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FIELDS, PARTICLES, AND NUCLEI

π^0, η, $\eta' \to \gamma\gamma$ Decays and the Explicit Chiral Symmetry Breaking

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Corrections to the Wess–Zumino–Witten anomaly caused by the explicit breaking of the $SU(3)\times SU(3)$ chiral symmetry are studied using the effective meson Lagrangian based on the Nambu–Jona-Lasinio model with the simultaneous expansion in derivatives, current quark masses, and $1/N_c$ powers. The leading contribution and the first correction for the amplitudes of the π^0, η, $\eta' \to \gamma\gamma$ decays and the contact term in the $\eta/\eta' \to \pi^+\pi^-\gamma$ amplitudes have been calculated. The results are compared with similar $1/N_c$ chiral perturbation calculations and existing experimental data.

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The electromagnetic decays of neutral pseudoscalar mesons π^0, η, and η' into two photons are possible due to the chiral symmetry breaking. The chiral symmetry in quantum chromodynamics (QCD) is violated by the light quark masses as well as by the non-Abelian $SU(3)_L \times SU(3)_R$ Wess–Zumino–Witten anomaly [1, 2]. The corresponding Lagrangian has an order of $O(p^4)$. Here, the standard counting rules accepted in the chiral perturbation theory (χPT) are implied [3, 4]. According to this theory, the low-energy dynamics of the octet of pseudo-Goldstone states is described by the effective Lagrangian that is an expansion in powers of low momenta p_μ and masses $m_{i=u,d,s} = \mathcal{O}(p^2)$ of current quarks.

The consistent inclusion of the η' meson in the theory requires additional arguments related to the $1/N_c$ QCD expansion [5–9] and, as a result, the extension of the chiral symmetry group to $U(3)_L \times U(3)_R$ transformations. The introduction of an additional parameter modifies the standard χPT expansion. The modified approach, which is briefly named the $1/N_c$ χPT, involves the effective Lagrangian [6]

$$\mathcal{L}_{\text{eff}} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \ldots, \quad (1)$$

where superscripts indicate the powers of the small parameter δ, which is introduced in order to represent the expansion in three small parameters $1/N_c = \mathcal{O}(\delta)$, $p^2 = \mathcal{O}(\delta)$, and $m_i = \mathcal{O}(\delta)$ as the expansion in the single small parameter δ. The Wess–Zumino–Witten Lagrangian in the $1/N_c$ χPT has an order of $\mathcal{O}(p^4 N_c) = \mathcal{O}(\delta)$ and is thereby included in $\mathcal{L}^{(1)}$.

This theory allows the systematization of the calculation of corrections due to the explicit chiral symmetry breaking by the light quark masses, which in particular provides the fundamental possibility to control the accuracy of theoretical estimates, including those of the two-photon decay widths of the π^0, η, and η' mesons [10–12]. Alternative calculations are based on the sum rule technique and give accurate theoretical estimates [5, 13, 14].

The current interest in the problem of the π^0, η, $\eta' \to \gamma\gamma$ decays is due to an increase in the accuracy of experiments carried out at the JLab η-meson factory. In particular, the record 1.5% accuracy in the measurement of the decay width of the neutral pion is reached with the PrimEx-I and PrimEx-II data [15]:

$$\Gamma(\pi^0 \to 2\gamma) = (7.802 \pm 0.052(\text{stat.}) \pm 0.105(\text{syst.}))\ \text{eV}.$$

The π^0–η–η' JLab physical program is reviewed in [16], where various theoretical methods used to describe the decays of pseudo-Goldstone states are also detailed.

In this work, the widths of the π^0, η, $\eta' \to 2\gamma$ decays are calculated in the $1/N_c$ Nambu–Jona-Lasinio (NJL) model [17–21]. The application of the NJL model for such calculations is of interest for a long time [5]. However, to implement this approach, it was necessary to derive the effective meson Lagrangian of the NJL model taking into account the explicit chiral symmetry breaking, which has recently been done in [22–24]. It is noteworthy that the expansion in powers of the light quark masses, which appears due to the inclusion of the counting rule $m_i = \mathcal{O}(\delta)$ in the NJL model makes this approach close but not identical to the $1/N_c$ χPT. In addition to the obvious difference between methods of the effective field theory used in the $1/N_c$ χPT and calculations in the model based on effective four-quark interactions, there are a number of differences, the main of which are presented below.

