RANDOM MATRICES

IN TOPOLOGICAL STRING THEORY

ACADEMISCH PROEFSCHRIFT

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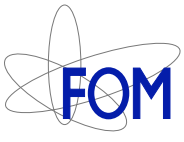

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Voor mijn moeder
Anneme adıyorum

PUBLICATIONS

This thesis is based on the following papers:

[1] R. Dijkgraaf, A. Sinkovics and M. Temurhan, "Matrix models and gravitational corrections", Adv. Theor. Math. Phys. **7** (2004) 1155 [arXiv:hep-th/0211241].

[2] R. Dijkgraaf, A. Sinkovics and M. Temurhan, "Universal correlators from geometry", J. High Energy Phys. JHEP **11** (2004) 012 [arXiv:hep-th/0406247].

CONTENTS

CHAPTER 1

INTRODUCTION

This thesis has been mostly motivated by the work of Dijkgraaf and Vafa, who wrote a series of papers in the summer of 2002 [3, 4, 5], where they made a connection between topological string theory and large N random matrix models on the one hand, and showed a way to calculate the effective superpotential of $\mathcal{N} = 1$ supersymmetric gauge theories from topological strings on the other hand.

These three subjects, topological strings, random matrices and $\mathcal{N} = 1$ supersymmetric gauge theories, are by themselves vast areas where one can get lost very easily. So it requires a certain amount of discipline and perseverance to get to the bottom of the Dijkgraaf-Vafa conjecture.

In this thesis, I will attempt to give an idea of these areas, first separately, with always in mind that they will be connected to each other, and then continue to some specific calculations that confirm the conjecture in one way or another.

In chapter 2, topological strings are introduced, including the topological A- and the B-model, which are topologically twisted versions of the sigma model. Also a notion of superspace and supersymmetry is given to prepare the way towards an understanding of the Dijkgraaf-Vafa conjecture, together with some mathematical tools that we will need later.

In chapter 3, random matrices are introduced as simple gauge theories, with the stress on the $N \times N$ one-matrix model and its large N limit where the 't Hooft coupling is fixed. This is connected to topological string theory through the open/closed string duality. Also the process of how the matrix model potential and free energy arise from the topological theory is shown. Multicut matrix models, as opposed to one-cut matrix models, have also been treated. The Gaussian matrix model is given

as an example.

In chapter 4, a gauge theoretic description of the conjecture is stated. Here it is shown that the effective $\mathcal{N} = 1$ superpotential as a function of the glueball superfield can be calculated exactly by means of a sum over the planar diagrams arising in this gauge theory. It is also shown that the conjecture extends to the deformed $\mathcal{N} = 2$ super Yang-Mills theory, where the $\mathcal{N} = 2$ supersymmetry is broken to $\mathcal{N} = 1$ supersymmetry by the presence of the effective superpotential. This is also related to Seiberg-Witten theory for flat spaces and to the Donaldson-Witten theory for curved spaces. The cubic matrix model is given as an example. A geometric description is also treated.

In chapter 5, the matrix model free energy at genus one is calculated directly from the matrix model loop equations using conformal field theory methods. This is then compared to the known results from topologically twisted $\mathcal{N} = 2$ super Yang-Mills theories and an agreement is found. This confirms the Dijkgraaf-Vafa conjecture at the nonplanar level.

In chapter 6, we shift our attention to a different description of matrix models from topological strings. In particular, instead of the 't Hooft limit, where a double scaling is taken, we take a look at Kontsevich-like matrix models for finite N. Here, a description of topological B-branes is given since they lead to these Kontsevich-like matrix models by inserting branes in the geometry. These B-branes are then treated as fermions. This is based on [6].

In chapter 7, we use our knowledge from the previous chapter of brane-insertions and calculate universal correlators in the matrix models. These correlators are shown to be related to universal correlators from Calabi-Yau crystals, which are treated as 3D random partitions.

Considering the huge amount of papers written in the context of the Dijkgraaf-Vafa conjecture, it was tempting to make an unreadable chapter as a summary references. However, it seemed a better choice to include the relevant references throughout the sections, whenever it seemed fit. Hopefully, this will make the reading of this thesis easier and more worthwile.

CHAPTER 2

TOPOLOGICAL STRINGS

In very general terms string theory is defined by a set of maps from the two-dimensional worldsheet Σ to a target space $\mathcal{M}$. This worldsheet is swept out by strings, one dimensional extended objects, as they move through spacetime. For a detailed introduction to string theory, please consult the standard books on the topic such as [7, 8].

Topological string theory is a simplification of physical string theory and the term "topological" is referring to the property that the theory is independent of the choice of a metric on the worldsheet. There are several detailed reviews on topological string theory, see for example [9, 10, 11, 12, 13].

What follows is meant to give an idea of what topological strings are and how they are related to random matrix models.

We are interested in certain topological strings since they can lead to simple gauge theories. For a special class of topological strings these gauge theories are just matrix models. This has been stated in a series of influential papers by Dijkgraaf and Vafa [3, 4, 5], see section 3.2 and chapter 4 for more on the Dijkgraaf-Vafa conjecture.

Another way in which topological string theory leads to matrix models is through the so-called B-branes. This methods brings us at Kontsevich like matrix models, as we will see in chapter 6.

What is a topological string?

To answer this question we first need to introduce some notions. As mentioned, string theory is defined by a set of maps $\phi : \Sigma \to \mathcal{M}$ from a Riemann surface Σ to a Riemannian target space $\mathcal{M}$. In this theory one integrates over all these maps ϕ and

over metrics g on Σ:

$$\int \mathcal{D}\phi\, \mathcal{D}g\, \mathrm{e}^{-S[g,\phi]},$$

where S is the Polyakov action

$$S[g,\phi] = \frac{T}{2}\int \mathrm{d}^2\sigma \sqrt{|\det g_{\alpha\beta}|}\, g^{\alpha\beta}\, \partial_\alpha \phi^\mu\, \partial_\beta \phi^\nu\, G_{\mu\nu}(\phi).$$

Here $\sigma = (\tau, \sigma)$ are the worldsheet coordinates, $G_{\mu\nu}(\phi)$ is the Riemannian background metric and T is the string tension. The path-integral over ϕ defines a two-dimensional quantum field theory known as a sigma model. Since in string theory one integrates over all the metrics as well, it can be thought of as a sigma model coupled to two-dimensional gravity.

Our focus will be on supersymmetric theories, which contain both bosonic and fermionic variables. The sigma model we have mentioned is a supersymmetric theory. First a few words about supersymmetry; for an introduction to supersymmetry the reader could take a look at, for example, [14, 10, 15, 16].

2.1 SUPERSYMMETRY

As an example, we will take a look at a one-dimensional supersymmetric theory with one bosonic variable x and its complex fermionic superpartner ψ, that has the Lagrangian

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}(W'(x))^2 + \frac{\mathrm{i}}{2}\left(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi\right) - W''(x)\,\bar{\psi}\psi,$$

where $\frac{1}{2}(W'(x))^2$ is the potential term and $W(x)$ is a real function of x. The fermionic variables are anti-commuting and $\bar{\psi} = \psi^\dagger$. The transformation rules

$$\begin{aligned}
\delta_\epsilon x &= \epsilon\bar{\psi} - \bar{\epsilon}\psi, \\
\delta_\epsilon \psi &= \epsilon\left(\mathrm{i}\dot{x} + W'(x)\right), \\
\delta_\epsilon \bar{\psi} &= \bar{\epsilon}\left(-\mathrm{i}\dot{x} + W'(x)\right),
\end{aligned} \tag{2.1}$$

with ϵ a complex variable and $\bar{\epsilon} = \epsilon^*$, change the Lagrangian only by a total derivative $\delta L = \frac{\mathrm{d}}{\mathrm{d}t}(\ldots)$ and the action is invariant, $\delta S = \int \delta L \mathrm{d}t = 0$, as long as the boundary variation vanishes. When this is the case the system has a symmetry generated by the above transformations, and since ϵ is fermionic, the symmetry is called fermionic symmetry.

For two of such transformations δ_{ϵ_1} and δ_{ϵ_2}, one can write down the relations

$$\begin{aligned}[\delta_{\epsilon_1}, \delta_{\epsilon_2}]x &= 2\mathrm{i}\,(\epsilon_1\bar{\epsilon}_2 - \epsilon_2\bar{\epsilon}_1)\,\dot{x},\\ [\delta_{\epsilon_1}, \delta_{\epsilon_2}]\psi &= 2\mathrm{i}\,(\epsilon_1\bar{\epsilon}_2 - \epsilon_2\bar{\epsilon}_1)\,\dot{\psi}.\end{aligned} \tag{2.2}$$

When these relations hold, such fermionic transformations are called supersymmetry and the classical system with the Lagrangian L is called supersymmetric.

Going from this simple example to more complicated ones, we will consider a two-dimensional supersymmetric theory with four real charges. Two of these charges have positive helicity and the other two have negative helicity under the Lorentz group. This symmetry is called $\mathcal{N} = (2,2)$ supersymmetry and will be important in topological string theory. In chapter 4, we will consider $\mathcal{N} = 1$ supersymmetric theories with two real supercharges. The superspace formalism will be similar to what we will see below.

2.1.1 SUPERSPACE

Since we will be working in two dimensions it is convenient to use the superspace formalism with the fermionic coordinates $\theta^{\pm}$, $\bar{\theta}^{\pm}$, added to the original coordinates z, $\bar{z}$ of $\mathbb{C}$. The fermionic coordinates are also complex and satisfy $(\theta^{\pm})^* = \bar{\theta}^{\pm}$ under complex conjugation. There are as many fermionic coordinates as there are supersymmetries, in this case there are four of them. Under the Lorentz group the fermionic coordinates transform as

$$\theta^{\pm} \mapsto \mathrm{e}^{\pm\mathrm{i}\alpha/2}\,\theta^{\pm}, \quad \bar{\theta}^{\pm} \mapsto \mathrm{e}^{\pm\mathrm{i}\alpha/2}\,\bar{\theta}^{\pm}.$$

They anti-commute with each other, $\theta^a\,\theta^b = -\theta^b\,\theta^a$, $\bar{\theta}^a\,\bar{\theta}^b = -\bar{\theta}^b\,\bar{\theta}^a$, $\theta^a\,\bar{\theta}^b = -\bar{\theta}^b\,\theta^a$ and, being Grassmann-valued, square to zero, $(\theta^{\pm})^2 = (\bar{\theta}^{\pm})^2 = 0$.

A superfield is a function in superspace and can be written as an expansion

$$\Phi(z, \bar{z}, \theta^{\pm}, \bar{\theta}^{\pm}) = \phi(z,\bar{z}) + \psi_+(z,\bar{z})\,\theta^+ + \psi_-(z,\bar{z})\,\theta^- + \cdots$$

where the fields ϕ, ψ_+, ψ_-, ... can be ordinary functions or Grassmann-valued functions on $\mathbb{C}$.

The integration measure

$$\mathrm{d}z\,\mathrm{d}\bar{z}\,\mathrm{d}\theta^+\,\mathrm{d}\theta^-\,\mathrm{d}\bar{\theta}^+\,\mathrm{d}\bar{\theta}^-$$

is invariant under certain transformations like the rotations and translations of the Poincaré group. Writing $z = x^1 + \mathrm{i}x^0$, in the Euclidean setting, the Hamiltonian and

the momentum operator can be written as

$$\begin{aligned} H &= -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}(\mathrm{i}x^0)} = -\mathrm{i}(\partial_+ - \partial_-), \\ P &= -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}(x^1)} = -\mathrm{i}(\partial_+ + \partial_-), \end{aligned}$$

with $\partial_+ = \partial_z$ and $\partial_- = \partial_{\bar{z}}$.

Acting on the fermionic coordinates there is a rotation generator M,

$$M = 2z\,\partial_+ - 2\bar{z}\,\partial_- + \theta^+\frac{\mathrm{d}}{\mathrm{d}\theta^+} - \theta^-\frac{\mathrm{d}}{\mathrm{d}\theta^-} + \bar{\theta}^+\frac{\mathrm{d}}{\mathrm{d}\bar{\theta}^+} - \bar{\theta}^-\frac{\mathrm{d}}{\mathrm{d}\bar{\theta}^-}\,. \tag{2.3}$$

The operators H, P and M form an algebra with the only nonzero relations

$$\begin{aligned} [M,H] &= -2P, \\ [M,P] &= -2H. \end{aligned}$$

There are also ferminonic operators leaving the measure invariant, which are differential operators and generators of the symmetry,

$$\begin{aligned} \mathcal{Q}_\pm =& \frac{\partial}{\partial\theta^\pm} + \mathrm{i}\bar{\theta}^\pm\,\partial_\pm, \\ \bar{\mathcal{Q}}_\pm =& -\frac{\partial}{\partial\bar{\theta}^\pm} - \mathrm{i}\theta^\pm\,\partial_\pm. \end{aligned}$$

These operators have fermionic charges $\pm\frac{1}{2}$. The covariant derivatives are given by

$$\begin{aligned} D_\pm =& \frac{\partial}{\partial\theta^\pm} - \mathrm{i}\bar{\theta}^\pm\,\partial_\pm, \\ \bar{D}_\pm =& -\frac{\partial}{\partial\bar{\theta}^\pm} + \mathrm{i}\theta^\pm\,\partial_\pm\,. \end{aligned}$$

Together with the operators H, P and M, they satisfy

$$\begin{aligned} \{\mathcal{Q}_\pm, \bar{\mathcal{Q}}_\pm\} &= -2\mathrm{i}\partial_\pm = P \pm H, \\ \{D_\pm, \bar{D}_\pm\} &= 2\mathrm{i}\partial_\pm = -P \mp H, \end{aligned}$$

and

$$\begin{aligned} [M, \mathcal{Q}_\pm] &= \mp\,\mathcal{Q}_\pm, \\ \left[M, \bar{\mathcal{Q}}_\pm\right] &= \mp\,\bar{\mathcal{Q}}_\pm, \\ [M, D_\pm] &= \mp\,D_\pm, \\ \left[M, \bar{D}_\pm\right] &= \mp\,\bar{D}_\pm. \end{aligned}$$

A superfield Φ satifying the conditions

$$\bar{D}_\pm \, \Phi = 0$$

is called chiral and has the general decomposition

$$\Phi = \phi(w, \bar{w}) + \theta^+ \, \psi_+(w, \bar{w}) + \theta^- \, \psi_-(w, \bar{w}) + \theta^+ \, \theta^- F(w, \bar{w}),$$

with $w = z - \mathrm{i}\theta^+ \, \bar{\theta}^+$, $\bar{w} = \bar{z} - \mathrm{i}\theta^- \, \bar{\theta}^-$ and F an auxiliary field.

An anti-chiral superfield has to satify

$$D_\pm \, \bar{\Phi} = 0,$$

and has a similar decomposition as a chiral superfield.

We would like to contstruct action functionals of superfields that are invariant under the following transformation:

$$\delta = \epsilon_+ \, \mathcal{Q}_- - \epsilon_- \, \mathcal{Q}_+ - \bar{\epsilon}_+ \, \bar{\mathcal{Q}}_- + \bar{\epsilon}_- \, \bar{\mathcal{Q}}_+ \, .$$

Integrals of type

$$\int \mathrm{d}^2 z \, \mathrm{d}^4 \theta \, K(\Phi, \bar{\Phi}),$$

where $K(\Phi, \bar{\Phi})$ is a scalar field, are invariant under this transformation and are called D-terms.

Integrals like

$$\int \mathrm{d}^2 z \, \mathrm{d}^2 \theta \, W(\Phi),$$

with $W(\Phi)$ a holomorphic function of Φ, thus independent of $\bar{\Phi}$, are also invariant and are called F-terms. Such terms usually give rise to a potential.

As an example we take a look at a theory with one chiral superfield. The D-term

$$S_{\mathrm{kin}} = \int \mathrm{d}^2 z \, \mathrm{d}^4 \theta \, \bar{\Phi} \, \Phi$$

takes the following form after integration over $\mathrm{d}^4\theta$, using the superfield expansion [10],

$$S_{\mathrm{kin}} = \int \mathrm{d}^2 z (|\partial_0 \phi|^2 - |\partial_1 \phi|^2 + \mathrm{i}\bar{\psi}_-(\partial_0 + \partial_1)\psi_- + \mathrm{i}\bar{\psi}_+(\partial_0 - \partial_1)\psi_+ + |F|^2). \quad (2.4)$$

This is the standard kinetic term for the complex scalar field ϕ and the Dirac fermion fields $\psi_\pm$, $\bar{\psi}_\pm$.

The F-term

$$S_{\mathrm{W}} = \int \mathrm{d}^2 z \, \mathrm{d}^2 \, \theta \, W(\Phi) + \mathrm{c.c.}$$

with the holomorphic function $W(\Phi)$, called the superpotential, gives after integrating over $\mathrm{d}^2\theta$

$$S_{\mathrm{W}} = \int \mathrm{d}^2 z (W'(\phi) F - W''(\phi)\psi_+\psi_- + \bar{W}' \, \bar{F} - \bar{W}''(\bar{\phi})\bar{\psi}_-\bar{\psi}_+). \tag{2.5}$$

The sum $S = S_{\mathrm{kin}} + S_{\mathrm{W}}$ can be rewritten in such a way that the terms involving the auxiliary field F, written as

$$|F + \bar{W}'(\bar{\phi})|^2,$$

can be eliminated by solving the equation of motion $F = -\bar{W}'(\bar{\phi})$. The remaining action is equal to the action for the scalar ϕ and the Dirac fermion $\psi_\pm$, $\bar{\psi}_\pm$ with a potential $|W'(\phi)|^2$ for ϕ and the fermion mass term $\bar{W}''(\bar{\phi})\bar{\psi}_-\bar{\psi}_+$.

This action is by definition invariant under the transformation δ which can be written as a variation of the component fields $\phi, \psi_\pm, \bar{\psi}_\pm, F$ of the superfield Φ. This is possible since the operators $\mathcal{Q}$ and D anti-commute and the variation $\delta\Phi$ will still be chiral,

$$\bar{D}_\pm \delta\Phi = \delta \bar{D}_\pm \Phi = 0.$$

This variation is given by

$$\begin{aligned} \delta\phi &= \epsilon_+ \, \psi_- - \epsilon_- \, \psi_+, \\ \delta\psi_\pm &= \pm 2\mathrm{i} \, \bar{\epsilon}_\mp \, \partial_\pm \, \phi + \epsilon_\pm \, F, \\ \delta F &= -2\mathrm{i} \, \bar{\epsilon}_+ \partial_- \, \psi_+ - 2\mathrm{i} \, \bar{\epsilon}_- \, \partial_+ \psi_- \, . \end{aligned} \tag{2.6}$$

plus their complex conjugates. Here F can again be eliminated.

Because of the anti-commutation relations $\{\mathcal{Q}_\pm, \bar{\mathcal{Q}}_\pm\} = -2\mathrm{i}\,\partial_\pm$, the variations for different parameters ϵ_1 and ϵ_2 satisfy

$$[\delta_1, \delta_2] = 2\mathrm{i}(\epsilon_{1-}\bar{\epsilon}_{2-} - \epsilon_{2-}\bar{\epsilon}_{1-})\partial_+ + 2\mathrm{i}(\epsilon_{1+}\bar{\epsilon}_{2+} - \epsilon_{2+}\bar{\epsilon}_{1+})\partial_- \, .$$

This is a generalized version of the supersymmetry relation Eq.2.2 with the transformation rules Eq.2.1. This means that the classical field theory given by the Eqs.2.4 + 2.5 has a supersymmetry generated by the variations in Eq.2.6.

Using the symmetry of the classical system, one can find the conserved Noether currents, defined by

$$\int (\partial_\mu \epsilon_+ G^\mu_- - \partial_\mu \epsilon_- G^\mu_+ - \partial_\mu \bar{\epsilon}_+ \bar{G}^\mu_- + \partial_\mu \bar{\epsilon}_- \bar{G}^\mu_+) \sqrt{g} \mathrm{d}^2 z,$$

and expressed as

$$\begin{aligned}
G^0_\pm &= 2\partial_\pm \bar{\phi}\,\psi_\pm \mp \mathrm{i}\bar{\psi}_\mp \bar{W}'(\bar{\phi}),\\
G^1_\pm &= \mp 2\partial_\pm \bar{\phi}\,\psi_\pm - \mathrm{i}\bar{\psi}_\mp \bar{W}'(\bar{\phi}),\\
\bar{G}^0_\pm &= 2\bar{\psi}_\pm\,\partial_\pm \phi \pm \mathrm{i}\psi_\mp W'(\phi),\\
\bar{G}^1_\pm &= \mp 2\bar{\psi}_\pm\,\partial_\pm \phi \pm \mathrm{i}\psi_\mp W'(\phi).
\end{aligned}$$

The conserved charges, which are supercharges, are

$$\mathcal{Q}_\pm = \int \mathrm{d}x^1\, G^0_\pm, \quad \bar{\mathcal{Q}}_\pm = \int \mathrm{d}x^1\, \bar{G}^0_\pm.$$

These charges transform as spinors under the Lorentz transformation

$$\mathcal{Q}_\pm \mapsto \mathrm{e}^{\mp \mathrm{i}\alpha/2}\,\mathcal{Q}_\pm, \quad \bar{\mathcal{Q}}_\pm \mapsto \mathrm{e}^{\mp \mathrm{i}\alpha/2}\,\bar{\mathcal{Q}}_\pm.$$

2.1.2 R-SYMMETRY

There are two independent transformations such that the chiral fields remain chiral after rotation

$$\begin{aligned}
R_A(\alpha) &: \Phi(z,\bar{z},\theta^\pm,\bar{\theta}^\pm) \mapsto \mathrm{e}^{\mathrm{i}\alpha q_A}\Phi(z,\bar{z},\mathrm{e}^{\mp\mathrm{i}\alpha}\theta^\pm,\mathrm{e}^{\pm\mathrm{i}\alpha}\bar{\theta}^\pm),\\
R_V(\alpha) &: \Phi(z,\bar{z},\theta^\pm,\bar{\theta}^\pm) \mapsto \mathrm{e}^{\mathrm{i}\alpha q_V}\Phi(z,\bar{z},\mathrm{e}^{-\mathrm{i}\alpha}\theta^\pm,\mathrm{e}^{+\mathrm{i}\alpha}\bar{\theta}^\pm),
\end{aligned}$$

where R_A denotes axial rotation and R_V denotes vector rotation and (q_V, q_A) are the charges of the superfield. The generating operators are

$$\begin{aligned}
F_V &= -\theta^+\frac{\mathrm{d}}{\mathrm{d}\theta^+} - \theta^-\frac{\mathrm{d}}{\mathrm{d}\theta^-} + \bar{\theta}^+\frac{\mathrm{d}}{\mathrm{d}\bar{\theta}^+} + \bar{\theta}^-\frac{\mathrm{d}}{\mathrm{d}\bar{\theta}^-},\\
F_A &= -\theta^+\frac{\mathrm{d}}{\mathrm{d}\theta^+} + \theta^-\frac{\mathrm{d}}{\mathrm{d}\theta^-} + \bar{\theta}^+\frac{\mathrm{d}}{\mathrm{d}\bar{\theta}^+} - \bar{\theta}^-\frac{\mathrm{d}}{\mathrm{d}\bar{\theta}^-}.
\end{aligned} \tag{2.7}$$

These operators have the following nonzero commutation relations with the other symmetry generators

$$\begin{aligned}
[F_V, \mathcal{Q}_\pm] &= \mathcal{Q}_\pm,\\
[F_V, \bar{\mathcal{Q}}_\pm] &= -\bar{\mathcal{Q}}_\pm,\\
[F_A, \mathcal{Q}_\pm] &= \pm\mathcal{Q}_\pm,\\
[F_A, \bar{\mathcal{Q}}_\pm] &= \mp\bar{\mathcal{Q}}_\pm.
\end{aligned}$$

Classically, transformations are symmetries if they leave the action invariant. The D-term S_{kin} and the F-term S_{W} are not mixed under the R-symmetries, so we can look at their invariance separately. An invariant integral has total R-charge 0 and

the charge of the measure is always opposite the charge of the integrand, such that they add up to 0.

Since $\mathrm{d}^4\theta$ is invariant under both R-symmetries, the D-term is invariant under $U(1)_V$ if one can assign vector R-charges to Φ in such a way that the integrand has vector R-charge equal to 0, which is the case for any R-charge. The same is true for the axial R-charge.

In the case of the F-term, $\mathrm{d}^2\theta$ has vector R-charge -2 and axial R-charge 0, so to be invariant we need $W(\Phi^i)$ to obtain an R-charge 2 under $U(1)_V$ and 0 under $U(1)_A$. For $U(1)_A$ this is the case for any R-charge, but for $U(1)_V$ it is only the case if W is a quasi-homogeneous function, *i.e.* $W(\lambda^q\Phi) = \lambda^2 W(\Phi)$, where q is the R-charge assignment we are looking for.

Summarizing, one can always find a $U(1)_A$-symmetry if all fields have R-charge 0, but $U(1)_V$ is a symmetry only if the superpotential is quasi-homogeneous.

2.1.3 QUANTUM CASE

The $\mathcal{N} = (2,2)$ supersymmetric theory we have been considering is a classical theory where the action is invariant under the supersymmetry transformations. In quantum theory one also requires the correlation functions to be invariant to have a symmetry,

$$\langle \delta\mathcal{O} \rangle = \int \mathcal{D}X \mathrm{e}^{iS}\mathcal{O} = 0,$$

and this is the case for an invariant path-integral measure,

$$\delta\left(\mathcal{D}X\mathrm{e}^{iS}\right) = 0.$$

When this is not the case, the symmetry is said to be anomalous.

Since the R-symmetries act on the fermions only,

$$R_A(\alpha) : \psi_\pm \mapsto \mathrm{e}^{\mp \mathrm{i}\alpha}\psi_\pm,$$
$$R_V(\alpha) : \psi_\pm \mapsto \mathrm{e}^{-\mathrm{i}\alpha}\psi_\pm,$$

we want to know whether the path-integral measure is invariant under these rotations.

With the map $\phi : \Sigma \to \mathcal{M}$, the fermions $\psi_\pm^i$ are considered as the components of the spinors $\psi_\pm$ with values in the pull-back of the tangent bundle $\phi^* T_\mathcal{M}$ on $\mathcal{M}$, and they are given by

$$\begin{aligned} \psi_\pm &\in \Gamma(S_\pm \otimes \phi^* T_\mathcal{M}^{(1,0)}), \\ \bar{\psi}_\pm &\in \Gamma(S_\pm \otimes \phi^* T_\mathcal{M}^{(0,1)}), \end{aligned}$$

where $S_\pm$ are the spinor bundles and $\in$ means "is a section of".

The Lagrangian in terms of the component fields is given by

$$L = -g_{i\bar{\jmath}}\,\partial^\mu\phi^i\partial_\mu\bar{\phi}^j - 2\mathrm{i}g_{i\bar{\jmath}}\,\bar{\psi}^j_-\Delta_+\psi^i_- - 2\mathrm{i}g_{i\bar{\jmath}}\,\bar{\psi}^j_+\Delta_-\psi^i_+ - R_{i\bar{\jmath}k\bar{l}}\,\psi^i_+\psi^k_-\bar{\psi}^j_+\bar{\psi}^l_-, \quad (2.8)$$

with

$$\begin{aligned} g_{i\bar{\jmath}} &\equiv \partial_i\bar{\partial}_j K(\Phi^i,\bar{\Phi}^i),\\ \Gamma^i_{jk} &\equiv g^{i\bar{\jmath}}\,\partial_j g_{k\bar{\jmath}},\\ \Delta_\pm\psi^i_\pm &\equiv \partial_\pm\psi^i_\pm + \Gamma^i_{jk}\partial_\pm\phi^j\psi^k_\pm. \end{aligned}$$

The Lagrangian is invariant under the supersymmetry transformations

$$\begin{aligned} \delta\phi^i &= \epsilon_+\psi^i_- - \epsilon_-\psi^i_+,\\ \delta\psi^i_+ &= 2\mathrm{i}\bar{\epsilon}_-\partial_+\phi^i + \epsilon_+F^i,\\ \delta\psi^i_- &= -2\mathrm{i}\bar{\epsilon}_+\partial_-\phi^i + \epsilon_-F^i, \end{aligned}$$

and their complex conjugates. F^i are auxiliary fields and can be integrated out by using the equation of motion and writing $F^i = \Gamma^i_{jk}\psi^j_+\psi^k_-$.

In the above Lagrangian, the operator Δ_+ acts on the fields ψ_- and Δ_- acts on the fields ψ_+. The eigenvectors of these operators with zero eigenvalues play a special role and are called zero modes. The index of these operators is defined as

$$k = \#(\text{zero modes}\,\Delta_+) - \#(\text{zero modes}\,\Delta_-^\dagger).$$

By the Riemann-Roch theorem, this index can be computed by

$$k = c_1(\mathcal{M})\cdot\beta + \dim\mathcal{M}\,(1-g). \quad (2.9)$$

Here g is the genus of the worldsheet Σ, β is the homology class of $\phi(\Sigma)$, with $\phi(\Sigma)$ the image of the worldsheet Σ in $\mathcal{M}$. Field configurations with $\beta \neq 0$ are also known as (worldsheet) instantons. $c_1(\mathcal{M})$ is the first Chern class of $\mathcal{M}$ defined as

$$c_1(\mathcal{M}) = \frac{\mathrm{i}}{2\pi}\mathrm{Tr}(R),$$

where R is the Ricci tensor on $\mathcal{M}$. This two-form is closed since $\mathrm{d}(\mathrm{Tr}R) = 0$. c_1 is a topological invariant and does not change under smooth deformations of the complex structure.

Supposing $k > 0$, the index theorem tells us that the number of ψ_--zero modes is larger than the number of $\bar{\psi}_-$-zero modes, by the amount of k, and the partition function becomes zero when one integrates over the zero modes

$$\int \mathcal{D}\psi\,\mathcal{D}\bar{\psi}\,\mathrm{e}^{-S[\psi,\bar{\psi}]} = 0.$$

To get a nonzero result, one can insert certain operators that can absorb these zero modes. Since our parameters are fermionic, we are doing Grassmann integrations and removing an integral over a zero mode corresponds to inserting another copy of this zero mode in the integral. We need to insert k copies to have a term containing all of the zero modes of ψ_- and $\bar{\psi}_-$ exactly once, and another k copies to cancel the zero modes of ψ_+ and $\bar{\psi}_+$. The new path-integral will then give a nonzero contribution.

This new integral is invariant under vector R-symmetry but it is not invariant under axial R-symmetry unless $k = 0$. This is the case when the first Chern class vanishes, $c_1(\mathcal{M}) = 0$, which is the condition that $\mathcal{M}$ is a Calabi-Yau space. For $c_1(\mathcal{M}) = 0$, the worldsheet instanton effects do not break the R-symmetry. This means that for a Calabi-Yau target space, there is no anomaly in the R-symmetry[1].

When there are no anomalies, the classical conserved charges correspond to the generators of the symmetry transformations in the quantum theory. The conserved charges $\mathcal{Q}_\pm, \bar{\mathcal{Q}}_\pm$ generate the supersymmetry transformation δ

$$\delta\mathcal{O} = [\hat{\delta}, \mathcal{O}]$$

with

$$\hat{\delta} = \mathrm{i}\epsilon_+\mathcal{Q}_- - \mathrm{i}\epsilon_-\mathcal{Q}_+ - \mathrm{i}\bar{\epsilon}_+\bar{\mathcal{Q}}_- + \mathrm{i}\bar{\epsilon}_-\bar{\mathcal{Q}}_+ \ .$$

The supersymmetry algebra is given by

$$\begin{aligned}
\mathcal{Q}_+^2 = \mathcal{Q}_-^2 =& \bar{\mathcal{Q}}_+^2 = \bar{\mathcal{Q}}_-^2 = 0, \\
\{\mathcal{Q}_\pm, \bar{\mathcal{Q}}_\pm\} &= P \pm H, \\
\{\mathcal{Q}_+, \mathcal{Q}_-\} =& \{\bar{\mathcal{Q}}_+, \bar{\mathcal{Q}}_-\} = 0, \\
\{\mathcal{Q}_+, \bar{\mathcal{Q}}_-\} =& \{\mathcal{Q}_-, \bar{\mathcal{Q}}_+\} = 0, \\
[M, \mathcal{Q}_\pm] = \mp\mathcal{Q}_\pm, \quad & [M, \bar{\mathcal{Q}}_\pm] = \mp\bar{\mathcal{Q}}_\pm, \\
[F_V, \mathcal{Q}_\pm] = \mathcal{Q}_\pm, \quad & [F_V, \bar{\mathcal{Q}}_\pm] = -\bar{\mathcal{Q}}_\pm, \\
[F_A, \mathcal{Q}_\pm] = \mp\mathcal{Q}_\pm, \quad & [F_A, \bar{\mathcal{Q}}_\pm] = \mp\bar{\mathcal{Q}}_\pm.
\end{aligned} \tag{2.10}$$

The hermiticity relation is $\mathcal{Q}_\pm^\dagger = \bar{\mathcal{Q}}_\pm$. The component fields of a superfield set up a representation of the $\mathcal{N} = 2$ supersymmetry algebra. For a chiral superfield Φ the component fields $\phi, \psi_\pm, F$ determine the chiral multiplet representation via Eq.2.6 where δ is replaced by $\hat{\delta}$ above. The lowest component ϕ of the chiral multiplet satisfies $[\bar{\mathcal{Q}}_\pm, \phi] = 0$.

Writing

$$\mathcal{Q}_A = \bar{\mathcal{Q}}_+ + \mathcal{Q}_- \quad \text{or} \quad \mathcal{Q}_B = \bar{\mathcal{Q}}_+ + \bar{\mathcal{Q}}_-$$

[1] Strictly speaking, this is only true at $g = 1$. For $g \neq 1$, there will be a *fixed* charge violation given by $\dim\mathcal{M}\,(1 - g)$.

for convenience, we define a chiral operator $\mathcal{O}$ satisfying

$$[\mathcal{Q}_B, \mathcal{O}] = 0$$

and a twisted chiral operator satisfying

$$[\mathcal{Q}_A, \mathcal{O}] = 0.$$

It follows from the supersymmetry algebra that for chiral (twisted chiral) operators the spacetime derivatives are $\mathcal{Q}_B$ ($\mathcal{Q}_A$) exact:

$$\frac{\mathrm{i}}{2}\left(\frac{\partial}{\partial(\mathrm{i}x^0)} + \frac{\partial}{\partial(x^1)}\right)\mathcal{O} = \{\mathcal{Q}_B, [\mathcal{Q}_+, \mathcal{O}]\},$$
$$\frac{\mathrm{i}}{2}\left(\frac{\partial}{\partial(\mathrm{i}x^0)} - \frac{\partial}{\partial(x^1)}\right)\mathcal{O} = \{\mathcal{Q}_B, [\mathcal{Q}_-, \mathcal{O}]\}.$$

which means that the $\mathcal{Q}_B$ ($\mathcal{Q}_A$) cohomology class of a (twisted) chiral operator is not changed by the spacetime translation.

