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Coherent states for generalized uncertainty relations and their cosmological implications

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Abstract. We study coherent states associated with a generalized uncertainty principle (GUP). Our particular focus is on the negative deformation parameter β. We show that the ensuing coherent state can be identified with Tsallis' probability amplitude with the non-extensivity parameter q being a monotonically increasing function of β. Furthermore, for $\beta < 0$, we reformulate the GUP in terms of a one-parameter class of Tsallis entropy based uncertainty relations, which are again saturated by the GUP coherent states. We argue that this combination of coherent states with Tsallis entropy offers a natural conceptual framework allowing to study the quasi-classical regime of GUP in terms of non-extensive thermostatistics. We bolster this claim by discussing a generalization of Verlinde's entropic force and the ensuing implications in the late-inflation epoch. The corresponding dependence of the β parameter on the cosmological time is derived for the reheating epoch. The obtained β is consistent with both values predicted by string-theory models and the naturalness principle.

1. Introduction

The Heisenberg Uncertainty Principle (HUP) – the cornerstone of quantum mechanics – provides an intrinsic limitation on the simultaneous knowledge of position and momentum of any quantum system. While working successfully in low-energy regime, it is anticipated that modifications will occur as we approach the Planck scale, likely due to quantum gravitational effects. Several models of quantum gravity, such as String Theory, Loop Quantum Gravity, Quantum Geometry and Doubly Special Relativity, have converged to the idea that the HUP should be generalized so as to account for the emergence of a minimal length at the Planck scale. The ensuing uncertainty relations are typically referred to as Generalized Uncertainty Principles (GUPs).

The simplest version of GUP can be obtained by adding a term quadratic in the momentum

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uncertainty to the right-hand side of HUP [1, 2, 3, 4, 5, 6], namely

$$\delta x\,\delta p \,\geq\, \frac{\hbar}{2}\left(1 \,+\, \beta\,\frac{\delta p^2}{m_p^2}\right), \tag{1}$$

where m_p stands for the Planck mass and the β parameter quantifies the departure from HUP. Here and throughout we set $c = 1$. Note that such term is not fixed by the theory, albeit it is generally assumed to be of order unity [1, 2, 3, 4]. Clearly, the traditional HUP form is recovered for $\beta \to 0$ and/or $\delta p \ll m_p$.

The symbol δ in Eq. (1) represents the uncertainty of a given observable, and it does not necessarily have to be associated with the standard deviation. In fact, in the original HUP δ can represent, e.g. Heisenberg's "ungenauigkeiten" (i.e., error-disturbance uncertainties caused by the back-reaction in simultaneous measurement) or $\delta p = \langle\psi||p||\psi\rangle \equiv \langle|p|\rangle_\psi$. Nevertheless, in cases where δ is identified with the standard deviation (henceforth denoted by Δ), the generalized uncertainty principle (1) can be derived from the deformed commutation relation (DCR)

$$[\hat{x},\hat{p}] \,=\, i\hbar\left(1+\beta\,\frac{\hat{p}^2}{m_p^2}\right), \tag{2}$$

via the Cauchy–Schwarz inequality [7, 8], provided one restricts the attention to *mirror symmetric* states, i.e. states satisfying $\langle\hat{p}\rangle_\psi = 0$. Commutator (2) is typically supplemented with the commutators

$$[\hat{x},\hat{x}] \,=\, [\hat{p},\hat{p}] \,=\, 0\,, \tag{3}$$

which, together with (2), satisfy Jacobi identities and determine the whole symplectic structure of the model. So far, the quadratic GUP (1) has been mostly used to study phenomenology of quantum gravity in many sectors, ranging from quantum mechanics [9, 10, 11] to particle physics [12, 13, 14, 15] and cosmology [16, 17]. In contrast, comparatively less research has been conducted on the quasi-classical domain of the GUP. Nevertheless, this domain has relevant and possibly observable implications for the early Universe cosmology and astrophysics [18, 19]. To explore physics in the quasi-classical realm, it is common to use coherent states (CSs). CSs are, in a sense, privileged quantum states in the description of the quantum-to-classical transition, as they are the only states that remain pure in the decoherence process [20, 21]. Since CSs are pure, they allow for maximal resolution in phase-space, thus appearing as the closest quantum counterparts of classical points. Additionally, the CS formalism offers a convenient description which can draw upon developments in quantum optics [22].