In the absence of external sources, the leading part of the $1/N_c$ χPT Lagrangian has the form

$$\mathcal{L}^{(0)} = \frac{F^2}{4}\langle \partial_\mu U \partial^\mu U^\dagger + \chi U^\dagger + \chi^\dagger U\rangle - \frac{\lambda_U}{2}\phi_0^2. \quad (2)$$

Here, $F = \mathcal{O}(\sqrt{N_c})$ is the decay constant of pseudo-Goldstone bosons in the chiral limit $m_i = 0$; the nonet of pseudoscalar fields $\phi = \sum_{a=0,\alpha}\phi_a\lambda_a$ takes values in the $U(3)$ Lie algebra, where $\lambda_0 = \sqrt{2/3}$ and λ_α are the Gell-Mann matrices; $U = \exp(i\phi)$ is the effective field corresponding to the exponential parameterization of the coset space of Goldstone modes at $N_c = \infty$; the angle brackets $\langle\ldots\rangle$ stand for the trace in the flavor space; $\chi = 2B_0 m$, where $B_0 = -\langle\bar{q}q\rangle/F^2$ is the second low-energy constant related to the quark condensate and $m = \mathrm{diag}(m_u, m_d, m_s)$; and the last term with the topological susceptibility $\lambda_U = \mathcal{O}(N_c^0)$ is the mass term of the η' meson introduced to solve the $U(1)$ problem [25–30]. It is noteworthy that the mass of the η' meson vanishes in the limit $N_c \to \infty$ [31]; as a result, the ninth Goldstone boson appears in the theory and the chiral symmetry group is extended to $U(3) \times U(3)$.

In the $1/N_c$ NJL model, the free Lagrangian of the neutral members of the pseudoscalar nonet follows from the calculations of quark one-loop diagrams; therefore, it depends on the masses M_i and m_i of the constituent and current quarks, respectively:

$$\mathcal{L}_{\phi^2} = \sum_{i=u,d,s}\left[\frac{\kappa_{Aii}}{16G_V}(\partial_\mu\phi_i)^2 - \frac{M_i m_i}{4G_S}\phi_i^2\right] - \frac{\lambda_U}{2}\phi_0^2. \quad (3)$$

Here, G_S and G_V are the constants characterizing the strength of the $U(3) \times U(3)$ chiral-symmetric four-quark interactions of spin-0 and spin-1, respectively, have a dimension of M^{-2}, and decrease as $\mathcal{O}(1/N_c)$ in the limit $N_c \to \infty$; and κ_{Aii} are the diagonal elements of the matrix κ_A determined by the relation

$$\kappa_{Aii}^{-1} = 1 + \frac{\pi^2}{N_c G_V M_i^2 J_1(M_i)}, \quad (4)$$

where

$$J_1(M) = \ln\left(1 + \frac{\Lambda^2}{M^2}\right) - \frac{\Lambda^2}{\Lambda^2 + M^2} \quad (5)$$

is the logarithmically divergent part of the quark one-loop diagram. Here, the covariant cutoff parameter $\Lambda = 1.1$ GeV $\approx 4\pi f_\pi$, which is related to the characteristic scale of spontaneous chiral symmetry breaking, is introduced to remove the divergence. The details of the derivation of the above expressions can be found in [18, 20, 24].

The application of the Lagrangian (3) in the considered approach requires the preliminary expansion of the $M_i(m_i)$ function in powers of $m_i = \mathcal{O}(\delta)$:

$$M_i(m_i) = M_0 + M'(0)m_i + \mathcal{O}(m_i^2), \quad (6)$$

where $M_0 = \mathcal{O}(\delta^0)$ is the solution of the gap equation at $m_i = 0$. The substitution of $M_i = M_0$ into Eq. (3) gives the leading contribution $\mathcal{L}_{\phi^2}^{(0)}$, which completely corresponds to the free part of the Lagrangian (2). In this case, the low-energy constants F and B_0 are expressed in terms of the parameters of the NJL model as

$$F = \sqrt{\frac{\kappa_{A0}}{4G_V}} = \mathcal{O}(\sqrt{N_c}), \quad (7)$$

$$B_0 = \frac{M_0}{2G_S F^2} = \frac{2G_V M_0}{G_S \kappa_{A0}} = -\frac{\langle\bar{q}q\rangle_0}{F^2} = \mathcal{O}(1). \quad (8)$$

Here and below, the subscript 0 of a function of the quark masses m_i means that this function is calculated in the limit $m_i \to 0$ as $\kappa_{A0}^{-1} = \lim_{m_i\to 0}(\kappa_A)_{ii}^{-1}$. In the leading approximation, the three parameters G_S, G_V, and Λ of the NJL model determine the constants F and B_0 because vectors and axial vector degrees of freedom are taken into account in the model. In particular, due to the elimination of the off-diagonal pseudoscalar–axial-vector transitions from the Lagrangian, the constant G_V appears in the expression for the constant F.