The product of two chiral operators is again a chiral operator. Since the charges $\mathcal{Q}_B$ anti-commute, one can compute the $\mathcal{Q}_B$-cohomology acting on the space of chiral superfields

$$\phi(z, \bar{z}) \equiv \phi(z, \bar{z}) + \{\mathcal{Q}_B, \psi(z, \bar{z})\}.$$

This equivalence relation defines, together with adding and multiplying operators of chiral fields, a ring structure, known as the chiral ring. For $\mathcal{Q}_A$ one gets the twisted chiral ring.

2.1.4 TWISTING

The flat spacetime we have been considering is in many cases not sufficient and one would like to formulate the supersymmetric theory on curved Riemann surfaces.

Defining supersymmetry on a curved Riemann surface Σ with metric g, one needs to choose a spin structure such that it will be possible to put spinors on this surface. Consider the variation of the supersymmetric action, given by

$$\delta S = \int_\Sigma (\nabla_\mu \epsilon_+ G^\mu_- - \nabla_\mu \epsilon_- G^\mu_+ - \nabla_\mu \bar{\epsilon}_+ \bar{G}^\mu_- + \nabla_\mu \bar{\epsilon}_- \bar{G}^\mu_+)\sqrt{g}\,\mathrm{d}^2 z.$$

The parameters $\epsilon_\pm, \bar{\epsilon}_\pm$ are spinors on Σ. For a flat metric these parameters can be chosen to be constant and the action will be invariant under the supersymmetry transformations. On a curved space, $\delta S = 0$ when these parameters are covariantly constant, $\nabla_\mu \epsilon_\pm = \nabla_\mu \bar{\epsilon}_\pm = 0$. However, there are no covariantly constant spinors on

a curved Riemann surface. A covariantly constant field should come back to itself after parallel transport around a closed loop, but when the space is curved it comes back to itself up to a rotation and is therefore not covariantly constant. Then there is no supersymmetry.

Since one is interested in having some fermionic symmetry and defining a path-integral over all possible metrics, one can make use of the twisting procedure which makes sure there is a conserved fermionic symmetry on a curved Riemann surface. After twisting, the theory still makes sense in flat space.

We are focusing on a theory with a $U(1)_V$ or $U(1)_A$ R-symmetry which we call $U(1)_R$ symmetry, and denote its generator by R. Twisting replaces the symmetry group $U(1)$ with the diagonal subgroup $U(1)' \in U(1) \times U(1)_R$. Its generator is $M' = M + R$ with M as in Eq.2.3. Then this symmetry group $U(1)'$ is treated as the new rotation group. Since there are two R-symmetries there are two ways to twist and they are denoted as

$$\begin{aligned} \mathrm{A-twist} : R = F_V, \quad U(1)_R = U(1)_V, \\ \mathrm{B-twist} : R = F_A, \quad U(1)_R = U(1)_A. \end{aligned}$$

After twisting the fields of the theory will be sections of different bundles over the Riemann surface. For example, take a look at a chiral superfield with trivial R-charges $q_V = q_A = 0$:

$$\Phi = \phi + \theta_+ \psi_+ + \theta_- \psi_- + \cdots .$$

In the untwisted case, the lowest component ϕ has both axial and vector R-charges as well as the charge under rotation M equal to zero. Therefore, twisting does not change its charge $M' = M = 0$ and ϕ is still a scalar after twisting. The ψ_- component has M charge 1 since it is a spinor field, a section of the spinor bundle $\sqrt{K}$ where K is the canonical bundle on Σ. Its R-charge is $q_V = -1, q_A = 1$. Doing the A-twist it gets $1 + q_V = 0$ for M' charge and becomes a scalar. It gets $1 + q_A = 2$ for M' charge after the B-twist and becomes a vector. It is then a one-form field and a section of the canonical bundle K. The other component fields get twisted charges as in the table below.

			Before	A-twist	B-twist
	$U(1)_V$	$U(1)_A$	$U(1)$	$U(1)'$	$U(1)'$
ϕ	0	0	0	0	0
ψ_-	-1	1	1	0	2
$\bar{\psi}_+$	1	1	-1	0	0
$\bar{\psi}_-$	1	-1	1	2	0
ψ_+	-1	-1	-1	-2	-2

Twisting does not affect the Riemann surfaces with flat metric, since the canonical bundles before and after twisting are the same, being equal to the trivial bundle $\underline{\mathbb{C}}$.

The topologically-twisted theory is different from the original one. Varying the metric changes the energy momentum tensor $T_{\mu\nu}$ since it is defined as the variation of the action with respect to the change of the metric. After twisting the energy momentum tensor is equal to:

$$T_{\mu\nu}^{\text{twisted}} = T_{\mu\nu} + \frac{1}{4}(\epsilon_{\mu}^{\lambda}\partial_{\lambda}J_{\nu}^{R} + \epsilon_{\nu}^{\lambda}\partial_{\lambda}J_{\mu}^{R}) \ ,$$

where J_{μ}^{R} is the $U(1)_R$ current which is defined as

$$\delta_{A^R}\langle\mathcal{O}\rangle = \left\langle \mathcal{O}\frac{1}{2\pi\mathrm{i}}\int *J^R \wedge \delta A^R \right\rangle ,$$

with respect to the variation of the gauge field A^R corresponding to the R-symmetry.

Twisting changes the spins of the supercharges. Looking at the commutation relations

$$\begin{array}{lllll} \mathrm{A-twist}: & [M', \mathcal{Q}_+] = -2\mathcal{Q}_+, & [M', \mathcal{Q}_-] = 0 \ , \\ & [M', \bar{\mathcal{Q}}_-] = 2\mathcal{Q}_-, & [M', \bar{\mathcal{Q}}_+] = 0 \ , \\ \mathrm{B-twist}: & [M', \mathcal{Q}_+] = -2\mathcal{Q}_+, & [M', \mathcal{Q}_-] = 2\mathcal{Q}_- \ , \\ & [M', \bar{\mathcal{Q}}_-] = 0, & [M', \bar{\mathcal{Q}}_+] = 0 \ , \end{array}$$

we see that the operator $\mathcal{Q}_A = \bar{\mathcal{Q}}_+ + \mathcal{Q}_-$ has become a scalar after the A-twist and that the operator $\mathcal{Q}_B = \bar{\mathcal{Q}}_+ + \bar{\mathcal{Q}}_-$ has become a scalar after the B-twist, see the table below. They are now independent of worldsheet coordinates. This means that the corresponding supersymmetry can be defined on an arbitrary worldsheet and this was what we were looking for.

			Before	A-twist	B-twist
	$U(1)_V$	$U(1)_A$	$U(1)$	$U(1)'$	$U(1)'$
$\mathcal{Q}_-$	-1	1	1	0	2
$\bar{\mathcal{Q}}_+$	1	1	-1	0	0
$\bar{\mathcal{Q}}_-$	1	-1	1	2	0
$\mathcal{Q}_-$	-1	-1	-1	-2	-2

Even though the Hilbert spaces and operators of the twisted and original theories are the same, there is a difference in the set of operators in Hilbert space that one considers "physical". After the twisting, physical operators are defined to commute with $\mathcal{Q}_A$ or $\mathcal{Q}_B$ and physical states are labeled by the $\mathcal{Q}_{A/B}$ cohomology elements. This means that physical operators of the A-twisted (B-twisted) theory are twisted chiral ring (chiral ring) elements and the twisted theory can be seen as a cohomological field theory.

Given a $\mathcal{Q}$-closed operator $\mathcal{O}^{(0)} = \mathcal{O}$, one can find a one-form operator $\mathcal{O}^{(1)}$ and a two-form operator $\mathcal{O}^{(2)}$ such that

$$\begin{aligned} 0 &= [\mathcal{Q}, \mathcal{O}^{(0)}], \\ \mathrm{d}\mathcal{O}^{(0)} &= \{\mathcal{Q}, \mathcal{O}^{(1)}\}, \\ \mathrm{d}\mathcal{O}^{(1)} &= [\mathcal{Q}, \mathcal{O}^{(2)}], \\ \mathrm{d}\mathcal{O}^{(2)} &= 0. \end{aligned} \tag{2.11}$$

These equations are called descent relations. For the B-twist this means

$$\begin{aligned} \mathcal{O}^{(1)} &= \mathrm{i}\mathrm{d}z\,[\mathcal{Q}_-, \mathcal{O}] - \mathrm{i}\mathrm{d}\bar{z}\,[\mathcal{Q}_+, \mathcal{O}], \\ \mathcal{O}^{(2)} &= \mathrm{d}z \wedge \mathrm{d}\bar{z}\,\{\mathcal{Q}_+, [\mathcal{Q}_-, \mathcal{O}]\}. \end{aligned}$$

To get the corresponding operators for the A-twist, one needs to replace $\mathcal{Q}_-$ by $\bar{\mathcal{Q}}_-$.

From the descent relations one can see that the operators

$$\int_\gamma \mathcal{O}^{(1)}, \qquad \int_\Sigma \mathcal{O}^{(2)},$$

with γ a closed one-cycle and Σ the worldsheet, are $\mathcal{Q}$-invariant.

The operators $\mathcal{Q}_{A/B}$ are nilpotent, $\mathcal{Q}_A^2 = \mathcal{Q}_B^2 = 0$, and the Hamiltonian and the momentum are $\mathcal{Q}_{A/B}$ exact. For many superconformal $\mathcal{N} = (2,2)$ theories, the twisted energy momentum tensor is also $\mathcal{Q}_{A/B}$ exact,

$$T_{\mu\nu}^{\text{twisted}} = \{\mathcal{Q}, G_{\mu\nu}\},$$

where $G_{\mu\nu}$ is a fermionic symmetric tensor. Consider the variation of the correlation functions as we vary the metric g

$$\begin{aligned} \delta_g \langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle &= \left\langle \frac{1}{4\pi} \int \mathrm{d}^2 z \sqrt{g} \delta g^{\mu\nu} T_{\mu\nu}^{\text{twisted}} \mathcal{O}_1 \cdots \mathcal{O}_s \right\rangle \\ &= \left\langle \frac{1}{4\pi} \int \mathrm{d}^2 z \sqrt{g} \delta g^{\mu\nu} \{\mathcal{Q}, G_{\mu\nu}\} \mathcal{O}_1 \cdots \mathcal{O}_s \right\rangle. \end{aligned}$$

This expression is equal to zero if all $\mathcal{O}_i$ are physical operators and commute with $\mathcal{Q}$, which means that the correlation functions of physical operators are independent of the choice of worldsheet metric. Because of this metric-independence the twisted theory is called a topological sigma model.

2.2 MODULI SPACES

Before we elaborate on the contents and meaning of the A/B-twisted $\mathcal{N} = (2,2)$ sigma models, we need to introduce some underlying structures.

We are mainly interested in Calabi-Yau manifolds since they provide the natural tools for topological string theories. Looking for a supersymmetry-preserving compactification from ten- to four-dimensional string theory, since we live in a four-dimensional spacetime, we find ourselved in the need of a manifold $\mathcal{M}$ with the properties of being complex, Kähler and Ricci-flat, which are the properties of a Calabi-Yau manifold.

Since it has turned out to be very difficult to find Ricci-flat Kähler metrics on Calabi-Yau manifolds, one often makes use of the theorem by Yau [17] which states that given that $\mathcal{M}$ admits a complex and Kähler metric, then it also admits a Ricci-flat metric if and only if its first Chern class vanishes, $c_1(\mathcal{M}) = 0$. A vanishing Chern class is equivalent to saying that there is always a nonvanishing holomorphic n-form on an n-dimensional Calabi-Yau manifold. So, given a certain topological space with a complex structure and Kähler class on it, there is always a unique Calabi-Yau metric and the space of all Calabi-Yau manifolds of a given topology is the same as the space of all possible Kähler classes and complex structures on this manifold. This space is called the moduli space of the Calabi-Yau manifold.

2.2.1 KÄHLER MODULI

The Kähler form

$$\omega = 2\mathrm{i}\, g_{i\bar{\jmath}}\, \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j$$

is a $(1,1)$-form and given a complex structure on $\mathcal{M}$, we can choose a cohomology class of degree $(1,1)$ which is an element of $H^{1,1}(\mathcal{M})$. Since any Kähler metric corresponds to a unique Kähler class, different cohomology classes of $(1,1)$-forms will result in different Calabi-Yau manifolds.

Globally, not all classes of $(1,1)$-forms are allowed as a Kähler class. A restriction comes from the basic requirement that volumes should have a positive value. So we choose cohomology classes such that all p-cycles on the manifold have a positive volume,

$$\int_{p-\text{cycles}} \underbrace{\omega \wedge \cdots \wedge \omega}_{p \text{ times}} > 0 \ .$$

The allowed Kähler classes form a vector space called the Kähler-cone with the properties that it includes the origin and any positive linear combination of its elements again belongs to the cone.

Locally, any allowed Kähler metric can be deformed by adding to its Kähler form a $(1,1)$-form which means that the space of all infinitesimal deformations to the Kähler moduli space is isomorphic to $H^{1,1}(\mathcal{M})$.

2.2.2 COMPLEX STRUCTURE MODULI

Given a Kähler class, we can choose a complex structure and split up the local coordinates of $\mathcal{M}$ into three holomorphic coordinates z^i and their complex conjugates $\bar{z}_i$. Due to Yau's theorem, we know that there will be a holomorphic, nowhere vanishing three-form Ω which can be written as

$$\Omega = \Omega_{ijk}(z)\, \mathrm{d}z^i \wedge \mathrm{d}z^j \wedge \mathrm{d}z^k,$$

with a holomorphic funtion $\Omega_{ijk}(z)$. This three-form is harmonic, $\partial\Omega = \bar{\partial}\Omega = 0$, and therefore represents a cohomology class in $H^{3,0}(\mathcal{M})$. For a Calabi-Yau manifold such a form is unique up to multiplication by a constant. The Hodge number $h^{3,0} = 1$ meaning that the cohomology class $H^{3,0}(\mathcal{M})$ is one-dimensional. It is a linear subspace of $H^3(\mathcal{M})$.

The variation of the complex structure is given by a Beltrami differential μ, which is defined as an anti-holomorphic one-form on $\mathcal{M}$ with values in the holomorphic tangent bundle, changing these local coordinates as

$$\mathrm{d}z^i \mapsto \mathrm{d}z^i + \mu^i_{\bar{j}} \mathrm{d}\bar{z}^{\bar{j}}.$$

Inserting this variation into Ω, the change of Ω takes the expression of a $(3,0)$-form plus $(2,1)$-form. To first order, it is then given by

$$\delta\Omega = \mu \cdot \Omega = \Omega_{ijk}(z)\, \mu^k_{\bar{\imath}}\, \mathrm{d}z^i \wedge \mathrm{d}z^j \wedge \mathrm{d}\bar{z}^{\bar{\imath}}\ .$$

Since $H^{3,0}(\mathcal{M})$ is one-dimensional and scalings do not change the complex structure, the only change to first order is adding cohomology classes of degree $(2,1)$ to the cohomology class of Ω. It turns out that the tangent space of infinitesimal deformations of the complex structure is isomorphic to $H^{2,1}(\mathcal{M})$. So, the dimension of the moduli space is equal to the Hodge number $h^{2,1}$.

$\mathcal{M}$ can be parametrized by taking a canonical basis for the homology group $H_3(\mathcal{M})$ given by the three-cycles (A^I, B_I) with $I = 0, \ldots, h^{2,1}$, with the intersection numbers

$$\begin{aligned} A^I \cap A^J &= 0, \\ A^I \cap B_J &= \delta^I_J, \\ B_I \cap B_J &= 0. \end{aligned}$$

One can think of intersection numbers as counting intersection points, the signs of which depend on the orientation of the cycles. Such intersection numbers are invariant under deformations of these cycles. The cohomology class of a three-form is determined completely by its period integrals

$$S^I = \int_{A^I} \Omega, \quad \mathcal{F}_I = \int_{B_I} \Omega, \tag{2.12}$$

with $I = 0, \ldots, h^{2,1}$. It can be shown that the S^I alone locally determine the complex structure, such that one can solve for the $\mathcal{F}_I$ in terms of the S^I, *i.e.* $\mathcal{F}_I = \mathcal{F}_I(S)$.

Expanding in $\mathrm{d}S$ and $\mathrm{d}\bar{S}$ one finds

$$\int_{\mathcal{M}} \delta\Omega = 0, \quad \int_{\mathcal{M}} \delta^2\Omega = 0,$$

which, together with the Riemann bilinear identity

$$\int_{\mathcal{M}} a \wedge b = \int_{A_i} a \int_{B^i} b - \int_{A_i} b \int_{B^i} a$$

for closed three-forms a and b, allows us to state the integrability condition

$$\frac{\partial \mathcal{F}_J}{\partial S^I} = \frac{\partial \mathcal{F}_I}{\partial S^J}.$$

With this condition, one can define a new holomorphic function $\mathcal{F}$

$$\mathcal{F}_I(S) = \frac{\partial \mathcal{F}}{\partial S^I}.$$

This function $\mathcal{F}$ depends on the scaling of Ω: for $\Omega \mapsto \epsilon\Omega$, $\mathcal{F}$ scales as $\mathcal{F} \mapsto \epsilon^2\mathcal{F}$. This means that $\mathcal{F}$ is homogeneous in S^I with degree two and that it is a section of a line bundle over the moduli space. Rescaling all S^I gives the same rescaling of $\mathcal{F}$ and this amounts to the same rescaling of Ω. This means that a rescaling of S^I does not change the complex structure. $\mathcal{F}$ can be written as

$$\mathcal{F}(z) = \frac{1}{2} \sum_{I=0}^{h^{2,1}} S^I \mathcal{F}_I \ .$$

Defined like this $\mathcal{F}$ is called the prepotential of the Calabi-Yau.

2.3 A/B-TWISTED SIGMA MODELS

To make the contents of the A/B-models clear, we will take a look at two examples, the A-twisted nonlinear sigma models and the B-twisted Calabi-Yau sigma models.

2.3.1 A-MODEL

Consider the map $\phi : \Sigma \mapsto \mathcal{M}$ where $\mathcal{M}$ is a Kähler manifold of dimension n and ϕ^i are the lowest components of the n chiral multiplet fields Φ^i. The fermionic

components $\psi_{\pm}^{i}$ are to be considered as spinors $\psi_{\pm}$ which take values in the pull-back of the tangent bundle of $\mathcal{M}$, $\phi^{*}T\mathcal{M}^{(1,0)}$.

The twisting is done by changing the spin of the fermions. Then ψ_{-} and $\bar{\psi}_{+}$ get spin zero and become scalars while ψ_{+} gets spin -2, becoming an anti-holomorphic one-form and $\bar{\psi}_{-}$ gets spin 2, becoming a holomorphic one-form. It is convenient to give these fields other names

$$\begin{aligned} \chi^{i} &\equiv \psi_{-}^{i}, \quad \chi^{\bar{\imath}} \equiv \bar{\psi}_{+}^{i}, \\ \rho_{\bar{z}}^{\bar{\imath}} &\equiv \bar{\psi}_{-}^{i}, \quad \rho_{z}^{i} \equiv \psi_{+}^{i}, \end{aligned}$$

and rewrite the Lagrangian Eq.2.8 in terms of these new fields as

$$L = -2t \left(g_{i\bar{\jmath}}\, \partial_{z}\phi^{i}\, \partial_{\bar{z}}\bar{\phi}^{j} + g_{i\bar{\jmath}}\, \partial_{\bar{z}}\phi^{i}\, \partial_{z}\bar{\phi}^{j} + \mathrm{i} g_{i\bar{\jmath}}\, \rho_{z}^{i}\, \Delta_{\bar{z}}\, \chi^{\bar{\jmath}} + \mathrm{i} g_{i\bar{\jmath}}\, \rho_{\bar{z}}^{\bar{\jmath}}\, \Delta_{z}\, \chi^{i} + \frac{1}{2} R_{i\bar{\jmath}k\bar{l}}\, \rho_{z}^{i}\, \rho_{\bar{z}}^{\bar{\jmath}}\, \chi^{k}\, \chi^{\bar{l}} \right),$$

where $\Delta_{z} = \Delta_{+}$, $\Delta_{\bar{z}} = \Delta_{-}$ and t is a coupling constant. Since all of the terms have a z and $\bar{z}$ index, this Lagrangian can be written as $\eta^{\mu\nu} L_{\mu\nu}$ such that it can be put in a covariant form and apply to any curved worldsheet with metric $g^{\mu\nu}$.

One would like to write the Lagrangian in such a way that it would be $\mathcal{Q}_{A}$-exact and independent of the coupling constant t. It turns out that one can write

$$L' = -\mathrm{i}t \{\mathcal{Q}_{A}, V\},$$

with

$$V = g_{i\bar{\jmath}} \left(\rho_{z}^{i}\, \partial_{\bar{z}}\bar{\phi}^{j} + \partial_{z}\phi^{i}\, \rho_{\bar{z}}^{\bar{\jmath}} \right),$$

which in turn can be written as

$$L' = L - 2t\, g_{i\bar{\jmath}} \left(\partial_{z}\phi^{i}\, \partial_{\bar{z}}\bar{\phi}^{j} - \partial_{\bar{z}}\phi^{i}\, \partial_{z}\bar{\phi}^{j} \right).$$

In geometric terms, this difference $L - L'$ has a special meaning since it can be rewritten as:

$$\begin{aligned} S - S' &= 2t \int_{\Sigma} \mathrm{d}^{2}z\; g_{i\bar{\jmath}} \left(\partial_{z}\phi^{i}\, \partial_{\bar{z}}\bar{\phi}^{j} - \partial_{\bar{z}}\phi^{i}\, \partial_{z}\bar{\phi}^{j} \right) \\ &= t \int_{\Sigma} \phi^{*}(\omega) = t \int_{\phi(\Sigma)} \omega, \end{aligned}$$

where ϕ^{*} is the pull-back to the worldsheet of the Kähler form $\omega = 2\mathrm{i}\, g_{i\bar{\jmath}}\, \mathrm{d}z^{i} \wedge \mathrm{d}\bar{z}^{j}$ in target space.

The Kähler form ω is closed, $\mathrm{d}\omega = 0$, and its integral depends only on the homology class of $\phi(\Sigma)$, which is denoted as β. Writing the integral over the homology class β as $\omega \cdot \beta$, the expression becomes

$$S - S' = t\, \omega \cdot \beta.$$

In the path-integral, for β fixed, this will give a number $\mathrm{e}^{-t\,\omega\cdot\beta}$ which is independent of the worldsheet metric $g_{\mu\nu}$ and the remaining part of the measure $\mathrm{e}^{-S'}$ will be $\mathcal{Q}_A$-exact: the theory is topological with respect to the worldsheet metric. The derivative $\frac{\mathrm{d}S'}{\mathrm{d}t}$ is also $\mathcal{Q}_A$-exact and $S - S'$ becomes independent of t. In the limit $t \to \infty$ the path integral can be computed exactly.

The A-model is independent of the complex structure since $S - S'$ does not depend on the complex structure, but it does depend on the Kähler class through ω.

A general local operator in the A-model looks like

$$\mathcal{O}_\omega = \omega_{i_1\cdots i_p\bar{\jmath}_1\cdots\bar{\jmath}_q}(\phi)\;\chi^{i_1}\cdots\chi^{i_p}\chi^{\bar{\jmath}_1}\chi^{\bar{\jmath}_q}.$$

One can only use the scalar fields ϕ and χ to contstruct a covariant zero-form operator of which the correlation functions would be nonzero. Including their derivatives or the component fields ψ, one would have to use the worldsheet metric to covariantize and the operator would not be topological.

To these operators one can associate the differential forms of $\mathcal{M}$ with the rule

$$\chi^i \leftrightarrow \mathrm{d}\phi^i, \quad \chi^{\bar{\imath}} \leftrightarrow \mathrm{d}\phi^{\bar{\imath}}.$$

Writing down the anti-commutators of the scalar fields with $\mathcal{Q}_A$,

$$\begin{aligned}
\{\mathcal{Q}_A, \phi^i\} &= -\chi^i, \\
\{\mathcal{Q}_A, \bar{\phi}^i\} &= -\chi^{\bar{\imath}}, \\
\{\mathcal{Q}_A, \chi^i\} &= 0, \\
\{\mathcal{Q}_A, \chi^{\bar{\imath}}\} &= 0,
\end{aligned}$$

one can see that $\mathcal{Q}_A$ is related to the de Rham cohomology operator:

$$\{\mathcal{Q}_A, \mathcal{O}_\omega\} = -\mathcal{O}_{\mathrm{d}\omega},$$

and $\mathcal{Q}_A$ can be viewed as the de Rham exterior derivative, $\mathrm{d} = \partial + \bar{\partial}$. Since the physical operators of a cohomological field theory are given by its cohomology classes, one can conclude that the physical operators of the A-model are in 1-to-1 correspondence with the elements of the de Rham cohomology on $\mathcal{M}$.

The twisted Lagrangian still has the same R-symmetries as the untwisted theory and the correlation function

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle_\beta \equiv \int_\beta \mathcal{D}\phi\,\mathcal{D}\chi\,\mathcal{D}\rho\,\mathrm{e}^{-S}\,\mathcal{O}_1\cdots\mathcal{O}_s$$

obeys certain selection rules. Assigning to each operator $\mathcal{O}_i$ a differential form of degree (p_i, q_i), it has a vector R-charge $q_V = -p_i + q_i$ and an axial R-charge $q_A =$

$p_i + q_i$. Since there is no vector R-anomaly, the correlator is only nonzero for $q_V = 0$, *i.e.* $\sum_{i=1}^{s} p_i = \sum_{i=1}^{s} q_i$. The axial R-anomaly is due to the difference in the number of zero modes of the operators Δ_z and $\Delta_{\bar{z}}$ acting on the fermions, given by $2k$ where k is given by Eq.2.9, with $\Delta_+ = \Delta_z$, $\Delta_- = \Delta_{\bar{z}}$. The axial rotation transforms the path-integral measure by $\mathrm{e}^{2\mathrm{i}\alpha k}$ which depends only on the homology class β. For fixed β the correlator has to obey the selection rule $\sum_{i=1}^{s}(p_i + q_i) = 2k$.

Together with the vector R-symmetry selection rule, the correlator gives a contribution for only

$$\sum_{i=1}^{s} p_i = \sum_{i=1}^{s} q_i = c_1(\mathcal{M}) \cdot \beta + \dim\mathcal{M}\,(1-g).$$

Due to the localization principle, which states that the path-integral gets localized to critical points where the variation of fermionic symmetry generators $\mathcal{Q}_A$ vanishes, one gets the condition $\partial_{\bar{z}}\phi^i = 0$ for the maps $\phi : \Sigma \to \mathcal{M}$, meaning that ϕ should be a holomorphic map for a fixed complex structure of Σ.

2.3.2 B-MODEL

Consider the map $\phi : \Sigma \mapsto \mathcal{M}$ where $\mathcal{M}$ is a compact Calabi-Yau manifold. The twisting is done by changing the spins of the fermions. Then $\bar{\psi}_\pm$ become scalars while $\psi_\pm$ become anti-holomorphic/holomorphic one-forms, with values in the pull back of their tangent bundle on $\mathcal{M}$. Denoting these fields as

$$\begin{aligned}
\eta^{\bar{\imath}} &= \bar{\psi}_+^i + \bar{\psi}_-^i, \\
\theta_i &= g_{i\bar{\jmath}}(\bar{\psi}_+^j - \bar{\psi}_-^j), \\
\rho_z^i &= \psi_+^i, \\
\rho_{\bar{z}}^i &= \psi_-^i,
\end{aligned}$$

the Lagrangian Eq.2.8 can be written as

$$L = -t\left(g_{i\bar{\jmath}}\,\eta^{\mu\nu}\,\partial_\mu\phi^i\,\partial_\nu\bar{\phi}^j + \mathrm{i}g_{i\bar{\jmath}}\,\eta^{\bar{\jmath}}\left(\Delta_{\bar{z}}\,\rho_z^i + \Delta_z\,\rho_{\bar{z}}^i\right) + \mathrm{i}\theta_i\left(\Delta_{\bar{z}}\,\rho_z^i - \Delta_z\,\rho_{\bar{z}}^i\right) + \frac{1}{2}R^l_{i\bar{\jmath}k}\,\rho_z^i\,\rho_{\bar{z}}^k\,\eta^{\bar{\jmath}}\,\theta_l\right).$$

Here t is again a coupling constant and the action can be covariantized as in the A-model.

One can write down a $\mathcal{Q}_B$-exact expression

$$L' = -\mathrm{i}t\,\{\mathcal{Q}_B, V\},$$

with

$$V = g_{i\bar{\jmath}}\left(\rho_z^i\,\partial_{\bar{z}}\bar{\phi}^j + \rho_{\bar{z}}^i\,\partial_z\bar{\phi}^j\right),$$

and find

$$L - L' = -t \left(\mathrm{i}\theta_i \left(\Delta_{\bar{z}}\, \rho_z^i - \Delta_z\, \rho_{\bar{z}}^i \right) + \frac{1}{2} R^l_{i\bar{j}k}\, \rho_z^i\, \rho_{\bar{z}}^k\, \eta^{\bar{j}}\, \theta_l \right).$$

Since this expression is anti-symmetric in the indices z and $\bar{z}$, it can also be written as a $(1,1)$-form. The integral of a $(1,1)$-form in two dimensions does not depend on the metric, and since $\{\mathcal{Q}_B, V\}$ is also independent of the metric, the theory is topological with respect to this metric. The dependence on t can be removed by absorbing it in the definition of θ.

In case of correlation functions which are homogeneous in θ, one can again take the limit $t \to \infty$ when calculating the exact path-integral since t will be contained in an overall factor. In the A-model, this was different since the t-dependence was for each separate homology class in the target space and the final answer did not have a simple overall t-dependence.

Computing the correlators, one finds

$$\{\mathcal{Q}_B, \phi^i\} = 0, \quad \{\mathcal{Q}_B, \bar{\phi}^i\} = -\eta^{\bar{\imath}},$$

where it is visible that the theory depends on the complex structure of the target space. However, one can show that there is no dependence on the Kähler moduli, since the cohomology class of the Kähler form ω has a variation that is $\mathcal{Q}_B$-exact.

A general local operator in the B-model can be written as

$$\mathcal{O}_\omega = \omega_{\bar{\imath}_1 \cdots \bar{\imath}_p}{}^{j_1 \cdots j_q}(\phi, \bar{\phi})\, \eta^{\bar{\imath}_1} \cdots \eta^{\bar{\imath}_p}\, \theta_{j_1} \cdots \theta_{j_q}.$$

It is convenient to make the identification

$$\eta^{\bar{\imath}} \leftrightarrow \mathrm{d}\phi^{\bar{\imath}},$$
$$\theta_i \leftrightarrow \frac{\partial}{\partial \phi^i},$$

because then the operator can be written as

$$\mathcal{O}_\omega = \omega_{\bar{\imath}_1 \cdots \bar{\imath}_p}{}^{j_1 \cdots j_q}\, \mathrm{d}\phi^{\bar{\imath}_1} \cdots \mathrm{d}\phi^{\bar{\imath}_p}\, \frac{\partial}{\partial \phi^{j_1}} \cdots \frac{\partial}{\partial \phi^{j_q}}.$$

This is an anti-holomorphic p-form with values in the anti-symmetrized product of q holomorphic tangent spaces $\wedge^q T^{(1,0)}\mathcal{M}$. Then the operator $\mathcal{Q}_B$ is identified with the Dolbeault cohomology operator $\bar{\partial}$. With the anti-commutation relations

$$\{\mathcal{Q}_B, \phi^i\} = 0,$$
$$\{\mathcal{Q}_B, \bar{\phi}^i\} = -\eta^{\bar{\imath}},$$
$$\{\mathcal{Q}_B, \theta_i\} = 0,$$
$$\{\mathcal{Q}_B, \eta^{\bar{\imath}}\} = 0,$$

one can see that

$$\{\mathcal{Q}_B, \mathcal{O}_\omega\} = -\mathcal{O}_{\bar{\partial}\omega}.$$

Using the same argument as in the A-model case, the physical operators of the B-model are in 1-to-1 correspondence with the elements of the Dolbeault cohomology on $\mathcal{M}$.

As in the A-model, because of the R-symmetry, the correlation function

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle = \int \mathcal{D}\phi\, \mathcal{D}\eta\, \mathcal{D}\theta\, \mathrm{e}^{-S}\, \mathcal{O}_1 \cdots \mathcal{O}_s$$

obeys the selection rules $\sum_{i=1}^s p_i = \sum_{i=1}^s q_i$ for the vector R-symmetry and

$$\sum_{i=1}^{s} (p_i + q_i) = 2\mathrm{dim}\mathcal{M}(1-g) = 2n(1-g)$$

for the axial R-symmetry.

For $g = 0$, the localization principle dictates that $\partial_\mu \phi^i = 0$, which means that ϕ is a constant map. The space of constant maps is the same as $\mathcal{M}$ and the path-integral reduces to an integral over $\mathcal{M}$. At this zero genus, the selection rule becomes $p = q = n$ with $p = \sum_{i=1}^s p_i$ and $q = \sum_{i=1}^s q_i$.

ω is a $(0,n)$-form with values in $\wedge^q T^{(1,0)}\mathcal{M}$, but to have an invariant integral, we need to integrate (n,n)-forms. ω is related to these (n,n)-forms as

$$\omega_{\bar{\imath}_1 \cdots \bar{\imath}_n}{}^{j_1 \cdots j_n} \mapsto \omega_{\bar{\imath}_1 \cdots \bar{\imath}_n}{}^{j_1 \cdots j_n}\, \Omega_{j_1 \cdots j_n}\, \Omega_{k_1 \cdots k_n}$$

over $\mathcal{M}$. Here Ω is the holomorphic n-form of the Calabi-Yau manifold $\mathcal{M}$. This also seems to follow from the definition of the path-integral.

We can conclude that the observables of the B-model are integrals of wedge products of forms over the target space $\mathcal{M}$. Considered like this, the B-model is simpler than the A-model since there one had to integrate over the moduli space.

The topological A- and B-models are related to one another by means of mirror symmetry. Since this is a vast subject, we will not go into it and refer the reader to [10].

2.4 TOPOLOGICAL STRING THEORY

Now that we have a topological field theory, we would like to construct a topological string theory.

These theories are very similar, except that in string theory one has besides a path-integral over all possible maps to the target space also a path-integral over the worldsheet metrics g. Maps from the worldsheet geometries to target space are viewed as Feynman diagrams in string theory and one can see the integrals over the Feynman diagrams as integrals over the complex structures of the Riemann surface. One refers to integrating over metrics as "quantum gravity" and so string theory becomes a study of quantum gravity theories on the worldsheet.

In topological string theory one wishes the topological sigma models to couple to worldsheet gravity, so in the context of the topological sigma models that we have considered, we want to integrate over the metrics on the worldsheet.

When one includes the metric in the Lagrangian of the two-dimensional sigma models, they become conformal field theories. Then one can use the conformal field theory methods and integrate over metrics that are conformally equivalent and then proceed with the remaining integral over the moduli space.