Our principal aim here is to address the issue of CSs for GUP and, in particular, to point out the role that Tsallis' entropy plays in this context. Being equipped with the ensuing CSs, we would like to discuss some simple consequences of GUP systems in their decoherence regime. To this end, we first introduce the Schrödinger–Nieto type of minimum-uncertainty CSs [7, 23] associated with GUP. Subsequently, we show that these states coincide with Tsallis' probability amplitudes. Furthermore, by using Bekner–Babenko inequality, we recast the GUP for $\beta < 0$ in terms of a one-parameter class of Tsallis entropy-power based uncertainty relations (EPUR), which are again saturated by the GUP CSs. Finally, we invoke the Maximum Entropy Principle (MEP), i.e. the principle which posits that the thermodynamic entropy is the statistical entropy evaluated at the maximum entropy distribution, to understand the quasi-classical domain of GUP in terms of non-extensive Tsallis thermostatistics. To further illustrate our point, we will examine two relevant examples from cosmology: a) the GUP generalization of Verlinde's entropic gravity force [24] and b) its connection with conformal gravity (CG) [25, 26, 27].

The remainder of the work is organized as follows: in Sec. 2 we derive coherent states for the quadratic GUP and reformulate the GUP in terms of Tsallis entropy-power based uncertainty

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relations. Within this setting, we discuss in Sec. 3 some illustrative examples from the early Universe cosmology by employing Verlinde's entropic gravity together with Tsallis' non-extensive thermostatistics. Conclusions and outlook are finally summarized in Sec. 4.

2. Coherent states for GUP

2.1. GUP (1) — closer look

Let us first briefly review the steps leading to (1) from the DCR (2). To this end, we shall quantify the uncertainty of an observable $\hat{\mathcal{O}}$ with respect to a state $|\psi\rangle$ via its standard deviation. In particular, for the variance (i.e., square of the standard deviation) we have

$$(\Delta\hat{\mathcal{O}})^2 \ = \ \langle\psi|\hat{\mathcal{O}}^2|\psi\rangle - \langle\psi|\hat{\mathcal{O}}|\psi\rangle^2 \ \equiv \ \langle\hat{\mathcal{O}}^2\rangle_\psi - \langle\hat{\mathcal{O}}\rangle_\psi^2 \,. \tag{4}$$

In the upcoming considerations, the subscript ψ in $\langle\cdots\rangle$ will be omitted. By specializing the analysis to the observables $\hat{x}$ and $\hat{p}$ and mirror states, it is easy to check that the DCR (2) holds. Indeed, by employing the Cauchy–Schwarz inequality, one can write

$$(\Delta x)^2(\Delta p)^2 \ = \ \langle\hat{x}^2\rangle\langle\hat{p}^2\rangle \ \geq \ |\langle\hat{x}\,\hat{p}\rangle|^2 \ . \tag{5}$$

At this point, we rewrite $\hat{x}\,\hat{p}$ as the sum of Hermitian and anti-Hermitian operators, so that

$$\hat{x}\,\hat{p} \ = \ \tfrac{1}{2}[\hat{x},\hat{p}]_+ \ + \ \tfrac{1}{2}[\hat{x},\hat{p}] \ = \ \tfrac{1}{2}[\hat{x},\hat{p}]_+ \ + \ \frac{i\hbar}{2}\left(1+\beta\,\frac{\hat{p}^2}{m_p^2}\right) . \tag{6}$$

Here $[,]_+$ denotes the anticommutator. By inserting this expression back into (5), we arrive at

$$(\Delta x)^2(\Delta p)^2 \ \geq \ \tfrac{1}{4}\left\langle[\hat{x},\hat{p}]_+\right\rangle^2 \ + \ \frac{\hbar^2}{4}\left[1+\beta\,\frac{(\Delta p)^2}{m_p^2}\right]^2 \ \geq \ \frac{\hbar^2}{4}\left[1+\beta\,\frac{(\Delta p)^2}{m_p^2}\right]^2 . \tag{7}$$

To obtain the last step, we have used the Robertson trick [8] and neglected the anticommutator part. Equation (7) clearly coincides with the GUP (1), with variances in place of error disturbances. In passing, we should mention that the inequality (7) based on (2) is valid not only for pure states, but holds also for mixed states [29].

2.2. Coherent states and GUP

Coherent states in quantum mechanics are quantum states that saturate Heisenberg uncertainty relations (URs). In fact, there are more defining properties that CSs could/should satisfy and there are many names used in the literature to denote states which have part but not all properties of the CSs, like the aforementioned minimal uncertainty states, or maximal localization states, or quasi-classical states, or weak coherent states. Here, we will concentrate on the so-called Schrödinger–Nieto type CSs where the only requirement is the saturation of URs [7, 23]. Such CSs are the most classical-like states in the sense that positions and momenta are in these states as well defined/localized as QM allows. We can mention yet another important property associated with the Schrödinger–Nieto CSs, namely that they most closely approximate the classical motion of a particle as they entangle least with the environment (and so they are least perturbed by it). It is this robustness with respect to decoherence that makes them an ideal tool for quantum optics applications.