The next term in expansion (1) has the order $\mathcal{O}(\delta)$. The corresponding Lagrangian $\mathcal{L}^{(1)}$ includes four dimensionless constants L_5, $L_8 = \mathcal{O}(N_c)$, Λ_1, and $\Lambda_2 = \mathcal{O}(1/N_c)$:

$$\mathcal{L}^{(1)} = L_5\langle\partial_\mu U^\dagger\partial^\mu U(\chi^\dagger U + U^\dagger\chi)\rangle + L_8\langle\chi^\dagger U\chi^\dagger U + \mathrm{H.c.}\rangle + \frac{1}{2}\Lambda_1 F^2\partial_\mu\phi_0\partial^\mu\phi_0 + \frac{i\Lambda_2}{2\sqrt{6}}F^2\phi_0\langle\chi^\dagger U - U^\dagger\chi\rangle + \mathcal{L}_{\mathrm{WZW}}. \quad (9)$$

Several remarks are in order.

(i) The Lagrangian $\mathcal{L}^{(1)}$ contains only two, L_5 and L_8, of ten structures of the order $\mathcal{O}(p^4)$ known in the standard χPT. The others have higher orders in δ; hence, the $1/N_c$ χPT is very appropriate in practice. The calculation of the low-energy constants L_5 and L_8 in the $1/N_c$ NJL model by the substitution of the expansion (6) into the Lagrangian (3) and the subsequent separation of the $\mathcal{O}(\delta)$ terms yields

$$L_5 = \frac{\bar{a} G_S F^4}{8M_0^2}, \quad L_8 = \frac{a G_S F^4}{16 M_0^2}. \tag{10}$$

Here,

$$a = M'(0) = \frac{\pi^2}{N_c G_S M_0^2 J_1^0} = \frac{G_V}{G_S}\left(\kappa_{A0}^{-1} - 1\right), \tag{11}$$

$$\bar{a} = 2a(1-\kappa_{A0})\left[1 - \frac{\Lambda^4}{J_1^0(\Lambda^2 + M_0^2)}\right], \tag{12}$$

where $J_1^0 \equiv J_1(M_0)$.

(ii) The terms with Λ_1 and Λ_2 violate the Okubo–Zweig–Iizuka rule; for this reason, their origin is attributed to the gluon exchange. Below, $\Lambda_1 = 0$ is set to reduce the number of independent parameters in the analysis. Qualitative reasons for this assumption are given in [5]. Thus, the Lagrangian $\mathcal{L}^{(1)}$ in the $1/N_c$ NJL model has only one low-energy constant Λ_2, which is unambiguously determined (together with the constant λ_U) by the experimental masses of the η and η' mesons.

(iii) Chiral logarithms appearing from one-loop diagrams constructed on the Lagrangian $\mathcal{L}^{(0)}$ have an order of $m_i/N_c \ln m_i = \mathcal{O}(\delta^2)$; i.e., their contribution can be neglected with an accuracy of $\mathcal{O}(\delta)$. In the $1/N_c$ NJL model, the third term in Eq. (6) has the same order. Consequently, to calculate it, the standard gap equation

$$M_i\left(1 - \frac{N_c G_S}{2\pi^2} J_0(M_i)\right) = m_i, \tag{13}$$

where

$$J_0(M_i) = \Lambda^2 - M_i^2 \ln\left(1 + \frac{\Lambda^2}{M_i^2}\right), \tag{14}$$

should be modified by including terms describing contributions from meson one-loop diagrams called tadpoles.

(iv) The Lagrangian $\mathcal{L}_{WZW}$ corresponds to the Wess–Zumino–Witten anomaly and removes the accidental $U \to U^{-1}$ symmetry of the Lagrangian $\mathcal{L}^{(0)} + \mathcal{L}^{(1)}$ that should not occur in QCD. The Lagrangian $\mathcal{L}_{WZW}$ breaks this discrete symmetry and is certainly (up to a common factor of N_c) determined by the topology of the mapping of the Minkowski space to the coset space of Goldstone fields by the matrix $U(x)$. In particular, the Lagrangian responsible for two-photon decays of the π^0, η, and η' mesons has the form

$$\mathcal{L}_{WZW} = -\frac{N_c \alpha}{4\pi} F_{\mu\nu} \tilde{F}^{\mu\nu} \langle Q^2 \phi \rangle + \ldots, \tag{15}$$

where $\tilde{F}^{\mu\nu} = \frac{1}{2} e^{\mu\nu\rho\sigma} F_{\rho\sigma}$, $\alpha = e^2/4\pi$ is the fine structure constant, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic strength tensor for the electromagnetic four-potential A_μ, and $Q = \mathrm{diag}(2/3, -1/3, -1/3)$ is the matrix of the electric charges of the quarks. Taking into account the count rule for the electric charge $e = \mathcal{O}(\sqrt{\delta})$, one can easily verify that these vertices have an order of δ.