In conformal field theory the energy momentum tensor is traceless, $T_{z\bar{z}} = T_{\bar{z}z} = 0$. Since the energy momentum tensor is a conserved Noether current, $\partial_\mu T^\mu{}_\nu = 0$, the component $T_{zz} \equiv T(z)$ is holomorphic and $T_{\bar{z}\bar{z}} \equiv \bar{T}(\bar{z})$ is anti-holomorphic in z. The Laurent expansion of these components lead to the Virasoro generators L_m (similar for $\bar{L}_m$),

$$T(z) = \sum L_m \, z^{-m-2},$$

which have the following commutation relations:

$$[L_m, L_n] = (m-n)\, L_{m+n} + \frac{c}{12} m\,(m^2-1)\,\delta_{m+n}.$$

Here c is the central charge of the system being considered. For $c = 0$ there is an agreement with the equation of motion,

$$\frac{\partial S}{\partial g^{\mu\nu}} = T_{\mu\nu} = 0,$$

which means that there is a restriction for the states $|\psi\rangle$ to be physical:

$$L_m \, |\psi\rangle = 0 \qquad \forall m \in \mathbb{Z}.$$

When $c \neq 0$, this does not agree with the Virasoro commutation relations, and there is a conformal anomaly. We would expect this anomaly to be a problem but it turns out, as we will see below, that this anomaly disappears after topologically twisting the theory.

At this point we need to make a distinction between open and closed strings.

For an open string, the worldsheet coordinates $\phi(\tau, \sigma)$ have to satisfy the Neumann boundary conditions

$$\partial_\sigma \phi^\mu(\tau, 0) = \partial_\sigma \phi^\mu(\tau, \pi) = 0,$$

where left- and right-moving waves are reflected at the ends and combine into standing waves. Here left-moving refers to holomorphic and right-moving refers to anti-holomorphic. Since there is a superposition of left- and right-moving terms, the holomorphic terms are related to anti-holomorphic terms.

For a closed string, the worldsheet coordinates $\phi(\tau, \sigma)$ have to satisfy the periodic boundary conditions

$$\phi^\mu(\tau, 0) = \phi^\mu(\tau, 2\pi),$$

and the left- and right-moving waves are independent of each other, and so are the holomorhic and anti-holomorphic terms like $T(z)$ and $\bar{T}(\bar{z})$.

Under the $U(1)$ R-symmetries there are conserved currents which are holomorphic, $J_z \equiv J(z)$ or anti-holomorphic, $J_{\bar{z}} \equiv \bar{J}(\bar{z})$, and independent of each other for a closed string. Using this independence for closed strings, one can write the $U(1)$ R-symmetry as a sum of left- and right-moving symmetries, generated by F_L and F_R. Using the definitions of F_A and F_V from Eq.2.7, one can write these new generators as

$$F_V = \frac{1}{2}(F_L + F_R), \qquad F_A = \frac{1}{2}(F_L - F_R).$$

The conserved current $J(z)$ can also be expanded in Laurent modes ,

$$J(z) = \sum J_m \, z^{-m-1}.$$

There is a similar expression for $\bar{J}(\bar{z})$. The nonzero commutation relations with the Virasoro operators are

$$\begin{aligned}
[L_m, L_n] &= (m-n)\, L_{m+n} + \frac{c}{12} m\,(m^2-1)\,\delta_{m+n}, \\
[L_m, J_n] &= -n\, J_{m+n}, \\
[J_m, J_n] &= \frac{c}{3} m\, \delta_{m+n},
\end{aligned}$$

where c is again the central charge.

Corresponding to this conserved current, one can formulate a conserved charge using the Noether principle,

$$F_L = \oint_{z=0} J(z)\, \mathrm{d}z = 2\pi \mathrm{i}\, J_0.$$

The contour around the origin is due to the fact that we are integrating $J(z)$ over a slice of the z-plane which is space-like, and since the string theory time direction is radial in this plane, a space-like slice is a curve around $z = 0$.

Twisting the theory like we did before, we introduce new Lorentz rotation generators $M' = M + R$ where R is the axial or vector R-symmetry generator.

$$\begin{aligned} \mathrm{A-twist}: R = F_V, \quad U(1)_R = U(1)_V, \\ \mathrm{B-twist}: R = F_A, \quad U(1)_R = U(1)_A. \end{aligned}$$

In string theory, the Lorentz rotation generator is equal to $M = 2\pi\mathrm{i}(L_0 - \bar{L}_0)$ and we can rewrite the twisting as

$$\begin{aligned} \mathrm{A-twist}: & L_{0,A} = L_0 - \frac{1}{2}J_0, \qquad \bar{L}_{0,A} = \bar{L}_0 + \frac{1}{2}\bar{J}_0 \\ \mathrm{B-twist}: & L_{0,B} = L_0 - \frac{1}{2}J_0, \qquad \bar{L}_{0,B} = \bar{L}_0 - \frac{1}{2}\bar{J}_0. \end{aligned}$$

Corresponding to the new holomorphic generator $L_0 - \frac{1}{2}J_0$, there is a conserved current [18, 19]

$$\tilde{T}(z) = T(z) + \frac{1}{2}\partial J(z),$$

where $\tilde{T}$ is again holomorphic, $\bar{\partial}\tilde{T} = 0$. Writing

$$\tilde{L}_m = L_m - \frac{1}{2}(m-n)\, J_m,$$

$\tilde{L}_0$ is equal to $L_{0,A}, L_{0,B}$. One can also do this with the other new generators. In the new algebra of the twisted theory, we find

$$[\tilde{L}_m, \tilde{L}_n] = (m-n)\, \tilde{L}_{m+n}.$$

Since there is no central charge there will be no conformal anomaly in the twisted theory. Absence of a central charge also means that there is no limiting condition on the number of dimensions and the theory will be valid in any number of dimensions. Since there is no anomaly, the integral over the metrics can be done without problems.

After integrating over conformally equivalent metrics, we want to integrate over the moduli of the target space. To this end we will consider only Calabi-Yau manifolds since only then will the axial R-anomaly be absent, as we have seen in chapter 2.1. We are especially interested in three-dimensional Calabi-Yau manifolds for the following reason:

Given a worldsheet Σ, the moduli space of the metrics on it has dimension $3(g-1)$, due to the Riemann-Roch theorem. Considering the holomorphic maps $\phi : \Sigma \mapsto \mathcal{M}$, for a fixed Σ, the moduli space of the Calabi-Yau manifold $\mathcal{M}$ has dimension $(\dim\mathcal{M})(1-g)$. There are only pairs (Σ, ϕ) on the space of these moduli spaces, when

$$\mathrm{virtual\,dimension} = \dim\mathcal{M}\,(1-g) + 3(g-1) = 0,$$

and this is only the case for a Calabi-Yau manifold with three complex dimensions.

To define topological string theory for Calabi-Yau three-folds we need to write down the expression for the twisted energy momentum tensor

$$T_{\mu\nu} = \{\mathcal{Q}, G_{\mu\nu}\}, \tag{2.13}$$

which is defined as

$$T_{\mu\nu} = \frac{1}{\sqrt{g}} \frac{\partial S}{\partial g^{\mu\nu}}.$$

Since we are dealing with the sigma model on a Calabi-Yau manifold, which is a conformal theory, the energy momentum tensor is traceless, $T^{\mu}_{\mu} = 0$, and the action is invariant under variations of the metric by rescaling. The remaning nonzero components are $T_{zz} \equiv T_{++}$ and $T_{\bar{z}\bar{z}} \equiv T_{--}$. Writing Eq.2.13 in these components we get

$$T_{++}(z,\bar{z}) = \{\mathcal{Q}, G_{++}(z,\bar{z})\}\,, \qquad T_{--}(z,\bar{z}) = \{\mathcal{Q}, G_{--}(z,\bar{z})\}\,.$$

with the currents G_{++}, G_{--}corresponding to the charges $\mathcal{Q}_+, \bar{\mathcal{Q}}_-$.

One can use these currents to define a measure on the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g:

$$\left\langle \prod_{i=1}^{3g-3} G_{++}(\mu^{(i)}) \prod_{i=1}^{3g-3} G_{--}(\bar{\mu}^{(i)}) \right\rangle$$

with

$$G_{++}(\mu^{(i)}) \equiv \int G_{zz}\, \mu^{z(i)}_{\bar{z}}\, \mathrm{d}^2 z$$

and a similar expression for $G_{--}(\bar{\mu})$. The Beltrami differentials μ enter the equation since the tangent to $\mathcal{M}_g$ at a point Σ correspondens to choices of Beltrami differentials on the Riemann surface Σ, $T\mathcal{M}_g|_\Sigma = H^1(T\Sigma)$. They span the complex tangent space to $\mathcal{M}_g$ at the point Σ.

This measure is *a priori* nonzero since the Gs each have axial charge -1 and the product has charge $(3-3g, 3-3g)$, canceling the axial anomaly. The genus $g > 1$ topological string amplitude is defined by

$$\mathcal{F}_g = \int_{\mathcal{M}_g} \prod_{i=1}^{3g-3} \mathrm{d}m_i\, \mathrm{d}\bar{m}_i \left\langle \prod_{i=1}^{3g-3} G_{++}(\mu^{(i)}) \prod_{i=1}^{3g-3} G_{--}(\bar{\mu}^{(i)}) \right\rangle. \tag{2.14}$$

Here $\mathrm{d}m_i$ are one-forms dual to the Beltrami differentials $\mu^{(i)}$.

The total axial R-charge of the path-integral measure is given by

$$6(g-1) - 2(\dim\mathcal{M})\,(g-1).$$

This axial R-anomaly vanishes if the Calabi-Yau target space has three complex dimensions and we have a nonzero path-integral for any g. The total vector R-charge of the measure remains zero.

2.4.1 FREE ENERGY

It has been found that the structure of the twisted $\mathcal{N} = 2$ superconformal algebra is isomorphic to that of the bosonic string [9, 20, 21]:

$$(G^{++}, J, T, G^{--}) \leftrightarrow (b, J_{\text{ghost}}, T, \bar{b}).$$

In the bosonic string there is a nilpotent BRST operator Q with the corresponding BRST charge Q. The energy-momentum tensor is Q-exact: $T(z) = \{Q, b(z)\}$. $\bar{b}$ is the anti-ghost corresponding to the diffeomorphism symmetry on the bosonic string worldsheet and it plays an important role in calculating the free energy of the bosonic string: one uses the anti-ghost to relate the conformal field theory correlators on the worldsheet Σ to string theory correlators. Because of this isomorphism one can identify the topological string amplitude Eq.2.14 with the string theory free energy where G^{--} hase been replaced by $\bar{b}$:

$$\mathcal{F}_g = \int_{\mathcal{M}_g} \prod_{i=1}^{3g-3} \mathrm{d}m_i \, \mathrm{d}\bar{m}_i \left\langle \prod_{i=1}^{3g-3} b(\mu^{(i)}) \prod_{i=1}^{3g-3} \bar{b}(\bar{\mu}^{(i)}) \right\rangle \tag{2.15}$$

with

$$b(\mu^{(i)}) = \int_{\Sigma} b_{zz} \, \mu_{\bar{z}}^{z\,(i)} \, \mathrm{d}^2 z.$$

The full topological string free energy is then defined as

$$\mathcal{F} = \sum_{g=0}^{\infty} g_s^{2g-2} \, \mathcal{F}_g, \tag{2.16}$$

where g_s is the string coupling constant. The topological string partititon function is then equal to

$$Z = \mathrm{e}^{\mathcal{F}}.$$

CHAPTER 3

RANDOM MATRIX MODELS

Before giving a short introduction into random matrix models, we will first make the connection to the last chapter visible and therefore we start with a description of matrix models from (topological) string theory.

3.1 OPEN/CLOSED STRING DUALITY

In closed string theory the studied objects are maps from a closed genus g Riemann surface Σ_g to a target manifold $\mathcal{M}$ and one is interested in calculating the genus g free energy $\mathcal{F}$, given by Eq.2.16. The free energy also depends on the geometric data of the target space. In the B-model it depends on the complex structure moduli of the Calabi-Yau manifold, and in the A-model on the Kähler moduli of the Calabi-Yau.

In open string theory, one looks at maps from an open, genus g Riemann surface $\Sigma_{g,h}$ with h holes to a target manifold $\mathcal{M}$, restricted by boundary conditions. Dirichlet boundary conditions provide a submanifold of $\mathcal{M}$, called a D-brane, on which the open strings can end.

For the open string theory to be topological, the boundaries of the worldsheet should preserve the $\mathcal{Q}$-symmetry. In the A-model the only allowed boundaries are three-dimensional D-branes that are wrapped around Lagrangian submanifolds of the Calabi-Yau manifold. Here, the term "Lagrangian" refers to the property that the Kähler form ω on this submanifold vanishes. In the B-model, the only allowed D-branes are the ones wrapping holomorphic submanifolds of $\mathcal{M}$.

In the physical superstring, D-branes are sources of flux. In the A-model this flux

corresponds to the Kähler two-form and in the B-model to the holomorphic three-form.

Consider a stack of D-branes. The endpoints of a string carry a charge under $U(1)$ gauge symmetry and for N coincident D-branes, the $U(1)$ gauge symmetry can be extended to $U(N)$ gauge symmetry. The fields will then have matrix indices. One can introduce a $U(N)$ gauge symmetry by using Chan-Paton indices to mark the different D-branes. The free energy is expressed as

$$\mathcal{F} = \sum_{g=0}^{\infty} \sum_{h=1}^{\infty} \mathcal{F}_{g,h}\, g_s^{2g-2}\, N^h,$$

where $\mathcal{F}_{g,h}$ are the open string amplitudes.

Open strings can sometimes be described by string field theory [22]. The topological string theory then simplifies to a $U(N)$ gauge theory. In this reduction N becomes the rank of the gauge group and g_s the gauge coupling constant. The reduced theory has only a finite number of gauge fields and the amplitudes $\mathcal{F}_{g,h}$ can then be calculated from this gauge theory by perturbation. This perturbation is done in the double line notation, invented by 't Hooft [23]. Using this notation, $\mathcal{F}_{g,h}$ can be written as coming from "fat" oriented Feynman diagrams (fatgraphs) with h holes and genus g.

The idea is to make use of the hidden variable N, the rank of the gauge group, which appears in the $U(N)$ gauge theories together with the coupling constants, and rewrite the Feynman diagrams by interchanging every single line with a double line, marked by indices, see fig.3.1. The fundamental field will then have two indices Φ_{ij} and one can keep track of the variable N. The new propagator will have the form

$$\langle \Phi_{ij} \Phi_{kl} \rangle = g_s \delta_{il} \delta_{jk}$$

and a vertex will look like

$$\frac{t_3}{g_s} \mathrm{Tr} \Phi_3 = \frac{t_3}{g_s} \sum_{i,j,k} \Phi_{ij} \Phi_{jk} \Phi_{ki}.$$

By contracting the indices one can build different fatgraph diagrams. Since the indices can be contracted in many ways, a usual Feynman diagram will correspond to several fatgraph diagrams. The indices i, j are distinguished because of the orientation of the open strings. By hermiticity, one has $\phi_{ij}^* = \phi_{ji}$.

Fatgraphs can be characterized topologically by the number of propagators (edges) E, the number of vertices with k legs V_k, and the number of closed loops h. Each propagator contributes a factor g_s, each vertex with k legs contributes a factor t_k/g_s

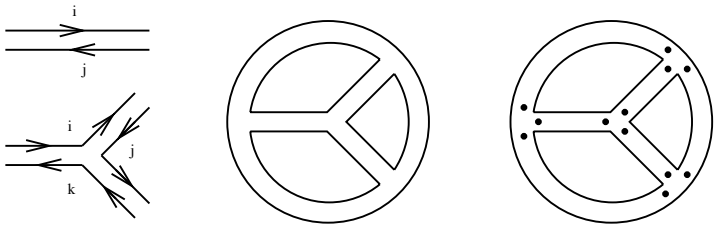

Figure 3.1: (a) The propagator and the vertex in the double line notation. Each line is labeled by one of the two matrix indices, marking the orientation. (b) A fatgraph in the perturbative expansion of the partition function. (c) Here each edge is marked by a dot and one can see that in this case $2E = 3V_3$. (From [13].)

and each loop contributes a factor N. The fatgraph will then give an overall factor

$$g_s^{E-V} N^h \prod_k t_k^{V^k}$$

with $V = \sum_k V_k$, the total number of vertices.

Such a fatgraph can also be thought of as a Riemann surface with holes (corresponding to the closed loops in the fatgraph). The genus g of this Riemann surface is given by

$$2g - 2 = E - V - h.$$

Introducing the 't Hooft parameter $S = N\, g_s$, one can express the fatgraph contribution as

$$g_s^{2g-2}\, S^h \prod_k t_k^{V_k}.$$

For $g = 0$ one gets planar diagrams (possible to draw on a flat surface, with non-crossing lines), and for higher genus one gets nonplanar diagrams. Planar diagrams can also be seen as a random triangulation of the surface, see fig.3.2.

Using this notation the free energy can be written as

$$\mathcal{F} = \sum_{g=0}^{\infty} \sum_{h=1}^{\infty} \mathcal{F}_{g,h}\, g_s^{2g-2}\, S^h.$$

We recognize this result as the free energy of the open string.

By defining the free energy at a fixed genus g to be $\mathcal{F}_g(S)$, summing over all h, we get the genus expansion of the free energy:

$$\mathcal{F} = \sum_{g=0}^{\infty} \mathcal{F}_g(S)\, g_s^{2g-2},$$

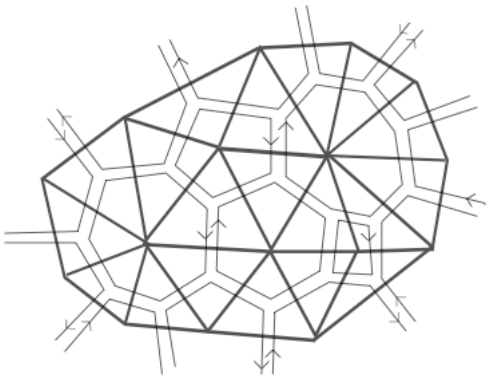

Figure 3.2: A piece of a random triangulation of a surface. (From [24].)

with

$$\mathcal{F}_g(S) = \sum_{h=1}^{\infty} \mathcal{F}_{g,h}\, S^h.$$

Comparing this answer with that from closed strings we see that open strings can lead to closed strings through fatgraphs by summing over all holes h. We will come back to this in section 3.6.

3.2 RELATION TO RANDOM MATRIX MODELS

In some cases the spacetime description of the open B-model reduces to a matrix model. To this end one should look at local Calabi-Yau three-folds $\mathcal{M}$, the simplest example of which is a genus g Riemann surface Σ_g with a bundle over it.

Compact vs. non-compact manifolds

The local Calabi-Yau three-fold is a non-compact manifold, while the manifolds we have been considering till now were compact manifolds. Fortunately, many of the properties we have used, also apply to the non-compact case. Yau's theorem that we mentioned in section 2.2 only applies to compact manifolds, and for a non-compact manifold, it needs to be supplemented by boundary conditions at infinity. For physical applications the Calabi-Yau manifold is not required to be compact and for non-compact manifolds, many topological string computations turn out to get simplified. Also the relation between topological strings and gauge theories turn out to arise for the non-compact manifolds, as we will see below.

One can embed Σ_g in $\mathcal{M}$, then the tangent bundle of $\mathcal{M}$ is written as

$$T\mathcal{M}|_{\Sigma_g} = T\Sigma_g \oplus N_{\Sigma_g}$$

where N_{Σ_g} is the normal bundle over Σ_g with the Calabi-Yau condition $c_1(\mathcal{M}) = 0$ giving

$$c_1(N_{\Sigma_g}) = 2g - 2.$$

For $g = 0$ and the two-sphere $\Sigma_g = \mathbb{P}^1$, we have line bundles $\mathcal{O}(n)$, $n \in \mathbb{Z}$, which are usually described in terms of two charts on $\mathbb{P}^1$ with coordinates (z, Φ) and (z', Φ') for the separate charts. These coordinates are related by

$$z' = \frac{1}{z}, \quad \Phi' = \frac{\Phi}{z^n} \quad .$$

With $c_1(\mathcal{O}(n)) = n$, it turns out that a two-sphere with a bundle over it can always be written as

$$\mathcal{O}(-a) \oplus \mathcal{O}(a-2) \to \mathbb{P}^1.$$

Considering string field theory of B-model topological strings on such Calabi-Yau manifolds, and requiring Dirichlet boundary conditions restricting $\mathbb{P}^1$ (meaning that there are N D-branes wrapping $\mathbb{P}^1$), spacetime can be described by the fields

$$A, \quad \Phi_0, \quad \Phi_1.$$

Here A is a $(0,1)$ $U(N)$ gauge potential on $\mathbb{P}^1$, Φ_0 is a section of $\mathcal{O}(-a)$ and Φ_1 is a section of $\mathcal{O}(a-2)$. Φ_0 and Φ_1 are in the adjoint representation of $U(N)$.

Setting $a = 0$ we get the Calabi-Yau manifold

$$\mathcal{O}(0) \oplus \mathcal{O}(-2) \to \mathbb{P}^1.$$

Now the field Φ_0 becomes a scalar field on $\mathbb{P}^1$ and Φ_1 becomes a $(1,0)$-form. The coordinates and fields on the two separate patches of $\mathbb{P}^1$ are now related by

$$z' = \frac{1}{z}, \quad \Phi_0' = \Phi_0, \quad \Phi_1' = z^2\Phi_1.$$

Since we want to find a geometry with n separate $\mathbb{P}^1$s at fixed positions, to make a connection to matrix models, we introduce a polynomial $W(\Phi_0)$ on Φ_0 of degree $n+1$ and the coordinate relations become

$$z' = \frac{1}{z}, \quad \Phi_0' = \Phi_0, \quad \Phi_1' = z^2\Phi_1 + W'(\Phi_0)z. \tag{3.1}$$

This is a different geometry than we had before, since it has been deformed by the polynomial $W(\Phi_0)$. In this new geometry we see n separate $\mathbb{P}^1$s at $W'(\Phi_0) = 0$.

Another way to look at these relations is by rewriting them as follows

$$\Phi_0 = x, \quad w = 2\Phi_1', \quad z = 2\Phi_1, \quad y = i(2z'\Phi_1' - W'(x)) \tag{3.2}$$

and

$$wz + y^2 + W'(x)^2 = 0. \tag{3.3}$$

The last line represents a geometry with singularities along $w = z = y = 0\ \forall x$ such that $W'(x) = 0$.

For $W'(x) = x$, and after rewriting the coordinates as $w = u - iv$, $z = u + iv$, this geometry becomes the conifold

$$u^2 + v^2 + x^2 + y^2 = 0.$$

For general $W(x)$, near the singularities, the geometry Eq.3.3 resembles that of the conifold. By blowing up the singularities in this geometry, (inflating a two-sphere $\mathbb{P}^1$ at each singularity) we get the manifold Eq.3.2, referred to as the resolved manifold since the singularities have been resolved by blowing them up.

Considering B-model topological strings on this resolved manifold, we again have the fields A, Φ_0, Φ_1 but the action has changed by the addition of the polynomial $W(x)$ [3, 25]

$$S = \frac{1}{g_s}\int_{\mathbb{P}^1} \mathrm{Tr}(\Phi_1 \bar{D}\Phi_0 + W(\Phi_0)\,\omega)$$

where ω is a Kähler two-form on $\mathbb{P}^1$ and $\bar{D} = \bar{\partial} + [A, \cdot]$, the anti-holomorphic covariant derivative.

The equations of motion lead to the geometry where N D-branes are wrapping n $\mathbb{P}^1$s. Since $[\Phi_0, \Phi_1] = 0$, Φ_0 and Φ_1 can be diagonalized simultaneously. Variation with respect to the eigenvalues of Φ_1 gives

$$\bar{\partial}\Phi_0 = 0.$$

Since $\mathbb{P}^1$ is compact, this means that Φ_0 is a constant. Variation with respect to Φ_0 leads to

$$\bar{\partial}\Phi_1 = W'(\Phi_0)\cdot\omega,$$

which means that $\Phi_1 = 0$ and that the eigenvalues of Φ_0 satisfy $W'(\Phi_0) = 0$. This implies that the eigenvalues are at the critical points of the polynomial.

Since Φ_1 appears linearly in the action, it can be integrated out and leads to the constraint $\bar{\partial}\Phi_0 = 0$. This means that Φ_0 is a constant $N \times N$ matrix, $\Phi_0(z) = \Phi = \mathrm{constant}$. Then the action reduces to the potential term and when one takes the integral of ω over $\mathbb{P}^1$, this leads to the matrix action

$$S(\Phi) = \frac{1}{g_s}\mathrm{Tr}\, W(\Phi).$$

With this action, the partition function is reduced to that of the matrix model,

$$Z = \int \mathcal{D}\Phi \, \mathrm{e}^{-\frac{1}{g_s}\mathrm{Tr}W(\Phi)}.$$

In summary, the total free energy of B-model topological strings on the Calabi-Yau manifold in Eq.3.2, the resolved manifold, is the same as the free energy of the matrix model with the above mentioned partition function. The $N = \sum_{i=1}^{n} N_i$ branes wrapping the n $\mathbb{P}^1$s correspond to $N = \sum_{i=1}^{n} N_i$ eigenvalues of the matrix model, located at the ith critical point of the matrix model potential $W(\Phi)$.

This equality has been derived by Dijkgraaf and Vafa [3] relating the B-model open topological string theory on a Calabi-Yau three-fold to a matrix model with potential $W(\Phi)$. Following [26], where a relation between Chern-Simons gauge theory and A-model topological string theory was stated, Dijkgraaf and Vafa developed a mirror version of this relation by finding the matrix model dual to B-model topological string theory [3].

Holomorphic vs. Hermitian matrix models

This is a good point to make a remark about the issue of holomorphic and Hermitian matrix models. In the random matrix models literature, it is quite common to use Hermitian matrices. However, holomorphic matrix models underlie the Dijkgraaf-Vafa conjecture [3], and it should be possible to formulate the matrix models in terms of holomorphic matrices.

When one uses the Hermitian approach to the Dijkgraaf-Vafa conjecture, one encounters several problems. One such problem is the fact that Hermitian one-matrix models with odd-degree polynomial potentials are not well-defined, because the real part of these potentials is not bounded from below on the real axis. In [27], this problem was encountered in checking the Dijkgraaf-Vafa conjecture for a cubic potential and was overcome in a pragmatic way. In [28], it has been shown that this pragmatic approach leads to the correct holomorphic result.

Another problem is that the matrix model resolvent

$$\omega(x) = \frac{1}{N}\mathrm{Tr}\frac{1}{x-\Phi},$$

can have cuts only on the real axis when Φ is a Hermitian matrix. This would imply that the Calabi-Yau manifold Eq.3.2 would be restricted. So one would like to have a holomorphic extension of the Hermitian approach such that one would get a large N distribution with support on curves in the complex plane.

In [28], these problems have been resolved by a direct analysis of holomorphic matrix models. A holomorphic one-matrix model is then constructed, similar to the

case of a Hermitian matrix model, and the hermiticity condition is relaxed to a general condition on the eigenvalues of the matrix. Then an eigenvalue representation is found by choosing a multi-dimensional contour. After this choice of contour, the matrix integral is formally equivalent to that of a Hermitian matrix integral, except that one integrates over eigenvalues in the complex plane. So it is justified to use Hermitian matrices as long as one keeps in mind that there is a contour dependence. See also [29, 30, 31].

3.3 INTRODUCTION TO RANDOM MATRICES

Matrix models are very simple gauge theories: they are zero-dimensional. The role of fields is played by $N \times N$ matrices Φ.

There is a huge amount of literature on the subject of random matrices. Therefore, we will just name a few and refer the reader to the detailed reviews on the subject such as [32, 33, 24, 34].

Large N matrix models became increasingly popular beginning with the seminal work of 't Hooft [23] in 1974, who showed that planar graphs with a large number of colors in QCD could be interpreted as Feynman diagrams for matrix models and that the size of the matrices could help keeping track of the topology of these Feynman diagrams. These ideas led to a paper by Brézin, Itzykson, Parisi and Zuber [35], who developed several tecniques to solve the matrix intergrals and were followed by many more. In 1990, a connection was made to two-dimensional quantum gravity by Brézin and Kazakov [36], Douglas and Shenker [37] and Gross and Migdal [38], which led to a new matrix revolution. All these developments had also a counter part in mathematics, due to papers by Witten [39, 40] and Kontsevich [41], where they described moduli spaces of punctured Riemann surfaces by using matrix models.

We will be focusing on the Hermitian one-matrix model for our purposes, with $\Phi^\dagger = \Phi$. The partition function for the Hermitian one-matrix model is given by the holomorphic, gauged matrix integral

$$Z = \frac{1}{\mathrm{vol}U(N)} \int \mathrm{d}\Phi \, \mathrm{e}^{-\frac{1}{g_s}\mathrm{Tr}W(\Phi)}, \tag{3.4}$$

with $W(\Phi)$ a polynomial in Φ of degree $n+1$

$$W(\Phi) = \sum_{k=1}^{\infty} \frac{t_k}{k} \Phi^k,$$

and g_s the string coupling constant, playing the role of $\hbar$ in this path integral.

To compute the matrix integral we reduce it by diagonalizing Φ,

$$\Phi = U\,\Lambda\,U^{-1} \quad \text{with} \quad \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N),$$

and taking the integral over the eigenvalues.

The measure $\mathrm{d}\Phi = \prod_{ij} \mathrm{d}\Phi_{ij}$ can be written in terms of these eigenvalues by using the $U(N)$ invariance $\Phi \to U\Phi U^\dagger$ and writing $U = \mathrm{e}^X$, with X hermitian and infinitesimal. Around $X = 0$ we find $U^{-1}\mathrm{d}U = \mathrm{d}X$ and the measure can be written as

$$\mathrm{d}\Phi = \mathrm{d}\Lambda + [\mathrm{d}X, \Lambda].$$

In terms of the eigenvalues the measure becomes

$$\prod_{ij} \mathrm{d}\Phi_{ij} = \mathrm{d}U \prod_i \mathrm{d}\lambda_i \prod_{i<j} (\lambda_i - \lambda_j)^2,$$

where $\mathrm{d}U$ is the Haar measure on $U(N)$. The integral over $\mathrm{d}U$ gives vol$U(N)$, which cancels against the factor in front of the matrix integral. The second product in the measure is the square of the Vandermonde determinant $\Delta(\lambda)$ coming from the Jacobian of the diagonalization procedure,

$$\Delta(\lambda) = \prod_{i<j} (\lambda_i - \lambda_j)$$

With this measure the partition function becomes

$$\begin{aligned} Z &= \int \prod \mathrm{d}\lambda_i \prod_{i<j} (\lambda_i - \lambda_j)^2 \exp\left(- \sum_i \frac{1}{g_s} W(\lambda_i) \right) \\ &= \int \prod \mathrm{d}\lambda_i \, \mathrm{e}^{-\frac{1}{g_s} S_{\text{eff}}(\lambda_1, \ldots, \lambda_N)}. \end{aligned}$$

The effective action S_{eff} is obtained by integrating out the angular variables in the matrix Φ and is equal to

$$S_{\text{eff}} = \sum_i W(\lambda_i) - 2g_s \sum_{i<j} \log|\lambda_i - \lambda_j|.$$

Minimizing the effective action with respect to the eigenvalues we get the saddle-point equation

$$W'(\lambda_i) - 2g_s \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} = 0, \quad i = 1, \ldots, N. \tag{3.5}$$

For classical vacua, where $g_s = 0$, one has the equation $W'(\lambda_i) = 0$.

By multiplying the saddle point equation by $\frac{1}{\lambda_i - x}$, and summing over i, one finds the loop equation [24]

$$\omega^2(x) - \frac{1}{N}\omega'(x) + \frac{1}{S}\omega(x)\, W'(x) - \frac{1}{4S^2} f(x) = 0, \tag{3.6}$$

where $S = g_s N$ is the 't Hooft coupling, $\omega(x)$ is the resolvent,

$$\omega(x) = \frac{1}{N}\sum_i \frac{1}{x - \lambda_i} = \frac{1}{N}\mathrm{Tr}\frac{1}{x - \Phi} = \frac{1}{N}\sum_{k=0}^{\infty} x^{-k-1}\mathrm{Tr}\,\Phi^k, \tag{3.7}$$

and $f(x)$ is a polynomial of degree $n - 1$,

$$f(x) = 4S\sum_i \frac{1}{N}\frac{W'(x) - W'(\lambda_i)}{x - \lambda_i}. \tag{3.8}$$

This function $f(x)$ determines the whole solution of the matrix integral through the loop equation.

It is convenient to introduce a single fixed eigenvalue $x = \lambda_{N+1}$ in the complex plane to study the large N dynamics and define

$$y(x) = \frac{\partial S_{\text{eff}}}{\partial x} = W'(x) - 2g_s \sum_i \frac{1}{x - \lambda_i},$$

where one can interpret the function $y(x)$ as the force exerted on this fixed eigenvalue by all of the other eigenvalues.

The matrix resolvent Eq.3.7 is the classical solution to the saddle point equation Eq.3.5. Using the resolvent the function $y(x)$ can be written as,

$$y(x) = W'(x) - 2S\,\omega(x). \tag{3.9}$$

3.3.1 HYPERELLIPTIC CURVE

One can associate an algebraic curve to the matrix model by considering the classical case where the vacua are given by $W'(\lambda_i) = 0$ and using the roots of this equation to define the algebraic curve. Such curves are usually defined by the position of their branch points and cuts in the complex plane. Setting $y^2 = W'(x)^2$ we get two branches $y = \pm W'(x)$. The curve has two complex planes that meet at the critical points a_i of the potential $W(x)$.

For the quantum case the equation gets a correction

$$y^2 = W'(x)^2 - f_{n-1}(x), \tag{3.10}$$

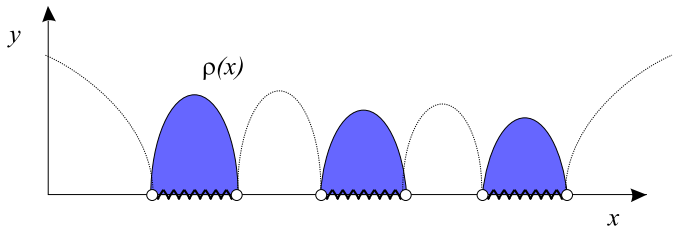

Figure 3.3: The eigenvalue density of a general one-matrix model is given by a hyperelliptic curve.

where $f_{n-1}(x)$ is a poynomial of degree $n-1$. To see this we take the square of the resolvent

$$\begin{aligned}\omega(x)^2 &= \frac{1}{N^2}\left(\sum_i \frac{1}{x-\lambda_i}\right)^2 \\ &= \frac{1}{N^2}\sum_{i\neq j}\frac{1}{(x-\lambda_i)(x-\lambda_j)} + \frac{1}{N^2}\sum_i \frac{1}{(x-\lambda_i)^2} \\ &= \frac{1}{N^2}\sum_{i\neq j}\frac{2}{(x-\lambda_i)(\lambda_i-\lambda_j)} - \frac{1}{N}\omega'(x),\end{aligned}$$

and drop the last term since ω is normalized to be finite in the large N limit. Substituting the saddle-point equation Eq.3.5 this becomes

$$\begin{aligned}\omega(x)^2 &\sim \frac{1}{N^2 g_s}\sum_i \frac{-W'(\lambda_i)}{x-\lambda_i} \\ &= \frac{1}{S}\left[\sum_i \frac{1}{N}\frac{W'(x)-W'(\lambda_i)}{x-\lambda_i} - W'(x)\sum_i \frac{1}{N}\frac{1}{x-\lambda_i}\right] \\ &= \frac{1}{4S^2}f(x) - \frac{1}{S}\omega(x)W'(x),\end{aligned} \tag{3.11}$$

with $f(x)$ as in Eq.3.8.

