mathrm{tr}_{\mathbf{F}}\, t^a t^b. \tag{1.38}$$

Here we use the conventional normalisation $\mathrm{tr}_{\mathbf{F}}(t^a t^b) = \delta^{ab}/2$ for the fundamental representation $\mathbf{F}$. The violation of the G_1 gauge current conservation (1.30) can be expanded as

$$\begin{aligned}\partial_\mu j^{\alpha\mu} = &-\frac{\mathrm{i}}{32\pi^2} \dim(\mathbf{r}_2) d^{\alpha\beta\gamma}(\mathbf{r}_2) \varepsilon^{\mu\nu\rho\sigma} F^\beta_{\mu\nu} F^\gamma_{\rho\sigma} - \frac{\mathrm{i}}{64\pi^2} (\mathrm{tr}_{\mathbf{r}_1} t^\alpha) C(\mathbf{r}_2) \varepsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^b_{\rho\sigma} \\ &- \frac{\mathrm{i}}{64\pi^2} (\mathrm{tr}_{\mathbf{r}_2} t^a)\, \varepsilon^{\mu\nu\rho\sigma} F^\alpha_{\mu\nu} F^a_{\rho\sigma}.\end{aligned} \tag{1.39}$$

The nonconservation of $j^{a\mu}$ can be written out in a similar manner. The first term in the expansion can be interpreted as anomalies purely in G_1, because the violation of the G_1 current conservation is given purely in terms of the G_1 field strength itself. The remaining terms constitute what we call the *mixed anomalies* where the conservation of the current in one factor is violated with the non-vanishing gauge field from the other factor. Note that mixed anomalies can only occur when either or both of the group factors contain $U(1)$. In practical calculation it is easier to compute pure and mixed anomalies separately.

As an example, consider a gauge theory with $G = SU(n) \times U(1)$ as a gauge group and a left-handed Weyl fermion transforming in the representation $\mathbf{r}_{+q}$. In addition to the anomaly in $SU(n)$, one also has

$$\partial_\mu j^\mu_{U(1)} = -\mathrm{i}\frac{q^3 \dim(\mathbf{r})}{32\pi^2}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} - \mathrm{i}\frac{qC(\mathbf{r})}{64\pi^2}\varepsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}, \tag{1.40}$$

where $F_{\mu\nu}, F^a_{\mu\nu}$ are $U(1)$ and $SU(n)$ gauge fields. Therefore, if the whole G is gauged and the fermion content is generalised so that it can be described by a set of indices $\{\ell(q,\mathbf{r})\}$ for each representation $\mathbf{r}_q$, all the anomalies cancel when the following conditions are satisfied:

$$\begin{aligned} \mathscr{A}_{U(1)} &= \sum \ell(q,\mathbf{r})\dim(\mathbf{r})q^3 = 0 \\ \mathscr{A}_{SU(n)} &= \sum \ell(q,\mathbf{r})A(\mathbf{r}) = 0 \\ \mathscr{A}_{\text{mixed}} &= \sum \ell(q,\mathbf{r})\,qC(\mathbf{r}) = 0. \end{aligned} \tag{1.41}$$

If one takes the existence of gauge anomalies from a chiral fermion as fundamental, then the ABJ anomaly is properly interpreted as a mixed anomaly. In our original example we have a Dirac fermion Ψ which can be decomposed into one left-handed and one right-handed Weyl fermions, denoted by ψ_L and ψ_R. The symmetry group is $G = U(1)_V \times U(1)_A$ where $U(1)_V$ is the gauge group and $U(1)_A$ is a global symmetry group (at least at the classical level). The charges of the fermions are given by

	$U(1)_V$	$U(1)_A$
ψ_L	$+1$	$+1$
ψ_R	$+1$	-1

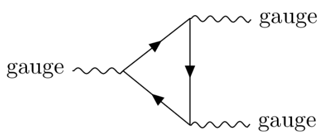
It is easy to see that the gauge anomaly coefficient $\mathscr{A}_V$ vanishes. There is an ’t Hooft anomaly in $U(1)_A$ because $\mathscr{A}_A \neq 0$ so it cannot be promoted to a gauge symmetry. But most importantly, there is a mixed anomaly

$$\mathscr{A}_{\text{V–A}} = \sum \ell(q_V, q_A)\, q_A q_V^2 = 1\cdot(+1)\cdot(+1)^2 + (-1)\cdot(-1)\cdot(+1)^2 = 2, \tag{1.42}$$

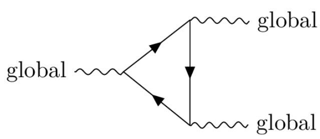
which means the axial current is not conserved even when we turn off the background gauge field for $U(1)_A$. Therefore $U(1)_A$ is not an actual symmetry of the theory at the quantum level.

Since there are many guises of anomalies depending on whether the symmetry group is gauged or not, it can be overwhelming reading about one after another in a long sequence. To aid the reader, here we provide a summary of consequences for each type of anomalies we have discussed so far. Suppose the symmetry group of a theory is a product between the gauge group and the global symmetry group, with currents $j^{a\mu}$ and $j^{\alpha\mu}$. Here the Latin indices belongs to the gauge group generators and the Greek indices belong to the global symmetry group generators. Then the anomalies that we have discussed so far lead to one of the following three consequences.

1. **Gauge anomaly**:
$$\partial_\mu j^{a\mu} \;\supset\; -\mathrm{i}\frac{d^{abc}}{32\pi^2}\,\varepsilon^{\mu\nu\rho\sigma}F^b_{\mu\nu}F^c_{\rho\sigma}. \tag{1.43}$$
If this anomaly is present, we have a breakdown of gauge invariance which cannot be compromised. Therefore, the net gauge anomaly must vanish for a theory to be mathematically consistent. The relevant triangle diagrams which reproduces this anomaly couple to the gauge currents at all corner as shown schematically right below. Note that we include the triangle diagram here only as a mnemonic for the symmetric tensor d^{abc} where each index corresponds to each vertex of the diagram.



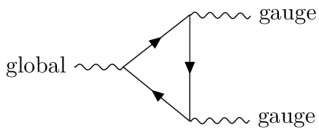
2. **'t Hooft anomaly**:
$$\partial_\mu j^{\alpha\mu} \;\supset\; -\mathrm{i}\frac{d^{\alpha\beta\gamma}}{32\pi^2}\,\varepsilon^{\mu\nu\rho\sigma}F^\beta_{\mu\nu}\wedge F^\gamma_{\rho\sigma}. \tag{1.44}$$
If this anomaly is present, the global symmetry F cannot be gauged, but it is still a good symmetry at the quantum level when the background field is turned off. The relevant triangle diagrams have global currents at the three corners as shown schematically below.



3. **ABJ anomaly**, or simply anomaly:

$$\partial_\mu j^{\alpha\mu} \supset -\mathrm{i}\frac{d^{\alpha bc}}{32\pi^2}\varepsilon^{\mu\nu\rho\sigma}F^b_{\mu\nu}F^c_{\rho\sigma}. \tag{1.45}$$

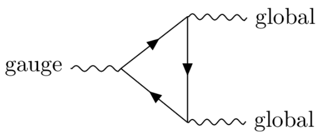
This is an anomaly in the earliest sense. The global current $j^{\alpha\mu}$ is never conserved because the gauge field strength fluctuates due to quantum effects and cannot be turned off. The classical global symmetry group therefore ceases to be a symmetry at the quantum level. The triangle diagrams reproducing this type of anomalies have the form



The same triangle diagrams also contribute to another term in the violation of the gauge current $j^{a\mu}$ in the form

$$\partial_\mu j^{a\mu} \supset -\mathrm{i}\frac{\mathrm{tr}\,t^\alpha}{32\pi^2}\varepsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^\alpha_{\rho\sigma}, \tag{1.46}$$

which suggests that gauge invariance is broken when the gauge field for F is turned on, providing a kind of 't Hooft anomalies. However, this is redundant because the global symmetry group that would have this 't Hooft anomaly is not existent thanks to the ABJ anomaly from the same triangle diagrams.



There is also another possibility that we have not discussed before which corresponds to the triangle diagrams shown schematically on the left. This gives rise to the nonconservation of the gauge current of the form

$$\partial_\mu j^{a\mu} \supset -\mathrm{i}\frac{d^{a\alpha\beta}}{32\pi^2}\varepsilon^{\mu\nu\rho\sigma}F^\alpha_{\mu\nu}\wedge F^\beta_{\rho\sigma}. \tag{1.47}$$

When this anomaly is present, the symmetry group F is a good symmetry if the background field is turned off, provided the global symmetry is not broken by the ABJ anomaly. However, it is not an 't Hooft anomaly in the usual sense; we cannot even couple a background field to the global symmetry without breaking gauge invariance.

Gravitational anomaly

One can also study quantum field theories on a non-trivial spacetime manifold. When the matter content contains Weyl fermions, general covariance could be spoiled, preventing us from making the metric dynamical. This is usually referred to as the *gravitational anomaly* [8, 151]. Although pure gravitational anomalies are present only when the spacetime dimension is 2 mod 4, there can be a mixed *gauge-gravitational anomaly* if the fermions are also coupled to a $U(1)$ symmetry [54, 56]. In the simplest case with only a systems of a signle left-handed Weyl fermion with a $U(1)$ global symmetry, the nonconservation is given by

$$\partial_\mu j^\mu = -\frac{\mathrm{i}}{384\pi^2}\frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}R_{\mu\nu\sigma\tau}R_{\alpha\beta}{}^{\sigma\tau}. \tag{1.48}$$

So $U(1)$ is not even a symmetry when we couple the theory to nontrivial gravity background. In flat space it is a good global symmetry but we are forbidden from gauging it because of the presence of an 't Hooft anomaly. Therefore, a $U(1)$ gauge theory with n left-handed Weyl fermions charged $q_1, q_2, \dots, q_n$ under the $U(1)$ gauge group is well-defined as a quantum gauge theory on general gravitational background when both the $U(1)$ gauge anomaly and the mixed $U(1)$-gravity anomaly vanish:

$$\sum_{i=1}^{n} q_i^3 = \sum_{i=1}^{n} q_i = 0. \tag{1.49}$$

1.1.2 Global anomalies

As has been shown in Section 1.1.1, anomalies arise because one encounters obstruction to defining the path integral $\det\left(\mathrm{i}\not{D}\right)$ consistently as a function over the space of gauge fields modulo gauge transformations. What we have seen so far are local obstructions that can be detected in perturbation theory. These obstructions can be global, that is, they appear from the global structure of the guage group and cannot be detected perturbatively. For example, an $SU(2)$ gauge theory with one chiral fermion in the fundamental representation is mathematically inconsistent just like in a theory with gauge group $SU(N)$ for N greater than 2, despite the vanishing rank 3 invariant tensor d^{abc}.

Consider a chiral fermion coupled to a gauge group G whose gauge field is denoted by a. The Euclidean partition function is given by

$$\mathscr{Z} = \int [\mathrm{d}a][\mathrm{d}\psi][\mathrm{d}\bar{\psi}]\mathrm{e}^{-\int \mathrm{d}^4x \left(\frac{1}{2g^2}\operatorname{tr} f_{\mu\nu}f^{\mu\nu} + \bar{\psi}\mathrm{i}\not{D}\psi\right)} = \int [\mathrm{d}a]Z[a]. \tag{1.50}$$

Since a gauge transformation $g : \mathbb{R}^4 \to G$ goes to the trivial transformation as $|x| \to \infty$, it can also be thought as a map from S^4, the 4-sphere which is a one-point compactification of $\mathbb{R}^4$, to G. Maps $g : S^4 \to G$ are classified by the fourth homotopy group $\pi_4(G)$. When the gauge group G is $SU(2)$, its fourth homotopy group is non-trivial ($\pi_4(SU(2)) = \mathbb{Z}_2$), so there exists a gauge transformation $U(x)$ which is not homotopically connected to the identity and cannot be reached by successive infinitesimal gauge transformations. Witten found that if the fermion is in the fundamental representation of $SU(2)$, then the partial partition function flips sign under such a nontrivial gauge transformation [149]

$$Z[a^U] = -Z[a], \tag{1.51}$$

where $a^U = U^{-1}aU - \mathrm{i}U^{-1}\mathrm{d}U$ is the gauge field after being gauged-transformed by U. Therefore, the full partition function vanishes and all correlation functions are ill-defined. This gives rise to an anomaly which does not show up at any order in perturbation theory. Again, the anomaly arises from failure to define the fermion path integral consistently for all gauge field configurations.

It is more instructive to define a path integral for Weyl fermions starting from a path integral for a Dirac fermion. Let Ψ be a Dirac fermion transforming in the doublet of the gauge group $SU(2)$ with the gauge field a_μ. Then the fermion path integral is

$$Z_{\mathrm{Dirac}}[a] = \int [\mathrm{d}\overline{\Psi}][\mathrm{d}\Psi] \exp\left(-\int \mathrm{d}^4x\,\overline{\Psi}\mathrm{i}\not{D}[a]\Psi\right), \tag{1.52}$$

where the Dirac operator twisted with the gauge field a_μ is $\mathrm{i}\not{D}[a] = \mathrm{i}\gamma^\mu(\partial_\mu - \mathrm{i}a_\mu)$. Defining the measure as before, the path integral is then given by the formal product of all eigenvalues. Moreover, note that since the Dirac operator is hermitian and anti-commute with γ_5, the eigenvalues come in pairs of positive and negative real numbers $\pm\lambda_n$ with $n \in \mathbb{Z}^+$. To fix the convention we order the eigenvalues by $\lambda_1 \leq \lambda_2 \leq \ldots$. Therefore, we can define the path integral (at least formally) as

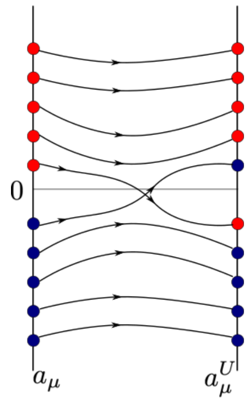
$$Z_{\mathrm{Dirac}}[a] = \prod_{n\geq 1} \lambda_n^2. \tag{1.53}$$

Now since our system of Weyl fermion in the doublet of $SU(2)$ is precisely half of the Dirac fermion system (by throwing out the right-handed components, for example). So we might think that we can define the path integral $Z_{\text{Weyl}}[a]$ as $(Z_{\text{Dirac}}[a])^{1/2}$, that is, as product of the positive eigenvalues

$$Z_{\text{Weyl}}[a] = \prod_{n\geq 1} \lambda_n, \tag{1.54}$$

without loss of generality. However, it turns out that there is no canonical prescription to pick the sign of the square root consistently for all gauge field configurations.

As we vary a_μ adiabatically into a_μ^U, the eigenvalues flow and rearrange in such a way that the last set of eigenvalues are exactly the same as in the beginning. It is possible that the flow of a positive eigenvalue crosses zero an odd number of times, that is, the mode associated with this positive eigenvalue in the beginning under $\mathrm{i}\not{D}[a]$ but now have negative eigenvalue under $\mathrm{i}\not{D}[a^U]$ as the figure on the right illustrates. The partition function changes sign if there are an odd number of positive modes that behave like this. Indeed, this is the case, as we demonstrate momentarily.



We realise the adiabatic variation of the gauge field configuration as follows. Construct a 5-dimensional cylinder $S^4 \times \mathbb{R}$ with $\tau \in \mathbb{R}$ parametrising the flow. We embed our 4-dimensional gauge field in the 5-dimensional gauge field A_i, $i = 1,\ldots,5$, such that A_τ always vanishes and $A_\mu = a_\mu$ or $A_\mu = a_\mu^U$ as $t \to -\infty$ or $t \to \infty$, respectively. This construction is usually called the *mapping torus* because we can identify the gauge field configuration at both ends through the gauge transformation U and define it on a 5-torus $S^4 \times S$ constructed from $S^4 \times \mathbb{R}$ by identifying $\tau = -$ with $\tau = \infty$. The number of zero-crossing of the eigenvalues can be computed in terms of the number of zero modes modulo 2 (called the mod-2 index) of a five-dimensional Dirac operator, *viz.*, solutions to the Dirac equation

$$\not{D}^{(5)}\Psi = \gamma^i(\partial_i - \mathrm{i}A_i)\Psi = 0. \tag{1.55}$$

Since A_i varies very slowly in τ, we can solve this via adiabatic approximation. If we write $\Psi = F(\tau)\phi^\tau(x^\mu)$ where $\phi^\tau(x^\mu)$ is an instantaneous mode of the 4-dimensional Dirac operator $\gamma^\tau\not{D}$ with eigenvalue $\lambda(\tau)$, then the Dirac equation $\not{D}^{(5)}\Psi = 0$ becomes $\mathrm{d}F/\mathrm{d}\tau = -\lambda(\tau)F(\tau)$ in this approximation. The solution is simply

$$F(\tau) = F(0)\exp\left(-\int_0^\tau \mathrm{d}s\,\lambda(s)\right), \tag{1.56}$$

which is normalisable only when $\lambda(\infty) > 0$ and $\lambda(-\infty) < 0$. The number of zero modes of $\not{D}^{(5)}$ is then given by the number of eigenvalues of $\mathrm{i}\not{D}$ that flow from negative to positive. BY symmetry, this is also the number of eigenvalues of $\mathrm{i}\not{D}$ that flows from positive to negative. So there is a one-to-one correspondence between the number of eigenvalue zero-crossings and the number of zero modes of $\not{D}^{(5)}$ in the adiabatic approximation. When corrections are included, the statement remains true only modulo 2.

The reason that only the number of zero modes modulo 2 is a topological invariant is as follows. The 5-dimensional Dirac operator $\not{D}^{(5)}$ is real and antisymmetric, so its eigenvalues are either zero or come in pairs of complex conjugate imaginary numbers. Therefore, as we vary the gauge field the nonzero eigenvalues can vanish only in pairs.

The mod 2 version of the Atiyah–Singer index theorem tells us that the mod-2 index of the Dirac operator $\not{D}^{(5)}$ is 1 mod 2. Therefore, there is an odd number of zero-crossings and the fermion path integral $Z_{\text{Weyl}}[a]$ changes sign, giving rise to a $\mathbb{Z}_2$ non-perturbative anomaly usually called the *Witten anomaly*.

However, the mapping torus argument presented above is not sufficient to take subtler global anomalies into account. Recent developments in condensed matter physics, especially the connection between symmetry-protected topological (SPT) phases in the bulk and global anomalies of the boundary theory show that one needs a method more refined than the mapping torus to take the role played by interactions into account. This is provided by the Dai-Freed theorem [51] and classification of SPT phases by cobordism [91]. In this language, when the perturbative anomaly vanishes, the global anomaly is given by a homomorphism from a bordism group $\Omega_5^H(BG)$, a certain mathematical object calculated from the underlying manifold structure H and gauge group G, to the phase $U(1)$. This theme will be explored in Chapter 3.

1.2 Dynamical Constraints from Anomalies

One can utilise the presence of anomalies in the global symmetry group of a quantum field theory to put strong constraints on the low energy effective description of the system. Known as *'t Hooft anomaly matching* [130], this technique is particularly powerful when the quantum field theory under consideration is strongly coupled since we have very little control of the dynamics.

Consider a strongly coupled gauge theory with gauge group G_s and global symmetry group G_F at high energy. It is possible that there is an anomaly $\mathscr{A}_{G_F}$ in G_F when we couple it to a background gauge field. This is the obstruction to gauging the symmetry, *viz.* to making the gauge field dynamical. This obstruction would be removed if one were to add auxiliary chiral massless fermions coupled only to G_{UV} in such a way that the anomaly contributed by these *spectator* fermions were exactly $-\mathscr{A}_{\mathrm{UV}}$. Let us now go along the RG flow to the low energy limit and ask what is the fate of the symmetry G_{UV}.

Since we can make the gauge coupling for G_F arbitrarily small such that it does not affect the dynamics of the original strongly coupled sector, the most obvious choice for the effective theory in the IR is a G_F gauge theory with the same spectator fermions. There must be new massless bound states charged under G_F emerging from the original degrees of freedom so that there is no gauge anomaly in G_F.

It is possible that one ends up unable to construct massless bound states that give a matching anomaly. If this is the case, then the assumption that G_F is the gauge group in the low energy theory might not be correct. In fact, G_F can be spontaneously broken.

One could also have a gapped, though non-trivial, system saturating the anomaly in the IR. This comes in the form of a TQFT, though it appears to be much rarer for this to happen compared to the previous two possibilities. For instance, [50] shows that in a certain class of theories, a TQFT with a particular anomaly does not exist if certain conditions are not satisfied.

1.2.1 Chiral symmetry breaking vs massless baryons

One of the clearest examples goes back to 't Hooft's original work in 1979. In [130], he studied low energy effective theory of QCD and used anomaly matching to rule out the massless baryon phase and show that there the global symmetry is spontaneously broken by the quark condensate.

In Nature, the up, the down, and the strange quarks, are so much lighter than the rest. Therefore it is a good idea to study a model where the three quark states are almost degenerate with an approximate flavour symmetry transforming them into one another. This is QCD with gauge group $SU(3)$ and 3 massless quarks. The global symmetry at the classical level is clearly $U(3)_L \times U(3)_R$, which transform the left-handed and right-handed components separately. However, the axial $U(1)$ symmetry has an anomaly as described earlier. So the global anomaly at the quantum level is simply

$$G_{\mathrm{UV}} = SU(3)_L \times SU(3)_R \times U(1)_V. \tag{1.57}$$

In general when there are n quarks, the global symmetry is given by

$$G_{\mathrm{UV}} = SU(n)_L \times SU(n)_R \times U(1)_V. \tag{1.58}$$

As mentioned in a general discussion above, the simplest way to saturate the 't Hooft anomalies is through massless bound states in the IR without any symmetry breaking. In the current problem the relevant fermionic bound states are baryons which are gauge invariant combinations of three quarks. Let us first analyse this scenario.

In the UV, there are three left-handed quarks (these are three colour components) each in the $\mathbf{n}$ representation of $SU(n)_L$ and also three right-handed quarks in the $\mathbf{n}$ representation of $SU(n)_R$, each with charge $+1$ under $U(1)_V$. Therefore, the pure $SU(n)_L$ anomaly and the mixed $[SU(n)_L]^2 - U(1)$ anomaly are given by

$$\begin{aligned} \mathscr{A}^{L}_{\text{UV}} &= 3A(\mathbf{n}) = 3 \\ \mathscr{A}^{L\times U(1)}_{\text{UV}} &= 3C(\mathbf{n}) = 3. \end{aligned} \tag{1.59}$$

Since there can be no massless particles charged under global symmetry with helicity greater than $\frac{1}{2}$ due to the Weinberg-Witten theorem [147], the only contributions to the anomalies are from the spin-1/2 representation, which comes in two chiralities. As the left-handed quarks are in the fundamental representation of $SU(n)_L$ and the right-handed quarks are in the fundamental of $SU(n)_R$, so naively the representations of $SU(n)_L \times SU(n)_R$ that are left-handed must contain an odd number of left-handed quarks:

$$\begin{array}{|c|c|c|}\hline L & L & L \\ \hline \end{array}\,, \qquad \begin{array}{|c|}\hline L \\ \hline L \\ \hline L \\ \hline \end{array}\,, \qquad \begin{array}{|c|}\hline L \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline R & R \\ \hline \end{array}\,, \qquad \begin{array}{|c|}\hline L \\ \hline \end{array} \otimes \begin{array}{|c|}\hline R \\ \hline R \\ \hline \end{array}\,, \qquad \begin{array}{|c|c|}\hline L & L \\ \hline L & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\,, \tag{1.60}$$

However, these representations can be dressed with gluons, which shift the helicity by ± 1, and we cannot be certain whether each species of the representaions listed above is left-handed or right-handed. To be on the safe side, we allow this possibility by using indices, the numbers of left-handed representaions minus the number of right-handed representations, to label these representations. As one can gap away a pair of left-handed and right-handed fermions through the Dirac mass term, a certain representation must contribute to the anomaly only through its index which is invariant under the gapping process. Let us call the indices ℓ_1, ℓ_2, ℓ_3, ℓ_4, and ℓ_5, respectively, for the 5 representations above.

There are more representations of $SU(n)_L \times SU(n)_R$ consisting of 4 quarks. They can be obtained by parity transformation, swapping left-handedness and right-handedness:

$$\begin{array}{|c|c|c|}\hline R & R & R \\ \hline \end{array}\,, \qquad \begin{array}{|c|}\hline R \\ \hline R \\ \hline R \\ \hline \end{array}\,, \qquad \begin{array}{|c|}\hline R \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline L & L \\ \hline \end{array}\,, \qquad \begin{array}{|c|}\hline R \\ \hline \end{array} \otimes \begin{array}{|c|}\hline L \\ \hline L \\ \hline \end{array}\,, \qquad \begin{array}{|c|c|}\hline R & R \\ \hline R & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\,, \tag{1.61}$$

with indices $-\ell_1$, $-\ell_2$, $-\ell_3$, $-\ell_4$, $-\ell_5$, respectively, assuming that parity is not spontaneously broken.

Let us first work out the pure $SU(n)_L$ anomaly in the infrared. This is given by

$$\begin{aligned}
\mathscr{A}^L_{\mathrm{IR}} &= \ell_1 A\left(\boxed{L}\boxed{L}\boxed{L}\right) + \ell_2 A\left(\begin{smallmatrix}\boxed{L}\\ \boxed{L}\\ \boxed{L}\end{smallmatrix}\right) + \ell_3\left[\dim\left(\boxed{R}\boxed{R}\right)A\left(\boxed{L}\right) - \dim\left(\boxed{R}\right)A\left(\boxed{L}\boxed{L}\right)\right] \\
&\quad + \ell_4\left[\dim\left(\begin{smallmatrix}\boxed{R}\\ \boxed{R}\end{smallmatrix}\right)A\left(\boxed{L}\right) - \dim\left(\boxed{R}\right)A\left(\begin{smallmatrix}\boxed{L}\\ \boxed{L}\end{smallmatrix}\right)\right] + \ell_5 A\left(\begin{smallmatrix}\boxed{L}\boxed{L}\\ \boxed{L}\end{smallmatrix}\right) \\
&= \frac{1}{2}\left((n+3)(n+6)\ell_1 + (n-3)(n-6)\ell_2\right) - \frac{1}{2}\left(n(n+7)\ell_3 + n(n-7)\ell_4\right) + \ell_5(n^2-9).
\end{aligned} \tag{1.62}$$

Similarly, the mixed anomaly is given by

$$\begin{aligned}
\frac{\mathscr{A}^{\mathrm{mixed}}_{\mathrm{IR}}}{3} &= \ell_1 C\left(\boxed{L}\boxed{L}\boxed{L}\right) + \ell_2 C\left(\begin{smallmatrix}\boxed{L}\\ \boxed{L}\\ \boxed{L}\end{smallmatrix}\right) + \ell_3\left[\dim\left(\boxed{R}\boxed{R}\right)C\left(\boxed{L}\right) - \dim\left(\boxed{R}\right)C\left(\boxed{L}\boxed{L}\right)\right] \\
&\quad + \ell_4\left[\dim\left(\begin{smallmatrix}\boxed{R}\\ \boxed{R}\end{smallmatrix}\right)C\left(\boxed{L}\right) - \dim\left(\boxed{R}\right)C\left(\begin{smallmatrix}\boxed{L}\\ \boxed{L}\end{smallmatrix}\right)\right] + \ell_5 C\left(\begin{smallmatrix}\boxed{L}\boxed{L}\\ \boxed{L}\end{smallmatrix}\right) \\
&= \frac{1}{2}\left((n+3)(n+2)\ell_1 + (n-3)(n-2)\ell_2\right) - \frac{1}{2}\left(n(n+3)\ell_3 + n(n-3)\ell_4\right) + \ell_5(n^2-3).
\end{aligned} \tag{1.63}$$

When n is a multiple of three, say $n = 3p$, the mixed anomaly matching conditions read

$$\begin{aligned}
1 &= \frac{9}{2}\left((p+1)(p+2)\ell_1 + (p-1)(p-2)\ell_2\right) - \frac{3p}{2}\left((3p+7)\ell_3 + (3p-7)\ell_4\right) \\
&\quad + 9(p^2-1)\ell_5,
\end{aligned} \tag{1.64}$$

which is impossible to satisfied because the right-hand side is an integral multiple of three.

More constraints are needed when n is not a multiple of three. These can be supplied from the *persistent mass condition*: if an elementary fermion gets any mass, then there should be no unbroken chiral symmetry preventing bound states containing this fermion from becoming massive [114]. In the present situation, the composite fermions containing the massive quark now transform in either the totally symmetric representation $\boxed{L}\boxed{L}$ or the totally antisymmetric representation $\begin{smallmatrix}\boxed{L}\\ \boxed{L}\end{smallmatrix}$ (and their parity conjugates) under the new chiral symmetry $SU(n-1)_L \times SU(n-1)_R \times U(1)$. Since a mass term can only form from a pair of left-handed and right-handed with the same quantum numbers, we need the indices of both representations to vanish in order to satisfy the persistent mass condition:

$$\ell\left(\boxed{L}\boxed{L}\right) = \ell_1 - \ell_3 + \ell_5 = 0, \tag{1.65}$$

$$\ell\left(\begin{smallmatrix}\boxed{L}\\ \boxed{L}\end{smallmatrix}\right) = \ell_2 - \ell_4 + \ell_5 = 0. \tag{1.66}$$

Solving these two new conditions together with the anomaly matching equations, $\mathscr{A}^L_{\text{IR}} = \mathscr{A}^L_{\text{UV}}$ and $\mathscr{A}^{\text{mixed}}_{\text{IR}} = \mathscr{A}^{\text{mixed}}_{\text{UV}}$, one obtains

$$\ell_1 = \ell_2 = \ell, \quad \ell_3 = \ell_4 = -\frac{1}{3} + 3\ell, \quad \ell_5 = -\frac{1}{3} + 2\ell, \tag{1.67}$$

with ℓ still undetermined. However, it is impossible to choose ℓ such that all indices are integer-valued. Hence the anomalies cannot be saturated by massless baryons when $n > 2$.

Therefore, when $n > 2$, one or more assumptions leading to the massless baryons scenario must be incorrect. Indeed, to end up with the massless baryons phase in the IR we have to assume that the global symmetry is not broken spontaneously; this process is not forbidden by any means and is a valid possibility.

The anomaly matching consideration reveals that the phase where the global symmetry remains unbroken in the IR is unattainable, since there is no way to match the 't Hooft anomaly in the UV with any massless bound states. Therefore, it must be spontaneously broken. The mechanism is provided by the quark condensation where the strong interaction binds the quarks and anti-quarks together. The condensate develops a vacuum expectation value (VEV) when the energy is well below the spontaneously generated energy scale Λ_{QCD},

$$\left\langle \bar{\Psi}_i \cdot \Psi_j \right\rangle \sim \Lambda^3_{\text{QCD}} \delta_{ij}, \tag{1.68}$$

which spontaneously breaks the global symmetry $SU(n)_L \times SU(n)_R \times U(1)$ down to its diagonal $SU(n)_{\text{diag}} \times U(1)$. There are no massless baryons as the quarks become massive through the condensate. The anomaly matching for the resulting global symmetry is satisfied: there is an equal number of left-handed quark and right-handed quarks coupled to the diagonal $SU(n)_{\text{diag}}$ global symmetry group in the UV.

Things are dramatically different, however, when $n = 2$. Due to the shortness of the $SU(2)$ representations, the left-handed bound state representations comprising of three quarks are the following three representations,

$$\begin{array}{|c|c|c|}\hline L & L & L \\ \hline\end{array}\,, \qquad \begin{array}{|c|}\hline L \\ \hline\end{array} \otimes \begin{array}{|c|c|}\hline R & R \\ \hline\end{array}\,, \qquad \begin{array}{|c|}\hline L \\ \hline\end{array} \otimes \begin{array}{|c|}\hline R \\ \hline R \\ \hline\end{array} = \begin{array}{|c|}\hline L \\ \hline\end{array} = \begin{array}{|c|c|}\hline L & L \\ \hline L & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \tag{1.69}$$

only, with corresponding indices ℓ_1, ℓ_2, ℓ_3 (with similar representations and indices for their parity conjugates). The anomaly matching conditions and the persistent mass condition become

$$20\ell_1 - 9\ell_2 + \ell_3 = 3, \tag{1.70}$$

$$10\ell_1 - 5\ell_2 + \ell_3 = 1, \tag{1.71}$$

$$\ell_1 - \ell_2 + \ell_3 = 0 \tag{1.72}$$

with the integral solution $\ell_1 = \ell_2 = 0$, $\ell_3 = 1$. Therefore, the 't Hooft anomaly matching constraints are not powerful enough to rule out the existence of the massless baryon phase when there are only 2 Dirac fermions. However, it is believed that the spontaneous chiral symmetry breaking scenario is more likely to occur based on evidence from lattice simulations (for instance, [60]).

1.2.2 Recent Developments

In recent years, more creative use of anomalies in constraining the dynamics of strongly coupled field theory has emerged. One particular work which seems to re-ignite interest in this sub-field is [71] by Gaiotto, Kapustin, Komargodski, and Seiberg. They make use of anomalies between time-reversal symmetry and higher-form symmetries to study the phase space of pure Yang-Mills with gauge group $SU(N)$ at finite temperature. More works in the same direction exploiting anomalies between discrete 1-form global symmetries, whose charged operators are line operators instead of local operators, and ordinary global symmetries to analyse the phase diagrams of Yang-Mills theory with matter content include [14, 5, 11, 110, 12, 13].

Constraints from anomalies are also found to be of use in other aspects of QCD. For instance, even though 't Hooft's original anomaly matching argument could not rule out the massless baryon phase in QCD when there are 2 generations of quarks, but one can use techniques of gauging higher-form symmetries to obtain a more refined 't Hooft anomaly matching condition (by taking discrete groups and quotients into account) and rule out the massless baryon phase in certain $SU(N)$ chiral gauge theories when N is even [39, 36, 38, 37]. Another example involves QCD at finite density [89, 87, 70], which is relevant in astrophysical studies such as the composition of neutron stars.

1.3 Outline

Hopefully, the discussion above should make it clear that the understanding of anomalies is essential in the study of quantum field theory, especially for strongly interacting theories where we have few handles on its dynamics. One can roughly divide the study of anomalies in quantum field theory into two stages. The first stage is to determine an anomaly in a given gauge theory and analyse its properties. The next stage is to analyse different phases of the theory and use the derived anomaly to constrain these phases. My work in this dissertation contains both of these stages.

Chapter 2 introduces the Standard Model of particle physics. Since it is a chiral gauge theory, various gauge and gravitational anomalies must cancel. We describe the intricacy of this cancellation and how it fixes the hypercharges of the Standard Model fermions. The Chapter concludes with a new perspective on anomaly cancellation in the Standard Model. Here I demonstrate how powerful the constraints can be. Assuming only the hypercharge quantisation, I show that the hypercharge assignment of the Standard Model fermions can be obtained solely from the gauge anomaly cancellation with a little help from Fermat's Last Theorem.

Having dealt with the perturbative anomaly cancellation, we need to worry about the global anomaly. In Chapter 3, I study the possibility of global anomalies in a variety of beyond Standard Model gauge theories through cobordism. The Standard Model Lagrangian is sensitive only to the Lie algebra $\mathfrak{g} = \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ of many possibilities of the gauge group G, while the matter content is consistent with four choices of the gauge group with this Lie algebra: $G = \tilde{G}/\Gamma$ where $\tilde{G} = SU(3) \times SU(2) \times U(1)$ and $\Gamma \in \{\mathbf{1}, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_6\}$. If the matter content is fixed, we can fit the theory with the gauge group $\tilde{G}/\mathbb{Z}_6$ in a grand unified theory with gauge group $SU(5)$ and show that there is no global anomaly. However, for beyond the Standard Model model-building, this type of global anomaly analysis is unsatisfactory because it does not allow us to alter the matter content of the theory. Therefore, it is necessary to determine whether a theory with a given gauge group can have a global anomaly regardless of the matter content. I use the Atiyah-Hirzebruch spectral sequence to calculate the bordism group $\Omega_5^{\text{Spin}}(BG)$ and show that there are no other global anomalies to worry about apart from the expected $\mathbb{Z}_2$ anomaly whenever the factor $SU(2)$ is present. I then extend the calculation to other gauge groups beyond the four choices of the Standard Model that are of interest to phenomenology.

In Chapter 4, I explore the interplay between local and global anomalies in $U(2)$ gauge theory. The fact that an $SU(2)$ gauge theory with a single fermion in the fundamental representation of the gauge group has a $\mathbb{Z}_2$ anomaly is also reflected by the fifth spin-bordism group $\Omega_5^{\text{Spin}}(BSU(2)) = \mathbb{Z}_2$. Therefore, in model building, there must be an even number of Weyl fermions if the gauge group contains the $SU(2)$ factor. When one similarly calculates the fifth spin-bordism group when the gauge group is $U(2)$ one finds that it vanishes, which suggests that there is no global anomaly. However, in practice application, one still requires an even number of Weyl fermions which are in the fundamental of the $SU(2)$ part of the gauge group. I show that the requirement is correct, but it comes from perturbative cancellation of the mixed $[SU(2)^2] \times U(1)$ anomaly rather than the global anomaly associated with $SU(2)$.

One can also study a similar system with fermions on a non-spin manifold like $\mathbb{CP}^2$ using the spin-$U(2)$ structure. It is found that $\Omega_5^{\text{Spin}-U(2)} = \mathbb{Z}_2$, which should be the analogue of the new $SU(2)$ anomaly discovered by Wang, Wen, and Witten [143]. Surprisingly, requiring that the mixed anomaly between $U(1)$ and $SU(2)$ factors and the mixed $U(1)$-gravity anomaly vanish is enough to ensure that this new $SU(2)$ anomaly cannot arise. The reason that the bordism group is non-trivial is because the transformation giving rise to the new anomaly involves a diffeomorphism of the underlying manifold and cannot be undone with a pure $U(1)$ gauge transformation.

In Chapter 5, I combine constraints from anomaly cancellation and other non-perturbative methods to study the quantum phase structure in variations of the Standard Model when the strengths of the strong and weak interactions are varied. The variations of the Standard Model studied in this chapter are all flavour-symmetric: there is no mixing between the fermions of different generations. I find that when there is only one generation of fermions present, there is no evidence of a phase transition as the relative strength between the strong and the weak interactions, given by the ratio of the $SU(3)$-generated scale Λ_{strong} and the $SU(2)$-generated scale Λ_{weak}, is varied. The spontaneous symmetry breaking patterns are the same on both sides of the phase diagram and the degrees of freedom in the IR have the same quantum numbers, even though they descend from different fermions in the UV. For example, in a variation with the hypercharge $U(1)$ factor in the gauge group, the symmetry group in the IR is a product of a $U(1)$ gauge group and a $U(1)$ global symmetry group (which can be identified with the usual $U(1)$ electromagnetism and the $\mathrm{B}-\mathrm{L}$ symmetry in the regime when $\Lambda_{\text{strong}} \gg \Lambda_{\text{weak}}$). The gapless fermionic degrees of freedom on one side comes from the left-handed neutrino, while it comes from a component of the right-handed down quarks on the other side of the phase diagram. However, it is impossible to distinguish them because they have identical quantum numbers under the IR symmetry group. On the other hand, the symmetry breaking patterns are different on both sides of the phase diagram when there is more than one generation of fermions. Therefore, there must be a (possibly first order) phase transition somewhere when the strong interaction and the weak interaction have roughly the same strength.

Finally, Chapter 6 brings the thesis to its conclusion. A few interesting questions that arise from various lines of investigation in this dissertation are discussed to provide a basis for further research in the future.

Chapter 2

Anomaly Cancellation in the Standard Model

Apart from being rich in interesting mathematical structures such as fibre bundles, strongly coupled gauge theory has a very important place in theoretical physics as the essential part of the Standard Model of particle physics. This model, which represented our most up-to-date understanding of the fundamental building blocks of the universe is described using the language of strongly interacting field theory. In particular, it is a Yang-Mills theory with a tightly constrained matter content. In this Chapter, we will review the basics of the Standard Model, how the fermion matter content cancel all possible gauge anomalies (including the mixed gauge-gravitational anomaly). Finally, we will show that the hypercharge assignments of the fermions is essentially the only possible choice if we impose hypercharge quantisation.

2.1 The Standard Model of Particle Physics

The Standard Model is a gauge theory with gauge group $G = SU(3) \times SU(2) \times U(1)_Y$ (in fact, the only thing we need here is that the Lie algebra of G is a direct sum $\mathfrak{g} = \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$. Subtleties involving the global structure of the possible gauge groups with the same Lie algebra $\mathfrak{g}$ given the fermion matter content of the Standard Model will be dealt with in Chapter 3 when we discuss global anomalies in the Standard Model). Note that the $U(1)$ gauge group appearing in G is not the same as the electromagnetic $U(1)$ that we see in low energy. To distinguish it it is customary to call it the *hypercharge* $U(1)$, denoted by the subscript Y. We also call the quantum number of a field charged under $U(1)_Y$ by the name hypercharge, denoted by Y.

The matter content of the Standard Model consists of three generations of fermions and a $SU(2)$ doublet of scalar fields. Each generation of fermions contains the left-handed quarks q_L^i, the left-handed leptons l_L^i, the right-handed quarks u_R^i and d_R^i, the right-handed electron e_R^i, and the right-handed neutrino ν_R^i. The index i running from 1 to 3 labels the generation. Each generation transforms under the gauge group as follows

	$SU(3)$	$SU(2)$	Y
q_L	$\mathbf{3}$	$\mathbf{2}$	$+1/6$
l_L	$\mathbf{1}$	$\mathbf{2}$	$-1/2$
u_R	$\mathbf{3}$	$\mathbf{1}$	$+2/3$
d_R	$\mathbf{3}$	$\mathbf{1}$	$-1/3$
e_R	$\mathbf{1}$	$\mathbf{1}$	-1

The doublet of scalar fields ϕ, called the Higgs field, transforms in the representation $(\mathbf{1},\mathbf{2})_{+1/2}$ of the gauge group. It is coupled to the fermion sector through the *Yukawa Lagrangian*:

$$\mathscr{L}_{\text{Yuk}} = Y_{dij} q_L^{i\dagger} \phi d_R^j + Y_{uij} \left(q_L^{i\dagger} \cdot \phi^\dagger \right) u_R^j + Y_{eij} l_L^{i\dagger} \phi e_R^j + \text{h.c.}, \tag{2.1}$$

with the Yukawa coupling matrices Y_d, Y_u, and Y_e, mixing fermions from different generations together.

The Higgs field acquires a vacuum expectation value below a certain energy scale, which we can put in the form

$$\langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{2.2}$$

without loss of generality. It triggers the Higgs mechanism [16, 61, 88, 82], breaking the electroweak gauge group from $SU(2) \times U(1)_Y$ down to a $U(1)$ subgroup which is identified with the gauge group of electromagnetism. Three gauge bosons, called the $W^\pm$-bosons and the Z^0-boson, become massive. The Yukawa term becomes the mass term for the fermions, with mass matrices given by $M = vY$ for each Yukawa matrix Y; the masses for the fermions can be obtained as eigenvalues of these matrices. However, it turns out that the mass matrices $M_d = vY_d$ and $M_u = vY_u$ in the quark sectors cannot be simultaneously diagonalised, so the quarks cannot be put in the mass basis. This complication leads to the violation of CP symmetry in the Standard Model [43, 94, 90], in addition to complete breaking of any non-abelian continuous global symmetry that rotate one generation into another.

Since the weak interaction only couples to left-handed quarks and leptons, the Standard Model is a chiral gauge theory. Therefore, there is a possibility for gauge anomaly. We will now make sure that this does not happen and the theory does not break gauge invariance due to anomaly.

First, consider the strong gauge group $SU(3)$. Since there is an equal number of left-handed and right-handed fermions in the fundamental representation, they can be paired and given a mass. So the gauge factor $SU(3)$ does not suffer from anomaly. Alternatively, when all the fermions are considered as left-handed (the original left-handed quarks and leptons together with the charge conjugated $u_R^{\mathsf{C}}, d_R^{\mathsf{C}}$, and e_R^{C}), they transform under the representation

$$\mathbf{R}_{SU(3)} = 2\cdot(\mathbf{3}\oplus\mathbf{1})\oplus 2\cdot(\bar{\mathbf{3}}\oplus\mathbf{1}), \tag{2.3}$$

which is a real representation. Hence, $d^{abc}(\mathbf{R}_{SU(3)})$ vanishes and there is no anomaly. There is no pure gauge anomaly in the $SU(2)$ factor, either. Since the fermions in the Standard Model only transform in the trivial and the fundamental representations of the weak gauge group $SU(2)$,which are real and pseudoreal representations, there is no perturbative anomaly in $SU(2)$. The theory is also free of global anomaly because the number of the fundamental representation is even: three from the left-handed quarks and one from the left-handed leptons. We now only have to deal with the cubic anomaly in $U(1)_Y$ and the mixed anomalies $[SU(2)]^2$-$U(1)_Y$ and $[SU(3)]^2$-$U(1)_Y$. It can be shown by direct computation that these anomalies perfectly cancel:

$$\mathscr{A}_Y = \left[6\cdot\left(\frac{1}{6}\right)^3 + 2\cdot\left(-\frac{1}{2}\right)^3\right] - \left[3\cdot\left(\frac{2}{3}\right)^3 + 3\cdot\left(-\frac{1}{3}\right)^3 + (-1)^3\right] = 0, \tag{2.4}$$

$$\mathscr{A}_{SU(2)} = 3\cdot(+1/6) + (-1/2) = 0, \tag{2.5}$$

$$\mathscr{A}_{SU(3)} = [2\cdot(+1/6)] - [(+2/3)+(-1/3)] = 0. \tag{2.6}$$

The theory admits a non-anomalous global $U(1)$ symmetry, under which the quarks q_L^i, u_R^i, d_R^i have charge $+1/3$ while the leptons l_L^i, e_R^i, ν_R^i have charge -1. This is known as the $\mathrm{B}-\mathrm{L}$ symmetry because it counts the number of baryons minus the number of leptons. The $\mathrm{B}-\mathrm{L}$ symmetry is free from any 't Hooft anomaly (including the mixed gauge-gravitational anomaly) only when the right-handed neutrinos are included even though the gauge anomaly cancellation does not require them because they are gauge neutral.

2.2 Hypercharge Quantisation and Fermat's Last Theorem

The delicate cancellation of gauge and mixed gauge-gravitational anomalies reveals the Standard Model to be a wonderfully elegant jigsaw, each piece interlocking perfectly with the others [41, 76, 79]. One could ask: is there another way to put the pieces together? In particular, are there other assignments of hypercharge that would also result in a consistent theory?

There are different ways of posing this question. For example, we could take the gauge group of the Standard Model to be,

$$G = \mathbb{R} \times SU(2) \times SU(3)$$

Here the unfamiliar factor of $\mathbb{R}$ reflects the fact that we do not impose any quantisation condition on the hypercharge. We take a single generation of fermions, sitting in the usual representations of the non-Abelian part of the gauge group, but with arbitrary hypercharges,

$$q_L : (\mathbf{2},\mathbf{3})_q\,, \quad l_L : (\mathbf{2},\mathbf{1})_l\,, \quad u_R : (\mathbf{1},\mathbf{3})_u\,, \quad d_R : (\mathbf{1},\mathbf{3})_d\,, \quad e_R : (\mathbf{1},\mathbf{1})_x$$

The resulting quantum field theory is consistent only if the hypercharges $\{q,l,u,d,x\}$, each of which is a real number, are constrained to obey three anomaly conditions. Two of these are linear, arising from the vanishing of the mixed anomalies between Abelian and non-Abelian gauge groups

$$2q - u - d = 0 \quad \text{and} \quad 3q + l = 0 \tag{1}$$

The third is a cubic equation arising from the Abelian triangle anomaly,

$$6q^3 + 2l^3 - 3u^3 - 3d^3 - x^3 = 0 \tag{2}$$

There are an infinite number of solutions to these equations with hypercharges valued in $\mathbb{R}$. In particular, there are an infinite number of solutions with x/q irrational. This means that if we do not impose quantisation of charge then the gauge anomaly constraints do not impose it for us.

In addition, we could quite reasonably ask that the Standard Model can be consistently coupled to gravity. This gives a further linear constraint, arising from the mixed gauge-gravitational anomaly [54, 56, 8],

$$6q + 2l - 3(u + d) - x = 0 \tag{3}$$

It is well known that there are two solutions to these anomaly equations, [Alvarez-Gaume, 75, 102, 146]. The first solution is somewhat trivial,

$$q = l = x = 0 \;\text{ and }\; u = -d \tag{4}$$

The second is, up to an overall rescaling, the charge assignment seen in Nature,

$$x = 2l = -3(u+d) = -6q \;\text{ and }\; u - d = \pm 6q \tag{5}$$

Both solutions result in a quantised hypercharge, in the sense that the ratios of all charges are rational. This means that the joint requirements of gauge and gravitational consistency imply charge quantisation, even though this wasn't imposed from the outset.

In this Section, we show the converse: charge quantisation, together with vanishing gauge anomalies, is sufficient to ensure cancellation of the gravitational anomaly. To this end, we take the gauge group of the Standard Model to be (omitting possible discrete quotients)

$$G = U(1) \times SU(2) \times SU(3)$$

with the $U(1)$ factor normalised so that all charges are integers. We now wish to find integer solutions to the gauge anomaly conditions (1) and (2). Such Diophantine equations are, in general, hard to solve. Recently, a number of methods have been developed to find integer solutions to the anomaly constraints in different quantum field theories [32, 104, 47, 137, 117, 6, 46]. For the Standard Model, with a single generation, it turns out that there is a remarkably quick way to find all solutions.