It is thus natural to ask how such CSs look like in a GUP-driven Universe. In order to find the states $|\psi\rangle$ that saturate the GUP inequality (7), we might observe from Eqs. (5) and (7) that two requirements must hold simultaneously [28], namely

$$\hat{p}|\psi\rangle \ = \ c\,\hat{x}|\psi\rangle\,, \qquad \langle[\hat{x},\hat{p}]_+\rangle \ = \ 0\,, \tag{8}$$

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with $c \in \mathbf{C}$. The first condition saturates the Cauchy–Schwarz inequality, whereas the second one saturates the Robertson trick. Note that from the second requirement it automatically follows that $c = -c^*$, so we can write $c = i\gamma$, with $\gamma \in \mathbf{R}$. Consequently, the state $|\psi\rangle$ that saturates the GUP inequality (7) must satisfy the equation

$$(\hat{p} - i\gamma\hat{x})\,|\psi\rangle \;=\; 0\,. \tag{9}$$

We now seek the solution to (9) in the momentum representation, i.e $|\psi\rangle \mapsto \psi(p) = \langle p|\psi\rangle$. It is not difficult to see that in momentum space the operators $\hat{x}$ and $\hat{p}$ have the form [29]

$$\hat{p}\,\psi(p) \;=\; p\,\psi(p)\,, \qquad \hat{x}\,\psi(p) \;=\; i\hbar\left(\frac{d}{dp} + \frac{\beta}{2\,m_p^2}\left[p^2, \frac{d}{dp}\right]_+\right)\psi(p)\,. \tag{10}$$

This form is chosen so that both $\hat{x}$ and $\hat{p}$ are manifestly symmetric. With this, we can cast Eq. (9) into a differential equation

$$\frac{d}{dp}\,\psi(p) \;=\; -\frac{\left(1 + \frac{\beta\gamma\hbar}{m_p^2}\right)}{\gamma\hbar\left(1 + \beta\frac{p^2}{m_p^2}\right)}\,p\,\psi(p)\,, \tag{11}$$

which has a generic solution of the form

$$\psi(p) \;=\; N\left[1 + (\beta\,p^2)/m_p^2\right]_+^{-\frac{m_p^2}{2\beta\gamma\hbar} - \frac{1}{2}}\,. \tag{12}$$

Here, $[z]_+ = \max\{z, 0\}$ guarantees that the wave functions (12) are single-valued. The normalization coefficient N ensures that $\int |\psi(p)|^2 dp = 1$ and for $\beta < 0$ it is

$$N_< \;=\; \sqrt{\sqrt{\frac{|\beta|}{m_p^2\pi}}\,\frac{\Gamma\left(\frac{1}{2} + \frac{m_p^2}{|\beta|\gamma\hbar}\right)}{\Gamma\left(\frac{m_p^2}{|\beta|\gamma\hbar}\right)}}\,, \tag{13}$$

with $\Gamma(x)$ being the Euler gamma function. In passing, we observe that as $\beta \to 0$ then Eq. (12) reduces to the usual minimum uncertainty Gaussian wave-packet (Glauber coherent state) associated with the Heisenberg uncertainty relation.

In order to determine the significance of the γ parameter, we note that (9) implies

$$0 \;=\; \langle\psi|\,(\hat{p} + i\gamma\hat{x})\,(\hat{p} - i\gamma\hat{x})\,|\psi\rangle \;=\; (\Delta p)^2 - \gamma\,|\langle\psi|\,[\hat{x},\hat{p}]\,|\psi\rangle| + \gamma^2\frac{|\langle\psi|\,[\hat{x},\hat{p}]\,|\psi\rangle|^2}{4(\Delta p)^2}\,, \tag{14}$$

where we utilized that (7) is saturated. This has a single solution for γ, which with the help of (2) can be written as

$$\gamma \;=\; \frac{2(\Delta p)^2}{\hbar\left[1 \;+\; \beta\,(\Delta p)^2/m_p^2\right]}\,. \tag{15}$$

We note that γ can be defined also in the limit $\Delta p \to \infty$, even though GUP (7) is in such a case meaningless. In addition, states (12) are clearly mirror symmetric.

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2.3. Tsallis' transition amplitude and Tsallis' entropies

Let us now take in (12) the following substitutions (valid for $\beta \lessgtr 0$):

$$q \;=\; \frac{\beta\gamma\hbar}{m_p^2 + \beta\gamma\hbar} + 1\,, \qquad b \;=\; \frac{2m_p}{\gamma\hbar} + \frac{2\beta}{m_p}\,. \tag{16}$$

With this, we can rewrite the two-parameter class of CSs (12) as

$$\psi(p) \;=\; \tilde{N}\left[1 - b\,(1-q)\frac{p^2}{2m_p}\right]_+^{\frac{1}{2(1-q)}}\,, \tag{17}$$

where $\tilde{N}$ is appropriately rescaled N. The above expression is nothing but the probability amplitude for the Tsallis distribution of a free, non-relativistic particle

$$q_T(p|q,b) \;=\; |\psi(p)|^2 \;=\; \frac{1}{Z}\left[1 - b\,(1-q)\frac{p^2}{2m_p}\right]_+^{\frac{1}{1-q}}, \tag{18}$$

with Z being the "partition function".