(v) The Lagrangian $\mathcal{L}^{(0)} + \mathcal{L}^{(1)}$ provides a reasonable approximation to the total effective Lagrangian $\mathcal{L}_{\text{eff}}$. It allows one to exactly describe the spectrum of the nonet of pseudo-Goldstone states, to calculate the ratios of the masses of light quarks avoiding an uncertainty introduced by the Kaplan–Manohar transformation [32], and to obtain the low-energy coupling constants and mixing angles. In the $1/N_c$ NJL model, all these results are achieved with the following main model parameters: $G_S = 6.6$ GeV^{-2}, $G_V = 7.4$ GeV^{-2}, $\Lambda = 1.1$ GeV, $m_u = 2.6$ MeV, $m_d = 4.6$ MeV, $m_s = 84$ MeV, $\lambda_U = (285\text{ MeV})^4$, and $\Lambda_2 = 0.46$ [19–21].

After these remarks, the two-photon decay widths can be directly calculated in the $1/N_c$ NJL model. To this end, it is necessary to pass from the bare dimensionless field ϕ in Eq. (15) to the variables corresponding to the π^0, η, and η' physical states. The detailed solution of this problem was presented in [20]. Using this result, we obtain

$$\langle Q^2 \phi \rangle = \frac{1}{9}(4\phi_u + \phi_d + \phi_s) = \frac{1}{3 f_{\pi^0}} \sum_{P=\pi^0,\eta,\eta'} c_P P, \tag{16}$$

where $c_P = c_P^{(0)} + c_P^{(1)}$. As a result, the two-photon decay width is given by the expression

$$\Gamma(P \to \gamma\gamma) = \frac{\alpha^2 m_P^3}{64\pi^3 f_{\pi^0}^2} c_P^2. \tag{17}$$

In the leading order (LO), we obtain

$$\begin{aligned} c_{\pi^0}^{(0)} &= 1 + \frac{1}{\sqrt{3}}\left[\epsilon_0(c_0 - \sqrt{8}s_0) + \epsilon_0'(s_0 + \sqrt{8}c_0)\right], \\ c_\eta^{(0)} &= \frac{1}{\sqrt{3}}\left(c_0 - \sqrt{8}s_0\right) - \epsilon_0, \\ c_{\eta'}^{(0)} &= \frac{1}{\sqrt{3}}\left(s_0 + \sqrt{8}c_0\right) - \epsilon_0', \end{aligned} \tag{18}$$

where $c_0 \equiv \cos\theta_0$, $s_0 \equiv \sin\theta_0$, and $\theta_0 = -14.97°$, $\epsilon_0 = 0.0177$, and $\epsilon_0' = 0.0033$ are the η–η', π^0–η, and π^0–η' mixing angles, respectively.

In the next-to-leading order (NLO), additional terms $c_P^{(1)}$ appear in Eqs. (18) due both to small corrections $\Delta\theta = -0.79°$, $\Delta\epsilon = -6.3\times 10^{-3}$, and $\Delta\epsilon' = -1.2 \times 10^{-3}$ to the mixing angles and to the linear dependence of the decay constants on the current quark masses $f_{i=u,d,s} = F\left(1 + \bar{a}m_i/(2M_0)\right)$. The corresponding expressions have the form

$$c_{\pi^0}^{(1)} = \frac{1}{\sqrt{3}}\Big[(\Delta\epsilon + \Delta\theta\epsilon_0')(c_0 - \sqrt{8}s_0) + (\Delta\epsilon' - \Delta\theta\epsilon_0)$$
$$\times (s_0 + \sqrt{8}c_0)\Big] - \frac{\bar{a}}{6M_0}\Big\{4m_u - m_d$$
$$+ \frac{5\hat{m}}{\sqrt{3}}\Big[\epsilon_0'(s_0 + \sqrt{2}c_0) + \epsilon_0(c_0 - \sqrt{2}s_0)\Big]$$
$$+ \sqrt{\frac{2}{3}}m_s\Big[\epsilon_0'(c_0 - \sqrt{2}s_0) - \epsilon_0(s_0 + \sqrt{2}c_0)\Big]\Big\},$$
$$c_{\eta}^{(1)} = -\Delta\epsilon - \frac{\Delta\theta}{\sqrt{3}}\left(s_0 + \sqrt{8}c_0\right) + \frac{\bar{a}}{6\sqrt{3}M_0}\Big[3\sqrt{3}\hat{m}\epsilon_0 \tag{19}$$
$$+ (4m_u + m_d)(\sqrt{2}s_0 - c_0) + \sqrt{2}m_s(s_0 + \sqrt{2}c_0)\Big],$$
$$c_{\eta'}^{(1)} = -\Delta\epsilon' + \frac{\Delta\theta}{\sqrt{3}}(c_0 - \sqrt{8}s_0) + \frac{\bar{a}}{6\sqrt{3}M_0}\Big[3\sqrt{3}\hat{m}\epsilon_0'$$
$$- (4m_u + m_d)(\sqrt{2}c_0 + s_0) - \sqrt{2}m_s(c_0 - \sqrt{2}s_0)\Big].$$

Here, the small terms $(m_d - m_u)^2$ are neglected and $\hat{m} = (m_u + m_d)/2$.