We will show that there are precisely two integer solutions to (1) and (2), namely (4) and (5). Each of these solutions automatically satisfies the mixed gauge-gravitational anomaly condition (3). In other words, insisting on a $U(1)$ gauge group, rather than $\mathbb{R}$, is sufficient to ensure consistency with gravity.

This statement is a little surprising. It is certainly not true that general chiral gauge theories with a $U(1)$ factor can be coupled to gravity. Indeed, the first consistent 4d chiral gauge theory was constructed by Ramanujan from his hospital bed in Putney and suffers a mixed gauge-gravitational anomaly[1].

To prove the claim, note that the first equation in (1) tells us that the sum of hypercharges $u+d$ is even. Therefore the difference is also even and we can write $u-d = 2y$. Using the second equation in (1) to set $l = -3q$, the remaining cubic equation (2) becomes

$$x^3 + 18qy^2 + 54q^3 = 0 \tag{6}$$

Our goal is to find integer solutions to this equation. There is the trivial solution with $x = q = 0$; this corresponds to (4). Any further solution necessarily has $q \neq 0$. Because (6) is

[1] $1729 = 1^3 + 12^3 = 9^3 + 10^3$. Ramanujan also constructed a two parameter family of integer solutions to $x^3 + y^3 + z^3 = w^3$. This is described on page 158 of [137]

a homogeneous polynomial we may, without loss of generality, rescale to set $q = 1$ and look for rational solutions to the curve

$$x^3 + 18y^2 + 54 = 0 \quad x, y \in \mathbb{Q} \tag{7}$$

This is a rather special elliptic curve. To see this, we introduce two new coordinates $v, w \in \mathbb{Q}$, defined by

$$x = -\frac{6}{v+w} \ , \ y = \frac{3(v-w)}{v+w}$$

This reveals the elliptic curve (7) to be the Fermat curve

$$v^3 + w^3 = 1 \tag{8}$$

Any non-trivial rational solution to this equation would imply a non-trivial integer solution to the equation $v^3 + w^3 = z^3$. There are none [62]. The trivial solutions to (8) are $v = 1, w = 0$ and $v = 0, w = 1$. These reproduce the hypercharge assignments (5) of the Standard Model.

We could also repeat the story above with a right-handed neutrino. With a Majorana mass, the right-handed neutrino is forbidden from carrying hypercharge and the results above are unchanged. In the absence of a Majorana mass, things are not so pretty. We ascribe hypercharge ν to the right-handed neutrino. With gauge group $\mathbb{R}$, it is simple to check that the combined gauge and gravitational anomalies no longer impose quantisation of charges. If, instead, we insist on gauge group $U(1)$ then equation (8) is replaced by the Fermat surface

$$v^3 + w^3 + t^3 = 1$$

where $\nu = 6t/(v+w)$. Now there are many non-trivial rational solutions, including the taxi-cab numbers. However, in this case cancellation of the mixed gauge-gravitational anomaly occurs only for the trivial solutions in which two of the numbers coincide. These solutions are given as a 2-parameter family of rational linear combination by the Standard Model hypercharge and $\mathrm{B}-\mathrm{L}$.

Chapter 3

Global anomalies in the Standard Model(s) and Beyond

3.1 Introduction

The Standard Model (SM) has been tremendously successful in explaining all the data collected from collider physics experiments such as at the LHC, with the gauge, flavour, and Higgs sectors having been tested at the per mille, per cent, and ten per cent levels respectively [131]. However, despite its successes, there are a number of unsolved problems in the SM. Some of these are experimental or observational in origin, such as the inability to account for the dark matter and dark energy that are observed by astrophysicists and cosmologists, while other problems appear to be more theoretical or aesthetic, such as the inability to describe physics beyond the Planck scale, and the (two) hierarchy problems associated with the two super-renormalisable operators in the SM lagrangian. It is clear that in order to offer a complete description of Nature, one must go beyond the Standard Model (BSM). In order to be a consistent quantum field theory, any BSM theory that we construct (as well as the SM itself) must not suffer from any anomalies associated with its gauge group.

In fact, before we consider going beyond the SM, it is important to emphasise that there is not even *an* unique SM, but many possible Standard Model*s*, all of which are consistent with the same experimental data. The experimentally-observed SM gauge bosons and their interactions, together with the representations of the SM fermion fields, tell us that the Lie algebra of the SM gauge group is $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$. The four gauge groups

$$G = \frac{G_{\text{SM}}}{\Gamma_n}, \qquad G_{\text{SM}} = SU(3) \times SU(2) \times U(1), \quad \Gamma_n \cong \mathbb{Z}_n, \quad n \in \{1,2,3,6\}, \tag{3.1}$$

all share this Lie algebra and have representations corresponding to the SM fermions,[1] and any one of these may be the gauge group of the SM.[2] Thus, in addition to the various deficiencies in the SM that necessitate its extension, there is also an ambiguity in the SM. The potential physical distinctions between the four options in Eq. (3.1) were studied recently in Ref. [136], and amount to different periodicities of the θ angle associated with the hypercharge factor, and different spectra of Wilson lines in the theory. Perhaps unsurprisingly, all of these effects have a topological flavour.

Another possible distinction, which is also topological in origin but which was not discussed in Ref. [136], is that some of these options might not in fact be consistent after closer inspection, in the sense that they might suffer from anomalies. Of course, since the four groups in Eq. (3.1) share the same Lie algebra the conditions for local anomaly cancellation will be the same, and thus all these SMs are free of local anomalies, as is well known. However, this does not rule out the possibility of more subtle global anomalies in the SMs associated with the topology of the gauge group, analogous to (but much more general in scope than) the $SU(2)$ anomaly discovered by Witten [149], which might render some of the SM variants recorded in Eq. (3.1) inconsistent. Our first goal in this Chapter is to investigate the possible global anomalies for each choice of discrete quotient in (3.1), for arbitrary fermion content.

To do so, we exploit the relation that arises in the absence of local gauge anomalies between the potential anomaly of the partition function (which arises in the phase) of a chiral gauge theory and the exponentiated η-invariant [23] (which is a regularized sum of positive eigenvalues minus negative eigenvalues) associated to an extension of the Dirac operator to a five-manifold that bounds spacetime. This relation, which was first suggested in Ref. [151], follows from a set of mathematical results due to Dai and Freed [51], which we briefly review in §4.5 (for a more detailed discussion, see [153–155]). To wit, one may show (via a vast generalisation of Witten's original 'mapping torus' argument [149]) that if $\exp 2\pi i\eta = 1$ on all closed five-manifolds that are equipped with a spin structure and a map to BG,[3] then there will be no anomalies on spacetimes which bound (in the sense that the requisite spin and gauge structures can be extended). Since $\exp 2\pi i\eta$ is invariant under bordism in the case that local anomalies vanish, this is guaranteed to be the case when the group $\Omega_5^{\mathrm{Spin}}(BG)$ (of

[1]The embeddings of the discrete subgroups Γ_n in G_{SM} are given by Eq. (3.30).

[2]Indeed, even this is far from an exhaustive list. What is true is that the connected component of the SM gauge group G is one of the four possibilities given in Eq. (3.1).

[3]To see why BG is relevant, note that a gauge field is defined by a connection on a principal G-bundle over a spacetime manifold Σ, and every such bundle corresponds to a map $\Sigma \to BG$; for global anomalies, the connection plays no role, and we have a one-to-one correspondence between G-bundles (without connection) and homotopy classes of maps $\Sigma \to BG$.

equivalence classes under bordism of five-manifolds equipped with a spin structure and a map to BG) vanishes.[4]

In this Chapter we begin by applying this criterion for global anomaly cancellation to the four versions of the SM given by Eq. (3.1). The computations we report in this Chapter build upon those of Ref. [74], which used the Atiyah-Hirzebruch spectral sequence to compute $\Omega^{\text{Spin}}_{d\leq 5}(BG)$ for a number of simple gauge groups G including $SU(n)$, $PSU(n)$, $USp(2k)$, and $SO(n)$, as well as for $U(1)$. From there it was argued in Ref. [74] that there are no global anomalies in the SMs, by exploiting the (perhaps fortuitous) fact that the particular fermion content of the SM can be embedded in an anomaly-free grand unified theory (GUT) with $G = SU(5)$ (which breaks down to G_{SM}/Γ_6 as we go below the GUT scale). Alternative derivations of this result can be found in Refs. [64, 141]. It turns out that this guarantees that all 4 versions of the SM in Eq. (3.1) are anomaly-free for the SM fermion content, or any other fermion representations that form representations of $SU(5)$.

We analyse the global anomalies in theories with one of the SM gauge groups by computing each $\Omega^{\text{Spin}}_5(BG)$ for the four gauge groups listed in Eq. (3.1) directly. At least in 3 out of the 4 cases (those in which $n \in \{1,2,3\}$), we can do this by first noting that the gauge group can be written as a product (for example, $G_{\text{SM}}/\Gamma_2 \cong U(2)\times SU(3)$). Next, we extend the methods of Ref. [74] to treat gauge groups which are products, by exploiting the fact that $B(G\times H) = BG\times BH$,[5] and using a Künneth formula in (co)homology. The 4th case, in which $G = G_{\text{SM}}/\Gamma_6$, succumbs to a slightly more sophisticated attack, which we describe in §3.4.5.

Our results for the four possible connected SM gauge groups can be applied, unlike those of Ref. [74], to any BSM theories with one of the SM gauge groups but with different fermion content (that do not necessarily fit inside any GUT with a simple gauge group). While one might have expected, given the much more general nature of the anomaly cancellation condition imposed, more constraints to appear beyond those required to cancel the familiar $SU(2)$ global anomaly discovered by Witten, one finds that in fact that the opposite happens: in some cases there are actually fewer constraints, due to a subtle interplay between global and local anomalies, which we describe in §3.4.6. This is related to the more mundane fact that for the gauge groups featuring quotients by $\Gamma_{n\neq 1}$ there are non-trivial constraints on the hypercharges of fermions depending on their representation. We give these constraints in §3.4.1.

We then turn our attention to global anomalies in a number of well-motivated BSM theories, which we analyse using the same bordism-based criteria. We demonstrate our

[4]In fact, there are reasons to believe that the vanishing of $\Omega^{\text{Spin}}_5(BG)$ is sufficient for the vanishing of global anomalies not only on spacetimes that bound, but also on those that do not – we discuss this at the end of §4.5.

[5]Similar ideas were used in the context of classifying higher-symmetry-protected topological phases [138].

methods in a wide variety of BSM examples, in the hope that readers can adapt the methods to analyse their own favourite models. In particular, we consider theories in which the SM gauge group is extended by products with arbitrary $U(1)$ factors, as well as a number of GUTs including Pati-Salam models and trinification models.

One might *a priori* expect all bets to be off when one goes beyond the SM, and that the possibility of $\Omega_5^{\mathrm{Spin}}(BG)$ being non-trivial might provide a variety of extra constraints on the fermion content of BSM models for the cancellation of new global anomalies. Interestingly, we will find that this is largely not the case. In all the four-dimensional examples we considered, we find that $\Omega_5^{\mathrm{Spin}}(BG)$ detects no new anomalies beyond the $\mathbb{Z}_2$-valued anomalies associated with $SU(2)$ (or more generally $Sp(r)$) factors in the gauge group. While we essentially arrive at a large collection of 'null results', we hope that the absence of any potential new anomalies in all of our examples will at least provide some assurance for the more conscientious BSM model-builders, who worry that their models might suffer from secret global anomalies.

We remark that in spacetime dimensions lower (or indeed higher) than four there are, however, potentially lots of new anomalies in theories with these gauge groups. We catalogue the relevant bordism groups in lower dimensions for the gauge groups we consider alongside the results of importance to the (B)SM case, in case they might be of interest to others (for example, in the condensed matter community). For ease of reference, all our bordism group results are collated across Tables 3.1, 3.3, and 3.4.

The outline of the rest of this Chapter is as follows. In §4.5 we review the so-called 'Dai–Freed theorem', and the arguments that underlie the bordism-based criterion for global anomalies that we use. In §3.3 we review the algebraic machinery of spectral sequences which we use to compute the bordism groups of interest to us. We then summarise and interpret our computations pertaining to global anomalies in the SMs in §3.4. In §3.5, we generalise the SM results to a 2-parameter family of theories that contains the SM, with gauge group $SU(N)\times Sp(M)\times U(1)$ for $N,\ M\in\mathbb{Z}$. We present the details of our computations for BSM theories in §3.6. Finally, we find that there are no global anomalies in a BSM theory in which the SM fermions are defined using a spin$_c$ structure, allowing also for arbitrary additional fermion content, by showing that $\Omega_5^{\mathrm{Spin}_c}(BG)=0$ for each choice of G in Eq. (3.1). Such a theory can be defined on all orientable four-manifolds (not only those that are spin), but requires an additional $U(1)$ symmetry be gauged such as $B-L$.

Note added: Ref. [139], which appeared after the paper [52] on which this Chapter is based was put on the ArXiv, confirms some of the bordism group calculations in this Chapter using the Adams spectral sequence.

3.2 Bordism and global anomalies

Both the local gauge anomalies first discovered by Adler, Bell, and Jackiw (ABJ) [3, 34] and the global anomalies first discovered by Witten [149] may arise in chiral gauge theories due to subtleties in defining the Dirac operator. To see how, and to motivate the more general bordism-based criterion for anomaly cancellation that we employ, it is helpful to first review some basic facts about chiral fermions, for which we largely follow the discussion in Ref. [153]. Other helpful references for this discussion are Refs. [154–156] (written with physicists in mind) and the original mathematical paper by Dai and Freed on which much of the discussion rests [51].

Firstly, we recall that defining a chiral gauge theory requires that any spacetime manifold be equipped with certain geometric structures. The important structures for our purposes are

- A form of spin structure to define fermions,
- A principal G-bundle to define gauge fields,
- A Dirac operator which couples fermions to gauge fields, whose determinant is a well-defined function on the background data if the theory is to be non-anomalous.

We work in four spacetime dimensions from the beginning, since that is the case of relevance to the particle physics applications we are interested in; however, all the material we review in this Section generalises straightforwardly to other numbers of dimensions. We always assume spacetime is euclideanised, and thus consider spacetime to be a smooth, compact, four-manifold Σ. At times it will be helpful to suppose Σ is equipped with a (riemannian) metric, but this shall not be especially important to our arguments.

In most of this Chapter, we assume that spacetime is orientable and that fermions are defined using an honest spin structure. It is possible, however, that fermions may be defined on an orientable spacetime using 'weaker' structures if there are gauge symmetries present, as is typically the case in particle physics. For example, the presence of a $U(1)$ gauge symmetry allows one to define fermions using only a spin$_c$ structure; note that all orientable four-manifolds are spin$_c$, but not all orientable four-manifolds are spin. In §3.7, we consider this possibility. In the presence of a larger gauge symmetry, such as $SU(2)$, one could get away with only a spin-$SU(2)$ structure to define fermions [143], and so on. A new kind of global anomaly has been recently discovered by Wang, Wen, and Witten [143] for an $SU(2)$ gauge theory formulated on all manifolds admitting such a spin-$SU(2)$ structure. They show that such a theory is anomalous if there is an odd number of fermion multiplets in spin $4r+3/2$ representations of $SU(2)$ (where $r \in \mathbb{Z}$). Of course, the more familiar $SU(2)$ global anomaly arises when the theory is defined on all spin manifolds, in which case there is an anomaly when $n_L - n_R = 1 \bmod 2$, where n_L (n_R) is the number of left-handed (right-handed) $SU(2)$ doublets [149]. In a time-reversal symmetric theory,[6] one could consider defining the theory also on *un*orientable spacetimes, in which case a form of pin structure could be used to define fermions. We describe how fermions can be defined using these various 'spin structures' in Appendix 3.A for reference; we also invite the reader to consult Appendix A of Ref. [153]. Throughout the main body of this Chapter, however, we assume that spacetime is orientable and equipped with a spin structure.

Defining gauge fields for some gauge group G requires the existence of a principal G-bundle over Σ. As we wrote before, the classifying space BG of the Lie group G has the property that the homotopy classes of maps from a space X to BG are in one-to-one correspondence with the set of (isomorphism classes of) principal G bundles over M.[7] Thus, we consider orientable spacetimes Σ equipped with a map $f : \Sigma \to BG$, in addition to a spin structure. We moreover insist that a gauge theory be defined on *all* manifolds admitting these structures, leading to a very broad notion of whether there is an 'anomaly' in the theory. Ultimately, these requirements are necessary to guarantee that the theory be consistent with locality.

3.2.1 Fermionic partition functions

One may define fermions and gauge fields on four-manifolds equipped with the given geometric structures. In a renormalisable four-dimensional chiral gauge theory, one couples the two via the lagrangian $\bar{\psi} i\not{D}\psi$, where $i\not{D}$ is an hermitian Dirac operator. We are now in a position to see how both the local and global anomalies can emerge in such a gauge theory.

The heart of the trouble in both kinds of anomaly lies in performing the functional integration over fermions. The result is a partition function $Z_\psi[A]$, which we consider to be a function of the background gauge field and also any other background fields or data such as a metric on spacetime.[8] Formally, $Z_\psi[A]$ is defined to be

$$Z_\psi[A] \equiv \int \mathscr{D}\psi \mathscr{D}\bar{\psi} e^{-\int d^4x \, \bar{\psi} i\not{D}\psi} = \det i\not{D}, \tag{3.2}$$

[6] We note that the SM is not time-reversal symmetric, since CP is explicitly broken by the phases appearing in the CKM and PMNS matrices, and in theory also by a non-zero QCD θ angle. Thus, in this Chapter we only consider theories with one of the SM gauge groups to be defined on orientable spacetimes.

[7] The classifying space BG is the quotient of a weakly contractible space EG by a proper free action of G. Any principal G-bundle over M is the pullback bundle f^*EG along a map $f : M \to BG$.

[8] Sometimes, we use 'A' to denote the background gauge field, while at others time we use 'A' to collectively denote all the background fields/data. Which of the two meanings is implied in a given instance ought to be clear from the context.

the determinant of the Dirac operator,[9] assumed to be appropriately regularized. The partition function $Z_\psi[A]$ of a non-anomalous quantum field theory is a kosher $\mathbb{C}$-valued function on the space of background data. For the case of coupling to background gauge fields, this means that $Z_\psi[A]$ must be a well-defined function on the space of connections on principal G-bundles modulo gauge transformations.

If this is not the case, G-invariance is anomalous, and since it is a gauge symmetry, the theory is not well-defined. This viewpoint sets the traditional ideas of local and global gauge anomalies in a more general context: in the case of a local anomaly, one has that $Z_\psi[A] \neq Z_\psi[A^g]$ even for a gauge transformation $A \to A^g$ with g infinitesimally close to the identity; for the original $SU(2)$ global anomaly [149], one finds $Z_\psi[A] = -Z_\psi[A^U]$ where the group element $U(x)$ corresponds to a gauge transformation in the non-trivial class of $\pi_4(SU(2))$. The partition 'function' of an anomalous theory is thus at best a section of a complex line bundle over the space of background data, called the determinant line bundle. Moreover, the modulus $|Z_\psi|$ of the partition function cannot suffer from anomalies, and the anomaly must come purely from the phase of Z_ψ. To see why, note that for any set of chiral fermions ψ, one can define a conjugate set $\widetilde{\psi}$ that transforms as the complex conjugate of ψ under all symmetries, and with an action that is the complex conjugate of the action for ψ. Thus, the functional integration over $\widetilde{\psi}$ yields precisely $\bar{Z}_\psi$, the complex conjugate of (3.2). Hence, for the combined system, the partition functon is $Z_\psi \bar{Z}_\psi = |Z_\psi|^2$. But given the complex conjugate set of fermions one can always write down mass terms for the set of fermions ψ, for which a Pauli-Villars regulator (which respects the symmetries of the lagrangian) is always available. Hence $|Z_\psi|^2$, and thus $|Z_\psi|$, cannot suffer from any anomalies.

With this realisation, one might first try to simply define the fermionic partition function to be equal to its modulus, and so construct an anomaly-free theory by *fiat*. But the modulus $|Z_\psi|$ on its own is not a smooth function of the background data A, just as $|w|$ is not a smooth function of the real or imaginary parts of a complex number w. The partition function must, however, depend smoothly on the background data, which includes gauge fields and metrics, otherwise correlation functions involving the stress-energy tensor and/or currents coupled to the gauge field would not be well-defined. Thus, one cannot evade anomalies in such a way, and one must instead consider carefully when Z_ψ is well-defined, and when it is not.

A set of mathematical results due to Dai and Freed [51] allow one to construct a candidate partition function, which is necessarily smooth on the space of background data, with which

[9] More generally, $Z_\psi[A]$ will be the Pfaffian of the Dirac operator. We essentially ignore this subtlety for the purpose of this discussion, by assuming fermions to be complex or pseudo-real.

to properly analyse anomalies. For brevity's sake, we refer collectively to these results as the *Dai–Freed theorem*. For an account written with physicists in mind, see Ref. [156].

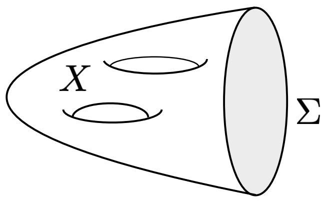


Fig. 3.1 The results of Dai and Freed give a prescription for writing down a fermionic partition function Z_ψ when spacetime Σ is the boundary of a five-manifold X.

The Dai–Freed theorem implies that a putative partition function $Z_\psi[A]$ that is smooth in A can always be defined when the four-dimensional spacetime Σ is the boundary of a five-manifold X, *viz.* $\Sigma = \partial X$ (as depicted in Fig. 3.1), to which the theory (and thus the spin structure and map to BG) must be extended. The five-manifold X must approach a 'cylinder' $(-\tau_0, 0] \times \Sigma$ near the boundary Σ, where the local coordinate $\tau \in (-\tau_0, 0]$ parametrises the fifth dimension. Moreover, the Dirac operator is extended to define a five-dimensional Dirac operator on X which we denote by $i\not{D}_X$, which near the boundary takes the form $i\not{D}_X = i\gamma^5(\partial_\tau + i\not{D})$, where $i\not{D}$ is the original Dirac operator on Σ.[10]

Schematically, the Dai–Freed definition of the putative partition function is then

$$Z_\psi[A] = |Z_\psi| \exp\left(-2\pi i \int_X I^0(F)\right) \exp\left(-2\pi i \eta_X\right), \tag{3.3}$$

where we have split the phase into two distinct contributions, which we will define shortly. Importantly, Dai and Freed showed that this construction varies smoothly with the background data.

The two contributions to the phase, as separated out in Eq. (3.3), correspond loosely to local and global anomalies. The first contribution to the phase of (3.3) is easier to understand. It is the integral of the anomaly polynomial $I^0(F)$ over the extended five-manifold X, which is a polynomial in the curvature F of the connection A defined such that

$$dI^0(F) = \hat{A}(R)\ \mathrm{tr}\ \exp\left(\frac{iF}{2\pi}\right)\bigg|_6, \tag{3.4}$$

[10] Special boundary conditions must be chosen to ensure that the operator $i\not{D}_X$ is hermitian throughout X. These are often referred to as '(generalised) APS boundary conditions', and we will not discuss them further, but rather refer the reader to *e.g.* Refs. [153, 156], in addition to the original papers of Atiyah, Patodi, and Singer [23–25].

where $\hat{A}(R)$ is the $\hat{A}$ genus (sometimes referred to as the 'Dirac genus'), with R the Riemann tensor. The bar and subscript '6' indicates that one should take only the six-form terms on the right-hand-side. This contribution to the phase is not necessarily invariant even under infinitesimal gauge transformations. Rather, its variation can be computed using Eq. (3.4), and requiring that this variation vanish after being integrated reproduces the familiar formulae for the cancellation of local anomalies (including gravitational and mixed gauge-gravitational anomalies). This type of anomaly is sometimes referred to as the perturbative anomaly, because one can derive it perturbatively by expanding the path integral around the zero background fields in flat spacetime.

The second contribution comes from the fermions on X, which one can think of as a kind of regulator for the system on Σ. The η-invariant is defined as the following sum over eigenvalues λ of the Dirac operator $i\not{D}_X$

$$\eta_X = \frac{1}{2}\left(\sum_{\lambda\neq 0} \operatorname{sign}(\lambda) + \operatorname{Dim}\ker(i\not{D}_X)\right), \tag{3.5}$$

which must of course be regularized.[11] This η-invariant was introduced by Atiyah, Patodi, and Singer (APS) in their generalisation of the Atiyah–Singer index theorem to manifolds with boundary [23–25]. It shall be useful in what follows to recall that the η-invariant possesses an important 'gluing' property, as follows: if two manifolds with boundary Y_1 and Y_2 are glued along a common boundary to give a manifold $Y_1 \cup Y_2$, then the exponentiated η-invariant factorizes, *i.e.*

$$\exp\left(2\pi i \eta_{Y_1\cup Y_2}\right) = \exp\left(2\pi i \eta_{Y_1}\right)\, \exp\left(2\pi i \eta_{Y_2}\right), \tag{3.6}$$

as illustrated in Fig. 3.2.

3.2.2 Global anomalies and the η-invariant

In order for (3.3) to describe an intrinsically four-dimensional theory on Σ, this putative definition for the fermionic partition function must be independent of the choice of five-manifold X and the extension to X of whatever structures are necessary to define the theory on Σ. Any dependence on X invariably leads to ambiguities and inconsistencies with locality and/or smoothness in the four-dimensional theory. Such inconsistencies are precisely what we call "anomalies".

[11] For example, in the original APS index theorem the sum over eigenvalues was regularized by replacing $\sum_{\lambda\neq 0}\operatorname{sign}(\lambda)$ with $\lim_{s\to 0}\sum_{\lambda\neq 0}\operatorname{sign}(\lambda)|\lambda|^{-s}$, which converges for large Re s, from which one can analytically continue to $s=0$ without encountering any poles.

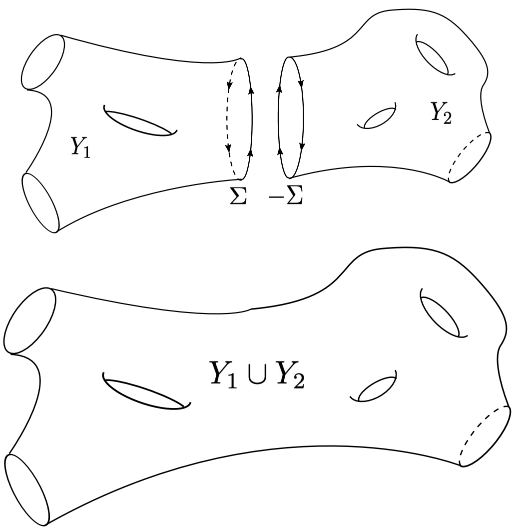


Fig. 3.2 Gluing of two manifolds Y_1 and Y_2 with a shared boundary component Σ, under which the exponentiated η-invariant factorizes.

It is worth mentioning here that, if the condition for anomaly cancellation is not satisfied, we can no longer use Eq. (3.3) as the partition function for our theory on the four-manifold Σ. Nonetheless, even in this context (3.3) remains a useful equation, because it precisely quantifies the anomalies in terms of anomaly inflow. Heuristically speaking, it tells us that we can make sense of an anomalous fermionic theory if it arises as a boundary degree of freedom of another theory in one dimension higher, where the anomalies at the boundary are precisely cancelled by the contribution from the bulk. This is captured solely by the η-invariant when there is no local anomaly, justifying our moniker of 'global' anomalies. This fact lies at the heart of our current understanding of topological insulators in condensed matter physics.

Let us return to our search for a criterion for anomaly-freedom. The putative partition function (3.3) is independent of the choice of five-manifold X if and only if

$$\exp\left(-2\pi i \eta_{\bar{X}}\right) = \exp\left(2\pi i \int_{\bar{X}} I^0(F)\right), \tag{3.7}$$

for all *closed* five-manifolds $\bar{X}$. To see this, consider a duplicate of our fermionic theory on Σ but extended to a different five-manifold X'. Let $-X'$ denote this five-manifold with its orientation reversed. It is then possible to glue the original system defined on (X,Σ) to that on $(-X',-\Sigma)$ along the mutual four-boundary Σ. The result is a fermionic theory on a closed five-manifold $\bar{X} \equiv X \cup (-X')$, as illustrated in Fig. 3.3. Since the two systems have the same fermionic theory on Σ, the moduli of the path integrals cancel, and the path integral of the

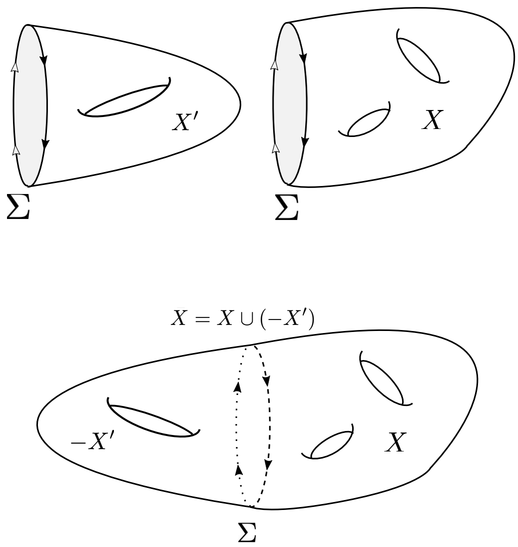


Fig. 3.3 Gluing of two manifolds X and X' with a shared boundary Σ into a closed manifold $\bar{X} = X \cup (-X')$.

combined system is the pure phase

$$Z_{\bar{X}} = \frac{Z_X}{Z_{X'}} = \exp\left(-2\pi i(\eta_X - \eta_{X'})\right) \exp\left(2\pi i \left(\int_X - \int_{X'}\right) I^0(F)\right). \tag{3.8}$$

Using the linearity property of integrals, together with the above gluing property for the η-invariant, we can rewrite the fermionic partition function on the closed five-manifold $\bar{X}$ as

$$Z_{\bar{X}} = \exp(-2\pi i \eta_{\bar{X}}) \exp\left(-2\pi i \int_{\bar{X}} I^0(F)\right),$$

which is trivial if and only if the condition (3.7) is satisfied. The triviality of $Z_{\bar{X}}$ for any closed five-manifold $\bar{X}$ implies that $Z_X = Z_{X'}$ for any pair of five-manifolds which share the same boundary theory Σ.

Thus, in the absence of local anomalies, *i.e.* when $I^0(F) = 0$, any residual global anomalies necessarily vanish, and the partition function describes an intrinsically four-dimensional theory, when $\exp(-2\pi i \eta_{\bar{X}}) = 1$ for all closed five-manifolds $\bar{X}$ (that admit a spin structure and a map to BG). Witten's mapping torus argument [149], by which the original $SU(2)$ global anomaly was first detected (for a fixed spacetime $\Sigma = S^4$), is equivalent to insisting that $\exp(-2\pi i \eta_{\bar{X}}) = 1$ on $\bar{X} = S^1 \times S^4$.

Moreover, when local anomalies cancel, such that $I^0(F) = 0$, it follows from the APS index theorem that $\exp(2\pi i\eta)$ is a bordism invariant.[12] By 'bordism' we mean (unless explicitly stated otherwise) the equivalence relation on compact p-manifolds equipped with a spin structure and a map to BG such that two manifolds are deemed equivalent if their disjoint union is the boundary of some compact $(p+1)$-manifold with the structures extended appropriately. By 'bordism invariant', we mean a well-defined homomorphism on the equivalence classes under bordism (or just bordism classes), which form an abelian group $\Omega_p^{\mathrm{Spin}}(BG)$. This means that $\exp(2\pi i\eta) = 1$ on any five-manifold that is null-bordant. Hence, when $I^0(F) = 0$ the η-invariant defines a homomorphism from the fifth spin bordism group to the phase of the partition function, or, in other words

$$\exp(2\pi i\eta) \in \mathrm{Hom}\left(\Omega_5^{\mathrm{Spin}}(BG), U(1)\right). \tag{3.9}$$

The group $\mathrm{Hom}\,(\Omega_5^{\mathrm{Spin}}(BG), U(1))$ clearly vanishes if $\Omega_5^{\mathrm{Spin}}(BG) = 0$. The vanishing of $\Omega_5^{\mathrm{Spin}}(BG)$ is in fact not only sufficient but also necessary for vanishing of $\mathrm{Hom}\,(\Omega_5^{\mathrm{Spin}}(BG), U(1))$, at least when $\Omega_5^{\mathrm{Spin}}(BG)$ is a finitely generated abelian group (as is the case for all the examples we examine here), which means it can be written as

$$\Omega_5^{\mathrm{Spin}}(BG) \cong \mathbb{Z}^r \times \mathbb{Z}_{p_1} \times \ldots \times \mathbb{Z}_{p_m}. \tag{3.10}$$

To see that this is the case, note that for each summand there exist non-trivial maps to $U(1)$ – for example, one can send $n \in \mathbb{Z}_p$ to $\exp(2\pi i n/p)$, or can send $k \in \mathbb{Z}$ to $\exp(\pi i k)$. Thus, as long as $\Omega_5^{\mathrm{Spin}}(BG) \neq 0$, the set of homomorphisms from the 5th spin bordism group to $U(1)$ is non-empty.

The exponentiated η-invariant is necessarily trivial when $\Omega_5^{\mathrm{Spin}}(BG)$ vanishes. Thus, if local anomalies cancel and if

$$\Omega_5^{\mathrm{Spin}}(BG) = 0, \tag{3.11}$$

then Eq. (3.7) implies there is a well-defined fermionic partition function which is independent of the choice of five-manifold X, and thus defines a sensible local quantum field theory.

In summary, the following precise statement, which follows from the Dai–Freed theorem, forms the basis of what follows:

> The path integral for a d-dimensional gauge theory with gauge group G with arbitrary matter content can be consistently formulated on null-bordant space-

[12]This fact was first used in the physics literature to analyse global anomalies in string theories [150].

time manifolds of dimension d using the Dai–Freed prescription if $I^0 = 0$ and $\Omega^{\text{Spin}}_{d+1}(BG) = 0$.

Two caveats are warranted here. Firstly, we still don't have a definition for spacetimes Σ that are not null-bordant. Such spacetimes appear regardless of the gauge group,[13] being generated by a K3 surface [120]. In general, locality forces such spacetimes to appear in the theory, and so one needs a general prescription for the fermionic partition function evaluated on spacetimes in non-trivial bordism classes, which goes beyond the original Dai–Freed theorem.

The second caveat is that, even if the Dai–Freed prescription cannot be made to work, it is still possible that some other suitable definition of the path integral might be found in cases where the condition (3.11) is violated.

In fact, recent developments in the mathematical field of topological field theory give hints that these two caveats can safely be struck out. Those developments suggest that an anomalous theory should be viewed as a special case of a relative field theory [67], namely a natural transformation between an extended field theory in one higher spacetime dimension (defined as a functor from some higher bordism category to some linear category) to the trivial extended field theory also in one dimension higher. Thus, part of the data of an anomalous field theory is a non-anomalous, non-trivial quantum field theory in one dimension higher. If there are no such theories, then there can be no anomalies.

The putative theory in one dimension higher is, in many cases (but see Refs. [67, 103]), both topological and invertible, meaning that it can be described by a classical topological action. It turns out that such actions can be classified by some Abelian group A corresponding to some (generalized) differential cohomology theory. The group is characterised by an exact sequence of Abelian groups $B \to A \to C$, where C corresponds here to the local anomaly and B to the global anomaly. In the case of ordinary differential cohomology (in which we have not bordism classes of manifolds with spin, but rather homology classes corresponding to smooth singular simplices), the group B is just the group $H^5(BG;U(1)) \cong \text{Hom}(H_5(BG),U(1))$ and so it is tempting to conjecture that the corresponding group here is indeed $\text{Hom}(\Omega^{\text{Spin}}_5(BG),U(1))$. Moreover, in the ordinary differential cohomology case, the exact sequence $B \to A \to C$ extends to a short exact sequence $0 \to B \to A \to C \to 0$, so that $A = 0$ iff. $B = C = 0$. If the same is true here, then we have a complete characterisation of the anomaly cancellation conditions, whose global part is $\text{Hom}(\Omega^{\text{Spin}}_5(BG),U(1)) = 0$.

[13] Furthermore, in the presence of a non-abelian gauge symmetry, for example in the case $G = SU(3)$, there exist additional spacetime manifolds that do not bound spin five-manifolds (to which the map to BG extends), generated by a manifold with instanton number one [155].

Indeed it is believed that [65, 155], as long as the object $Z_{\bar{X}}$ defined by (3.8) equals one for all closed five-manifolds $\bar{X}$, a prescription for the partition function on non-nullbordant spacetimes can be given, that is consistent with the principles of unitarity and locality and free of anomalies, by assigning an arbitrary theta angle to each generator of $\Omega_4^{\text{Spin}}(BG)$. There is no quantum field theory principle that can be used to fix the arbitrary theta angles, which correspond to an element in $\text{Hom}(\Omega_4^{\text{Spin}}(BG), U(1))$, because any such element equals a partition function for an invertible topological field theory (in four dimensions) to which the theory may be consistently coupled. In the context of string theory these statements are well-known, with the assignment of theta angles sometimes referred to as "setting the quantum integrand" [152, 66].

3.3 Methodology

It remains to explain how we actually compute a bordism group of the form $\Omega_5^{\text{Spin}}(BG)$, for a specific G. As is so often the case in algebraic topology, one is faced with a calculation that is seemingly impossible, no matter how simple the choice of G, but which turns out to be possible for almost any G, provided one knows enough tricks. The main tricks in the case at hand are the Atiyah-Hirzebruch spectral sequence [22] (see Refs. [84, 99] for introductions to spectral sequences) and the use of cohomology operations (see Ref. [44]). We follow, essentially verbatim, the method set out in Ref. [74], but we feel it might be helpful to readers to give a more pedestrian description, as follows.

Spectral sequences are an important calculational tool in algebraic topology. So, what is a spectral sequence? In essence, a spectral sequence is a collection of abelian groups $E^r_{p,q}$ indexed by three non-negative integers r, p, and q, together with a collection of group homomorphisms between them. Perhaps more appealingly, one can picture a spectral sequence to be a 'book' consisting of (infinitely) many pages, labelled by a 'page number' r, with a two-dimensional array of abelian groups $E^r_{p,q}$ on each page. There are maps (called 'boundary maps' or 'differentials') between the groups within a given page of the form[14]

$$d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}, \qquad \text{such that} \qquad d^r_{p-r,q+r-1} \circ d^r_{p,q} = 0, \tag{3.12}$$

which endows the groups $E^r_{p,q}$ on the corresponding 'diagonals' of a given page with the structure of a *chain complex*. The first few pages are illustrated schematically in Fig. 3.4. Moreover, one passes from one page to the next by 'taking the homology' with respect to the

[14]Note that we are here describing the *homological* version of a spectral sequence, which shall also be the kind we employ in our bordism computations. There is an analogous *co*homological version, in which the boundary maps go in the opposite directions.

differentials, specifically

$$E^{r+1}_{p,q} \cong \ker(d^r_{p,q})/\mathrm{im}(d^r_{p+r,q-r+1}). \tag{3.13}$$

As we keep 'turning the pages' in this way, the abelian group appearing in any given (p,q) position will eventually stabilise (because there are only a finite number of differentials going 'in' and 'out' for any (p,q)). It is conventional to refer to the 'last page', after which all entries of the AHSS have stabilised, as $E^\infty_{p,q}$. Important topological information will be contained in this last page.

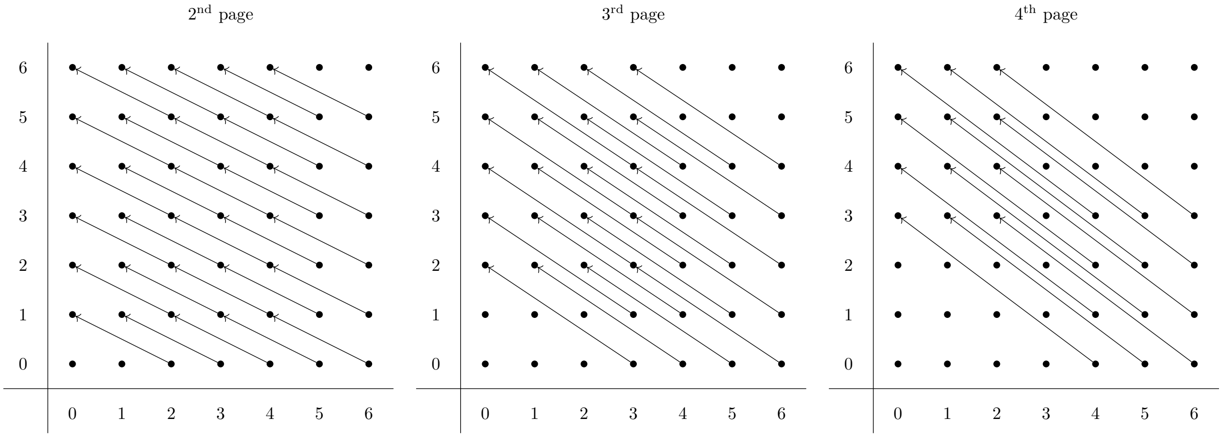


Fig. 3.4 Schematic illustration of a spectral sequence

For example, the Serre spectral sequence can be used to compute the (co)homology groups of a topological space X appearing as the total space in a fibration $F \to X \to B$, from the (co)homology of the two spaces F and B, where we take B to be simply connected. For the Serre spectral sequence, we can in fact ignore the first page, and begin at the second page, whose entries are given by the peculiar formula $E^2_{p,q} = H_p(B;H_q(F;A))$; in words, the homology groups of the base space with coefficients valued in the homology groups of the fibre (for some coefficient group A). We then proceed to turn the pages using the differentials (3.12), until we get to the last page at which all the entries have stabilised. Then the nth homology group of the total space X can be pieced together for each n, using $H_n(X;A) = \bigoplus_p E^\infty_{p,n-p}$, in others words, by taking the direct sum of all the groups on the nth diagonal of the last page of the Serre spectral sequence.[15]

The Atiyah-Hirzebruch spectral sequence (AHSS) is a generalisation of the Serre spectral sequence just described, in which ordinary (co)homology is replaced by generalised (co)homology. The bordism groups $\Omega^{\mathrm{Spin}}_5(BG)$ that we want to compute to classify global

[15] This is in fact a simplification, and only holds when the coefficient group A is a field. Otherwise, a non-trivial group extension problem must be solved.

anomalies are examples of generalised homology groups, and so the AHSS provides an appropriate tool for our computation, if we can fit BG into a useful fibration

$$F \to BG \to B. \tag{3.14}$$

Given such a fibration, the AHSS is then constructed in a similar fashion to the Serre spectral sequence. We begin at the second page, whose entries are now the homology groups

$$E^2_{p,q} = H_p(B; \Omega_q^{\text{Spin}}(F)). \tag{3.15}$$

If the singular homology groups $H_p(B;\mathbb{Z})$ are *free* (*i.e.* do not contain torsion) then this simplifies to

$$E^2_{p,q} = H_p(B; \Omega_q^{\text{Spin}}(F)) = H_p(B;\mathbb{Z}) \otimes \Omega_q^{\text{Spin}}(F). \tag{3.16}$$

If this is not the case, then the *universal coefficient theorem* (in homology) must be used to calculate (3.15). This second page comes equipped with differentials as specified in Eq. (3.12), and if the differentials are known we can turn to the next page. If we are able to continue turning pages until all the entries with $p+q=5$ are stabilised, then we can use these entries to extract $\Omega_5^{\text{Spin}}(BG)$. Analogous to the example of the Serre spectral sequence, it shall be the case in all the examples we consider that $\Omega_5^{\text{Spin}}(BG)$ shall simply be the direct sum of the entries $E^\infty_{p,q}$ with $p+q=5$.[16]

The simplest fibration involving BG, which we shall employ most frequently, is the trivial one in which BG is fibred over itself, such that the fibre is a point which we denote by pt, *i.e.* we consider

$$\text{pt} \longrightarrow BG \longrightarrow BG. \tag{3.17}$$

In this case, computing the elements (3.16) of the second page of the AHSS requires two ingredients: (i) the singular homology groups of the classifying space, $H_p(BG;\mathbb{Z})$, and (ii) the bordism groups (preserving the spin structure) equipped with maps to a point; in other words, simply the equivalence classes (under bordism) of spin five-manifolds. Fortunately

[16]While there is a straightforward condition telling us when this is the case for the Serre sequence - namely, when the coefficient group A is a field - there is (as far as we are aware) no similarly straightforward condition pertaining to the AHSS and our bordism calculations. Rather, one must refer to the definition of the spectral sequence in terms of *filtrations* of the bordism groups we are trying to compute, using which the answer can often be extracted unambiguously from the last page. In particular, this was the case in all the examples we present in the sequel.

for us, these bordism groups are well known in low dimensions [15]:

$$\begin{array}{c|ccccccccccc} n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline \Omega_n^{\text{Spin}}(\text{pt}) & \mathbb{Z} & \mathbb{Z}/2 & \mathbb{Z}_2 & 0 & \mathbb{Z} & 0 & 0 & 0 & \mathbb{Z}^2 & (\mathbb{Z}_2)^2 & (\mathbb{Z}_2)^3 \end{array} \tag{3.18}$$

The other ingredients we need are the homology groups of the classifying space of any gauge group G we want to consider. As we have advertised above, we will consider many examples where G is a product and our strategy here will be to build up the homology groups of such groups from the homology groups of their factors. We shall make frequent use of the fact that

$$B(G \times H) = BG \times BH, \tag{3.19}$$

which follows from the definition of the classifying space of a group (see, for example, Chapter 16, §5 of [98]). Thence, we shall use the Künneth theorem to compute the homology of the product space $BG \times BH$ with coefficients in $\mathbb{Z}$. In the absence of torsion,[17] this is simply

$$H_p(BG \times BH; \mathbb{Z}) \cong \bigoplus_{m+n=p} H_m(BG; \mathbb{Z}) \otimes H_n(BH; \mathbb{Z}). \tag{3.21}$$

The classifying spaces (and their homology rings) for some elementary groups are well-known; for example, $BU(1) = \mathbb{C}P^\infty$, with

$$H_p(BU(1) = \mathbb{C}P^\infty; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{when } p = 0 \bmod 2\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{3.22}$$

and $BSU(2) = \mathbb{H}P^\infty$, with

$$H_p(BSU(2) = \mathbb{H}P^\infty; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{when } p = 0 \bmod 4\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{3.23}$$

While the homology groups for these two examples are known in all degrees, it is often enough for our purposes to know the groups $H_p(BG; \mathbb{Z})$ in sufficiently low dimensions; for

[17] If there is torsion, the correct statement of the Künneth theorem is that there is a short exact sequence

$$0 \to \bigoplus_{m+n=p} H_m(BG; \mathbb{Z}) \otimes H_n(BH; \mathbb{Z}) \to H_p(BG \times BH; \mathbb{Z}) \to \bigoplus_{m+n=p-1} \mathrm{Tor}\left(H_m(BG;\mathbb{Z}), H_n(BH;\mathbb{Z})\right) \to 0, \tag{3.20}$$

and that this sequence splits (although not canonically).

instance, the result

$$H_p(BSU(n);\mathbb{Z}) = \{\mathbb{Z},0,0,0,\mathbb{Z},\dots\} \tag{3.24}$$

(for $n>1$) shall be useful for our consideration of gauge theories relevant to particle physics.

Unfortunately for our purposes, results are usually quoted for *co*homology groups of classifying spaces, not least because of their starring role in the theory of characteristic classes. But one can obtain the homology groups using some universal coefficient theorem.

Turning the pages

We have now proposed how to obtain all the ingredients with which to write down the second page of the AHSS associated with the fibration (3.17); but we do not yet know how to turn to the next page of the AHSS, which requires knowledge of the differential maps introduced in Eq. (3.12). One thing we know for certain is that the differentials are *group homomorphisms*, and in many cases this shall turn out to be enough to deduce the image and/or kernel of many differentials unambiguously; for example, we make frequent use of the fact that $\mathrm{Hom}(\mathbb{Z}_n,\mathbb{Z})\cong 0$. Similarly, for any pair of finite integers n and m, we may use the fact that $\mathrm{Hom}(\mathbb{Z}_n,\mathbb{Z}_m)\cong\mathbb{Z}_{\gcd(n,m)}$.