Two-parameter class of probability distributions (18) decays asymptotically following power law rather than exponential law. If we keep variance and mean as the only statistical observables, power-law type distributions are incompatible with the conventional maximum entropy prescription (MaxEnt) applied to Shannon–Gibbs entropy. It is, however, the Shannon–Gibbs' MaxEnt that provides Gaussian probability distributions associated with canonical (or Glauber) CSs. In this respect, it is interesting to observe that distributions (18) are maximizers for Tsallis (differential) entropy [30, 31, 32], i.e., entropy of the form

$$\mathcal{S}_q^T(\mathcal{F}) \;=\; \frac{1}{(1-q)}\left(\int_{\mathbb{R}} dp\, \mathcal{F}^q(p) - 1\right)\,, \tag{19}$$

where $\mathcal{F}$ is a probability density function. It can easily be seen (by applying L'Hopital's rule) that in the limit $q \to 1$ the Tsallis entropy tends to the Shannon entropy

$$\mathcal{S}^S(\mathcal{F}) \;=\; -\int_{\mathbb{R}} dp\, \mathcal{F}(p) \log_2 \mathcal{F}(p)\,, \tag{20}$$

and the ensuing maximizer, i.e. distribution (18), tends to a Gaussian distribution.

2.4. Connection with entropic uncertainty relations

Now, having the entropy that is maximized by CSs, we can rephrase the original GUP in terms of entropies. The strategy is basically the same as in conventional quantum mechanics.

To illustrate this point we start with the canonical commutation relation $[\hat{x},\hat{p}] = i\hbar$. This induces ordinary representations of $\hat{x}$ and $\hat{p}$. For instance, in momentum representation we have

$$(\hat{x})^{\mathcal{P}} \;=\; i\hbar\frac{d}{dp} \quad \text{and} \quad (\hat{p})^{\mathcal{P}} \;=\; p\,. \tag{21}$$

Since eigenstates $\langle p|\psi_x\rangle \equiv \psi_x(p)$ and $\langle x|\psi_p\rangle \equiv \psi_p(x)$ are plane waves, generic states in momentum and position representation are connected via the Fourier transform

$$\psi(x) \;=\; \int_{\mathbb{R}} e^{ip\cdot x/\hbar}\,\hat{\psi}(p)\,\frac{dp}{(2\pi\hbar)^{1/2}}\,.$$

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By using Beckner–Babenko's inequality for the Fourier transform duals (basically inequality between their norms) [34, 35], one can derive the Shannon entropy-based UR [37]

$$N(|\psi|^2)N(|\hat{\psi}|^2) \;\geq\; \frac{\hbar^2}{4}\,, \tag{22}$$

which for white Gaussian distributions (i.e. Gaussian distributions whose covariance matrix is proportional to the identity matrix) reduces to $\sigma_x\sigma_p = \frac{\hbar}{2}$. The function $N(\ldots)$ is the so-called Shannon entropy power, which is defined as [36]

$$\mathcal{S}^S\left(\mathcal{X}\right) \;=\; -\int d^D x\, \mathcal{F}(x) \log \mathcal{F}(x) \;=\; \mathcal{S}^S\left(\sqrt{N(\mathcal{X})}\cdot \mathcal{Z}^G\right)\,. \tag{23}$$

Here, $\{\mathcal{Z}_i^G\}$ is a Gaussian random vector with zero mean and unit covariance matrix and D is the dimension of the random vector. So, $N(\ldots)$ is the variance of a would-be Gaussian distribution that has the same (Shannon's) information content as the random variable $\mathcal{X}$ in question. In (23), we have employed the short-hand notation $\mathcal{S}^S(A) \equiv \mathcal{S}^S(\mathcal{F}_A)$, with $\mathcal{F}_A$ being a probability density function associated with the random variable describing the system A. Equation (23) has the unique solution [36]

$$N(\mathcal{X}) \;=\; \frac{1}{2\pi e}\exp\left(\frac{2}{D}\mathcal{S}(\mathcal{X})\right)\,. \tag{24}$$

A few observation are in order: a) entropy-power UR represented by (22) is saturated only for $\tilde{\psi}$ and ψ that are Gaussian (and white for higher dimensional random vectors), b) Shannon's entropy power is defined only for distributions with continuous spectrum (integral calculus is essential in the proof) and c) the above EPUR indicates that Shannon entropy is a pertinent entropy in the semi-classical regime of conventional quantum mechanics.