Using $y(x) = W'(x) - 2S\,\omega(x)$ and comparing its square with Eq.3.11, we get the curve in Eq.3.10. This is a hyperelliptic curve with a two-fold cover over the x-plane and the covers meet along the cuts where the eigenvalues are distributed. The curve has the form of a classical curve with the addition of a polynomial $f(x)$ of degree $n-1$, which plays the role of the quantum correction. In the quantum case, we have the branch points α_i, β_i giving rise to the curve

$$y^2 = \prod_i (x-\alpha_i)(x-\beta_i), \tag{3.12}$$

where the eigenvalues are distributed along the cut $[\alpha_i, \beta_i]$. The curves have $2n$ moduli, n of which are parametrized by the coupling constants of $W'(x)$ and the other n by the coefficients of the polynomial $f(x)$.

3.3.2 ONE-CUT SOLUTION

In the saddle point approximation, the eigenvalue distribution is defined as

$$\rho(\lambda) = \frac{1}{N}\sum_{i=1}^{N}\delta(\lambda - \lambda_i),$$

where λ_i are solutions of the saddle point equation Eq.3.5. In the limit $N \to \infty$, this distribution is assumed to become a continuous function with compact support and one also assumes that it will be zero outside a certain interval $\mathcal{C}$. This is the one-cut assumption. We use the normalization

$$\int_{\mathcal{C}} \rho(\lambda)\, \mathrm{d}\lambda = 1.$$

In terms of this eigenvalue density, the saddle point equation Eq.3.5 becomes

$$\frac{1}{2S}W'(\lambda) = \fint \frac{\rho(\lambda')}{\lambda - \lambda'}\, \mathrm{d}\lambda'. \tag{3.13}$$

For a given potential $W(\lambda)$, one could find $\rho(\lambda)$ in terms of the 't Hooft parameter S and the coupling constants t_k . From the solution for $\rho(\lambda)$, one could find the planar free energy $\mathcal{F}_0$ using

$$\frac{1}{N^2}\mathcal{F} = S_{\text{eff}}(\rho) + \mathcal{O}(\frac{1}{N^2}),$$

with

$$S_{\text{eff}}(\rho) = \frac{1}{S}\int_{\mathcal{C}} \rho(\lambda)\, W(\lambda)\, \mathrm{d}\lambda - \int_{\mathcal{C}\times\mathcal{C}} \rho(\lambda)\, \rho(\lambda')\, \log|\lambda - \lambda'|\, \mathrm{d}\lambda\, \mathrm{d}\lambda'. \tag{3.14}$$

Then the planar free energy is given by

$$\mathcal{F}_0(S) = S^2 S_{\text{eff}}(\rho) = \frac{S}{2}\int_{\mathcal{C}} \rho(\lambda)\, W(\lambda)\, \mathrm{d}\lambda,$$

where S_{eff} is evaluated at ρ, which is the solution to Eq.3.13.

The resolvent Eq.3.7 has a genus expansion [42]

$$\omega(x) = \sum_{g=0}^{\infty} g_s^{2g}\omega_g(p),$$

and the planar part can be written in terms of the eigenvalue density,

$$\omega_0(x) = \int \mathrm{d}\lambda \, \frac{\rho(\lambda)}{x-\lambda}.$$

This function is analytic on the whole complex plane, except on the interval $\mathcal{C}$ where the function has a singularity for $\lambda = x$, with $\lambda \in \mathcal{C}$. The discontinuity of $\omega_0(x)$ over the interval $\mathcal{C}$ can be computed since at the singularity $\lambda = x$,

$$\rho(x) = -\frac{1}{2\pi \mathrm{i}} \left(\omega_0(x + \mathrm{i}\epsilon) - \omega_0(x - \mathrm{i}\epsilon)\right).$$

When the planar resolvent is known, one can calculate the eigenvalue density from this equation and find the planar free energy.

The planar resolvent can also be written in a closed form [43],

$$\omega_0(x) = \frac{1}{2S} \oint_{\mathcal{C}} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{W'(z)}{x-z} \left(\frac{(x-a)(x-b)}{(z-a)(z-b)}\right)^{\frac{1}{2}}, \tag{3.15}$$

where the contour $\mathcal{C}$ is given by the interval $b \leq \lambda \leq a$. This can be seen by using the relation from Eq.3.13 near the singularity,

$$\omega_0(x + \mathrm{i}\epsilon) + \omega_0(x - \mathrm{i}\epsilon) = \frac{1}{2S} W'(x),$$

and writing it as a discontinuity relation

$$\frac{\omega_0(x + \mathrm{i}\epsilon)}{\sqrt{(x-a)(x-b)}} - \frac{\omega_0(x - \mathrm{i}\epsilon)}{\sqrt{(x-a)(x-b)}} = -\frac{1}{S} \frac{W'(x)}{\sqrt{(x-a)(x-b)}}.$$

This way, one can calculate $\omega_0(x)$ up to regular terms. However, since

$$\omega_0(x) \sim \frac{1}{x}, \quad x \to \infty, \tag{3.16}$$

there are no regular terms, and one has the whole solution for $\omega_0(x)$, given by Eq.3.15.

One can expand Eq.3.15 for large x and the asymptotic behavior gives then the following restrictions on the endpoints of the cut, a and b,

$$\begin{aligned} \oint_{\mathcal{C}} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{W'(z)}{\sqrt{(z-a)(z-b)}} &= 0, \\ \oint_{\mathcal{C}} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{zW'(z)}{\sqrt{(z-a)(z-b)}} &= 2S. \end{aligned} \tag{3.17}$$

With these conditions, a and b can be solved completely in terms of the 't Hooft coupling S and the coupling constants t_k in the potential W.

When $W(z)$ is a polynomial, the contour can be deformed in such a way that the integral Eq.3.15 gets poles at $z = x$ and $z = \infty$ and we can write the resolvent as

$$\omega_0(x) = \frac{1}{2S}\left(W'(x) - \sqrt{(x-a)(x-b)}M(x)\right), \tag{3.18}$$

with

$$M(x) = \oint_0 \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{W'(1/z)}{1-xz} \frac{1}{\sqrt{(1-az)(1-bz)}}.$$

Together with the conditions for the endpoints, one can now solve the one-cut one-matrix model completely.

3.3.3 Example: the Gaussian matrix model

For $W(\Phi) = \frac{1}{2}\Phi^2$ the matrix model becomes Gaussian which is the simplest example of the Hermitian matrix model and can be solved exactly.

The equations Eq.3.17 for the endpoints become

$$\begin{aligned} \oint_0 \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{1}{z^2} \frac{1}{\sqrt{(1-az)(1-bz)}} &= 0, \\ \oint_0 \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{1}{z^3} \frac{1}{\sqrt{1-a^2z^2}} &= 2S. \end{aligned}$$

The contour is around $z = 0$ and $a+b = 0$, since the potential is symmetric. The second equation gives $a = 2\sqrt{S}$ and we have the interval $\mathcal{C} = [-a, a] = [-2\sqrt{S}, 2\sqrt{S}]$.

The planar resolvent Eq.3.18 becomes

$$\omega_0(x) = \frac{1}{2S}\left(x - \sqrt{x^2 - 4S}\right),$$

and the eigenvalue density is

$$\rho(x) = \frac{1}{2\pi S}\sqrt{4S - x^2}.$$

This is also known as Wigner's semi-circle law, see fig.3.4.

The hyperelliptic curve Eq.3.10 is given by

$$y^2 = x^2 - 4S.$$

This is also the answer for a genus zero curve.

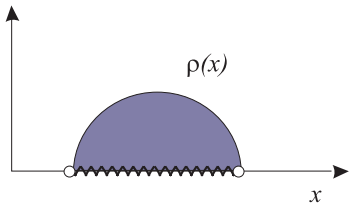

Figure 3.4: Wigner's semi-circle.

3.4 SCHWINGER-DYSON/QUANTUM LOOP EQUATION

The matrix model with $N \times N$ matrices Φ

$$Z = \int \mathrm{d}\Phi \, \exp\left(-\frac{1}{g_s}\mathrm{Tr}\, W(\Phi)\right),$$

with a generic potential

$$W(\Phi) = \sum_k t_k \, \Phi^k,$$

can also be solved in a different way using the Schwinger-Dyson equations. These equations express the reparametrization invariance of the path-integral. As an example we will take a single scalar ϕ and reparametrize the zero-dimensional path-integral

$$\int \mathrm{d}\phi \, \mathrm{e}^{-S(\phi)/\hbar},$$

with $\delta\phi = \xi(\phi)$. The change in the path-integral is then

$$\begin{aligned}\delta S &= \int \delta(\mathrm{d}\phi)\mathrm{e}^{-S(\phi)/\hbar} + \int \mathrm{d}\phi \frac{-1}{\hbar}\frac{\partial S}{\partial \phi}\mathrm{e}^{-S(\phi)/\hbar} \\ &= \int \mathrm{d}\phi(\partial\xi - \frac{1}{\hbar}\partial S\xi)\mathrm{e}^{-S(\phi)/\hbar}.\end{aligned}$$

The invariance of the path-integral gives the Schwinger-Dyson equation,

$$\langle \xi\partial S - \hbar\partial\xi \rangle = 0.$$

For the simple Gaussian action $S = \frac{m^2}{2}\phi^2$, this equation reads

$$\left\langle \xi m^2\phi - \hbar\partial\xi \right\rangle = 0$$

When we choose a reparametrization $\xi = \epsilon\phi$, it becomes

$$\epsilon \left\langle m^2\phi^2 - \hbar \right\rangle = 0 \Rightarrow \left\langle \phi^2 \right\rangle = \frac{\hbar}{m^2} \left\langle 1 \right\rangle .$$

which is the correct correlator.

This example makes the use of the Schwinger-Dyson equation obvious: by using a symmetry of the path-integral, we can compute expectation values of operators. We can also apply this method to the matrix model.

We take a reparametrization of the $N \times N$ matrix Φ equal to

$$\delta\Phi = \Phi^{n+1}, \quad n \geq -1.$$

For this choice we get an invariant path-integral but the action and the measure will not be invariant on their own. The change in the measure $\mathrm{d}\Phi$ is

$$\delta \mathrm{d}\Phi = \left(\frac{\partial}{\partial \Phi_{ij}} \delta\Phi_{ji}\right) \mathrm{d}\Phi = \mathrm{Tr}\left(\frac{\partial}{\partial \Phi} \delta\Phi\right) \mathrm{d}\Phi.$$

Taking the reparametrization $\delta\Phi = \epsilon\,\Phi$, we get

$$\delta \mathrm{d}\Phi = \mathrm{Tr}\epsilon\, \mathrm{d}\Phi = N\epsilon\, \mathrm{d}\Phi,$$

and the measure becomes

$$\mathrm{d}\Phi \to \mathrm{e}^{N\epsilon}\, \mathrm{d}\Phi.$$

Picking another reparametrization $\delta\Phi = \Phi^{n+1}$, the variation gives

$$\begin{aligned} 0 &= \int \mathrm{d}\Phi \frac{\partial}{\partial \Phi_{ij}} \left(g_s \Phi_{ji}^{n+1} \mathrm{e}^{-\frac{1}{g_s}\mathrm{Tr}\, W(\Phi)}\right) \\ &= \left\langle \mathrm{Tr}(\Phi^{n+1} W'(\Phi)) - g_s \sum_{p=0}^{n} \mathrm{Tr}\,\Phi^p \mathrm{Tr}\,\Phi^{n-p} \right\rangle, \end{aligned}$$

where we used

$$\frac{\partial}{\partial \Phi_{ij}} \Phi_{kl}^{n+1} = \frac{\partial}{\partial \Phi_{ij}} (\underbrace{\Phi_{km}\Phi_{mr}\cdots\Phi_{sl}}_{n}) = \sum_{p=0}^{n} \Phi_{ki}^{p} \delta_{ij} \Phi_{jl}^{n-p}.$$

We see that the path-integral is invariant under this reparametrization.

Taking the derivative of the path-integral with respect to the couplings t_k, brings down a power of Φ of the operator $\mathrm{Tr}\,\Phi^k$

$$g_s \frac{\partial}{\partial t_k} Z = \left\langle \mathrm{Tr}\,\Phi^k \right\rangle$$

and with this we can rewrite the Schwinger-Dyson equation as a differential equation in these couplings:

$$L_n Z = 0, \qquad n \geq -1,$$

where

$$L_n = \sum_k k\, t_k \frac{\partial}{\partial t_{k+n}} + g_s^2 \sum_{p=0}^{n} \frac{\partial^2}{\partial t_p\, \partial t_{n-p}}.$$

$Z(t)$ is then considered as a formal generating function

$$Z(t) = \left\langle \mathrm{e}^{-\frac{1}{g_s}\sum_k t_k \mathrm{Tr}\,\Phi^k} \right\rangle.$$

Multiplying the path-integral with the variables t_k remakes a factor of $\mathrm{Tr}\,\Phi^k$, and we can see that these two operators satisfy the canonical commutation relations

$$\left[\frac{\partial}{\partial t_m}, t_n\right] = \delta_{mn}.$$

The operators L_n satisfy the positive part of the Virasoro algebra

$$[L_n, L_m] = (n-m)\, L_{n+m},$$

and therefore the equations $L_n Z = 0$ are called Virasoro constraints. This is no surprise since

$$L_n = \Phi^{n+1} \frac{\partial}{\partial \Phi}$$

generate matrix reparameterizations of the variable Φ.

Now we would like to derive the loop equations and we will see that they are equal to the Schwinger-Dyson equations. It is convenient to use the resolvent

$$\omega = \frac{1}{N}\left\langle \mathrm{Tr}\frac{1}{x-\Phi}\right\rangle = \frac{1}{N}\sum_k x^{-k-1}\left\langle \mathrm{Tr}\Phi^k\right\rangle.$$

Considering the variation $\delta\Phi = \frac{1}{x-\Phi}$, the Schwinger-Dyson equations become

$$\begin{aligned} 0 &= \int \mathrm{d}\Phi \frac{\partial}{\partial \Phi_{ij}} \left(g_s \left(\frac{1}{x-\Phi}\right)_{ji} \mathrm{e}^{-\frac{1}{g_s}\mathrm{Tr}\,W(\Phi)} \right) \\ &= \left\langle \mathrm{Tr}\frac{W'(\Phi)}{x-\Phi} - g_s \left(\mathrm{Tr}\frac{1}{x-\Phi}\right)^2 \right\rangle. \end{aligned}$$

This can be simplified in the large N limit since products in the correlator factorize:

$$\left\langle \mathrm{Tr}\Phi^p\, \mathrm{Tr}\Phi^q\right\rangle = \left\langle \mathrm{Tr}\Phi^p\right\rangle \left\langle \mathrm{Tr}\Phi^q\right\rangle.$$

This is a consequence of the fact that the disconnected part of this correlator scales like N^2, while the connected part scales like N. With this factorization, the Schwinger-Dyson equation becomes the planar loop equation:

$$\left\langle \mathrm{Tr}\frac{W'(\Phi)}{x-\Phi}\right\rangle - g_s \left\langle \mathrm{Tr}\frac{1}{x-\Phi}\right\rangle^2 = 0.$$

We can see their equivalence in the following way: define the quantum correction as

$$f(x) = 4g_s \left\langle \mathrm{Tr} \frac{W'(x) - W'(\Phi)}{x - \Phi} \right\rangle,$$

and introduce the variable

$$y(x) = W'(x) - 2g_s \left\langle \mathrm{Tr} \frac{1}{x - \Phi} \right\rangle.$$

Computing $y(x)^2$ and writing it in terms of $W'(x)$ and $f(x)$, we get

$$\begin{aligned} y(x)^2 &= W'(x)^2 - 4g_s^2 \left\langle \mathrm{Tr} \frac{1}{x - \Phi} \right\rangle + 4g_s \left\langle \mathrm{Tr} \frac{W'(\Phi)}{x - \Phi} \right\rangle \\ &= W'(x)^2 - 4g_s \left\langle \mathrm{Tr} \frac{W'(x) - W'(\Phi)}{x - \Phi} \right\rangle. \end{aligned}$$

Substituting the expression for the quantum correction we get the spectral curve

$$y^2 = W'(x)^2 - f(x).$$

The funtion $y(x)$ can also be written as the derivative of a scalar field φ:

$$y(x) = \partial\varphi(x)$$

with the collective field

$$\varphi = W(x) - 2g_s \mathrm{Tr} \log(x - \Phi) = S_{\mathrm{eff}}(x, \widetilde{\Phi}). \tag{3.19}$$

Here $\widetilde{\Phi}$ denotes the saddle-point approximation to Φ.

The effective action in this formula is similar to the action for the eigenvalues with Coulomb interaction and we can write

$$\langle \partial\varphi(x) \rangle = \left[\sum_{k>0} k\, t_k\, x^{k-1} - 2g_s \sum_{k \geq 0} x^{-(k+1)} \frac{\partial}{\partial t_k} \right] Z.$$

The energy-momentum tensor corresponding to the collective field is then written as

$$y^2 = (\partial\varphi)^2 = T(x) = \sum L_n\, x^{-n-2},$$

and the loop equation becomes

$$\langle T(x) \rangle = W'(x)^2 - f(x).$$

Using the Virasoro constraints $L_n Z = 0$, $n \geq -1$, this is equal to

$$\langle T(x) \rangle_< \equiv \sum_{n \geq -1} x^{-n-2}\, L_n\, Z = 0.$$

We can write $y^2 = W'(x)^2 - f(x)$ as the hyperelliptic curve

$$y^2 = \prod_{i=1}^{2N}(x - x_i). \tag{3.20}$$

Since the classical field $\varphi(x)$ changes sign around the branch points, the expectation values of the bosonic field are given by two branches of $y(x)$. Then $\varphi(x)$ can be thought of as a single bosonic field defined on the branched covering given by the hyperelliptic curve Eq.3.20. The classical value of the collective field is given by

$$\partial\varphi_c(x) = \prod_i (x - x_i)^{1/2}. \tag{3.21}$$

3.5 MULTICUT MATRIX MODELS

As we have seen in section 3.2, the topological B-model on the Calabi-Yau manifold Eq.3.1 with N branes wrapping around n spheres has the free energy

$$\mathcal{F}(g_s, N_i) = \sum_{g=0}^{\infty} \sum_{h_1,\ldots,h_n=1}^{\infty} \mathcal{F}_{g,h_1,\ldots,h_n}\, g_s^{2g-2}\, N_1^{h_1} \cdots N_n^{h_n},$$

and the corresponding matrix model is given by the matrix integral

$$Z = \int \mathrm{d}\Phi\, \mathrm{e}^{-\frac{1}{g_s}\mathrm{Tr}W(\Phi)}.$$

One can compute the coefficients $\mathcal{F}_{g,h_1,\ldots,h_n}$ from the matrix model perturbatively.

Corresponding to the brane configuration with $N = \sum_i N_i$ branes, the matrix model has N_i eigenvalues at the ith critical point of the matrix model potential W. In the saddle point approximation, this results in a solution with more than one cut: the multicut solution. Here the eigenvalues are to be found at any extremum, thus at all (local) minima and maxima, while in the one-cut solution all of the eigenvalues were to be found at the global minimum. The number of eigenvalues at any of the extrema is fixed at N_i for the ith cut. Fixing the number of eigenvalues in a cut, one actually makes a choice for the classical vacuum. Different filling fractions will give different answers for the classical vacuum. This has to do with the fact that the matrix model gives only an effective description of the brane configuration.

It is convenient to use the fixed filling fraction ν defined as $\nu_i = N_i/N$. With this definition, one can write the partial 't Hooft coupling as $S_i = g_s N_i = S\nu_i$, with $S = \sum_i S_i$. Since the eigenvalue density at each cut is also fixed,

$$\int_{2x_i}^{2x_{i-1}} \mathrm{d}\lambda\, \rho(\lambda) = \nu_i, \quad i = 1, \ldots, n, \tag{3.22}$$

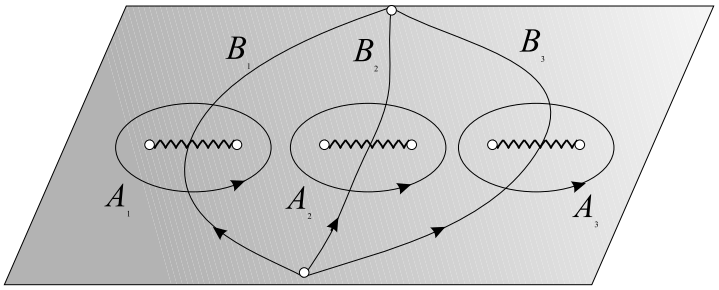

Figure 3.5: The A and B cycles drawn in the x-plane.

it can be written as

$$S_i = \frac{1}{2\pi \mathrm{i}} \oint_{A_i} \mathrm{d}\lambda \, y(\lambda), \quad i = 1, \ldots, n,$$

with

$$y(\lambda) = W'(\lambda) - 2S \, \omega_0(\lambda),$$

as in equation Eq.3.9. Here the cycle A_i corresponds to the hyperelliptic curve

$$y^2 = W'(x)^2 - f(x)$$

going around the ith cut $\mathcal{C}_i$.

Since $y(x)\,\mathrm{d}x$ defines a meromorphic one-form on the hyperelliptic curve, one can consider the cycles A_i and B_i, which are canonically conjugated cycles on the hyperelliptic Riemann surface, see figs.3.5 and 3.6. The A_i-cycles are closed curves around each of the n branch cuts, and the B_i-cycles run from one of the branch-points to the point at infinity.

From the gauge theory point of view, taking the integrals of the one-form over the A_i-cycles gives the vacuum expectation values of the gluino condensates S_i. The function $y(x)$ has single poles at the position of the eigenvalues, and therefore we can compute the integral over the A_i-cycle by using Cauchy's theorem

$$\frac{1}{2\pi \mathrm{i}} \oint_{A_i} y(x)\,\mathrm{d}x = \frac{1}{2\pi \mathrm{i}} \oint_{A_i} \left(W'(x) - \frac{2S}{N} \sum_j \frac{1}{x - \lambda_j} \right) \mathrm{d}x = g_s \, N_i = S_i. \tag{3.23}$$

The B_i-cycles are non-compact, and doing the integral over these cycles, we need a cut-off at $x = \Lambda$,

$$\int_{B_i} y(x)\,\mathrm{d}x = \int_{a_i}^{\Lambda} \mathrm{d}S_{\text{eff}} = \frac{\partial \mathcal{F}_0}{\partial S_i}. \tag{3.24}$$

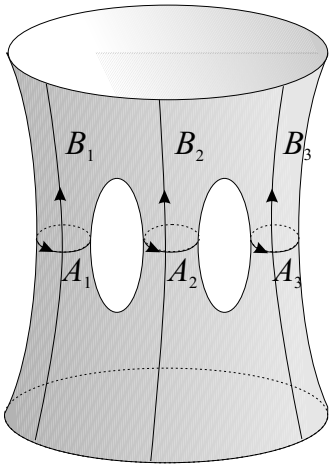

Figure 3.6: The A and B cycles drawn on the Riemann surface that is a double cover of the x-plane.

This corresponds to moving one eigenvalue from the cut to infinity, and B_i stands for the path of this eigenvalue. We will see later in sections 3.6 and 4.1.3 that these period integrals also emerge from string theory.

Determining the endpoints of the cuts is done in a similar way to the one-cut problem, see section 3.3.2, except that now the contour $\mathcal{C}$ is different. It is again assumed that the density of the eigenvalues has a compact support near $N = \infty$. The support is given by

$$\sigma \equiv \cup_{i=1}^{n}[x_{2i}, x_{2i-1}], \quad x_1 > x_2 > \ldots > x_{2n}.$$

In the large N limit, one can again use the saddle-point approximation to find the equation for the eigenvalue density $\rho(\lambda)$. However, because of the restriction Eq.3.22, we need to add an extra term to the effective action, for which we introduce the Lagrange multipliers Γ_i, $i = 1, \ldots, n-1$. In the planar limit, the effective action Eq.3.14 is then given by [44, 45]

$$\begin{aligned} S_{\text{eff}}(\rho) &= \frac{1}{S}\int_{\mathcal{C}} \rho(\lambda)\, W(\lambda)\, \mathrm{d}\lambda - \int_{\mathcal{C}\times\mathcal{C}} \rho(\lambda)\, \rho(\lambda')\, \log|\lambda - \lambda'|\, \mathrm{d}\lambda\, \mathrm{d}\lambda' \\ &\quad + \sum_{i=1}^{n-1} \Gamma_i \left(\nu_i - \int_{\mathcal{C}_i} \rho(\lambda)\, \mathrm{d}\lambda \right), \end{aligned}$$

with $\sum_i \nu_i = 1$ and $\mathcal{C} = \sum_i \mathcal{C}_i$. The derivative of the action with respect to $\rho(\lambda)$ gives

$$\frac{1}{S} W(\lambda) - 2\int_{\mathcal{C}} \rho(\lambda')\, \log|\lambda - \lambda'|\, \mathrm{d}\lambda' - \Gamma = 0, \qquad \forall \lambda \in \mathcal{C},$$

with $\Gamma = \sum_i \Gamma_i$. This implies that the effective potential

$$W_{\text{eff}}(\lambda) = \frac{1}{S} W(\lambda) - 2 \int_{\mathcal{C}} \rho(\lambda') \, \log|\lambda - \lambda'| \, \mathrm{d}\lambda'$$

is constant on the support σ and equal to Γ_i at each interval,

$$W_{\text{eff}}(\lambda) = \Gamma_i \qquad \text{if } \lambda \in \mathcal{C}_i.$$

However, the corresponding eigenvalue density $\rho_c(\lambda)$ and the effective action $S_{\text{eff},c}$ still depend on the filling fractions ν because of

$$S_{\text{eff},c}(\nu) = \frac{1}{2} \left(\int_{\mathcal{C}} \rho_c(\lambda) \, W(\lambda) \, \mathrm{d}\lambda + \sum_i \Gamma_i \, \nu_i \right).$$

Taking the derivative of the action with respect to the filling fractions, we get

$$\sum_{i=1}^{n-1} \Gamma_i = 0,$$

which fixes the value of the filling fractions. For a two-cut matrix model, this determines $\rho_c(\lambda)$ uniquely with $\Gamma_1 = \Gamma_2$ and the large N free energy is then given by

$$\mathcal{F}_0 = S^2 \, S_{\text{eff},c}(\nu).$$

For a polynomial potential W of degree $n+1$ and with fixed ν, the planar resolvent for the two-cut case is given by

$$\omega_0(\lambda, \nu) = \frac{1}{2S} \left(W'(\lambda) - M(\lambda) \, L(\lambda) \right) \quad \text{with } L(\lambda) = \sqrt{(\lambda - a)(\lambda - b)(\lambda - c)(\lambda - d)},$$

where $M(\lambda)$ is a polynomial of degree $n - 2$.

$\rho(\lambda, \nu)$ is given by the discontinuity of ω_0, as in section 3.3.2. The coefficients of M and the endpoints a, b, c, d are completely determined by the asymptotic behavior of ω_0 given by Eq.3.16 and by the restrictions

$$\nu = \int_a^b \rho(\lambda, \nu) \, \mathrm{d}\lambda = \frac{1}{2\pi} \int_a^b |M(\lambda, \nu)| \, L(\lambda)$$

and

$$\Gamma_1 = \Gamma_2,$$

which leads to

$$\int_b^c \mathrm{d}\lambda \, (2S \, \omega_0 - W'(\lambda)) = - \int_b^c |M(\lambda, \nu)| \, L(\lambda).$$

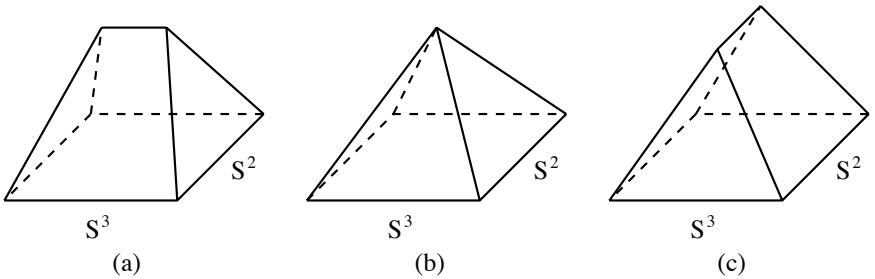

Figure 3.7: (a) The deformed conifold, (b) the ordinary conifold, (c) the resolved conifold. In each case, the square at the base represents the slice $S^3 \times S^2$. In (b), the tip of the cone is a singularity; in (a) it is replaced by an S^3 , and in (c) by an S^2. (From [13].)

3.6 GEOMETRIC TRANSITION

We have seen that the hyperelliptic curve Eq.3.10 is the planar solution of a multicut matrix model, which in turn computes the open string amplitudes on the Calabi-Yau manifold Eq.3.1.

The period integrals S_i and $\frac{\partial \mathcal{F}_0}{\partial S_i}$ in Eqs.3.23 and 3.24 look very similar to the expressions in Eq.2.12, which were obtained from the B-model with a special geometry. We will see below that this is no coincidence and that it is related to the smoothing out of the singularities in the Calabi-Yau geometry.

As we have seen in section 3.1, closed string free energy can be obtained from open string amplitudes by using the double line notation of 't Hooft and summing over all holes in the fatgraphs.

This can be explained from the gauge theory point of view [26, 46] and open strings living on a Calabi-Yau manifold are then related to closed strings living on another Calabi-Yau manifold.

This is done by smoothing out singularities in the Calabi-Yau geometry, either by resolving or by deforming them. Resolving is achieved by blowing up the singularities into two-spheres, like in section 3.2, whereas deforming is done by inflating n three-spheres in the geometry, one for each singularity, see fig.3.7.

The manifold

$$wz + y^2 + W'(x)^2 = f(x) \tag{3.25}$$

is non-compact and for a non-compact manifold $\mathcal{M}$ the first Chern class $c_1(\mathcal{M})$ is not well-defined. Recall that for a Calabi-Yau manifold we had the condition $c_1(\mathcal{M}) = 0$,

and therefore, generally, one cannot speak of a Calabi-Yau manifold. However, it was shown in [47] that a non-compact manifold *can* carry a Ricci-flat Calabi-Yau metric that looks conical at infinity as long as it has a holomorphic form with coordinates of degree d_i such that

$$\sum d_i > 1.$$

The Calabi-Yau manifold Eq.3.25 is deformed by a polynomial $f(x)$ of degree $n-1$ and is called the deformed manifold. The deformation corresponds to n three-spheres in the geometry. The Calabi-Yau three-form corresponding to this non-compact manifold is given by

$$\Omega = \frac{\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}u}{v}.$$

One can consider the inflated three-spheres as two-spheres which are fibered over an interval in the x-plane. To see this, consider the deformed conifold

$$x^2 + y^2 + u^2 + v^2 = \mu, \tag{3.26}$$

where we substituted $wz = u^2 + v^2$ in Eq.3.25 and μ is real. The three-sphere in this geometry has real-valued variables. Take x to be in the interval $[-\sqrt{\mu}, \sqrt{\mu}]$. For this fixed value of x, we get a two-sphere with radius $\sqrt{\mu - x^2}$. At the endpoints of the integral, the two-sphere collapses. The total geometry of the two-sphere together with the interval $[-\sqrt{\mu}, \sqrt{\mu}]$ gives a three-sphere. In the general case with n cuts and $2n$ endpoints, there are n three-spheres with two-sphere fibrations over the cuts.

Consider the B-model closed topological strings on the deformed manifold. For this three-form we can choose a symplectic basis of three-cycles $\hat{A}_i$, $\hat{B}^j$ with the conditions

$$\begin{aligned}
\hat{A}_i \cap \hat{A}_j &= 0, \\
\hat{A}_i \cap \hat{B}^j &= \delta_i^j, \\
\hat{B}^i \cap \hat{B}^j &= 0.
\end{aligned}$$

It turns out that the $\hat{A}_i$ cycles correspond to the A_i cycles around the ith cut in the x-plane and that $\hat{B}_i$ cycles correspond to the B_i cycles on the x-plane [46]. Given this duality, the periods of Ω

$$S_i = \frac{1}{2\pi} \oint_{\hat{A}_i} \Omega, \quad \frac{\partial \mathcal{F}_0}{\partial S_i} = \int_{\hat{B}^i} \Omega$$

reduce to the periods

$$S_i = \frac{1}{2\pi \mathrm{i}} \oint_{A_i} \mathrm{d}x\, y(x), \quad \frac{\partial \mathcal{F}_0}{\partial S_i} = \int_{B^i} \mathrm{d}x\, y(x)$$

on the hyperelliptic curve

$$y^2 = W'(x)^2 - fx).$$

To see the connection, consider again the deformed conifold Eq.3.26 with one three-sphere. This three-sphere is a two-sphere fibration over the cut $[-\sqrt{\mu}, \sqrt{\mu}]$. So the A-period can be calculated by first doing the integral over the two-sphere and then the integral over the cut,

$$S = \frac{1}{4\pi} \int_{-\sqrt{\mu}}^{\sqrt{\mu}} \int_{\mathbf{S}^2} \frac{\mathrm{d}y \, \mathrm{d}x}{\sqrt{\mu - x^2 - y^2 - u^2}} = \frac{1}{2\pi} \int_{-\sqrt{\mu}}^{\sqrt{\mu}} \sqrt{\mu - x^2} \, \mathrm{d}x = \frac{1}{2\pi} \int_{-\sqrt{\mu}}^{\sqrt{\mu}} y(x) \, \mathrm{d}x,$$

with $y^2 + x^2 = \mu$. This is equivalent to the A-period in Eq.3.23, with $y \to -\mathrm{i}y$. This example can be extended to the general case with n three-spheres. So the special geometry of the deformed manifold leads to the planar solution of the matrix model with multiple cuts and the corresponding hyperelliptic curve and periods.

What this physically means is that open topological strings on the resolved manifold Eq.3.2, where N D-branes wrap the n spheres, are related to closed topological strings on the deformed manifold Eq.3.25 without any D-branes. As we have seen, the 't Hooft couplings S_i in the open topological string theory are equivalent to the periods arising from the geometry in the closed topological string theory. This idea is supported by the fact that open topological strings can be described by a matrix model, see section 3.1.

CHAPTER 4

GAUGE THEORIES AND MATRIX MODELS

4.1 DIJKGRAAF-VAFA CONJECTURE

The Dijkgraaf-Vafa conjecture [3] entails that for certain classes of $\mathcal{N} = 1$ supersymmetric gauge theories in four dimensions the effective superpotential as a function of the glueball superfield can be calculated exactly by summing up the planar diagrams of this gauge theory. This calculation is done perturbatively and it reduces to a matrix model with an action corresponding to the tree-level superpotential of the gauge theory. The derivation of this correspondence directly from field theory was shown in [48, 49].