However, simple algebraic arguments like this will seldom be enough to determine all the differentials in the AHSS. Fortunately, we can make use of the fact that some of the differentials *on the second page* $E^2_{p,q}$ are known for the case of the spin bordism groups Ω^{Spin}_q. In particular, we have that the differential

$$d^2_{p,0}: H_p(B;\Omega^{\mathrm{Spin}}_0)\to H_{p-2}(B;\Omega^{\mathrm{Spin}}_1) \tag{3.25}$$

is the composition of the (homology) dual of the Steenrod square and followed by reduction modulo 2 [133, 134], and that the differential

$$d^2_{p,1}: H_p(B;\Omega^{\mathrm{Spin}}_1)\to H_{p-2}(B;\Omega^{\mathrm{Spin}}_2) \tag{3.26}$$

is the dual of the Steenrod square [133, 134]. The Steenrod square, Sq^2, is an operation on mod 2 *co*homology classes, $\mathrm{Sq}^2: H^n(X;\mathbb{Z}_2)\to H^{n+2}(X;\mathbb{Z}_2)$, whose particular action on the generators of H^n are known for the classifying spaces of Lie groups, thanks to Borel and Serre [40]. We will make regular use of their results in what follows. We note here for future reference that Sq^2 is an example of more general Steenrod squares, $\mathrm{Sq}^k: H^n(X;\mathbb{Z}_2)\to H^{n+k}(X;\mathbb{Z}_2)$ which are operations on mod 2 cohomology rings satisfying the following properties

$$\begin{aligned}
&1) \quad \mathrm{Sq}^0(x) = x, \\
&2) \quad \mathrm{Sq}^k(x) = 0 \quad \text{if } k > \deg(x), \\
&3) \quad \mathrm{Sq}^{\deg(x)}(x) = x \cup x, \\
&4) \quad \mathrm{Sq}^k(x \cup y) = \sum_{i+j=k} \mathrm{Sq}^i(x) \cup \mathrm{Sq}^j(y) \quad \text{(Cartan's formula)}
\end{aligned} \tag{3.27}$$

Moreover, the Steenrod squares, being natural transformations of cohomology functors, have the property that they commute with the map $f^* : H^\bullet(Y;\mathbb{Z}_2) \to H^\bullet(X;\mathbb{Z}_2)$ induced on cohomology by a map $f : X \to Y$. Thus we have $f^*\mathrm{Sq}^k_Y = \mathrm{Sq}^k_X f^*$.

By virtue of this naturality, the Steenrod squares' action on $H^\bullet(BG_1 \times BG_2;\mathbb{Z}_2)$, which we denote by $\mathrm{Sq}^k_\times$ for clarity, are fully determined by their action on $H^\bullet(BG_1;\mathbb{Z}_2)$ and $H^\bullet(BG_2;\mathbb{Z}_2)$, denoted by Sq^k_1 and Sq^k_2. To see this, consider a projection $\pi_i : BG_1 \times BG_2 \to BG_i$, with $i = 1,2$. Let $c_i \in H^\bullet(BG_i;\mathbb{Z}_2)$ be a generator. By naturality we have $\mathrm{Sq}^k_\times(\pi^* c_i) = \pi^*(\mathrm{Sq}^k_i c_i)$. But since $\pi_i^* c_i$ is naturally identified with c_i through the Künneth theorem for cohomology, this gets simplified to

$$\mathrm{Sq}^k_\times c_i = \mathrm{Sq}^k_i c_i. \tag{3.28}$$

With help from Cartan's formula (3.27), the Steenrod squares' action on any generator of $H^\bullet(BG_1 \times BG_2;\mathbb{Z}_2)$ can be subsequently worked out.

3.4 Global anomalies in the Standard Model(s)

Now that we have laid the groundwork and described the computational tools we use to identify potential global anomalies, we are ready to report our computations. We begin with a gauge theory of indisputable importance to particle physics phenomenology, namely the Standard Model(s). Our results for the SM gauge groups are summarised in Table 3.1.

The Standard Model (SM) of particle physics is a four-dimensional gauge theory, with gauge group

$$G = \frac{G_{\mathrm{SM}}}{\Gamma_n}, \qquad G_{\mathrm{SM}} = SU(3) \times SU(2) \times U(1), \quad \Gamma_n \cong \mathbb{Z}_n, \quad n \in \{1,2,3,6\}. \tag{3.29}$$

Here, the $\mathbb{Z}_6$ quotient in the case of Γ_6 is generated by the element

$$\xi = (\omega, \eta, e^{2\pi i/6}) \in G_{\mathrm{SM}}, \tag{3.30}$$

where ω is the generator of the $\mathbb{Z}_3$ centre of $SU(3)$ (with $\omega^3 = \mathbf{1} \in SU(3)$), and η is the generator of the $\mathbb{Z}_2$ centre of $SU(2)$ (with $\eta^2 = \mathbf{1} \in SU(2)$). The Γ_3 quotient in (3.29) is generated by ξ^2, and the Γ_2 quotient by ξ^3. The fermion content of the SM consists of quarks and leptons, which are chiral fermions transforming in the following representations of G

$$Q \sim (\mathbf{3},\mathbf{2})_{1/6}, \quad U^c \sim (\overline{\mathbf{3}},\mathbf{1})_{-2/3}, \quad D^c \sim (\overline{\mathbf{3}},\mathbf{1})_{1/3}, \quad L \sim (\mathbf{1},\mathbf{2})_{-1/2}, \quad E^c \sim (\mathbf{1},\mathbf{1})_{1},$$

where here all the fields indicated are left-handed.

We compute the fifth bordism group (preserving spin structure) for all four groups listed in Eq. (3.29), and so identify potential global anomalies in these theories. Recall that in Refs. [74, 64], it was argued that there are no global anomalies in the SM with any of these four gauge groups, by fitting all four possibilities inside an $SU(5)$ GUT which is easily shown to be anomaly-free (since the computation of the bordism group for $SU(n)$ is straightforward). What we shall prove is a more general result, since it shall apply to gauge theories with one of these four gauge groups, but with *arbitrary* fermion content. Thus, the results we find shall apply immediately to any BSM theories in which the gauge group is that of the SM, but in which there are additional chiral fermion fields.

3.4.1 Hypercharge constraints

Before we start computing bordism groups, it is important to point out that if we extend the SM by adding extra fermions, one must make sure that such fermions transform in *bona fide* representations of whichever gauge group from Eq. (3.29) is being considered. In the cases where $G = G_{\text{SM}}/\Gamma_n$ with $n \in \{2,3,6\}$ there are constraints on the possible hypercharges fermions can take, depending on their representation under the $SU(3) \times SU(2)$ factor of G_{SM}. Since the derivations of these constraints involve a digression into representation theory, we relegate them to Appendix 3.D. In this Section we simply record what these constraints are – specifically, see Eqns (3.33, 3.36, 3.38). (Needless to say, the SM fermion representations satisfy these constraints.)

The Γ_2 quotient case

Given the $\mathbb{Z}_2$ quotient in the case $G = G_{\text{SM}}/\Gamma_2$ is generated by ξ^3, where ξ is given in Eq. (3.30), we can write this particular quotient of the SM gauge group as

$$\frac{G_{\text{SM}}}{\Gamma_2} = SU(3) \times \frac{SU(2) \times U(1)}{\mathbb{Z}_2} \cong SU(3) \times U(2). \tag{3.31}$$

In addition to its use in deriving the hypercharge constraints, writing the gauge group in this way (*i.e.* as a product) is crucial to our strategy for computing its bordism groups, in §3.4.3. Focussing on the $U(2) = (SU(2) \times U(1))/(\mathbb{Z}_2)$ factor of G, a representation of $U(2)$ corresponds to a representation of $SU(2) \times U(1)$, which in this subsection we denote by (j,q) where j denotes the isospin-j representation of $SU(2)$ (which has dimension $2j+1$) and $q \in \mathbb{Z}$ is the integer-normalised $U(1)$ charge, with some restrictions imposed.

To see how these constraints arise, let us first consider a field ψ transforming in the representation $(\frac{1}{2}, q)$, *i.e.* in the fundamental representation of $SU(2)$, since this is the simplest case. This means that $\psi \mapsto \psi' = \exp(iq\theta)\, \sigma \cdot \psi$ under the action of the $U(2)$ group element corresponding to $(\sigma, \exp i\theta) \in SU(2) \times U(1)$. For this to be a kosher representation of $U(2)$, one must identify the action of $(\mathbf{1}, \exp i\pi)$ and $(-\mathbf{1}, 1)$, which gives us the constraint $\exp iq\pi = -1$. Therefore, any $SU(2)$ doublet must have hypercharge

$$q = 1 \bmod 2, \tag{3.32}$$

i.e. an odd integer.[18] This is the case in the SM, where the doublet representations Q and L carry hypercharges 1 and -3 respectively, using an integer normalisation in which the smallest charge (that belonging to Q) is set to one.

If one wishes to add additional electroweak doublets, choosing the gauge group (3.31), one must ensure they too have odd hypercharges.

If one adds additional BSM fields transforming in larger representations of $SU(2)$, there are similar constraints on their hypercharges if they are to embed in representations of $U(2)$. To wit, for a field transforming in the (j,q) representation, the hypercharge must satisfy

$$q = 2j \bmod 2. \tag{3.33}$$

In other words, the charge must be even for all integer isospin representations (including, of course, any $SU(2)$ singlets), and odd for all half-integer isospin representations. For the proof of this general statement, we refer the reader to Appendix 3.D.

[18] Similar restrictions on $U(1)$ charges appear in the context of defining fermions on manifolds that are not necessarily spin, by using the $U(1)$ gauge symmetry to define a spin$_c$ structure. In that context, such charge restrictions depend on the representations of fermions under the Lorentz group, and are thus referred to as 'spin-charge relations' [122]. We consider these spin-charge relations more in §3.7.

The Γ_3 quotient case

Given the $\mathbb{Z}_3$ quotient in the case $G = G_{\text{SM}}/\Gamma_3$ is generated by the element ξ^2, we can write this variant of the SM gauge group in the more useful form

$$\frac{G_{\text{SM}}}{\Gamma_3} = \frac{SU(3)\times U(1)}{\mathbb{Z}_3} \times SU(2) \cong U(3)\times SU(2), \tag{3.34}$$

In this case, we obtain hypercharge constraints on any fields transforming non-trivially under $SU(3)$, by requiring that they embed in representations of $U(3)$.

Consider the simplest case of a field ψ transforming in the fundamental triplet representation of $SU(3)$ (*a.k.a.* a quark) and with charge q under $U(1)$. Under the action of $\exp(iq\theta)g \in U(3)$, for some $g \in SU(3)$, we have that $\psi \mapsto \psi' = \exp(iq\theta)g\cdot\psi$. To be a *bona fide* representation of $U(3)$ means that $(\exp^{2\pi i/3}, \mathbf{1}_3)$ and $(1, \omega = e^{2\pi i/3}\mathbf{1}_3)$ are identified in $SU(3)\times U(1)$, giving the constraint $e^{2q\pi i/3} = e^{2\pi i/3}$. Hence, any colour triplet must have hypercharge

$$q = 1 \bmod 3. \tag{3.35}$$

The SM quark fields Q, U, and D have hypercharges $+1$, $+4$, and -2 respectively, all of which are indeed equal to 1 mod 3.

One might consider adding fermions in other representations of $SU(3)$, and for each representation there is a corresponding hypercharge constraint. Irreducible representations of $SU(3)$ correspond to Young diagrams with two rows, and so can be labelled by a pair integers (λ_1, λ_2) corresponding to the number of boxes in each of the two rows, with $\lambda_1 \geq \lambda_2 \geq 0$. In Appendix 3.D, we prove that the hypercharge q of a field transforming in the (λ_1, λ_2) representation of $SU(3)$ must satisfy

$$q = (\lambda_1 + \lambda_2) \bmod 3, \tag{3.36}$$

if the gauge group is $U(3)\times SU(2)$. Note in particular that any colour singlets must have charge $q \in 3\mathbb{Z}$, as is the case for the SM leptons.

The Γ_6 quotient case

Finally, we discuss the case with gauge group $G = G_{\text{SM}}/\Gamma_6$. Consider a field in an arbitrary representation of this gauge group, corresponding to the (λ_1, λ_2) representation of $SU(3)$, the isospin-j representation of $SU(2)$, and with $U(1)$ charge q. The hypercharge constraint is that

$$q = 2j \bmod 2 = (\lambda_1 + \lambda_2) \bmod 3 \tag{3.37}$$

(see Appendix 3.D). For example, for a field with $j = 1/2$ and $(\lambda_1, \lambda_2) = (1,0)$, *i.e.* corresponding to the bifundamental representation of $SU(3) \times SU(2)$, this constraint reduces to

$$q = 1 \bmod 6. \tag{3.38}$$

The only SM fermion transforming in the bifundamental representation of $SU(3) \times SU(2)$ is the left-handed quark doublet Q, and sure enough the charge of Q is one.

Having established these constraints on the hypercharges of fermion fields for these four versions of the SM gauge group, we now turn to our main concern, which is to compute the bordism groups of BG for each of the four possible gauge groups G, which detect potential global anomalies theories with these gauge groups. We begin with the simplest case.

3.4.2 $\Omega_5^{\mathbf{Spin}}(BG_{\mathbf{SM}})$

For the simplest case where $G = G_{\mathrm{SM}} = SU(3) \times SU(2) \times U(1)$ with a regular spin structure, we use the AHSS associated with the fibration (3.17) to compute the bordism groups $\Omega_{d\leq 5}^{\mathrm{Spin}}(BG_{\mathrm{SM}})$.

To begin, we have that

$$B\left[SU(3) \times SU(2) \times U(1)\right] = BSU(3) \times BSU(2) \times BU(1). \tag{3.39}$$

Together with the Künneth formula in cohomology, this means that the cohomology ring of BG_{SM} is generated by the Chern classes associated with each factor of the gauge group,

$$H^\bullet\left(BG_{\mathrm{SM}};\mathbb{Z}\right) \cong \mathbb{Z}\left[x, c_2', c_2, c_3\right], \tag{3.40}$$

where $x \in H^2\left(BG_{\mathrm{SM}};\mathbb{Z}\right)$ indicates the first Chern class associated with the $U(1)$ factor, $c_2' \in H^4\left(BG_{\mathrm{SM}};\mathbb{Z}\right)$ indicates the second Chern class of $SU(2)$, and $c_2 \in H^4\left(BG_{\mathrm{SM}};\mathbb{Z}\right)$ and $c_3 \in H^6\left(BG_{\mathrm{SM}};\mathbb{Z}\right)$ indicate the second and third Chern classes respectively of the $SU(3)$ factor. We thus have the following low dimension cohomology groups

$$\begin{aligned}
H^0\left(BG_{\mathrm{SM}};\mathbb{Z}\right) &\cong \mathbb{Z}, \\
H^2\left(BG_{\mathrm{SM}};\mathbb{Z}\right) &\cong \mathbb{Z}, \\
H^4\left(BG_{\mathrm{SM}};\mathbb{Z}\right) &\cong \mathbb{Z}^3, \\
H^6\left(BG_{\mathrm{SM}};\mathbb{Z}\right) &\cong \mathbb{Z}^4,
\end{aligned} \tag{3.41}$$

with all cohomology groups in odd degrees vanishing. Because of this, and because these groups are all torsion-free, there is a (non-canonical) isomorphism

$$H_{2k}\left(BG_{\mathrm{SM}};\mathbb{Z}\right) \cong H^{2k}\left(BG_{\mathrm{SM}};\mathbb{Z}\right), \tag{3.42}$$

yielding the homology groups that we need to populate the entries of the second page of the AHSS relevant for computing the bordism groups $\Omega_d^{\mathrm{Spin}}(BG_{\mathrm{SM}})$ up to $d=5$, since we know that

$$E^2_{p,q} = H_p(BG_{\mathrm{SM}};\Omega_q^{\mathrm{Spin}}(\mathrm{pt})) = H_p(BG_{\mathrm{SM}};\mathbb{Z}) \otimes \Omega_q^{\mathrm{Spin}}(\mathrm{pt}), \tag{3.43}$$

where the bordism groups of a point $\Omega_q^{\mathrm{Spin}}(\mathrm{pt})$ are as listed in Eq. (3.18). The entries of the second page are shown in Fig. 3.5.

E^2 page

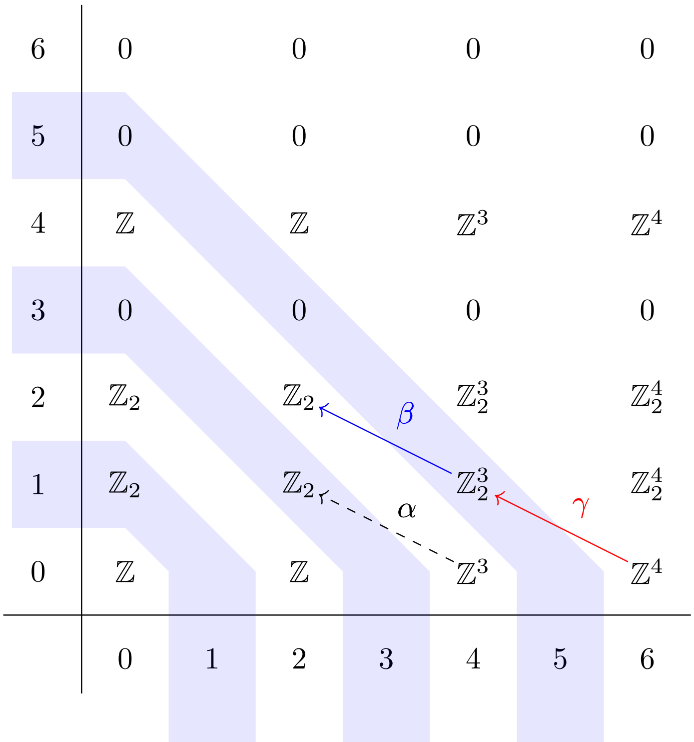


Fig. 3.5 The E^2 page of the Atiyah-Hirzebruch spectral sequence for $G = G_{\mathrm{SM}}$. We see that there is only a single entry relevant to the computation of $\Omega_5^{\mathrm{Spin}}(BG_{\mathrm{SM}})$, with a map (γ) going in and a map (β) going out.

Since the action of the Steenrod square on the generators of $H^\bullet(BSU(n);\mathbb{Z}_2)$, which are the universal Chern classes, is given by the formula [74]

$$\mathrm{Sq}^2(c_i) = (i-1)\,c_{i+1}$$

the Steenrod square action on each of the generators of the cohomology ring (3.40) is then given by

$$\begin{aligned} \mathrm{Sq}^2(x) &= x^2, \\ \mathrm{Sq}^2(c_2') &= 0, \\ \mathrm{Sq}^2(c_2) &= c_3, \\ \mathrm{Sq}^2(c_3) &= 0, \end{aligned} \tag{3.44}$$

where x^2 is a shorthand notation for $x \cup x$, the cup product of cohomology classes. This follows from the third line of Eq. (3.27) and naturality of the Steenrod squares, as discussed at the end of §3.3. We see from Fig. 3.5 that there is only a single entry on the diagonal $p+q=5$ which is thus relevant to the computation of $\Omega_5^{\mathrm{Spin}}(BG_{\mathrm{SM}})$, and that is $E^2_{4,1}$. We need to compute what this stabilises to, so we begin by turning to the third page, which requires us to compute the differentials labelled β and γ in Fig. 3.5.

Using the Steenrod squares (3.44), together with Eqs. (3.26) and the fact that $\Omega_1^{\mathrm{Spin}}(\mathrm{pt}) = \Omega_2^{\mathrm{Spin}}(\mathrm{pt}) \cong \mathbb{Z}_2$, we have that the differential labelled β in Fig. 3.5 is the dual of the Steenrod square

$$\begin{aligned} \mathrm{Sq}^2 : H^2(BG_{\mathrm{SM}};\mathbb{Z}_2) &\longrightarrow H^4(BG_{\mathrm{SM}};\mathbb{Z}_2) \\ x &\mapsto x^2. \end{aligned} \tag{3.45}$$

Let us denote the generators of $E^2_{4,1} \cong \mathbb{Z}_2^3$ as $\widetilde{x^2}$, $\widetilde{c_2'}$, and $\widetilde{c_2}$, which are dual to the generators x^2, c_2', $c_2 \in H^4(BG_{\mathrm{SM}};\mathbb{Z}_2)$ by the Kronecker pairing (denoted $\langle\cdot,\cdot\rangle$) between homology and cohomology. Then we see that

$$\begin{aligned} \left\langle \widetilde{\mathrm{Sq}^2}\widetilde{x^2}, x \right\rangle &= \left\langle \widetilde{x^2}, x^2 \right\rangle = 1, \\ \left\langle \widetilde{\mathrm{Sq}^2}\widetilde{c_2'}, x \right\rangle &= \left\langle \widetilde{c_2'}, x^2 \right\rangle = 0, \\ \left\langle \widetilde{\mathrm{Sq}^2}\widetilde{c_2}, x \right\rangle &= \left\langle \widetilde{c_2}, x^2 \right\rangle = 0, \end{aligned} \tag{3.46}$$

where $\widetilde{\mathrm{Sq}^2}$ denotes the dual Steenrod square. Hence, the kernel of β is $\ker\beta \cong \mathbb{Z}_2^2$, generated by $\widetilde{c_2'}$ and $\widetilde{c_2}$.

The differential labelled γ in Fig. 3.5 is the composition of the dual Steenrod square and the reduction mod 2:

$$\gamma : \mathbb{Z}^4 \xrightarrow{\text{mod } 2} \mathbb{Z}_2^4 \xrightarrow{\widetilde{\mathrm{Sq}^2}} \mathbb{Z}_2^3, \tag{3.47}$$

where the relevant Steenrod square is

$$\begin{aligned} \mathrm{Sq}^2 : H^4\left(BG_{\mathrm{SM}};\mathbb{Z}_2\right) &\longrightarrow H^6\left(BG_{\mathrm{SM}};\mathbb{Z}_2\right) \\ x^2 &\mapsto 2x^3 = 0 \bmod 2, \\ c_2' &\mapsto 0, \\ c_2 &\mapsto c_3, \end{aligned} \tag{3.48}$$

where to deduce $x^2 \mapsto 2x^3$ we have used Cartan's formula (3.27) and the fact that $\mathrm{Sq}^1(x) = 0$ as H^3 is trivial. Again using the Kronecker pairing, we deduce that $\widetilde{\mathrm{Sq}^2}$ kills $\widetilde{x^3}$, $\widetilde{c_2 \cup x}$, $\widetilde{c_2' \cup x}$, and sends $\widetilde{c_3}$ to $\widetilde{c_2}$. Therefore $\mathrm{im}\gamma \cong \mathbb{Z}_2$, generated only by $\widetilde{c_2}$. We can then take the homology with respect to the differentials β and γ to turn the page of the AHSS and deduce the $(4,1)$ element of the third page,

$$E^3_{4,1} = \frac{\ker\beta}{\mathrm{im}\gamma} \cong \frac{\mathbb{Z}_2^2}{\mathbb{Z}_2} \cong \mathbb{Z}_2. \tag{3.49}$$

Since the entries in every odd column vanish, there are no non-trivial differentials on the third page, and so we can turn to the fourth page with $E^4_{p,q} = E^3_{p,q}$ for all (p,q).

On the fourth page the only differential relevant to computing $\Omega_5^{\mathrm{Spin}}(BG_{\mathrm{SM}})$ is $d^4 : E^4_{4,1} \to E^4_{0,5}$, which is a homomorphism from $\mathbb{Z}_2$ to $\mathbb{Z}$ and is thus trivial. So the $(4,1)$ entry stabilises to $E_\infty^{4,1} \cong \mathbb{Z}_2$. Since this is the only non-zero element on the $p+q=5$ diagonal it follows that

$$\Omega_5^{\mathrm{Spin}}\left(BG_{\mathrm{SM}}\right) \cong \mathbb{Z}_2, \tag{3.50}$$

where we can identify the potential global anomaly in this theory with the Witten anomaly associated to the $SU(2)$ factor.

To see that this must be the case, consider a theory with gauge group G_{SM} and a single fermion transforming as a doublet under $SU(2)$ and a singlet under both $SU(3)$ and hypercharge. Using the Dai–Freed prescription for the fermionic partition function one obtains an anomalous theory because $\exp 2\pi i\eta = -1$ on $S^4 \times S^1$. This must therefore correspond to the non-trivial class in $\Omega_5^{\mathrm{Spin}}\left(BG_{\mathrm{SM}}\right)$.

We can continue to compute the bordism groups of BG_{SM} in lower degrees in a similar fashion. From Fig. 3.5 we can immediately read off

$$\Omega_0^{\text{Spin}}(BG_{\text{SM}}) \cong \mathbb{Z}, \qquad \text{and} \qquad \Omega_1^{\text{Spin}}(BG_{\text{SM}}) \cong \mathbb{Z}_2, \tag{3.51}$$

and it is straightforward to show that

$$\Omega_2^{\text{Spin}}(BG_{\text{SM}}) \cong \mathbb{Z} \times \mathbb{Z}_2. \tag{3.52}$$

Next, to compute $\Omega_3^{\text{Spin}}(BG_{\text{SM}})$, we need the differential

$$\alpha : \mathbb{Z}^3 \xrightarrow{\text{mod } 2} (\mathbb{Z}_2)^3 \xrightarrow{\widetilde{\text{Sq}^2}} \mathbb{Z}_2, \tag{3.53}$$

as well as the map $d^2_{2,1} : \mathbb{Z}_2 \to \mathbb{Z}_2$. The dual Steenrod square is precisely the same as for the map β, which maps $\widetilde{x^2} \mapsto \widetilde{x}$, and the other generators to zero, so we have that $\text{im}\,\alpha \cong \mathbb{Z}_2$. Then, we do not need to compute the map $d^2_{2,1}$ to deduce that its kernel must be $\mathbb{Z}_2$, because we know that $\text{im}\,\alpha \subset \ker d^2_{2,1}$. Hence, taking the homology, we deduce that $E^\infty_{2,1} = 0$. All elements on the $p+q=3$ diagonal thus stabilise to zero and we have that

$$\Omega_3^{\text{Spin}}(BG_{\text{SM}}) = 0. \tag{3.54}$$

To compute $\Omega_4^{\text{Spin}}(BG_{\text{SM}})$, we know from above that the map β into $E^2_{2,2}$ has image $\text{im}\,\beta \cong \mathbb{Z}_2$, generated by the element $\widetilde{x} \in H_2(BG_{\text{SM}};\mathbb{Z}_2)$. The map out of $E^2_{2,2}$ is to zero and so its kernel is $\mathbb{Z}_2$; turning to the next page, this element therefore stabilises at $\mathbb{Z}_2/\mathbb{Z}_2 = 0$. More care is required to deduce $\ker\alpha$, as follows. We have that $\widetilde{c_2'}$ and $\widetilde{c_2}$ certainly map to zero, where note that the elements $\widetilde{x^2}$, $\widetilde{c_2'}$, and $\widetilde{c_2}$ are here valued in integral homology (rather than in homology with coefficients in $\mathbb{Z}_2$). Thus, while $\widetilde{x^2} \in H^4(BG_{\text{SM}};\mathbb{Z})$ maps to the non-zero element $\widetilde{x} \in H^2(BG_{\text{SM}};\mathbb{Z}_2)$, the element $2\widetilde{x^2} \in H^4(BG_{\text{SM}};\mathbb{Z})$ maps to zero in $H^2(BG_{\text{SM}};\mathbb{Z}_2)$. Hence, the map α has a kernel $\ker\alpha \cong \mathbb{Z}^3$ (which may look strange given its image is non-zero), and so we deduce $E^\infty_{4,0} \cong \mathbb{Z}^3$. Given also that $E^\infty_{0,4} \cong \mathbb{Z}$, we compute

$$\Omega_4^{\text{Spin}}(BG_{\text{SM}}) \cong \mathbb{Z}^4, \tag{3.55}$$

thus concluding our computation of the bordism groups $\Omega^{\text{Spin}}_{d\leq 5}(BG_{\text{SM}})$ for the SM gauge group without a quotient. This result, along with others, is summarized in Table 3.1.

G	$\Omega_d^{\text{Spin}}(BG)$ 0	1	2	3	4	5
$U(1)\times SU(2)\times SU(3)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_2$	0	$\mathbb{Z}^4$	$\mathbb{Z}_2$
$(U(1)\times SU(2)\times SU(3))/\Gamma_2$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_2$	0	$\mathbb{Z}^4$	0
$(U(1)\times SU(2)\times SU(3))/\Gamma_3$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_2$	0	$\mathbb{Z}^4$	$\mathbb{Z}_2$
$(U(1)\times SU(2)\times SU(3))/\Gamma_6$	$\mathbb{Z}$	$\mathbb{Z}_2$	$e(\mathbb{Z}_3,\mathbb{Z}\times\mathbb{Z}_2)$	0	$e(\mathbb{Z}_3,e(\mathbb{Z}_3,\mathbb{Z}^4))$	0

Table 3.1 Summary of results from our bordism computations for the four possible SM gauge groups. We tabulate the bordism groups in degrees zero through five.

3.4.3 $\Omega_5^{\textbf{Spin}}(B(G_{\textbf{SM}}/\Gamma_2))$

We now turn to compute the bordism groups for the variants of the SM involving quotients of G_{SM} by discrete subgroups of its center, as listed in Eq. (3.1). Recall from §3.4.1 that

$$\frac{G_{\text{SM}}}{\Gamma_2} \cong SU(3)\times U(2). \tag{3.56}$$

Hence $B(G_{\text{SM}}/\Gamma_2) = BU(2)\times BSU(3)$ using (3.19). This is useful, because the cohomology ring of the classifying space of the groups $U(n)$ is well-known.

Using the usual fibration pt $\longrightarrow B(G_{\text{SM}}/\Gamma_2) \longrightarrow B(G_{\text{SM}}/\Gamma_2)$, the second page of the AHSS is given by $E^2_{p,q} = H_p\left(BU(2)\times BSU(3);\Omega_q^{\text{Spin}}(\text{pt})\right)$, as shown in figure 3.6. Recall that the relevant cohomology rings are

$$\begin{aligned} H^\bullet\left(BSU(3);\mathbb{Z}\right) &\cong \mathbb{Z}[c_2,c_3] \\ H^\bullet\left(BU(2);\mathbb{Z}\right) &\cong \mathbb{Z}[c_1',c_2'] \end{aligned} \tag{3.57}$$

where c_i, c_i' are the ith Chern classes (which are cohomology classes in degree $2i$) for $SU(3)$ and $U(2)$, respectively. Thus, we have the integral cohomology groups

$$\begin{aligned} &H^0\left(B(G_{\text{SM}}/\Gamma_2);\mathbb{Z}\right) \cong \mathbb{Z}, \\ &H^2\left(B(G_{\text{SM}}/\Gamma_2);\mathbb{Z}\right) \cong \mathbb{Z}, \quad \text{generated by} \quad c_1', \\ &H^4\left(B(G_{\text{SM}}/\Gamma_2);\mathbb{Z}\right) \cong \mathbb{Z}^3, \quad \text{generated by} \quad c_1'^2,\ c_2',\ c_2, \\ &H^6\left(B(G_{\text{SM}}/\Gamma_2);\mathbb{Z}\right) \cong \mathbb{Z}^4, \quad \text{generated by} \quad c_1'^3,\ c_1'c_2',\ c_1'c_2,\ c_3. \end{aligned} \tag{3.58}$$

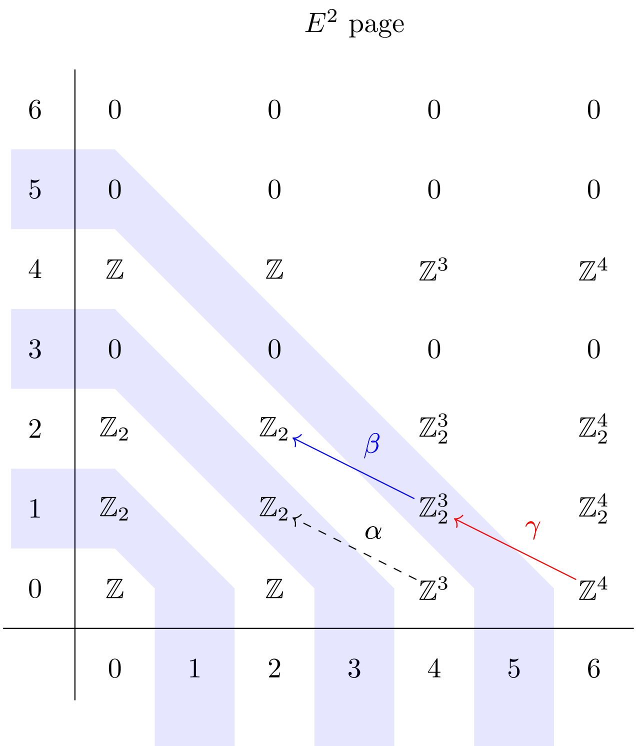


Fig. 3.6 The E^2 page of the Atiyah-Hirzebruch spectral sequence for $G = U(2) \times SU(3)$, with differentials relevant to the computation of the fourth and fifth bordism groups labelled.

Again, because these are torsion-free and the cohomology groups all vanish in odd degrees, we deduce from these the integral homology groups,

$$H_{2k}\left(B(G_{\text{SM}}/\Gamma_2);\mathbb{Z}\right) \cong H^{2k}\left(B(G_{\text{SM}}/\Gamma_2);\mathbb{Z}\right). \tag{3.59}$$

Thus far, this appears superficially identical to the case of no discrete quotient considered above, and indeed the second page of the AHSS is populated by the same groups; however, the action of the Steenrod squares is subtly different, meaning the action of the differentials (and, specifically, the maps α, β, and γ) is not necessarily the same as above. It turns out that an important difference shall be in the map γ. In particular, since the action of the Steenrod square on the generators c_i of $H^\bullet(BU(n);\mathbb{Z}_2) \cong \mathbb{Z}_2[c_1,\ldots,c_n]$ is given by [40]

$$\text{Sq}^2(c_i) = c_1 \cup c_i + (i-1)c_{i+1}, \tag{3.60}$$

we have that its action on the generators of the cohomology ring of $B(U(2)\times SU(3))$ is

$$\begin{aligned}
\mathrm{Sq}^2(c_1') &= c_1'^2, \\
\mathrm{Sq}^2(c_2') &= c_1' \cup c_2', \\
\mathrm{Sq}^2(c_2) &= c_3, \\
\mathrm{Sq}^2(c_3) &= 0.
\end{aligned} \tag{3.61}$$

Notice the second line in particular, to be contrasted with the second line in Eq. (3.44). As before, this follows from naturality of the Steenrod square.

The differentials relevant to the calculation of $\Omega_4^{\mathrm{Spin}}(B(G_{\mathrm{SM}}/\Gamma_2))$ and $\Omega_5^{\mathrm{Spin}}(B(G_{\mathrm{SM}}/\Gamma_2))$ are again given by

$$\begin{aligned}
\alpha &= \widetilde{\mathrm{Sq}^2} \circ \rho, \\
\beta &= \widetilde{\mathrm{Sq}^2}, \\
\gamma &= \widetilde{\mathrm{Sq}^2} \circ \rho,
\end{aligned} \tag{3.62}$$

where ρ denotes reduction modulo 2. Since $\mathrm{Sq}^2 : H^2 \to H^4$ maps $c_1' \mapsto c_1'^2$, we see that both α, β map $\widetilde{c_1'^2} \mapsto \widetilde{c_1'}$ and others to zero. Moreover, α maps $2\widetilde{c_1'^2}$ to zero. So we have, using similar arguments as before, that

$$\ker\alpha \cong \mathbb{Z}^3, \quad \mathrm{im}\alpha \cong \mathbb{Z}_2, \qquad \ker\beta = (\mathbb{Z}_2)^2, \quad \mathrm{im}\beta \cong \mathbb{Z}_2, \tag{3.63}$$

which is as it was in the previous case.

We now turn to the map γ. The relevant Steenrod square is here

$$\begin{aligned}
\mathrm{Sq}^2 : H^4(B(G_{\mathrm{SM}}/\Gamma_2);\mathbb{Z}_2) &\longrightarrow H^6(B(G_{\mathrm{SM}}/\Gamma_2);\mathbb{Z}_2) \\
c_1'^2 &\mapsto 2c_1'^3 \equiv 0 \bmod 2, \\
c_2' &\mapsto c_1' \cup c_2', \\
c_2 &\mapsto c_3,
\end{aligned} \tag{3.64}$$

where the third line should be contrasted with that in Eq. (3.48). So γ maps $\widetilde{c_1' \cup c_2'} \mapsto \widetilde{c_2'}$ and $\widetilde{c_3} \mapsto \widetilde{c_2}$, while mapping other generators to zero. This gives $\mathrm{im}\gamma \cong (\mathbb{Z}_2)^2$. Then

$$E^3_{4,1} = \frac{\ker\beta}{\mathrm{im}\gamma} = 0, \tag{3.65}$$

to be contrasted with the non-zero result in Eq. (3.49). Thus, this entry stabilises, and there are no non-zero entries on the diagonal $p+q=5$ of the last page of this AHSS. Hence, we deduce

$$\Omega_5^{\mathrm{Spin}}(B(G_{\mathrm{SM}}/\Gamma_2)) = 0, \tag{3.66}$$

and thus that this version of the SM has no global anomalies, no matter what the fermion content. One can compute the bordism groups in lower degrees using the same methods as in the previous example, and one finds no other differences in the results, which are again recorded in Table 3.1.

We thus arrive at a seemingly curious result; there are no global anomalies in this version of the SM, for arbitrary fermion content. The reader might wonder what has happened to the Witten anomaly, and the condition that there must be an even number of $SU(2)$ doublets in the theory. We discuss the resolution to this puzzle (which also occurs in the case $G = G_{\mathrm{SM}}/\Gamma_6$) in §3.4.6. For now, it might be useful to remark on what goes wrong with the argument of the previous Section, in which we considered a theory with a single fermion in the spin-$\frac{1}{2}$ representation of $SU(2)$ (and a singlet under both $SU(3)$ and $U(1)$), and claimed $\exp 2\pi i\eta = -1 \neq 1$ on $S^1 \times S^4$. We cannot use such an argument when $G = G_{\mathrm{SM}}/\Gamma_2$, because the hypercharge constraints presented in §3.4.1 mean there is no such representation of the gauge group, because any $SU(2)$ doublet fermion must have *odd* (and thus non-zero) hypercharge. We must then take care to ensure that *local* anomalies associated with hypercharge cancel, before we turn to the global anomalies. We return to this issue in §3.4.6.

3.4.4 $\Omega_5^{\mathbf{Spin}}(B(G_{\mathbf{SM}}/\Gamma_3))$

Our approach for tackling this variant of the SM is qualitatively very similar to that employed for the $\mathbb{Z}_2$ quotient in the previous Subsection. Recall from §3.4.1 that the gauge group here may written as

$$\frac{G_{\mathrm{SM}}}{\Gamma_3} \cong U(3) \times SU(2). \tag{3.67}$$

One may tackle this variant of the SM using the same methods employed for the $\mathbb{Z}_2$ quotient in the previous Subsection. Thus, to avoid repetition, we relegate the calculations for this gauge group to Appendix 3.C. The upshot is that we find

$$\Omega_5^{\mathrm{Spin}}(B(G_{\mathrm{SM}}/\Gamma_3)) \cong \mathbb{Z}_2, \tag{3.68}$$

corresponding to the Witten anomaly associated with the $SU(2)$ factor in (3.67). The lower-degree bordism groups are tabulated in Table 3.1.

For this gauge group, an alternative fibration exists which we can also use to compute the bordism groups, based on the Puppe sequence. Reassuringly, using this other fibration yields the same bordism groups, and we include the details of both methods in Appendix 3.C. We will need to employ such a Puppe-induced fibration shortly in §3.4.5 to compute the bordism groups of $B(G_{\text{SM}}/\Gamma_6)$.

3.4.5 $\Omega_5^{\text{Spin}}(B(G_{\text{SM}}/\Gamma_6))$

The $\mathbb{Z}_6$ quotient in the case $G = G_{\text{SM}}/\Gamma_6$ is generated by the element ξ given by (3.30), and there is no straightforward way to write the group G_{SM}/Γ_6 as a product, as we did in the previous two cases. This means a direct attempt to use the AHSS to compute the bordism groups of G_{SM}/Γ_6 seems unlikely to work, given we do not know how the differentials on the second page act.

Instead, we consider the following fibration[19]

$$\mathbb{Z}_3 \longrightarrow U(2) \times SU(3) \longrightarrow G_{\text{SM}}/\Gamma_6. \tag{3.69}$$

This induces the fibration $B(\mathbb{Z}_3) \to B(U(2) \times SU(3)) \to B(G_{\text{SM}}/\Gamma_6)$, which turns into the following, more useful, fibration after we invoke the Puppe sequence (we here follow a similar strategy to that used in Ref. [80]):

$$B(U(2) \times SU(3)) \longrightarrow B(G_{\text{SM}}/\Gamma_6) \longrightarrow K(\mathbb{Z}_3, 2), \tag{3.70}$$

where $K(\mathbb{Z}_3, 2) = B(B(\mathbb{Z}_3))$ is an Eilenberg-Maclane space.

The second page of the AHSS associated with this fibration is given by

$$E^2_{p,q} = H_p\left(K(\mathbb{Z}_3, 2); \Omega_q^{\text{Spin}}(B(U(2) \times SU(3)))\right). \tag{3.71}$$

While this may look like a rather unwieldy expression, note that the bordism groups $\Omega_q^{\text{Spin}}(B(U(2) \times SU(3)))$ are precisely those that we have already computed in our study of global anomalies for the case $G = G_{\text{SM}}/\Gamma_2$, as recorded in the second line of Table 3.1. These groups only feature factors of $\mathbb{Z}$ and $\mathbb{Z}_2$, and the homology groups of the Eilenberg-Maclane

[19] We note, to avoid confusion, that there also exists a fibration of the group $U(2) \times SU(3)$ over $U(2) \times PSU(3)$ (which cannot be the gauge group of the Standard Model because $PSU(3)$ does not admit a triplet representation) with the same homotopy fibre. While this fibration would be written using the same notation as (3.69), the maps are, of course, different.

space $K(\mathbb{Z}_3,2)$ valued in $\mathbb{Z}$ and $\mathbb{Z}_2$ are [42]

$$\begin{array}{r|cccccc} i & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline H_i(K(\mathbb{Z}_3,2);\mathbb{Z}) & \mathbb{Z} & 0 & \mathbb{Z}_3 & 0 & \mathbb{Z}_3 & 0 \\ H_i(K(\mathbb{Z}_3,2);\mathbb{Z}_2) & \mathbb{Z}_2 & 0 & 0 & 0 & 0 & 0. \end{array} \tag{3.72}$$

We can thence compute all the entries (3.71) in the second page of the AHSS. These are shown in Fig. 3.7.

E^2 page

5	0		0		0	0
4	$\mathbb{Z}^4$		$\mathbb{Z}_3^4$		$\mathbb{Z}_3^4$	0
3	0		0		0	0
2	$\mathbb{Z}\times\mathbb{Z}_2$		$\mathbb{Z}_3$		$\mathbb{Z}_3$	0
1	$\mathbb{Z}_2$		0		0	0
0	$\mathbb{Z}$		$\mathbb{Z}_3$		$\mathbb{Z}_3$	0
	0	1	2	3	4	5

Fig. 3.7 The second page of the Atiyah-Hirzebruch spectral sequence corresponding to the fibration (3.70). The entries relevant to the computation of $\Omega_5^{\mathrm{Spin}}(BG_{\mathrm{SM}}/\Gamma_6)$ are highlighted, all of which vanish already on the second page.

Somewhat fortunately (for the sake of being able to perform the computation), all the entries on the $p+q=5$ diagonal relevant for the computation of $\Omega_5^{\mathrm{Spin}}(BG_{\mathrm{SM}}/\Gamma_6)$ vanish already on the second page. This is just as well, because for this fibration we do not know any formulae for the action of the differentials (with which to turn to the next page) in terms of Steenrod squares (or indeed any other operation on (co)homology).[20] We thus conclude

[20] Note that the similar-looking fibration $\mathbb{Z}_2 \longrightarrow U(3)\times SU(2) \longrightarrow G_{\mathrm{SM}}/\Gamma_6$ does not yield such simplifications, and so cannot be used to compute the relevant bordism group because there are unknown differentials on the second page. This is roughly because the homology of $K(\mathbb{Z}_2,2)$ is 'more complicated' than that of $K(\mathbb{Z}_3,2)$.

that

$$\Omega_5^{\mathrm{Spin}}(B(G_{\mathrm{SM}}/\Gamma_6)) = 0. \tag{3.73}$$

Since all relevant homomorphisms are trivial, all entries $E_{p,q}$ with $p+q<5$ stabilise on the second page. We can then compute the remaining bordism groups with degree lower than 5 without ambiguities apart from $\Omega_2^{\mathrm{Spin}}(B(G_{\mathrm{SM}}/\Gamma_6))$ and $\Omega_4^{\mathrm{Spin}}(B(G_{\mathrm{SM}}/\Gamma_6))$ due to non-splitting extensions. They are given by

$$\begin{aligned}
\Omega_2^{\mathrm{Spin}}(B(G_{\mathrm{SM}}/\Gamma_6)) &\cong e(\mathbb{Z}_3, \mathbb{Z}\times\mathbb{Z}_2),\\
\Omega_4^{\mathrm{Spin}}(B(G_{\mathrm{SM}}/\Gamma_6)) &\cong e(\mathbb{Z}_3, e(\mathbb{Z}_3,\mathbb{Z}^4)).
\end{aligned} \tag{3.74}$$

The notation $e(A,B)$ denotes a group extension of A by B, that is, a group that fits into the following short exact sequence

$$0 \longrightarrow B \longrightarrow e(A,B) \longrightarrow A \longrightarrow 0. \tag{3.75}$$

We tabulate our results in Table 3.1.

Note added: since this article appeared in preprint form, the Adams spectral sequence has been used to resolve the ambiguities we found (using the AHSS) in Eq. (3.74) [139]. It was therein found that

$$\Omega_2^{\mathrm{Spin}}(B(G_{\mathrm{SM}}/\Gamma_6)) \cong \mathbb{Z}\times\mathbb{Z}_2. \tag{3.76}$$

Comparing with our result (3.74), this corresponds to the non-trivial extension

$$0 \longrightarrow \mathbb{Z}\times\mathbb{Z}_2 \longrightarrow \mathbb{Z}\times\mathbb{Z}_2 \longrightarrow \mathbb{Z}_3 \longrightarrow 0, \tag{3.77}$$

where the first map is multiplication by 3 on the first factor and the identity on the second. In Ref. [139] it was also found that

$$\Omega_4^{\mathrm{Spin}}(B(G_{\mathrm{SM}}/\Gamma_6)) \cong \mathbb{Z}^4, \tag{3.78}$$

also corresponding to a non-trivial solution to the extension problem (3.74).

3.4.6 Interplay between global and local anomalies

It is interesting that there are no possible global anomalies in the cases with quotients by $\mathbb{Z}_2$ and $\mathbb{Z}_6$, whereas in the case of a quotient by $\mathbb{Z}_3$ (or the case with no quotient at all) there is a $\mathbb{Z}_2$ global anomaly which we have identified with the familiar Witten anomaly associated with the $SU(2)$ factor.

This might at first appear puzzling. We know that cancellation of the Witten anomaly in an $SU(2)$ gauge theory, and in the SM, requires $n_L - n_R = 0 \bmod 2$ if there are n_L (n_R) left-handed (right-handed) fermions in $SU(2)$ doublets. More generally, the Witten anomaly receives contributions from any fermions in $SU(2)$ representations with isospin $2r+1/2$, $r \in \mathbb{Z}$. Does the fact that we have computed that there are no such conditions for global anomaly cancellation in two variants of the SM mean that in these cases we can dispense with Witten's condition, and consider extensions of the SM with odd numbers of $SU(2)$ doublets? The answer is no, due to a subtle interplay between global and local anomaly cancellation, which we now describe.

The key point is that taking discrete quotients of G_{SM} changes the set of representations that fermions can carry, since every fermion must be in a *bona fide* representation of the group G. This leads to constraints on the possible hypercharges for fermions transforming as electroweak doublets. As we derived in §3.4.1, when we quotient G_{SM} by $\mathbb{Z}_2$ or $\mathbb{Z}_6$, any field transforming in the (j,q) representation of the $SU(2) \times U(1)$ factor must satisfy the isospin-charge relation

$$q = 2j \bmod 2. \tag{3.79}$$

Of course, one is free to perform an overall rescaling of all the $U(1)$ charges in the theory, so the precise statement is that there must exist a normalisation of the $U(1)$ gauge coupling such that the charge constraints (3.79) are possible. We assume such a normalisation for the $U(1)$ charges in the following.[21]

Now consider the cancellation of local anomalies. Suppose we have N_j fermions transforming in the $SU(2)$ representation with isospin j, and that these have charges denoted $\{q_j^{(a)}\}$, where $a = 1, \dots N_j$, and $q_j^{(a)} = 2j \bmod 2$. We assume that all fermions have left-handed chirality. The $SU(2)^2 \times U(1)$ anomaly coefficient is then proportional to

$$\sum_j T(j) \sum_{a=1}^{N_j} q_j^{(a)} = 0, \tag{3.80}$$

where the sum over j is over the different values of isospin, and $T(j)$ denotes the Dynkin index (defined such that $\text{Tr}\left(t_j^a t_j^b\right) = \frac{1}{2} T(j) \delta_{ab}$, where $\{t_j^a\}$ denotes a basis for $\mathfrak{su}(2)$ in the isospin$-j$ representation), which is given by the formula

$$T(j) = \frac{2}{3} j(j+1)(2j+1). \tag{3.81}$$

[21]Note that the local anomaly cancellation equations are homogeneous polynomials in rational charges, and thus are properly defined on a projective rational variety; thus, we are free to fix an overall normalisation as we wish.

This formula implies that $T(j)$ is odd when $j = 2r + 1/2$, $r \in \mathbb{Z}$, and is even otherwise.

When the anomaly condition (4.16) is reduced mod 2, only the contributions to (4.16) from isospins $2r + 1/2$ remain, since it is only these irreps for which both $T(j)$ and the charges $q_j^{(a)}$ are necessarily odd. We thus obtain

$$\sum_{j \in 2\mathbb{Z}+1/2} N_j = 0 \bmod 2. \tag{3.82}$$

In other words, in the theories with gauge groups G_{SM}/Γ_2 or G_{SM}/Γ_6, the total number of fermions transforming in isospin $2r + 1/2$ representations must be even, in order for the local $SU(2)^2 \times U(1)$ anomaly to cancel – even though there is no global anomaly in either of these cases. This is equivalent to the condition, in the $SU(2) \times U(1)$ case, that the usual Witten anomaly vanishes. This anomaly interplay will be explored more deeply in Chapter 4.