Let us now repeat the above steps for our GUP; in particular, we concentrate on $\beta < 0$. In this case, the canonical commutation relation reads $[\hat{x},\hat{p}] = i\hbar\left(1 - |\beta|\frac{\hat{p}^2}{m_p^2}\right)$, which induces in momentum space the representation

$$(\hat{x})^{\mathcal{P}} \;=\; i\hbar\left(\frac{d}{dp} - \frac{|\beta|}{2m_p^2}\left[p^2, \frac{d}{dp}\right]_+\right) \quad \text{and} \quad (\hat{p})^{\mathcal{P}} \;=\; p\,. \tag{25}$$

Eigenstates $\langle p|\psi_x\rangle \equiv \psi_x(p)$ are not plane waves but rather they have the form

$$\psi_x(p) \;=\; B_x\,\frac{e^{-ixm_p \operatorname{arctanh}\left(p\sqrt{|\beta|}/m_p\right)/\hbar\sqrt{|\beta|}}}{\sqrt{m_p^2 \;-\; p^2|\beta|}}\,, \quad B_x = \sqrt{m_p^2/2\pi\hbar}\,. \tag{26}$$

Note that $\psi_x(p)$ are not quadratically integrable, namely $\psi_x(p) \notin L^2((-m_p/\sqrt{|\beta|}, m_p/\sqrt{|\beta|}))$, but instead $\psi_x(p) \in \mathcal{S}^{1\prime}((-m_p/\sqrt{|\beta|}, m_p/\sqrt{|\beta|}))$, i.e. they belong to the space of complex-valued tempered distributions (like plane waves). It can be further shown [29] that for $\beta < 0$ the operator $(\hat{x})^{\mathcal{P}}$ is self-adjoint and its spectrum is continuous.

Generic states in momentum and position representation are thus not connected via the Fourier transform, but via the *Abel transform*

$$\psi(x) \;=\; \int_{-m_p/\sqrt{|\beta|}}^{m_p/\sqrt{|\beta|}} dp\; \psi_p(x)\tilde{\psi}(p) \;=\; \int_{-m_p/\sqrt{|\beta|}}^{m_p/\sqrt{|\beta|}} \frac{dp}{\sqrt{2\pi\hbar}}\,\frac{e^{ixm_p \operatorname{arctanh}\left(p\sqrt{|\beta|}/m_p\right)/\hbar\sqrt{|\beta|}}}{\sqrt{1 - p^2|\beta|/m_p^2}}\,\tilde{\psi}(p)\,, \tag{27}$$

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where $\tilde{\psi}(p) = \langle p|\psi\rangle$. Analogous relation holds for the inverse transformation.

At this stage, we define the entropy power based on Tsallis entropy as

$$\mathcal{S}_q^T(\mathcal{X}) \;=\; \mathcal{S}_q^T\left(\sqrt{M_q^T(\mathcal{X})}\cdot\mathcal{Z}_{2-q}^T\right). \tag{28}$$

Here $\{\mathcal{Z}_i^T\}$ is a Tsallisian random vector (i.e. random vector distributed according to Tsallis distribution) with zero mean and unit covariance matrix. Eq. (28) has a unique solution [29, 37]

$$M_q^T(\mathcal{X}) \;=\; A_q\,\left[\exp_q(\mathcal{S}_q^T(\mathcal{X}))\right]^{2/D} \;=\; A_q\,\exp_{1-(1-q)D/2}\left(\frac{2}{D}\,\mathcal{S}_q^T(\mathcal{X})\right), \tag{29}$$

with A_q being a q-dependent constant and D dimension of the random vector. In (29) we have employed the q-exponential $e_q^x = [1+(1-q)x]^{1/(1-q)}$. As a consistency check, we might notice that in the $q\to 1$ limit we recover Shannon's EP, namely

$$\lim_{q\to 1} M_q^T(\mathcal{X}) \;=\; \frac{1}{2\pi e}\exp\left(\frac{2}{D}\mathcal{S}^S(\mathcal{X})\right) \;=\; N(\mathcal{X}). \tag{30}$$

At this stage one can employ the fact that the Abel transform (27) can be equivalently rewritten as the Fourier transform of the function

$$\bar{\psi}(z) \;=\; \frac{\tilde{\psi}(m_p\tanh(z\sqrt{|\beta|}/m_p)/\sqrt{|\beta|})}{\cosh(z\sqrt{|\beta|}/m_p)}, \tag{31}$$

and use again Beckner–Babenko's inequality for the Fourier transform duals. After some algebra, one obtains the Tsallis entropy-based URs [29]

$$M_{q/2}^T(|\psi|^2)M_{q'/2}^T(|\tilde{\psi}|^2) \;\geq\; \frac{\hbar^2}{4}, \tag{32}$$

where q' and q are Hölder conjugates, i.e. $1/q+1/q'=1$ with $q\in\mathbb{R}^+$. This one-parameter class of entropy power URs clearly resembles the form of Shannon's entropy power UR (22) by having the unique irreducible lower bound.