We discuss the results.

(a) In Eq. (16), the general constant F is changed to $f_{\pi^0} = (92.277 \pm 0.095)$ MeV. This fixes the normalization of the anomaly. As is known [1], the Ward identities determine the effective vertices of the anomalous part of the Lagrangian up to an arbitrary constant F, which is usually related to the decay constant of the neutral pion f_{π^0} and its value can be determined from the weak decay constant of the charged pions $f_{\pi^\pm}$. They differ from F only in the next order of the chiral expansion, which is insignificant when considering the leading contribution. In particular, the known result for the decay width of the π^0 meson:

$$\Gamma(\pi^0 \to 2\gamma) = \frac{\alpha^2 m_{\pi^0}^3}{64\pi^3 f_{\pi^0}^2} = 7.750 \pm 0.016 \text{ eV}, \tag{20}$$

follows from Eq. (18) if mixing effects are neglected, i.e., at $c_{\pi^0}^{(0)} = 1$. The error in Eq. (20) is due to the error in the determination of the constant f_{π^0}.

The inclusion of mixing gives $c_{\pi^0}^{(0)} = 1.022$, which increases the decay width by 4.4% to $\Gamma(\pi^0 \to 2\gamma) = (8.094 \pm 0.017)$ eV. The dominant contribution to this increase comes from the π^0–η mixing, which gives 3.4% of the indicated 4.4%; the π^0–η' mixing gives the remaining 1.0%. The same picture is also observed in the $1/N_c$ χPT [10]. Alternative calculations in the χPT with the inclusion of effects of quark masses and dynamic photons give an estimate of $\Gamma(\pi^0 \to 2\gamma) = (8.06 \pm 0.02 \pm 0.06)$ eV [33]. Thus, the general tendency is that the mixing in the leading approximation increases the decay width of the π^0 meson by about 4.5%.

This contradicts the experimental value $\Gamma(\pi^0 \to 2\gamma) = (7.72 \pm 0.12)$ eV reported by the Particle Data Group [34], as well as the joined result of the PrimEx-I and PrimEx-II Collaborations presented above. Phenomenological data confirm the chiral anomaly prediction (20) with a high accuracy. Consequently, it is important to study higher corrections in the used theoretical schemes. This goal becomes even more relevant because the previous estimates indicate that the LO result remains valid with the NLO corrections [10]. The only known exception is the result of the join application of the dispersion relations and sum rules. The corresponding estimate is $\Gamma(\pi^0 \to 2\gamma) = (7.93 \pm 0.12)$ eV [13], which on the one hand has an accuracy corresponding to the accuracy of the PrimEx Collaboration measurements and on the other hand indicates that mixing effects increase the decay width (20) only by 2.3%. This result is in agreement with the experimental data within the presented errors.

We now discuss the NLO contribution given by Eqs. (19) to the picture described above. Since all model parameters in Eqs. (19) are known, we obtain

$$c_{\pi^0}^{(1)} = -\left(6.15 + 1.66 - 0.3 + 8.6\right) \times 10^{-3} = -0.016. \tag{21}$$

This result includes the contributions from the corrections $\Delta\epsilon$, $\Delta\epsilon'$, and $\Delta\theta$ to the mixing angles $\epsilon = \epsilon_0 + \Delta\epsilon$, $\epsilon' = \epsilon_0' + \Delta\epsilon'$, and $\theta = \theta_0 + \Delta\theta$, as well as a contribution that explicitly depends on the current quark masses and is proportional to $\bar{a}$, and in the sum with the leading contribution yields $c_{\pi^0} = 1.006$ or

$$\Gamma(\pi^0 \to 2\gamma) = (7.84 \pm 0.02) \text{ eV}. \tag{22}$$

It is important that the correction $c_{\pi^0}^{(1)}$ is negative and significantly (by 3.1%) suppresses mixing effects in the pion decay amplitude; the pion decay width is finally in agreement both with the experimental data [15, 34] and with the sum rule prediction [13]. The error in Eq. (22) includes only the spread in f_{π^0}. The uncertainty in the estimate of chiral corrections is not considered here.

The suppression of mixing effects is not accidental. This suppression can be prevented by a contribution $\propto\Delta\theta$, but the correction to the η–η' mixing angle in the $1/N_c$ NJL model is negligibly small (this correction in the $1/N_c$ χPT is about 50%). A similar situation occurs in the $\eta \to 3\pi$ decay [21], where the $U(1)$ axial anomaly plays a special role in the suppression of the η–η' mixing. The $\pi^0 \to \gamma\gamma$ decay is another example, where the NLO corrections suppress the significant isospin symmetry breaking in the leading approximation.