3.5 A generalisation of the SM

The Standard Model with gauge group $G_{\text{SM}} = SU(3) \times SU(2) \times U(1)$ is the starting point of a 2-parameter family of anomaly-free chiral gauge theories [137, 96]. The gauge group for this family of generalised Standard Model theories is

$$G_{\text{GSM}} = SU(N) \times Sp(M) \times U(1), \qquad N > 2 \text{ and odd}, \ M \geq 1 \tag{3.83}$$

It will be shown in Chapter 5 that theories in this family have the same phase structure as the Standard Model when one varies the relative strength between the strong force and the weak force. It is also not far-fetched to assume that this family of theories exhibits similar features in the infrared. This generalisation subjects the Standard Model to the framework of large-N expansion, which could potentially be used to analyse the dynamics of this family of chiral gauge theories perturbatively in a more controlled fashion.

The left-handed doublets of fermions that couple to the weak force in the Standard Model now become $2M$-tuplets in the fundamental representation of $Sp(M)$. Since there are $N + 1$ chiral fermions in the fundamental representation of $Sp(M)$, we need N to be odd to cancel the $\mathbb{Z}_2$ global anomaly. In order to have sufficient number of chiral fermions to cancel the local anomalies, the right-handed fermions must proliferate, and we end up with M copies each of right-handed electrons E_α, right-handed down quarks D_α, right-handed up quarks U_α, and right-handed neutrinos N_α, with $\alpha = 1, \ldots, M$. There are also M copies of the Higgs field, H_α. The matter content of this generalised theory and its representations under the gauge group G_{GSM} is given in full in Table 3.2. The simplest case with $M = 1$ and $N = 3$ gives the Standard Model.

	$U(1)$	$Sp(M)$	$SU(N)$
Q	$+1$	**2M**	**N**
L	$-N$	**2M**	**1**
D^c_α	$(2\alpha-1)N-1$	**1**	$\overline{\mathbf{N}}$
U^c_α	$-(2\alpha-1)N-1$	**1**	$\overline{\mathbf{N}}$
E^c_α	$2\alpha N$	**1**	**1**
N^c_α	$-(2\alpha-2)N$	**1**	**1**
H_α	$(2\alpha-1)N$	**2M**	**1**

Table 3.2 Matter content in the generalised Standard Model. In this table, the boldface characters denote the dimensions of the respective representations, with **2M** denoting the fundamental representation of $Sp(M)$ and **N** denoting the fundamental of $SU(N)$.

The hypercharges given in Table 3.2 are chosen so that the theory is free of local anomalies, and the theory is moreover free of Witten anomalies associated with the $Sp(M)$ factor. It is natural to ask whether this generalisation is really consistent for every (N, M) by considering our more general criterion for global anomalies, detected by $\Omega_5^{\text{Spin}}(BG_{\text{GSM}})$. Fortunately, we do not need to repeat our calculation of the spin bordism group for this new gauge group as it is the same as the calculation in §3.4.2. To see this, first recall that the relevant entries on the second page of the AHSS are given by

$$E^2_{p,q} = H_p(BSU(N) \times BSp(M) \times BU(1); \Omega_q^{\text{Spin}}(\text{pt}))$$

with $p + q \leq 6$. The Künneth theorem for homology then tells us that these entries depend only on $H_r(BSp(M))$ and $H_r(BSU(N))$ with $r \leq 6$. But note that the homology groups in low dimensions of $BSp(M)$ and $BSU(N)$ are given by,

$$H_p(BSp(M); \mathbb{Z}) = \{\mathbb{Z}, 0, 0, 0, \mathbb{Z}, 0, 0, \ldots\},$$
$$H_p(BSU(N); \mathbb{Z}) = \{\mathbb{Z}, 0, 0, 0, \mathbb{Z}, 0, \mathbb{Z}, \ldots\}.$$

which are the same as those of $SU(2)$ and $SU(3)$, respectively. Therefore, the relevant entries on the second page of the AHSS are still given by Fig. 3.5. Moreover, the action of the Steenrod square on the generators of lowest degrees of the cohohomology rings of $BSp(M)$ and $BSU(N)$ are the same as in the Standard Model case, giving rise to the same relevant differentials in Fig. 3.5. The calculation given in §3.4.2 then goes through unaltered. We then have that

$$\Omega_5^{\text{Spin}}(BG_{\text{GSM}}) \cong \mathbb{Z}_2 \tag{3.84}$$

implying that there is no additional global anomaly except the usual Witten anomaly associated with the $Sp(M)$ factor of the gauge group (for any choice of M).

	$\Omega_d^{\text{Spin}}(BG)$					
G	0	1	2	3	4	5
$SU(N)\times Sp(M)\times U(1), N>2$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_2$	0	$\mathbb{Z}^4$	$\mathbb{Z}_2$

Table 3.3 The bordism groups pertaining to a generalisation of the SM gauge group.

3.6 Global anomalies in BSM theories

In this Section, we show how to extend these methods to compute whether there are any potential global anomalies in BSM theories, by considering various popular examples. Firstly, we consider extensions of the SM by an arbitrary product of gauged $U(1)$ symmetries (such as in theories featuring heavy Z' gauge bosons). We then turn to a number of grand unified theories, namely the Pati-Salam model and two trinification models.

3.6.1 Multiple Z' extensions of the SM

We consider a four-dimensional gauge theory with gauge group

$$G_m \cong U(1)^m \times SU(2) \times SU(3), \qquad m \geq 2, \tag{3.85}$$

corresponding to an extension of the (usual) SM gauge group by arbitrary $U(1)$ factors, with *a priori* arbitrary fermion content. The corresponding Z' bosons in such a theory have been posited to address many phenomenological questions – for a review, see *e.g.* Ref. [95]. We will compute whether there are potential global anomalies in such a BSM theory.

The cohomology ring for BG_m is

$$H^\bullet(BG_m;\mathbb{Z}) \cong \mathbb{Z}\left[x_1,\ldots,x_m,c_2',c_2,c_3\right], \tag{3.86}$$

where x_k is the first Chern class associated with the kth $U(1)$ factor, and the remaining Chern classes are defined as in Eq. (3.40). In particular, we have the following low-dimensional cohomology groups

$$
\begin{aligned}
&H^0\left(BG_m;\mathbb{Z}\right) \cong \mathbb{Z}, \\
&H^2\left(BG_m;\mathbb{Z}\right) \cong \mathbb{Z}^m, \\
&H^4\left(BG_m;\mathbb{Z}\right) \cong \mathbb{Z}^{m'}, \quad m' = \binom{m+1}{2} + 2, \\
&H^6\left(BG_m;\mathbb{Z}\right) \cong \mathbb{Z}^{m''}, \quad m'' = \binom{m+2}{3} + 2m + 1,
\end{aligned}
\tag{3.87}
$$

with all cohomology groups in odd degrees vanishing, which of course coincides with the SM case when $m = 1$. Again, these groups are isomorphic to the corresponding groups in homology, with which we can deduce the entries $E^2_{p,q}$ of the AHSS, which are shown in Fig. 3.8.

We task ourselves here with the computation of $\Omega_5^{\mathrm{Spin}}(BG_m)$, which measures the potential global anomalies in the four-dimensional gauge theory we are interested in from the point of view of BSM. The relevant entries of the AHSS, lying on the $p+q=5$ diagonal, are highlighted in Fig. 3.8. To turn to the third (and thence fourth) page, we thus need to compute the differentials here labelled α and β.

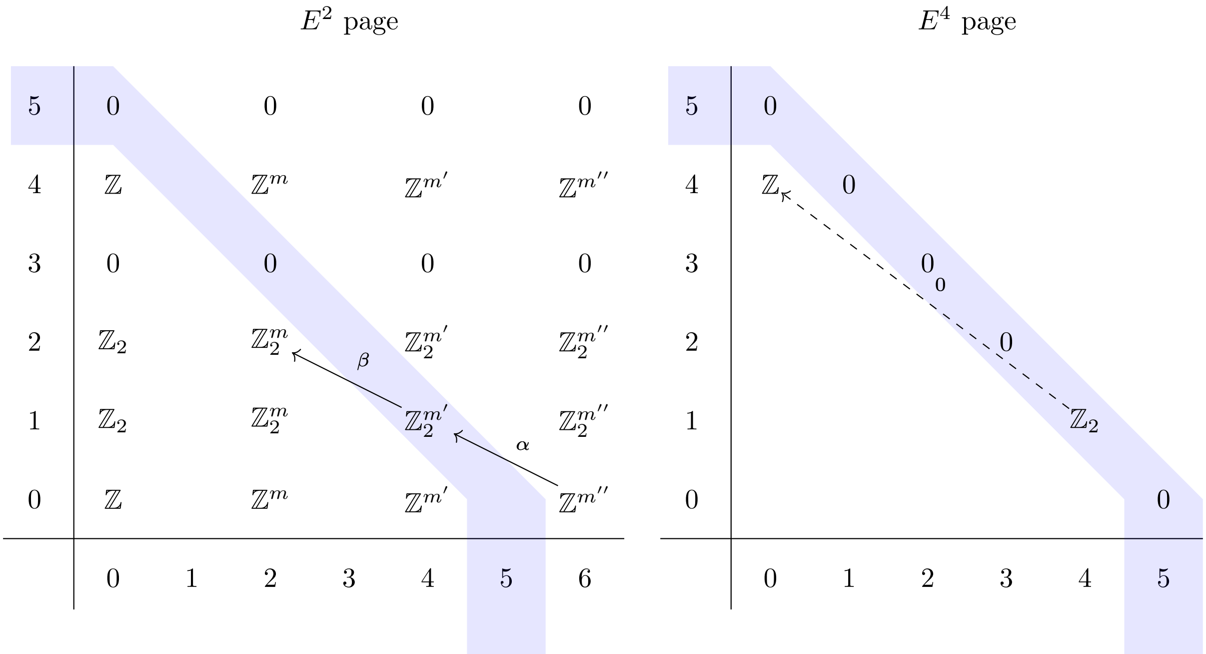


Fig. 3.8 The E^2 and E^4 pages of the Atiyah-Hirzebruch spectral sequence for $G = G_m = U(1)^m \times SU(2) \times SU(3)$ with all elements and differentials relevant to the calculation of Ω_5^{Spin} highlighted.

This is again similar to the case of the SM considered above. The map β is the dual to the Steenrod square

$$\begin{aligned} \mathrm{Sq}^2 : H^2(BG_m;\mathbb{Z}_2) &\longrightarrow H^4(BG_m;\mathbb{Z}_2) \\ x_i &\mapsto x_i^2, \end{aligned} \tag{3.88}$$

so the kernel of β is spanned by $\widetilde{c_2}$, $\widetilde{c_2'}$, and $\widetilde{x_i \cup x_j}$ with $i<j$. Hence $\ker\beta \cong (\mathbb{Z}_2)^{\frac{1}{2}m(m-1)+2}$. To calculate $\mathrm{im}\alpha$, where $\alpha = \widetilde{\mathrm{Sq}^2} \circ \rho$, we first look at the corresponding Steenrod square

$$\begin{aligned} \mathrm{Sq}^2 : H^4(BG_m;\mathbb{Z}_2) &\longrightarrow H^6(BG_m;\mathbb{Z}_2) \\ x_i^2 &\mapsto 2x_i^3 \equiv 0 \bmod 2, \\ x_i x_j &\mapsto x_i^2 x_j + x_i x_j^2, \\ c_2 &\mapsto c_3, \\ c_2' &\mapsto 0. \end{aligned} \tag{3.89}$$

Thus the image of $\widetilde{\mathrm{Sq}^2}$, and also of α, is spanned by $\widetilde{c_2}$ and $\widetilde{x_i x_j}$, for $i<j$. Thus $\mathrm{im}\alpha \cong (\mathbb{Z}_2)^{\frac{1}{2}m(m-1)+1}$. Taking the quotient then yields

$$E^3_{4,1} = E^4_{4,1} \cong \mathbb{Z}_2. \tag{3.90}$$

On the E_4 page (see Fig. 3.8) the only relevant differential must be trivial as it is a homomorphism from $\mathbb{Z}_2$ to $\mathbb{Z}$, so the $(4,1)$ entry stabilises to $E^\infty_{4,1} \cong \mathbb{Z}_2$ and it follows that

$$\Omega_5^{\mathrm{Spin}}(B(U(1)^m \times SU(2) \times SU(3))) \cong \mathbb{Z}_2, \tag{3.91}$$

where we can again identify the potential global anomaly in this theory with the Witten anomaly associated to the $SU(2)$ factor. Thus we find that there are no potential new global anomalies associated with extending the usual SM gauge group by an arbitrary torus, and indeed by arbitrary fermion content coupled to such a gauge group. There have been a number of recent studies [59, 6, 46] attempting to classify the space of $U(1)$ extensions of the SM that are free of *local* anomalies; here, we show that all such models are automatically free also of global anomalies, provided of course that there is no Witten anomaly associated with $SU(2)$. It is also straightforward to calculate the lower-degree bordism groups for this example, which we simply tabulate in the first line of Table 3.4. We find that the additional $U(1)$ factors do indeed affect the bordism groups in lower degrees, in particular in degrees two and four.

G	$\Omega_d^{\mathrm{Spin}}(BG)$ 0	1	2	3	4	5
$U(1)^m \times SU(2) \times SU(3)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}^m \times \mathbb{Z}_2$	0	$\mathbb{Z}^{3+\frac{1}{2}m(m+1)}$	$\mathbb{Z}_2$
$SU(4) \times SU(2)_L \times SU(2)_R$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}^4$	$\mathbb{Z}_2^2$
$SU(3)_C \times SU(3)_L \times SU(3)_R$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}^4$	0
$\dfrac{SU(3)_C \times SU(3)_L \times SU(3)_R}{\mathbb{Z}_3}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_3$	0	$\mathbb{Z}^4$ or $\mathbb{Z}^4 \times \mathbb{Z}_3$	0

Table 3.4 Summary of results from our bordism computations of relevance to BSM physics. The first row corresponds to theories with multiple Z' bosons, the second row to a Pati-Salam model, and the last two rows to trinification models.

3.6.2 Pati-Salam models

Here we consider the simplest incarnation (for our purposes) of the Pati-Salam model, in which the SM gauge group is embedded in the larger group

$$\mathrm{PS} \equiv SU(2)_L \times SU(2)_R \times SU(4). \tag{3.92}$$

The cohomology ring for $B(\mathrm{PS})$ is

$$H^\bullet\left(B(\mathrm{PS});\mathbb{Z}\right) \cong \mathbb{Z}\left[c_2^L, c_2^R, c_2', c_3', c_4'\right], \tag{3.93}$$

where $c_2^{L/R}$ denote the second Chern classes of the $SU(2)_{L/R}$ factors, and c_i' denotes the ith Chern class of $SU(4)$. A notable difference between this example and all those considered previously is that the second homology group is here vanishing. This only serves to simplify the computation of the AHSS, and so we choose to omit the details for brevity. The upshot is that we find

$$\Omega_5^{\mathrm{Spin}}\left(B(\mathrm{PS})\right) \cong \mathbb{Z}_2 \times \mathbb{Z}_2. \tag{3.94}$$

We identify the two $\mathbb{Z}_2$-valued global anomalies with the Witten anomalies associated with each $SU(2)$ factor in the Pati-Salam group, a result that follows straightforwardly from Witten's original arguments. We quote the remaining results of our calculations for all bordism groups $\Omega_{d\leq 5}^{\mathrm{Spin}}\left(B(\mathrm{PS})\right)$ in Table 3.4.

We note in passing that there are variants on the Pati-Salam gauge group that involve various discrete factors, which complicate the computation of the bordism groups. For example, left-right symmetric models have been proposed in which $G = \mathrm{PS} \rtimes \mathbb{Z}_2$, and there are also models featuring a quotient by a $\mathbb{Z}_2$ subgroup. Unfortunately, neither of the bordism computations for these gauge groups succumb to attack using the simple fibrations considered in this Chapter.

3.6.3 Trinification models

In trinification models of grand unification[77], the underlying gauge group is either

$$G = SU(3)_C \times SU(3)_L \times SU(3)_R \quad \text{or} \quad G = \frac{SU(3)_C \times SU(3)_L \times SU(3)_R}{\mathbb{Z}_3}, \tag{3.95}$$

where the $\mathbb{Z}_3$ quotient is the diagonal subgroup of the $(\mathbb{Z}_3)^3$ centre symmetry. In both cases, the SM quarks are packaged into representations $(\bar{\mathbf{3}}, \mathbf{1}, \mathbf{3})$ and $(\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1})$, with the leptons transforming in the $(\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}})$. The model also contains multiple Higgs fields transforming in the $(\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}})$ representation (each of which contains three SM-like Higgs doublets), needed to break the gauge symmetry down to a SM subgroup; the first option in (3.95) is broken down to G_{SM}/Γ_2, while the second is broken to G_{SM}/Γ_6. Like Pati-Salam models, trinification models are attractive in part because all the gauge, Yukawa, and quartic couplings in the lagrangian can be run to arbitrarily high energies without hitting any Landau poles, thereby exhibiting 'total asymptotic freedom' [106].

No quotient

To find out whether there are potential global anomalies when the gauge group is $SU(3)^3$, we compute $\Omega_d^{\text{Spin}}\left(BSU(3)^3\right)$. Since the method is very similar to that used in previous Sections, we will only quote the results here to avoid repetition. We find

i	0	1	2	3	4	5
$\Omega_i^{\text{Spin}}(BSU(3)^3)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}^4$	0.

Since $\Omega_5^{\text{Spin}}\left(BSU(3)^3\right) = 0$, the trinification models based on this gauge group are free of any global anomalies, regardless of the fermion content.

$\mathbb{Z}_3$ quotient

Now let us consider the option involving a permutation symmetry among the three $SU(3)$ factors, *i.e.* where $G = SU(3)^3/\mathbb{Z}_3$. We have the fibration $\mathbb{Z}_3 \to SU(3)^3 \to G$, which we can

E^2 page

q \ p	0	1	2	3	4	5
5	0		0		0	0
4	$\mathbb{Z}^4$		$\mathbb{Z}_3^4$		$\mathbb{Z}_3^4$	0
3	0		0		0	0
2	$\mathbb{Z}_2$		0		0	0
1	$\mathbb{Z}_2$		0		0	0
0	$\mathbb{Z}$		$\mathbb{Z}_3$		$\mathbb{Z}_3$	0

Fig. 3.9 The E^2 page of the Atiyah-Hirzebruch spectral sequence for trinification models featuring a $\mathbb{Z}_3$ quotient of the gauge group.

use the Puppe sequence to turn into the following fibration

$$BSU(3)^3 \longrightarrow BG \longrightarrow B^2\mathbb{Z}_3 \cong K(\mathbb{Z}_3, 2). \tag{3.96}$$

Using this fibration, we can now form the AHSS to find $\Omega_5^{\text{Spin}}(BG)$. The second page, as we have seen so many times, is given by

$$E^2_{p,q} = H_p\left(K(\mathbb{Z}_3,2); \Omega_q^{\text{Spin}}(BSU(3)^3)\right)$$

which can be constructed using the results for $\Omega_{\text{pt}}^{\text{Spin}}\left(BSU(3)^3\right)$, which were already calculated in this Subsection. It is displayed in Fig. 3.9. One can see immediately that all entries with $p+q=5$ stabilise already at this page. We can again conclude that

$$\Omega_5^{\text{Spin}}\left(B\left(\frac{SU(3)_C \times SU(3)_L \times SU(3)_R}{\mathbb{Z}_3}\right)\right) = 0.$$

The other entries with $p+q<5$ also stabilise on this page because all relevant homomorphisms are trivial. The spin bordism groups of lower degrees can be calculated uniquely apart from Ω_4^{Spin} which involves non-splitting group extensions. It is given by

$$\Omega_4^{\mathrm{Spin}}\left(B\left(\frac{SU(3)_C\times SU(3)_L\times SU(3)_R}{\mathbb{Z}_3}\right)\right)\cong e(\mathbb{Z}_3,\mathbb{Z}^4). \tag{3.97}$$

The full results are given in Table 3.4.

3.7 (B)SM theories with spin$_c$ structures

Part of the motivation for the bordism-based criterion for anomaly cancellation that we have used in this Chapter is the desire to define the SM (or our favourite BSM extension) on arbitrary four-manifolds, or at least within some suitable class of four-manifolds. Such a requirement can be motivated by locality, and is certainly a requirement in a quantum theory of gravity in which the geometry (and thus topology) of spacetime cannot be held fixed.

In order to define fermions, one needs to equip spacetime with a spin structure, or a variant thereof with which to stitch together locally-valued spinor fields into globally-defined ones. It is well known that not all orientable four-manifolds admit a spin structure (with $\mathbb{C}P^2$ being a well-known example of an orientable four-manifold that is not spin). The obstruction to being spin is measured by the second Stiefel-Whitney class which takes values in $H^2(\Sigma;\mathbb{Z}_2)$. While $H^2(\Sigma;\mathbb{Z}_2)=0$ for all orientable manifolds in dimension three or fewer, it does not vanish for all four manifolds. One might therefore ask whether the SM and related theories we have explored in this Chapter can be defined on *all* orientable four-manifolds, by not assuming the presence of a spin structure. We invite the reader to consult Appendix 3.A, in which we provide more details regarding the definitions of spin structures and the like.

As we noted in §4.5, in the presence of a $U(1)$ gauge symmetry it becomes possible to define spinors using only a spin$_c$ structure on spacetime. The transition functions on a spin$_c$ bundle over an oriented four-manifold Σ are valued in the group Spin$_c(4)$, which can be defined by the short exact sequence

$$0\to U(1)_A\to \mathrm{Spin}_c(4)\to SO(4)\to 0, \tag{3.98}$$

where $U(1)_A$ denotes a gauged symmetry. Since all orientable four-manifolds admit a spin$_c$ structure (the obstruction here being in the third Stiefel-Whitney class), one can in principal try to define a four-dimensional gauge theory on all orientable four manifolds by using a spin$_c$ structure. These observations were first made back in 1977 [85], motivated by the authors' desire to define a theory of quantum gravity on all orientable spacetimes.

In order to define all fermions using a spin$_c$ structure, for a particular non-abelian gauge theory (such as one of the SMs), requires there exists a $U(1)$ subgroup of the gauge symmetry, here denoted by $U(1)_A$, such that all fermions in the theory transform in *bona fide* representations of the group (3.98). Using similar arguments to those given in §3.4.1, this results in constraints on the allowed $U(1)_A$ charges of fermions, which here depend on their *spin*. We begin our discussion by recapping what these 'spin-charge relations' are, which was recently discussed (in the context of defining similar theories on spin$_c$ manifolds) in Ref. [122].

3.7.1 Spin-charge relations

To derive the spin-charge relations, we require that the SM fermions transform in *bona fide* representations of both $\text{Spin}_c(4)$ and G, where G is one of the four SM gauge groups listed in Eq. (3.1). It is here helpful to write

$$\text{Spin}_c(4) \cong \frac{\text{Spin}(4) \times U(1)_A}{\mathbb{Z}_2} \cong \frac{SU(2)_L \times SU(2)_R \times U(1)_A}{\mathbb{Z}_2}, \tag{3.99}$$

A Weyl fermion transforms in the $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$ representation of the $SU(2)_L \times SU(2)_R$ factor. So, when considering Weyl fermions we may restrict our attention to a subgroup of $\text{Spin}_c(4)$ isomorphic to

$$\frac{SU(2) \times U(1)_A}{\mathbb{Z}_2} \cong U(2). \tag{3.100}$$

Thus, by the same argument we used in §3.4.1, one deduces that there exists a normalisation of charges such that all Weyl fermion have *odd* charges under $U(1)_A$, in order to define the theory using this spin$_c$ structure.

The question then is, is there any $U(1)_A$ subgroup of G in which all the SM fermions have odd charges? It turns out the answer is no. To see why, consider $U(1)_A$ to be generated by

$$X = aY + b\tilde{T}_3 + cT_3 + dT_8, \tag{3.101}$$

where Y is the generator of hypercharge,

$$\tilde{T}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is the Cartan generator of (electroweak) $SU(2)$,

$$T_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad T_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

are the Cartan generators of $SU(3)$ (in a non-standard normalisation which is convenient for our purposes). Eq. (3.101) defines a general $U(1)_A$ subgroup of G. [22]

We then need to decompose all the SM fermion fields into eigenstates of (3.101). To wit, consider the left-handed doublet of quarks, Q. This needs both an $SU(2)$ index (which we denote by an upper Greek index $\alpha \in \{1,2\}$) and an $SU(3)$ index (which we denote by a lower Latin index $i \in \{1,2,3\}$). In this notation, $Q_{\alpha i}$ denotes $2\times 3 = 6$ Weyl fermions. We thus denote the SM fermion content by the fields $\{Q_i^\alpha,\, L^\alpha,\, U_i,\, D_i,\, E\}$, which number fifteen in total.

The charges of all the SM fields under the generator (3.101) are then

Field	Charge
Q_1^1	$a+b+c+d$
Q_1^2	$a-b+c+d$
Q_2^1	$a+b-c+d$
Q_2^2	$a-b-c+d$
Q_3^1	$a+b-2d$
Q_3^2	$a-b-2d$
L^1	$-3a+b$
L^2	$-3a-b$
U_1	$4a+c+d$
U_2	$4a-c+d$
U_3	$4a-2d$
D_1	$-2a+c+d$
D_2	$-2a-c+d$
D_3	$-2a-2d$
E	$-6a$

(3.102)

[22] Different inclusions of $U(1)$ in the non-abelian factors are related to our choice simply by a change of basis.

There are no rational values for a, b, c, and d such that all the charges in this table are odd numbers. To see why, note firstly that the oddness of the charge of e requires that $a = (2n+1)/2$. But then there is no value of d such that both d_3 and u_3 have odd charge.

We hereby see the restrictiveness of the spin-charge relations: there is in fact no $U(1)$ gauge symmetry in the SM which one can use to define the theory using a spin$_c$ structure. This fact was pointed out in Ref. [74]. Hence, given only the gauge symmetries and the fermion content of the SM, one cannot define it on all four-manifolds using a spin$_c$ structure.[23]

3.7.2 Gauging $B-L$

One can instead define a theory on all orientable four-manifolds in which the SM gauge group is extended by an additional $U(1)$ gauge symmetry for which the spin-charge relations are satisfied, such as gauging $B-L$,[24] where B is baryon number and L is lepton number. Under $U(1)_{B-L}$ all the SM fermions have odd charges (either -1 or 3), and so this gauge symmetry can be used to define a spin$_c$ structure [74].

Of course, $B-L$ is free of local ABJ-type anomalies. Here we consider global anomalies in SM$\times U(1)$ theories defined on all spin$_c$ manifolds, such as gauged $B-L$, by computing the bordism groups $\Omega_5^{\mathrm{Spin}_c}(BG)$, for the SM gauge groups listed in Eq. (3.1). These bordism groups can be computed using the AHSS associated to a fibration of the form $F \to BG \to B$. For example, given the 'trivial' fibration pt $\to BG \to BG$, the second page of the AHSS is now

$$E^2_{p,q} = H_p(B; \Omega_q^{\mathrm{Spin}_c}(F)), \tag{3.103}$$

where the bordism groups of spin$_c$ q-manifolds equipped with maps to a point are [28]

$$\begin{array}{c|ccccccccccc} q & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline \Omega_q^{\mathrm{Spin}_c}(\mathrm{pt}) & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z}^2 & 0 & \mathbb{Z}^2 & 0 & \mathbb{Z}^4 & 0 & \mathbb{Z}^4. \end{array} \tag{3.104}$$

[23] Note that it may still be possible to define the SM consistently on all four-manifolds, but using an even weaker structure than spin$_c$. For example, one may use a spin$-SU(2)$ structure, or a spin$-H$ structure in general where H is any subgroup of G. We do not consider such possibilities here.

[24] We note that, in this setup, the vector field A_μ that couples to $B-L$ is not technically an abelian gauge field, since its field strength will not satisfy the Dirac quantisation condition (the corresponding Chern class is only half-integral). Thus, it is not technically correct to describe such a theory as a theory with gauge symmetry $G_{\mathrm{SM}} \times U(1)$. Rather, the vector field A_μ defines a spin$_c$ connection on Σ.

Interestingly, these groups do not feature any torsion, and moreover they vanish in all odd degrees, at least up to $\Omega_9^{\mathrm{Spin}_c}(\mathrm{pt})$. It then follows immediately that

$$\Omega_d^{\mathrm{Spin}_c}(BG_{\mathrm{SM}}) = \Omega_d^{\mathrm{Spin}_c}(BG_{\mathrm{SM}}/\Gamma_2) = \Omega_d^{\mathrm{Spin}_c}(BG_{\mathrm{SM}}/\Gamma_3) = 0 \qquad \text{for all odd } d \leq 9, \tag{3.105}$$

because non-zero entries in $E^2_{p,q}$ can only appear when $p+q$ is even (since $H_p(BG;\mathbb{Z})$ also vanishes in all odd degrees for these gauge groups). In particular, these groups vanish in degree $d=5$, so there are no possibilities of global anomalies in any of these theories.

The case where $G = G_{\mathrm{SM}}/\Gamma_6$ is only slightly less straightforward. We may as before proceed via the Puppe sequence to deduce the fibration

$$B(U(2)\times SU(3)) \to B(G_{\mathrm{SM}}/\Gamma_6) \to K(\mathbb{Z}_3, 2), \tag{3.106}$$

and write down the corresponding AHSS, from which one immediately sees that

$$\Omega_5^{\mathrm{Spin}_c}(BG_{\mathrm{SM}}/\Gamma_6) = 0, \tag{3.107}$$

so again such a theory is automatically free of global anomalies. These conclusions hold when the SM fermion content is extended arbitrarily.

Appendix 3.A Spin structures and the like

In this Appendix, we consider fermions defined on a p-dimensional smooth spacetime manifold Σ^p. Fermions are usually defined to be spinors on Σ^p. Defining spinors requires a spin structure on spacetime. To explain what a spin structure is, we first assume that Σ^p is orientable. A spinor is then a section of a so-called spinor bundle over Σ^p, whose structure group is the group $\mathrm{Spin}(p)$, the double cover of $SO(p)$ (which is the structure group of the tangent bundle). What this means is that two locally-valid descriptions of a spinor field, Ψ_α (defined on an open set U_α of Σ^p) and Ψ_β (defined on U_β), are related by $\Psi_\alpha = T_{\alpha\beta}\Psi_\beta$, for some matrix $T_{\alpha\beta} \in \mathrm{Spin}(p)$ defined on the double-overlap $U_\alpha \cup U_\beta \equiv U_{\alpha\beta}$.[25] In order to be able to define spinors globally, we must be able to piece together locally-valid descriptions on open sets $\{U_\alpha\}$ consistently. This requires a set of $\mathrm{Spin}(p)$-valued transition functions defined on every double overlap $U_{\alpha\beta}$, which moreover satisfy a consistency condition on triple overlaps, *viz.* $T_{\alpha\beta}\cdot T_{\beta\gamma}\cdot T_{\gamma\alpha} = \mathbf{1}$ on $U_{\alpha\beta\gamma}$. A consistent set of $\{T_{\alpha\beta}\}$ is called a spin structure on Σ^p.

[25] The spin-valued matrices $T_{\alpha\beta}$ are moreover obtained by lifting the transition functions from the tangent bundle, which are valued in the (orientation-preserving) structure group $SO(p)$.

Not every Riemannian manifold admits such a collection of Spin(p)-valued transition functions that satisfy the consistency condition. An orientable manifold admits a spin structure, which can be used to define spinors, if and only if both the first and second Stiefel-Whitney classes (which take values in $H^1(\Sigma^p;\mathbb{Z}_2)$ and $H^2(\Sigma^p;\mathbb{Z}_2)$ respectively) vanish. If this is the case, Σ^p is called a spin manifold. For example, all orientable manifolds in dimension $p \leq 3$ are spin; whereas four-manifolds are not, necessarily. The Spin(p)-valued $T_{\alpha\beta}$ then define transition functions on a vector bundle $S \to \Sigma^p$, called a spinor bundle, of which a fermion field is a section.

This is not the only way to define a geometric object which behaves as a fermion. If spacetime is non-orientable, alternative structures (called pin structures) may still be used to define an analogue of the spinor,[26] and hence to define fermions. The idea here is very similar to defining spinors in the case that Σ^p was orientable, except that now the transition functions of the tangent bundle are valued in $O(p)$, rather than $SO(p)$, because they need not preserve orientation. Consequently, the structure group of the 'pinor' bundle is a double cover of $O(p)$, which is called a Pin(p) group. But now there is not just one such double cover of $O(p)$, but two possible choices called Pin$^+$ and Pin$^-$, as follows. One may choose a spatial reflection $\mathbf{R}$ to satisfy $\mathbf{R}^2 = 1$ when acting on spinors, which defines the double cover Pin$^+$, or choose $\mathbf{R}^2 = -1$, which defines the double cover Pin$^-$. A pin structure is then defined in a similar way to a spin structure; the $O(p)$-valued transition functions of the tangent bundle are lifted to (say) Pin$^+$-valued functions, which must satisfy a consistency relation on triple overlaps. A non-orientable manifold that admits a (say) pin$^+$ structure is, not surprisingly, called a pin$^+$ manifold. Again, there are topological obstructions (involving Stiefel-Whitney classes) to defining such pin structures, which are different for pin$^+$ and pin$^-$ structures. Notably, every non-orientable 2-manifold and 3-manifold admits a pin$^-$ structure, but not necessarily a pin$^+$ structure.[27]

In both the orientable and non-orientable cases, one may in fact still define fermions using weaker structures on Σ^p, provided there are additional gauge symmetries acting on the fermions. For example, a manifold that is not spin may nonetheless admit a spin$_c$ structure, which is defined analogously to a spin structure, but where the transition functions can be valued in the Spin$_c(p)$ group rather than Spin(p). The group Spin$_c(p)$ can be defined by the short exact sequence $0 \to U(1) \to \text{Spin}_c(p) \to SO(p) \to 0$; in an intuitive sense, this "allows" the transition functions to vary by a (local) $U(1)$-valued phase, which can be used to "stitch together" transition functions where a spin structure might not be possible. If a fermion is acted upon by a $U(1)$ gauge symmetry, then it is invariant under such local $U(1)$ rephasings,

[26] In the unorientable case, the fermion might better be called a 'pinor'.

[27] For example, the manifold $\mathbb{R}P^2$ admits only pin$^-$ structures.

and so will be well-defined using only the spin$_c$ structure. The obstruction to a manifold admitting a spin$_c$ structure now lies in its *third* Stiefel-Whitney class valued in $\mathbb{Z}$ (rather than $\mathbb{Z}_2$). Importantly, all orientable manifolds in dimension $p \leq 4$ are spin$_c$.[28] Analogously defined pin$_c$ structures may be used to define fermions on non-orientable spacetimes with a $U(1)$ gauge symmetry.

Appendix 3.B Computation of $H_6(K(\mathbb{Z}_3,2),\mathbb{Z})$

In Ref. [74], a theorem from Ref. [108] was used to show that the homology groups $H_i(K(\mathbb{Z}_3,2);\mathbb{Z})$ are given by where C is an abelian group of exponent less than or equal to 6,

i	0	1	2	3	4	5	6
$H^i(K(\mathbb{Z}_3,2);\mathbb{Z})$	$\mathbb{Z}$	0	$\mathbb{Z}_3$	0	$\mathbb{Z}_3$	0	$C\times\mathbb{Z}_9$

Table 3.5 Integral homology groups of $K(\mathbb{Z}_3,2)$

i.e., the degree of any element in C does not exceed 6. This means that, *a priori*, it has the form

$$C \cong \mathbb{Z}_2^{h_2}\times\mathbb{Z}_3^{h_3}\times\mathbb{Z}_4^{h_4}\times\mathbb{Z}_5^{h_5}\times\mathbb{Z}_6^{h_6} \tag{3.108}$$

with $h_i \geq 0$. We will use the Serre spectral sequence to show that C must be of the form

$$C \cong \mathbb{Z}_3^n, \qquad n \geq 0. \tag{3.109}$$

Recall that for a fibration $F \to X \to B$, the (p,q) entry on the second page of the Serre spectral sequence is given by [83]

$$E^2_{p,q} = H_p\left(B;H_q(F)\right), \tag{3.110}$$

where we use the shorthand notation $H_q(F) = H_q(F;\mathbb{Z}_2)$. The spectral sequence converges to $H_\bullet(X)$, that is, the homology groups of X is determined from the last page of the spectral sequence by[29]

$$H_n(X) = \bigoplus_{p=0}^{n} E^\infty_{p,n-p} \tag{3.111}$$

[28]Even 'weaker' structures have been used to define fermions on general spacetimes in the quantum gravity literature, using the idea of spin-G structures for various Lie groups G [27, 26]. The use of spin-$SU(2)$ structures, for an $SU(2)$ gauge theory, has recently been used to derive a new kind of global anomaly [143].

[29]To be precise, we need to phrase this in terms of filtrations, and solve extension problems to determine the homology groups. However, since the spectral sequence we are interested in converges to 0 for $p+q>0$, as we shall see momentarily, it follows that all extensions split.

Just like in [74], we consider the fibration

$$K(\mathbb{Z}_3,1) \longrightarrow \star \longrightarrow K(\mathbb{Z}_3,2) \tag{3.112}$$

where $\star$ is a contractible space. The second page of the Serre spectral sequence is given in figure 3.10.

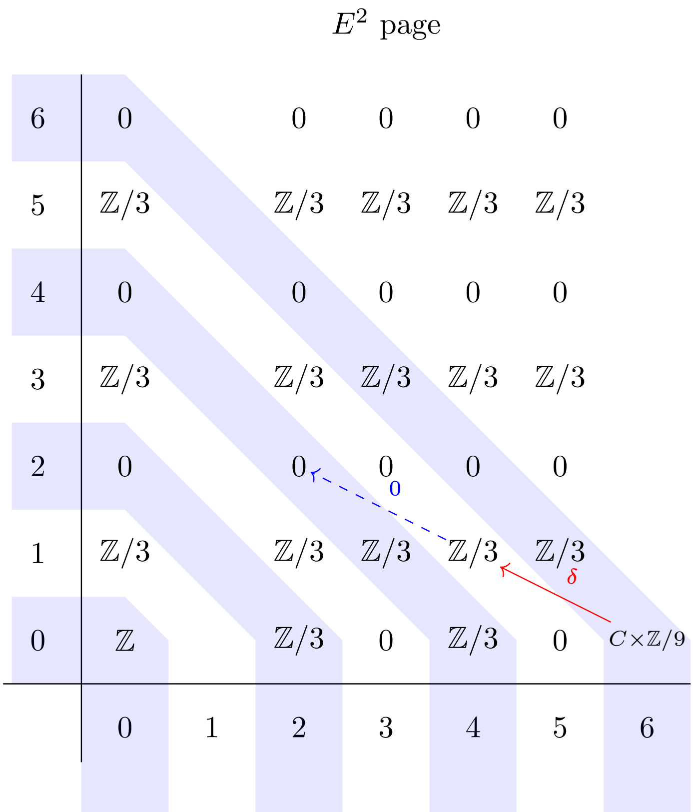


Fig. 3.10 The E_2 page of the Serre spectral sequence for the fibration (3.112)

Since $H_i(\star) = 0$ for $i > 0$, any entry in the Serre spectral sequence apart from $E_{0,0}$ must stabilise to 0. In particular, the entry $E_{6,0}$ must stabilise to 0. Since the differential δ acts trivially on $\mathbb{Z}_2, \mathbb{Z}_4$, and $\mathbb{Z}_5$, these factors would be present in $E^{\infty}_{6,0}$ unless $h_2 = h_4 = h_5 = 0$.

We can also see that $h_6 = 0$ by a similar argument. Suppose that $h_6 \neq 0$. Let δ_6 be a homomorphism from $\mathbb{Z}_6$ to $\mathbb{Z}_3$. There are three choices depending on where it sends the element 1. The first choice is $\delta_6(1) = 0$, which is the trivial homomorphism, in which case the kernel is $\mathbb{Z}_6$. The second choice and third choice are sending 1 to 1 or 2, both of which result in the same kernel: $\ker \delta_6 \cong \mathbb{Z}_2$. In subsequent pages, the homomorphisms from the $(6,0)$ entry go into either 0 or $\mathbb{Z}_3$, and can never result in a trivial kernel. Therefore, $E^{\infty}_{6,0} \neq 0$, which is a contradiction. Hence $h_6 = 0$. This is enough for our purpose: we have determined that

$$H_6(K(\mathbb{Z}_3,2);\mathbb{Z}) \cong \mathbb{Z}_3^n \times \mathbb{Z}_9, \qquad n \geq 0. \tag{3.113}$$

Appendix 3.C Two derivations of the fifth spin-bordism group of

In this Appendix we give the details of the computation of the spin bordism groups of the SM quotient by $\mathbb{Z}_3$. We present two methods, associated with two different fibrations.

Method 1

Firstly, we use the AHSS associated to the fibration

$$\text{pt} \to U(3) \times SU(2) \to U(3) \times SU(2), \tag{3.114}$$

for which the second page of the AHSS is given by $E^2_{p,q} = H_p(B(U(3) \times SU(2)); \Omega_q^{\text{Spin}}(\text{pt}))$. The relevant cohomology rings are

$$\begin{aligned} H^\bullet(BU(3);\mathbb{Z}) &\cong \mathbb{Z}[c_1, c_2, c_3] \\ H^\bullet(BSU(2);\mathbb{Z}) &\cong \mathbb{Z}[c_2'] \end{aligned} \tag{3.115}$$

where c_i, c_i' are the ith Chern classes for $BU(3)$ and $BSU(2)$, respectively. From this, together with the Künneth formula in cohomology, we find that $H^2(B(G_{\text{SM}}/\Gamma_3))$ is generated by c_1, $H^4(B(G_{\text{SM}}/\Gamma_3))$ by c_1^2, c_2, c_2', and $H^6(B(G_{\text{SM}}/\Gamma_3))$ by $c_1^3, c_1c_2', c_1c_2, c_3$, and again the absence of torsion means these cohomology groups are isomorphic to the corresponding groups in homology.

We again form the AHSS associated to the trivial fibration over a point. The entries on the second page of the AHSS are identical to those of the previous two cases, albeit with different action of the differentials, so we choose not to reproduce the diagram for a third time. Again, the difference to the previous cases shall enter in the action of the differential labelled γ.

The differentials relevant to the calculation of $\Omega_4^{\text{Spin}}(B(G_{\text{SM}}/\Gamma_3))$ and $\Omega_5^{\text{Spin}}(B(G_{\text{SM}}/\Gamma_3))$ may be labelled precisely as in Eq. (3.62) above. Since $\text{Sq}^2 : H^2 \to H^4$ maps $c_1 \mapsto c_1^2$, we see that both α, β maps $\widetilde{c_1^2} \mapsto \widetilde{c_1}$ and others to zero, and moreover α maps $2\widetilde{c_1^2}$ to zero as before. So we again have $\ker\alpha \cong \mathbb{Z}^3$, $\text{im}\alpha \cong \mathbb{Z}_2$, $\ker\beta \cong \mathbb{Z}_2^2$, and $\text{im}\beta \cong \mathbb{Z}_2$.

We turn to the action of γ. The relevant Steenrod square is here

$$\begin{aligned}\mathrm{Sq}^2 : H^4\left(B(G_{\mathrm{SM}}/\Gamma_3);\mathbb{Z}_2\right) &\longrightarrow H^6\left(B(G_{\mathrm{SM}}/\Gamma_3);\mathbb{Z}_2\right)\\ c_1^2 &\mapsto 2c_1^3 \equiv 0 \bmod 2,\\ c_2' &\mapsto 0,\\ c_2 &\mapsto c_1c_2 + c_3.\end{aligned} \tag{3.116}$$

So γ maps $\widetilde{c_1c_2} \mapsto \widetilde{c_2}$ and $\widetilde{c_3} \mapsto \widetilde{c_2}$, while mapping other generators to zero. This gives $\mathrm{im}\gamma \cong \mathbb{Z}_2$, and hence

$$E^3_{4,1} = \frac{\ker\beta}{\mathrm{im}\gamma} \cong \mathbb{Z}_2, \tag{3.117}$$

and this entry stabilises. This is the only non-vanishing entry on the $p+q=5$ diagonal, and so we find

$$\Omega_5^{\mathrm{Spin}}(B(G_{\mathrm{SM}}/\Gamma_3)) \cong \mathbb{Z}_2. \tag{3.118}$$

Since the discrete $\mathbb{Z}_3$ quotient is here embedded 'orthogonally' to the $SU(2)$ factor in G, we feel safe in suggesting that this $\mathbb{Z}_2$ captures the Witten anomaly coming from the $SU(2)$ factor. As for the previous example, the lower-degree bordism groups are unchanged (see Table 3.1).

Method 2

We provide here an alternative proof that $\Omega_5^{\mathrm{Spin}}(B(G_{\mathrm{SM}}/\Gamma_3)) \cong \mathbb{Z}_2$ using an alternative fibration,

$$\mathbb{Z}_3 \longrightarrow G_{\mathrm{SM}} \longrightarrow G_{\mathrm{SM}}/\Gamma_3. \tag{3.119}$$

After we apply the Puppe sequence, this fibration turns into

$$BG_{\mathrm{SM}} \longrightarrow B(G_{\mathrm{SM}}/\Gamma_3) \longrightarrow K(\mathbb{Z}_3, 2) \tag{3.120}$$

Using the results for the homology groups of $K(\mathbb{Z}_3, 2)$ up to degree 6 given in Appendix 3.B, we can work out the E^2 page of the Atiyah-Hirzebruch spectral sequence, given in Figure 3.11. Moreover, we can deduce that the differential d in the E^6 page must be trivial, since it is a homomorphism from a product of $\mathbb{Z}_m$ factors with m odd to $\mathbb{Z}_2$. All the entries $E_{p,q}$ with $p+q=5$ now stabilise, and we can read off the spin bordism group as

$$\Omega_5^{\mathrm{Spin}}(B(G_{\mathrm{SM}}/\Gamma_3)) \cong \mathbb{Z}_2, \tag{3.121}$$

as claimed.

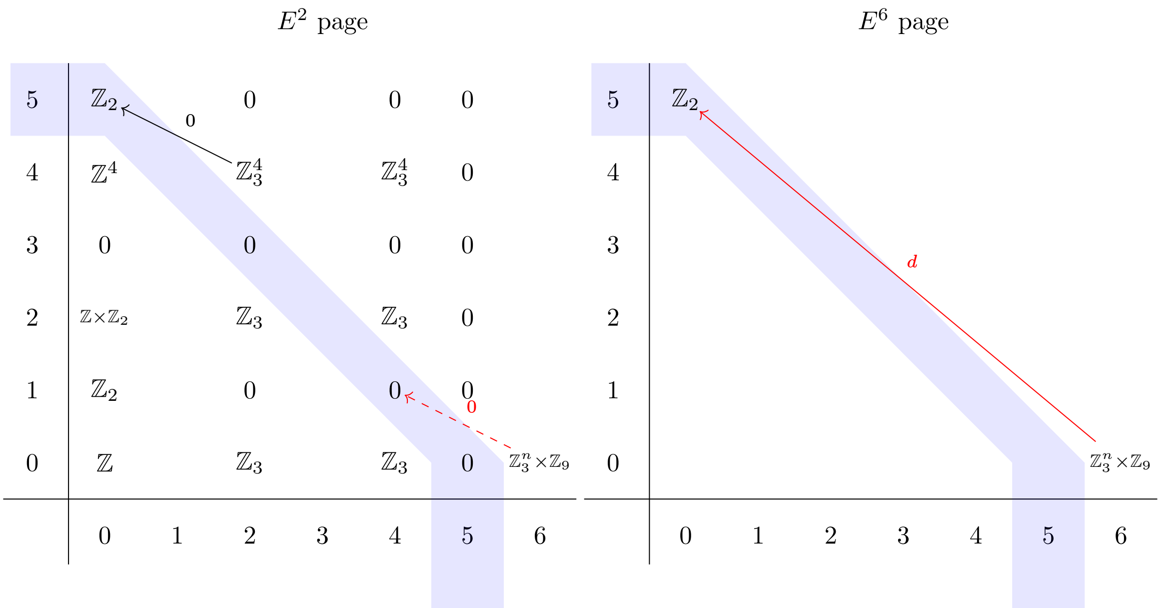


Fig. 3.11 The E^2 and E^6 pages of the Atiyah-Hirzebruch spectral sequence for $G = G_{\mathrm{SM}}/\Gamma_3$ from the fibration (3.120).

Appendix 3.D Decomposing $U(n)$ irreducible representations

The purpose of this Appendix is to decompose an irreducible representation of $U(n) \cong \frac{SU(n)\times U(1)}{\mathbb{Z}_n}$ in terms of the $U(1)$ charge and $SU(n)$ irreducible representation using character theory, from which we extract the charge constraints presented in §3.4.1.