It is now worth noting a few points; a) the inequality (32) is saturated if and only if ψ and $\tilde{\psi}$ are CSs of GUP (1), b) the above entropy power URs can be formulated only for $\beta<0$ (as in such cases the spectrum is continuous) and c) entropic URs (32) indicate that Tsallis entropy is a pertinent entropy functional in the GUP framework. CSs saturating UR belong to the class of so-called pointer states, i.e. those states that are least affected by the interaction with the environment [38, 39]. Such states are particularly pertinent to the quasi-classical domain of quantum theory, as they are maximally predictable despite decoherence [40, 41]. Among all pointer states in the would-be GUP-driven Universe, only CSs (18) saturate both the "x-p" GUPs and ensuing Tsallis entropy power URs. Moreover, the existence of Tsallis entropy power UR indicates that Tsallis' entropy should be a relevant entropy in the GUP context. If we couple this observation with the fact that CSs (18) extremize Tsallis entropy, we might invoke (similarly as in conventional statistical physics) MEP, but this time with Tsallis entropy (in place of Shannon–Gibbs entropy) to discuss a statistical physics of an ensemble of non-interacting GUP-governed particles in their quasi-classical regime. In this context, non-extensive thermostatistics of Tsallis (NTT) [30] should provide the necessary mathematical framework that ought to be employed to describe the borderline between classical and quantum physics in the GUP Universe.

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3. Applications

3.1. Verlinde's entropic gravity

Particularly interesting applications of the aforementioned observation can be expected in the early-Universe gravity. To do this, we must extend our one-dimensional analysis to three dimensions. This can be done, e.g., by extending the algebra (2)-(3) as, cf. [6, 42]

$$[\hat{\mathbf{x}}_i, \hat{\mathbf{p}}_j] \ = \ i\hbar\delta_{ij}\left(1 \ + \ \beta\frac{\mathbf{p}^2}{m_p^2}\right), \quad [\hat{\mathbf{x}}_i, \hat{\mathbf{x}}_j] \ = \ 2i\hbar\frac{\beta}{m_p^2}\left(\hat{\mathbf{x}}_i\hat{\mathbf{p}}_j - \hat{\mathbf{x}}_j\hat{\mathbf{p}}_i\right), \quad [\hat{\mathbf{p}}_i, \hat{\mathbf{p}}_j] \ = \ 0\,, \tag{33}$$

which in the momentum space representation (in terms of symmetric operators) yields

$$\hat{\mathbf{p}}_i\,\psi(\mathbf{p}) \ = \ \mathbf{p}_i\,\psi(\mathbf{p})\,, \qquad \hat{\mathbf{x}}_i\,\psi(\mathbf{p}) \ = \ i\hbar\left(\frac{d}{d\mathbf{p}_i} + \frac{\beta}{2\,m_p^2}\left[\mathbf{p}^2, \frac{d}{d\mathbf{p}_i}\right]_+\right)\psi(\mathbf{p})\,. \tag{34}$$

By following a similar route as in 1 dimension, the system of commutators admits only one (normalized) mirror-symmetry solution for $\psi \in L^2(\mathbb{R}^3)$, namely

$$\psi(\mathbf{p}) \ = \ N\left[1 \ + \ (\beta\,\mathbf{p}^2)/m_p^2\right]_+^{-\frac{m_p^2}{2\beta\gamma\hbar}-\frac{1}{2}} \qquad \text{with} \quad N_<^2 \ = \ \frac{\beta^{3/2}}{2\pi m_p^3\,B(5/2, m_p^2/|\beta|\gamma\hbar)}\,. \tag{35}$$

Here, $B(x, y)$ is the beta function. At this stage, we can ask what modifications to Newton's law should be expected in the quasi-classical limit of the GUP-based Universe. To answer this question, we can combine Verlinde's idea that gravity is an entropy-driven phenomenon — entropic gravity (EG), with the non-extensive thermostatistics of Tsallis.