We will consider actions in superspace, as in section 2.1.1, but with four bosonic and four fermionic coordinates and similar F-terms and D-terms,

$$\int \mathrm{d}^4x\, \mathrm{d}^2\theta\, W(\Phi), \quad \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, K(\Phi, \bar{\Phi}).$$

We want to couple the chiral superfield Φ to a supersymmetric gauge theory with gauge group G and therefore we introduce the adjoint valued vector superfield $V(x, \theta, \bar{\theta})$. The physical modes of this vector mutiplet consists of the gauge boson A_μ and the gaugino λ. We have the reality condition for this vector superfield, $V^\dagger = V$.

Consider the minimal coupling to the chiral superfield given by

$$\int \mathrm{d}^4\theta\, \bar{\Phi}\, \mathrm{e}^V\, \Phi,$$

where we assume that Φ transforms in a representation R of gauge group G. The spinor field strength is defined as

$$\mathcal{W}_\alpha = -\mathrm{i}\bar{D}^2\,\mathrm{e}^{-V}\,D_\alpha\,\mathrm{e}^{V}.$$

This is a covariantly chiral superfield, with the lowest components given by,

$$\mathcal{W}_\alpha = \lambda_\alpha + \theta^\beta F_{\alpha\beta} + \dots\ .$$

Here $F_{\alpha\beta} = F_{\beta\alpha}$ is the self-dual part of the field strength $F_{\mu\nu}$,

$$F_{\alpha\beta} = (\gamma^\mu\gamma^\nu)_{\alpha\beta}F_{\mu\nu}.$$

We define the glueball superfield S as

$$S = \frac{1}{32\pi^2}\mathrm{Tr}_{\mathrm{adj}}\mathcal{W}^2,$$

and with this we can write down the action of the super Yang-Mills theory,

$$\int \mathrm{d}^2\theta\, 2\pi\mathrm{i}\,\tau\, S + c.c.$$

with

$$S = \frac{1}{32\pi^2}\left(\mathrm{Tr}_{\mathrm{adj}}\lambda^2 + \ldots + \mathrm{Tr}_{\mathrm{adj}}F^2_{\alpha\beta}\,\theta^\alpha\,\theta^\beta\right).$$

and the complexified gauge coupling

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi\mathrm{i}}{g^2},$$

with the theta-angle θ and the gauge coupling g. Here the lowest component is the gluino bilinear $\mathrm{Tr}\,\lambda^2$, and the highest component is the self-dual part of the operator $\mathrm{Tr}\,F^2$.

The chiral ring for a simple gauge group G is generated by S. Classically, one has the relation [50]

$$S^h = 0,$$

where h is the dual Coxeter number, which is the second Casimir in the adjoint representation. For $G = SU(N)$ the Coxeter number is N. In the quantum theory one has for the chiral ring

$$S^h = \Lambda^{3h}.$$

Here Λ is the dynamically generated mass scale which emerges through the renormalization procedure, even when the original theory has no dimensional parameters. This phenomenon is called dimensional transmutation [51]. One can write it in terms of the complexified gauge coupling τ as

$$\Lambda^{3h} = \mu^{3h}\mathrm{e}^{2\pi\mathrm{i}\tau},$$

with μ the energy scale parameter of the theory.

To study the vacuum structure of the theory, we need to find the low energy quantum effective action. This action was derived by Veneziano and Yankielowicz [52] and is equal to

$$\int \mathrm{d}^4x\, \mathrm{d}^2\theta\, W_{\mathrm{eff}}(S).$$

This effective action should describe the vacuum structure of the quantum super Yang-Mills theory and reproduce the correct vacuum expectation value for S. It can be derived by integrating the chiral anomaly. When we change the glueball superfield by a chiral transformation, then, by definition, the change in the effective action should be equal to the anomaly. The transformation $S \to \mathrm{e}^{2\pi\mathrm{i}\alpha}S$ changes the effective action by

$$W_{\mathrm{eff}} + \delta S = W_{\mathrm{eff}} + 2\pi\mathrm{i}\alpha\, S.$$

The effective superpotential that is consistent with this anomaly was found to be

$$W_{\mathrm{eff}}(S) = hS\left(1 - \log(S/\Lambda^3)\right).$$

For $\Lambda^3 = \Lambda_0^3\mathrm{e}^{2\pi\mathrm{i}\tau_0}$ the effective energy becomes

$$W_{\mathrm{eff}}(S) = hS\left(1 - \log(S/\Lambda_0^3)\right) + 2\pi\mathrm{i}\tau_0 hS,$$

where τ_0 is the bare gauge coupling. One can check that this is the correct vacuum expectation value for S by minimizing W_{eff} with respect to S which results in

$$\log(S/\Lambda^3)^h = 0, \quad S = \mathrm{e}^{2\pi\mathrm{i}n/h}\Lambda^3, \quad n \in \mathbb{Z}_h.$$

So the Veneziano-Yankielowicz action reproduces the correct vacuum expectation value for S.

In the original paper [52], the Veneziano-Yankielowicz term was derived by integrating the chiral anomaly. In [53, 54] this term was derived by using the Konishi anomaly (see section 4.1.5), in the presence of a tree-level superpotential. A diagrammatic derivation of the Veneziano-Yankielowicz term is given in [55]. The full superpotential, including the Veneziano-Yankielowicz term and the kinetic terms, was discussed in [56].

The chiral ring was studied in several papers in the context of the Dijkgraaf-Vafa conjecture, among which are [49, 57, 58, 50, 59, 60].

4.1.1 Deformed $\mathcal{N} = 2$ super Yang-Mills theory

As an example we will take a look at deformed $\mathcal{N} = 2$ super Yang-Mills theories. The $\mathcal{N} = 2$ vector multiplet consists of a gauge boson, two Weyl spinors and a

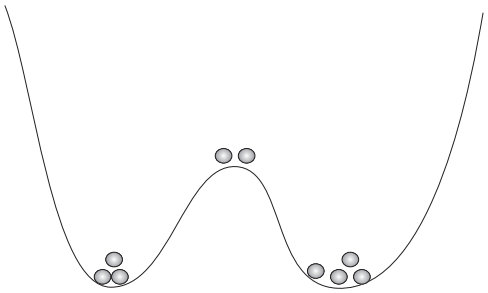

Figure 4.1: A fourth order superpotential W with eigenvalues distributed over the critical points. Note that the superpotential $W(\Phi)$ appears as $|W'(\varphi)|^2$ in the scalar potential, so the eigenvalues are localized at the minima of the scalar potential.

complex scalar, $(A_\mu, \lambda, \psi, \varphi)$. In $\mathcal{N} = 1$ language, the $\mathcal{N} = 2$ multiplet consists of an $\mathcal{N} = 1$ vector multiplet V and an $\mathcal{N} = 1$ chiral multiplet Φ. Since $\mathcal{N} = 2$ supersymmetry does not allow a superpotential term in the action, turning on a superpotential breaks the extended supersymmetry to $\mathcal{N} = 1$ supersymmetry. Such theories are known as deformed $\mathcal{N} = 2$ theories.

Consider a deformed $\mathcal{N} = 2$ theory with gauge group $G = U(N)$ with field content consisting of an $\mathcal{N} = 1$ vector multiplet and one adjoint chiral multiplet $\Phi(x, \theta)$, which is an $N \times N$ matrix.

Consider the tree-level superpotential

$$\int \mathrm{d}^2\theta \,\mathrm{Tr}\, W(\Phi),$$

to be a polynomial in the chiral superfield

$$W(x) = \sum_{k=1}^{n+1} \frac{t_k}{k} x^k.$$

Assume that the superpotential has n inequivalent, isolated critical points

$$W'(x) = \prod_{k=1}^{n} (x - a_k).$$

To obtain the classical vacua, the F- and D-terms are set to zero and solved for Φ. In this deformed $\mathcal{N} = 2$ theory the D-term for Φ is

$$V_D = \mathrm{Tr}[\Phi, \Phi^\dagger]^2,$$

and Φ should be a diagonal matrix

$$\Phi_{\rm cl} = U\,\mathrm{diag}(\lambda_1,\ldots,\lambda_N)\,U^{-1}.$$

The F-term is equal to $\mathrm{Tr}W(\Phi)$ and the eigenvalues λ_i are taken to be the roots a_i of $W'(\Phi)$.

We can distribute the eigenvalues over the roots and make a partititon with filling numbers,

$$N = N_1 + \cdots + N_n,$$

where N_k is the number of eigenvalues λ_i equal to a_k.

With this vacuum, the $U(N)$ symmetry is broken to

$$U(N) \to U(N_1) \times \cdots \times U(N_n).$$

When the strong coupling becomes dominant, the quantum effects will become important and one will expect to see confinement. This will break down the symmetry further to

$$U(N_i) \cong U(1) \times SU(N_i) \to U(1).$$

In perturbation theory, the effective superpotential $W_{\rm eff}(S_1,\ldots,S_n)$ can be computed exactly and is equal to [5]

$$W_{\rm eff} = \sum_i \left(N_i \frac{\partial \mathcal{F}_0}{\partial S_i} + 2\pi \mathrm{i} \tau_0 S_i \right) \tag{4.1}$$

where

$$\mathcal{F}_0(S) = \sum_i -\frac{1}{2} S_i^2 \log(S_i/\Lambda_0^3) + \sum_k a_k S^k.$$

The sum is over n colored planar Feynman diagrams (or, as in section 3.1, over oriented fatgraphs), which are given by the tree-level action $W_{\rm tree}$. With k colored holes, $i_1,\ldots,i_k \in \{0,\ldots,n-1\}$, the graph gets multiplied by

$$S^k = S_{i_1} \cdots S_{i_k}.$$

One can derive the Feynman rules from the superpotential $\mathrm{Tr}\,W(\Phi)$ by considering it as the bosonic action of a zero-dimensional field theory, which is a matrix theory.

4.1.2 EXAMPLE: THE CUBIC SUPERPOTENTIAL

Consider for example the cubic superpotential

$$W(\Phi) = \frac{m}{2}\Phi^2 + \frac{g}{3}\Phi^3.$$

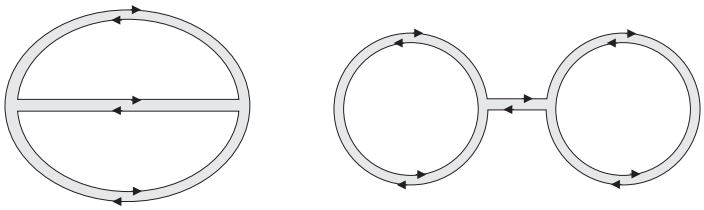

Figure 4.2: The two planar two-loop diagrams, with combinatorial weight $\frac{1}{6}$ and $\frac{1}{2}$, that contribute to the order S^3 term in the free energy $\mathcal{F}_0(S)$.

Its critical points are $\Phi = 0$ and $\Phi = -g/m$. In the vacuum, $\Phi = 0$ and the $SU(N)$ part of the gauge group $U(N)$ confines

$$U(N) \to U(1).$$

The Feynman rules can be derived easily since we are in a zero-dimensional field theory and the momentum dependence can be dropped. For a propagator one gets $1/m$ and for a cubic vertex $g/3$. In addition, every boundary loop gives a factor of S. Since Φ is in the adjoint representation, we can use the 't Hooft double line notation, as in section 3.1. Up to two loops, we get the expression

$$\mathcal{F}_0 = -\frac{1}{2}S^2 \log\left(\frac{S}{\Lambda^3}\right) + \frac{1}{2}S^2 \log\left(\frac{m}{\Lambda}\right) + \frac{2}{3}\frac{g^2}{m^3}S^3 + \mathcal{O}(S^4).$$

The first term results in the Veneziano-Yankielowicz term. The second term comes from the single one-loop diagram and the third term is coming from the sum of the two two-loop diagrams, see fig.4.2.

To obtain the free energy $\mathcal{F}$, one can use an auxiliary matrix model. Take Φ to be an $\widetilde{N} \times \widetilde{N}$ matrix and consider it as an element of the Lie algebra of $U(\widetilde{N})$. The generating function for the quantum effective superpotential is a matrix integral

$$Z = \frac{1}{\mathrm{vol}\, U(\widetilde{N})} \int \mathrm{d}\Phi\, \mathrm{e}^{-\frac{1}{g_s}\mathrm{Tr}\, W(\Phi)} = \exp \sum_{g\geq 0} g_s^{2g-2}\, \mathcal{F}_g(S).$$

We can distribute the eigenvalues of the matrix Φ over the critical points of $W'(S)$ with the partition

$$\widetilde{N} = \widetilde{N}_1 + \ldots + \widetilde{N}_n.$$

The 't Hooft coupling for the matrix model is defined as

$$S_i = g_s\, \widetilde{N}_i.$$

In the 't Hooft large $\widetilde{N}$ limit, we send $g_s \to 0, \widetilde{N}_i \to \infty$, keeping the S_i fixed. Then the only contribution to the path-integral is coming from planar diagrams. Here, one should not confuse $\widetilde{N}$, the size of the matrix, with N, the rank of the gauge group.

Once the path-integral is solved, we will also have computed the planar free energy $\mathcal{F}_0$ and use it to determine the quantum effective superpotential for S_i exactly.

Till now, we have only considered the theory with the gauge group $U(N)$. Of course, one could also consider other gauge groups that allow a large N description, such as $SO(N)$ or $Sp(N)$. This has been a subject of much interest with the papers [61, 62, 63, 64, 65, 66] as result.

4.1.3 STRING THEORY/GEOMETRIC INTERPRETATION

The effective superpotential can also be computed from string theory using certain dualities in string theory.

In topological open string theory, the effective superpotential is computed by means of planar diagrams, see section 3.1. To calculate the superpotential to order S^{h-1}, consider a sphere with h holes cut out, see fig.4.3. This corresponds to an open string diagram with $h-1$ loops. Then we insert $h-1$ pairs of λ in this diagram. These are coming from the zero-mode analysis since only in this case do we get an effective action of the right form, $\int \mathrm{d}^2\theta\, W(S)$. There are h possible ways to do this insertion. Summing over all gauge group indices, we get S^{h-1} for internal loops and N for the outer loop, when the gauge group is $U(N)$. In total, the diagram contributes

$$N\, \mathcal{F}_{0,h}\, h\, S^{h-1} = N\, \frac{\partial}{\partial S}\, S^h\, \mathcal{F}_{0,h}.$$

The effective superpotential in Eq.4.1 also has a non-polynomial term, leading to the Veneziano-Yankielowicz term. Here it is added by hand, but it can also come from the $U(N)$ volume factor in the matrix model path-integral [5, 67, 68].

The quantum hyperelliptic curve Eq.3.12 in section 3.3.1, which emerged from the matrix model, can be used in the gauge theory to formulate a geometric interpretation.

Recall that the function $y(x)$ was defined as

$$y(x) = \frac{\partial S_{\text{eff}}}{\partial x} = W'(x) - \frac{2S}{\widetilde{N}} \sum_i \frac{1}{x - \lambda_i},$$

and $y(x)\, \mathrm{d}x$ defines a meromorphic one-form on the hyperelliptic curve.

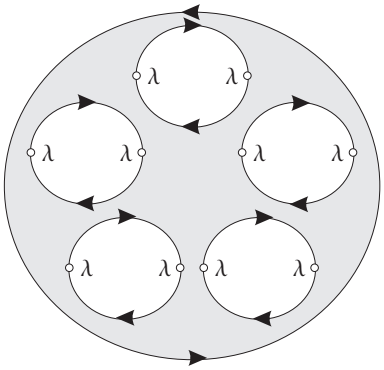

Figure 4.3: An open string worldsheet diagram with $h-1$ loops, it contributes to the S^{h-1} term of the effective superpotential. In this case $h=6$.

As we mentioned in section 3.5, the period integrals

$$S_i = \frac{1}{2\pi \mathrm{i}} \oint_{A_i} \mathrm{d}x\, y(x), \quad \frac{\partial \mathcal{F}_0}{\partial S_i} = \int_{B_i} \mathrm{d}x\, y(x)$$

can also be derived from string theory. From the gauge theory point of view, the A-period integrals give the vacuum expectation values of the gluino condensates S_i [69] and the B-period integrals correspond to moving one eigenvalue from the cut to infinity, where B_i stands for the path of this eigenvalue.

Using these expressions we can write the effective superpotential as

$$W_{\text{eff}}(S) = \sum_i \left[N_i \int_{B_i} y\, \mathrm{d}x - \tau_0 \oint_{A_i} y\, \mathrm{d}x \right]. \tag{4.2}$$

This can be compared to the results from special geometry of the Calabi-Yau dual to this gauge theory. It can be realized as a compactification on a Calabi-Yau three-fold $\mathcal{M}$ with Ramond-Ramond (RR) and Neveu-Schwarz (NS) fluxes in type IIB string theory.

With the three-form flux

$$H = H_{RR} + \tau_0\, H_{NS},$$

and the holomorphic (3,0) form Ω, the Gukov-Vafa-Witten superpotential can be written as [70, 71, 72]

$$W_{\text{eff}} = \int_{\mathcal{M}} H \wedge \Omega.$$

Taking the canonical basis for H_3 and using the Riemann bilinear identity, as in

section 2.2.2, this is equal to

$$W_{\text{eff}} = \sum_i \left[\oint_{A_i} H \oint_{B_i} \Omega - \oint_{B_i} H \oint_{A_i} \Omega \right].$$

The flux of the H-field through the A_i- and B_i-cycles is related to the rank of the gauge group, N_i, and the cut-off dependent couplings τ_i, corresponding to the bare gauge coupling τ_0,

$$\int_{A_i} H = N_i, \qquad \int_{B_i} H = \tau_i.$$

The A_i-periods of the holomorphic three-form Ω give the variables S_i, which correspond to the moduli of the complex structure of the Calabi-Yau manifold. The B_i-periods give the supergravity prepotential,

$$S_i = \oint_{A_i} \Omega, \qquad \frac{\partial \mathcal{F}_0}{\partial S_i} = \int_{B_i} \Omega.$$

Inserting the periods of Ω and H_3 into Eq.4.2 we get Eq.4.1 for the effective superpotential.

The effective geometry that describes the effective superpotential has also been derived from the matrix model point of view in [73], where besides the gauge group $U(N)$ also $SO(N)$ and $Sp(\frac{N}{2})$ have been considered. In [74] the $\mathcal{N} = 2$ prepotential has been derived for four-dimensional gauge theories that have been compactified from six dimensions, using the three different methods of matrix models, geometric engineering and instanton calculations.

The claim, that the effective superpotential for deformed $\mathcal{N} = 2$ theories can be obtained from matrix models, can also be made within field theory [48, 75]. One then needs to show that it is sufficient to take planar zero-momentum Feynman diagrams, to calculate $W_{\text{eff}}(S)$.

The general argument is as follows. Consider a chiral superfield Φ in a external gauge field background $\mathcal{W}$. The contribution of the S^l term in the superpotential is given by fat Feynman diagrams with $2l$ insertions of $\mathcal{W}$. The zero-mode analysis of this diagram tells us that only planar diagrams contribute to the superpotential. When such diagrams are evaluated, it turns out that the momentum contribution of the fermions cancels exactly against that of the bosons.

4.1.4 MATRIX MODEL VS. GAUGE THEORY

We would like to relate the matrix model quantities to the ones obtained from the gauge theory. Unfortunately, the expectation values in these theories turn out not to

be equal [76, 77],

$$\langle \Phi^n \rangle_{\text{matrix}} \neq \langle \Phi^n \rangle_{\text{gauge}} .$$

To see this, consider the matrix model with the potential

$$W(x) = \frac{m}{2}x^2 + \frac{g}{4}x^4,$$

and calculate the correlator $\langle \Phi^4 \rangle$.

In the matrix model, we need to take the Φ^4 vertex and pair the endpoints to make loops. Each index loop gets a factor $S = g_s\widetilde{N}$. Including the outer loop, there are in total three loops and we find the contribution

$$g_s \ \langle \Phi^n \rangle_{\text{matrix}} = C\, S^3,$$

where $C = \frac{1}{2}$ is the combinatorial factor of the diagram.

In the gauge theory, we need to insert an external field $\mathcal{W}$ for each internal loop, to get a planar diagram. For the two-loop diagram there are three ways to do this and we find

$$\langle \Phi^4 \rangle_{\text{gauge}} = 3\, C\, S^2.$$

This is different from the matrix model answer but it can be fixed as follows: if we insert two extra $\mathcal{W}$'s

$$\langle \operatorname{Tr} \mathcal{W}^2\, \Phi^4 \rangle_{\text{gauge}} ,$$

we will get an extra factor of S, and since there is only one way to distribute the six $\mathcal{W}$'s over the diagram, we don't get the extra factor of three. So with the extra $\mathcal{W}$ insertions, we arrive at the matrix model result

$$\langle \operatorname{Tr} \mathcal{W}^2\, \Phi^4 \rangle_{\text{gauge}} = C\, S^3 = g_s \langle \operatorname{Tr} \Phi^4 \rangle_{\text{matrix}} .$$

Therefore we can state that

$$g_s \langle \operatorname{Tr} \Phi^n \rangle_{\text{matrix}} = \langle \operatorname{Tr}(\mathcal{W}^2\, \Phi^n) \rangle_{\text{gauge}} .$$

In [78], a discrepancy was found between the field theory and matrix model methods beyond h (Coxeter number) loops.

4.1.5 KONISHI ANOMALY

In the gauge theory, the loop equations are derived by using the invariance of the path-integral under reparametrizations of the chiral superfield Φ. These loop equations can also be derived by using the Konishi anomaly. This derivation is done without referring to any matrix model [49].

Classically, the path-integral is invariant under the infinitesimal transformation

$$\delta\Phi = \epsilon\,\Phi, \qquad \delta\bar{\Phi} = \bar{\epsilon}\,\bar{\Phi},$$

and this is a symmetry when the tree-level superpotential is not turned on. However, the chiral measure $\mathrm{d}\Phi$ is not invariant under this transformation and will lead to an anomaly, named after its discoverer K. Konishi [79, 80]. In terms of a supercurrent

$$J = \mathrm{Tr}(\bar{\Phi}\,\mathrm{e}^{-V}\,\Phi\,\mathrm{e}^{V}) = \mathrm{Tr}\,\widetilde{\Phi}\,\Phi,$$

the anomaly is given by the divergence of this current,

$$\bar{D}^2 J = \frac{1}{32\pi^2}\mathrm{Tr}_{\mathrm{adj}}\mathcal{W}^2.$$

To find the loop equation, one turns on the tree-level superpotential $W(\Phi)$ and uses the fact that, in the chiral ring, the expectation value of $\bar{D}^2 J$ must be zero,

$$0 = \left\langle \bar{D}^2 J \right\rangle = \left\langle \mathrm{Tr}(\Phi\,\frac{\partial W}{\partial \Phi}) + 2NS - w^2 \right\rangle,$$

with $w_\alpha = \frac{1}{4\pi}\mathrm{Tr}\,\mathcal{W}_\alpha$. However, to get the right expression, one needs to use the generalized supercurrent

$$J_f = \mathrm{Tr}(\widetilde{\Phi}\,f(\phi,\mathcal{W})),$$

with the generalized variation

$$f = \frac{\mathcal{W}^2}{x-\Phi}.$$

Then the expectation value of $\bar{D}^2 J_f$,

$$\bar{D}^2 J_f = \frac{\partial f_{ij}}{\partial \Phi_{ij}}\,\bar{D}^2(\widetilde{\Phi}_{ji}\,\Phi_{kl}),$$

leads to the loop equation,

$$\begin{aligned}
\left\langle \bar{D}^2 J_f \right\rangle &= \left\langle \mathrm{Tr}\left(\frac{\mathcal{W}^2}{x-\Phi}W'(\Phi)\right)\right\rangle_{\mathrm{gauge}} + \left\langle \left(\mathrm{Tr}\frac{\mathcal{W}^2}{x-\Phi}\right)^2\right\rangle_{\mathrm{gauge}} \\
&= \left\langle \mathrm{Tr}\left(\frac{\mathcal{W}^2}{x-\Phi}W'(\Phi)\right)\right\rangle_{\mathrm{gauge}} + \left\langle \mathrm{Tr}\frac{\mathcal{W}^2}{x-\Phi}\right\rangle^2_{\mathrm{gauge}} \\
&= g_s\left\langle \mathrm{Tr}\left(\frac{\mathcal{W}^2}{x-\Phi}W'(\Phi)\right)\right\rangle_{\mathrm{matrix}} + g_s^2\left\langle \mathrm{Tr}\frac{\mathcal{W}^2}{x-\Phi}\right\rangle^2_{\mathrm{matrix}}.
\end{aligned}$$

Here we used the correspondence between the gauge theory and the matrix model, mentioned in the last section. The factorization of the correlators in the gauge theory is due to the properties of the chiral ring. In the matrix model, the factorization takes place at large $\widetilde{N}$.

The Konishi-anomaly approach was used in many papers to confirm the claim by Dijkgraaf and Vafa Eq.4.1, see for example [81, 82, 83, 84, 85].

4.2 SEIBERG-WITTEN THEORY

The $\mathcal{N}=1$ theories that we have been considering are deformations of pure $\mathcal{N}=2$ Yang-Mills theories. The Seiberg-Witten theory deals with pure $\mathcal{N}=2$ super Yang-Mills theory and we would expect to recover this theory when the deformation of the $\mathcal{N}=1$ theory is turned off.

$\mathcal{N}=2$ super Yang-Mills theories are asymptotically free and therefore, their large distance behavior is equivalent to their low energy behavior.

At low energies, $\mathcal{N}=2$ super Yang-Mills theories behave as Abelian gauge theories [86]. In the case of the gauge group $SU(2)$, the effective low energy theory is parametrized by a complex parameter u defined as

$$u = \operatorname{Tr}\phi^2 = 2a^2, \quad \phi = \frac{1}{2} a\,\sigma_3,$$

where σ_3 is the third Pauli matrix. u is a good coordinate of the classical moduli space since it labels the gauge-inequivalent vacua of the theory. u can be complex and since it is a coordinate on the moduli space, this space is often called the complex u-plane.

The Lagrangian of $\mathcal{N}=2$ super Yang-Mills theory can be written in terms of the prepotential $\mathcal{F}$. The classical prepotential of the theory is

$$\mathcal{F} = \frac{1}{2}\tau_0 \Psi^2,$$

with τ_0 the bare coupling. Here Ψ is the chiral superfield of the of $\mathcal{N}=2$ super Yang-Mills theory (not to be confused with the $\mathcal{N}=1$ chiral superfield Φ) in terms of which the prepotential is holomorphic.

In $\mathcal{N}=1$ superspace, the Lagrangian is given by

$$L = \frac{1}{4\pi}\operatorname{Im}\operatorname{Tr}\left[\int \mathrm{d}^4\theta \frac{\partial \mathcal{F}(S)}{\partial S}\bar{S} + \int \mathrm{d}^2\theta \frac{1}{2}\frac{\partial^2 \mathcal{F}(S)}{\partial S^2}\mathcal{W}^\alpha \mathcal{W}_\alpha\right],$$

where S is the chiral $\mathcal{N}=1$ superfield that contains the fields (ϕ,ψ), and $\mathcal{W}$ is a chiral spinor superfield that contains the non-Abelian gauge field and its $\mathcal{N}=1$ superpartner (A_μ,λ). All these fields take values in the adjoint representation of our gauge group $SU(2)$. The potential for the complex scalar ϕ is given by

$$V(\phi) = \operatorname{Tr}([\phi,\phi^\dagger]^2).$$

The vacua are parametrized by $u = \operatorname{Tr}\phi^2$. For $u \neq 0$, there is a spontaneous symmetry breaking $SU(2) \to U(1)$ and the spectrum splits up into two massive $\mathcal{N}=2$

vector multiplets. One of them contains the massive $W^{\pm}$ bosons and their superpartners, and the other contains the $\mathcal{N}=2$ photon and its superpartners. For $u=0$, one gets the full $SU(2)$ symmetry classically.

It turns out that there are points on this complex u-plane where certain matter fields become massless. These singular points of the vacuum moduli space correspond to $u=\pm\Lambda^2$, with Λ the dynamically generated scale of the theory. At $u=\Lambda^2$, one has an $\mathcal{N}=2$ supersymmetric Abelian gauge theory with a massless monopole and at $u=-\Lambda^2$ one has this theory with a massless dyon. At the singular points, these theories are related by a $\mathbb{Z}_2$ symmetry in the u-plane.

In the quantum case, one gets a prepotential that depends only on an Abelian multiplet. Seiberg and Witten [86] found that (*i*) the $SU(2)$ symmetry does not get restored, (*ii*) the u-plane is a complex one-dimensional Kähler manifold, (*iii*) the prepotential has singularities at $u=\pm\Lambda^2$, (*iv*) the singularities correspond to coupling to a massless monopole at $u=\Lambda$ and a massless dyon at $u=-\Lambda$, (*v*) which should be included in the effective action near one of the singularities.

The local coordinates a, $\bar{a}\equiv a^*$ on the moduli space are appropriate to describe the low-energy action. They are not valid globally because of violations of the positivity of the metric on the moduli space. In the regions where they are not the good coordinates, one uses their duals a_D, $\bar{a}_D$ with the relation

$$a_D=\frac{\partial\mathcal{F}(a)}{\partial a}.$$

To determine the exact non-perturbative low-energy effective action, one needs to find $a(u)$ and $a_D(u)$, then convert $a(u)$ to $u(a)$, substitute this into $a_D(u)$ to get $a_D(a)$. Then one can integrate $a_D(a)$ with respect to a, which gives $\mathcal{F}(a)$, and with this the low-energy effective action. The $a(u)$ and $a_D(u)$ were found to be [86]

$$a_D=\oint_{\gamma_1}\lambda,\quad a=\oint_{\gamma_2}\lambda,\quad \lambda=\frac{\sqrt{2}}{2\pi}\frac{\sqrt{x-u}}{\sqrt{x^2-1}}\,\mathrm{d}x. \tag{4.3}$$

Here the cycle γ_2 goes once around the cut (-1,1) and γ_1 goes from 1 to u on the first Riemann sheet and from u to 1 on the second Riemann sheet, of the corresponding Riemann surface. This geometry has the topology of a torus and has therefore genus one. The τ-parameter

$$\tau(u)=\frac{\mathrm{d}a_D/\mathrm{d}u}{\mathrm{d}a/\mathrm{d}u} \tag{4.4}$$

describes the complex structure of the torus.

From another point of view, Seiberg and Witten showed [86] that the u-plane with punctures at 1, -1 and ∞ can be described by the quotient of the upper half plane

H by $\Gamma(2)$ (which is a subgroup of $Sl(2,\mathbb{Z})$) and that this quotient $H/\Gamma(2)$ is the moduli space of the family of elliptic curves of the form

$$y^2 = (x^2 - 1)(x - u).$$

This is again the mentioned genus-one Riemann surface.

Matrix model description

Matrix model methods can be used to find the solution to the pure $\mathcal{N} = 2$ super Yang-Mills theory of Seiberg and Witten. In [86] Seiberg and Witten used duality arguments to obtain exact results for many $\mathcal{N} = 2$ super Yang-Mills theories and determined their moduli space of vacua. To describe the $SU(N)$ gauge theory, one breaks the supersymmetry down to $\mathcal{N} = 1$ by introducing a degree $N + 1$ tree-level superpotential $\mathrm{Tr}\, W(\Phi)$ of the adjoint chiral multiplet Φ and picks the breaking pattern

$$U(N) \to U(1)^N,$$

by distributing the N eigenvalues of Φ among the N critical points of W such that each root is occupied exactly once,

$$N = 1 + \cdots + 1.$$

One further decouples the diagonal $U(1)$ by putting the overall bare coupling $\tau_0 = 0$. The effective superpotential then simplifies to

$$W_{\text{eff}}(S) = \sum_i \frac{\partial \mathcal{F}_0}{\partial S_i}. \tag{4.5}$$

The planar diagrams can be summed exactly and the solution can be written in terms of period integrals on the associated hyperelliptic Riemann surface,

$$y^2 = P(x)^2 - f(x),$$

with

$$P(x) = W'(x) = \sum_{i=0}^{N} u_i\, x^{N-i}.$$

The definition of the variables $S_1, \ldots, S_N$ here is subtle, since they are defined in terms of the traceless piece of a $U(1)$ gauge field. They vanish classically, but make sense quantum mechanically as operators. The dependence on the S_i is implicit in terms of the quantum deformation $f(x)$, which is a polynomial of degree $N-1$. After solving the constraint $\mathrm{d}W_{\text{eff}}(S) = 0$, this curve takes the form of the Seiberg-Witten curve,

$$y^2 = P(x)^2 - \Lambda^{2N}. \tag{4.6}$$

To obtain the original $\mathcal{N}=2$ theory, one can now scale the tree-level superpotential as

$$W \to \epsilon\, W,$$

and take the limit $\epsilon \to 0$. There are two obvious quantities that by a scaling argument do not depend on ϵ and can therefore be straightforwardly extracted from the $\mathcal{N}=1$ solution . First, there is the coupling matrix

$$\tau_{ij} = \frac{\partial^2 \mathcal{F}_0}{\partial S_i \partial S_j}$$

of the $U(1)^N$ low-energy effective Abelian theory. Geometrically, this is given by the period matrix of the curve Eq.4.6. Minimizing the superpotential Eq.4.5 gives the condition

$$\sum_i \tau_{ij} = 0.$$

Then τ_{ij} takes the following form at the extremum [87]

$$\begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix} = \tau \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

where τ is the effective gauge coupling for the off-diagonal $U(1)$. By setting $\tau_0 = 0$ in Eq.4.5, we automatically eliminated the diagonal $U(1)$-factor.

The second ϵ-invariant quantity is the genus-one free energy $\mathcal{F}_1$ that gives the gravitational correction, see chapter 5.

4.3 DONALDSON-WITTEN THEORY

As we mentioned in section 4.2, matrix models lead to the solution of the pure $\mathcal{N}=2$ super Yang-Mills theory of Seiberg and Witten. However this only works for flat manifolds. When the theory is considered on a curved manifold, in addition to the Seiberg-Witten solution, gravitational terms appear. These corrections have been computed in the topologically twisted $\mathcal{N}=2$ theory that computes Donaldson invariants.

Donaldson theory can be formulated as a topologically twisted version of $\mathcal{N}=2$ supersymmetric Yang-Mills theory. This was first done by Witten [88] and this new theory is called Donaldson-Witten theory. The vacuum expectation values of the observables of this theory are Donaldson invariants for four-manifolds. Donaldson invariants are topological invariants for four-manifolds which depend on the differentiable structure of the manifold [89].

Using the results of [86], Witten applied them to topologically twisted theories and obtained an expression for Donaldson invariants in terms of the new Seiberg-Witten invariants [90].