Let G be a group and V a d-dimensional representation of G. An element $g \in G$ is represented by a $d \times d$ matrix $R_V(g)$. The character of g in the representation V, denoted by $\chi_V(g)$, is defined by

$$\chi_V(g) = \frac{1}{\dim V}\,\mathrm{Tr}\; R_V(g). \tag{3.122}$$

(We use the normalised character where we have $\chi_V(e) = 1$ for all finite irreducible representation V.) From this definition, it is easy to see that the character of g is a class function, that is, it only depends on the conjugacy class of g

$$\chi_V(g) = \chi_V(hgh^{-1}), \qquad \text{for any } h \in G \tag{3.123}$$

We now specialise to the case $G = U(n)$. Since any unitary matrix can be diagonalised by a unitary matrix, any element $g \in U(n)$ is conjugate to a diagonal matrix of the forms

$$g \sim \text{ diag } (z_1, z_2, \dots, z_n), \qquad |z_i| = 1. \tag{3.124}$$

Therefore, a $U(n)$ character can be thought of as a function $\chi_V^{U(n)} : T^n \to \mathbb{C}$, where T^n is the maximal torus of $U(n)$.

Characters of irreducible representations of $U(n)$ are given by a certain type of symmetric functions called Schur's functions. Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be an array of integers satisfying

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n. \tag{3.125}$$

Note that if $\lambda_n \geq 0$ this is the partition $\boldsymbol{\lambda}$ of the non-negative integer $|\boldsymbol{\lambda}| = \lambda_1 + \dots + \lambda_n$. In fact, we can write $\boldsymbol{\lambda}$ in terms of an integer m and a *bona fide* partition $\boldsymbol{\mu} = (\mu_1, \dots, \mu_{n-1})$, with $\mu_i \in \mathbb{Z}$ and

$$\mu_1 \geq \mu_2 \geq \dots \geq \mu_{n-1} \geq 0, \tag{3.126}$$

by writing $\lambda_i = m + \mu_i$ for $i = 1, \dots, n-1$ and $\lambda_n = m$. We denote this decomposition by $\boldsymbol{\lambda} = (m)^n + \boldsymbol{\mu}$. $\boldsymbol{\mu}$ can be represented by a Young diagram consisting of $|\boldsymbol{\mu}|$ boxes in total, with m_i boxes in the i$^{\text{th}}$ row. We now define Schur's function in n variables $\mathbf{z} = (z_1, \dots, z_n)$ by

$$s_{\boldsymbol{\lambda}}(\mathbf{z}) = \frac{\begin{vmatrix} z_1^{\lambda_1+n-1} & \dots & z_n^{\lambda_1+n-1} \\ z_1^{\lambda_2+n-2} & \dots & z_n^{\lambda_2+n-2} \\ \vdots & \ddots & \vdots \\ z_1^{\lambda_n} & \dots & z_n^{\lambda_n} \end{vmatrix}}{\begin{vmatrix} z_1^{n-1} & \dots & z_n^{n-1} \\ z_1^{n-2} & \dots & z_n^{n-2} \\ \vdots & \ddots & \vdots \\ z_1^{0} & \dots & z_n^{0} \end{vmatrix}} \tag{3.127}$$

The irreducible characters $\chi_V^{U(n)}(\mathbf{z})$ of $U(n)$ are precisely the Schur functions $s_{\boldsymbol{\lambda}}(\mathbf{z})$ [135].

One gets a similar result for the irreducible characters of $\tilde{g} \in SU(n)$. Since $\det \tilde{g} = 1$, it is conjugate to the diagonal matrix of the form

$$\tilde{g} \sim \text{ diag } \left(y_1, y_2 y_1^{-1}, y_3 y_2^{-1}, \dots, y_{n-1} y_{n-2}^{-1}, y_{n-1}^{-1}\right). \tag{3.128}$$

Any irreducible representation of $SU(n)$ can be labelled by a partition $\boldsymbol{\mu}$, and the associated character is given by

$$\chi_{\boldsymbol{\mu}}^{SU(n)}(y_1,\dots,y_n) = s_{\boldsymbol{\mu}}(y_1, y_2 y_1^{-1},\dots, y_{n-1} y_{n-2}^{-1}, y_{n-1}^{-1}). \tag{3.129}$$

where y_i, $i=1,\dots,n-1$ parametrises the maximal torus $\tilde{T}^{n-1}$ of $SU(n)$.

A $U(n)$ irreducible representation labelled by $\boldsymbol{\lambda} = (m)^n + \boldsymbol{\mu}$ can be written uniquely in terms of the $SU(n)$ irreducible representation $V(\boldsymbol{\lambda})$ and the $U(1)$ charge $q(\boldsymbol{\lambda})$ as follows.

$$(V(\boldsymbol{\lambda}), q(\boldsymbol{\lambda})) = (\boldsymbol{\mu}, nm + |\boldsymbol{\mu}|). \tag{3.130}$$

To see this, we first write $g \in U(n)$ in terms of a $U(1)$ element $e^{i\theta}$ and an element $\tilde{g} \in SU(n)$ as $g = e^{i\theta}\tilde{g}$. Then the coordinates $\mathbf{z}$ of T^n is given in terms of θ and the coordinates $\mathbf{y}$ of $\tilde{T}^{n-1}$ by

$$z_1 = e^{i\theta} z_1, \quad z_2 = e^{i\theta} y_2 y_1^{-1}, \quad \dots \quad , \quad z_{n-1} = e^{i\theta} y_{n-1} y_{n-2}^{-1}, \quad z_n = e^{i\theta} y_{n-1}^{-1}. \tag{3.131}$$

In the representation (q,V), g is represented by $e^{iq\theta} R_V(\tilde{g})$. This can be phrased in terms of characters as

$$\chi_V^{U(n)}(z_1,\dots,z_n) = e^{iq\theta}\, \chi_{\tilde{V}}^{SU(n)}(y_1,\dots,y_{n-1}), \tag{3.132}$$

By direct substitution of (3.131) into (3.127), it is easy to show that

$$s_{\boldsymbol{\lambda}}(\mathbf{z}) = e^{i(nm+|\boldsymbol{\mu}|)\theta}\, s_{\boldsymbol{\mu}}\left(y_1, y_2 y_1^{-1}, \dots, y_{n-1}^{-1}\right), \tag{3.133}$$

whence our claim that $(V,q) = (\boldsymbol{\mu}, nm + |\boldsymbol{\mu}|)$ follows.

Therefore, for an irreducible representation $(\boldsymbol{\mu}, q)$ of $SU(n) \times U(1)$ to be a *bona fide* irreducible representation of $U(n)$, we need q to be equal to the number of boxes in $\boldsymbol{\mu}$ modulo n.

This result can be applied to a more complicated scenario. As an example, we consider the group $G = G_{\mathrm{SM}}/\Gamma_6$ which can be realised as $G = (U(3) \times U(2))/U(1)$, where we identify the overall $U(1)$ factor in $U(3)$ with the one in $U(2)$. Our result (3.130) tells us that, for a representation $(\boldsymbol{\nu}, \boldsymbol{\mu}, q)$ of $SU(3) \times SU(2) \times U(1)$ to be a *bona fide* representation of G, we must have

$$q = |\boldsymbol{\mu}| \bmod 2, \qquad \text{and} \qquad q = |\boldsymbol{\nu}| \bmod 3. \tag{3.134}$$

Chapter 4

Anomaly interplay in $U(2)$ gauge theories

4.1 Introduction

An $SU(2)$ chiral gauge theory in four dimensions suffers from a non-perturbative global anomaly when there is an odd number of fermion multiplets in isospin $2r+1/2$ representations, for $r \in \mathbb{Z}_{\geq 0}$ [149]. Such a theory is anomalous because the (Euclidean) partition function changes sign under an $SU(2)$ gauge transformation that corresponds to the non-trivial element in $\pi_4(SU(2)) = \mathbb{Z}_2$. Equivalently, the anomaly can be seen from a constant gauge transformation by the central element $-\mathbf{1} \in SU(2)$, in the background of a single instanton, as we review in §4.2.

One might be forgiven for guessing that a $U(2)$ chiral gauge theory suffers from a similar global anomaly, given that $\pi_4(U(2)) = \mathbb{Z}_2$ also, and given that $U(2)$ is locally equivalent to $SU(2) \times U(1)$ which has a global anomaly associated with the $SU(2)$ factor. It turns out that this is not the case. A quick way of reaching this conclusion is to recall that global anomalies are detected by the exponentiated η-invariant [151, 51],[1] which becomes a bordism invariant when perturbative anomalies vanish. Because the spin-bordism group

$$\Omega_5^{\mathrm{Spin}}(BU(2)) = 0 \tag{4.1}$$

(which can be straightforwardly adapted from calculations in [52, 139]), the exponentiated η-invariant must be trivial on all closed spin five-manifolds equipped with a $U(2)$ gauge

[1]Here we refer to the η-invariant of an extension of the Dirac operator $i\not{D}$ to a five-manifold that bounds spacetime. The η-invariant of a Dirac operator is a regularized sum of its positive eigenvalues minus its negative eigenvalues, as introduced by Atiyah, Patodi, and Singer [25].

bundle, which means that there can be no global anomalies in the 4d $U(2)$ gauge theory when perturbative anomalies cancel. In contrast $\Omega_5^{\mathrm{Spin}}(BSU(2)) = \mathbb{Z}_2$, which allows for a possible global anomaly in the $SU(2)$ theory.

In this Chapter, our first goal is to explain why there is no global anomaly in a $U(2)$ gauge theory, defined with a choice of spin structure. This is the subject of §4.3. The argument is simple enough to summarise in this Introduction. Recall firstly that $U(2)$ may be written as

$$U(2) \cong \frac{SU(2) \times U(1)}{\mathbb{Z}_2}, \tag{4.2}$$

where the $\mathbb{Z}_2$ quotient is generated by the central element $(-\mathbf{1}, e^{i\pi}) \in SU(2) \times U(1)$. As for the $SU(2)$ case, one could make a constant gauge transformation by the element $(-\mathbf{1}, 1) \in SU(2) \times U(1)$ in the background of a single instanton, and might thus be tempted to reach the same conclusion that there can be a global anomaly. However, this gauge transformation is equivalently described by the element $(\mathbf{1}, e^{i\pi}) \in SU(2) \times U(1)$. Thus, the anomalous transformation is in fact a local $U(1)$ transformation, and we can compute the variation of the fermionic partition function using the appropriate counterterms in the effective action. The non-invariance of the path integral measure (when there is an odd number of multiplets with isospin $2r + 1/2$) arises simply because there is a mixed triangle anomaly.

We show explicitly that the (perturbative) mixed triangle anomaly can vanish only if there is an even number of multiplets with isospin $2r + 1/2$, by reducing the anomaly cancellation condition modulo 2.[2] Note that this is only true when the global structure of the gauge group is strictly $U(2)$. The argument does not follow for the (locally isomorphic) gauge group $SU(2) \times U(1)$, even though the formula for the perturbative anomaly is the same, because not every representation of $SU(2) \times U(1)$ corresponds to a representation of $U(2)$. Having realised that the apparently global $SU(2)$ anomaly is manifest in $U(2)$ rather as a local anomaly, we may conclude from (4.1) that there can be no other new global anomalies in a $U(2)$ theory (defined with a spin structure).

Understanding the absence of global anomalies in a $U(2)$ gauge theory, but nonetheless the necessity of the condition on isospin $2r + 1/2$ multiplets, is of some phenomenological interest, because $U(2)$ could be the gauge group for the electroweak theory [136]. For example, anomaly cancellation in such a theory provides constraints on the electroweak quantum numbers of field content in the context of going beyond the Standard Model.

We then turn to the more subtle case of a $U(2)$ gauge theory defined without a spin or spin$_c$ structure, and perform a similar analysis relating to the 'new $SU(2)$ (global) anomaly'

[2] In §4.5 we arrive at the same conclusion by directly computing the η-invariant using the Atiyah–Patodi–Singer (APS) index theorem [25].

that afflicts an $SU(2)$ gauge theory that is similarly defined without a spin structure [143]. Recall that fields in such a theory are instead defined using a spin-$SU(2)$ structure, which requires that all fermions (bosons) have half-integer (integer) isospin. The $SU(2)$ theory is anomalous if there is an odd number of fermion multiplets with isospin $4r+3/2$, for $r \in \mathbb{Z}_{\geq 0}$. The partition function for such a theory, defined on certain manifolds that are not spin (in particular, on $\mathbb{C}P^2$), changes sign under the combined action of a diffeomorphism φ and an $SU(2)$ gauge transformation W. This is the new $SU(2)$ anomaly, which we shall recap in §4.2.

The second goal of this Chapter is to understand what happens to the new $SU(2)$ anomaly in the analogous situation in which the gauge group is enlarged from $SU(2)$ to $U(2)$. If the field content is such that all fermions (bosons) have half-integer (integer) isospins and odd (even) $U(1)$ charges, then the $U(2)$ gauge theory can be defined without a spin structure, using this time a spin-$U(2)$ structure to parallel transport fields. Again, one might expect that a global anomaly should afflict such a theory, corresponding to the new $SU(2)$ anomaly; and again, this turns out not to be the case, as we show in §4.4.

The new $SU(2)$ anomaly enjoys a similar but subtly different fate to the old one. This time, because of the crucial role played by the diffeomorphism φ in deriving the new $SU(2)$ anomaly, we find that the anomalous combination of φ and W cannot be replaced by a local $U(2)$ gauge transformation, as was the case for the 'old' $SU(2)$ anomaly. However, the anomalous combined action of φ and W has the same effect on the fermionic partition function as a local $U(2)$ gauge transformation with determinant -1. This gives rise to a local anomaly, that is a combination of the mixed triangle anomaly (corresponding to a Feynman diagram with two external $SU(2)$ currents and one $U(1)$ current) with the gauge-gravity anomaly for the $U(1)$ current. By considering this particular combination of perturbative anomalies reduced modulo 4, we find that the $U(2)$ gauge theory defined using a spin-$U(2)$ structure can only be anomaly-free when there is an even number of fermion multiplets with isospin $4r+3/2$.

It is important to stress that, in the $U(2)$ theory, this condition on isospin $4r+3/2$ multiplets must be satisfied simply for perturbative anomalies to cancel; thus, unlike the new $SU(2)$ anomaly, this condition persists even if we choose to restrict our attention to spin manifolds.

In §4.2 we review the pair of global anomalies in $SU(2)$ gauge theory. In §4.3 we discuss the $U(2)$ theory defined using a spin structure, before turning to the case without spin structure in §4.4. Finally, in §4.5 we interpret our results in terms of cobordism invariants. We thence explain why there are no other global anomalies in the $U(2)$ theory defined using a spin-$U(2)$ structure.

4.2 Review of the $SU(2)$ global anomalies

The old anomaly

We first review the global anomaly that occurs for an $SU(2)$ gauge theory defined on a four-manifold M (which we take to be Euclidean) using a spin structure [149]. Consider a single fermion transforming in the isospin-j representation, coupled to a background $SU(2)$ gauge field with curvature F. Let n_+ (n_-) denote the number of fermion modes with positive (negative) chirality (*i.e.* eigenvalue under γ^5). The Atiyah–Singer index theorem tells us that

$$n_+ - n_- = -\frac{1}{8\pi^2}\int_M \mathrm{Tr}\, F\wedge F = -T(j)\; p_1(F), \tag{4.3}$$

where $p_1(F)\in\mathbb{Z}$ is the first Pontryagin number (or instanton number), and

$$T(j) = \frac{2}{3}j(j+1)(2j+1) \tag{4.4}$$

is the Dynkin index defined via $\mathrm{Tr}(t_j^a t_j^b) = \frac{1}{2}T(j)\delta^{ab}$. Here $\{t_j^a\}$ denotes a basis for the isospin-j representation of $\mathfrak{su}(2)$. Because $n_+ - n_-$ is congruent to $n_+ + n_- \equiv \mathcal{N}_j$ modulo 2, the total number of fermion zero modes satisfies

$$\mathcal{N}_j \equiv T(j)\; p_1(F) \pmod 2. \tag{4.5}$$

If $\mathcal{N}_j$ is odd, then the partition function will change sign under the action of $(-1)^F$, where F is the fermion number. But since $(-1)^F$ is equivalent to applying a gauge transformation by the central element $-\mathbf{1}\in SU(2)$, this implies that $SU(2)$ is anomalous in such a scenario.

Only fermions with isospin $j = 2r+1/2$ can contribute to this anomaly, and only in backgrounds with odd instanton number, because it is only for these values of j that the Dynkin index (4.4) is odd. Thus, the anomaly vanishes if and only if the following holds

> Condition 1: There is an even number of fermions transforming in representations with isospin $2r+1/2$, for $r\in\mathbb{Z}_{\geq 0}$. (4.6)

This is the familiar $SU(2)$ anomaly discovered by Witten [149].

The new anomaly

Suppose now that there is no spin structure available, and that fermions are instead defined using a weaker spin-$SU(2)$ structure.[3] The transition functions for a spin-$SU(2)$ bundle are valued in the group

$$\mathrm{Spin}_{SU(2)}(4) \equiv \frac{\mathrm{Spin}(4) \times SU(2)}{\mathbb{Z}_2}, \tag{4.7}$$

where the $\mathbb{Z}_2$ quotient is generated by the central element $-\mathbf{1}$ of $SU(2)$ paired with the element $(-1)^F \in \mathrm{Spin}(4)$. All fields must transform in representations of this group, which requires that all fermions have half-integer isospin, and all bosons have integer isospin. Such a theory can be defined on all orientable four-manifolds, including those that are not spin such as $\mathbb{C}P^2$.[4]

In the simpler case that we discussed above, we saw how the usual $SU(2)$ anomaly could be seen from the action of $(-1)^F$ on the path integral measure, since $(-1)^F$ is equivalent to an $SU(2)$ gauge transformation by $-\mathbf{1} \in SU(2)$. The new $SU(2)$ anomaly is more subtle, and cannot be seen from a pure gauge transformation. Rather, the new $SU(2)$ anomaly is the non-invariance of the path integral under a transformation $\hat{\varphi}$ which is a combined diffeomorphism φ of M (for certain non-spin manifolds M) with an $SU(2)$ gauge transformation W.

To see this anomaly one may take M to be $\mathbb{C}P^2$, and $\varphi : z_i \mapsto z_i^*$ to act by complex conjugation on the homogeneous complex coordinates $\{z_i\}$ of $\mathbb{C}P^2$. A spin-$SU(2)$ connection A may be defined by embedding a spin$_c$ connection a in $\mathfrak{su}(2)$, *viz.* $A = \sigma^3 a$, where σ^3 is the diagonal Pauli matrix. The spin$_c$ connection a obeys the following quantisation condition

$$\int_S \frac{da}{2\pi} \equiv \frac{1}{2} \int_S w_2(TM) \pmod 1, \tag{4.8}$$

for any closed oriented 2-manifold $S \subset M$, where $w_2(TM)$ is the second Stiefel–Whitney class, which is such that $2a$ defines a properly-normalised $U(1)$ gauge field. In particular, choose a spin$_c$ connection a such that

$$\int_{\mathbb{C}P^1} \frac{da}{2\pi} = \frac{1}{2} \tag{4.9}$$

[3] The idea of using such 'spin-G' structures, for various Lie groups G (going beyond the case where $G = U(1)$), was introduced in Refs. [26, 27].

[4] It was first observed that a fermionic theory can be defined on $\mathbb{C}P^2$, using a spin$_c$ structure, in Ref. [85]. Indeed, every orientable four-manifold admits a spin$_c$ structure – but one must assume that M is equipped with a spin-$SU(2)$ structure, and not a spin$_c$ structure, in order to see the new $SU(2)$ anomaly.

for some $\mathbb{C}P^1 \subset \mathbb{C}P^2$. Such a spin$_c$ connection reverses sign under the diffeomorphism φ. The spin-$SU(2)$ connection A, however, is invariant under the combined action of φ with any $SU(2)$ gauge transformation W which also flips its sign, such as $W = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

An anomaly in the transformation $\hat{\varphi}$ has to arise from the path integral over the fermion zero modes. On $\mathbb{C}P^2$ the number of zero modes $\mathcal{N}_j$ equals the index of the Dirac operator $\mathfrak{I}_j$ (they are not only congruent modulo 2 as before).[5] For a single fermion multiplet in the isospin-j representation coupled to the background spin-$SU(2)$ connection A defined above, the Atiyah–Singer index theorem implies the index is [143]

$$\mathfrak{I}_j = \mathcal{N}_j = \frac{1}{24}(4j^2-1)(2j+3). \tag{4.10}$$

The zero modes come in pairs with eigenvalues $+1$ and -1 under $\hat{\varphi}$. Hence, the fermionic partition function $Z[A]$ transforms under the action of $\hat{\varphi}$ by

$$Z[A] \xrightarrow{\hat{\varphi}} (-1)^{\mathfrak{I}_j/2} Z[A]. \tag{4.11}$$

The index $\mathfrak{I}_j$ is even for all half-integer values of j, but is congruent to 2 mod 4 only when $j = 4r + 3/2$ for $r \in \mathbb{Z}_{\geq 0}$. For all other half-integer values of j, the index $\mathfrak{I}_j$ is divisible by 4. Hence, the partition function is invariant under $\hat{\varphi}$, and the theory is non-anomalous, if and only if the following condition holds:

Condition 2: there is an even number of fermions transforming in representations with isospin $4r + 3/2$, for $r \in \mathbb{Z}_{\geq 0}$. (4.12)

This is the new $SU(2)$ anomaly recently discovered by Wang, Wen, and Witten [143].

4.3 $U(2)$ gauge theory with a spin structure

We now turn to $U(2)$ gauge theory. We begin with the simpler case of a $U(2)$ gauge theory defined with a spin structure, for which the vanishing of the bordism group (4.1) implies there are no global anomalies. We will here give a physical explanation of this fact, previously noted in Refs. [52, 139], which demonstrates the subtle interplay between local and global anomalies in $U(2)$.

The representation theory of $U(2)$ plays a crucial role in the arguments used in this Chapter. Recall that an irreducible representation of $U(2) \cong (SU(2) \times U(1))/\mathbb{Z}_2$ is labelled an irreducible representation of $SU(2)$, itself labelled by an isospin j, together with a $U(1)$

[5] This is because on $\mathbb{C}P^2$ the Dirac operator only has zero modes of one chirality.

charge q, subject to a restriction relating q and j. Namely, q and j must satisfy the following 'isospin-charge relation'[6]

$$q \equiv 2j \pmod 2, \tag{4.13}$$

in convenient units where both gauge couplings are set to one.

Consider a theory with a single fermion with isospin j and charge q (satisfying (4.13)), coupled to a background $U(2)$ gauge field with curvature F and defined on S^4. Recall that the usual $SU(2)$ anomaly occurs when the fermionic partition function changes sign under the gauge transformation by $-\mathbf{1} \in SU(2)$. Embedding $SU(2) \subset U(2)$, this global $SU(2)$ transformation is equivalent to a $U(1)$ gauge transformation by $e^{i\pi}$, which is a local gauge transformation.

The variation of the partition function $Z[A]$ under a potentially anomalous $U(1)$ gauge transformation can be computed using the appropriate counterterms in the effective action (see *e.g.* [113]). For a $U(1)$ transformation by angle θ, we have that

$$\begin{aligned} Z[A] \to Z[A] \exp\left[-\frac{iq\theta}{8\pi^2}\int_{S^4} \mathrm{Tr}\, F\wedge F + \text{gravitational piece}\right] \\ = Z[A]\exp\left[-iq\theta\, T(j)\; p_1(F) + \text{gravitational piece}\right], \end{aligned} \tag{4.14}$$

where the gravitational piece is proportional to the integral of Tr $R\wedge R$ which vanishes for S^4. Setting $\theta = \pi$ and the instanton number $p_1(F) = 1$, this reduces to

$$Z[A] \to (-1)^{qT(j)} Z[A]. \tag{4.15}$$

We see that the path integral is invariant under this transformation if and only if $qT(j)$ is even.

Recall that the Dynkin index $T(j)$ is only odd for isospins $j \in 2\mathbb{Z}_{\geq 0} + 1/2$. The isospin-charge relation (4.13) means that q is also odd for these representations. Hence, there is necessarily an anomaly if there is an odd number of fermions in multiplets with isospin $2r + 1/2$; in other words, precisely when condition (4.6) is violated. Thus, we find that the $SU(2)$ global anomaly manifests itself rather as a perturbative anomaly when $SU(2)$ is embedded in $U(2)$. There are no global anomalies in the $U(2)$ theory.

Indeed, one can directly derive that condition (4.6) must hold for a $U(2)$ gauge theory by considering the equations for perturbative anomaly cancellation. Suppose that we have N_j fermions transforming in isospin-j representations of $U(2)$, with charges $\{q_{j,\alpha}\}$, where

[6] We note in passing that this isospin-charge relation (4.13) is satisfied by all the SM fermion fields, where $U(1)$ corresponds to hypercharge. Hence the electroweak gauge symmetry could be either $SU(2)\times U(1)$ or $U(2)$.

$\alpha = 1, \ldots N_j$. We assume without loss of generality that all fermions have left-handed chirality. The mixed triangle anomaly (that is, the triangle anomaly involving two $SU(2)$ gauge bosons and one $U(1)$ gauge boson) is proportional to

$$\mathscr{A}_{\text{mix}} \equiv \sum_j T(j) \sum_{\alpha=1}^{N_j} q_{j,\alpha} = 0, \tag{4.16}$$

The fact that $T(j)$ is odd only for $j \in 2\mathbb{Z}_{\geq 0} + 1/2$, together with the isospin-charge relation, means that reducing mod 2 immediately yields

$$\sum_{j \in 2\mathbb{Z}+1/2} 1 \equiv 0 \pmod 2, \tag{4.17}$$

and hence that condition (4.6) must be satisfied to avoid a perturbative mixed anomaly. It is possible to give a unified discussion of the perturbative and non-perturbative anomalies in this theory by computing the η-invariant explicitly. We give such an account in §4.5.

4.4 $U(2)$ gauge theory without a spin structure

We now turn to the case where a spin structure is not available. Instead, we can use a spin-$U(2)$ structure to parallel transport fields, provided that all fields transform in representations of the group

$$\text{Spin}_{U(2)} \equiv \frac{\text{Spin}(4) \times U(2)}{\mathbb{Z}_2}. \tag{4.18}$$

The $\mathbb{Z}_2$ quotient is generated by the product of the element $(-1)^F \in \text{Spin}(4)$ with the central element $-\mathbf{1} \in U(2)$. Recalling also the effects of the $\mathbb{Z}_2$ quotient within $U(2)$, we have the following constraints on the allowed representations:

$$\begin{aligned} \text{fermion} &\longleftrightarrow j \in (2\mathbb{Z}+1)/2 &&\longleftrightarrow q \text{ odd}, \\ \text{boson} &\longleftrightarrow \qquad j \in \mathbb{Z} &&\longleftrightarrow q \text{ even}, \end{aligned} \tag{4.19}$$

where (q, j) label the $U(2)$ representations as before.

In the analogous $SU(2)$ theory, the new $SU(2)$ anomaly is associated with a transformation $\hat{\varphi}$ that is a combined diffeomorphism φ plus gauge transformation W, as we reviewed in §4.2. Recall that $\hat{\varphi}$ acts on the partition function as

$$Z[A] \xrightarrow{\hat{\varphi}} (-1)^{\mathfrak{I}_j/2} Z[A]. \tag{4.20}$$

Let us first analyse the behaviour of the $U(2)$ theory under this same transformation. To that end, again take M to be $\mathbb{C}P^2$, and as in §4.2 define $\hat{\varphi}$ to be the combination of the complex conjugation diffeomorphism $\varphi : z_i \mapsto z_i^*$ with the $U(2)$ gauge transformation $W = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Moreover, we define a spin-$U(2)$ connection $A = \sigma^3 a$, where a is the spin$_c$ connection satisfying Eqs. (4.8, 4.9), which is invariant under $\hat{\varphi}$.

The diffeomorphism φ (on its own) is such that $\varphi^2 = -1$ when acting on fermions. More specifically, φ can be thought of as a certain spatial rotation through an angle π, corresponding (in certain coordinates) to the following transformation on a 2-component Weyl fermion ψ_a:

$$\psi_a \overset{\varphi}{\mapsto} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \psi_a, \tag{4.21}$$

where the index labels Lorentz $SU(2)$ indices of the spin-$1/2$ fermion. Because the matrix appearing in (4.21) is not proportional to the identity, this diffeomorphism cannot therefore be subsumed by the $U(1)$ phase degree of freedom in $U(2)$. Thus, as in the $SU(2)$ case, the transformation $\hat{\varphi}$ is necessarily not equivalent to a pure $U(2)$ gauge transformation. Since $\hat{\varphi}$ is inequivalent to a local gauge transformation, in contrast to the situation in §4.3, we might suspect that this new $SU(2)$ global anomaly will stick around in the $U(2)$ theory.

However, what we can do instead is construct a local $U(2)$ gauge transformation whose action on the fermionic partition function $Z[A]$ is identical to (4.20). Consequently, cancellation of perturbative anomalies shall guarantee that the suspected global anomaly in fact vanishes. To wit, consider a gauge transformation by

$$\tilde{W}(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \in U(2), \qquad \theta \notin \pi\mathbb{Z}, \tag{4.22}$$

i.e. by a pure $U(1)$ phase. Note that $\det \tilde{W} \neq 1$ for $\theta \notin \pi\mathbb{Z}$, so that there is no corresponding gauge transformation in $SU(2)$ by design. Let us now compute the transformation of $Z[A]$ under $\tilde{W}(\theta)$, for a single fermion multiplet with isospin-j and charge q coupled to the spin-$U(2)$ connection A. This time the gravitational contribution will be non-vanishing because $\mathbb{C}P^2$ has non-zero signature. Taking into account the contributions from both the mixed gauge anomaly and the gauge-gravity anomaly, the shift in the Euclidean partition function, for now on a general four-manifold M with metric g, is

$$Z[A] \to Z[A] \exp\left(-S_{\text{gauge}} - S_{\text{grav}}\right), \tag{4.23}$$

where

$$S_{\text{gauge}} = -\frac{i\theta}{16\pi^2} q \int_M \text{Tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu} d^4x, \tag{4.24}$$

in which the trace is only over the $SU(2)$ gauge indices (we here choose to keep Lorentz indices explicit for clarity), and

$$S_{\text{grav}} = -\frac{i\theta}{16\pi^2}\frac{\text{Tr}(Q)}{24}\int_M R_{\mu\nu\sigma\tau}\tilde{R}^{\mu\nu\sigma\tau}\sqrt{g}\,d^4x, \tag{4.25}$$

where Q is the generator of the $U(1)$ factor in $U(2)$, and the trace sums over all $2j+1$ components of the isospin-j representation. Recall that $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\sigma\tau}F_{\sigma\tau}$ and $\tilde{R}^{\mu\nu\sigma\tau} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}R_{\alpha\beta}{}^{\sigma\tau}$, where $R_{\mu\nu\sigma\tau}$ are the components of the Riemann tensor.

We can relate both these integrals to characteristic classes of bundles over M, taking care with the various normalisation factors. Noting that $\tau^a = \sigma^a/2$ are the generators of the $SU(2)$ factor of $U(2)$, the choice $A = \sigma^3 a$ implies that $F^a_{\mu\nu} = 2\delta^{a3}f_{\mu\nu}$, where $f = da$ is the curvature of the spin$_c$ connection a. We can thus reduce (4.24) to an integral over the spin$_c$ connection,

$$S_{\text{gauge}} = -\frac{iq\theta}{4\pi^2}\left(\frac{T(j)}{2}\right)\int_M f_{\mu\nu}\tilde{f}^{\mu\nu}d^4x = -\frac{iq\theta}{4\pi^2}T(j)\int_M f\wedge f. \tag{4.26}$$

The normalisation (4.9) of the spin$_c$ connection determines its first Pontryagin class in terms of the signature σ of M, *viz.*

$$\frac{1}{2}\int_M \frac{f\wedge f}{(2\pi)^2} = \frac{1}{8}\sigma. \tag{4.27}$$

Since $\sigma = 1$ for $\mathbb{C}P^2$ we have that, when $M = \mathbb{C}P^2$,

$$S_{\text{gauge}} = -\frac{i\theta}{4}T(j)q. \tag{4.28}$$

For the gravitational contribution, we use the fact that

$$-\frac{1}{16\pi^2}\int_M R_{\mu\nu\sigma\tau}\tilde{R}^{\mu\nu\sigma\tau}\sqrt{g}d^4x = \frac{1}{2}\int_M \frac{\text{Tr}\,R\wedge R}{(2\pi)^2} = p_1[M] = 3\sigma(M), \tag{4.29}$$

and that $\text{Tr}(Q) = (2j+1)q$ to deduce that

$$S_{\text{gravity}} = +\frac{i\theta}{8}(2j+1)q \tag{4.30}$$

when $M = \mathbb{C}P^2$.

The partition function therefore shifts by

$$Z[A] \to Z[A]\exp\left[-\frac{i\theta}{4}\left(T(j) - \frac{1}{2}(2j+1)\right)q\right]. \tag{4.31}$$

Using the expression (4.4) for the Dynkin index, we find that the factor in square brackets is nothing but $-i\theta\mathfrak{I}_j q$, where $\mathfrak{I}_j$ is the same index from (4.10) that detected the new $SU(2)$ anomaly. Therefore, setting $\theta = \pi/2$ gives

$$Z[A] \xrightarrow{\tilde{W}(\pi/2)} (-1)^{\mathfrak{I}_j q/2} Z[A]. \tag{4.32}$$

Recalling that all fermions in this theory have half-integral isospin j and odd charge q, and that $T(j) \equiv 2 \pmod 4$ only when $j \in 4\mathbb{Z} + 3/2$, we see that there is a perturbative $U(2)$ anomaly when there is an odd number of fermion multiplets with isospin $j \in 4\mathbb{Z}_{\geq 0} + 3/2$; in other words, precisely when condition (4.12) is violated.

Another way to see that the $U(2)$ gauge transformation by $\tilde{W}(\pi/2)$ has the same action on the path integral as the action $\hat{\varphi}$ of the diffeomorphism φ plus $SU(2)$ gauge transformation W is to consider the composition $\hat{\varphi}(\pi/2) \equiv \hat{\varphi} \cdot \tilde{W}(\pi/2)$ of these two transformations. In other words, consider the combined action on $Z[A]$ of the diffeomorphism φ plus a $U(2)$ gauge transformation by $\tilde{W}(\pi/2) \cdot W = iW$. The argument proceeds almost exactly as the argument for the new $SU(2)$ anomaly, as summarised in §4.2; the only difference is that now the fermion zero modes transform in pairs under $\hat{\varphi}(\pi/2)$ with eigenvalues $+i$ and $-i$ (rather than $+1$ and -1) whose product is now $+1$ (rather than -1 as before). Thus, since there is an even number of zero modes, the action of $\hat{\varphi} \cdot \tilde{W}(\pi/2)$ is always non-anomalous, and so each of $\hat{\varphi}$ and $\tilde{W}(\pi/2)$ must contribute the same mod 2 anomaly.

As we saw in §4.3 for the old $SU(2)$ anomaly, we can again deduce the necessity of condition (4.12) directly from the equations for perturbative anomaly cancellation. This time, however, we also need to use the cancellation of the gauge-gravity anomaly,

$$\mathscr{A}_{\text{grav}} \equiv \sum_j (2j+1) \sum_{\alpha=1}^{N_j} q_{j,\alpha} = 0. \tag{4.33}$$

If we take a particular linear combination of local anomaly equations, *viz.* $\frac{1}{4}$[(4.16)-$\frac{1}{2}$(4.33)], we obtain

$$\sum_{j \text{ half integer}} \mathfrak{I}_j \sum_\alpha q_{j,\alpha} = 0. \tag{4.34}$$

Reducing this equation modulo 4, and using the properties of $\mathfrak{I}_j$ noted above, we immediately obtain

$$\sum_{j=4r+3/2} 1 \equiv 0 \pmod 2, \tag{4.35}$$

recovering the condition (4.12) that, in the $SU(2)$ case, is required to cancel the new $SU(2)$ anomaly.

4.4.1 Interpretation of the $U(2)$ anomalies

We have now seen how both conditions (4.6) and (4.12), for the cancellation of the old and new $SU(2)$ anomalies, do not correspond to global anomalies when $SU(2)$ is embedded as a subgroup of $U(2)$. The arguments used for the two anomalies were, however, qualitatively different. In the case of the old $SU(2)$ anomaly, for a theory defined using a spin structure, the global transformation in $SU(2)$ corresponds to a local transformation in $U(2)$, for which there is an associated perturbative anomaly if there are an odd number of multiplets with isospin $j \in 2\mathbb{Z}_{\geq 0} + 1/2$.

For the new $SU(2)$ anomaly, however, the mixed diffeomorphism plus gauge transformation is not equivalent to a local transformation in $U(2)$. It nonetheless transpires to be equivalent to a local transformation in $U(2)$ at the level of its action on the fermionic partition function. In this sense, the condition (4.12) emerges somewhat coincidentally from perturbative anomaly cancellation in the $U(2)$ theory, which should be thought of as 'trivialising' the new $SU(2)$ global anomaly; for the old $SU(2)$ anomaly, the correct interpretation is rather that there is no global anomaly at all in $U(2)$.

As a result, the condition (4.12) enjoys a different 'status' in the $SU(2)$ theory versus the $U(2)$ theory. It is important to recall that the new $SU(2)$ anomaly is no barrier to the consistency of an $SU(2)$ gauge theory when formulated only on spin manifolds.[7] In contrast, the constraint (4.35) on the $U(2)$ theory is required by $U(2)$ gauge invariance, and so its violation, like the violation of the original Witten anomaly, would render the $U(2)$ theory inconsistent (even on spin manifolds).

4.4.2 Disentangling the anomaly interplay

It is possible to make rigorous the claim that the condition (4.12) emerges only coincidentally in the $U(2)$ theory without spin structure. In fact, in this Section we show that, at least at the level of effective field theory, the perturbative anomaly may be cancelled to leave behind a theory with the 'new' type of global anomaly, thereby disentangling the anomaly interplay described above.

For instance, if one interprets the $U(2)$ gauge theory described in Section 4.4 as an effective field theory of the light excitations that is valid only up to some momentum

[7] In fact, the new $SU(2)$ anomaly is not an insurmountable barrier to consistency on non-spin manifolds either; in this case, one can couple to a topological quantum field theory (tQFT), in the same 4d bulk, which has the same anomaly theory (specifically, this anomaly theory has 5-form lagrangian given by the product $w_2 w_3$ of Stiefel–Whitney classes), and thereby cancel the $\mathbb{Z}_2$-valued global anomaly. This kind of anomaly cancellation mechanism was introduced as a 'topological Green–Schwartz mechanism' in [73]. Note that the tQFT to which we couple has no propagating degrees of freedom that would alter the phenomenology of the theory.

cutoff scale Λ, then Wess–Zumino (WZ) terms may be included in the lagrangian which cancel anomalies in the low-energy theory.[8] If we consider again a general spectrum with N_j fermions transforming with isospin-j and with charges $\{q_{j,\alpha}\}$, then let us modify the effective lagrangian by adding the pair of WZ terms [113, 148, 8]

$$\mathscr{L} \to \mathscr{L} + \mathscr{L}_{\text{WZ}}, \qquad \mathscr{L}_{\text{WZ}} = \frac{i\mathscr{A}_{\text{mix}}}{32\pi^2}\phi F^a_{\mu\nu}\tilde{F}^{a\mu\nu} + \frac{i\mathscr{A}_{\text{grav}}}{384\pi^2}\phi\sqrt{g}R_{\mu\nu\sigma\tau}\tilde{R}^{\mu\nu\sigma\tau}, \tag{4.36}$$

where $\phi(x)$ is a dimensionless (circle-valued) pseudoscalar field which enjoys a shift symmetry under the $U(1)$ factor in $U(2)$, *viz.* $\phi(x) \to \phi(x) + \theta$ for $g = e^{i\theta}$,[9] and is a singlet under the $SU(2)$ part. These WZ terms conveniently encode the effects of integrating out a "mirroring" set of heavy chiral fermions, which transform in the same set of $U(2)$ representations but with opposite chirality.[10]

One can check explicitly that under any $U(2)$ gauge transformation, including generic $U(1)$ transformations of the form (4.22), the effective lagrangian is now invariant; the shifts of the WZ terms precisely cancel the shift in the effective action due to the non-invariance of the path integral measure for the chiral fermions, as is the purpose of the construction. However, gauge invariance comes at a price, which is that the full $U(2)$ symmetry is no longer linearly-realised. To see this, note that invariance under local $U(1)$ gauge transformations $\phi(x) \to \phi(x) + \theta(x)$, for a smooth function $\theta(x)$, requires that the pseudoscalar ϕ should have a kinetic term of the Stueckelberg form, that is

$$\mathscr{L} \supset \frac{1}{2}|d\phi - b|^2, \tag{4.37}$$

where b is the $U(1)$ component of the spin-$U(2)$ connection, which transforms as $b \to b + d\theta$.[11] Thus, the component b becomes massive, meaning that at low-energies only a subgroup $SU(2) \subset U(2)$ is linearly-realised.

Interestingly, adding WZ terms to the effective lagrangian is not guaranteed to cancel the more subtle global anomalies. In the presence of the WZ terms (4.36), one may now consider fermion content which violates condition (4.12) without violating perturbative anomaly

[8] The mechanism we describe here for cancelling anomalies at low-energies might also be referred to as a 'Green–Schwartz mechanism' [78], a terminology that stems from a famous application to cancelling mixed anomalies in string theory.

[9] We remark that these WZ terms are well-defined even though ϕ is circle-valued; under the 'large gauge transformation' $\phi(x) \to \phi(x) + 2\pi$, the phase of the exponentiated action shifts by an integer multiple of 2π and so the path integral is unchanged, for any orientable 4-manifold M and for any fermion content.

[10] We might imagine that heavy masses could arise from Yukawa-like interactions with a Higgs field. However, the precise construction of a suitable Yukawa sector is not immediately obvious, and we do not venture the details of a UV completion here.

[11] Locally, b behaves like a $U(1)$ gauge field.

cancellation. For such a theory, we should reconsider its behaviour under the combined diffeomorphism plus gauge transformation, denoted $\hat{\varphi}$, that led to the new $SU(2)$ anomaly on $M = \mathbb{C}P^2$.

How do the pair of WZ terms transform under $\hat{\varphi}$? Recall that $\hat{\varphi}$ is the combination of a complex conjugation diffeomorphism φ with a $U(2)$ gauge transformation by $W = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The spin-$U(2)$ connection $A = \sigma^3 a$ defined earlier in this Section, which should now be interpreted as a spin-$SU(2)$ connection due to the massive $U(1)$ component b decoupling, is invariant under $\hat{\varphi}$, and hence so is the field strength F. The Pontryagin class $\frac{\mathrm{Tr}\, R \wedge R}{8\pi^2}$, being a topological invariant [105], is invariant under the diffeomorphism φ and hence invariant under $\hat{\varphi}$. Finally, given φ is locally equivalent to a spatial rotation (in four dimensions), and given also that ϕ is an $SU(2)$-singlet, the pseudoscalar ϕ is invariant under $\hat{\varphi}$. So both WZ terms in (4.36) are invariant under the action of $\hat{\varphi}$.

We already know how the partition function varies under $\hat{\varphi}$ due to the chiral fermion contribution, which is precisely the variation given in Eq. (4.11). Hence, we conclude that if condition (4.12) is violated, in other words if there is an odd number of fermions with isospins $j \in 4\mathbb{Z}_{\geq 0} + 3/2$, then the effective field theory, which is free of perturbative anomalies by virtue of the effective WZ term, does indeed suffer from a $\mathbb{Z}_2$-valued global anomaly in $\hat{\varphi}$. Up to the effects of the WZ terms, we have arrived at precisely the $SU(2)$ theory defined with spin-$SU(2)$ structure that was introduced by Wang, Wen, and Witten to illustrate the new $SU(2)$ anomaly [143].

In this way, one can in fact disentangle the effects of perturbative anomalies in the $U(2)$ gauge theory with spin-$U(2)$ structure, and isolate an effective theory that suffers from the new $SU(2)$ anomaly at low energies. But it is important to emphasize that this can only be achieved by including WZ terms (or something similar), which enriches the dynamics of the theory – for instance, in the gauge we have chosen one must include the effects of a pseudoscalar field ϕ. The global anomaly that remains would then have precisely the same physical interpretation as the new $SU(2)$ anomaly; it presents a barrier to defining the theory on non-spin manifolds such as $\mathbb{C}P^2$, at least in the absence of couplings to topological degrees of freedom. This fact that the new $SU(2)$ anomaly, unlike the old one, is in a sense still there in $U(2)$, may also be understood from the perspective of cobordism, as we explain in §4.5.2. We remark that a similar trick cannot be performed to restore the old $SU(2)$ global anomaly in the $U(2)$ theory.

It is worth spelling out the fact that, as is the case for the new $SU(2)$ anomaly, this residual global anomaly can always be cancelled by coupling to a tQFT (and considerations of cobordism in §4.5.2 reveal that there can be no further global anomalies). Unlike the WZ term, such topological degrees of freedom would not alter the dynamics of the theory, but would rather embue the theory with topological order in the deep infrared. We postpone such considerations for future work.

One might distil the various ideas at play in this Section into the following statement:

> it is possible to write down a consistent $U(2)$ theory of a single isospin-3/2 fermion, that can be defined on non-spin manifolds using a spin-$U(2)$ structure, if one includes a pair of WZ terms to cancel the perturbative anomalies, and couples to a tQFT to cancel the residual global anomaly.

4.5 Cobordism and the absence of $U(2)$ global anomalies

Finally, we discuss the connection between our results and cobordism invariants in five dimensions. Such considerations will also enable us to conclude that there are no further anomalies in the $U(2)$ gauge theories we have considered, defined either with or without a spin structure.

4.5.1 Case I: with a spin structure

For an $SU(2)$ gauge theory defined on a four-manifold M equipped with spin structure, the original $SU(2)$ anomaly is detected by the bordism group

$$\Omega_5^{\mathrm{Spin}}(BSU(2)) = \mathbb{Z}_2. \tag{4.38}$$

There is a corresponding cobordism invariant, namely the η-invariant, which reduces in this case to a 5d mod 2 index because the fermions are in real representations. Let $\mathscr{I}_{1/2}$ denote this 5d mod 2 index for a single fermion with isospin-$1/2$. For anomalous fermion content, $\mathscr{I}_{1/2}$ is non-vanishing on the mapping torus $M \times S^1$ [149, 143].

When $SU(2)$ is embedded in $U(2)$, a fermion with isospin-$1/2$ is necessarily in a non-trivial representation of $U(1)$ by (4.13), and thus in a complex representation. Hence, the η-invariant no longer reduces to a mod 2 index in this case. But this does not matter in the end, because one may calculate the bordism group directly to find that [139, 52]

$$\Omega_5^{\mathrm{Spin}}(BU(2)) = 0. \tag{4.39}$$

Hence, in the case that perturbative anomalies vanish and the η-invariant becomes a cobordism invariant, there are no cobordism invariants and thus the η-invariant must be trivial. We therefore deduce that there are no global anomalies in this theory. This is consistent with our explicit calculation in §4.2, which realised the potentially anomalous global $SU(2)$ gauge transformation to be equivalent to a local $U(2)$ gauge transformation.

These statements can be seen from a slightly different perspective. The exponentiated η-invariant captures both the global and perturbatives anomalies [153, 156, 155]. In the current case, this can be seen quite explicitly. The vanishing of the fifth bordism group of $BU(2)$ means that any closed spin five-manifold X equipped with a $U(2)$-bundle structure is a boundary of a six-manifold Y with the $U(2)$ and spin structures extended appropriately. The direct relationship between the η-invariant on such a five-manifold and the anomaly polynomial I_6 is then fixed by the Atiyah–Patodi–Singer (APS) index theorem [25]

$$\mathrm{ind}\left(i\not{D}\right) = \int_Y I_6 - \eta_X. \tag{4.40}$$

Whenever the perturbative anomaly vanishes, $\exp(-2\pi i\eta_X)$ becomes trivial on all closed spin five-manifolds and so there can be no additional anomaly.

On the other hand, when the perturbative anomaly doesn't vanish, we can use (4.40) to compute the η-invariant explicitly, from the anomaly polynomial I_6. We may choose the closed five-manifold X to be the mapping torus $X = M \times S^1$. This is the boundary of a six-manifold $Y = M \times D^2$ to which the $U(2)$ bundle may be extended, where D^2 is a hemisphere (topologically a disc) whose equator coincides with the original S^1. Note that, importantly, this cannot be done in general for $SU(2)$, or indeed for $SU(2)\times U(1)$, bundles. To see this, let A denote an $SU(2)$ gauge field on M with instanton number one and let $U(x)$ denote a gauge transformation in the non-trivial class of $\pi_4(SU(2))$. Recall that a 5d gauge field on the mapping torus $X = M \times S^1$ of the form $A_\phi = (1-\phi/2\pi)A + (\phi/2\pi)A^U$, where ϕ parametrises the S^1, cannot be extended to any bounding six-manifold. However, if such an $SU(2)$ configuration is embedded in $U(2)$, we may consider a connection $\mathbf{A} = a + A_\phi$ extended to $Y = M \times D^2$, where A_ϕ is the $SU(2)$ connection written above (supported only on the boundary $X = \partial Y$), and a is a $U(1)$ gauge field supported only on the D^2 factor. In particular, take a to be the connection for a Dirac monopole with twice the smallest unit of charge placed at the centre of the hemisphere. Because $a \sim d\phi$ on the equator, $\mathbf{A}$ is gauge equivalent to A_ϕ on the boundary $X = \partial Y$.