(b) We now discuss the results obtained for the two-photon decays of the η and η' mesons. These processes significantly differ from the pion decay in the role of mixing effects. They are small for the $\pi^0 \to \gamma\gamma$ decay and the theory should explain the mechanism of this suppression. On the contrary, if the mixture is disregarded to describe the η, $\eta' \to \gamma\gamma$ decays, Eqs. (18) give the known values $c_\eta^{(0)} = 1/\sqrt{3} \simeq 0.58$ and $c_{\eta'}^{(0)} = \sqrt{8}/\sqrt{3} \simeq 1.63$ corresponding to the $U(3)$ symmetry, which are significantly different from the experimental estimates $c_\eta = 0.997 \pm 0.017$ and $c_{\eta'} = 1.243 \pm 0.028$, respectively [34]. Therefore, the theoretical description of two-photon decays of the η and η' mesons is an important step to understand the mechanism of the explicit chiral symmetry breaking in QCD.

In the leading order, Eqs. (18) give $c_\eta^{(0)} = 0.962$ and $c_{\eta'}^{(0)} = 1.425$; i.e., the inclusion of the η–η' mixing noticeably improves the result: the observed amplitudes differ from the predicted values by no more than 13%. The NLO corrections given by Eqs. (19) improve agreement: $c_\eta = 1.10$ and $c_{\eta'} = 1.24$. The dominant contribution to the constants $c_\eta^{(1)} = 0.137$ and $c_{\eta'}^{(1)} = -0.185$ comes from the terms with the factor $\bar{a}$, which give 0.111 and –0.173, respectively. The theoretical estimate of the decay width $\Gamma(\eta' \to 2\gamma) = (4.26 \pm 0.01)$ keV is in complete agreement with the value $\Gamma(\eta' \to 2\gamma) = (4.28 \pm 0.19)$ keV given by the Particle Data Group, whereas the decay width of the η meson $\Gamma(\eta \to 2\gamma) = (0.626 \pm 0.001)$ keV is larger than the experimental value $\Gamma(\eta \to 2\gamma) = (0.515 \pm 0.018)$ keV.

Thus, the $1/N_c$ NJL model gives good results for all three two-photon decays. Further progress here can be achieved beyond the NLO approximation [43] or with the inclusion of a new interaction violating the Zweig rule [36]. The latter is possible in the presence of off-diagonal terms in the kinetic part of the effective Lagrangian, whose diagonalization requires two η–η' mixing angles. In both cases, the number of free parameters increases and, thereby, additional possibilities appear to successfully describe two photon decays [37].

Finally, we calculate the contact part of the amplitudes of the $\eta/\eta' \to \pi^+\pi^-\gamma$ decays, which is described by the Lagrangian

$$\mathcal{L}_{\rm WZW} = \frac{ieN_c}{24\pi^2} e^{\mu\nu\rho\sigma} A_\mu \langle Q \partial_\nu \phi \partial_\rho \phi \partial_\sigma \phi \rangle. \quad (23)$$

The WASA-at-COSY, ARGUS, KLOE, MARK II, JADE, CELLO, PLUTO, WA76, TASSO, TPC, Crystal Barrel, and BESIII collaborations studying the $\eta/\eta' \to \pi^+\pi^-\gamma$ radiative decay always pay attention to the contribution from the box anomaly. On the one hand, these modes make it possible to test the contact term of the non-Abelian Wess–Zumino–Witten anomaly; on the other hand, effects of the explicit symmetry breaking in the contact interaction can be studies with the high statistics of events. This aspect of studies is of no less interest because it promotes a deeper insight into the fine details of the mechanism of the explicit chiral symmetry breaking in QCD.

Success can hardly be achieved here without reliable calculation methods (both analytical and lattice). The problem is complicated because it is necessary to carefully take into account strong interactions responsible for the production of a $\pi^+\pi^-$ pair. To describe the $\eta/\eta' \to \pi^+\pi^-\gamma$ decays, methods of the chiral perturbation theory [38], as well as the effective meson Lagrangians including the contribution from vector particles [39, 40], are used. Dispersion methods combined with the effective chiral field theory are also applied [41]. This approach allows one to reproduce the analytic properties of the amplitude, to include the interaction of pions in the final state, and to take into account effects of the explicit isospin symmetry breaking due to the ρ—ω mixing. As a result, data obtained by the BESIII Collaboration with a very high statistics (9.7×10^5 $\eta' \to \pi^+\pi^-\gamma$ events) were described with a high accuracy [42]. The most accurate description of spectral data on two-pion events is achieved by fitting with the contact contribution α_0 as a free parameter. This fit ensures the minimum $\chi^2 = 1.74$ and gives the result [41]

$$\alpha_0 = (18.41 \pm 0.19)\ \text{GeV}^{-3}. \quad (24)$$

At the same time, the quantity α_0 can be calculated theoretically as

$$\alpha_0 = \frac{\sqrt{2}N_c}{18\sqrt{3}\pi^2 f_{\pi^0}^3 \Omega_1^1(4m_\pi^2)} c_{\eta'\pi^+\pi^-}, \quad (25)$$