Following Verlinde [24], we suppose that the true (unknown) microscopic degrees of freedom in any given part of space are stored in discrete bits on the holographic screen that surrounds them. A holographic screen can be considered to be spherically symmetric of area $A = 4\pi R^2$. Outside of the screen is the *emergent world*, so the screen acts as an interface between known and unknown physics. When a test particle moves away from the screen, it feels an effective force F satisfying $F\delta x = T\delta S$, where T and δS are the temperature and the entropy change on the holographic surface, respectively, and δx is the distance of the particle from the screen. In NTT the heat one-form $T\delta S$ must be replaced with [43]: $T\delta\mathcal{S}_q^T/[1 + ((1-q)/k_B)\mathcal{S}_q^T]$ (in our context $\mathcal{S}_q \mapsto \mathcal{S}_{2-q}$ as it is $\mathcal{S}_{2-q}$ that is extremized by $q_T(\mathbf{p}|q, b)$). If L is a (dimensionless) characteristic length scale (e.g. radius R/ℓ_p) then the Bekenstein–Hawking entropy $S_{BH} = \ln W(L) \propto L^2$, which implies that the total number of internal configurations W behaves as $W(L) = \phi(L)\nu^{L^2}$ for $L \gg 1$ (ϕ is a positive function satisfying $\lim_{L\to\infty} \ln\phi/L^2 = 0$ and $\nu > 1$ constant). Hence, from the outside, the holographic screen has Tsallis'entropy

$$\mathcal{S}_{2-q}^T = \ k_B \ln_{2-q} W(L) = \frac{k_B}{q-1}\left[\left(\phi(L)\nu^{L^2}\right)^{q-1} - 1\right]. \tag{36}$$

Consequently, the entropic force follows from the relation

$$F\delta x \ \ = \ \ \frac{T\delta S_{2-q}^T}{1 + (q-1)\left(\omega_3 L^3 + \omega_2 L^2 + \cdots\right) + \cdots}\,, \tag{37}$$

where $\omega_2, \omega_3 > 0$ are intensive coefficients — so-called Hills' coefficients, known from entropy expansion in (conventional) thermodynamics of small and mesoscopic systems [44]. To comply with Hills' expansion, we have formally included the term $\omega_3 L^3$, even if it is not supported by the EG prescription. It will be seen that such a term is cosmologically unfeasible in the quasi-classical regime, so that $\omega_3 \approx 0$.

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By holographic scaling, the energy residing inside the holographic screen is related with the on-screen degrees of freedom via the equipartition theorem $E = Nk_BT/2$, with $E = M$ being the total mass enclosed by the surface and $N = A/(G\hbar)$ the number of bits connected with the area by the holographic principle [24]. EG paradigm posits that the minimum possible increase in the screen entropy (equivalent to one bit of Shannon's information) happens if a particle of radius of Compton wave length λ_C is added to a holographic sphere [24, 45]. This happens when a point-like quantum particle appears at the distance λ_C from the screen (note that λ_C is the minimal distance at which a particle stil retains its single-particle picture). By setting $\delta x = \lambda_C = \hbar/m$ and using the non-extensive version of Landauer principle [46, 47], which states that the erasure of information leads to an entropy increase $\delta S_q^T = 2\pi k_B/(3-2q)$ per erased bit, we derive the following modified Newton's law

$$F(R) \ = \ \frac{GMm}{wR^2}\frac{1}{1-\kappa_3\varepsilon_q R^3-\kappa_2\varepsilon_q R^2}\,, \tag{38}$$

with $\varepsilon_q = 1-q$, $w = 1+2\varepsilon_q$ and $\kappa_n = \omega_n/\ell_p^n$ ($\ell_p = \hbar/m_p \approx 1.6\times 10^{-35}$m is the Planck length), $n = 2, 3$. Since $2\varepsilon_q$ is small (see below), we can set $w = 1$. The ensuing gravitational potential up to the first-order in ε_q then reads

$$V(R) \ = \ \frac{r_s}{2}\left[-\frac{1}{R} \ + \ \varepsilon_q\kappa_2 R \ + \ \frac{\varepsilon_q\kappa_3}{2}\,R^2\right], \tag{39}$$

where $r_s = 2GM$ is the Schwarzschild radius. The gravitational potential (39) coincides with the Mannheim–Kazanas gravitational potential of a static, spherically symmetric source of mass M in conformal Weyl gravity (CWG) [26, 27]. Strictly speaking, in CWG a given local gravitational source generates only a gravitational potential

$$V_{MK}(R) \ = \ -\frac{r_s}{2R} \ + \ \frac{\chi}{2}R \ = \ \frac{r_s}{2}\left[-\frac{1}{R} \ + \ (1-q)\kappa_2\,R\right], \tag{40}$$

where $\kappa_2 = \omega_2/\ell_p^2$ ($\omega_2 = \pi$ is the second Hill's coefficient [29, 44]). The would-be term $\propto R^2$ corresponds to a trivial vacuum solution of CG and hence does not couple to matter sources [25, 26, 27]. Fitting with CWG thus implies $\omega_3 \approx 0$.