We then have that

$$\int_{M\times D^2} I_6 = \frac{1}{2}\int_{M\times S^2} \hat{A}(\mathscr{R})\ \mathrm{Tr}\ \exp\left(\frac{\mathscr{F}}{2\pi}\right)\Bigg|_6, \tag{4.41}$$

where we have expressed the anomaly polynomial explicitly in terms of the $\hat{A}$-genus (sometimes called the 'Dirac genus') and the $U(2)$ gauge field $\mathscr{F}$. This can be expanded out to give

$$\int_{M\times D^2} I_6 = \frac{1}{2}\int_{M\times S^2}\left[\frac{1}{24}p_1(\mathscr{R})\ \mathrm{Tr}\,\frac{\mathscr{F}}{2\pi}+\frac{1}{3!}\ \mathrm{Tr}\left(\frac{\mathscr{F}}{2\pi}\right)^3\right], \tag{4.42}$$

where $p_1(\mathscr{R})$ is the first Pontryagin class of the tangent bundle. Now, $\int_M p_1(\mathscr{R})$ is a multiple of 48 when the (orientable) four-manifold M is spin, due to a signature theorem of Rochlin, so we can ignore the contribution to $\exp(-2\pi i\eta_X)$ coming from the first term in Eq. (4.42) and focus only on the second term. For a fermion with charge q under the $U(1)$ part and isospin-j under the $SU(2)$ part of the gauge group $U(2)$, we can write the $U(2)$ gauge field $\mathscr{F}$ in terms of the $U(1)$ gauge field f and the $SU(2)$ gauge field $F = F^a t^a_j$ as

$$\mathscr{F} = fq\mathbf{1}_{2j+1} + F. \tag{4.43}$$

To see the anomaly, we can choose $\mathscr{F}$ such that f has unit magnetic flux through S^2 and F is a one-instanton on M, whence we obtain

$$\int_{M\times D^2} I_6 = \frac{1}{2}q\int_{S^2}\frac{f}{2\pi}\int_M \frac{1}{8\pi^2}\ \mathrm{Tr}\,F\wedge F = \frac{1}{2}qT(j), \tag{4.44}$$

and thereby conclude that $\exp(-2\pi i\eta_X) = (-1)^{qT(j)}$. Recall that any fermion with isospin $j\in 2\mathbb{Z}_{\geq 0}+1/2$ necessarily has odd charge q. We thus arrive at the same physical outcome as in the usual $SU(2)$ global anomaly, only that it is now the perturbative anomaly that contributes to the η-invariant (as we saw already in §4.3).

4.5.2 Case II: without a spin structure

Recall that for the $SU(2)$ gauge theory defined without spin structure the corresponding bordism group is [65, 81, 138]

$$\Omega_5^{\frac{\mathrm{Spin}\times SU(2)}{\mathbb{Z}_2}} = \mathbb{Z}_2\times\mathbb{Z}_2. \tag{4.45}$$

A possible basis is given by $\mathscr{I}_{1/2}$ and $\mathscr{I}_{3/2}$, the 5d mod 2 indices associated with a single fermion with isospin-$1/2$ or $3/2$ respectively [143]. The former corresponds to the old $SU(2)$ anomaly, and the latter corresponds to the new one.

Now consider the case of a $U(2)$ gauge theory formulated without a spin structure, but rather using a spin-$U(2)$ structure, as was the subject of §4.4. In Appendix 4.A we calculate using the Adams spectral sequence that

$$\Omega_5^{\frac{\mathrm{Spin}\times U(2)}{\mathbb{Z}_2}} = \mathbb{Z}_2. \tag{4.46}$$

What is the interpretation of this 5d mod 2 cobordism invariant? And does it signify a possible new global anomaly that we have so far missed?

Fermions in either the isospin-$1/2$ or $3/2$ representations must have odd and thus non-vanishing charge under $U(1)$. Thus, it is not clear how to relate the η-invariant for this theory to a mod 2 index such as $\mathscr{I}_{1/2}$ or $\mathscr{I}_{3/2}$. Moreover, unlike in §4.5.1, we cannot use the APS index theorem to compute the η-invariant for an arbitrary closed five-manifold with spin-$U(2)$ structure, because Eq. (4.46) implies that not all such manifolds are bordant to zero. Fortunately, we may follow Ref. [143] in identifying a mod 2 cobordism invariant dual to the generator of (4.46) to be

$$J(Y) = \int_Y w_2(TY)w_3(TY), \tag{4.47}$$

where Y is a closed 5-manifold, and $w_{2,3}(TY)$ are Stiefel–Whitney classes. The crucial point is that $J(Y)$ is a mod 2 cobordism invariant of 5-manifolds with no further structure defined. Indeed, the fact that the new $SU(2)$ anomaly can be cancelled by the topological Green–Schwartz mechanism, as noted in footnote 7 above, follows essentially from this fact. Hence, $J(Y)$ is automatically a cobordism invariant of 5-manifolds with spin-$U(2)$ structure, albeit one that can only be detected on non-spin 5 manifolds. For example,

$$J\left(\frac{\mathbb{C}P^2\times S^1}{\mathbb{Z}_2}\right) = 1, \tag{4.48}$$

and thus the Dold manifold $(\mathbb{C}P^2\times S^1)/\mathbb{Z}_2$ is a suitable generator for the bordism group (4.46). Here the $\mathbb{Z}_2$ acts as complex conjugation on $\mathbb{C}P^2$, and is the antipodal map on S^1. Because $J(Y)$ vanishes trivially on spin manifolds, it does not appear in either (4.38) or (4.39).

In Ref. [143], the cobordism invariant $J(Y)$ was identified, for any five-manifold with spin-$SU(2)$ structure, with the mod 2 index $\mathscr{I}_{3/2}$, and thus with the new $SU(2)$ anomaly, since the Dold manifold corresponds precisely to the action of the diffeomorphism plus gauge transformation $\hat{\varphi}$ on $\mathbb{C}P^2$. Since the action of $\hat{\varphi}$ on the corresponding $U(2)$ theory is equivalent, at the level of the partition function, to a local $U(2)$ transformation as described in §4.4, the potential global anomaly corresponding to this cobordism invariant necessarily vanishes by perturbative anomaly cancellation. That said, as we saw in §4.4.2, by including

WZ terms to cancel the perturbative anomalies in the low energy effective theory, it is possible to reveal a low-energy theory which does indeed suffer from this 'new $U(2)$ anomaly', which corresponds to the $\mathbb{Z}_2$ in (4.46). Since there are no other independent cobordism invariants, we conclude that there are no other possible global anomalies in the $U(2)$ gauge theory defined using a spin-$U(2)$ structure.

Appendix 4.A Spin-$U(2)$ bordism

In this section we calculate the bordism group $\Omega_5^{\frac{\mathrm{Spin}\times U(2)}{\mathbb{Z}_2}}(\mathrm{pt})$, using the Adams spectral sequence. For a guide to using the Adams sequence to compute bordism groups, we recommend Ref. [33].

To make the presentation clearer, we will write U_n and SO_n for $U(n)$ and $SO(n)$, as well as $H^\bullet(X)$ for $H^\bullet(X;\mathbb{Z}_2)$, in the rest of this Appendix.

When there is no odd-torsion involved, the bordism group $\Omega^G_{t-s}(\mathrm{pt})$ can be evaluated via the Adams spectral sequence

$$\mathrm{Ext}^{s,t}_{\mathscr{A}}(H^\bullet(MTG),\mathbb{Z}_2) \Rightarrow \Omega^G_{t-s}(\mathrm{pt}), \tag{4.49}$$

where $\mathscr{A}$ is the Steenrod algebra and MTG is the Madsen–Tillmann spectrum defined in terms of the Thom spectrum by $MTG = \mathrm{Thom}(BG,-V)$, with V a stable bundle of virtual dimension 0 pulled back from the tautological stable bundle over BO by $BG \to BO$. In our case, MTG can be written as

$$MTG = M\mathrm{Spin} \wedge X_G, \tag{4.50}$$

with X_G a Thom spectrum to be determined. For $t-s<8$, this simplifies the Adams spectral sequence above to

$$\mathrm{Ext}^{s,t}_{\mathscr{A}_1}(H^\bullet(X_G),\mathbb{Z}_2) \Rightarrow \Omega^G_{t-s}(\mathrm{pt}), \tag{4.51}$$

by the Anderson-Brown-Peterson theorem. Here $\mathscr{A}_1$ denotes the subalgebra of $\mathscr{A}$ generated by the Steenrod operations Sq^1 and Sq^2.

Calculation of X_G

We will now show that the Thom spectrum X_G when $G = (\mathrm{Spin}\times U_2)/\mathbb{Z}_2$ is given by $X_G = \Sigma^{-5} MSO_3 \wedge MU_1$. We follow the calculation of related examples in Refs. [138, 139], whose method was based on Ref. [65].

The fibration $\mathbb{Z}_2 \longrightarrow G \longrightarrow SO \times SO_3 \times U_1$ gives rise to the following fibration sequence of classifying spaces

$$BG \xrightarrow{(f,f',f'')} BSO \times BSO_3 \times BU_1 \xrightarrow{w_2+w_2'+w_2''} K(\mathbb{Z}_2,2), \tag{4.52}$$

where $w_2 \in H^2(BSO), w_2' \in H^2(BSO_3)$, and $w_2'' \in H^2(BU_1)$ are the second Stiefel–Whitney classes for BSO, BSO_3, and BU_1, respectively. The fibration sequence (4.52) arises as a Puppe sequence, so the composite map

$$w_2 \circ f + w_2' \circ f' + w_2'' \circ f'' : BG \to K(\mathbb{Z}_2,2)$$

is null-homotopic. Moreover, since these classes are valued modulo 2, this is equivalent to saying that the map $w_2 \circ f$ is homotopy equivalent to $w_2' \circ f' + w_2'' \circ f''$. Therefore, the following diagram

$$\begin{tikzcd} BG \arrow[r, "(f',f'')"] \arrow[d, "f"] & BSO_3 \times BU_1 \arrow[d, "w_2'+w_2''"] \\ BSO \arrow[r, "w_2"] & K(\mathbb{Z}_2,2) \end{tikzcd} \tag{4.53}$$

is a homotopy pullback square, which we also use to define the map $V: BG \xrightarrow{f} BSO \hookrightarrow BO$.

Equivalently, BG fits into the homotopy pullback

$$\begin{tikzcd} BG \arrow[r] \arrow[d, "(f,f',f'')"] & B\mathrm{Spin} \arrow[d, "g"] & \\ BSO \times BSO_3 \times BU_1 \arrow[r, "h"] & BSO \arrow[r, "w_2"] & K(\mathbb{Z}_2,2) \end{tikzcd} \tag{4.54}$$

where $w_2 \circ g$ is null-homotopic and h is to be determined. This can be seen by finding a suitable map h, as follows. Since BG fits into the homotopy pullback (4.53), we can think of its element as a triplet of vector bundles $(V,V_3,V_2) \in BSO \times BSO_3 \times BU_1$, such that $w_2(V) = w_2(V_3) + w_2(V_2)$. We take the map h from BG to BSO to be

$$(V,V_3,V_2) \mapsto V + V_3 + V_2 - 5, \tag{4.55}$$

which sends three bundles into a stable SO-bundle of virtual dimension 0. Using the Whitney product formula, the second Stiefel–Whitney class of the virtual bundle $V+V_3+V_2-5$ is given by

$$w_2(V+V_3+V_2-5) = w_2(V) + w_2(V_3) + w_2(V_2) = 0 \tag{4.56}$$

where we obtain the last equality using the pullback square (4.53). Therefore, the stable SO-bundle $V+V_3+V_2-5$ can be lifted to a stable spin bundle, denoted by W, establishing the existence of a homotopy pullback (4.54).

Therefore, the map $-V : BG \to BSO$ is homotopy equivalent to the map $-W+V_3+V_2-5$ from $B\mathrm{Spin} \times BSO_3 \times BU_2$ into BSO, giving rise to the identification of the Thom spectrum $MTG = \mathrm{Thom}(BG; -V)$ with

$$\mathrm{Thom}(B\mathrm{Spin} \times BSO_3 \times BU_1; -W+V_3+V_2-5) = \Sigma^{-5} M\mathrm{Spin} \wedge MSO_3 \wedge MU_1. \tag{4.57}$$

$\mathscr{A}_1$-module structure of $H^\bullet(X_G)$ and Adams spectral sequence

We will now work out the $\mathscr{A}_1$-module structure of the spectrum X_G. Recall that

$$H^\bullet(BSO_3) \cong \mathbb{Z}_2[w_2', w_3'] \quad \text{and} \quad H^\bullet(BU_1) \cong \mathbb{Z}_2[w_2''], \tag{4.58}$$

where w_2', w_3' are the Stiefel–Whitney classes, with w_2'' being the first Chern class modulo 2, which coincides with the second Stiefel–Whitney class. By the Thom isomorphism, we have the identifications

$$H^\bullet(MSO_3) \cong \mathbb{Z}_2[w_2', w_3']\{U\} \quad \text{and} \quad H^\bullet(MU_1) \cong \mathbb{Z}_2[w_2'']\{V\}, \tag{4.59}$$

where the Thom classes U and V are in $H^3(MSO_3)$ and $H^2(MU_1)$ respectively. The Künneth theorem for the cohomology ring of a Thom space implies that

$$\begin{aligned} H^\bullet(\Sigma^{-3} MSO_3 \wedge \Sigma^{-2} MU_1) &\cong \Sigma^{-5} H^\bullet(MSO_3) \otimes H^\bullet(MU_1) \\ &\cong \Sigma^{-5} \mathbb{Z}_2[w_2', w_3', w_2'']\{UV\}. \end{aligned} \tag{4.60}$$

Using the relations between Thom classes, the Steenrod squares, and the Stiefel–Whitney classes, we find that the $\mathscr{A}_1$-module structure of $H^\bullet(X_G)$ up to degree 5 can be expressed as the cell diagram shown in Fig. 4.1, with the corresponding Adams chart for $\mathrm{Ext}^{s,t}_{\mathscr{A}_1}(H^\bullet(X_G), \mathbb{Z}_2)$ shown in Fig. 4.2. In the Adams chart, each dot corresponds to a $\mathbb{Z}_2$ generator. A line joining two generators α_s and α_{s+1} of the same $t-s$ but with $\Delta s = 1$ means that the generator α_{s+1} is given by $\alpha_{s+1} = h_0 \alpha_s$, where h_0 is the generator of $\mathrm{Ext}^{1,1}_{\mathscr{A}_1}(\mathbb{Z}_2, \mathbb{Z}_2)$. In the range of our interest ($t-s<6$), the entries are too sparse and all the differentials are trivial, apart from a possible non-trivial differential d_r from the entry $(s, t-s) = (0,5)$ to the entries $(s, t-s) = (r, 4)$. However, using the fact that d_r commutes with h_0, it can be shown that these differentials are trivial, too. Therefore, the Adams spectral sequence collapses already at the E_2 page for $t-s<6$.

Finally, the rule for extracting the bordism groups can be roughly summarised as follows: an h_0-tower containing m dots gives a factor of $\mathbb{Z}_2^m$, and an infinite h_0-tower gives a factor of $\mathbb{Z}$. With this rule, the bordism groups of degree lower than six can be read off from the chart in Fig. 4.2 to be

$$\Omega_0^G = \mathbb{Z}, \quad \Omega_1^G = 0, \quad \Omega_2^G = \mathbb{Z}, \quad \Omega_3^G = 0, \quad \Omega_4^G = \mathbb{Z}^3, \tag{4.61}$$

and, crucially for us,

$$\Omega_5^G = \mathbb{Z}_2. \tag{4.62}$$

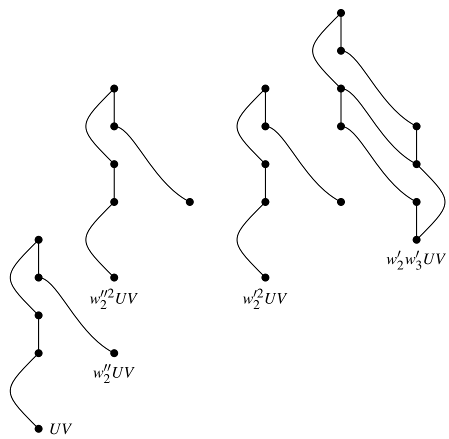


Fig. 4.1 The $\mathscr{A}_1$-module structure for $\mathbb{Z}_2[w_2', w_3', w_2'']\{UV\}$, up to degree ten.

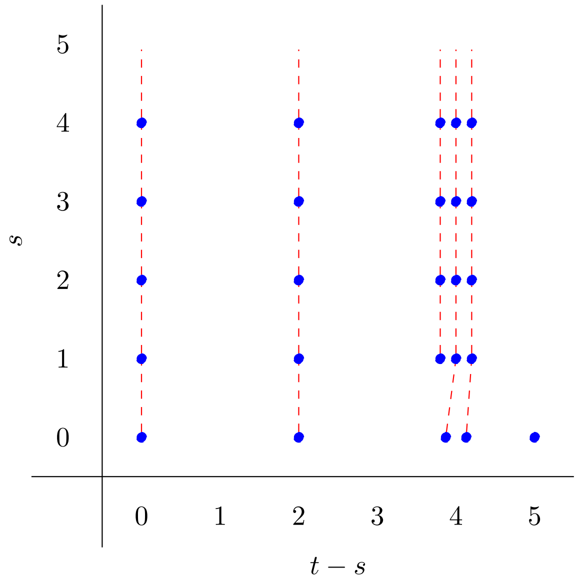


Fig. 4.2 The E_2 page of the Adams spectral sequence (4.51), from which one can read off the bordism groups $\Omega_{d\leq 5}^{\frac{\mathrm{Spin}\times U(2)}{\mathbb{Z}_2}}(\mathrm{pt})$.

Chapter 5

If the weak were strong and the strong were weak

5.1 Introduction

There are two energy scales in the Standard Model. In combination with a handful of dimensionless couplings and some simple, yet intricate, dynamics, these give birth to the wondrous diversity of scales, spanning many orders of magnitude, that emerge in nuclear physics, atomic physics and condensed matter physics.

The two scales are the Higgs expectation value v, and the scale $\Lambda_{\rm strong}$, usually called Λ_{QCD}, at which the strong force lives up to its name. They take values

$$v \approx 250 \text{ GeV} \quad \text{and} \quad \Lambda_{\rm strong} \approx 250 \text{ MeV}$$

There is also a third, counterfactual scale in the Standard Model, which doesn't play any role in our world. This is the infra-red scale at which the weak force would become strong if other effects didn't first intervene. It is a rather academic exercise to specify this scale but if we were to ignore electroweak symmetry breaking and run the $SU(2)$ beta function down, assuming a single massless generation, it is given by[1]

$$\Lambda_{\rm weak} \approx 3 \times 10^{-3} \text{ eV}$$

[1]Obviously, if the Higgs mechanism turns off then all three generations become massless, together with any further generations that lie beyond our current reach. This slows the running of the beta functions. With three massless generations, and the dimensionless couplings fixed to their values at 80 GeV, we have $\Lambda_{\rm strong} \approx 40$ MeV and $\Lambda_{\rm weak} \approx 2 \times 10^{-15}$ eV.

The purpose of this Chapter is to understand the phase structure and possible quantum phase transitions of the theory as the three scales, v, Λ_{strong} and Λ_{weak}, vary.

The question of what happens if the Higgs mechanism is turned off, and the strong force dominates, has been well studied. This situation occurs in the regime,

$$v \ll \Lambda_{\text{weak}} \ll \Lambda_{\text{strong}} \tag{5.1}$$

It was pointed out long ago that the chiral condensate of QCD transforms under electroweak symmetry. This means that the pions act as a substitute for the Higgs boson, giving a mass of order f_π to the W- and Z-bosons, an observation that motivated the subsequent development of technicolor models [145, 129]. The phenomenology of this regime was described in [119] and, in more detail, by Quigg and Shrock [115].

In this Chapter, we are interested in what happens as we vary the couplings, to interpolate from (5.1) to the regime

$$v \ll \Lambda_{\text{strong}} \ll \Lambda_{\text{weak}} \tag{5.2}$$

The pattern of chiral symmetry breaking in this regime was mentioned briefly in [115] and explored further in [127] and will be reviewed in some detail below. First the weak force with $SU(2)$ gauge group confines, with a particular pattern of chiral symmetry breaking. This condensate breaks the strong gauge group, $SU(3) \to SU(2)$, which itself subsequently confines, breaking chiral symmetry yet further. The resulting physics shares some similarities with the early work of Abbott and Farhi [2, 1, 45], exploring the possibility that the $SU(2)$ weak force is actually confining, rather than spontaneously broken.

Our goal is to understand the spectrum of massless fermions and Goldstone bosons of the Standard Model, and a closely related chiral gauge theory, in the two regimes (5.1) and (5.2). Our primary motivation for undertaking this work is simple: we thought it was a fun question. More generally, this Chapter sits within a larger programme aimed at understanding the dynamics of chiral gauge theories. Early work on this topic is summarised in the review article [29]. Since then, a number of articles have studied the dynamics and phase structure of large classes of chiral gauge theories [17, 18, 20, 126, 125, 124, 39, 118, 36, 10]. A number of proposals have been made for lattice regularisations of chiral gauge theories [58, 111, 140, 142, 141].

Summary of Results

As we vary the coupling constants of the theory, to interpolate from regime (5.1) to regime (5.2), the physics depends strongly on the number of generations, which we denote as N_f.

Perhaps the biggest surprise arises when we have just a single generation, $N_f = 1$. In this case, the massless fermion spectrum is comprised of a single, left-handed neutrino in the regime (5.1), a fact that is familiar from our world[2]. In contrast, in regime (5.2) the massless fermion spectrum contains a single colour component of the right-handed down quark. Furthermore, the unbroken symmetries are identical in the two regimes. In particular, the massless down quark that emerges when the weak force dominates is neutral under electromagnetism, a fact that can be understood by noting that the $U(1)_Q$ electromagnetic subgroup twists within the $SU(2) \times SU(3)$ gauge group as we vary the ratio $\Lambda_{\text{strong}}/\Lambda_{\text{weak}}$. These facts suggest that the two regimes sit in the same phase, and the neutrino morphs smoothly into the down quark[3].

With $N_f \geq 2$ generations, there is a similar story: in regime (5.1), one finds massless left-handed leptons, while in regime (5.2) there are massless, neutral right-handed quarks. This time, however, the symmetry breaking structure in the two regimes differs, ensuring that there is a phase transition between the two. The exact structure of the symmetry breaking depends on the fields and couplings that are present, and we consider a number of variations of the Standard Model, both with and without hypercharge and Yukawa couplings.

In Section 5.3, we introduce a novel chiral gauge theory, based on the gauge group

$$G = U(1)_Y \times Sp(r) \times SU(N)$$

When coupled to specific fermion and scalar fields, this can be thought of as a two parameter extension of the Standard Model. We again explore the phase structure as the relative couplings of the two non-Abelian gauge groups are varied and find a pattern analogous to that of the Standard Model.

Finally, we include two extended Appendices which describe a number of features of *vacuum alignment*, the dynamical process that determines the vacuum structure of theories with chiral symmetry breaking and multiple gauge groups [107, 112]. This, it turns out, is important in order to understand the structure of chiral symmetry breaking in regime (5.2).

[2]This statement holds in the absence of a right-handed neutrino. However, as we describe in the bulk of the paper, the essential physics remains unchanged by the addition of a right-handed neutrino.

[3]This conclusion is based only on the breaking pattern of the continuous global symmetries. It may well be that more subtle symmetries, such as the higher form symmetries described in [71, 72] give a finer classification of the phases. These ideas were applied to bi-fundamental, but non-chiral, gauge theories in [132, 92]. We hope to return to this question in the future

Note Added

As we started to write the paper [96] that this Chapter is based on, we became aware of the [35] by Berger, Long and Turner which asks essentially the same questions, motivated by early universe baryogenesis. Our results largely agree where there is overlap.

5.2 Variations on the Standard Model

We start by discussing a simple chiral gauge theory with gauge group

$$G = SU(2) \times SU(3)$$

We stick to convention and refer to $SU(2)$ as the *weak force* and $SU(3)$ as the *strong force*. However, we will be interested both in situations where these names are appropriate, and also in situations where the weak is strong and the strong is weak. We will encounter a large number of different groups below, both gauge and global; where there is a possibility of confusion, we will refer to the gauge groups as $SU(2)_{\text{weak}}$ and $SU(3)_{\text{strong}}$.

We couple four Weyl fermions to G: these are the left-handed quarks q_L, left-handed leptons l_L, and right-handed quarks $q_R = (d_R, u_R)$. At this stage, we include neither the right-handed electron, nor right-handed neutrino since both are singlets under G. These will be introduced in Sections 5.2.2 and 5.3 respectively.

We start by considering just a single generation of fermions; we then turn to multiple generations in Section 5.2.1. The non-anomalous global symmetry is

$$F = SU(2)_R \times U(1)_V \tag{5.3}$$

where here, and elsewhere, we ignore discrete factors. Here $U(1)_V$ is the familiar $B - L$ symmetry of the Standard Model. The transformations of the various fermions under the gauge and global symmetries are summarised as

	G		F	
	$SU(2)$	$SU(3)$	$SU(2)_R$	$U(1)_V$
q_L	**2**	**3**	**1**	$+1$
l_L	**2**	**1**	**1**	-3
q_R	**1**	**3**	**2**	$+1$

Both baryon and lepton number are anomalous, meaning that these quantum numbers are not individually conserved in the quantum theory. We will see a dramatic illustration of this fact as we adiabatically vary the coupling constants.

There are 't Hooft anomalies for both $U(1)_V^3$ and $SU(2)_R$ [130]. (The latter is a $\mathbb{Z}_2$ anomaly [149].) This ensures that either these symmetries are spontaneously broken in the infra-red, or there are massless fermions. We will find that the anomalies are saturated in the infra-red by massless fermions, albeit with different microscopic representatives playing the role in different regimes.

Both gauge group factors are asymptotically free. This means that the gauge couplings become large in the infra-red where, left to the their own devices, they result in two dynamically generated scales. We refer to the scales for $SU(2)$ and $SU(3)$ as Λ_{weak} and Λ_{strong} respectively. We have little understanding of the dynamics when $\Lambda_{\text{strong}} \approx \Lambda_{\text{weak}}$. However, in the two limits $\Lambda_{\text{strong}} \gg \Lambda_{\text{weak}}$ and $\Lambda_{\text{weak}} \gg \Lambda_{\text{strong}}$, some simple intuition about confinement and chiral symmetry breaking is enough to understand what happens. We will then try to match the two regimes.

$\Lambda_{\text{strong}} \gg \Lambda_{\text{weak}}$

The limit where the strong force dominates is well studied [145, 129, 119, 115]. The strong dynamics results in a quark condensate which takes the form

$$\langle q_{L\,i}^\dagger q_{R\,j} \rangle \sim \Lambda_{\text{strong}}^3 \delta_{ij} \quad i,j = 1,2$$

If we ignore the weak force, then this is a condensate in two-flavour QCD. The $SU(2)_L \times SU(2)_R$ flavour symmetry is spontaneously broken to the diagonal subgroup $SU(2)_{\text{diag}}$, resulting in three Goldstone bosons.

Turning on the weak force, we identify $SU(2)_L$ with the $SU(2)_{\text{weak}} \subset G$ gauge group. The condensate acts as a Higgs field, completely breaking the $SU(2)$ gauge group. All three of the would-be Goldstone bosons are eaten, giving mass to the W-bosons. This mass is of order f_π, the pion decay constant.

A global symmetry survives, formed from a diagonal combination $SU(2)_{\text{diag}} \subset SU(2)_{\text{weak}} \times SU(2)_R$, and the infra-red global symmetry takes the same form as the ultra-violet symmetry (5.3),

$$F_{\text{strong}} = SU(2)_{\text{diag}} \times U(1)_V$$

The quark bound states are all massive and form representations of F_{strong}. Meanwhile, the leptons l_L remain massless, transforming as

	$SU(2)_{\text{diag}}$	$U(1)_V$
l_L	$\mathbf{2}$	-3

These massless leptons saturate the 't Hooft anomalies of the global symmetry. In the UV, the fields q_L, l_L, and q_R are in the representations $3 \cdot \mathbf{2}_{+1}, \mathbf{2}_{-1}$, and $3 \cdot \mathbf{2}_{+1}$ of $SU(2)_{\text{diag}} \times U(1)_V$, respectively. In particular, the left-handed and the right-handed quarks are in the same representation so their contributions to the anomalies cancel each other out. Therefore, the only fields that contribute nontrivially to the anomalies in the UV are the leptons and we see that the anomalies must match.

$\Lambda_{\text{weak}} \gg \Lambda_{\text{strong}}$

When the weak force dominates, we expect a condensate of left-handed fermions to form. There are four such left-handed fermions, each a doublet of $SU(2)_{\text{weak}}$. We write these as

$$\psi_m = (q_{L,1}, q_{L,2}, q_{L,3}, l_L) \quad m = 1,2,3,4 \tag{5.4}$$

The condensate takes the general form

$$\langle \varepsilon_{\alpha\beta} \psi_m^\alpha \psi_n^\beta \rangle \sim \Lambda_{\text{weak}}^3 J_{mn} \tag{5.5}$$

where $\alpha, \beta = 1,2$ are $SU(2)_{\text{weak}}$ indices, and J_{mn} is a 4×4 anti-symmetric matrix.

If we ignore the strong force, then the $SU(2)$ gauge theory enjoys an $SU(4)$ global symmetry, under which ψ transforms in the $\mathbf{4}$. The condensate breaks[4] this to $Sp(2)$, resulting in $\dim SU(4) - \dim Sp(2) = 15 - 10 = 5$ Goldstone bosons.

Now we turn the strong force back on, and see the effect of the condensate (5.5). This was discussed previously in [127]. It turns out that the choice of J_{mn} does not affect the physics in this case. (This statement no longer holds when we discuss multiple generations; we will discuss this in more detail in Section 5.2.1 and in much more detail in the Appendices.) For any choice of J_{mn}, the condensate (5.5) includes a quark-bilinear of the form

$$\langle q_{La} \cdot q_{Lb} \rangle \sim \Lambda_{\text{weak}}^3 \varepsilon_{abc} \sigma^c \tag{5.6}$$

where $a, b, c = 1,2,3$ are $SU(3)_{\text{strong}}$ indices and we've now suppressed the $SU(2)_{\text{weak}}$ index structure which remains as in (5.5). For any choice of σ^c, the condensate acts as Higgs field

[4] We use the convention $Sp(1) \equiv SU(2)$.

for the strong force, breaking

$$SU(3)_{\text{strong}} \to SU(2)_{\text{strong}}$$

All 5 of the would-be Goldstone bosons described above are eaten by the now-massive gluons.

Without loss of generality, we choose $\sigma^c = (0,0,1)$, so that the condensate (5.6) involves only the q_a with $a = 1,2$ coloured quarks. We denote the remaining quark as $\hat{q}_L = q_{L3}$. It forms a condensate with the lepton

$$\langle \hat{q}_L \cdot l_L \rangle \sim \Lambda^3_{\text{weak}} \tag{5.7}$$

where the anti-symmetry of (5.5) is assured because the condensate is symmetrised over both spinor and weak indices, leaving the Grassmann nature of the fermions to do its job.

Both condensates (5.6) and (5.7) would appear to break the $U(1)_V$ symmetry of the original theory; they have charges $+2$ and -2 respectively. However, it is straightforward to find a global $U(1)$ symmetry that survives by locking $U(1)_V$ with a suitable $SU(3)_{\text{strong}}$ gauge transformation. If we denote the generator of $U(1)_V$ as Q_V, then the generator of the surviving global symmetry is defined as

$$Q_{\hat{V}} = Q_V + \text{diag}(-1,-1,+2)_{\text{strong}} \tag{5.8}$$

At this point, the left-handed quarks q_L and leptons l_L have become gapped. We're left just with the right-handed quarks q_R, which now transform under the unbroken $SU(2)_{\text{strong}}$ gauge group. Under the combined symmetry breaking

$$SU(3)_{\text{strong}} \times SU(2)_R \times U(1)_V \to SU(2)_{\text{strong}} \times SU(2)_R \times U(1)_{\hat{V}}$$

the right-handed quarks decompose as

$$q_R : (\mathbf{3},\mathbf{2})_{+1} \to (\mathbf{2},\mathbf{2})_0 \oplus (\mathbf{1},\mathbf{2})_{+3} \tag{5.9}$$

Now we let $SU(2)_{\text{strong}}$ flow to the infra-red where it too confines. The $(\mathbf{2},\mathbf{2})_0$ quarks above will form a condensate

$$\langle q_{Rai} q_{Rbj} \rangle \sim \Lambda^3_{\text{strong}} \epsilon_{abc} \hat{\sigma}^c \epsilon_{ij}$$

Here $\hat{\sigma}^c$ specifies the direction in $SU(3)$ colour space determined by the weak condensate (5.6). Importantly this new condensate breaks neither $SU(2)_R$ nor $U(1)_{\hat{V}}$. We're left with the infra-red symmetry which, once again, is unchanged in form from the ultra-violet (5.3),

$$F_{\text{weak}} = SU(2)_R \times U(1)_{\hat{V}}$$

There is a single massless fermion transforming under F_{weak}; this is the right-handed quark $\hat{q}_R$ that transforms in the $(\mathbf{1},\mathbf{2})_{+3}$ representation in the decomposition (5.9), or

	$SU(2)_R$	$U(1)_{\hat{V}}$
$\hat{q}_R$	$\mathbf{2}$	$+3$

Once again this saturates the 't Hooft anomalies. There is an odd number of doublets coupling to $SU(2)_R$ both in the IR ($\hat{q}_R$) and the UV (3 doublets of q_R) so Witten's $\mathbb{Z}_2$ anomaly matches. The perturbative anomalies in the IR are given by

$$\begin{aligned} [U(1)_{\hat{V}}]^3 : &\quad \mathscr{A}_{\hat{V}}^{\text{IR}} = -2(+3)^3 = -54, \\ [SU(2)_R]^2 \times U(1)_{\hat{V}} : &\quad \mathscr{A}_{\text{mixed}}^{\text{IR}} = -3, \end{aligned}$$

Since the UV fields transform under $SU(2)_R \times U(1)_{\hat{V}}$ as

$$\begin{aligned} q_L : &\quad 2(2\cdot\mathbf{1}_0 \oplus \mathbf{1}_{+3}) \\ l_L : &\quad 2\cdot\mathbf{1}_{-3} \\ q_R : &\quad 2\cdot\mathbf{2}_0 \oplus \mathbf{2}_{+3}, \end{aligned}$$

the UV perterbative anomalies are

$$\begin{aligned} [U(1)_{\hat{V}}]^3 : &\quad \mathscr{A}_{\hat{V}}^{\text{UV}} = 2(+3)^3 + 2(-3)^3 - 2(+3)^3 = -54, \\ [SU(2)_R]^2 \times U(1)_{\hat{V}} : &\quad \mathscr{A}_{\text{mixed}}^{\text{UV}} = -3, \end{aligned}$$

which clearly match the IR anomalies. Note, in particular, that the $U(1)$ charge $+3$ (as opposed to -3 seen when $\Lambda_{\text{strong}} \gg \Lambda_{\text{weak}}$) is compensated by the fact that we have a massless right-handed fermion, rather than left-handed. We see that the massless lepton l_L in the regime $\Lambda_{\text{strong}} \gg \Lambda_{\text{weak}}$ has transmuted into a massless right-handed quark in the regime $\Lambda_{\text{strong}} \ll \Lambda_{\text{weak}}$. This provides a striking example of the lack of individual baryon and lepton number conservation in the theory. However, this transmutation occurs without violating the $B-L$ symmetry, a feat which is made possible by the twisting (5.8) which means that the infra-red gauge-invariant down quark $\hat{q}_R$ carries a different $B-L$ quantum number from its gauge-dependent microscopic parent.

No Phase Transition?

For a single generation, the global symmetry group of the theory remains unbroken both when $\Lambda_{\text{strong}} \gg \Lambda_{\text{weak}}$ and when $\Lambda_{\text{weak}} \gg \Lambda_{\text{strong}}$. While it is true that the UV symmetry group is locked with different gauge symmetries in each case, there is no gauge invariant way to distinguish them. This suggests that there is no phase transition as we vary the ratio $\Lambda_{\text{strong}}/\Lambda_{\text{weak}}$, and the massless lepton transforms smoothly into the massless quark.

This picture resonates with an old story. Recall that QCD with two flavours is one of the few cases where the 't Hooft anomalies can be saturated by massless baryons [130]. There is a complementary way of viewing this [116, 55, 31], in which a $\langle q_L q_L \rangle$ condensate forms, Higgses $SU(3)_{\text{strong}} \to SU(2)_{\text{strong}}$, and leaving behind a massless quark. The fact that there is no phase transition between the Higgs and confining phases means that the massless baryon can be viewed as a continuously connected to this massless quark.

Ultimately, the physics described in the above paragraph is thought not to occur for QCD. However, it does occur in the regime $\Lambda_{\text{weak}} \gg \Lambda_{\text{strong}}$, where the $\langle q_L q_L \rangle$ quark condensate (5.6) is encouraged by the $SU(2)_{\text{weak}}$ force rather than $SU(3)_{\text{strong}}$. This suggests that as we head into the regime $\Lambda_{\text{strong}} \approx \Lambda_{\text{weak}}$, it may be appropriate to better think of the massless quark $\hat{q}_R$ as a massless baryon. Indeed, the baryon $B \sim q_L \cdot q_L \cdot q_R$ has the same quantum numbers as the massless fermion. This means that, starting from the regime $\Lambda_{\text{strong}} \gg \Lambda_{\text{weak}}$, the massless lepton can mix with the baryon, and ultimately emerge in the other regime $\Lambda_{\text{weak}} \gg \Lambda_{\text{strong}}$ as a massless quark.

5.2.1 Multiple Generations

We now repeat the analysis of the previous section, but with N_f generations of fermions. The gauge group remains $G = SU(2) \times SU(3)$, but the global symmetry group is now (again omitting discrete factors)

$$F = SU(N_f)_{L'} \times SU(N_f)_L \times SU(2N_f)_R \times U(1)_V \tag{5.10}$$

The quantum numbers of the fermions are

	G		F			
	$SU(2)$	$SU(3)$	$SU(N_f)_{L'}$	$SU(N_f)_L$	$SU(2N_f)_R$	$U(1)_V$
q_L	**2**	**3**	**1**	$\mathbf{N}_f$	**1**	$+1$
l_L	**2**	**1**	$\mathbf{N}_f$	**1**	**1**	-3
q_R	**1**	**3**	**1**	**1**	$\mathbf{2N}_f$	$+1$

There are now numerous 't Hooft anomalies for F. This time we will see that some of these symmetries are broken, with the 't Hooft anomalies in the surviving symmetries saturated by massless fermions.

Both $SU(2)_{\rm weak}$ and $SU(3)_{\rm strong}$ remain asymptotically free for $N_f \leq 5$. (This bound comes from the weak force; the strong force remains asymptotically free up to $N_f = 8$ generations.)

For suitably large N_f, the individual gauge theories sit in a conformal window while, for suitably low N_f, they undergo chiral symmetry breaking. The lower end of the conformal window is not well understood, but it is thought that it sits around 8 Dirac fermions for $SU(3)$ [19] and around 6 Dirac fermions for $SU(2)$ [93, 86, 9].

We analyse the theory in the regime in which both gauge groups undergo chiral symmetry breaking. This means that our analysis is restricted to $N_f = 2$ and, possibly, $N_f = 3$ which is a marginal case for $SU(2)_{\rm weak}$.

$\Lambda_{\rm strong} \gg \Lambda_{\rm weak}$

Once again, the limit where the strong force dominates is well understood. The usual QCD condensate forms,

$$\langle q^\dagger_{Li} q_{Rj}\rangle \sim \Lambda^3_{\rm strong}\Sigma_{ij} \quad i,j = 1,2N_f \tag{5.11}$$

with a moduli space parameterised by Σ_{ij}. If we ignore the weak force, then this condensate breaks the $SU(2N_f)_L \times SU(2N_f)_R \to SU(2N_f)_{\rm diag}$ flavour symmetry in the usual fashion, resulting in $4N_f^2 - 1$ Goldstone bosons.

We now turn on the $SU(2)_{\rm weak}$ gauge coupling. Often in such situations, different points on the moduli space give rise to different symmetry breaking patterns and one must work harder to determine which of the original possible vacua becomes the true vacuum [107, 112]. We will see a number of examples of this shortly. However, in the present situation this issue does not arise. Instead, each point in the moduli space breaks the $SU(2)_{\rm weak}$ gauge symmetry completely.

The condensate (5.11) breaks the global symmetry group (5.10) to

$$F_{\rm strong} = SU(N_f)_{L'} \times SU(2)_{\rm diag} \times SU(N_f)_{\rm diag} \times U(1)_V \tag{5.12}$$

where $SU(N_f)_{\rm diag} \subset SU(N_f)_L \times SU(2N_f)_R$ and, as in the previous section, $SU(2)_{\rm diag} \subset SU(2)_{\rm weak} \times SU(2N_f)_R$. This results in a moduli space of Goldstone modes,

$$\mathscr{M}_{\rm strong} = \frac{SU(N_f)_L \times SU(2N_f)_R}{SU(2) \times SU(N_f)_{\rm diag}} \tag{5.13}$$

There are $\dim \mathscr{M}_{\text{strong}} = 4(N_f^2 - 1)$ Goldstone bosons. Note that this is three fewer than the counting before we turned on $SU(2)_{\text{weak}}$; these three were sacrificed on the altar of the Higgs mechanism.

As in the previous section, the leptons remain massless. They transform under the surviving symmetry group F_{strong} as

	$SU(N_f)_{L'}$	$SU(2)_{\text{diag}}$	$SU(N_f)_{\text{diag}}$	$U(1)_V$
l_L	$\mathbf{N}_f$	$\mathbf{2}$	$\mathbf{1}$	-3

It is simple to check that these massless leptons saturate the 't Hooft anomalies of the surviving global symmetry F_{strong}. In the UV, the fields transform under F_{strong} as

$$\begin{aligned} q_L &: \quad 3(\mathbf{1},\mathbf{2},\mathbf{N}_f)_{+1} \\ l_L &: \quad (\mathbf{N}_f,\mathbf{2},\mathbf{1})_{-3} \\ q_R &: \quad 3(\mathbf{1},\mathbf{2},\mathbf{N}_f)_{+1}. \end{aligned}$$

Note that the left-handed and the right-handed quarks have the same representation. Therefore, their contributions to the anomaly cancel. We are left with the left-handed leptons and the anomalies trivially match.

<u>$\Lambda_{\text{weak}} \gg \Lambda_{\text{strong}}$</u>

When the weak force dominates, we again expect a condensate of left-handed fermions to form. We write the collection of $SU(2)_{\text{weak}}$ doublets as ψ_{mi}, with $m = 1,2,3,4$ labelling the quarks and leptons as in (5.4), and $i = 1,\ldots,N_f$, the flavour index. The condensate takes the general form

$$\langle \psi_{mi} \cdot \psi_{nj} \rangle \sim \Lambda^3_{\text{weak}} \bar{J}_{mi,nj} \tag{5.14}$$

where we have suppressed the $SU(2)_{\text{weak}}$ indices and $\bar{J}_{mi,nj}$ is a $4N_f \times 4N_f$ anti-symmetric matrix.

If we ignore the strong force, then the $SU(2)_{\text{weak}}$ gauge theory has an $SU(4N_f)$ global symmetry which is broken by the condensate to $Sp(2N_f)$, resulting in an intermediate moduli space

$$\mathscr{M}_0 = \frac{SU(4N_f)}{Sp(2N_f)}$$

This is parameterised by $8N_f^2 - 2N_f - 1$ Goldstone modes. The question that we want to answer is: what becomes of these modes when we turn on $SU(3)_{\text{strong}}$?

This time there is a slightly involved calculation to do. Different choices of $\bar{J}_{mi,nj}$ give different symmetry breaking patterns for $SU(3)_{\rm strong}$ and a different mass spectrum for the resulting gauge bosons. For example, if the condensate forms in a flavour-diagonal fashion, with $\bar{J}_{mi,nj} = J_{mn}\delta_{ij}$, then it breaks the strong gauge group to

$$SU(3)_{\rm strong} \to SU(2)_{\rm strong} \tag{5.15}$$

which is the same symmetry breaking pattern that we saw in the $N_f = 1$ case. Such a condensate also breaks the global symmetry (5.10) to

$$\tilde{F} = SO(N_f) \times SU(2N_f)_R \times U(1)_{\hat{V}} \tag{5.16}$$

where $SO(N_f) \subset SU(N_f)_{L'} \times SU(N_f)_L$ and $U(1)_{\hat{V}} \subset U(1)_V \times SU(3)_{\rm strong}$ as in (5.8).

Alternatively, a condensate $\bar{J}_{mi,nj}$ which is off-diagonal in the flavour basis will break the $SU(3)_{\rm strong}$ gauge group completely and further break $\tilde{F}$ [127]. (We provide a number of specific examples in Appendix 5.B.2.) The question that we must ask is: what is the preferred choice of breaking?

The tools to answer this question were introduced long ago by Peskin [107] and Preskill [112]. They showed how introducing a second gauge group induces a potential on the moduli space M_0. The true ground state of the system is determined by the minimum of this potential. We review this mechanism in some detail in Appendix 5.A. Furthermore in Appendix 5.B.1 we show that the flavour-diagonal condensate, with symmetry breaking (5.15) and (5.16) is a local, stable minimum of the potential. Although we have been unable to prove, in generality, that there are not other local minima, we argue that generically one expects all other condensates to exhibit tachyonic modes and, in Appendix 5.B.2, we show this explicitly for a number of putative vacua with different symmetry breaking patterns. The upshot is that the flavour-diagonal symmetry breaking pattern (5.15) and (5.16) appears to be dynamically preferred.

With the global symmetry F defined in (5.10) broken to $\tilde{F}$ in (5.16), the moduli space of ground states arising from the weak dynamics, is

$$\mathscr{M}'_{\rm weak} = \frac{SU(N_f)_{L'} \times SU(N_f)_L}{SO(N_f)} \tag{5.17}$$

Correspondingly, there are $\dim \mathscr{M}'_{\rm weak} = \frac{3}{2}N_f^2 + \frac{1}{2}N_f - 2$ Goldstone bosons. Note that, for $N_f > 1$, the difference between $\dim \mathscr{M}_0$ and $\dim \mathscr{M}'_{\rm weak}$ is greater than the 5 Goldstone bosons eaten by the Higgs mechanism (5.15). This reflects the existence of a potential on $\mathscr{M}_0$ induced by gauge symmetry $SU(3)_{\text{strong}}$. We compute the masses of the resulting pseudo-Goldstone bosons in Appendix 5.B.1.

We're still left with the dynamics of the unbroken $SU(2)_{\text{strong}}$ gauge symmetry to contend with. Under the residual symmetry $SU(2)_{\text{strong}} \times SU(2N_f)_R \times U(1)_{\hat{V}}$, the remaining quarks q_R decompose as

$$q_R : (\mathbf{3}, \mathbf{2N}_f)_{+1} \to (\mathbf{2}, \mathbf{2N}_f)_0 \oplus (\mathbf{1}, \mathbf{2N}_f)_{+3} \tag{5.18}$$

When $SU(2)_{\text{strong}}$ confines, the quarks in the $(\mathbf{2}, \mathbf{2N}_f)_0$ representation condense, further breaking the $SU(2N_f)_R$ global symmetry to $Sp(N_f)_R$. The final surviving global symmetry is

$$F_{\text{weak}} = SO(N_f) \times Sp(N_f)_R \times U(1)_{\hat{V}} \tag{5.19}$$

and Goldstone bosons parameterise the space

$$\mathscr{M}_{\text{weak}} = \frac{SU(N_f)_{L'} \times SU(N_f)_L}{SO(N_f)} \times \frac{SU(2N_f)}{Sp(N_f)} \tag{5.20}$$

As in the previous section, the massless fermion is now identified with $\hat{q}_R$, corresponding to the $(\mathbf{1}, \mathbf{2N}_f)_{+3}$ representation in (5.18). This transformation properties of this fermion are

	$SO(N_f)$	$Sp(N_f)_R$	$U(1)_{\hat{V}}$
$\hat{q}_R$	$\mathbf{1}$	$\mathbf{2N}_f$	$+3$

Again, the fermion $\hat{q}_R$ saturates the surviving 't Hooft anomalies. More explicitly, the anomalies in the IR consist of a Witten anomaly in $Sp(N_f)$ and the perturbative anomalies

$$\begin{aligned} [U(1)_{\hat{V}}]^3 : &\quad \mathscr{A}_{\hat{V}}^{\text{IR}} = -2N_f(+3)^3 = -54N_f, \\ [Sp(N_f)_R]^2 \times U(1)_{\hat{V}} : &\quad \mathscr{A}_{\text{mixed}}^{\text{IR}} = -3, \end{aligned} \tag{5.21}$$

and no anomaly involving $SO(N_f)$. On the other hand, since the UV fields transform under F_{weak} in the following representations

$$\begin{aligned} q_L : &\quad 4(\mathbf{N}_f, \mathbf{1})_0 \oplus 2(\mathbf{N}_f, \mathbf{1})_{+3} \\ l_L : &\quad 2(\mathbf{N}_f, \mathbf{1})_{-3} \\ q_R : &\quad 2(\mathbf{1}, \mathbf{2N}_f)_0 \oplus (\mathbf{1}, \mathbf{2N}_f)_{+3} \end{aligned}$$

There is no pure anomaly in $SO(N_f)$ because the vector representation is a real representation and can always be given a mass term. The mixed $[SO(N_f)]^2 \times U(1)_{\hat{V}}$ anomaly contributions from q_L and l_L cancel each other. There is a Witten anomaly in $Sp(N_f)$ because there are 3 multiplets of q_R in the fundamental representation of $Sp(N_f)$. The remaining perturbative anomalies are given by

$$\begin{aligned} [U(1)_{\hat{V}}]^3: &\quad \mathscr{A}_{\hat{V}}^{\mathrm{UV}} = 2 \cdot N_f \cdot (-3)^3 = -54 N_f, \\ [Sp(N_f)_R]^2 \times U(1)_{\hat{V}}: &\quad \mathscr{A}_{\mathrm{mixed}}^{\mathrm{UV}} = -3. \end{aligned} \tag{5.22}$$

Note that in the UV, the nontrivial contribution to $\mathscr{A}_{\hat{V}}^{\mathrm{UV}}$ comes from the left-handed leptons while the contribution to $\mathscr{A}_{\mathrm{mixed}}^{\mathrm{UV}}$ comes from the right-handed quarks.