where $c_{\eta'\pi^+\pi^-} = \sin\theta_P + \sqrt{2}\cos\theta_P$, θ_P is the η–η' mixing angle, and $\Omega_1^1(s)$ is the Omnes function, which appears when the interaction of pions in the final state is taken into account. The choice $s = 4m_\pi^2$ is the chiral fit, which reduces the number of the parameters in the amplitude; correspondingly, $\Omega_1^1(4m_\pi^2) = 1.159$. The substitution of this value and the mixing angle $\theta_P = -21.37°$ into Eq. (25) yields $\alpha_0 = 14.37$ GeV^{-3}. The approach with two mixing angles gives $\alpha_0 = 15.17$ GeV^{-3} [41], which is closer to Eq. (24). The authors of [41] used the parameters obtained in the NNLO approximation of the $U(3)$ χPT [43].

The parameter α_0 can be determined in the $1/N_c$ NJL model in the NLO approximation. The corresponding contributions to the coefficient $c_{\eta'\pi^+\pi^-} = c^{(0)}_{\eta'\pi^+\pi^-} + c^{(1)}_{\eta'\pi^+\pi^-}$ have the form

$$c^{(0)}_{\eta'\pi^+\pi^-} = s_0 + \sqrt{2}c_0 - \sqrt{3}\epsilon_0',$$

$$c^{(1)}_{\eta'\pi^+\pi^-} = \Delta\theta(c_0 - \sqrt{2}s_0) - \sqrt{3}\Delta\epsilon' + 3\sqrt{3}\frac{\hat{m}\bar{a}}{2M_0}\epsilon_0' - \frac{\bar{a}}{2M_0}(2m_u + m_d)(\sqrt{2}c_0 + s_0). \quad (26)$$

The substitution of Eqs. (26) into Eq. (25) gives

$$\alpha_0 = (15.60 \mp 0.05)\ \text{GeV}^{-3}. \quad (27)$$

Within the indicated errors, this value is 14% below the value presented in Eq. (24); nevertheless, the value given in Eq. (27) is the closest among the presented theoretical estimates to the BESIII data. It is seen that the NLO correction reduces the LO result $\alpha_0^{(0)} = 16.69 \mp 0.05\ \text{GeV}^{-3}$. The other terms in Eq. (26) make the following contributions: $\alpha_0^{(1)} = (-0.28 + 0.03 + 0.00 - 0.84 = -1.09)\ \text{GeV}^{-3}$. As expected, the contributions responsible for the $SU(3)$ symmetry breaking dominate. Corrections due to the isospin symmetry breaking are smaller than 3%. The NNLO calculations in the $1/N_c$ NJL model can further improve agreement with Eq. (24).

For completeness, similar calculations for the $\eta \to \pi^+\pi^-\gamma$ decay give $c_{\eta\pi^+\pi^-} = c^{(0)}_{\eta\pi^+\pi^-} + c^{(1)}_{\eta\pi^+\pi^-}$, where

$$c^{(0)}_{\eta\pi^+\pi^-} = c_0 - \sqrt{2}s_0 - \sqrt{3}\epsilon_0,$$

$$c^{(1)}_{\eta\pi^+\pi^-} = -\Delta\theta(s_0 + \sqrt{2}c_0) - \sqrt{3}\Delta\epsilon + 3\sqrt{3}\frac{\hat{m}\bar{a}}{2M_0}\epsilon_0 + \frac{\bar{a}}{2M_0}(2m_u + m_d)(\sqrt{2}s_0 - c_0). \quad (28)$$

The existing experimental data on the $\eta \to \pi^+\pi^-\gamma$ decay are yet obtained with a statistics insufficient to determine the contact contribution with a high accuracy. For this reason, we calculate the contact term only by Eq. (25) (with the obvious substitutions $c_{\eta'\pi^+\pi^-} \to c_{\eta\pi^+\pi^-}$ and $\alpha_0 \to \alpha_{0\eta}$), which provides an additional test for the $1/N_c$ NJL model compared to future precise measurements. According to Eqs. (28),

$$\alpha_{0\eta} = (19.11 \mp 0.06)\ \text{GeV}^{-3}. \quad (29)$$

Here, the NLO result is smaller than that in the case of the η' meson, but the tendency remains the same: the first correction $\alpha_{0\eta}^{(1)} = (0.23 + 0.16 + 0.03 - 1.01 = -0.59)\ \text{GeV}^{-3}$ reduces the leading contribution $\alpha_{0\eta}^{(0)} = (19.70 \mp 0.06)\ \text{GeV}^{-3}$. It is noteworthy that the isospin symmetry breaking makes a noticeable contribution and provides about 24% of the NLO correction.

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CONFLICT OF INTEREST

The author declares that he has no conflicts of interest.