The results from $\mathcal{N} = 2$ super Yang-Mills theories at low energies can be used in the context of the topologically twisted quantum field theories for the following reason: In the twisted theory, the gauge coupling g can be regarded as a rescaling of the metric. When g goes to infinity, the rescaling of the metric becomes very large and one can do calculations in terms of vacua corresponding to $\mathbb{R}^4$. Since $\mathcal{N} = 2$ super Yang-Mills theories are asymptotically free and therefore, their large distance behavior is equivalent to their low energy behavior, one can use the Seiberg-Witten formalism in this limit.

In the next chapter, we will use the results for topologically twisted $\mathcal{N} = 2$ super Yang-Mills theories to compare them to the results for a one-matrix model at higher genus.

The gauge theory we have considered is four dimensional. One can also consider three-dimensional theories, see for example [91, 92, 93, 94, 95]. Here one uses the fact that, as Seiberg and Witten have shown, the low-energy effective superpotential in four dimensions can be computed by compactifying the theory on a circle. In such a three-dimensional theory it turns out to be simpler to calculate the superpotential.

In [96] it has been shown that any $(4+k)$-dimensional supersymmetric gauge theory can be written in terms of four-dimensional $\mathcal{N} = 1$ superfields on a k-dimensional manifold. The tree-level superpotential of such a theory is then given by a k-dimensional bosonic action and the four-dimensional effective superpotential can be written in terms of the effective action of the k-dimensional bosonic gauge theory. This generalizes the relation between four-dimensional gauge theories and matrix models. For $k = 1$, this idea leads to matrix quantum mechanics.

CHAPTER 5

MATRIX MODELS AND GRAVITATIONAL CORRECTIONS

As we have seen in the last chapter, the $\mathcal{N} = 1$ effective superpotential $W_{\text{eff}}(S)$ can be computed exactly in terms of the glueball superfield S. In the large N limit, this superpotential is given by summing just the planar diagrams. Non-planar diagrams will in general contribute to gravitational corrections and that is what we will consider in this chapter.

Diagrams with genus one topology, that give the leading $1/N^2$ correction $\mathcal{F}_1$ to the matrix model free energy, contribute to an effective curvature term of the form

$$\frac{1}{16\pi^2}\int d^4x\, \mathcal{F}_1(S)\, (R_+)_{ab} \wedge (R_+)_{ba}$$

with $R_+ = \frac{1}{2}(R + R^*)$ the self-dual part of the Riemann curvature tensor. This induced gravitational correction measures the back-reaction of the field theory when it is placed in a curved background.

For exactly solvable matrix models, the summation of the diagrams of any fixed topology in closed form can be done in principle. However, the techniques become increasingly difficult for high genus. One can then try to compare these exact answers to known properties of four-dimensional supersymmetric gauge theories.

In this chapter we will compare the results for a one-matrix model to the gravitational corrections that have been computed for topological field theories that are twisted versions of $\mathcal{N} = 2$ super Yang-Mills theories [97, 98, 99, 100, 101, 102, 103]. As we mentioned in section 4.3, these topological field theories are used to compute

the Donaldson and Seiberg-Witten invariants of four-manifolds. In this chapter we will demonstrate how these terms can also be computed using loop equations of matrix models.

As we have seen in section 3.6, for an exactly solvable matrix model, the emergent geometry is related to a dual geometry in the B-model topological string theory. This dual geometry arises from a geometrical transition from an open to a closed string description and takes the form of a non-compact Calabi-Yau three-fold. Topological closed strings propagating on such a Calabi-Yau three-fold give rise to a genus-one partition function $\mathcal{F}_1$ that can be expressed as a generalized Ray-Singer analytic torsion [21],

$$\mathcal{F}_1 = \sum_{p,q=0}^{3} p\,q\,(-1)^{p+q}\,\log\det\Delta_{p,q},$$

where $\Delta_{p,q}$ is the Laplacian acting on (p,q)-forms.

In the simple class of matrix models that we consider in this chapter, the effective geometry is given by an affine algebraic curve, and we expect an expression of the form

$$\mathcal{F}_1 = -\frac{1}{2}\log\det\Delta_0$$

with Δ_0 the scalar Laplacian on the algebraic curve acting on the collective bosonic field. We will verify that this is indeed the case in some cases by explicit computation. This relation between matrix models and two-dimensional conformal collective field theory is a much more general feature [104].

5.1 SUPERPOTENTIALS AND GRAVITATIONAL COUPLINGS

According to [5] the effective superpotential is given by

$$W_{\text{eff}}(S) = \sum_i \left(N_i\,\frac{\partial\mathcal{F}_0}{\partial S_i} + 2\pi\tau_0\,S_i \right)$$

where τ_0 is the bare coupling constant and $\mathcal{F}_0(S_i)$ is the free energy of the corresponding matrix model, obtained in a semi-classical expansion around the classical vacuum.

The corresponding matrix model takes the form of an integral over an $\widetilde{N}\times\widetilde{N}$ matrix Φ

$$\frac{1}{\operatorname{vol}\mathrm{U}(\widetilde{\mathrm{N}})}\int \mathrm{d}\Phi\,\exp\left(-\frac{1}{g_s}\operatorname{Tr}W(\Phi)\right) \sim \exp\left(-\sum_{g\geq 0} g_s^{2g-2}\mathcal{F}_g(S_i)\right).$$

The semi-classical expansion of this integral around the saddle points gives then the expression at the right hand side. Here we have the identification $S_i = g_s\widetilde{N}_i$ which is kept fixed in the 't Hooft limit while $g_s \to 0$, $\widetilde{N}_i \to \infty$. More precisely, we have

$$\mathcal{F}_0(S) = \frac{1}{2}S^2 \log(S/\Lambda_0^3) + \mathcal{F}_0^{\rm pert}(S). \tag{5.1}$$

The first term gives rise to the Veneziano-Yankielowicz effective action of the pure Yang-Mills theory [52],

$$W_{\rm eff}(S) = N\, S\, \log(S/\Lambda^3).$$

In the matrix model this contribution to Eq.5.1 is reproduced as the large $\widetilde{N}$ volume of the unitary group.

The second term in Eq.5.1 is given by a sum over planar diagrams that appear in the perturbative expansion of the matrix model. See [87] for a careful description of this expansion around a vacuum with a spontaneous broken gauge symmetry. A diagram with l index loops comes with a factor of S^l. The actual physical values of $W_{\rm eff}$ and the condensates S_i in the quantum vacua are given by extremizing $W_{\rm eff}$ with respect to the glueball fields S_i.

This relation is not restricted to planar diagrams. There is also an interpretation of the higher genus diagrams that give the corrections $\mathcal{F}_g$ in terms of the coupling to a supergravity background [5]. The induced gravitational effective action obtained by putting the field theory on a curved space-time contains the F-term of the form

$$\frac{1}{16\pi^2}\int d^4x\, \mathcal{F}_1(S)\, (R_+)_{ab} \wedge (R_+)_{ba} \tag{5.2}$$

with $R_+ = \frac{1}{2}(R + R^*)$ the self-dual part of the Riemann curvature tensor. There is of course a similar anti-holomorphic term $\bar{\mathcal{F}}_1$ multiplying $(R_-)_{ab} \wedge (R_-)_{ba}$ with $R_- = \frac{1}{2}(R - R^*)$.

If we consider the partition function on a Euclidean four-manifold $\mathcal{M}^4$, then this gravitational coupling induces a term

$$\exp\left[\mathcal{F}_1(S)\left(\frac{1}{2}\chi - \frac{3}{4}\sigma\right)\right] \tag{5.3}$$

with χ the Euler number and σ the Hirzebruch signature of $\mathcal{M}$, defined by, respectively,

$$\chi(\mathcal{M}) = \int_{\mathcal{M}} e(\mathcal{M}),$$

with $e(\mathcal{M})$ the Euler form given by the r-fold wedge product

$$e(\mathcal{M}) = \frac{1}{(-4\pi)^m r!}\, \epsilon^{k_1 \ldots k_r}\, R_{k_1 k_2} \ldots R_{k_{r-1} k_r}, \quad r = 2m = \dim\mathcal{M},$$

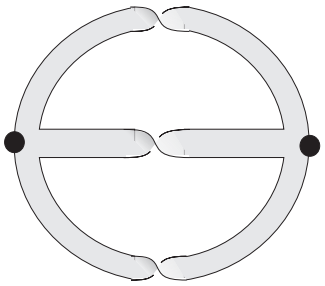

Figure 5.1: The simplest genus one, non-planar diagram in a cubic theory — the leading perturbative contribution to $\mathcal{F}_1$. The $\bullet$'s indicate insertions of the background gauge field $\mathcal{W}_\alpha$.

where R_{ab} is the $SO(k)$-valued curvature two-form of $\mathcal{M}$, and

$$\sigma(\mathcal{M}) = \int_{\mathcal{M}} L(\mathcal{M}),$$

where $L(\mathcal{M})$ is the Hirzebruch L-polynomial

$$L(\mathcal{M}) = \prod_j \frac{x_j}{\tanh x_j} = 1 + \frac{1}{3}p_1 + \frac{1}{45}(7p_2 - p_1^2) + \dots$$

with p_i the Pontrjagin classes of degree $4i$. The Pontrjagin class for a four-dimensional Riemannian manifold $\mathcal{M}$ is given by

$$p_1 = -\frac{1}{8\pi^2} R_{ab} \wedge R_{ba}.$$

Now one can then find the expressions for $\chi(\mathcal{M})$ and $\sigma(\mathcal{M})$:

$$\chi(\mathcal{M}) = \frac{1}{32\pi^2} \int_{\mathcal{M}} \epsilon_{abcd}\, R_{ab} \wedge R_{cd} = \frac{1}{16\pi^2} \int \mathrm{Tr}\, R^* \wedge R,$$

and

$$\sigma(\mathcal{M}) = -\frac{1}{24\pi^2} \int_{\mathcal{M}} R_{ab} \wedge R_{ba} = -\frac{1}{24\pi^2} \int \mathrm{Tr}\, R \wedge R.$$

When we fill in these expressions in Eq.5.2, we get Eq.5.3.

Evaluating the term $\mathcal{F}_1(S)$ in perturbation theory, one finds that it is given exactly by the sum of diagrams with topology genus one, *i.e.* the diagrams that give the leading $1/\widetilde{N}^2$ corrections in the large $\widetilde{N}$ limit of the matrix model. More precisely, $\mathcal{F}_1(S)$ is given as

$$\mathcal{F}_1(S) = -\frac{1}{12} \sum_i \log(S_i/\Lambda_0^3) + \mathcal{F}_1^{\mathrm{pert}}(S). \tag{5.4}$$

This expression is the gravitational analogue of Eq.5.1.

Assuming confinement, so that the only field accounted for is S, the first term has an interpretation as an integrated form of the gravitational contribution to the $U(1)$ R-anomaly,

$$\partial_\mu J_5^\mu = \frac{1}{16\pi^2}\left[\frac{1}{2}F_{ab}\wedge F_{ba} - \frac{1}{12}R_{ab}\wedge R_{ba}\right]. \tag{5.5}$$

If one assumes that the low energy dynamics of the gauge system is described by an effective action in which the glueball superfields S_i can be treated as elementary fields, the anomalous behaviour under the R-symmetry

$$S \to \mathrm{e}^{\mathrm{i}\theta} S$$

is reproduced by the combination of the Veneziano-Yankielowicz contribution $N\,S\log S$ to $W_{\rm eff}(S)$, together with the $-\frac{1}{12}\log S$-term multiplying the gravitational correction. Including the complex conjugated term that multiplies $\log\bar{S}$, we pick up the anomaly Eq.5.5.

The perturbative contribution $\mathcal{F}_1^{\rm pert}$ to Eq.5.4 is given by summing all genus one diagrams. For example, in a cubic theory with superpotential $W(\Phi) = m\Phi^2 + g\Phi^3$, the leading diagram is given by fig.5.1 and this gives

$$\mathcal{F}_1^{\rm pert}(S) = \frac{1}{2}\frac{g^2}{m^3}S + \mathcal{O}(S^2).$$

In the physical vacua, all these expressions for $\mathcal{F}_1$ have to be evaluated for those values of the S_i that minimize the effective superpotential given by the planar contribution.

The genus zero diagrams also contribute, but their contribution can be shown to cancel at the critical point as Dijkgraaf, Grisaru, Ooguri, Vafa and Zanon have done in [105]. Following [106, 107], they have shown that the gravitational correction for the genus zero diagrams takes the form

$$\mathrm{Tr}\,(R_+\wedge R_+)\sum_{i,j} N_i\,N_j\,\frac{\partial^2\mathcal{F}_0}{\partial S_i\,\partial S_j},$$

where $\mathcal{F}_0$ is the planar partition function.

As was shown in [5], extremizing the superpotential W with respect to the glueball superfield S_i gives the following equation:

$$N_i\,\partial_i\partial_j\,\mathcal{F}_0 + \tau = 0.$$

Here τ is the bare gauge coupling of the $U(N)$ gauge theory. One can then include these solutions into the genus zero gravitational correction and find

$$-\mathrm{Tr}\,(R_+\wedge R_+)\sum_j N_j\,\tau. \tag{5.6}$$

This is proportional to the universal contribution $N\,\tau$ with $N = \sum_j N_j$ and depends only on the rank N of the gauge group. So, this contribution does not depend on any couplings.

Consider now the case of spacetime-filling D-branes, wrapping two-cycles of a Calabi-Yau [69, 46]. Then, for each brane, there will be gravitational corrections proportional to $\mathrm{Tr}(R_+ \wedge R_+)$ [108]. The volume of the D-branes is given by τ which yields a term $\tau\,\mathrm{Tr}\,(R_+ \wedge R_+)$ in four dimensions, and for N branes one has

$$\mathrm{Tr}\,(R_+ \wedge R_+)\,N\,\tau.$$

This is the induced curvature term on the branes. In the context of string theory, it has to be added to the contribution Eq.5.6 from the sum of the planar diagrams in gauge theory. We see then that they cancel one another completely.

As we have mentioned in section 3.6, open topological strings on certain Calabi-Yau with D-branes are dual to closed topological strings on another Calabi-Yau without any D-branes. In this closed string dual, there should be no genus zero correction to the curvature term. The only contribution should come from the genus one correction and this is the case as we have seen above [21, 109]

5.2 Matrix models and $\mathcal{N} = 2$ theories

5.2.1 Gravitational coupling from topological field theory

On flat spacetime the $\mathcal{N} = 2$ $SU(2)$ gauge theory is described by the Seiberg-Witten solution. Putting the theory on a curved manifold additional gravitational terms appear in the low energy effective action. This gravitational correction has been directly computed in the topologically twisted $\mathcal{N} = 2$ theory that computes Donaldson invariants. In the twisted version one modifies the action of the Lorentz group

$$SO(4) \cong SU(2)_+ \times SU(2)_-.$$

One replaces $SU(2)_+$ with the diagonal subgroup of $SU(2)_+ \times SU(2)_R$, where the last factor is the $\mathcal{N} = 2$ internal R-symmetry group [88].

In the twisted topological theory considered on a curved four-manifold $\mathcal{M}$, these interactions are restricted to the topological terms:

$$\frac{1}{16\pi^2}\int_{\mathcal{M}} \mathrm{Tr}\,(R^* \wedge R) \quad \text{or} \quad -\frac{1}{24\pi^2}\int_{\mathcal{M}} \mathrm{Tr}\,(R \wedge R).$$

The terms $\mathrm{Tr}\,(R^*\wedge R)$ and $\mathrm{Tr}\,(R\wedge R)$ are densities and their integrals are proportional to the Euler characteristic $\chi(\mathcal{M})$ and the Hirzebruch signature $\sigma(\mathcal{M})$, respectively, as we have seen in the previous section. The gravitational couplings contribute to the partition function with the factor [98, 99]

$$\exp\left(b(u)\chi + c(u)\sigma\right),$$

where $b(u)$ and $c(u)$ are functions of the parameter u on the gauge theory moduli space. The precise form of the functions $b(u)$, $c(u)$ can be found from analyzing the modular transformation properties of the quantum theory on the curved manifold [98]:

$$\begin{aligned} \mathrm{e}^{b(u)} &= \alpha\left((u^2-1)\frac{\mathrm{d}\tau}{\mathrm{d}u}\right)^{\chi/4}, \\ \mathrm{e}^{c(u)} &= \beta(u^2-1)^{\sigma/8}, \end{aligned}$$

where α and β are constant coefficients independent of $\mathcal{M}$.

The contribution to the path-integral measure is then given by [98],

$$A^\chi B^\sigma = \alpha^\chi \beta^\sigma \left((u^2-1)\frac{\mathrm{d}\tau}{\mathrm{d}u}\right)^{\chi/4} \left(u^2-1\right)^{\sigma/8}.$$

Following [99], this expression becomes

$$A^\chi B^\sigma = \left(\frac{\mathrm{d}u}{\mathrm{d}a}\right)^{\chi/2} \Delta^{\sigma/8},$$

where $\Delta = u^2 - 1$ and a is the "electric" period of the Seiberg-Witten curve, see Eq.4.3.

To connect these computations in topological field theory to the physical theory, we recall that for manifolds with metrics of $SU(2)$ holonomy (hyper-Kähler manifolds) the topological twist is invisible since there is no holonomy in $SU(2)_+$. We can therefore directly compare to the physical gauge theory. In that case the metric is pure self-dual, and we have

$$\sigma = -\frac{2}{3}\chi.$$

For example one could take $\mathcal{M} = K3$ for which $\chi = 24$ and $\sigma = -16$. So the overall contribution to the path-integral is

$$\exp\left(\left(b(u) - \frac{2}{3}c(u)\right)\chi\right).$$

If we compare this to Eq.5.3, where we use that for a self-dual geometry $\frac{1}{2}\chi - \frac{3}{4}\sigma = \chi$, we have the following identification between the matrix model and gauge theory

quantities,

$$\mathcal{F}_1(S) = b(u) - \frac{2}{3}c(u).$$

We will now check this relation in a number of cases. We will for convenience put $\chi = 1$.

5.2.2 THE $\mathcal{N} = 2$ $SU(2)$ THEORY

In this case the Seiberg-Witten geometry can be described by deforming the $\mathcal{N} = 2$ theory with a tree level superpotential,

$$W'(\Phi) = \epsilon(\Phi^2 - u). \tag{5.7}$$

As described above, extremization of the effective glueball superpotential gives the Seiberg-Witten curve for $SU(2)$

$$y^2 = (x^2 - u)^2 - 1. \tag{5.8}$$

Here the scale Λ is set to 1 for convenience and the factor ϵ is absorbed. As we mentioned in section 4.2, the physical quantities $\mathcal{F}_1$ and the coupling matrix τ_{ij} are independent of the deformation parameter ϵ. The curve has four branch points at

$$x_i = \pm\sqrt{u \pm 1}.$$

It is described by the two-cut solution of the matrix model with the potential $W(\Phi)$ given by Eq.5.7.

The genus-one free energy for two-cut solutions in matrix models have been explicitly computed. Here we use the relevant solution of Akemann [110], which is an elaboration of the methods of [111],

$$\begin{aligned}\mathcal{F}_1 &= -\frac{1}{24}\sum_{i=1}^{4}\log M_i - \frac{1}{2}\log|K(k)| \\ &\quad -\frac{1}{12}\Delta + \frac{1}{4}\log|(x_1 - x_3)(x_2 - x_4)|.\end{aligned}$$

This solution was derived by an iterative genus expansion of the loop equation; we discuss this further in the next section. Here Δ is the discriminant of the elliptic curve Eq.5.8,

$$\Delta = \prod_{i<j}(x_i - x_j)^2 = 64(u^2 - 1),$$

and $K(k)$ is the complete elliptic integral, where the nome k is expressed in the modulus τ of the Seiberg-Witten curve Eq.4.4. The solution also depends on the first moments of the potential that are generally defined as

$$M_i = \frac{1}{2}\pi \mathrm{i} \oint_{C_\infty} \mathrm{d}x \frac{W'(x)}{(x-x_i)\sqrt{\prod_{i=1}^{4}(x-x_i)}}.$$

For the simple potential Eq.5.7, the contour can be deformed to infinity, and one gets $M_i = \epsilon$.

For comparison with the gauge theory result, it is useful to express $\mathcal{F}_1$ in terms of the $SU(2)$ moduli space parameter u. The elliptic parameterization of the Seiberg-Witten curve Eq.5.8 can be written in terms of the Jacobi θ-functions as

$$u = \frac{\theta_2^4 + \theta_3^4}{2(\theta_2\theta_3)^2}, \qquad u^2 - 1 = \frac{\theta_4^8}{4(\theta_2\theta_3)^4}$$

where the definition of the θ-functions is as usual,

$$\begin{aligned} \theta_2 &= \sum_{n\in\mathbf{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2}, \\ \theta_3 &= \sum_{n\in\mathbf{Z}} q^{\frac{1}{2}n^2}, \\ \theta_4 &= \sum_{n\in\mathbf{Z}} (-1)^n q^{\frac{1}{2}n^2}, \end{aligned}$$

with $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$. A useful identity they satisfy is $\theta_2^4 + \theta_4^4 = \theta_3^4$. The complete elliptic integral $K(k)$ can also be expressed in the θ-functions as

$$K(k) = \frac{\pi}{2}\theta_3^2.$$

With this elliptic parameterization, the matrix model answer for the two-cut solution to $\mathcal{F}_1$ can be written as

$$\mathcal{F}_1 = -\frac{1}{6}\log\epsilon + \frac{1}{4}\log\frac{4}{\pi(\theta_2\theta_3)^2} - \frac{1}{12}\log\frac{16\,\theta_4^8}{(\theta_2\theta_3)^4}. \tag{5.9}$$

The factor $\log\epsilon$ can be absorbed in the measure.

5.2.3 Comparison to the gauge theory

We now have to compare this result to the topological field theory answer that reads

$$\begin{aligned} \mathrm{e}^{b(u)} &= \alpha\left((u^2-1)\frac{\mathrm{d}\tau}{\mathrm{d}u}\right)^{1/4}, \\ \mathrm{e}^{c(u)} &= \beta(u^2-1)^{1/8}, \end{aligned}$$

where α and β are constant coefficients. This contribution to the partition function should match with the matrix model computation for the corresponding genus-one contribution. To check this, it helps to rewrite the gauge theory contribution as

$$Z_{\text{gauge}} = \mathrm{e}^{b(u)-\frac{2}{3}c(u)} = A^{-1/2}\Delta^{-1/12},$$

with

$$A = \frac{\mathrm{d}a}{\mathrm{d}u}, \qquad \Delta = 64(u^2-1).$$

Here,

$$(u^2-1)\frac{\mathrm{d}\tau}{\mathrm{d}u} = \frac{i}{4\pi}\left(\frac{\mathrm{d}u}{\mathrm{d}a}\right)^2$$

is rewritten in terms of the "electric" period of the Seiberg-Witten curve a, see Eq.4.3. Substituting the modular parameterization of the curve in terms of the θ-functions, we find

$$\Delta = \frac{16\theta_4^8}{(\theta_2\theta_3)^4}, \qquad A = \frac{\mathrm{d}a}{\mathrm{d}u} = \frac{1}{2}\theta_2\theta_3.$$

Comparing this with the matrix model contribution Eq.5.9, we find perfect agreement.

5.2.4 $SU(N)$ GENERALIZATION

The gauge theory computation for the partition function can be generalized for the $SU(N)$ theory. The generalization is based on a similar analysis of anomalies as for the $SU(2)$ case.

At a generic point on the Coulomb branch, where the gauge symmetry is broken to $U(1)^{N-1}$, the $SU(N)$ theory can be described by the hyperelliptic curve

$$\begin{aligned} y^2 &= P(x)^2 - 1 = \prod_{i=1}^{2N}(x-x_i), \\ P(x) &= \sum_{i=0}^{N} u_i x^{N-i}, \quad \sum_{i=1}^{N} u_i = 0. \end{aligned} \tag{5.10}$$

Here the u_i's are the symmetric polynomials of the roots of $P(x)$, and x_i are the branch points of the curve. The hyperelliptic curve is a Riemann surface of genus $g = N-1$.

For a genus g Riemann surface, one takes a basis of $2g$ homology cycles (A_i, B_i) with the canonical intersection product. The periods of the curve are then related to a set of dual holomorphic one-forms $\omega_i = x^{i-1}\mathrm{d}x/y$ as

$$A_{ij} = \oint_{A_i}\omega_j = \frac{\partial a_i}{\partial u_{j+1}}, \qquad B_{ij} = \oint_{B_i}\omega_j = \frac{\partial a_{D,i}}{\partial u_{j+1}}.$$

The period (or coupling) matrix τ_{ij} is given as

$$\tau_{ij} = \frac{\partial a_{D,i}}{\partial a_j} = (BA^{-1})_{ij}.$$

The partition function for $SU(N)$ is a direct generalization of the corresponding $SU(2)$ contribution. We will write it as, discarding overall constants,

$$Z_{\text{gauge}} = A^{-\chi/2}\Delta^{\sigma/8},$$

with

$$A = \det A_{ij}, \qquad \Delta = \prod_{i<j}^{2N}(x_i - x_j)^2.$$

Putting a self-dual metric and $\chi = 1$ we get

$$Z_{\text{gauge}} = A^{-1/2}\Delta^{-1/12}. \tag{5.11}$$

We will now compare this result to the genus one free energy of the matrix model.

5.3 MULTICUT SOLUTIONS AND CONFORMAL FIELD THEORY

For the one-loop free energy for the $SU(N)$ theory, we have to solve the corresponding matrix model with the tree-level superpotential $W(\Phi)$ with $W'(\Phi) = \epsilon P(\Phi)$ and with the maximum number of cuts.

The method of using loop equations to obtain the $1/\widetilde{N}$ corrections in matrix models was developed in [111]. The planar solution for the multicut matrix model is given in section 3.5. As we have seen in section 3.4, matrix model results can also be computed using conformal field theory. Here we will follow [104] that gives a good general exposition of the relation of loop equation methods to conformal field theory techniques.

5.3.1 SUBLEADING CORRECTIONS AND TWIST FIELDS

The subleading term $\mathcal{F}_1$ in the free energy is given by the Gaussian fluctuations around the classical solution. As we mentioned in the introduction, this term is equal to $-\frac{1}{2}\log\det\Delta_0$, the Laplace operator on the Riemann surface, and additional (dressing) terms arising from the fluctuation of the branch points.

Instead of thinking of the collective field $\varphi(x)$ as a field living on the hyperelliptic Riemann surface Eq.5.10, we can also think of it on the complex x-plane in the presence of twist operators $\sigma(x_i)$ associated with the branch points x_i. In the neighborhood of such a twist field, the current $\partial\varphi(x)$ is no longer single-valued but has a branch cut in its operator product,

$$\partial\varphi(x)\cdot\sigma(x_i)\sim(x-x_i)^{-1/2}\kappa(x_i).$$

Here $\sigma(x)$ and $\kappa(x)$ are conformal fields of dimension $1/16$ and $9/16$ respectively. A naive expression for the genus one contribution to the matrix model would now be given by

$$Z_{\text{twist}}=\left\langle\prod_{i=1}^{2N}\sigma(x_i)\right\rangle.$$

In this expression the chiral projection is done by putting the loop momenta of the field $\varphi(x)$ to zero. The chiral twist field correlation function is well-known [112, 113, 114],

$$Z_{\text{twist}}=A^{-\frac{1}{2}}\prod_{i<j}(x_i-x_j)^{-1/8}. \tag{5.12}$$

Here

$$A=\det(A_{ij})$$

is the determinant of the period matrix, related to the integral of the one-forms ω_i over the A-cycles

$$A_{ij}=\oint_{A_i}\frac{x^{j-1}\mathrm{d}x}{y}.$$

For example, in the case of a two-point function we get the familiar result

$$\langle\sigma(x_1)\sigma(x_2)\rangle=(x_1-x_2)^{-1/8},$$

expressing the fact that the conformal dimension of a $\mathbb{Z}_2$ twist field is $1/16$.

Formula Eq.5.12 also can be expressed as the chiral determinant of the Laplace operator Δ_0 of the twisted boson on the hyperelliptic curve

$$Z_{\text{twist}}=(\det\,\Delta_0)^{-1/2}.$$

5.3.2 STAR OPERATORS

However, Eq.5.12 is not the full answer, since this expression does not solve the Virasoro constraints. An elegant solution to this has been given by Kostov [104] in terms of star operators.

We can associate a Hilbert space with the local complex variable near each branch point and solve the Virasoro constraint in the vicinity of the branch point. We have to look for an operator which creates a conformally invariant state near the branch point. The twist operator itself does not satisfy all the Virasoro constraints, in particular it does not satisfy L_{-1}. Therefore we will look for a new operator which satisfies all constraints. Such operators are called star operators [115], and they are constructed from the modes of the twisted bosonic field near the branch point.

The twisted bosonic current near the branch point x_i is now decomposed into a classical and quantum part

$$\partial\varphi(x) = \partial\varphi_c(x) + \sum_{r\in\mathbb{Z}+\frac{1}{2}} \alpha_r (x-x_i)^{-r-1}.$$

The expansion of the classical current Eq.3.21 is

$$\partial\varphi_c(x) = \sum_{r\geq\frac{1}{2}} \mu_r(x_k)\cdot(x-x_k)^{r-1}.$$

This defines the coefficients $\mu_r(x_i)$. The Fock vacuum for such a twist field is defined as

$$|0_i\rangle = \sigma(x_i)\,|0\rangle\,,$$

and it satisfies

$$\alpha_r\,|0_i\rangle = 0, \quad r>0.$$

Since it depends on the position of the branch points, it is not translationally invariant. To make it invariant, one introduces the star operator,

$$S(x_i) = \mathrm{e}^{s(x_i)}\,\sigma(x_i),$$

and assumes that it is defined perturbatively by a mode expansion

$$s(x_i) = \sum_{n\geq 0}\frac{1}{n!}\sum_{r_1\ldots r_n} s_{r_1\ldots r_n}(x_i)\,\alpha_{-r_1}\ldots\alpha_{-r_n}.$$

The coefficients in the mode expansion are determined by imposing the conditions of conformal invariance

$$L_n\,\mathrm{e}^{s(x_i)}\,|0_i\rangle = 0, \quad n\geq -1.$$

Up to corrections of order $1/\widetilde{N}^2$ one finds simply an extra multiplicative factor [104],

$$S(x_i) = \big(\mu_{3/2}(x_i)\big)^{-1/24}\,\sigma(x_i).$$

The full genus one contribution to the free energy, obtained by solving the loop equation including the order $1/\widetilde{N}^2$ corrections, is therefore given by the correlation function of star operators, not the twist operators,

$$\begin{aligned}\mathcal{F}_1 &= \log\left\langle\prod_{i=1}^{2N} S(a_i)\right\rangle \\ &= -\frac{1}{24}\sum_{i=1}^{2N}\log\mu_{3/2}(x_i) + \log Z_{\rm twist},\end{aligned}$$

where $Z_{\rm twist}$ is the correlation function of the $2N$ twist fields Eq.5.12. From the expansion of the classical current $\partial\varphi_c(z)$ we get

$$-\frac{1}{24}\sum_{i=1}^{2N}\log\mu_{3/2}(x_i) = -\frac{1}{24}\log\prod_{i<j}(x_i - x_j)$$

So the final result for $Z_{\rm matrix}$ is then

$$Z_{\rm matrix} = \mathrm{e}^{\mathcal{F}_1} = A^{-1/2}\Delta^{-1/12}$$

This result is in complete agreement with the gauge theory partition function $Z_{\rm gauge}$ Eq.5.11.

In [27] Klemm, Mariño and Theisen also calculated the gravitational corrections to the free energy in $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetric gauge theories from topological strings and compared them to the matrix model computaions. However, they followed a different route then we did.

They have used the fact that the relevant correlation functions in the A-model topological string theory are given by sums of the worldsheet instanton contributions of a given genus, and then used mirror symmetry to compute these sums exactly from the B-model topological strings. In particular, they have checked the Dijkgraaf-Vafa conjecture for $g=1$ for a cubic superpotential. They have also found agreement when they compared their instanton corrections with those found by Nekrasov [116] for the nth instanton contribution to the $\mathcal{N}=2$, $SU(N)$ prepotential.

As was noted in [5], $\mathcal{F}_1$ can be written in terms of modular forms and is then equal to

$$\mathcal{F}_1 = -\log\eta(\tau),$$

where $\eta(\tau)$ is the Dedekind function and τ is the modular parameter of the Seiberg-Witten curve, written in the $\Gamma(2)$ description of [86]. This was confirmed in [27] from the point of view of the perturbative instanton sums. Gravitational F-terms were also considered in [117, 118, 119, 120], where the Konishi-anomaly approach has been used to compute the F-terms.

In [121], the $g = 1$ corrections for a multicut Hermitian matrix model have been derived directly from the loop equations, and shown agreement with the geometric results. In [122], it is claimed that the free energy for this matrix model can be calculated for *all* genera in the $1/\widetilde{N}$ expansion, using Feynman graph techniques.

CHAPTER 6

BRANES AND MATRIX MODELS

It is known that non-critical bosonic strings[1], which can be viewed as deformations of topological strings, have two different matrix model descriptions:

- i a double scaling limit of a matrix model, in which the string worldsheets emerge through the 't Hooft fatgraphs as triangulations [3, 4, 5],
- ii a finite N matrix model, introduced by Kontsevich [41], in which the matrix diagrams can be considered as open string field theory diagrams that triangulate moduli space [39].

These viewpoints can be unified into a single one through the topological B-model on Calabi-Yau three-folds [6]. We will now consider this.

In the following, we will consider non-compact Calabi-Yau three-folds that can be seen as a hypersurface given by

$$z\,w - H(y,x) = 0 \tag{6.1}$$

with $z, w, x, y \in \mathbb{C}$. Even though they are non-compact, these geometries allow a Ricci-flat metric that is conical at infinity [124, 125, 126, 47]. The holomorphic three-form is given by

$$\Omega = \frac{\mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y}{z}. \tag{6.2}$$

The geometry Eq.6.1 can be reduced to one complex-dimension by looking at the perturbation of $H(x,y)$ and keeping the dependence on w, z fixed. Then the Calabi-Yau $\mathcal{M}$ can be considered as a fibration over the (y,x)-plane, where the fiber is given

[1] By non-critical we refer to the number of dimensions of the worldsheet, which is less than ten. See [123] for a review on non-critical bosonic strings.

by the curve Eq.6.1. This fiber develops a node on the locus

$$H(y,x)=0, \tag{6.3}$$

which is a non-compact curve.

By Cauchy's theorem, the periods of the three-form Ω over the three-cycles on $\mathcal{M}$ reduce to integrals of the two-form

$$\int_D \mathrm{d}x \wedge \mathrm{d}y,$$

where D is a real two-dimensional domain in the complex two-dimensional (y,x)-plane. One then has $\partial D \subset \Sigma$, with the Riemann surface Σ given by the curve Eq.6.3. Then, by Stokes' theorem, these integrals themselves reduce to

$$\int_\gamma y\,\mathrm{d}x,$$

where $\gamma = \partial D$ is a one-cycle on the Riemann surface Σ. From this it follows [21] that the complex structure deformations of $H(y,x)$ depend on the one-form

$$\lambda = y\,\mathrm{d}x.$$

The natural set of two-branes that lives in this geometry [127] is parametrized by a fixed point (y_0,x_0) in the (y,x)-plane. These branes are given by the subspace (z,w,y_0,x_0) with the restriction

$$z\,w = H(y_0,x_0).$$

This corresponds to the situation that the brane wraps the fiber of the fibration of $\mathcal{M}$ over the (y,x)-plane. From now on, we will drop the subscript 0 for convenience and write (y,x).