In contrast to the case with $N_f = 1$, the symmetry breaking pattern (5.12) and (5.19) differs in the two regimes, meaning that there is certainly a quantum phase transition as we vary the relative strengths of $\Lambda_{\mathrm{strong}}$ and Λ_{weak}. It is natural to ask the order of this phase transition.

Sadly, the symmetry breaking structure gives little guidance. Note that neither of the surviving symmetry groups, F_{strong} and F_{weak}, is a subgroup of the other, reflecting the fact that the order parameters associated to the two phases are different. Most phase transitions in Nature that exhibit this property are first order; indeed, this “sub-group criterion" is sometimes stated to be a clear indication of a first order phase transition. However, there is nothing that guarantees that this has to be the case. The two phases could be reached by two second order phase transitions, with an intermediate phase in between. This intermediate phase must have a global symmetry group that contains both F_{weak} and F_{strong} as subgroups, for example the UV global symmetry F.

It is also possible that the transition proceeds through a single, continuous phase transition. In Landau theory, this requires tuning to a multi-critical point. However, more exotic phase transitions, in which a gauge symmetry emerges and no fine tuning is needed, are also possible [123]. Needless to say, it would be interesting to better understand the nature of the transition.

5.2.2 Adding Hypercharge and Yukawa Couplings

We now extend our study by including $U(1)_Y$ hypercharge and Yukawa couplings. The gauge group is

$$G = U(1)_Y \times SU(2) \times SU(3)$$

To ensure cancellation of anomalies, we must now also include a right-handed electron e_R in our theory. We further include a single Higgs field, ϕ. We will omit the right-handed neutrino for now, but revisit this in Section 5.3.

We include N_f generations of fermions, coupled to the Higgs through the Yukawa couplings

$$\mathscr{L}_{\text{Yuk}} = \lambda_d\, q_{Li}^\dagger \phi\, d_{Ri} + \lambda_u (q_{Li}^\dagger \cdot \phi^\dagger)\, u_{Ri} + \lambda_e\, l_{Li}^\dagger \phi\, e_{Ri} \tag{5.23}$$

Here the flavour index $i = 1, \ldots, N_f$ is summed over so that, in contrast to the Standard Model, there is an independent $SU(N_f)$ flavour symmetry for quarks and leptons, as well as the $B-L$ symmetry that we denote as $U(1)_V$

$$F = SU(N_f)_q \times SU(N_f)_l \times U(1)_V \tag{5.24}$$

The representations of the fields under G and F are shown in the table.

	G			F		
	$U(1)_Y$	$SU(2)$	$SU(3)$	$SU(N_f)_q$	$SU(N_f)_l$	$U(1)_V$
q_L	$+1$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{N}_f$	$\mathbf{1}$	$+1$
l_L	-3	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{N}_f$	-3
d_R	-2	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{N}_f$	$\mathbf{1}$	$+1$
u_R	$+4$	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{N}_f$	$\mathbf{1}$	$+1$
e_R	-6	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{N}_f$	-3
ϕ	$+3$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0

Because we haven't included a right-handed neutrino, the $SU(N_f)_l \times U(1)_V$ symmetries have various 't Hooft anomalies, all of which arise from the leptons. The contribution to the 't Hooft anomalies from quarks vanish.

We are interested in this theory in the regime

$$v \ll \Lambda_{\text{weak}}, \Lambda_{\text{strong}}$$

where the Higgs expectation value, v, is much smaller than all other scales so that the dynamics is dominated by the gauge interactions. We now repeat the analysis of previous sections. As before, we assume that N_f is sufficiently small so that both gauge groups undergo chiral symmetry breaking; $N_f = 2$ appears to surely be safe; $N_f = 3$ is unclear.

$\underline{\Lambda_{\text{strong}} \gg \Lambda_{\text{weak}}}$

When the strong force dominates, a condensate (5.11) forms as before. In terms of the up and down quarks, this reads

$$\langle q_{L1i}^\dagger d_{Rj} \rangle \sim \Lambda_{\text{strong}}^3 \delta_{ij} \quad \text{and} \quad \langle q_{L2i}^\dagger u_{Rj} \rangle \sim \Lambda_{\text{strong}}^3 \delta_{ij}$$

where the $1,2$ labels on q_L are $SU(2)_{\text{weak}}$ indices. As in the Standard Model, this condensate breaks

$$U(1)_Y \times SU(2)_{\text{weak}} \to U(1)_Q$$

where the generator of $U(1)_Q$ is related to the generator of $U(1)_Y$ by

$$Q = \frac{1}{6}Y + \frac{1}{2}\text{diag}(1,-1)_{\text{weak}}$$

This, of course, is the usual symmetry breaking pattern of electroweak down to electromagnetism.

As the theory no longer has a chiral symmetry, the full global symmetry (5.24) survives in the infra-red

$$F_{\text{strong}} = SU(N_f)_q \times SU(N_f)_l \times U(1)_V$$

Because the full symmetry group survives, there are no Goldstone bosons. There are, however, light pion modes. These are the usual massless Goldstone bosons arising from the chiral symmetry breaking of QCD, which get a mass through the Yukawa couplings. (Even in the absence of a Higgs vev, the pions get a mass through mixing with ϕ.) Some aspects of these pions, as well as the associated baryons, were discussed in [115].

The leptons l_L and e_R remain unaffected by the gauge dynamics. They are distinguished by their charges under $U(1)_Q$; the left-handed lepton splits into e_L with charge $Q=-1$ and ν_L with charge $Q=0$. Meanwhile, the right-handed electron e_R has charge $Q=-1$. The electron pair gets a mass through the Yukawa coupling, while the left-handed neutrino remains massless, transforming as

	$U(1)_Q$	$SU(N_f)_q$	$SU(N_f)_l$	$U(1)_V$
ν_L	0	$\mathbf{1}$	$\mathbf{N}_f$	-3

This saturates the 't Hooft anomalies of F.

$\Lambda_{\text{weak}} \gg \Lambda_{\text{strong}}$

When the weak force dominates, the condensate (5.14) forms. When we subsequently turn on both $SU(3)_{\text{strong}}$ and $U(1)_Y$ gauge groups, we must again determine the correct vacuum.

One might be tempted to think that since $U(1)_Y$ is free in the infra-red, it does not affect the vacuum state described in the previous section. This, it turns out, is correct but it takes a calculation to show it. Indeed, in [112], Preskill gave examples of chiral symmetry breaking where a subsequent gauging of a $U(1)$ subgroup changes the vacuum structure when the

$U(1)$ coupling constant becomes sufficiently strong. In Appendix 5.B.3 we show that this doesn't happen in the present case.

The upshot of this argument is that the condensate (5.14) that minimises the potential remains unchanged by $U(1)_Y$. The quarks once again condense in a flavour-diagonal basis, as in (5.6), to

$$\langle q_{Lai} \cdot q_{Lbj} \rangle \sim \Lambda^3_{\text{weak}} \varepsilon_{abc} \sigma^c \delta_{ij} \tag{5.25}$$

with $a,b,c = 1,2,3$ colour indices and $i,j = 1,\dots N_f$ flavour indices. The remaining condensate pairs the $\hat{q}_{Li} = \sigma^a q_{Lai}$ quark with the leptons as in (5.7)

$$\langle \hat{q}_{Li} l_{Lj} \rangle \sim \Lambda^3_{\text{weak}} \delta_{ij} \tag{5.26}$$

The condensate breaks the global symmetry F in (5.24) to

$$F_{\text{weak}} = SO(N_f) \times U(1)_{\hat{V}} \tag{5.27}$$

where $SO(N_f) \subset SU(N_f)_q \times SU(N_f)_l$ and, as previously, $U(1)_{\hat{V}} \subset U(1)_V \times SU(3)_{\text{strong}}$.

The two condensates break the remaining gauge group to

$$SU(3)_{\text{strong}} \times U(1)_Y \to SU(2)_{\text{strong}} \times U(1)_{\hat{Q}} \tag{5.28}$$

We have seen the breaking to $SU(2)_{\text{strong}}$ previously. To see that a $U(1)_{\hat{Q}}$ survives, note that both of the condensates (5.25) and (5.26) carry $U(1)_Y$ charge $+2$. If we pick $\sigma^c = (0,0,1)$ in the condensate (5.25), then we can construct the unbroken gauge generator

$$\hat{Q} = \frac{1}{6} Y - \frac{1}{6} \text{diag}\,(1,1,-2)_{\text{strong}}$$

The surviving $SU(2)_{\text{strong}}$ gauge symmetry is coupled to the right-handed quarks. Under the breaking

$$SU(3)_{\text{strong}} \times U(1)_Y \times U(1)_V \to SU(2)_{\text{strong}} \times U(1)_{\hat{Q}} \times U(1)_{\hat{V}}$$

the right-handed fermions decompose as

$$\begin{aligned}
d_R: &\quad \mathbf{3}_{[-2,+1]} \to \mathbf{2}_{[-\frac{1}{2},0]} \oplus \mathbf{1}_{[0,+3]} \\
u_R: &\quad \mathbf{3}_{[+4,+1]} \to \mathbf{2}_{[+\frac{1}{2},0]} \oplus \mathbf{1}_{[1,+3]} \\
e_R: &\quad \mathbf{1}_{[-6,-3]} \to \mathbf{1}_{[-1,-3]}
\end{aligned}$$

The fermions that transform as doublets under $SU(2)_{\text{strong}}$ condense and become gapped as the gauge group becomes strong. The resulting condensate does not further break F_{weak} from (5.27). This means that, in contrast to the regime $\Lambda_{\text{strong}} \gg \Lambda_{\text{weak}}$, there is now a moduli space of Goldstone

$$\mathscr{M}_{\text{weak}} = \frac{SU(N_f)_q \times SU(N_f)_l}{SO(N_f)}$$

We're left with three gapless Weyl fermions, which were singlets under $SU(2)_{\text{strong}}$. Two of these, arising from u_R and e_R, carry equal and opposite $U(1)_{\hat{Q}} \times U(1)_{\hat{V}}$ charge. Although these are not coupled directly through the Yukawa coupling (5.23), there is nothing to prohibit this pair becoming gapped as they interact with the scalar field. This leaves $\hat{d}_R$, the neutral component of the down quark, as the surviving massless fermion. It transforms as

	$U(1)_{\hat{Q}}$	$SO(N_f)$	$U(1)_{\hat{V}}$
$\hat{d}_R$	0	$\mathbf{N}_f$	+3

It is noticeable that in the UV theory, the quarks did not appear to play any role in the computation of 't Hooft anomalies. Yet, by the time we flow to the infra-red, the sole remaining fermion is a quark and saturates the surviving 't Hooft anomalies of F_{weak}.

Note that in both $\Lambda_{\text{strong}} \gg \Lambda_{\text{weak}}$ and $\Lambda_{\text{weak}} \gg \Lambda_{\text{strong}}$ regimes, there is a surviving $U(1)$ gauge symmetry that we may identify with electromagnetism, and a surviving $U(1)$ global symmetry that we may identify with $B-L$. These symmetries are twisted with different gauge symmetries in the two regimes, but this does not impede us from identifying them. This conclusion differs from [115] were it is claimed that both electromagnetic and $B-L$ symmetries are broken in the $\Lambda_{\text{weak}} \gg \Lambda_{\text{strong}}$ regime.

The addition of hypercharge and Yukawa couplings does not change the conclusions of our earlier sections. If we have $N_f = 1$ generation of fermions, then it seems plausible that the transition between the two regimes proceeds without a phase transition. Meanwhile, for $N_f \geq 2$, a phase transition must occur.

However, in contrast to the situation in Section 5.2.1, there is a fairly simple symmetry breaking pattern between the two regimes, with $SU(N_f)_q \times SU(N_f)_l$, which survives when the strong force dominates, breaking to $SO(N_f)_{\text{diag}}$ when the weak force dominates, suggesting that a mean-field description of the phase transition in terms of Landau theory may be appropriate.

5.3 A Novel Chiral Gauge Theory

In this section, we extend our analysis to a chiral gauge theory with gauge group

$$G = U(1)_Y \times Sp(r) \times SU(N)$$

Anomaly considerations, to be described below, mean that we must take N odd. For the simplest values of $r = 1$ and $N = 3$ this gauge group coincides with that of the Standard Model.

The chiral fermion content is a natural extension of that of the Standard Model: we take left-handed fermions q_L and l_L to transform under $Sp(r)$, while the right-handed fermions are singlets under $Sp(r)$. One key difference is that we must take r copies of each of the right-handed fermions, including r copies of the right-handed neutrinos ν_R. We introduce an index $\alpha = 1,\ldots,r$ to distinguish these fields. For later convenience, we also introduce r distinct Higgs fields ϕ_α at this time too. The full set of fermions and scalars and their transformations is given by

	$U(1)_Y$	$Sp(r)$	$SU(N)$
q_L	$+1$	$\mathbf{2r}$	$\mathbf{N}$
l_L	$-N$	$\mathbf{2r}$	$\mathbf{1}$
$d_{R\alpha}$	$-(2\alpha-1)N+1$	$\mathbf{1}$	$\mathbf{N}$
$u_{R\alpha}$	$+(2\alpha-1)N+1$	$\mathbf{1}$	$\mathbf{N}$
$e_{R\alpha}$	$-2\alpha N$	$\mathbf{1}$	$\mathbf{1}$
$\nu_{R\alpha}$	$(2\alpha-2)N$	$\mathbf{1}$	$\mathbf{1}$
ϕ_α	$(2\alpha-1)N$	$\mathbf{2r}$	$\mathbf{1}$

with $\alpha = 1,\ldots,r$. It is straightforward to show that, with these charge assignments, all gauge anomalies vanish. The mixed anomalies, as well as the mixed gauge-gravitational anomaly,

$$\begin{aligned}
[\mathrm{Sp}(r)]^2 \times U(1)_Y: &\quad N\cdot(+1)+1\cdot(-N)=0,\\
[SU(N)]^2 \times U(1)_Y: &\quad 2r\cdot(+1)-\sum_{\alpha=1}^{r}[(-(2\alpha-1)+1)N+((2\alpha-1)N+1)]=0,\\
[\mathrm{grav}]^2 \times U(1)_Y: &\quad -\sum_{\alpha=1}^{r}\{N\cdot[-(2\alpha-1)N+1+(2\alpha-1)N+1]-2\alpha N+(2\alpha-2)N\}\\
&\quad +\{2rN\cdot(+1)+2r\cdot(-N)\}=0
\end{aligned}$$

can be easily seen to cancel. Only the other hand, the cubic hypercharge anomaly is more complicated:

$$\mathscr{A}_{Y^3} = \left\{2rN\cdot(1)^3 + 2r\cdot(-N)^3\right\} - \sum_{\alpha=1}^{r}\left\{N(1-(2\alpha-1)N)^3 - N(1+(2\alpha-1)N)^3\right\}$$
$$- \sum_{\alpha=1}^{r} N^3\left\{(2\alpha-2)^3 + (-2\alpha)^3\right\}, \tag{5.29}$$

but it can also be shown to vanish using the sum of cubes formula. The $\mathbf{Z}_2$ anomaly of $Sp(r)$ is vanishing only for N odd. Notice that the first right-handed neutrino is decoupled from the gauge fields, as in the Standard Model, but the other $r-1$ carry $U(1)_Y$ charge.

The $U(1)_Y$ charge assignments also allow us to construct Yukawa interactions. For a single generation, we have

$$\mathscr{L}_{\text{Yuk}} = \lambda_d\, q_L^\dagger \phi_\alpha d_{R\alpha} + \lambda_u (q_L^\dagger\cdot\phi_\alpha^\dagger) u_{R\alpha} + \lambda_e\, l_L^\dagger \phi_\alpha e_{R\alpha} + \lambda_\nu (l_L^\dagger\cdot\phi_\alpha^\dagger)\nu_{R\alpha} \tag{5.30}$$

Here $(q_L^\dagger\cdot\phi^\dagger)$ denotes the $Sp(r)$ singlet that one can construct from these two fields. (This is analogous to the $SU(2)$ singlet constructed using ε_{ab}.)

The generalisation of the Standard Model with $SU(3)$ gauge group replaced by $SU(N)$ is fairly well explored. (See, for example, [128].) The generalisation with $SU(2)$ replaced by $Sp(r)$, with the anomaly-free charge assignments shown in the table above, appears to be novel. We note the possibility that such a theory with gauge group $U(1)\times Sp(r)\times SU(3)$ may describe our world, with the additional Higgs fields ϕ_α, $\alpha = 2,\ldots,r$, breaking $Sp(r)$ to $SU(2)$ at some high scale. Moreover, this two parameter extension of the Standard Model may lend itself to a large r, large N expansion; we leave this possibility to future work. (A large N expansion of certain chiral gauge theories was previously proposed in [57, 21].)

We will adopt the convention of the Standard Model and refer to the $Sp(r)$ gauge group as *weak* and the $SU(N)$ gauge group as *strong*. As in the previous section, we will be interested in the phase diagram of the theory, with the two asymptotic regimes in which one of the gauge groups dominates over the other. We will study different variants of this problem, both with and without hypercharge interactions and Yukawa couplings.

Beta Functions

We will discuss the chiral theory coupled to N_f generations of fermions and focus on situations where both gauge groups are asymptotically free . The $SU(N)$ gauge group is coupled to $2rN_f$ Dirac fermions, each in the fundamental representation, and is asymptotically free provided

$$11N > 4rN_f$$

Meanwhile, the $Sp(r)$ factor is coupled to $N_f(N+1)$ Weyl fermions, each in the pseudo-real fundamental representation. If we ignore the Higgs fields for now, $Sp(r)$ is asymptotically free provided

$$11(r+1) > (N+1)N_f$$

For $N_f \geq 6$, at least one of the gauge groups is infra-red free. In contrast, for any $N_f \leq 5$, there are always choices of N and r for which both gauge groups become strongly coupled in the infra-red. This conclusion persists in the presence of Higgs fields.

As before, our analysis will rely on chiral symmetry breaking in the regime where one or the other gauge group becomes strongly coupled. This takes place for suitably low N_f, below the conformal window. The lower-edge of the conformal window is not well established. The $SU(N)$ gauge factor has $2rN_f$ Dirac fermions in the fundamental representation, and undergoes chiral symmetry breaking for

$$2rN_f < C_\star N$$

for some $C_\star$ which is expected to sit somewhere around 3 to 4. Meanwhile the $Sp(r)$ gauge factor has $N_f(N+1)$ Weyl fermions in the pseudo-real fundamental representation, and is expected to undergo chiral symmetry breaking when

$$N_f(N+1) < \hat{C}_\star(r+1)$$

where $\hat{C}_\star$ is around 6 to 8. (See, for example, [63].) For $N_f \leq 2$ there are an infinite number of choices of N and r for which chiral symmetry breaking occurs, while for $N_f = 4$ it seems likely there are none. The situation for $N_f = 3$ is, in all cases, more murky.

5.3.1 $Sp(r) \times SU(N)$

We start by neglecting the $U(1)_Y$ factor and focussing only on the gauge group

$$G = Sp(r) \times SU(N)$$

Because the right-handed electrons e_R and neutrinos ν_R are singlets under the non-Abelian part of the gauge group, we may ignore them for the purpose of this discussion. We will also discard the Higgs field for now, focussing only on the fermions. As in the case of the Standard Model, here we will find the richest symmetry breaking patterns, unconstrained by hypercharge assignments and Yukawa couplings. We will then reintroduce both of these in Section 5.3.2

With N_f generations of fermions, the global, non-anomalous, symmetry group is

$$F = SU(N_f)_{L'} \times SU(N_f)_L \times SU(2rN_f)_R \times U(1)_V \tag{5.31}$$

Under the gauge and global symmetry groups, the fermions transform as

	G		F			
	$Sp(r)$	$SU(N)$	$SU(N_f)_{L'}$	$SU(N_f)_L$	$SU(2rN_f)_R$	$U(1)_V$
q_L	$\mathbf{2r}$	$\mathbf{N}$	$\mathbf{1}$	$\mathbf{N}_f$	$\mathbf{1}$	$+1$
l_L	$\mathbf{2r}$	$\mathbf{1}$	$\mathbf{N}_f$	$\mathbf{1}$	$\mathbf{1}$	$-N$
q_R	$\mathbf{1}$	$\mathbf{N}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2rN}_f$	$+1$

with $q_R = (u_R, d_R)$, the two right-handed quarks now undistinguished by hypercharge. There are numerous 't Hooft anomalies between the various subgroups of F.

$\Lambda_{\text{strong}} \gg \Lambda_{\text{weak}}$

When the $SU(N)_{\text{strong}}$ force dominates, the usual quark condensate (5.11) forms and the quarks become gapped, leaving behind a number Goldstone bosons.

If we ignore the $Sp(r)$ weak force, the theory has a $SU(2rN_f)_L \times SU(2rN_f)_R \times U(1)_V$ global symmetry, broken by the condensate to $SU(2rN_f)_{\text{diag}} \times U(1)_V$. Gauging $Sp(r)_{\text{weak}}$, means that the global symmetry F in (5.31) breaks to

$$F_{\text{strong}} = SU(N_f)_{L'} \times Sp(r)_{\text{diag}} \times SU(N_f)_{\text{diag}} \times U(1)_V$$

Here $Sp(r)_{\text{diag}} \subset Sp(r)_{\text{weak}} \times SU(2rN_f)_{\text{diag}}$ arises from a simultaneous gauge transformation and surviving $SU(2rN_f)_{\text{diag}}$ rotation. This ensures that the $Sp(r)$ gauge symmetry is fully broken, with only this "weak-flavour-locked" global symmetry surviving. The remaining $SU(N_f)_{\text{diag}}$ is the centraliser of $Sp(r)$ in $SU(2rN_f)_{\text{diag}}$.

The Goldstone bosons therefore parameterise the moduli space

$$\mathscr{M}_{\text{strong}} = \frac{SU(N_f)_L \times SU(2rN_f)_R}{Sp(r) \times SU(N_f)_{\text{diag}}}$$

There are $\dim \mathscr{M}_{\text{strong}} = 4r^2N_f^2 - 2r^2 - r - 1$ of them.

With the $Sp(r)_{\text{weak}}$ gauge group fully Higgsed, the left-handed leptons remain massless. They transform under F_{strong} as

	$SU(N_f)_{L'}$	$Sp(r)_{\text{diag}}$	$SU(N_f)_{\text{diag}}$	$U(1)_V$
l_L	$\mathbf{N}_f$	$\mathbf{2r}$	$\mathbf{1}$	$-N$

These saturate the 't Hooft anomaly of F_{strong}. Again this is because the left- and the right-handed quarks in UV have the same representation $(\mathbf{1},\mathbf{2r},\mathbf{N}_f)_{+1}$ under F_{strong} so they don't contribute to the anomaly.

$\Lambda_{\text{weak}} \gg \Lambda_{\text{strong}}$

When the $Sp(r)_{\text{weak}}$ force dominates, the $(N+1)N_f$ left-handed fermions condense. Collectively, we refer to these as ψ_{mi}, with $m = 1,\ldots,(N+1)$ labelling the quarks and leptons in a single generation, and $i = 1,\ldots,N_f$ the flavour index. The condensate takes the form

$$\langle \psi_{mi} \cdot \psi_{nj} \rangle \sim \Lambda^3_{\text{weak}} \bar{J}_{mi,nj} \tag{5.32}$$

with $\psi_{mi} \cdot \psi_{nj}$ a $Sp(r)_{\text{weak}}$ singlet. (The gauge group indices are contracted using the $Sp(r)$ invariant anti-symmetric tensor) and $\bar{J}_{mi,nj}$ an anti-symmetric matrix.

Once again, we must determine the choice of $\bar{J}_{mi,nj}$ that minimizes the potential induced by gauging the $SU(N)_{\text{strong}}$ group. This is a fairly involved calculation and is presented in Appendix 5.B.4, where we show that the flavour-diagonal condensate is again a (local) minimum of the potential, with no tachyonic modes. This means that the dynamically preferred vacuum condensate breaks the gauge group to

$$SU(N)_{\text{strong}} \to Sp((N-1)/2)_{\text{strong}} \tag{5.33}$$

generalising the earlier result (5.15). At the same time, the global symmetry F is broken to

$$\tilde{F} = SO(N_f) \times SU(2rN_f)_R \times U(1)_{\hat{V}} \tag{5.34}$$

where $SO(N_f) \subset SU(N_f)_{L'} \times SU(N_f)_L$ and $U(1)_{\hat{V}} \subset U(1)_V \times SU(N)_{\text{strong}}$ is defined in analogy with (5.8).

We're still left with the right-handed quarks q_R, which are now coupled to the surviving $Sp(\frac{1}{2}(N+1))$ gauge group. Under the symmetry breaking

$$SU(N)_{\text{strong}} \times U(1)_V \to Sp((N-1)/2)_{\text{strong}} \times U(1)_{\hat{V}}$$

these right-handed quarks decompose as

$$q_R : \mathbf{N}_{+1} \to (\mathbf{N}-\mathbf{1})_0 \oplus \mathbf{1}_{+N} \tag{5.35}$$

As $Sp(\frac{1}{2}(N-1)$ becomes strong and confines, those quarks transforming in the $\mathbf{N}-\mathbf{1}$ representation condense. This further breaks the flavour symmetry group to

$$F_{\text{weak}} = SO(N_f) \times Sp(rN_f)_R \times U(1)_{\hat{V}} \tag{5.36}$$

The final result is that we have a moduli space of vacua,

$$\mathscr{M}_{\text{weak}} = \frac{SU(N_f) \times SU(N_f)}{SO(N_f)} \times \frac{SU(2rN_f)}{Sp(rN_f)}$$

generalising our earlier result (5.20). Meanwhile, the singlet fermions in (5.35) remain massless, transforming under F_{weak} as

	$SO(N_f)$	$Sp(rN_f)_R$	$U(1)_{\hat{V}}$
$\hat{q}_R$	$\mathbf{1}$	$\mathbf{2rN}_f$	$+N$

The fermion content can be shown to saturate the 't Hooft anomalies in the same way demonstrated explicitly at the end of Section 5.2.1. For a single generation of fermions, $N_f = 1$, we again see that the infra-red global symmetry in the two regimes coincides: both are $Sp(r) \times U(1)$. This now differs from the UV symmetry (5.31), meaning that there are Goldstone bosons even in this case. Nonetheless, it appears plausible that there is no phase transition for $N_f = 1$ as we vary the gauge coupling constants. As in Section 5.2, the left-handed lepton in one regime transmutes into a right-handed quark in the other.

For $N_f \geq 2$, the symmetry breaking patterns on either side differ and there must be a phase transition as we vary between them. Once again, neither of the symmetries F_{strong} and F_{weak}, defined in (5.34) and (5.36), are subgroups of the other.

5.3.2 Adding Hypercharge and Yukawa Couplings

Finally we discuss the theory introduced at the beginning of this section, replete with $U(1)_Y$ coupling and flavour-diagonal Yukawa interactions (5.30). The gauge symmetry is

$$G = U(1)_Y \times Sp(r) \times SU(N)$$

and the global symmetry is

$$F = SU(N_f)_q \times SU(N_f)_l \times U(1)_V \tag{5.37}$$

where the matter fields transform as

	G			F		
	$U(1)_Y$	$Sp(r)$	$SU(N)$	$SU(N_f)_q$	$SU(N_f)_l$	$U(1)_V$
q_L	$+1$	$\mathbf{2r}$	$\mathbf{N}$	$\mathbf{N}_f$	$\mathbf{1}$	$+1$
l_L	$-N$	$\mathbf{2r}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{N}_f$	$-N$
$d_{R\alpha}$	$-(2\alpha-1)N+1$	$\mathbf{1}$	$\mathbf{N}$	$\mathbf{N}_f$	$\mathbf{1}$	$+1$
$u_{R\alpha}$	$+(2\alpha-1)N+1$	$\mathbf{1}$	$\mathbf{N}$	$\mathbf{N}_f$	$\mathbf{1}$	$+1$
$e_{R\alpha}$	$-2\alpha N$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{N}_f$	$-N$
$\nu_{R\alpha}$	$(2\alpha-2)N$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{N}_f$	$-N$
ϕ_α	$(2\alpha-1)N$	$\mathbf{2r}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0

This time, the presence of the right-handed neutrinos ensure that the global symmetries F suffer no 't Hooft anomalies. The arguments of the previous section allow us to quickly determine the symmetry breaking pattern in the two regimes.

$\Lambda_{\text{strong}} \gg \Lambda_{\text{weak}}$

When the strong force dominates, the full UV symmetry (5.37) survives. There are no Goldstone bosons. Although the leptons remain massless after the gauge interactions become strong, they interact with the Higgs fields and, indirectly, with the mesons and there is nothing to prevent them gaining a mass, suppressed by the Yukawa coupling. For generic values of the Yukawa couplings, we therefore expect the fermions to be gapped.

$\Lambda_{\text{weak}} \gg \Lambda_{\text{strong}}$

When the weak force dominates, the condensate (5.14) forms. A computation of the correct vacuum alignment can be found in Appendix 5.B.4, which shows that the ground state preserves the global symmetry

$$F_{\text{weak}} = SO(N_f) \times U(1)_{\hat{V}}$$

with $SO(N_f) \subset SU(N_f)_{\text{diag}} \subset SU(N_f)_q \times SU(N_f)_l$. The condensate also breaks the $SU(N)_{\text{strong}}$ gauge group as in (5.33). As the surviving subgroup of $SU(N)_{\text{strong}}$ confines, the resulting condensate does not further break the global symmetry F_{weak}. Once again, no symmetry principle ensures massless fermions and, generically, none are expected to survive. Instead, the gapless modes are supplied by the Goldstone bosons which parameterise

$$\mathscr{M}_{\text{weak}} = \frac{SU(N_f)_q \times SU(N_f)_l}{SO(N_f)} \tag{5.38}$$

The story is, by now, familiar. There is no evidence of a phase transition in the symmetry breaking pattern when $N_f = 1$. Such a phase transition must occur for $N_f \geq 2$ although now the symmetry breaking pattern suggests that such a phase transition can be captured by a mean field Landau-Ginzburg description. It would surely be interesting to gain a better understanding of the nature of the phase transition, both here and in other examples.

Appendix 5.A Vacuum Alignment

When a gauge theory spontaneously breaks chiral symmetry, the resulting Goldstone bosons parameterise a moduli space of vacua $\mathscr{M}_0$. If this theory is subsequently coupled to a second gauge group, which becomes strong at a lower scale, then much of the the vacuum moduli space $\mathscr{M}_0$ is lifted. The preferred ground state is chosen dynamically in a process known as *vacuum alignment*.

The physics of vacuum alignment was explained in two beautiful papers by Peskin [107] and Preskill [112]. In this appendix, we review the results of these papers. In Appendix 5.B, we then apply these results to understand the ground states in situations of interest in Sections 5.2 and 5.3.

We consider a general gauge theory with gauge group as

$$G = G_1 \times G_2$$

with the convention that the gauge group G_1 will always run to strong coupling before G_2, meaning that the dynamically generated scales are ordered as

$$\Lambda_1 \gg \Lambda_2$$

We couple our gauge theory to fermions. The full theory will have a global symmetry group that we denote as F. However, if we first turn off the second gauge group G_2 by setting its coupling to zero, the global symmetry group of remaining theory, with only G_1, will be larger: we denote this global symmetry group as K.

We are interested in situation where the confinement of G_1 and subsequent condensation of fermion bilinears breaks this global symmetry to a smaller subgroup

$$K \longrightarrow H$$

The are three basic symmetry breaking patterns, suggested by the maximally attractive channel hypothesis, that can arise with a single gauge group G_1 [107]. Only two of them will be needed in the bulk of the paper, but we list all three for completeness:

- If there are n massless Dirac fermions in a complex representation of G_1, we have the global symmetry $K = SU(n)_L \times SU(n)_R$, with the two factors acting on left- and right-handed Weyl fermions which we denote as ψ_L and ψ_R. The condensate takes the general form
$$\langle \psi^\dagger_{Ri} \psi_{Lj} \rangle \sim \Lambda_1^3\, (U_R^\dagger U_L)_{ij}$$
with $i, j = 1, \ldots, n$ and $U_{L/R} \in SU(N)_{L/R}$. The subgroup $H = SU(n)_{\text{diag}}$ leaves the condensate untouched, meaning that we have the familiar QCD-like breaking pattern
$$SU(n)_L \times SU(n)_R \longrightarrow SU(n)$$
This form of condensate arises in the bulk of the paper when the $SU(N)$ gauge group, with $N \geq 3$, first becomes strong.

- If there are $2n$ Weyl fermions in a pseudo-real representation of G_1 then we have a global symmetry $K = SU(2n)$. The condensate forms through the invariant anti-symmetric tensor, ε^{ab}. It takes the general form
$$\left\langle \varepsilon^{ab} \psi_{ai} \psi_{bj} \right\rangle \sim \Lambda_1^3\, (U^T J U)_{ij} \qquad \text{with } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
Here $U \in SU(2n)$ and J is the $2n \times 2n$ anti-symmetric matrix given in block form above. The resulting symmetry breaking pattern is
$$SU(2n) \longrightarrow Sp(n)$$
This form of the condensate arises in the bulk of the paper when the gauge group $Sp(r)$ or $SU(2)$ becomes strong.

- If there are $2n$ Weyl fermions in a real representation of G_1, then the global symmetry is again $K = SU(2n)$. This time the condensate forms through the invariant symmetry tensor of the reperesentation, d^{ab}. It takes the general form

$$\left\langle d^{ab}\psi_{a\,i}\psi_{b\,j}\right\rangle \sim \Lambda_1^3\,(U^T DU)_{ij} \qquad \text{with } D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Again, $U \in SU(2n)$ and D is a $2n \times 2n$ symmetric matrix given in block form above. Now, the symmetry breaking pattern is

$$SU(2n) \longrightarrow O(2n)$$

This form of the condensate does not play a direct role in this Chapter although, as we show in Section 5.2.1, we have a similar symmetry breaking pattern when first $Sp(r)$ and subsequently $SU(N)$ becomes strong.

Each of the symmetry breaking patterns described above results in a vacuum moduli space

$$\mathscr{M}_0 = K/H$$

Each point of $\mathscr{M}_0$ corresponds to a different orientation of $H \subset K$.

5.A.1 A Potential Over the Moduli Space

We now turn on the coupling for the second gauge group $G_2 \subset K$. The global symmetry of the theory is reduced to F. Correspondingly, the symmetry breaking pattern $K \to H$ is reduced. Different orientations of H in K descend to different symmetry breaking patterns, each of the form

$$G_2 \times F \longrightarrow \tilde{G} \times \tilde{F} \tag{5.39}$$

The question we need to address is: what symmetry breaking pattern is preferred? This is the question of *vacuum alignment*.

As explained in [107, 112], the choice of vacuum is determined dynamically. To see why this is the case note that, after gauging G_2, there are three different fates for the would-be Goldstone modes in $\mathscr{M}_0$. Some will be charged under G_2; these act as Higgs bosons, breaking G_2 to the smaller group $\tilde{G} \subset G_2$ and are eaten by the Higgs mechanism. Other scalars in $\mathscr{M}_0$ are not eaten, but are no longer protected by symmetry constraints; they will gain a mass, as we explain more fully below, and are referred to as *pseudo-Goldstone bosons*. Finally, some scalars remain exactly massless; these are Goldstone modes of the full theory, whose moduli space includes the factor

$$\mathscr{M} \subset F/\tilde{F}$$

Note that this need not be the full moduli space because when the surviving gauge group $\tilde{G}$ becomes strong, it too may break some chiral symmetry, resulting in further Goldstone bosons.

We now describe how the potential is generated over $\mathscr{M}_0$, following [107, 112]. The minimum determines the locus of ground states in $\mathscr{M}_0$ and, correspondingly, the surviving symmetries $\tilde{G}$ and $\tilde{F}$ in (5.39). The potential is generated by the coupling

$$\delta\mathscr{L} = \sum_{\alpha} A^{\alpha}_{\mu} J^{\alpha\mu}$$

with $\alpha = 1,\ldots,\dim G_2$. The current is given by

$$J^{\alpha}_{\mu} = i\bar{\psi}\gamma_{\mu}G^{\alpha}\psi$$

where G^{α} are the generators of G_2.

At one-loop, exchange of the W-bosons gives rise to a potential on the moduli space. A point $\Omega \in \mathscr{M}_0$ corresponds to a putative vacuum state $|\Omega\rangle$. The energy of this state is given by

$$V(\Omega) = -\frac{g_2^2}{2}\Delta^{\mu\nu}(x)\,\langle\Omega|\,J^{\alpha}_{\mu}(x)J^{\alpha}_{\nu}(0)\,|\Omega\rangle \tag{5.40}$$

This correlation function, and those below, are time-ordered. Here g_2 is the gauge coupling associated to the gauge group G_2 and the gauge propagator $\Delta_{\mu\nu}(x)$ is defined in the usual way by

$$\langle\Omega|\,A^{\alpha}_{\mu}(x)A^{\beta}_{\nu}(0)\,|\Omega\rangle = -i\delta^{\alpha\beta}\Delta_{\mu\nu}(x)$$

It will prove to be useful to change perspective, somewhat analogous to the shift from an active to passive viewpoint. To this end, we fix a reference vacuum state $|0\rangle$. A general ground state $|\Omega\rangle$ is given by the unitary action

$$|\Omega\rangle = U\,|0\rangle$$

with $U \in K/H$. (Strictly speaking, U is a unitary representation of K acting on the Hilbert space.) We now parameterise the point in $\mathscr{M}_0$ by $U \in K/H$. We define the rotated currents,

$$\mathscr{J}^{\alpha}_{\mu} = i\bar{\psi}\gamma_{\mu}U^{\dagger}G^{\alpha}U\psi \tag{5.41}$$

In this notation, the potential (5.40) becomes

$$V(U) = -\frac{g_2^2}{2}\Delta^{\mu\nu}(x)\,\langle 0|\,\mathscr{J}^{\alpha}_{\mu}(x)\,\mathscr{J}^{\alpha}_{\nu}(0)\,|0\rangle \tag{5.42}$$

In the vacuum $|0\rangle$, there is a particular embedding of the unbroken subgroup $H \subset K$. We introduce the following notation for the generators of the Lie algebra of K and its subalgebras[5]

- Let T^m, $m = 1, \ldots, \dim K$, denote the generators of K
- Let H^a, $a = 1, \ldots, \dim H$, denote the generators of $H \subset K$.
- Let X^i, $i = 1, \ldots, \dim K - \dim H$, denote the generators of K/H.

Any generator, T of K, can be decomposed into components projected along the two subalgebras H and $X = K/H$. We write the projection along H as T_H and the projection along X as T_X, so we have

$$T = T_H + T_X := \mathrm{Tr}(TH^a)H^a + \mathrm{Tr}(TX^i)X^i$$

with $T_H^a = \mathrm{Tr}(TH^a)$ the projection along H and $T_X^i = \mathrm{Tr}(TX^i)$ the projection along X. Implementing a decomposition of this kind for the current (5.41), we have

$$\mathscr{J}_\mu^\alpha = \mathrm{Tr}(U^\dagger G^\alpha U H^a)\,\mathscr{J}_{H\mu}^a + \mathrm{Tr}(U^\dagger G^\alpha U X^i)\,\mathscr{J}_{X\mu}^i$$

with $\mathscr{J}_{H\mu}^a = i\bar{\psi}\gamma_\mu H^a\psi$ the currents that lie in the unbroken $H \subset K$ and $\mathscr{J}_{X\mu}^i = i\bar{\psi}\gamma_\mu X^i\psi$ the currents that lie in the broken K/H. Substituting this decomposition into the potential (5.42), we have three terms: $\mathscr{J}_H^2$, $\mathscr{J}_X^2$ and $\mathscr{J}_H\mathscr{J}_X$. The cross-term $\mathscr{J}_H\mathscr{J}_X$ vanishes. The other two terms also simplify. In particular, using the fact that K/H is a symmetric space, we have

$$\langle 0|\,\mathscr{J}_{X\mu}^i(x)\mathscr{J}_{X\nu}^j(0)\,|0\rangle = \ \mathrm{Tr}\ (X^iX^j)\,\langle 0|\,\mathscr{J}_{X\mu}(x)\mathscr{J}_{X\nu}\,|0\rangle$$

where $\mathscr{J}_X$ denotes any choice of normalised generator, e.g. $\mathscr{J}_X = \mathscr{J}_X^1$; the exact choice doesn't matter precisely because it's a symmetric space. We use a similar convention for $\mathscr{J}_H$. The potential (5.42) can then be written as

$$\begin{aligned} V(U) = -\frac{g_2^2}{2}\Delta^{\mu\nu}(x)\sum_\alpha \Big[& \mathrm{Tr}\ (U^\dagger G^\alpha U)_H^2\ \langle 0|\,\mathscr{J}_{H\mu}(x)\mathscr{J}_{H\nu}(0)\,|0\rangle \\ & + \ \mathrm{Tr}\ (U^\dagger G^\alpha U)_X^2\ \langle 0|\,\mathscr{J}_{X\mu}(x)\mathscr{J}_{X\nu}(0)\,|0\rangle \Big] \end{aligned} \tag{5.43}$$

We can further simplify this using

$$\begin{aligned} \mathrm{Tr}\ (G^\alpha)^2 = \ \mathrm{Tr}\ (U^\dagger G^\alpha U)^2 &= \ \mathrm{Tr}\ ((U^\dagger G^\alpha U)_X + (U^\dagger G^\alpha U)_H)^2 \\ &= \ \mathrm{Tr}\ (U^\dagger G^\alpha U)_X^2 + \ \mathrm{Tr}\ (U^\dagger G^\alpha U)_H^2 \end{aligned}$$

[5] To avoid an explosion of notation, we denote the Lie group and Lie algebra by the same letter.

Both terms in the potential (5.43) can then be written in terms of Tr $(U^\dagger G^\alpha U)_X^2$, giving

$$V(U) = V_0 + \frac{g_2^2}{2}\frac{f_\pi^2 M^2}{4\pi}\sum_\alpha \text{Tr } (U^\dagger G^\alpha U)_X^2 \tag{5.44}$$

where V_0 is independent of U. Here we've introduced f_π, the characteristic energy scale associated to chiral symmetry breaking, defined in the usual manner as

$$\langle 0 | \mathscr{J}^i_{X\mu} \left| \pi^j \right\rangle = i f_\pi p_\mu \delta^{ij}$$

The mass scale M^2 in (5.44) is given in terms of broken and unbroken current by

$$M^2 = \frac{4\pi}{f_\pi^2}\Delta^{\mu\nu}(x)\,\langle 0| \mathscr{J}_{H\mu}(x)\mathscr{J}_{H\nu}(0) - \mathscr{J}_{X\mu}(x)\mathscr{J}_{X\nu}(0)\,|0\rangle \tag{5.45}$$

Importantly, M^2 can be shown to be positive [107, 112]. We postpone the derivation of this result to Appendix 5.A.2 below.

The expression for the potential (5.44) has a particularly elegant interpretation: the group theoretic factor is simply the sum of the G_2 gauge boson masses,

$$\sum_\alpha \text{Tr } (U^\dagger G^\alpha U)_X^2 \sim \sum (\text{gauge boson mass})^2$$

We see that, with $M^2 > 0$, the minimum of the potential $V(U)$ occurs when the gauge group is broken the least, in the sense that the sum of the gauge boson masses is smallest.

In practice, life is simplest if we are able to pick the reference state $|0\rangle$ to be a local minimum. For this to be the case, the generators G^α of $G_2 \subset K$ must obey a number of properties. To see this, we parameterise the vacua in the neighbourhood of $|0\rangle$ by

$$U(\rho) = \exp\left(i\rho_i X^i\right)$$

To leading order in ρ, the potential (5.44) then reads

$$V(\boldsymbol{\rho}) = V_0 + \frac{g_2^2}{2}\frac{f_\pi^2 M^2}{4\pi}\sum_\alpha \Big(\text{ Tr } \left(G_X^\alpha\right)^2 + i\rho_i \text{ Tr } (G_X^\alpha \left[G^\alpha, X^i\right]_X) + \dots \Big)$$

For $\rho = 0$ to be an extremum of V, we need

$$\frac{\partial}{\partial \rho_i} V(0) \sim \sum_\alpha \text{ Tr } G_X^\alpha \left[G^\alpha, X^i\right]_X = \sum_\alpha \text{ Tr } G_X^\alpha \left[G_H^\alpha, X^i\right] = \sum_\alpha \text{ Tr } X^i \left[G_H^\alpha, G_X^\alpha\right] = 0$$

where the second equality follows from the fact that, for K/H a symmetric space, $[H,H] \sim iH$ and $[H,X] \sim iX$, and $[X,X] \sim iT$. The third equality is of course the cyclic property of trace. We learn that the reference vacuum $|0\rangle$ is a stationary point of V provided that

$$[G_H^\alpha, G_X^\alpha] = 0 \quad \text{for each } \alpha \quad \text{(no sum)} \tag{5.46}$$

Next we must ensure that $|0\rangle$ is a local minimum, as opposed to a maximum or saddle point. For this, we must compute the Hessian of V. In a mass-diagonal basis for the broken generators X^i, one can show that the mass eigenstates are given by

$$m_X^2 = \frac{g_2^2 M^2}{4\pi} \sum_\alpha \Big[\text{ Tr } [G_H^\alpha, [G_H^\alpha, X]] X - \text{ Tr } [G_X^\alpha, [G_X^\alpha, X]] X \Big] \tag{5.47}$$

This combination will show up regularly in what follows; we denote it as

$$m_X^2 = \frac{g_2^2 M^2}{4\pi} \sum_\alpha \mathscr{C}_X (G^\alpha) \tag{5.48}$$

We see that we have a local minimum only if $m_X^2 > 0$ for each of the pseudo-Goldstone bosons X. In contrast, if there is any direction with $m_X^2 < 0$ then there is a tachyonic mode which destabilises the would-be vacuum.

In fact, life is not quite as simple as we have described. We will encounter a number of situations in which the leading order result (5.47) gives $m_X^2 = 0$ for some pseudo-Goldstone boson X, even though there is no symmetry protecting the mass. In this case, we must work harder and look to the second-order terms.

5.A.2 Second Order Corrections to the Potential

To compute the second order corrections to the masses of pseudo-Goldstone bosons, we need a little bit of non-perturbative information. Fortunately, this information is available in the form of sum rules, first derived by Weinberg [144]. Moreover, this machinery is precisely what's required to prove that M^2, defined in (5.45), is positive definite. We now review this, following [107, 112].

The spectral function $\rho_H(s)$, corresponding to the unbroken currents $\mathscr{J}_H$ is defined by

$$\langle 0 |\, \mathscr{J}_H^\mu(x) \mathscr{J}_H^\nu(0) \,| 0 \rangle = \int_0^\infty \mathrm{d}s \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \rho_H(s) \frac{-ie^{-ip\cdot x}}{p^2 - s + \mathrm{i}\varepsilon} \left(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right)$$

For the broken currents $\mathscr{J}_X$, the corresponding spectral function ρ_X has an extra term,

$$\langle 0|\, \mathscr{J}_X^\mu(x)\mathscr{J}_X^\nu(0)\,|0\rangle = \int_0^\infty \mathrm{d}s \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\,\frac{-ie^{-ip\cdot x}}{p^2-s+\mathrm{i}\varepsilon}\left[\rho_X(s)\left(\eta^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right)-f_\pi^2\delta(p^2)p^\mu p^\nu\right]$$

Importantly, the spectral functions obey a number of sum rules [144],

$$\begin{aligned}\int_0^\infty \mathrm{d}s\,(\rho_H(s)-\rho_X(s)) &= 0\\ \int_0^\infty \frac{\mathrm{d}s}{s}\,(\rho_H(s)-\rho_X(s)) &= f_\pi^2\end{aligned} \tag{5.49}$$

The mass M^2 can be written in terms of the two spectral functions (see, for example, [112]) as

$$M^2 = \frac{3}{4\pi f_\pi^2}\int_0^\infty \mathrm{d}s\,\log\frac{s_0}{s}\,(\rho_H(s)-\rho_X(s))$$

where s_0 is a regularisation scale. To simplify this further, we must assume that the spectral functions are dominated by the lowest lying mesons, and are correspondingly approximated by delta-functions

$$\rho_H \simeq \lambda_1^2\,\delta(s-M_H^2) \qquad \text{and} \qquad \rho_X \simeq \lambda_2^2\,\delta(s-M_X^2)$$

Here M_H and M_X are the masses of the lowest-lying spin-1 mesons coupled to the unbroken and broken currents respectively[6] and λ_1, λ_2 are the strengths of the couplings. The two sum rules (5.49) then tell us that $\lambda_1^2 \simeq \lambda_2^2 = \lambda$ from the second equation and

$$\frac{1}{M_H^2}-\frac{1}{M_X^2} = \frac{f_\pi^2}{\lambda^2}$$

Since the right-hand side is positive, we learn that $M_X^2 > M_H^2$. This is sufficient to guarantee positivity of M^2 which can be related to the meson masses as

$$M^2 = \frac{3\lambda^2}{4\pi f_\pi^2}\log\left(\frac{M_X^2}{M_H^2}\right)$$

The machinery of spectral functions is also needed to get an expression for the second-order correction to the masses of the pseudo-Goldstone bosons. The essence of the idea is simple: the masses M^2 are computed using the gapless gauge boson propagator $\Delta^{\mu\nu}(x)$ in

[6] For orientation, in QCD with $N_f = 2$ flavours, the broken and unbroken generators arise from the chiral symmetry breaking pattern $SU(2)_L \times SU(2)_R \to SU(2)_{\text{diag}}$. Here the ρ meson, with $M_H = 770$ MeV couples to $\mathscr{J}_H$ while the a_1 meson, with mass $M_X = 1260$ MeV couples to $\mathscr{J}_X$.