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REFERENCES

1. J. Wess and B. Zumino, Phys. Lett. B **37**, 95 (1971).
2. E. Witten, Nucl. Phys. B **223**, 422 (1983).
3. S. Weinberg, Phys. A (Amsterdam, Neth.) **96**, 327 (1979).
4. J. Gasser and H. Leutwyler, Nucl. Phys. B **250**, 465 (1985).
5. B. Moussallam, Phys. Rev. D **51**, 4939 (1995).
6. H. Leutwyler, Phys. Lett. B **374**, 163 (1996).
7. H. Leutwyler, Phys. Lett. B **374**, 181 (1996).
8. P. Herrera-Siklódy, J. I. Latorre, P. Pascual, and J. Taron, Nucl. Phys. B **497**, 345 (1997).
9. R. Kaiser and H. Leutwyler, Eur. Phys. J. C **17**, 623 (2000).
10. J. L. Goity, A. M. Bernstein, and B. R. Holstein, Phys. Rev. D **66**, 076014 (2002).
11. A. M. Bernstein and B. R. Holstein, Rev. Mod. Phys. **85**, 49 (2013).
12. P. Bickert and S. Scherer, Phys. Rev. D **102**, 074019 (2020).
13. B. L. Ioffe and A. G. Oganesian, Phys. Lett. B **647**, 389 (2007).
14. S. Khlebtsov, Y. Klopot, A. Oganesian, and O. Teryaev, Phys. Rev. D **104**, 016011 (2021).
15. I. Larin, Y. Zhang, A. Gasparian, et al. (PrimEx-II Collab.), Science (Washington, DC, U. S.) **368**, 506 (2020).
16. L. Gan, B. Kubis, E. Passemar, and S. Tulin, Phys. Rep. **945**, 1 (2022).

17. A. A Osipov, JETP Lett. **115**, 305 (2022).
18. A. A. Osipov, JETP Lett. **115**, 371 (2022).
19. A. A. Osipov, Phys. Rev. D **108**, 016014 (2023).
20. A. A. Osipov, Phys. Rev. D **108**, 036012 (2023).
21. A. A. Osipov, JETP Lett. **117**, 898 (2023).
22. A. A. Osipov, JETP Lett. **113**, 413 (2021).
23. A. A. Osipov, Phys. Lett. B **817**, 136300 (2021).
24. A. A. Osipov, Phys. Rev. D **104**, 105019 (2021).
25. G. Veneziano, Nucl. Phys. B **159**, 213 (1979).
26. C. Rosenzweig, J. Schechter, and G. Trahern, Phys. Rev. D **21**, 3388 (1980).
27. P. di Vecchia and G. Veneziano, Nucl. Phys. B **171**, 253 (1980).
28. K. Kawarabayashi and N. Ohta, Nucl. Phys. B **175**, 477 (1980).
29. P. di Vecchia, F. Nicodemi, R. Pettorino, and G. Veneziano, Nucl. Phys. B **181**, 318 (1981).
30. K. Kawarabayashi and N. Ohta, Prog. Theor. Phys. **66**, 1709 (1981).
31. E. Witten, Nucl. Phys. B **156**, 269 (1979).
32. D. B. Kaplan and A. V. Manohar, Phys. Rev. Lett. **56**, 2004 (1986).
33. B. Ananthanarayan and B. Moussallam, J. High Energy Phys., No. 05, 052 (2002).
34. R. L. Workman, V. D. Burkert, V. Crede, et al. (Particle Data Group), Prog. Theor. Exp. Phys. **2022**, 083C01 (2022).
35. X.-K. Guoa, Z.-H. Guoa, J. A. Oller, and J. J. Sanz-Cillerod, J. High Energy Phys., No. 06, 175 (2015).
36. J. Schechter, A. Subbaraman, and H. Weigel, Phys. Rev. D **48**, 339 (1993).
37. R. Escribano and J-M. Frère, J. High Energy Phys. **0506**, 029 (2005).
38. J. Bijnens, A. Bramon, and F. Cornet, Phys. Lett. B **237**, 488 (1990).
39. M. Benayoun, P. David, L. Del Buono, Ph. Leruste, and H. B. O'Connell, Eur. Phys. J. C **31**, 525 (2003).
40. A. A. Osipov, A. A. Pivovarov, M. K. Volkov, and M. M. Khalifa, Phys. Rev. D **101**, 094031 (2020).
41. L.-Y. Dai, X.-W. Kang, U.-G. Meißner, X.-Y. Song, and D.-L. Yao, Phys. Rev. D **97**, 036012 (2018).
42. M. Ablikim, M. N. Achasov, S. Ahmed, et al. (BESIII Collab.), Phys. Rev. Lett. **120**, 242003 (2018).
43. X. K. Guo, Z. H. Guo, J. A. Oller, and J. J. Sanz-Cillero, J. High Energy Phys. **175**, 1506 (2015).

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