Eq.6.3 corresponds to a surface with a number of boundaries. Near each boundary, one can choose a local coordinate x with the condition $x \to \infty$ at the boundary. For every x, one has a canonical partner $y(x)$ with the one-form λ as above [127]. λ is holomorphic in Σ but it will have singularities at infinity.

To study the variation of the complex structure at $x=\infty$, we introduce a scalar with φ with the property

$$\lambda = \partial\varphi,$$

and repeat this at every boundary patch. One can then state that $y(x) = \partial_x\varphi$. If we denote by $y_{\mathrm{cl}}(x)$ the solution to $H(y_{\mathrm{cl}}(x),x)=0$ near $x=\infty$, we can write

$$\partial_x\varphi_{\mathrm{cl}} = y_{\mathrm{cl}}(x), \tag{6.4}$$

with the classical expectation value $\langle\varphi(x)\rangle = \varphi_{\rm cl}(x)$.

If we think of the function $H(y,x)$ as a Hamiltonian on the phase space (y,x), then the curve Σ, that is given by Eq.6.3, can be interpreted as a Fermi surface in the corresponding fermion theory. In this sense, the one-form λ is the Liouville form $y\,\mathrm{d}x$ and Eq.6.4 is the Hamilton-Jacobi relation which gives the classical action $\varphi(x)$ for $H=0$.

This is connected to matrix models as we will see below.

6.1 B-BRANES

B-branes are a kind of D-branes that preserve B-type supersymmetry, compatible with the vector R-symmetry

$$[F_V, \mathcal{Q}_B] = \mathcal{Q}_B, \quad [F_V, \bar{\mathcal{Q}}_B] = -\bar{\mathcal{Q}}_B,$$

with F_V and $\mathcal{Q}_B$ given by Eq.2.10. Similarly, D-branes that preserve A-type supersymmetry, compatible with the axial R-symmetry, are called A-branes, which we will not discuss here.

As we have mentioned in section 3.1, branes in the B-model are holomorphic cycles. In the A-model they are special Lagrangians, where Lagrangian refers to the vanishing of the Kähler form on the brane.

Consider a D-brane wrapped on a submanifold γ of the target space manifold $\mathcal{M}$. The boundary of the worldsheet Σ is then mapped into γ,

$$\varphi : \partial\Sigma \to \gamma.$$

The preservation of B-type supersymmetry on a D-brane turns out to constrain γ in such a way that it must be a complex submanifold of $\mathcal{M}$ and the superpotential W of the theory must be locally constant on γ, see [10] for a detailed explanation.

In the context of superstring target space, branes are defined by their impact on gravitational modes, as sources for certain fields. Similarly, this also the case for branes in topological strings. If we consider a one complex-dimensional subspace inside the Calabi-Yau $\mathcal{M}$ with N B-branes wrapped over it, then this affects the closed string modes by changing the periods of the holomorphic threeform Ω with [128],

$$\Delta \int_{\mathcal{C}} \Omega = N\, g_s, \tag{6.5}$$

where $\mathcal{C}$ is a three-cycle.

6.1.1 COMPACT B-BRANES

Consider a Calabi-Yau $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1$, with N compact B-branes wrapped over $\mathbb{P}^1$. The three-cycle surrounding $\mathbb{P}^1$ in this case is an S^3 and one has [128]

$$\int_{S^3} \Omega = N\, g_s.$$

This suggests that the Calabi-Yau undergoes a geometric transition where $\mathbb{P}^1$ shrinks and S^3 grows, with the size of S^3 is given by [26]

$$S = N\, g_s.$$

Since we want to find a geometry with n separate $\mathbb{P}^1$s at fixed positions, to make a connection to matrix models, we introduce a polynomial $W(x)$ of degree $n+1$. The class of Calabi-Yaus with this geometry is given by a hypersurface

$$z\, w - y^2 + W'(x)^2 = 0.$$

Near each of the critical points $W'(x) = 0$, there is a conifold singularity that can be blown up to a $\mathbb{P}^1$. One can then wrap N_i B-branes, with $i = 1, \ldots, n$, around each of the $\mathbb{P}^1$'s, and this geometry undergoes a transition to another geometry where n S^3's replace the n $\mathbb{P}^1$'s [46]

$$z\, w - y^2 + W'(x)^2 - f(x) = 0.$$

Here $f(x)$ is a polynomial of degree $n-1$ in x, with its coefficients fixed by the size of each S^3, given by

$$S_i = N_i\, g_s.$$

This duality has been worked out in the context of topological strings [3]. The open string field theory was then identified with a two-dimensional holomorphic Chern-Simons theory, which was shown to reduce to a matrix model with the action

$$S = \mathrm{Tr}\, W(\Phi)/g_s.$$

This led to the Dijkgraaf-Vafa conjecture we considered in chapters 3 and 4:

$$Z_{\mathrm{closed}}(S_i) = \int \mathcal{D}\Phi\ \exp\left(-\mathrm{Tr}\, W(\Phi)/g_s\right).$$

6.1.2 NON-COMPACT B-BRANES

We now consider non-compact branes, with moduli parameterized by a point on the surface Eq.6.3. In the presence of non-compact B-branes, Eq.6.5 implies a change in

the integral of the one-form $\lambda = y\,\mathrm{d}x$. This one-form is the reduction of the three-form Ω in Eq.6.2, integrated along two of the normal directions. Integrating the period of λ around the point P on the Riemann surface where the brane intersects, one has the contribution

$$\oint_P \lambda = \oint_P \partial\varphi = g_s.$$

If we denote the operator creating a brane at the point P on the surface by $\psi(P)$, then we have the identity

$$\left\langle \cdots \oint_P \partial\varphi\, \psi(P) \cdots \right\rangle = g_s \left\langle \cdots \psi(P) \cdots \right\rangle .$$

This implies that the brane creation operator affects the closed string sectors by

$$\psi(z) = \exp\left(\varphi(z)/g_s\right). \tag{6.6}$$

This means that the brane is the fermion associated to the chiral scalar bosonic field φ by the standard bosonization rules [128]. Eq.6.6 is also consistent with the fact that the classical action for the fermionic brane at position z is given by [127]

$$S(z) = \frac{1}{g_s}\int^z \lambda = \varphi(z)/g_s.$$

Similarly, the anti-brane is defined such that it gives the opposite change in the period integral, so it is given by the conjugate fermion

$$\psi^*(z) = \exp(-\varphi(z)/g_s).$$

In the Kodaira-Spencer theory of gravity [21], the observables correspond to variations of the complex structure at infinity. The theory has several patches of asymptotic infinity and in each patch one has a chiral boson $\varphi(x)$,

$$\partial\varphi(x) = y(x),$$

that describes these variations.

We now consider this theory with the geometry described by the deformation of a Riemann surface $H(y,x) = 0$. Denoting a coordinate for asymptotic infinity of the Riemann surface as $x \to \infty$, we put branes at positions x^i near this asymptotic patch. Then the gravitational back-reaction is given by

$$\left\langle \prod_i \psi(x^i) \right\rangle = \left\langle \prod_i \exp(\varphi(x^i)/g_s) \right\rangle \sim \prod_{i\neq j}(x^i - x^j) \exp\left(\sum_i \varphi_{\mathrm{cl}}(x^i)/g_s\right),$$

where the approximation is for the region where $x^i \to x^j$ and φ_{cl} is the classical part of the boson.

The expectation value $\langle \partial\varphi(x) \rangle$ in this background is then equal to

$$\langle \partial\varphi(x) \rangle = g_s \sum_i \frac{1}{x - x_i} = g_s \sum_{n>0} x_i^{-n} x^{n-1}. \tag{6.7}$$

This implies that we have turned on a background that is given by the couplings

$$t_n = \frac{g_s}{n} \sum_i (x^i)^{-n}, \quad n > 0. \tag{6.8}$$

The momentum of the bosonic field φ is also shifted by the number of fermions we put in. Putting anti-branes would give a similar formula, except for

$$t_n = \frac{g_s}{n} \sum \pm (x^i)^{-n}.$$

Here the sign $\pm$ depends on whether we put a brane or an anti-brane at x^i. There is also a factor of $(x^i - x^j)^{\pm 1}$ that signifies whether we put branes of the same type or opposite type at the patches.

We can also write this in terms of branes. If we put N branes at positions $x^1, \ldots, x^N$ in the local x-patch, then the correlation function is given by

$$\langle N | \prod_{i=1}^{N} \psi(x^i) \, | V \rangle = \mathrm{e}^{\mathcal{F}(t)} \prod_{i<j} (x^i - x^j).$$

6.2 WAVE FUNCTIONS

We have seen that inserting a fermion $\psi(x)$ at a point on the Riemann surface corresponds to inserting a B-brane in the Calabi-Yau manifold. The B-branes are objects that are globally well defined and this would seem to imply that the fermions are free. This is not totally true as we will see below.

To see this, consider the world-volume theory on the brane and insert a B-brane in an asymptotic patch $x_i \to \infty$. The D-brane is non-compact and its partition function is given by a wave function

$$Z_{\text{open}}(x_i) = \Psi(x_i).$$

It depends on x_i, because this is what we fix at infinity on the world-volume of the D-brane.

In the Kodaira-Spencer theory of gravity [21] this corresponds to inserting a fermion $\psi = \mathrm{e}^{\varphi/g_s}$ at $x = x_i$, and we have

$$\Psi(x_i) = \langle \psi(x_i) \rangle .$$

$\Psi(x_i)$ transforms like a wave function when we move from patch to patch. In this patch, we have a symplectic pair of variables (x_i, y_i). In terms of the theory on the B-branes, the variables x_i and y_i correspond to the zero modes of fields that are canonically conjugate,

$$[y_i, x_i] = g_s.$$

Consider another patch with symplectic pair of coordinates (y_j, x_j) and the corresponding fermion. Than (y_i, x_i) and (y_j, x_j) are related by a canonical transformation that preserves the symplectic form

$$\mathrm{d}x_i \wedge \mathrm{d}y_i = \mathrm{d}x_j \wedge \mathrm{d}y_j,$$

with a generating function $S(x_i, x_j)$. Then $\psi_i(x_i)$ must transform as

$$\psi_j(x_j) = \int \mathrm{d}x_i \, \mathrm{e}^{-S(x_i,x_j)/g_s} \, \psi_i(x_i).$$

Using the bosonization/fermionization formulae, we can express the one-point function of the brane creation operator

$$\Psi(x) = \langle \psi(x) \rangle$$

as

$$\Psi(x) = \exp\left(\frac{1}{g_s}\int^x y(x)\,\mathrm{d}x\right) = \exp\left(\frac{1}{g_s}\varphi(x)\right). \tag{6.9}$$

We recognize this as the WKB approximation for a wave function.

The boson $\varphi(x)$ has a classical piece, that corresponds to the background geometry, and a fluctuating quantum piece

$$\varphi(x) = \varphi_{\mathrm{cl}}(x) + \varphi_{\mathrm{qu}}(x).$$

The classical piece is given by the integral of the canonical one-form on the Riemann surface,

$$\varphi_{\mathrm{cl}}(x) = \int^x y\,\mathrm{d}x.$$

For the fermions that create the branes/anti-branes we have

$$\psi(x) = \mathrm{e}^{\varphi_{\mathrm{cl}}(x)/g_s}\,\psi_{\mathrm{qu}}(x), \qquad \psi^*(x) = \mathrm{e}^{-\varphi_{\mathrm{cl}}(x)/g_s}\,\psi^*_{\mathrm{qu}}(x).$$

For small quantum fluctuations one can approximate Eq.6.9 by a Gaussian integral. Then ψ_{qu} transforms as

$$\psi^{\mathrm{qu}}_j(x_j)(\mathrm{d}x_j)^{\frac{1}{2}} = \psi^{\mathrm{qu}}_i(x_i)(\mathrm{d}x_i)^{\frac{1}{2}}.$$

So, the fermions transform as global spin $\frac{1}{2}$-fields on the Riemann surface, as we would expect fermions to do from quantum mechanics.

We also note that the one-point function $\Psi(x) = \langle\psi(x)\rangle$ satisfies the Schrödinger equation

$$H(y,x)\,\Psi(x) = 0.$$

So we can think of the function $H(y,x)$ as a Hamiltonian on the phase space (y,x), with the operator

$$y = -g_s\frac{\partial}{\partial x}.$$

Then the curve $H = 0$ can be interpreted as a Fermi surface in the corresponding fermion theory. In this sense, $y\,\mathrm{d}x$ is the Liouville form and

$$\partial_x\varphi_{\mathrm{cl}}(x) = y(x)$$

is the Hamilton-Jacobi relation which gives the classical action $\varphi_{\mathrm{cl}}(x)$ for $H = 0$.

6.3 KONTSEVICH-LIKE MATRIX MODELS

In the patch with local coordinates x_j, which is parameterized by the couplings t_n^i Eq.6.8, one can view the closed string deformation of the geometry as being induced by branes at $x^i = x_j^i$ with $j = 1,\dots N$. In terms of the string theory description, one then has [6]

$$Z_{\mathrm{closed}}(0)\,Z_{\mathrm{open}}(x_j^i) = Z_{\mathrm{closed}}(t_n^i),$$

where $Z_{\mathrm{closed}}(t_n^i)$ denotes the partition function of closed strings and Z_{open} that of the open strings. The t_n^i are determined from the geometry of the branes, which are located at positions x_i^j in each asymptotic geometry $x^i \to \infty$. These parameters are related to each other by

$$t_n^i = \frac{g_s}{n}\sum_j (x_j^i)^{-n}.$$

This means that the t_n^i dependence of $Z_{\mathrm{closed}}(t_n^i)$ can be computed by integrating the field theory living on the brane, which is given by $Z_{\mathrm{open}}(x_j^i)$. So one can compute the closed string amplitudes using the open string sector which deforms it.

The open string sector action has been studied in [127]. The idea is, simplified for a single brane probe, is as follows. Consider a brane probe located at $x = x_1$ near the asymptotic region $x \to \infty$. The tree-level action is a function of varying the position of the brane to a generic point x. It is given by integrating the one-form $\lambda = x\,\mathrm{d}y$,

$$S(y) = \frac{1}{g_s}\left(\int^y x(y')\,dy' - x_1\,y\right) = \frac{1}{g_s}\left[W(y) - x_1 y\right],$$

where $x(y)$ is the solution to $H(y,x)$ and

$$W(y) = \int^y x(y')\,\mathrm{d}y'.$$

The classical solution is given by the extremum of the action

$$\frac{\mathrm{d}S}{\mathrm{d}y} = 0 \rightarrow x(y) = x_1,$$

and this is consistent with the classical position of the brane. For more than one brane, say N branes at positions

$$\Lambda = (x_1, \ldots, x_N),$$

one can write a matrix model version of this action. Then y is treated as an $N \times N$ matrix, P, and Λ is treated as a diagonal $N \times N$ matrix. We then have a matrix action

$$S(P) = \frac{1}{g_s}\mathrm{Tr}\left[W(P) - \Lambda P\right].$$

This is a classical action and there could be quantum corrections that one should consider. In some cases there is no quantum correction and one is left with a simple matrix model description for the closed string amplitude,

$$Z_{\mathrm{closed}}(t_n) = \int \mathcal{D}P\ \exp\left[\frac{1}{g_s}\mathrm{Tr}(W(P) - \Lambda P)\right]$$

This is a Kontsevich-like matrix model, where the source term contains the data about the classical positions of the branes. It satisfies

$$t_n = \frac{g_s}{n}\mathrm{Tr}\,\Lambda^{-n}.$$

Here it is assumed that $\partial\varphi$ has integral expansion powers in x, like in Eq.6.7. In cases where one may have a monodromy in $\partial\varphi$ as $x \rightarrow \mathrm{e}^{2\pi\mathrm{i}}x$, the expansion modes n will be fractional.

For $W(P) = \frac{1}{3}P^3$, one has the traditional Kontsevich model with the action

$$S(P) = \frac{1}{g_s}\mathrm{Tr}\left[\frac{1}{3}P^3 - \Lambda P\right].$$

This result can also be described from a different perspective, where the quantum corrections are included. Consider N B-branes in a patch with the coordinates (x_i, y_i) and relate these branes to B-branes in different coordinates (x_j, y_j), related to the original coordinates by a symplectic transformation generated by $S(x_i, x_j)$,

$$S(x_i, x_j) = (-\frac{d}{2}x_i^2 + x_i x_j - \frac{a}{2}x_j^2)/c.$$

The amplitude

$$\langle N|\ \psi(x_{j,1})\dots\psi(x_{j,N})\ |V\rangle = \prod_{n<m}(x_{j,n}-x_{j,m})\,\mathrm{e}^{\mathcal{F}(x_j)}$$

can then be written in terms of

$$\langle N|\ \psi(x_{i,1})\dots\psi(x_{i,N})\ |V\rangle$$

as

$$\langle N|\ \psi(x_{j,1})\dots\psi(x_{j,N})\ |V\rangle = \int\prod_{n=1}^{N}\mathrm{d}x_{i,n}\,\mathrm{e}^{-\sum_{n=1}^{N}S(x_{i,n},x_{j,n})}\ \langle N|\ \psi(x_{i,1})\dots\psi(x_{i,N})\ |V\rangle\,.$$

This is equal to an eigenvalue representation of the matrix integral

$$Z(X_j) = \int \mathcal{D}X_i\,\mathrm{e}^{-\mathrm{Tr}\,S(X_i,X_j)}\,Z(X_i) \tag{6.10}$$

with $Z(X_i)=\mathrm{e}^{\mathcal{F}(X_i)}$ written in terms of

$$\mathrm{Tr}\,X_i^n = \sum_m x_{i,m}^n.$$

We have used here the expression for the Itzykson-Zuber integral [129]. In cases where one can find the variables (y_i,x_i) such that $Z(X_i)$ is simple, then Eq.6.10 is a good way to find the amplitudes for other patches.

6.4 EXAMPLE: THE GAUSSIAN MATRIX MODEL

As we have seen, putting a brane in the matrix model corresponds to shifting the couplings in the superpotential

$$t_k \to t_k + \frac{g_s}{k}x^{-k},$$

and the brane creation operator is given by

$$\psi(x) = \mathrm{e}^{\varphi(x)/g_s} = \mathrm{e}^{W(x)/g_s}\det(x-\Phi).$$

Partition functions of branes can then be written as correlation functions of characteristic polynomials in the matrix model. An M-point function, for example, is given by

$$\langle M|\,\psi_{\mathrm{qu}}(x_1)\cdots\psi_{\mathrm{qu}}(x_M)\,|V\rangle = \langle\det(x_1-\Phi)\cdots\det(x_M-\Phi)\rangle\,.$$

Here $\langle\cdots\rangle$ denotes the normalized expectation value in the matrix model.

We will use the methods of orthogonal polynomials to evaluate such correlators, and write $\{P_i(x)\}_{i\geq 0}$ for a basis of orthogonal polynomials for the matrix model

$$Z = \frac{1}{\mathrm{vol}U(N)} \int \mathrm{d}\Phi \, \exp\left[\frac{1}{g_s}\mathrm{Tr}W(\Phi)\right],$$

with the normalization $P_i(x) = x^i + \ldots$. The corresponding "wave functions" will be denoted by

$$\Psi_i(x) = P_i(x)\, \mathrm{e}^{W(x)/2g_s}.$$

Then one can express the brane n-point function as an $n \times n$ Slater determinant [130]

$$\langle \det(x_1 - \Phi) \cdots \det(x_M - \Phi) \rangle = \frac{\det P_{N+j-1}(x_i)}{\Delta(x)},$$

with the Vandermonde determinant given by

$$\Delta(x) = \det x_i^{j-1}.$$

We add the classical contribution and get

$$\langle \psi(x_1) \cdots \psi(x_M) \rangle = \frac{\det \Psi_{N+j-1}(x_i)}{\Delta(x)}.$$

The one-point function is then given by

$$\langle \psi(x) \rangle = \Psi_N(x).$$

For the simple Gaussian matrix model, one has

$$Z = \frac{1}{\mathrm{vol}\, U(N)} \int \mathrm{d}\Phi \cdot \exp\left[-\frac{1}{2g_s}\mathrm{Tr}\, \Phi^2\right].$$

As we have seen in chapter 3, this corresponds to the deformed geometry

$$H(y, x) = y^2 + x^2 - 4S = 0$$

with $S = g_s\, N$. In this case, the orthogonal polynomials are eigenfunctions of a Schrödinger operator. The polynomials $P_i(x)$ are the Hermite polynomials, and the corresponding wave functions satisfy

$$\left(-g_s^2 \frac{\partial^2}{\partial x^2} + x^2\right) \Psi_k(x) = g_s\, (k + \frac{1}{2})\, \Psi_k(x).$$

In this case we can also verify that the one-point function $\langle \psi(x) \rangle = \Psi_N(x)$ satisfies

$$H\Psi_N = 0,$$

up to the quantum shift $N \to N + \frac{1}{2}$.

For general non-Gaussian matrix models, the corresponding Riemann surface has genus $g > 0$ and therefore there are non-trivial loop momenta or fluxes

$$\frac{1}{2\pi} \oint_{A_i} y \, \mathrm{d}x = g_s \, N_i.$$

Because of these moduli, the free fermion formulation becomes less straightforward.

We can now derive the Kontsevich matrix model description of this Gaussian model. We then put M non-compact B-branes in the geometry

$$zw = y \, (y - x) - S.$$

near $x \to \infty$. This is described by the Kontsevich-like matrix model with the action

$$S(P) = \mathrm{Tr} \left[\Lambda P - \int X(P) dP \right],$$

where $x(y)$ solves $y \, (y - x) = S$. We get the matrix model

$$\int_{M \times M} \mathcal{D}P \, \det(P)^N \, \mathrm{e}^{\frac{1}{g_s} \mathrm{Tr} \left[\Lambda P - \frac{1}{2} P^2 \right]}, \tag{6.11}$$

with $S = g_s \, N$.

CHAPTER 7

UNIVERSAL CORRELATORS IN MATRIX MODELS

As we have seen in the last chapter, non-compact B-branes also have a matrix model description. In this chapter we will again be considering non-compact Calabi-Yau manifolds that are given as the hypersurface

$$zw - H(y, x) = 0,$$

where $z, w, y, x \in \mathbb{C}$. The non-compact branes are then parameterized by a fixed point in the (x, y) plane that lies on the curve

$$\mathcal{C} : \ H(x, y) = 0,$$

and they extend in the coordinate z or w.

The B-model describes the complex structure deformations of the complex curve $\mathcal{C}$ or, equivalently, of the "Hamiltonian" $H(x, y)$. The variations of the complex structure at infinity can be introduced by a chiral boson, $\varphi(x)$. In a local coordinate patch this chiral boson is defined by

$$y(x) = \partial\varphi(x),$$

and it describes the variation of the curve through its parameterization $y(x)$.

The target space theory of the B-model on this geometry is the Kodaira-Spencer theory where $\partial\varphi$ is the dimensional reduction of the Kodaira-Spencer field A. The B-branes can be thought of as defects for the Kodaira-Spencer field $\varphi(x)$. The operator $\psi(x)$ that creates or annihilates a brane turns out to be a free fermion field and is

related to the chiral boson via the bosonization formula

$$\psi(x) = \mathrm{e}^{\mathrm{i}\varphi(x)/g_s}.$$

Similarly the field $\psi^*(x) = \mathrm{e}^{-\mathrm{i}\varphi(x)/g_s}$ creates/annihilates an anti D-brane.

In terms of the matrix model, the chiral boson is the collective field of the eigenvalues of the matrix, see Eq.3.19. The fermions are basically free fermions, but transform between the different patches as wave functions with generalized Fourier transformation. We will make use of this fact by choosing good coordinates, and later Fourier transforming back to the original coordinates.

We will use the fermionic formulation to study certain correlators in the matrix models. In random matrix models, it has been understood for a long time that eigenvalue correlators have an interesting behavior at short scales, called "universality". In particular, the joint probability of n eigenvalues is given by a determinant of a single kernel

$$\rho(\lambda_1, \dots \lambda_n) = \det_{n\times n} K(\lambda_i, \lambda_j).$$

In the limit N large, while keeping the rescaled distances $N(\lambda_i - \lambda_j)$ fixed, the kernel K takes the form

$$K(\lambda_i, \lambda_j) \sim \frac{\sin N\pi\, \rho(\bar{\lambda})(\lambda_i - \lambda_j)}{N\pi\,(\lambda_i - \lambda_j)},$$

where $\rho(\bar{\lambda})$ is the density of eigenvalues at the mean $\bar{\lambda}(\lambda_i + \lambda_j)/2$. The formula is called universal because it does not depend explicitly on the form of the matrix model potential, thus it has the same form for any potential. It has only a functional dependence on the potential through the scaling factor of the mean density.

There are many ways to derive this kernel in random matrix models, the usual one relies on introducing orthogonal polynomials. With the use of brane insertions, we provide a simple derivation by writing the kernel in terms of the correlator of free fermions as

$$K(x_1, x_2) = \langle \psi(x_1)\psi^*(x_2)\rangle \sim \frac{\mathrm{e}^{i(\varphi(x_1)-\varphi(x_2))/g_s}}{x_1 - x_2},$$

where the second equality is by bosonization. Here $\varphi(x)$ is the chiral boson which is related to the geometry associated to the matrix model. In this formula, the use of good coordinates, which are single-valued, is an essential point. That is, it only takes this simple form if the coordinate x parametrizes a unique point on the curve $\mathcal{C}$. It is only in these coordinates that the branes, or free fermions, can be inserted at a definite position.

As we will explain, to find the kernel in the usual double-valued coordinates, we can use Fourier transformation. In fact, we are just applying standard techniques of the semiclassical WKB approximation. The kernel will then be given as a weighted

sum of contributions, coming from the "images" of the brane in the multivalued coordinates. In the multicover coordinates, the position of the brane cannot be fixed unambigously, and the brane insertion is best defined via the Fourier transformation. A version of this idea first appeared in [104] using methods of conformal field theory. Here we make it more explicit and reformulate it in the language of brane insertions as we have explained in the last chapter, based on [6].

The universal correlator is an interesting quantity to consider, since it probes the geometry at short distances. We can investigate it with the idea of inserting probe branes in the geometry. Brane probes can also be used in the more general context of the Calabi-Yau crystal [131, 132, 133].

In this chapter, we will study certain correlators in the crystal which are the three-dimensional analogues of the eigenvalue correlators of matrix models. We find that such correlators can be described by fermion insertions in the geometry. These fermions are "generalized" in the sense they depend on two parameters. They are the usual one dimensional chiral fermions, but inserted at an arbitrary slice of the crystal, giving an additional parameter. Thus the fermions are effectively two dimensional objects probing the three-dimensional structure of the crystal.

The generalized fermions do not correspond to the Lagrangian brane probes in [133]. They are different objects probing the interior of the crystal. Finding the precise brane interpretation is an interesting open problem.

It is natural to ask whether the free fermion correlators of the crystal also show a similar universal behaviour when scaling parameters appropriately. Such correlators have also been computed in the mathematics literature of random partitions [134], and a similar structure of universal correlators is found.

When taking a two dimensional slice of the crystal, in the scaling region we find the same sine-kernel as in the case of random matrices. It is well-known in the mathematics of partitions that one can define a suitable probability measure which is the discretized version of the measure for Gaussian matrix models. The three dimensional analogue of the sine-kernel is a more complicated object, written in terms of an incomplete beta-function. We conjecture that this beta-function is the universal scaling limit of a certain two-matrix model with a unitary measure, which appears in the context of Chern-Simons theory [135], and the topological vertex [136].

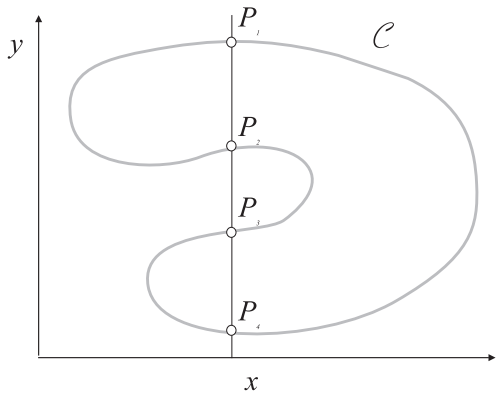

Figure 7.1: The graph of the function $y(x)$ is given by the orbit $\mathcal{C}$ determined by $H(x,y)=0$. In general it has locally several branches: to a single value of x correspond the geometric points $P_1, P_2, \ldots$.

7.1 BRANES AND THE WKB APPROXIMATION

To understand the relation between eigenvalues in matrix models and brane insertions in topological strings, let us first review some familiar facts about semi-classical quantization. We consider (x,y) as coordinates on a two-dimensional phase space. The curve $\mathcal{C}$ given by $H(x,y)=0$ can be viewed as the orbit that corresponds to the classical ground state of the system described by the Hamiltonian H. To such a one-dimensional curve one can associate the (global) action variable S given by the contour integral

$$S=\frac{1}{2\pi}\oint_{\mathcal{C}} y\,\mathrm{d}x.$$

The Bohr-Sommerfeld rule quantizes this as $S=(N+\frac{1}{2})g_s$, where we identify $\hbar$ with the string coupling constant g_s. It is useful to think of the Hamiltonian $H(x,y)$, and therefore also the curve $\mathcal{C}$, as part of a family of systems parametrized by the modulus S.

In general, if we want to use x as a coordinate and solve for y as a function of x on the curve $H(x,y)=0$, there can be several branches $y_I(x)$. This is because the line of constant x will intersect the curve $\mathcal{C}$ in different points P_I, see fig.7.1.

All these points at these different sheets will contribute to the semi-classical wave function $\psi(x)$ associated to the orbit $\mathcal{C}$. The phase of this wave function is given by the local action

$$\varphi_I(x)=\int_{x_0}^{x} y_I(x')\,\mathrm{d}x'$$

with x_0 some arbitrary reference point. The full WKB expression for the wave function reads with some overall normalization c

$$\psi(x) = c \cdot \sum_I \left(\frac{\partial^2 \varphi_I}{\partial x \, \partial S} \right)^{1/2} \exp\left[\mathrm{i} \frac{\varphi_I(x)}{g_s} \right]$$

Here all the intersection points P_I contribute. For example, for the harmonic oscillator with

$$H(x,y) = x^2 + y^2 - 4S$$

we have two such points. Indeed the wave function is well-known to be given in the WKB approximation by (for the "allowed zone")

$$\psi(x) \sim (4S - x^2)^{-\frac{1}{4}} \cos\left(\int^x \mathrm{d}x' \sqrt{4S - x'^2} - \frac{\pi}{4} \right).$$

In the context of the matrix models for the topological B-model, the geometric points P_I on the curve $\mathcal{C}$ correspond to non-compact D-branes on the Calabi-Yau space. The variable x appears naturally in the matrix model as the eigenvalue, but it will not be a good coordinate, since the spectral curve $\mathcal{C}$ will always be a multiple cover over the x-plane (a double cover for a single matrix integral). The natural fermion operators $\psi(x)$, that appear in the matrix model and correspond to creating a single eigenvalue, are therefore not represented by a single geometric brane. Instead, they are a superposition of branes inserted at the inverse images P_I of x, just as we have in the formula for the WKB wave function.

For the local geometry with two sheets, where we forget about all the x-dependence and therefore can write the local geometry as

$$y^2 = p^2,$$

we have branes inserted at the points P_1 and P_2 given by $y_{1,2} = \pm p$. The corresponding action functions are $\varphi_1(x) = p\,x$ and $\varphi_2(x) = -p\,x$. The wave function is naturally given as the sum

$$\psi(x) \sim p^{-1/2} \left[\exp \mathrm{i} \left(\frac{p\,x}{g_s} + \frac{\pi}{4} \right) + \exp \mathrm{i} \left(-\frac{p\,x}{g_s} - \frac{\pi}{4} \right) \right].$$

That is, we have in terms of brane insertions the natural identification

$$\psi(x) = \mathrm{e}^{\frac{i\pi}{4}} \psi(P_1) + \mathrm{e}^{-\frac{i\pi}{4}} \psi(P_2).$$

Similarly the anti-brane is inserted by

$$\psi^*(x) = \mathrm{e}^{\frac{i\pi}{4}} \psi^*(P_1) + \mathrm{e}^{-\frac{i\pi}{4}} \psi^*(P_2).$$

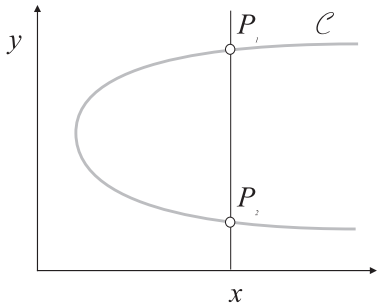

Figure 7.2: Close to a turning point, given by $y^2 = x$, the good coordinate is y not x.

Note that this is not the complex conjugated expression.

With this translation we can easily compute the multi-fermion correlation function. Here we should remind ourselves that we will have only contraction of operators that create or annihilate branes on the same sheet. For example, we have the two-point function

$$\langle \psi(x)\, \psi^*(x')\rangle = \mathrm{e}^{\frac{i\pi}{2}}\, \langle \psi(P_1)\, \psi^*(P_1')\rangle - \mathrm{e}^{-\frac{i\pi}{2}}\, \langle \psi(P_2)\, \psi^*(P_2')\rangle\,.$$

This can be evaluated in the limit $x \to x'$ as

$$\langle \psi(x)\psi^*(x')\rangle \sim \frac{\sin\left(\sqrt{S}(x-x')/g_s\right)}{\sqrt{S}(x-x')}$$

which gives the famous sine-kernel.

This description becomes problematic at caustics or turning points where the curve $\mathcal{C}$ is perpendicular to the x-direction and two of the branches coalesce as in fig.7.2. The right description at these points is to use the coordinate y instead. The new wave function is then determined by a Fourier transform

$$\psi(y) = \int \mathrm{d}x\; \mathrm{e}^{\mathrm{i}xy/g_s}\psi(x).$$

As we will see later, this naturally leads to the Airy kernel.

7.2 UNIVERSAL CORRELATOR IN RANDOM MATRIX MODELS

We now turn to a detailed investigation of the universal correlators in matrix models using the geometric picture of brane insertions just described. Consider an $N \times N$

matrix model with partition function

$$Z = \int \mathrm{d}\Phi \, \mathrm{e}^{-\frac{1}{g_s}\mathrm{Tr}W(\Phi)}$$

and let us define the density of n of its eigenvalues as

$$\rho(\lambda_1, \dots \lambda_n) = \frac{(N-n)!}{N!} \left\langle \prod_{k=1}^{n} \mathrm{Tr}\, \delta(\lambda_k - \Phi) \right\rangle .$$

For the random matrix model this describes the probability of measuring n eigenvalues at the same time. More precisely, the interesting part of this correlation function is its connected part. In the large N limit, the multicorrelator factorizes into one-point functions, and therefore only its connected component contains new information.