(5.45). However, the condensate partially breaks the gauge group G_2, giving some of the gauge bosons a mass. For these gauge bosons, the propagator should be replaced by the massive propagator.

Here for simplicity, we assume that each of the massive G_2 gauge bosons has the same mass, which we denote as μ^2. (In the examples of Appendix 5.B, this is too naive and there will be gauge bosons with different masses. This adds an extra complication, but here we deal with just the simplest case.) The mass μ^2 changes the propagator of the gauge boson and, correspondingly, shifts the mass M^2 to M_μ^2, which we will compute shortly. Note that the mass μ^2 will be proportional to g_2^2, meaning that M_μ^2 differs from M^2 only at order g_2^4.

With this correction, the mass of the pseudo-Goldstone bosons (5.48) becomes

$$m_X^2 = \frac{g_2^2}{4\pi}\left(M^2 \sum_{\text{unbroken}} \mathscr{C}_X(G^\alpha) + M_\mu^2 \sum_{\text{broken}} \mathscr{C}_X(G^\alpha)\right)$$

Our interest lies in those pseudo-Goldstone bosons whose mass m_X^2 vanishes at leading order. For these, it must be the case that $\sum_{\text{unbroken}} \mathscr{C}_X(G^\alpha) = \sum_{\text{broken}} \mathscr{C}_X(G^\alpha)$, so we can write

$$m_X^2 = \frac{g_2^2}{4\pi} \sum_{\text{unbroken}} \mathscr{C}_X(G^\alpha)\left(M^2 - M_\mu^2\right) \tag{5.50}$$

Note, however, that for an unbroken generator G^α we have, by definition, $G_X^\alpha = 0$ (since G_X^α is the projection onto the broken part). Using the definition of $\mathscr{C}_X(G^\alpha)$ in (5.47) and (5.48), we see that

$$\sum_{\text{unbroken}} \mathscr{C}_X(G^\alpha) > 0$$

This is the key to showing that the second order correction to the mass terms is positive. (It is also the step that needs revisiting when the broken gauge bosons have different masses.) Invoking the spectral representation, the mass (5.50) can be written as

$$\begin{aligned} m_X^2 &= \frac{3g_2^2}{(4\pi)^2 f_\pi^2}\left(\sum_{\text{unbroken}} \mathscr{C}_X(G^\alpha)\right) \int ds \left(\log\frac{s_0}{s} - \frac{s}{s-\mu^2}\log\frac{\mu^2}{s}\right)(\rho_H - \rho_X) \\ &\approx \frac{3g_2^2}{(4\pi)^2 f_\pi^2}\left(\sum_{\text{unbroken}} \mathscr{C}_X(G^\alpha)\right) \int ds\, \frac{\mu^2}{s}\log\frac{s}{\mu^2}\,(\rho_H(s) - \rho_X(s)) \\ &\approx \frac{3g_2^2\lambda^2}{(4\pi)^2 f_\pi^2}\left(\sum_{\text{unbroken}} \mathscr{C}_X(G^\alpha)\right)\left(\frac{\mu^2}{M_H^2}\log\frac{M_H^2}{\mu^2} - \frac{\mu^2}{M_X^2}\log\frac{M_X^2}{\mu^2}\right) \end{aligned} \tag{5.51}$$

The fact that this is positive definite follows once again from the observation that $M_X^2 > M_H^2$. This ensures that those pseudo-Goldstone bosons that remain massless at leading order receive a positive mass at the next order. Note also that the gauge boson mass is of order $\mu^2 \sim g_2^2 f_\pi^2$, ensuring that this mass m_X^2 is indeed of order g_2^4 as expected.

As we stressed above, this calculation assumed that the massive G_2 gauge bosons have a common mass μ^2. This allowed us to write the second-order correction to the massless pseudo-Goldstone bosons as (5.50). Below we will meet situations in which this step needs revisiting, and the positivity of the mass correction is no longer so straightforward. Nonetheless, we will see that the positivity remains.

Appendix 5.B Examples

We now apply the results of Appendix 5.A to the models considered in the bulk of the paper.

5.B.1 Vacuum Alignment for $SU(2)\times SU(3)$

We start by applying the ideas above to the chiral gauge theory with gauge group $G = SU(2)\times SU(3)$, coupled to N_f generations of fermions. For now, we include neither hypercharge nor Yukawa interactions. This is the theory described in Section 5.2.1.

When $\Lambda_{\text{strong}} \gg \Lambda_{weak}$, so the strong force dominates, the original chiral symmetry breaking gives rise to a moduli space $\mathscr{M}_0 = [SU(2N_f)_L \times SU(2N_f)_R]/SU(2N_f)$. In this case, there is no calculation to do: each point in $\mathscr{M}_0$ breaks the $SU(2)$ gauge group completely. As described in the main text, the true moduli space of the theory is $\mathscr{M}_{\text{strong}}$ defined in (5.13). We have $\dim \mathscr{M}_0 - \dim \mathscr{M}_{\text{strong}} = 3$, with this difference accounted for by the Higgs mechanism which means that three pions are eaten when $SU(2)$ is broken. This simple counting means that there are no pseudo-Goldstone bosons in this case and no potential over $\mathscr{M}_0$ is generated.

The regime $\Lambda_{\text{weak}} \gg \Lambda_{\text{strong}}$ is more involved. When the $SU(2)$ gauge group becomes strong, the resulting condensate (5.14) allows for a number of different symmetry breaking patterns. These include $SU(3)_{\text{strong}} \to SU(2)_{\text{strong}}$, and $SU(3)_{\text{strong}} \to \varnothing$. We show here that the former symmetry breaking pattern is a (local) minimum of the potential. In Appendix 5.B.2 we show that putative vacua in which $SU(3)_{\text{strong}}$ is completely broken have a tachyon and are unstable.

We denote the fermions as

$$\psi_{mi} = \left(q_{Li}^1, q_{Li}^3, q_{Li}^2, l_{Li}\right) \quad \text{with } m = 1,2,3,4 \text{ and } i = 1,\dots N_f$$

(Note that the colour components q^3 and q^2 are exchanged compared to the main text. This doesn't change the conclusions, but makes some of the generators below a little simpler.) If we ignore the $SU(3)_{\text{strong}}$ gauge fields, we have a moduli space of vacua given by

$$\mathscr{M}_0 = K/H = \frac{SU(4N_f)}{Sp(2N_f)}$$

We now turn on the $SU(3)_{\text{strong}}$ gauge fields. We will show that the flavour diagonal ground state

$$\langle \psi_{mi} \cdot \psi_{nj} \rangle \sim \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix}_{mn} \delta_{ij} \tag{5.52}$$

is a minimum of the resulting potential.

It is trivial to show that this vacuum is an extremum of the potential, with the generators obeying (5.46); this follows from the flavour-diagonal nature and the fact that there is no vacuum alignment problem for a single generation. It remains to show that the masses (5.47) of the pseudo-Goldstone bosons are non-tachyonic. For this, we will need explicit expressions for the generators of $G_2 = SU(3)_{\text{strong}}$ and $X \in SU(4N_f)/Sp(2N_f)$.

First the gauge generators. Since the vacuum (5.52) breaks $SU(3)_{\text{strong}} \longrightarrow SU(2)_{\text{strong}}$, it makes sense to classify the generators in terms of their representation under $SU(2)_{\text{weak}}$. They decompose as $\mathbf{8} \longrightarrow \mathbf{3} \oplus 2(\mathbf{2}) \oplus \mathbf{1}$. The triplet

$$G_{\mathbf{3}}^{\alpha} = \frac{1}{\sqrt{2N_f}} \sigma^{\alpha} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbf{1}_{N_f} \quad \alpha = 1,2,3$$

are the generators of the unbroken $SU(2)_{\text{strong}}$ where, here and below, the generators are normalised as $\operatorname{Tr} G^{\alpha} G^{\beta} = \delta^{\alpha\beta}$. The two pairs of generators transforming in the doublet are

$$G_{\mathbf{2}}^{\alpha} \in \left\{ \frac{1}{\sqrt{2N_f}} \left(\begin{array}{cc|cc} & 1 & & \\ 1 & & & \\ \hline & & & \\ & & & \end{array}\right) \otimes \mathbf{1}_{N_f}, \frac{1}{\sqrt{2N_f}} \left(\begin{array}{cc|cc} & -i & & \\ i & & & \\ \hline & & & \\ & & & \end{array}\right) \otimes \mathbf{1}_{N_f} \right\},$$

$$\left\{ \frac{1}{\sqrt{2N_f}} \left(\begin{array}{cc|cc} & & & 0 \\ & & 1 & \\ \hline & 1 & & \\ 0 & & & \end{array}\right) \otimes \mathbf{1}_{N_f}, \frac{1}{\sqrt{2N_f}} \left(\begin{array}{cc|cc} & & & 0 \\ & & -i & \\ \hline & i & & \\ 0 & & & \end{array}\right) \otimes \mathbf{1}_{N_f} \right\}$$

Finally the singlet is given by

$$G_{\mathbf{1}} = \frac{1}{\sqrt{6N_f}}(1,-2,1,0)\otimes \mathbf{1}_{N_f}$$

All gauge generators are singlets under the unbroken $SO(N_f)$ flavour group.

Next, the (pseudo)-Goldstone modes. Under the original symmetry breaking $SU(4N_f) \longrightarrow Sp(2N_f)$, the broken generators transform in the traceless antisymmetric rank-2 tensor representation of $Sp(2N_f)$, denoted by $\mathscr{A}$. We have

$$\dim \mathscr{A} = (2N_f-1)(4N_f+1)$$

After gauging $SU(3)_{\text{colour}}$, the global group $H = Sp(2N_f)$ is broken to

$$Sp(2N_f) \longrightarrow SU(2)_{\text{strong}} \times SO(N_f)\times U(1)_{\hat{V}}$$

Under this decomposition, the branching rule for the anti-symmetric representation $\mathscr{A}$ reads

$$\mathscr{A} \longrightarrow 2\,(\mathbf{2},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{1})_0 \oplus 2\,(\mathbf{2},S)_0 \oplus (\mathbf{1},S)_0 \oplus (\mathbf{1},S\oplus A)_0 \oplus (\mathbf{3},A)_0 \oplus (\mathbf{2},A)_{\pm 3} \oplus (\mathbf{1},A)_{\pm 6}$$

where S and A are the traceless symmetric and the antisymmetric rank-2 tensor representation of $SO(N_f)$, respectively; they have dimensions

$$\dim S = \frac{N_f}{2}(N_f+1)-1 \quad \text{and} \quad \dim A = \frac{N_f}{2}(N_f-1)$$

The five generators sitting in singlet representations of $SO(N_f)$, namely $2(\mathbf{2},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{1})_0$ are the only generators that remain in the case $N_f = 1$. These are the five Goldstone modes that become the longitudinal modes of the massive gauge bosons as $SU(3)_{\text{strong}} \to SU(2)_{\text{strong}}$.

We need explicit forms for the remaining generators. This is aided by the observation that, for the condensate (5.52) with symmetry breaking pattern $SU(4N_f) \to Sp(2N_f)$, the unbroken and broken generators take the form

$$H = \left(\begin{array}{c|c} A & B \\ \hline B^T & -A^T \end{array}\right) \quad \text{and} \quad X = \left(\begin{array}{c|c} C & D \\ \hline D^T & C^T \end{array}\right)$$

with A Hermitian, B symmetric, C traceless Hermitian and D anti-symmetric. In their full glory, the broken generators are:

- The pair of $(\mathbf{2},S)_0$ representations are generated by matrices of the form

$$X_{(\mathbf{2},S)} = \frac{1}{2\sqrt{N_f}}\left(\begin{array}{cc|cc} & z & & \\ z^* & & & \\ \hline & & & z^* \\ & & z & \end{array}\right)\otimes S \quad \text{and} \quad \frac{1}{2\sqrt{N_f}}\left(\begin{array}{cc|cc} & & & -z^* \\ & & z^* & \\ \hline & z & & \\ -z & & & \end{array}\right)\otimes S$$

with S a traceless, symmetric matrix and $z \in \{1, i\}$.

- The representation $(\mathbf{1},S)_0$ is generated by matrices of the form

$$X_{(\mathbf{1},S)} = \frac{1}{2\sqrt{N_f}}\operatorname{diag}(1,-1,1,-1)\otimes S$$

- The representation $(\mathbf{1},S\oplus A)_0$ is generated by matrices of the form

$$X_{(\mathbf{1},S\oplus A)} = \frac{1}{\sqrt{2N_f}}\operatorname{diag}(0,1,0,1)\otimes L$$

with L a traceless, Hermitian matrix.

- The representation $(\mathbf{3},A)_0$ is generated by matrices of the form

$$X_{(\mathbf{3},A)} = \frac{1}{\sqrt{2N_f}}\sigma^i \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes A$$

with A an anti-symmetric Hermitian matrix.

- The pair of representations $(\mathbf{2},A)_{\pm 3}$ is generated by matrices of the form

$$X_{(\mathbf{2},A)} = \frac{1}{2\sqrt{N_f}}\left(\begin{array}{cc|cc} & z & & \\ z^* & & & \\ \hline & & & -z^* \\ & & -z & \end{array}\right)\otimes A \quad \text{and} \quad \frac{1}{2\sqrt{N_f}}\left(\begin{array}{cc|cc} & & & z^* \\ & & z^* & \\ \hline & z & & \\ z & & & \end{array}\right)\otimes A$$

again with $z \in \{1, i\}$.

- Finally, the pair of representations $(\mathbf{1},A)_{\pm 6}$ is generated by matrices of the form

$$X_{(\mathbf{1},A)} = \frac{1}{\sqrt{2N_f}} \left(\begin{array}{c|c} & z^* \\ \hline z & \end{array} \right) \otimes A$$

Each of the $N_f \times N_f$ matrices above is normalised such that

$$\mathrm{Tr}\, L^2 = \mathrm{Tr}\, S^2 = \mathrm{Tr}\, A^2 = N_f$$

ensuring that the generators have normalisation $\mathrm{Tr}\, X^2 = 1$. Note that in the basis given above, $(\mathbf{2},A)_{\pm 3}$ and $(\mathbf{1},A)_{\pm 6}$ are not $U(1)_{\hat{V}}$ diagonal, but they can be made diagonal in the full $SU(2)_{\mathrm{strong}} \times SO(N_f) \times U(1)_{\hat{V}}$ under a unitary change of basis.

With these explicit expressions, it is now a simple matter to compute the masses of the various generators using (5.47). We find three of the representations have mass,

$$m^2_{(\mathbf{3},A)} = \frac{g_s^2 M^2}{\pi N_f}, \quad m^2_{(\mathbf{2},A)} = \frac{g_s^2 M^2}{2\pi N_f}, \quad m^2_{(\mathbf{1},A)} = \frac{g_s^2 M^2}{6\pi N_f} \tag{5.53}$$

with g_s the gauge coupling of $SU(3)_{\mathrm{strong}}$. Each of these is positive, as is required for a stable ground state. The remaining three generators are massless

$$m^2_{(\mathbf{2},S)} = m^2_{(\mathbf{1},S)} = m^2_{(\mathbf{1},S\oplus A)} = 0 \tag{5.54}$$

Of these massless generators, $(\mathbf{1},S)_0$ and $(\mathbf{1},S\oplus A)_0$ are neutral under the $SU(2)_{\mathrm{strong}}$ gauge group and so we do not expect them to receive any further corrections. Indeed, these generators correspond to the exact Goldstone bosons of the theory. We can confirm this with some simple counting,

$$\left(\frac{1}{2}N_f(N_f+1)-1\right) + (N_f^2-1) = \frac{3}{2}N_f^2 + \frac{N_f}{2} - 2$$

which coincides $\dim \mathscr{M}'_{\mathrm{weak}}$ defined in (5.17), the expected number of exact Goldstone modes.

This leaves us with the fate of the pair of $(\mathbf{2},S)_0$ representations unaccounted for. These are not exact Goldstone bosons, so we expect that the vanishing of the mass is an artefact of working to leading order in perturbation theory; we must look to second order to see if the resulting mass-squared is positive or negative.

Second Order Corrections

We now adapt the results of Appendix 5.A.2 to determine the second order correction to the $(\mathbf{2},S)_0$ states. As explained previously, the relevant physics comes from taking into account the mass splitting of the broken $SU(3)_{\text{strong}}$ gauge generators. One key difference with the results of Appendix 5.A.2 is that now these gauge bosons have different masses.

It will be useful to describe the general case, in which the broken gauge generators sit in more than one irreducible representations of the unbroken gauge group $\tilde{G}$,

$$\mathbf{r}_{\text{broken}} = \bigoplus_{i\in I} \mathbf{r}_i$$

The masses of gauge bosons in each representation $\mathbf{r}_i$ will, in general, differ. We denote these masses as μ_i^2. The analysis of Appendix 5.A.2 then proceeds, with the final result (5.51) replaced by

$$m_X^2 \approx -\frac{3g_s^2\lambda^2}{(4\pi)^2 f_\pi^2}\sum_{i\in I}\left(\sum_\alpha \mathscr{C}_X(G^\alpha_{\mathbf{r}_i})\right)\left(\frac{\mu_i^2}{M_H^2}\log\frac{M_H^2}{\mu_i^2} - \frac{\mu_i^2}{M_X^2}\log\frac{M_X^2}{\mu_i^2}\right) \tag{5.55}$$

Note that, in contrast to (5.51), we are now summing over the *broken* generators, rather than the unbroken generators. This is compensated by the overall minus sign that sits in front. The fact that $M_X^2 > M_H^2$ ensures that the log terms are positive definite. To ensure stability, we now need that the group theory factor is negative, to cancel the overall minus sign.

Our example of interest has $\tilde{G} = SU(2)$ and the broken generators sit in $\mathbf{1}\oplus\mathbf{2}\oplus\mathbf{2}$. The masses of the corresponding gauge bosons are given by

$$\mu_{\mathbf{1}}^2 = \frac{2}{3}g_s^2 f_\pi^2 \quad \text{and} \quad \mu_{\mathbf{2}}^2 = \frac{1}{2}g_s^2 f_\pi^2$$

We should then apply (5.55) to the worrisome pseudo-Goldstone mode $X = (\mathbf{2},A)$. Happily it turns out that there are no tricky cancellations between group theory factors; instead one finds

$$\sum_{\alpha=1}^{4}\mathscr{C}_X(G^\alpha_{\mathbf{2}}) = -1 \quad \text{and} \quad \mathscr{C}_X(G_{\mathbf{1}}) = -\frac{1}{2}$$

The fact that each of these is negative, means that they both contribute positively to the mass m_X^2. We see the vacuum (5.52) remains stable at second order.

5.B.2 Unstable Vacua

We have shown that the flavour-diagonal condensate (5.52) is a local minimum of the potential. We have not, however, shown that it is global minimum.

This, it appears, is a challenging problem. The moduli space is large, and there may be many saddle points. However, the simple observation that the potential (5.44) is proportional to the sum of the W-boson masses means that those ground states which break the gauge symmetry the least are favoured. For this reason, it seems likely that the flavour-diagonal vacuum (5.52) is, in fact, the true ground state of the system.

In this appendix, we give some calculations to back up this intuition. We have not been able to find other local minima of the potential. Instead, we will show that a number of obvious candidates for ground states have tachyonic modes and so are unstable. We work with $N_f = 2$ and give two examples of putative ground states, each with different symmetry breaking patterns, which turn out to be saddle points.

For the first example, consider a condensate of the form

$$\langle q^a_{1L} \cdot q^b_{2L} \rangle \sim \Lambda^3_{\text{weak}} \delta^{ab} \quad , \quad \langle l_{1L} \cdot l_{2L} \rangle \sim \Lambda^3_{\text{weak}} \tag{5.56}$$

where $a = 1, 2, 3$ is the $SU(3)_{\text{strong}}$ colour index, and the 1,2 labels on the quarks and leptons refer to flavour. With such a condensate, the gauge groups breaks as

$$SU(3)_{\text{strong}} \longrightarrow SO(3)_{\text{strong}}$$

If we choose to order the 8 Weyl spinors as $\psi = \left(q^1_{1L}, q^2_{1L}, q^3_{1L}, l_{1L} | q^1_{2L}, q^2_{2L}, q^3_{2L}, l_{2L}\right)^{\text{T}}$, then the $SU(3)_{\text{strong}}$ generators act as

$$G^\alpha = \frac{1}{2} \left(\begin{array}{cc|cc} \lambda^\alpha & & & \\ & 0 & & \\ \hline & & \lambda^\alpha & \\ & & & 0 \end{array} \right)$$

with λ^α are the usual 3×3 Gell-Mann matrices. It is straightforward to show that the broken and unbroken generators obey (5.46), so this condensate is at least a saddle point of the potential. However, one finds that this condensate has a higher energy than (5.52). More importantly, there are also tachyonic modes. One of this is associated to the would-be

Goldstone bosons transforming in the octet of $SU(3)_{\rm strong}$

$$X_{\mathbf{8}}^{\alpha} = \frac{1}{2}\left(\begin{array}{cc|cc} \lambda^{\alpha} & & & \\ & 0 & & \\ \hline & & \lambda^{\alpha\,\mathrm{T}} & \\ & & & 0 \end{array}\right)$$

The mass matrix for these 8 generators can be found using (5.47); it is diagonal, and given by

$$m^2_{\alpha\beta} = -\frac{g_s^2 M^2}{2\pi}\,(0,1,0,0,1,0,1,0)$$

The overall minus sign means that this vacuum is unstable.

As a second example, we consider the condensate that arises in the single flavour case, but now with two flavours which align differently within the $SU(3)_{\rm strong}$ gauge group. For the first generation we take,

$$\langle q^1_{1L}\cdot q^2_{1L}\rangle \sim \Lambda^3_{\rm weak} \quad , \quad \langle l_{1L}\cdot q^3_{1L}\rangle \sim \Lambda^3_{\rm weak} \tag{5.57}$$

which picks out the $a=3$ colour direction. Meanwhile, for the second generation we take

$$\langle q^1_{2L}\cdot q^3_{2L}\rangle \sim \Lambda^3_{\rm weak} \quad , \quad \langle l_{2L}\cdot q^2_{2L}\rangle \sim \Lambda^3_{\rm weak} \tag{5.58}$$

which picks out the $a=2$ colour direction. The combination of both condensates breaks the $SU(3)_{\rm strong}$ gauge group completely.

To compute the masses, it is simplest to note that the condensate is related to (5.52) by a permutation of the ψ components. There are two ways to proceed; we could fix the action of the gauge generators G^{α} on ψ, in which case the permutation acts as conjugation on the broken generators X defined in Appendix 5.B.1. Alternatively, we could fix the action of the unbroken generators, in which case the permutation acts by conjugation of G. In either case, a simple calculation shows that the condensates (5.57) and (5.58) are indeed a saddle point of the potential, but with energy higher than both the local minimum (5.52) and the unstable vacuum (5.56).

It is a little more involved to demonstrate that the condensates (5.57) and (5.58) are unstable and we refrain from giving all the details. Because the gauge group is broken completely, many of the broken generators X given in Appendix 5.B.1 now mix. Diagonalising the resulting mass matrix, one finds that there are massive modes, massless modes and, crucially, two tachyonic modes. This vacuum is unstable.

5.B.3 Adding Hypercharge

We now repeat the calculation of Appendix 5.B.1 in the presence of a $U(1)_Y$ hypercharge interaction. This corresponds to an additional gauge generator which, in the notation of Appendix 5.B.1, takes the form

$$G_Y = \frac{1}{2\sqrt{3N_f}}(1,1,1,-3) \otimes \mathbf{1}_{N_f}$$

The unfamiliar normalisation factor ensures that this generator obeys $\mathrm{Tr}\, G_Y^2 = 1$.

There is a simple generalisation of the mass formula (5.47) in which the different gauge generators are summed over, weighted with their gauge couplings. We denote the gauge coupling associated to $U(1)_Y$ as g_Y. Then the masses of the pseudo-Goldstone bosons (5.53) are replaced by

$$m^2_{(\mathbf{3},A)_0} = \frac{g_s^2}{\pi N_f} M^2, \quad m^2_{(\mathbf{2},A)_{\pm 3}} = \frac{g_s^2 M^2}{2\pi N_f}, \quad m^2_{(\mathbf{1},A)_{\pm 6}} = (g_s^2 + 2g_Y^2)\frac{M^2}{6\pi N_f}$$

These modes are not destabilised by the hypercharge interaction. Meanwhile, the massless modes (5.54) remain massless at leading order,

$$m^2_{(\mathbf{2},S)_0} = m^2_{(\mathbf{1},S)_0} = m^2_{(\mathbf{1},S\oplus A)_0} = 0$$

The $(\mathbf{1},S)_0$ and $(\mathbf{1},S\oplus A)_0$ states remain as exact Goldstone bosons as previously. There is, however, a correction to the second-order mass of the $(\mathbf{2},S)_0$ states due to hypercharge. To see this, we need the usual mixing of the generators $G_{\mathbf{1}}$ and G_Y which yield the Z-boson and the photon. The mass-matrix for the corresponding gauge fields is

$$\mu^2 = \frac{f_\pi^2}{3}\begin{pmatrix} 2g_s^2 & \sqrt{2}g_s g_Y \\ \sqrt{2}g_s g_Y & g_Y^2 \end{pmatrix}$$

which has eigenvalues

$$\mu_Z^2 = \left(2g_s^2 + g_Y^2\right)\frac{f_\pi^2}{3} \quad \text{and} \quad \mu_\gamma^2 = 0$$

Correspondingly, the generators mix and take the form

$$G_Z = \frac{1}{\sqrt{2g_s^4 + g_Y^4}}\left(\sqrt{2}g_s^2 G_{\mathbf{1}} + g_Y^2 G_Y\right), \quad G_\gamma = \frac{1}{\sqrt{3}}\left(\sqrt{2}G_Y - G_{\mathbf{1}}\right)$$

Armed with these results, we can now revisit the calculation of Appendix 5.B.1. At leading order, only the mass of $(\mathbf{1},A)$ is modified by the hypercharge to

$$m^2_{(\mathbf{1},A)} = \frac{(g_s^2 + 2g_Y^2)M^2}{6\pi N_f}$$

While the mass of $(\mathbf{2},S)$ is modified by replacing the contribution from $G_\mathbf{1}$ in (5.55) by the contribution from the Z-boson, which is

$$\mathscr{C}(G_Z) = -\frac{g_s^4 + 2g_s^2 g_Y^2}{2g_s^4 + g_Y^4}$$

The group theoretic factor remains negative and the mass remains positive.

The upshot of these short calculations is that hypercharge does not destabilise the vacuum. Indeed, it is clear from the calculations above why this is: both the chosen vacuum, and the hypercharge, are flavour-diagonal. The analysis of [107, 112] shows that the flavour-diagonal vacuum is likely to be destabilised only by the introduction of $U(1)$ gauge symmetry under which different generations carry different charges.

5.B.4 Vacuum Alignment for $Sp(r) \times SU(N)$

The analysis of vacuum alignment for $Sp(r) \times SU(N)$ follows that of Section 5.B.1; only the group theory is a little more involved.

Before we turn on the $SU(N)_{\text{strong}}$ gauge symmetry, the chiral condensate induces the $K \to H$ symmetry breaking pattern expected of a pseudo-real representation,

$$SU((N+1)N_f) \longrightarrow Sp((N+1)N_f/2)$$

We now gauge $SU(N)_{\text{strong}}$. We postulate that vacuum is again formed by a flavour-diagonal condensate, under which the gauge group is broken to

$$SU(N)_{\text{strong}} \longrightarrow Sp(\nu)_{\text{strong}} \quad \text{with } \nu = \frac{N-1}{2}$$

More generally, as explained in Section 5.3, the surviving global symmetry H is broken to

$$H = Sp((N+1)N_f/2) \longrightarrow Sp(\nu)_{\text{strong}} \times SO(N_f) \times U(1)_{\hat{V}} \tag{5.59}$$

As before, it will prove useful to decompose all generators into representations of this unbroken group. The calculation is very similar to that in Appendix 5.B.1 apart from one complication arising from the presence of the traceless antisymmetric tensor representation of $Sp(\nu)_{\text{strong}}$ which is absent when $\nu = 1$.

The $SU(N)_{\text{strong}}$ generators decompose as

$$\mathbf{ad} \longrightarrow \hat{\mathscr{S}} \oplus \hat{\mathscr{A}} \oplus 2\left(\hat{\mathscr{F}}\right) \oplus \mathbf{1} \tag{5.60}$$

where we use the hat to distinguish these $Sp(\nu)_{\text{strong}}$ representations from similar $SO(N_f)$ representations that we will meet below. Here $\hat{\mathscr{S}}$ is the symmetric tensor representation with dimensions $\nu(2\nu+1)$, $\hat{\mathscr{A}}$ is the traceless antisymmetric tensor representation with dimensions $(\nu-1)(2\nu+1)$, and $\hat{\mathscr{F}}$ is the fundamental representation $\mathbf{2\nu}$. All are singlets under $SO(N_f)$ and neutral under $U(1)_{\hat{V}}$.

The (pseudo)-Goldstone modes again sit in the traceless antisymmetric rank-2 tensor representation $\mathscr{A}$ of $Sp((N+1)N_f/2)$. Under (5.59), the branching rules are

$$\begin{aligned} \mathscr{A} \longrightarrow & (\hat{\mathscr{A}},\mathbf{1})_0 \oplus 2(\hat{\mathscr{F}},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{1})_0 \oplus (\hat{\mathscr{A}},S)_0 \oplus 2(\hat{\mathscr{F}},S)_0 \oplus (\mathbf{1},S)_0 \oplus (\mathbf{1},S\oplus A)_0 \\ & \oplus (\hat{\mathscr{S}},A)_0 \oplus (\hat{\mathscr{F}},A)_{\pm N} \oplus (\mathbf{1},A)_{\pm 2N} \end{aligned}$$

Now we can track the fate of each of these representations.

The representations $(\hat{\mathscr{A}},\mathbf{1})_0$, $(\hat{\mathscr{F}},\mathbf{1})_0$, and $(\mathbf{1},\mathbf{1})_0$, each of which is a singlet under $SO(N_f)$, are eaten by the Higgs mechanism, and absorbed as longitudinal modes of the massive gauge bosons that arise when $SU(N)_{\text{strong}}$ is broken to $Sp(\nu)_{\text{strong}}$.

The representations $(\mathbf{1},S)_0$ and $(\mathbf{1},S\oplus A)_0$, which are both singlets under the unbroken $Sp(\nu)_{\text{strong}}$ gauge group, are exact Goldstone bosons. They parameterise the moduli space $\mathscr{M}_{\text{weak}} = SU(N_f)_q \times SU(N_f)_l/SO(N_f)$ of the theory defined in (5.38).

The remaining representations are pseudo-Goldstone bosons. At leading order, we can compute their masses using the formula (5.47). However, the full calculation is a little involved, so here we only quote the final result to aid the readers:

$$m^2_{(\mathbf{1},A)} = \frac{N-1}{2N}\frac{g_s^2M^2}{2\pi N_f} \quad , \quad m^2_{(\hat{\mathscr{S}},A)} = \frac{g_s^2M^2}{\pi N_f} \quad , \quad m^2_{(\hat{\mathscr{F}},A)} = \frac{5N-7}{2(N-1)}\frac{g_s^2M^2}{4\pi N_f}$$

Meanwhile, at leading order, two of the pseudo-Goldstone bosons remain massless,

$$m^2_{(\hat{\mathscr{A}},S)} = m^2_{(\hat{\mathscr{F}},S)} = 0$$

We write these as $m^2_{\mathscr{A}}$ and $m^2_{\mathscr{F}}$ for short. As in previous examples, to see whether these massless modes will destabilise the vacuum we need to look at higher order.

The broken gauge generators sit in the $\hat{\mathscr{A}}$, $\hat{\mathscr{F}}$ and singlet representations in (5.60). The masses of the corresponding gauge bosons are given by

$$\mu_{\mathscr{A}}^2 = g_s^2 f_\pi^2, \quad \mu_{\mathscr{F}}^2 = g_s^2 \frac{1}{2} f_\pi^2, \quad \mu_{\mathbf{1}}^2 = \frac{N+1}{2N} g_s^2 f_\pi^2$$

We now use the formula (5.55); each of the $X = (\hat{\mathscr{A}}, S)$ and $X = (\hat{\mathscr{F}}, S)$ pseudo-Goldstone modes get mass at second order, given by

$$m_X^2 \approx -\frac{3 g_s^2 \lambda^2}{(4\pi)^2 f_\pi^2} \sum_{i \in \{\mathscr{A}, \mathscr{F}, \mathbf{1}\}} \left(\sum_\alpha \mathscr{C}_X (G_{\mathbf{r}_i}^\alpha) \right) \left(\frac{\mu_i^2}{M_H^2} \log \frac{M_H^2}{\mu_i^2} - \frac{\mu_i^2}{M_X^2} \log \frac{M_X^2}{\mu_i^2} \right)$$

Once again, each of the representations contributes a positive contribution to the mass. For the $X = (\hat{\mathscr{A}}, S)$ state, we have

$$\sum_\alpha \mathscr{C}_{(\hat{\mathscr{A}}, S)} (G_{\mathscr{A}}^\alpha) = -(N-5) \quad , \quad \sum_\alpha \mathscr{C}_{(\hat{\mathscr{A}}, S)} (G_{\mathscr{F}}^\alpha) = -\frac{5N-13}{N-3} \quad , \quad \mathscr{C}_{(\hat{\mathscr{A}}, S)} (G_{\mathbf{1}}) = 0$$

Note that there is no $\mathscr{A}$ representation when $N = 3$, and these formulae hold only for $N \geq 5$. Meanwhile, for the $X = (\hat{\mathscr{F}}, S)$ state, we have

$$\sum_\alpha \mathscr{C}_{(\hat{\mathscr{F}}, S)} (G_{\mathscr{A}}^\alpha) = -\frac{N-3}{2} \quad , \quad \sum_\alpha \mathscr{C}_{(\hat{\mathscr{F}}, S)} (G_{\mathscr{F}}^\alpha) = -\frac{3N-5}{2(N-1)} \quad , \quad \mathscr{C}_{(\hat{\mathscr{F}}, S)} (G_{\mathbf{1}}) = -\frac{1}{N-1}$$

The fact that each of these is negative ensures that the masses of the pseudo-Goldstone bosons are positive and the vacuum is stable. One can further show that the vacuum is not destabilised by the addition of hypercharge.

Chapter 6

Conclusion and Outlook

Our theoretical understanding of elementary particle physics has been stagnant for several years. Hence, it is imperative to get hints of what lies beyond in whatever way possible. One way forward is to learn about various constraints placed on the Standard Model due to its Nature as a gauge field theory and how to utilise them to construct similar theories. Another is to explore other phases of the theory in the parameter space other than that of our universe. These are the two directions that this dissertation has offered to explore, first by re-examining both the perturbative and global anomalies in the Standard Model, and second by investigating the phase diagram of the Standard Model while dialling the strengths of the weak and the strong nuclear force.

It is challenging to construct anomaly-free gauge theories. The fact that the Standard Model, with its complicated matter content, is one of them is therefore rather striking and demands a more in-depth explanation. The deeper explanation comes in the form of gravity. Physicists have recognised for an almost equally long time that if we require the mixed hypercharge-gravitational anomaly, as well as all the other gauge anomalies, to vanish, then the hypercharge assignment for the fermions in the Standard Model must be as is given in Nature.

It is still gravity in our different take on the subject that offers an additional constraint on the matter content. The starting point of our investigation in Chapter 2 is to assume that the gauge group $U(1) \times SU(2) \times SU(3)$ is compact. In particular, the hypercharge of each particle must be an integral multiple of an elementary charge, that is, it must be quantised. We can then resolve all the gauge anomaly cancellation conditions to obtain a single cubic equation in three variables of the form

$$x^3 + y^3 = z^3. \tag{6.1}$$

Hypercharge quantisation forces us to find a solution (x,y,z) in $\mathbb{Z}^3$. Remarkably, Fermat's Last Theorem informs us right away that the only solutions are the trivial ones where one of the variables vanishes. These solutions coincide with the actual hypercharge assignment of the Standard Model.

It is a remarkable fact that hypercharge quantisation and gauge anomaly cancellation give the essentially unique hypercharge assignment of the the Standard Model fermions that is consistent with gravity. There are many ways to interpret this. For example, one could say that consistency with gravity requires the unification of the gauge group, for the most natural way to obtain a compact abelian gauge group is by embedding the Standard Model gauge group in a bigger compact non-abelian gauge group like $SU(5)$ GUT. Others might interpret this as unsurprising since it is believed that consistency with quantum gravity requires abelian symmetry groups to be compact [109, 30]. The Standard Model is just a special case for which a unique hypercharge assignment can be obtained from consistency with gravity either through hypercharge quantisation or through gauge-gravitational anomaly cancellation. Whether this special property of the Standard Model has some bearing in the search for new physics is still unclear and requires further investigation.

We would be construed as naive if we only studied one side of the coin, that is if we studied local anomalies without touching global anomalies. We would like to know whether there is any subtler global anomaly that is not captured by Witten's mapping torus argument, given our modern knowledge that we can describe global anomalies in terms of the eta-invariant and cobordism. In this new framework, we expressed the fermion path integral as the formal determinant of the Dirac operator. The phase of the path integral is then the eta-invariant η_Y associated with Y, the extension of our spacetime in 5 dimensions. The eta-invariant η_Y is defined to be half the regularised version of the number of positive modes minus the number of negative modes of the Dirac operators on Y, with all gauge fields and metric extended appropriately. There is no anomaly when our result does not depend on the choice of the extension Y. This condition translates into the statement that

$$\exp\left(2\pi i \eta_{\bar{Y}}\right) = 1, \tag{6.2}$$

for any closed 5-manifold $\bar{Y}$. When all perturbative anomalies are gone, the exponentiated eta-invariant $\exp(2\pi i \eta_{\bar{Y}})$ is a bordism invariant. What this means is that the exponentiated eta-invariant is trivial when the closed 5-manifold $\bar{Y}$ is a boundary of a 6-manifold, with all the relevant structures like the spin structure, the gauge bundle, and the metric, extended appropriately. In a more mathematical term, the exponentiated eta-invariant is a homomorphism from the bordism group with appropriated structures into $U(1)$.

We use the Atiyah-Hirzebruch spectral sequence as our main tool to compute the relevant bordism groups, with some help from the Serre spectral sequence to work out the unknown homology and cohomology groups. As we would like to define a fermion in our theory, the appropriate extra structure is the spin structure. The choices for the gauge group of the Standard Model consistent with the Lie algebra and the matter content are

$$G_n = \frac{U(1)\times SU(2)\times SU(3)}{\mathbb{Z}_n}, \quad n = 1,2,3,6. \tag{6.3}$$

For each choice of the gauge group, we find in Chapter 3 that the fifth spin-bordism group is given by

$$\Omega_5^{\mathrm{Spin}}(BG_1) = \Omega_5^{\mathrm{Spin}}(BG_3) = \mathbb{Z}_2, \quad \Omega_5^{\mathrm{Spin}}(BG_2) = \Omega_5^{\mathrm{Spin}}(BG_6) = 0. \tag{6.4}$$

In other words, there is a mod 2 global anomaly for G_1 and G_3 but not for G_2 and G_6. The mod 2 anomaly here can be identified with the usual Witten anomaly in the gauge group $SU(2)$. In other examples involving gauge groups for various Beyond Standard Model (BSM) theories, we only obtain the Witten $\mathbb{Z}_2$ anomaly associated with a factor of $SU(2)$ or $\mathrm{Sp}(n)$ in the guage group at most.

The curious absent of the global anomaly when the gauge group factor $U(1)\times SU(2)$ is replaced by $U(2)$ prompt us to dive deeper and investigate in Chapter 4 what is going on, since practically we still impose the condition that chiral fermions must come in pairs in the fundamental representation of $SU(2)$. More generally, fermions in the isospin $2r+1/2$ representation of $SU(2)$, with r a non-negative integer, contribute to the mod 2 Witten anomaly. Therefore, the necessary condition for the global anomaly to vanish is to have an even number of fermions in these representations. In a $U(2)$ gauge theory, we find that exactly the same necessary condition still applies. However, the condition is derived from the cancellation of the mixed anomaly between the $U(1)$ factor and $SU(2)$ factor inside $U(2)$ rather than from a global anomaly consideration. The reason we have exactly the same condition is because the $SU(2)$ gauge transformation that reveals the Witten anomaly for isospin $2r+1/2$ chiral fermions is identified with a $U(1)$ transformation when $SU(2)$ is embedded inside $U(2)$.

Recently, a different kind of global anomalies has been discovered by Wang, Wen, and Witten [143]. This new anomaly again occurs in an $SU(2)$ gauge theory, but only visible when the theory is defined on a non-spin manifold. Instead, fermions are defined using the spin-SU(2) structure which can be given to any orientable closed 4-manifold. Suppose we couple a fermion with isospin s to this gauge theory, Then, in a certain background gauge field, the fermion path integral changes sign under a symmetry transformation that is a result of a gauge transformation combined with a diffeomorphism if and only if $s = 4r + 3/2$ where r is a non-negative integer. In the cobordism framework, the existence of this extra anomaly can be seen from the fact that

$$\Omega_5^{\mathrm{Spin}-SU(2)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2, \tag{6.5}$$

instead of just $\mathbb{Z}_2$. It is then natural to ask whether this new $SU(2)$ anomaly can be shifted to the mixed anomaly with the $U(1)$ factor when the theory is defined using the spin-U(2) instead. The answer to this question is double-layered and more complicated than the ordinary $U(2)$ case. Given that we do not have any fermions in the isospin $2r + 1/2$ of the $SU(2)$ factor. We first find that in order to cancel the mixed anomaly between the $U(1)$ factor and $SU(2)$, as well as the mixed gauge-gravitational anomaly, there must be an even number of fermions in the isospin $4r + 3/2$ representations. Therefore, we can detect no global anomaly as long as we cancel the local ones. On the other hand, we use the Adam spectral sequence to calculate that

$$\Omega_5^{\mathrm{Spin}-U(2)} \cong \mathbb{Z}_2, \tag{6.6}$$

which strongly hints us that the new $SU(2)$ anomaly must be temporarily masked by the local anomaly cancellation, but does not equate itself to a certain combination of local anomalies. Indeed, this is the case. It is possible to construct a $\mathrm{Spin} - U(2)$ gauge theory with an anomalous fermion content together with an appropriate topological Wess-Zumino term that exactly absorbs the perturbative anomalies from the fermions. In such a theory, the perturbative anomalies all vanish by construction, leaving us to see the new $SU(2)$ anomaly in the clear.

In our investigation of gauge anomalies in the Standard Model from the framework of cobordism of Chapter 3, we have only looked at one aspect of constraints from anomalies, that of searching for subler global gauge anomalies. We have, however, ignored a potentially very interesting prospect when discrete symmetries are brought into discussion. It has been discussed in the litereature (for example, in Reference [74]) that if there is a $\mathbb{Z}_4$ discrete symmetry whose order 2 element can be identified with the Fermion number operator $(-1)^{\mathsf{F}}$, then a more refined anomaly is found to be $\mathbb{Z}_{16}$. Physically, it means we can give a mass to a set of 16 Majorana fermions without breaking a certain chiral symmetry in the theory. The Standard Model has exactly 16 Majorana fermions per generation, and this prospect of *symmetric mass generation* has excited physicists from across the board, ranging from condensed matter physicists to lattice field theorists. This will be a really exciting direction to extend to in a future work.

Another interesting aspect that is worth looking into is how general the anomaly interplay is. In Chapter 4, we see how the finite type anomaly in a theory with gauge group $SU(2) \subset U(2)$ is deduced from the perturbative anomaly cancellation in a theory with gauge group $U(2)$. Similar anomaly interplay can also be seen in Section 4.3 of [74] where the $\mathbb{Z}_{16}$ finite type anomaly of a theory with a $\text{spin}_{\mathbb{Z}_4} \subset \text{spin}_c$ structure is similarly obtained from the perturbative anomaly cancellation in a theory with a spin_c structure. In both examples, the bordism group with the bigger structure vanishes so one only has to deal with the local anomalies. Further work is needed to find more examples like these two, and to investigate whether it can lead to a new way to obtain bounds on finite type anomalies when the relevant bordism groups are too difficult to be calculated directly.

Having studied more or less exhaustively the restriction imposed on the Standard model from anomalies, the picture would be incomplete if we stopped there and not use them to study properties of the theory in the IR. However, due to the complexity of strongly coupled field theories, even a tool as powerful as the anomaly cancellation and 't Hooft anomaly matching are not enough to extract useful information regarding the phase space of the theory at low energy, and we have to utilise other technique to get a fuller picture.

The parameter space of the gauge theory underlying the Standard Model is vast, ranging from the gauge couplings to the Yukawa couplings, from the number of fermion generations to the theta angles. Only a small corner of this space is explored in Chapter 5: the weak and strong gauge couplings are varied with respect to each other while the Yukawa couplings are insensitive to different generations. Even so, we obtain interesting results where phase transition does not occur when there is only one generation. Leptons transmute smoothly into quarks as we gradually tuning the strength of the weak force higher and higher compared to the strong force.

More precisely, when the dynamically generated scale Λ of the strong $SU(3)$ gauge group is much larger than the weak scale Λ of $SU(2)$, quarks form a condensate, breaking the remaining gauge group $U(1) \times SU(2)$ spontaneously down to $U(1)_{\text{em}}$ electromagnetism. The global symmetry is the vector $U(1)_{\text{B}-\text{L}}$. The remaining massless fermion is the gauge-neutral left-handed neutrino of charge -3 under $U(1)_{\text{B}-\text{L}}$. On the other extreme of the phase diagram when $\Lambda_{\text{strong}} \ll \Lambda_{\text{weak}}$, the $SU(2)$ gauge group confines first before the effect of the $SU(3)$ kicks in. The left-handed fermions form condensates, breaking $SU(3)$ down to $SU(2)$, as well as breaking $U(1)_Y$ and $U(1)_{\text{B}-\text{L}}$ completely. However, there is a different combination of $U(1)_Y$ (and $U(1)_{\text{B}-\text{L}}$) with an $SU(3)$ generator that remain unbroken by the condensates, leaving a different $U(1)$ gauge group and a different $U(1)$ global symmetry group in the IR which we denote by $\hat{U}(1)_{\text{em}}$ and $\hat{U}(1)_{\text{B}-\text{L}}$, respectively. Subsequently, $SU(2)$ confines and most of the remaining fermions condense. At the end of the day, the only massless fermion left is a colour component of the right-handed down quark. It is neutral under the $\hat{U}(1)_{\mathrm{em}}$ gauge group while charged under the global $\hat{U}(1)_{\mathrm{B-L}}$ with charge $+3$. Taking into account the opposite helicities of the two remaining massless fermions in the IR on either side, we find that they cannot be distinguished in any meaningful way.

On the other hand, phase transition definitely happens when more generations of fermions are present. However, we could not determine the order of the transition within the current framework. From the point of view of Landau's paradigm, the transition must be of first order because the symmetry group on one side of the phase diagram cannot be embedded as a subgroup of the symmetry group on the other side, and vice versa.

However, developments in condensed matter physics over the past few years suggest that this might not be the only choice that could occur, especially when there is a gauge theory lurking in the background. The symmetry groups on either side can be different subgroup of some bigger symmetry group (for example, the global symmetry group of the UV physics). The phase transition is not first order but becomes a critial point at which extra gauge degrees of freedom emerge. This is known as *deconfined criticality* [123]. The theory we look at fits the conditions so we cannot rule out this possibility at this stage. It would be interesting to see if it is actually realised and, if so, how the extra gauge symmetry can be described in terms of what we know from the UV theory and the effective descriptions of the IR theory at either extreme of the phase diagram.

Lastly, it must be emphasised that in this investigation of the Standard Model phase space we have left out a rather big and important corner by taking the Yukawa couplings to be flavour-symmetric, despite their important role in CP violation of the actual Standard Model. Our excuse is that we do not want to distract ourselves from the main effects produced by the gauge couplings by studying a much higher dimensional parameter space instead of just 1 dimension. It then comes as a matter of course that we should next explore how the theory behaves as we move around the space of Yukawa couplings. There are many directions that one can pursue here. One aspect is to study the topological structure of the parameter space and the locus of CP preserving Yukawa couplings. Aside from being an interesting mathematical problem in itself, it also provides a foundation to subsequent investigation such as the possibility of anomalies in coupling constants [121, 48, 49]. Because of its many-faceted importance, it is really worth exploring in the immediate future.

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