3.2. Some cosmological consequences

In CWG, the parameter in front of the linear term is identified with the inverse Hubble length R_H (more precisely with $1/(2R_H)$) [48]. It s quite intriguing that, for present macroscopic scales (i.e., $R_H \sim 10^{26}$m), the Mannheim–Kazanas solution has been successful in fitting more than two hundred galactic rotation curves with no adjustable parameters (other than the galactic mass-to-light ratios) with no need for dark matter or other exotic modifications of gravity [26, 27]. Despite the fact that macroscopic-scale gravity does not fall within the assumed quasi-classical regime, the idea that the coefficient in front of a linear term in (40) should be associated with the inverse Hubble length is valid even in the early Universe cosmology. This is because the argument of CWG leading to this result is independent of an actual Universe epoch [49].

In conventional cosmology, it is expected that a quasi-classical description becomes pertinent at the late-inflation epoch (after the first Hubble radius crossing) and perhaps even after its end during reheating [51, 50]. So, in this period the NTT should be a suitable framework for the description of an "inflaton gas". For instance, by viewing the "inflaton gas" as an ideal gas, the NTT predicts that the inflaton pressure for $0 < q < 1$ should satisfy a polytrope relation [52, 53]

$$p \ = \ \frac{2\pi\hbar^2}{m_i\,e^{5/3}}\,\rho^{5/3}\,, \tag{41}$$

where $\rho = N/V$ is the particle density. In this connection, it should be stressed that the relation (41) holds for $0 < q < 1$ but not in the limit $q \to 1$, see, e.g. [53]. In fact, at $q = 1$ one has the familiar pressure relation $p \propto \rho_E$. So, the NTT and extensive limits are not interchangeable. The polytrope relation of the type (41) often appears in studies on late inflation, see, e.g. [54, 55].

In order to gain information about β, we employ the CWG observation that the cosmologically viable linear term in (40) should have its parameter associated with $1/R_H$. According to CWG, the Newtonian potential (40) should dominate on short scales, while the linear one becomes prominent at large scales. Both potentials get equal at R_H, which in our case implies that $q = 1 - \ell_p^2/(\pi R_H^2)$. Note that this is compatible with the condition that $r_s = R_H$. By combining the latter expression for q with (15) and (16), we obtain $|\beta| \simeq m_p^2\,\ell_p^2/(2\pi\,(\Delta p)_\psi^2\,R_H^2)$.

To see how such β explicitly depends on a cosmological time t, we first write $R_H(t) = H^{-1}(t) = a(t)/\dot{a}(t)$, where H is the Hubble parameter and $a(t)$ is the scale factor. The latter can be evaluated, e.g., from the Vilenkin–Ford inflationary model [56], where $a(t) = A\sqrt{\sinh{(B\,t)}}$, with $B = 2\sqrt{\Lambda/3}$ (Λ is the cosmological constant). We then use the result from the relativistic equipartition theorem $(\Delta p)_\psi^2 \simeq 12\,(k_B T)^2$ and after simple algebraic manipulations, we obtain

$$|\beta| \;\equiv\; |\beta(t)| \;=\; \frac{m_p^2\,\ell_p^2\,\Lambda}{72\pi\,(k_B T)^2\,\tanh^2\left(2t\sqrt{\Lambda/3}\right)}\,. \tag{42}$$

For concreteness' sake, let us consider the reheating epoch, i.e. time scale $t \simeq 10^{-33}$s. By taking the mass of the inflaton $m_i = 10^{12} \div 10^{13}$GeV, T of the order of the reheating temperature $T_R \simeq 10^7 \div 10^8$GeV and the presently known value of the cosmological constant $\Lambda \simeq 10^{-52}\text{m}^{-2}$, we obtain $|\beta| \sim 10^{-2} \div 1$, which is in agreement with the values predicted by string-theory models, cf. e.g. [1, 3, 4]. This result is also consistent with the naturalness principle that dictates that not so far from the Planck scale the β should not be too large nor too small.

4. Conclusions

In this paper, we have unified two seemingly unrelated concepts, namely generalized uncertainty principle and Tsallis thermostatistics. On the one hand, the GUP strives to explore consequences of the existence of a minimal length scale. On the other hand, Tsallis statistics is a theoretical concept that accounts for systems with long-range correlations or long-time memory, for which the conventional central limit theorem does not apply. A merger of these two concepts presented here is intriguing from both a conceptual perspective and from a phenomenological point of view.

In order to substantiate our point, we have employed the NTT to generalize Verlinde's entropic force. Apart from obtaining a modified Newtonian (basically Mannheim–Kazanas-like) potential, we have argued that such a generalization should be phenomenologically pertinent at the late-inflation epoch. The corresponding dependence of the GUP β parameter on the cosmological time t has also been derived for the reheating epoch. The β parameter obtained is consistent both with values predicted by string-theory models and with the naturalness principle.

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