Introducing the orthogonal polynomials

$$\int \mathrm{d}\lambda \, \mathrm{e}^{-\frac{W(\lambda)}{g_s}} \, P_n(\lambda) \, P_m(\lambda) = \delta_{nm},$$

the joint eigenvalue distribution can be expressed as a determinant of a single two-point kernel

$$\rho_n(\lambda_1, \dots \lambda_n) = \frac{N^n (N-n)!}{N!} \det_{n \times n} K(\lambda_i, \lambda_j).$$

In terms of the wave functions

$$\Psi_n(\lambda_i) = P_n(\lambda_i) \, \mathrm{e}^{-\frac{W(\lambda)}{2g_s}}$$

this kernel can be written as

$$K(\lambda_i, \lambda_j) = \frac{1}{N} \sum_{n=0}^{N-1} \Psi_n(\lambda_i) \, \Psi_n(\lambda_j) = \frac{\Psi_N(\lambda_i) \, \Psi_{N-1}(\lambda_j) - \Psi_{N-1}(\lambda_i) \, \Psi_N(\lambda_j)}{\lambda_i - \lambda_j}. \tag{7.1}$$

Here we dropped the overall normalization factors, and at the second equality used the Darboux-Christoffel formula for the orthogonal polynomials. This formula can also be interpreted as a Slater determinant for second quantized fermions

$$K(\lambda_i, \lambda_j) = \left\langle \Psi^\dagger(\lambda_i) \, \Psi(\lambda_j) \right\rangle .$$

The eigenvalues of the matrix model are usually located along a cut on the real axis. When scaling close to a fixed point in the middle of the cut, this kernel satisfies universal properties

$$K(\lambda_i, \lambda_j) \sim \frac{\sin\left[N\pi(\lambda_i - \lambda_j)\rho(\bar{\lambda})\right]}{N\pi(\lambda_i - \lambda_j)}, \tag{7.2}$$

in the region where $(\lambda_i - \lambda_j) \sim O(1/N)$. Here $\bar{\lambda} = (\lambda_i + \lambda_j)/2$, and $\rho(\bar{\lambda})$ is the single eigenvalue density.

When we take a scaling close to an endpoint of the cut, the kernel has a different expression in terms of Airy functions

$$K(\lambda_i, \lambda_j) \sim \frac{\mathrm{Ai}(\lambda_i)\,\mathrm{Ai}'(\lambda_j) - \mathrm{Ai}(\lambda_j)\,\mathrm{Ai}'(\lambda_i)}{\lambda_i - \lambda_j},$$

where the λ's are now suitably scaled variables close to the endpoint.

Since the form of the kernel does not depend on the specific form of the potential, we can restrict our attention to the case of the Gaussian potential, $V(x) = x^2/2$. For the Gaussian potential, the sine and the Airy kernel can be derived by using the direct asymptotic expansion of the wave functions, which in this case are the Hermite polynomials. We include this standard computation in the Appendix.

7.3 UNIVERSAL CORRELATOR FROM GEOMETRY

In the context of topological string theory, matrix models are directly related to the geometry of the Calabi-Yau, see section 3.2. In the limit $N \to \infty$, $g_s \to 0$, $S = g_s N$ fixed, the Calabi-Yau geometry associated to the matrix model is

$$zw + y(x)^2 + (W'(x))^2 - f(x) = 0.$$

Here the quantum deformation $f(x)$ is a polynomial computed from the potential as

$$f(x) = \frac{4S}{N} \sum_i \frac{W'(x) - W'(\lambda_i)}{x - \lambda_i},$$

and

$$y(x) = \partial\varphi(x)$$

is the chiral boson from the collective field, as in Eq.3.19

$$\varphi(x) = W'(x) - 2g_s \mathrm{Tr}\log(x - \Phi).$$

The Riemann surface associated with the Gaussian matrix model in this way is given by

$$H(y, x) = y^2 + x^2 - 4S = 0.$$

As discussed in [6], the geometry can be probed by inserting non-compact branes. Taking an asymptotic patch in the geometry, and considering the worldvolume action

of the non-compact brane one finds that x and y are a symplectic pair of variables with canonical commutation relations $[x, y] = ig_s$. In terms of operators,

$$y = -\mathrm{i}g_s \frac{\partial}{\partial x}.$$

Inserting a non-compact brane in this geometry corresponds to adding a fermion $\psi(x) = \mathrm{e}^{\mathrm{i}\varphi(x)/g_s}$ with wave function

$$\Psi(x) = \langle \psi(x) \rangle = \mathrm{e}^{\mathrm{i}\varphi_{\mathrm{cl}}(x)/g_s}.$$

This wave function satisfies the Schrödinger equation

$$H\left(-\mathrm{i}g_s \frac{\partial}{\partial x}, x\right) \Psi(x) = 0.$$

For the Gaussian matrix model these wave functions are the Hermite polynomials. In fact in the Gaussian case these are the same wave functions as the orthogonal polynomial wave functions of the previous section. In the general case the orthogonal polynomial wave functions do not satisfy the Schrödinger equation, due to more complicated normal ordering issues.

In this case, we can simply derive the kernel by finding the wave functions from the geometry, and inserting in the Darboux-Christoffel formula Eq.7.1. Let us first show this for the Airy kernel. When close to the branch points, the curve is simply

$$y^2 + x = 0,$$

where we rescaled x for convenience and put the branch point at $x = 0$. We can insert a brane in a definite position in the y-coordinate which is single-valued. In the y-coordinate the wave function solves

$$\left(y^2 + \mathrm{i}g_s \frac{\partial}{\partial y}\right) \Psi(y) = 0,$$

so it is given by

$$\Psi(y) = \mathrm{e}^{\mathrm{i}\frac{y^3}{3g_s}},$$

where we dropped the overall constant. Transforming back to the x-coordinate we get

$$\Psi(x) = \int \mathrm{d}y\, \mathrm{e}^{\frac{\mathrm{i}xy}{g_s}} \Psi(y) = {g_s}^{1/3} \mathrm{Ai}\left[{g_s}^{-2/3} x\right].$$

In the large N limit the Darboux-Christoffel formula Eq.7.1 becomes

$$K(x_1, x_2) \sim \frac{\Psi(x_1)\Psi'(x_2) - \Psi(x_2)\Psi'(x_1)}{x_1 - x_2},$$

where $\Psi(x)$ are now the wave functions defined from the planar geometry, keeping N large and $S = g_s N$ fixed. So from the Darboux-Christoffel formula we immediately have

$$K(x_1, x_2) \sim \frac{\mathrm{Ai}(x_1)\mathrm{Ai}'(x_2) - \mathrm{Ai}(x_2)\mathrm{Ai}'(x_1)}{x_1 - x_2}.$$

where $x_{1,2}$ are rescaled coordinates measuring distances from the branch points.

The sine-kernel can be derived in a similar way by scaling to a fixed point in the cut away from the branch points. In this case the curve reduces to

$$y^2 + \bar{x}^2 - 4S + 2x\bar{x} = 0,$$

where $\bar{x}$ is the point we scale to, and x is the distance to that point. Let us work again in the y-coordinate. If we drop the small x-term, the wave function solves

$$(y^2 - p(\bar{x})^2)\Psi(y) = 0,$$

where $p(\bar{x}) = \sqrt{4S - \bar{x}^2}$ is the classical momentum at $\bar{x}$. The wave function is a sum of δ-functions

$$\Psi(y) = \mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\delta(y - p) + \mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}\delta(y + p).$$

The phases are fixed using the WKB approximation discussed in section 7.1. Fourier transforming back,

$$\Psi(x) = \mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\mathrm{e}^{\frac{\mathrm{i}px}{g_s}} + \mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}\mathrm{e}^{-\frac{\mathrm{i}px}{g_s}}.$$

Choosing the symmetric combination and substituting back in the Slater determinant formula we get the sine-kernel.

A more precise way is to keep the small x-term,

$$\left(y^2 - p(\bar{x})^2 + 2\mathrm{i}g_s\bar{x}\frac{\partial}{\partial y}\right)\Psi(y) = 0.$$

The corrected solution is then

$$\Psi(y) = \exp\left[\frac{i}{2g_s\bar{x}}\left(\frac{y^3}{3} - p(\bar{x})^2 y\right)\right].$$

Transforming back to x, the solution is again the Airy function

$$\Psi(x) = (2g_s\bar{x})^{1/3}\mathrm{Ai}\left[\left(\frac{2\bar{x}}{g_s^2}\right)^{1/3}\left(x - \frac{p(\bar{x})^2}{2\bar{x}}\right)\right].$$

Now we still have to expand in x, which is the small distance to the fixed point $\bar{x}$. Doing a saddle-point expansion of the Airy integral, we pick up the contribution from the two saddle points

$$y_{1,2} = \pm\sqrt{\bar{p}^2 - 2x\bar{x}}.$$

Substituting in the Darboux-Christoffel formula Eq.7.1

$$K(x_1, x_2) \sim \frac{\sin p(x_1 - x_2)}{x_1 - x_2},$$

we again get the sine-kernel.

7.4 CORRELATOR FROM TWO BRANE INSERTIONS

An even shorter way to derive the universal correlators is to insert two branes in the geometry. In the single-valued coordinates, where the branes can be inserted in a definite position, the kernel can be expressed as [104]

$$K(y_1, y_2) = \langle \psi(y_1)\psi^*(y_2) \rangle \,. \tag{7.3}$$

An intuitive derivation of this formula would express the joint correlators in the collective field, and then rewrite it with bosonization. However, the x-coordinates here are really inconvenient, because they are double-valued and an inconvenient averaging is involved. This averaging was first described in [104], and can be understood as summing over the images of the brane in the multicover coordinate, as discussed in detail in section 7.1.

In the single-valued y-coordinate, we start with the formula Eq.7.3, which can now be interpreted as an insertion of a brane and an antibrane in the geometry. The branes inserted are free fermions, consisting of a classical wave function, and a quantum part

$$\psi(y) = \Psi(y)\psi_{\rm qu}(y) = \mathrm{e}^{\frac{\mathrm{i}\varphi_{\rm cl}(y)}{g_s}}\psi_{\rm qu}(y).$$

So the short-distance correlator in the y-coordinates is

$$\langle \psi(y_1)\psi^*(y_2) \rangle = \frac{\mathrm{e}^{\frac{i}{g_s}(\varphi(y_1)-\varphi(y_2))}}{y_1 - y_2}.$$

Consider now again the geometry close to the branch point,

$$y^2 + x = 0.$$

The classical wave function from the geometry is

$$\Psi(y) = \mathrm{e}^{\frac{\mathrm{i}}{g_s}\varphi(y)} = \mathrm{e}^{\frac{\mathrm{i}}{g_s}y^3/3}.$$

Transforming back to the x-coordinate

$$\langle \psi(x_1)\psi^*(x_2) \rangle = \int \mathrm{d}y_1 \, \mathrm{d}y_2 \, \frac{\exp \frac{\mathrm{i}}{g_s}\left(x_1 y_1 - x_2 y_2 + y_1^3/3 - y_2^3/3\right)}{y_1 - y_2}.$$

The Fourier transform is easily evaluated by noting that

$$e^{ix_1y_1-ix_2y_2} = \frac{-i}{x_1-x_2}\left(\partial_{y_1}+\partial_{y_2}\right) e^{ix_1y_1-ix_2y_2}$$

Substituting and after a partial integration in rescaled coordinates we arrive at

$$\langle\psi(x_1)\psi^*(x_2)\rangle \sim \frac{\mathrm{Ai}(x_1)\,\mathrm{Ai}'(x_2)-\mathrm{Ai}(x_2)\,\mathrm{Ai}'(x_1)}{x_1-x_2}.$$

Let us expand this formula in the limit where, while close to the endpoint, the distance between the two points is very small, so $x_1 = \bar{x}+x$, $x_2 = \bar{x}-x$, where x is the small distance. Doing a saddle-point expansion of the Airy integral, we pick up the contribution of the two saddle points $y = \pm\sqrt{\bar{x}}$. Substituting, we arrive at the sine-kernel formula

$$\langle\psi(x_1)\psi^*(x_2)\rangle \sim \frac{\sin x\sqrt{\bar{x}}}{x}.$$

This is consistent with the previous expression for the sine-kernel, since close to the endpoint, $y^2+x=0$, the density scales as $\rho(\bar{x}) \sim \sqrt{\bar{x}}$. Doing the saddle-point expansion of the Fourier transform, and picking up the contributions from the two saddle points, we automatically introduced an averaging in the double valued x-coordinates.

We could also consider the more general geometry

$$y^m + x = 0.$$

In this case the wave functions in the y-patch are

$$\Psi(y) = \exp\left(\frac{iy^{m+1}}{g_s(m+1)}\right).$$

Fourier transforming in the x-patch, this gives

$$\Psi(x) = \int dy\, \exp\frac{1}{g_s}\left(ixy + i\frac{y^{m+1}}{(m+1)}\right).$$

The two-point correlator and its scaling limits can be derived similarly.

7.5 CORRELATORS IN CRYSTAL MELTING

In [131, 132] a connection was made between the statistical mechanics picture of crystal melting and A-model topological strings on Calabi-Yau manifolds. The crystal describes the toric base of the Calabi-Yau, with lattice spacing of order g_s. The

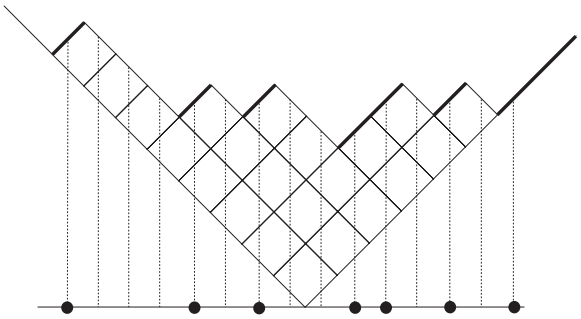

Figure 7.3: Wedge diagram of the partition $(8, 5, 4, 2, 2, 1)$. (From [134].)

temperature of the melting crystal is $1/g_s$. At very large scales the Calabi-Yau is described in terms of the classical geometry. Decreasing the distance scale to the string scales, the geometry takes the form of a smooth limit shape. At even smaller distances a "gravitational foam" picture emerges.

The structure of the crystal can be probed by putting probe branes in the geometry. The effect of adding a single brane probe was first studied in [132], where the connection to the partition function was discussed.

Addition of multiple branes and antibranes was considered in [133]. In this paper, a statistical interpretation of non-compact probe branes as defects in the Calabi-Yau crystal is given, and correlators for inserting branes are derived. These results are in accordance to that non-compact brane probes can be thought of as adding free fermions with a certain geometric transformation law [6], at the B-model side.

Similarly as for the random matrices, we will investigate certain correlators by inserting branes in the geometry. The correlators we are interested in are the 3D crystal analogues of the eigenvalue correlators of random matrices.

7.5.1 PARTITIONS ON THE PLANE

Let us first consider partitions on the plane. A partition is a nonincreasing sequence

$$\mu = \{\mu_1 \geq \mu_2 \geq \mu_3 \geq \ldots \geq 0\}$$

of non-negative integers such that $\mu_i = 0$ for all sufficiently large i. A plane or 2D partition is a matrix of non-negative integers with nonincreasing rows and columns and finitely many nonzero entries. A partition is best understood in terms of its diagram, like in fig.7.3. This diagram corresponds to the partition $(8, 5, 4, 2, 2, 1)$.

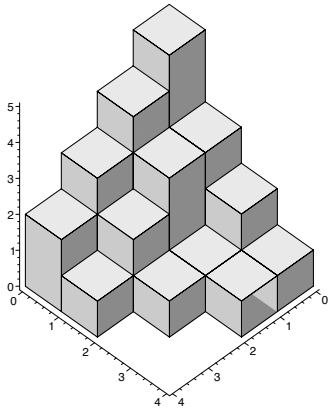

Figure 7.4: 3D diagram of the plane partition π Eq.7.4. (From [134].)

The diagram of a plane partition looks like a number of cubes stacked in a corner, fig.7.4. The corresponding partition is given by

$$\pi = \begin{pmatrix} 5 & 3 & 2 & 1 \\ 4 & 3 & 1 & 1 \\ 3 & 2 & 1 & \\ 2 & 1 & & \end{pmatrix}. \tag{7.4}$$

If we draw the diagram of a partition of N, and rescale it with $\sqrt{N}$, a smooth limit curve of the diagram emerges [134, 137], fig.7.5. This is the two-dimensional analogue of the limit shape of the melting crystal. In general, the boundary is a graph of a piecewise linear function, consisting of jumps up and down. Correlators in this context measure the probability of a given pattern of up and down jumps. In fact it is enough to consider the probabilities to measure the set of down jumps, since this already gives the complete information about the shape of the partition. When scaling N large, so that the elements in the set of down jumps x_i are

$$\frac{x_i}{\sqrt{N}} \in [-2, 2],$$

such probabilities are given by the determinant of a single discrete sine-kernel [134, 137]

$$K_{\sin}(x_i, x_j) = \frac{\sin\left(\varphi_i(x_i - x_j)\right)}{\pi(x_i - x_j)}, \quad \varphi_i = \arccos\left(\frac{x_i}{2\sqrt{N}}\right).$$

This is the same sine-kernel as for random matrices emerges. The important difference however is that in this case x_i are discrete, while in the random matrix case we had a continuous distribution. Also the scaling factor φ is different from the random

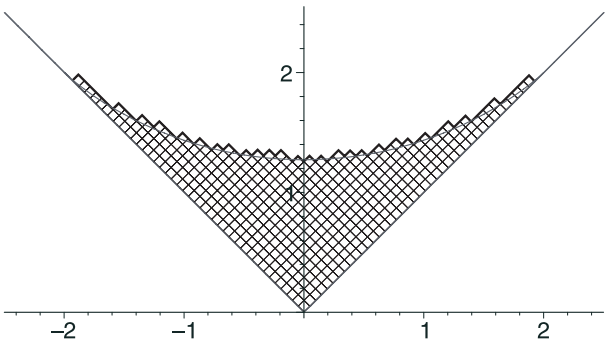

Figure 7.5: Smooth limit curve of the plane partition. (From [134].)

matrix case where we had a scaling with the density, but such scalings can always be absorbed in the redefinition of the variables.

The endpoint formula for the random matrix correlator in terms of the Airy function also has a counterpart in random partitions.

7.5.2 3D PARTITIONS

We will now study similar correlators in 3D partitions. In the 3D generalization we can build a partition from a sequence of diagonal slices, which satisfy the interlacing condition [132, 131, 134] which we will state in the next section.

Another way of looking at the partition is viewing it from the top, and projecting it to a two-dimensional tiling pattern, fig.7.6. If the partition is located in the positive corner of (x, y, z) plane, the position of plaquettes in the tiling pattern is given in the discrete coordinates

$$\begin{aligned} t &= y - x, \\ h &= z - \frac{1}{2}(y + x). \end{aligned}$$

One can then ask for the probability of measuring a set of fixed plaquettes, at positions $\{(t_1, h_1) \dots (t_n, h_n)\}$. These correlators are given as the determinant of a 3D kernel [138]

$$\begin{aligned} &K_{3D}((t_i, h_i), (t_j, h_j)) = \\ &= \frac{1}{(2\pi \mathrm{i})^2} \int_{|s|=1\pm\epsilon} \int_{|w|=1\mp\epsilon} \mathrm{d}s\, \mathrm{d}w\, \frac{\Phi_{3D}(t_1, s)\Phi_{3D}^{-1}(t_2, w)}{s - w} \frac{1}{s^{h_i + \frac{|t_i|}{2} + \frac{1}{2}} w^{-h_j - \frac{|t_j|}{2} + \frac{1}{2}}}. \end{aligned}$$

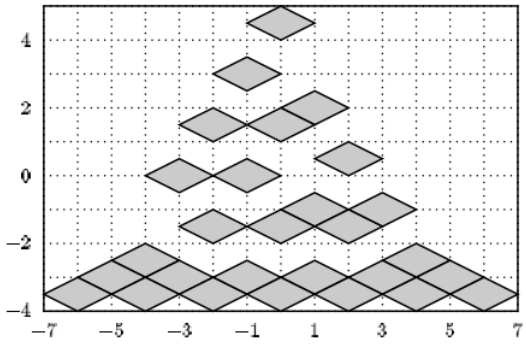

Figure 7.6: A 3D partition viewed from the top. (From [134].)

Here in the contour integral the top/bottom $\pm$ signs are valid when $t_1 \geq t_2$ and and $t_1 < t_2$ respectively, $\Phi_{3D}^{-1} = 1/\Phi_{3D}$, and

$$\Phi_{3D}(s,t) = \frac{\prod_{m>\max(0,-t)} (1 - q^m/s)}{\prod_{m>\max(0,t)} (1 - q^m s)}, \quad m \in \mathbb{Z} + \frac{1}{2}.$$

This kernel measures the probability of finding two fixed plaquettes in the random tiling pattern. In fact, this two-point correlator can be thought of as a non-compact brane and anti-brane probe insertion in the geometry.

The A-model geometry is related by mirror symmetry to B-model on the mirror Calabi-Yau [6]

$$zw - \mathrm{e}^{-u} - \mathrm{e}^{v} + 1 = 0.$$

Inserting a single brane, the wave function is an eigenfunction of the Hamiltonian

$$\begin{aligned} H(u,v)\, L(u,q) \;&=\; 0, \\ H(u,v) \;&=\; q^{-1/2}\mathrm{e}^{-u} + \mathrm{e}^{-g_s \partial_u} - 1, \quad q = \mathrm{e}^{-g_s}. \end{aligned}$$

The one-point function which satisfies this equation is the quantum dilogarithm

$$L(u,q) = \prod_{n=0}^{\infty} (1 - \mathrm{e}^{-u} q^{n+\frac{1}{2}}). \tag{7.5}$$

Similarly, the one-point function for an anti-brane is $L^*(u) = 1/L(u)$.

Let us introduce the new variables

$$\begin{aligned} s &= \mathrm{e}^{u}, \\ w &= \mathrm{e}^{\tilde{u}}, \\ \tilde{h}_i &= h_i - \frac{1}{2} + \frac{|t_i|}{2}, \\ \tilde{\tilde{h}}_j &= h_j + \frac{1}{2} + \frac{|t_j|}{2}. \end{aligned}$$

In terms of these variables, the kernel can be rewritten as

$$K_{3D}(\tilde{h}_i, \tilde{\tilde{h}}_j) = \frac{1}{(2\pi i)^2} \int \mathrm{d}u\, \mathrm{d}\tilde{u}\, \frac{1}{\mathrm{e}^{u} - \mathrm{e}^{\tilde{u}}} \mathrm{e}^{-u\tilde{h}_i}\, \mathrm{e}^{\tilde{u}\tilde{\tilde{h}}_j}\, \Phi_{3D}(u,t)\, \Phi_{3D}^{-1}(\tilde{u},t).$$

This is essentially the same form as the two-point free-fermion function of a brane and anti-brane considered in [6], now Fourier-transformed in the variables $(\tilde{h}_i, \tilde{\tilde{h}}_j)$. Here Φ_{3D} and Φ_{3D}^{-1} are the brane and anti-brane one-point functions,

$$\begin{aligned} \Phi_{3D}(u,t) &= \begin{cases} \frac{L(u,q)}{L(-u+tg_s,q)} & t \geq 0, \\ \frac{L(u-tg_s,q)}{L(-u,q)} & t < 0 \end{cases} \\ \Phi_{3D}^{-1}(\tilde{u},t) &= \frac{1}{\Phi_{3D}(u,t)}. \end{aligned}$$

The formula shows that shifting the coordinate t shifts us to another slice of the crystal, parametrized by t. In particular the wave function at the slice $t = 0$ is

$$\Phi_{3D}(u,0) = \frac{L(u,q)}{L(-u,q)}.$$

7.5.3 BRANE PICTURE

It is a natural question to ask what the precise brane configuration is that reproduces the 3D correlators, and how these branes are inserted in the geometry. To understand this, we consider the picture of [133] where Lagrangian branes were described as defects in the crystal.

Consider now the 3D crystal constructed from a sequence of interlacing 2D partitions, $\mu(t)$. A single 2D partition is a nonincreasing sequence of non-negative integers $\mu = \{\mu_1, \mu_2, \ldots\}$. To build a 3D partition, such sequences are labeled by an integer t, and must satisfy the interlacing condition

$$\begin{aligned} &\mu(t) < \mu(t+1), \quad t < 0, \\ &\mu(t+1) < \mu(t), \quad t \geq 0, \end{aligned}$$

where $\mu(t+1) > \mu(t)$, if

$$\mu_1(t+1) \geq \mu_1(t) \geq \mu_2(t+1) \geq \mu_2(t) \dots .$$

A single two-dimensional slice can be described as a state in the fermionic Fock-space. By introducing the set of down jumps of the tableaux and its transpose

$$\begin{aligned} a_i &= \mu_i - i + \frac{1}{2}, \\ b_i &= \mu_i^T - i + \frac{1}{2}, \end{aligned}$$

the fermionic state describing the tableaux consisting of d boxes is given by

$$|\mu\rangle = \prod_{i=1}^{d} \psi^*_{-a_i} \psi_{-b_i}.$$

Using bosonization, we introduce

$$q^{L_0} \, |\mu\rangle = q^{|\mu|} \, |\mu\rangle \, ,$$

where $|\mu|$ denotes the number of boxes, and the creation/annihilation operators are given by

$$\Gamma_{\pm}(z) = \mathrm{e}^{\varphi_{\pm}(z)}.$$

Here $\varphi_{\pm}(z)$ are the positive and negative mode part of the chiral boson $\varphi(z)$, which is related to the complex fermion by bosonization

$$\psi(z) =: \mathrm{e}^{\varphi(z)} : .$$

In this language, introducing a D-brane corresponds to the insertion of a fermion. Discarding the zero mode part, the D-brane operator can be written as

$$\Psi_D(z) = \Gamma_-^{-1}(z)\Gamma_+(z).$$

When one wants to define a quantum theory on a non-compact D-brane, one needs to specify boundary conditions at infinity. This is called the "framing" of the D-brane [139]. We use here and throughout the chapter the standard framing $p = 0$.

The partition function of the crystal with the operators can be written as

$$Z(q) = \langle 0| \prod_{n=1}^{\infty} \Gamma_+(q^{n-\frac{1}{2}}) \prod_{m=1}^{\infty} \Gamma_-(q^{-m+\frac{1}{2}}) \, |0\rangle \, ,$$

where $q = \mathrm{e}^{-g_s}$. With the commutation relation

$$\Gamma_+(z)\,\Gamma_-(z') = (1 - z/z')^{-1}\Gamma_-(z')\,\Gamma_+(z),$$

it is straightforward to show that the partition function is the McMahon function

$$Z(q) = \prod_{n=1}^{\infty} (1-q^n)^{-n} = M(q). \tag{7.6}$$

In [133], Lagrangian D-brane insertions were considered. These branes end on the axis, and have the geometry

$$y = x + a = z + a, \quad a > 0,$$

where we chose the brane to end at $y = a$. In [133] it was shown that introducing such a Lagrangian brane at a position $a = g_s(N_0 + \frac{1}{2})$, corresponds to inserting a fermion D-brane operator $\Psi_D(q^{-(N_0+\frac{1}{2})})$ at the slice $t = N_0 + 1$,

$$\begin{aligned} Z_D(q, N_0) &= \langle 0| \prod_{n=1}^{\infty} \Gamma_+(q^{n-\frac{1}{2}}) \times \\ &\quad \prod_{m=1}^{N_0+1} \Gamma_-(q^{-m+\frac{1}{2}})\, \Psi_D(q^{-(N_0+\frac{1}{2})}) \prod_{m=N_0+2}^{\infty} \Gamma_-(q^{-m+\frac{1}{2}})\ |0\rangle\,. \end{aligned}$$

Commuting through the operators gives the brane one-point function

$$\begin{aligned} Z_D(q, N_0) &= M(q)\, \xi(q) \prod_{n=1}^{\infty} \left(1 - \mathrm{e}^{-g_s(N_0+\frac{1}{2})} q^{n-\frac{1}{2}}\right) \\ &= M(q)\, \xi(q)\, L(g_s(N_0 + \frac{1}{2}), q), \end{aligned}$$

where we have expressed the wave function in terms of the overall McMahon function $M(q)$ Eq.7.6 and the quantum dilogarithm $L(u, q)$ Eq.7.5. Here $\xi(q)$ is a normalization factor with respect to the string theory answer, so that $Z_{\text{crystal}} = \xi(q)\, Z_{\text{string}}$. Then we have

$$\xi(q) = \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

These brane insertions depend on the single parameter N_0, expressing the fact that the branes correspond to the Lagrangian geometry. Comparing with the brane insertions for the 3D correlators in the previous section, we see that our brane insertions are of a more general kind, since they depend on the two independent parameters t and u. In fact, they correspond to the insertion of a D-brane operator $\Psi_D(z)$ at an arbitrary slice t.

Let us insert a fermionic operator $\Psi_D(q^{-(N+\frac{1}{2})})$ at the slice $t = N_0 + 1$. Note that N

and N_0 are now independent. This is given by the operator expression

$$Z_D(q,N,N_0) \;=\; \langle 0| \prod_{n=1}^{\infty} \Gamma_+(q^{n-\frac{1}{2}}) \;\times \\ \prod_{m=1}^{N_0+1} \Gamma_-(q^{-m+\frac{1}{2}})\, \Psi_D(q^{-(N+\frac{1}{2})}) \prod_{m=N_0+2}^{\infty} \Gamma_-(q^{-m+\frac{1}{2}}) \;|0\rangle\,.$$

Substituting $\Psi_D(z) = \Gamma_-^{-1}(z)\Gamma_+(z)$ we obtain

$$Z_D(q,N,N_0) = \langle 0| \prod_{n=1}^{\infty} \Gamma_+(q^{n-\frac{1}{2}}) \;\times \\ \prod_{m=1}^{N_0+1} \Gamma_-(q^{-m+\frac{1}{2}})\, \Gamma_-^{-1}(q^{-(N+\frac{1}{2})})\, \Gamma_+(q^{-(N+\frac{1}{2})}) \prod_{m=N_0+2}^{\infty} \Gamma_-(q^{-m+\frac{1}{2}}) \;|0\rangle\,.$$

Commuting first the $\Gamma_+(q^{-(N+\frac{1}{2})})$ to the right gives a factor

$$\prod_{m=N_0+2}^{\infty} (1-q^{m-N-1})^{-1} = \frac{1}{L(-g_s(N+\frac{1}{2})+(N_0+1)g_s,q)}.$$

The leftover $\Gamma_-^{-1}(q^{-(N+\frac{1}{2})})$ cancels then the Nth Γ_- in the sequence. The remainder is then the McMahon function Eq.7.6, up to an extra factor making up for the missing Γ_- at the place N. This extra factor is

$$\prod_{n=1}^{\infty}(1-q^{n+N}) = L(g_s(N+\frac{1}{2}),q).$$

Thus

$$Z_D(q,N,N_0) = M(q)\,\frac{L(g_s(N+\frac{1}{2}),q)}{L(-g_s(N+\frac{1}{2})+(N_0+1)g_s,q)}.$$

Comparing with the wave functions appearing in the 3D correlators we obtain

$$Z_D(q,N,N_0) = M(q)\;\Phi_{3D}(g_s(N+\frac{1}{2}),N_0+1).$$

Since we have started with a fermionic insertion at $u = g_s(N+1/2)$ and at the slice $t = N_0+1$, there is a precise agreement. The branes inserted in the 3D correlator with wave function $\Phi_{3D}(u,t)$ correspond to the fermionic insertions $\Psi(\mathrm{e}^{g_s u})$ at the slice t. Note that we have the choice to insert the fermion at a slice $t \geq 0$, or at $t < 0$. Inserting at $t < 0$ precisely reproduces $\Phi_{3D}(u,t)$ for $t < 0$. The branes are generalized branes, in the sense that they depend on the two parameters u and t.

It is interesting to note that the Fourier transform of the wave function of the generalized branes $\Phi_{3D}(u,t)$ is the Wigner function. Let us take $t > 0$. Using that the

generalized brane wave function is a product of a brane and a shifted antibrane wave function, the Fourier transform gives

$$W(g_s t, y) = \int \mathrm{d}u\, L(u,q)\, L^*(-u + g_s t, q)\, \mathrm{e}^{-\mathrm{i}uy},$$

where we recognize the Wigner function.

7.5.4 UNIVERSALITY FOR 3D PARTITIONS

It is interesting to observe that a universal scaling behaviour also appears in the case of the correlators of the 3D random partitions. In the same scaling limit as in the two-dimensional case, the 3D kernel reduces to the incomplete beta-kernel [134]

$$K_\beta(t,h) = \frac{1}{2\pi i}\int_{\bar{\eta}}^{\eta} \mathrm{d}z\, (1-z)^t \frac{1}{z^{h+t/2+1}}.$$

where the distances kept fixed are $t = t_i - t_j$ and $h = h_i - h_j$, and η is a density scaling parameter. This reduces to the sine-kernel on two different slices. If we take $t = 0$, or $x = y$ in the original coordinates,

$$K_\beta(h) = \frac{\sin h\varphi}{\pi h},$$

we obtain the sine-kernel in h. Taking $h + t/2 + 1 = 0$, or $x - z = 1$, and changing variables $z \to (1 - z)$

$$K_\beta(t, \widetilde{\phi}) = \frac{\sin \widetilde{\phi}(t+1)}{\pi(t+1)},$$

the kernel reduces to a sine-kernel in $t' = t+1$. Here $\widetilde{\phi}$ is a redefined scaling variable. This means the kernel is likely connected to a two-matrix model. Taking a Gaussian two-matrix model with

$$\int \mathrm{d}X\, \mathrm{d}Y\, \mathrm{e}^{-\frac{1}{g_s}(\frac{1}{2}X^2 + \frac{1}{2}Y^2 + V(X,Y))},$$

the universal kernel has the form of $K(x_i - x_j, y_i - y_j) = K(x,y)$. Reducing to $y = 0$ or $x = 0$ we have

$$\begin{aligned} K(x,0) &\sim \frac{\sin \pi\rho x}{\pi x}, \\ K(0,y) &\sim \frac{\sin \pi\rho y}{\pi y}. \end{aligned}$$

Thus reducing to the two slices would correspond to restricting to the eigenvalues of only X or only Y in the two-matrix model. It is clear that the corresponding two-matrix model must have an interaction term, otherwise the full correlator would be

just a product of the two single-matrix (or single-slice) correlators. This is apparently not the case for the beta-kernel.

The two-matrix model in question is most likely the one related to the Chern-Simons partition function [135] and the topological vertex [140].

To understand the full kernel, we need the four-point function in this two-matrix model, and its scaling limit. Such kernels have been considered in the context of Brownian motion [141] for the usual two-matrix models. Using the context of Brownian motion [142] could lead to a more precise identification.

D-brane insertions have also been related to black holes. Four-dimensional BPS black holes arise in compactifications of strings on Calabi-Yau three-folds and counting the BPS microstates lead to topological string amplitudes [143] with the relation

$$Z_{\rm bh} = |Z_{\rm top}|^2,$$

where $Z_{\rm bh}$ is the black hole partition function and $Z_{\rm top}$ is the topological string partition function. $Z_{\rm top}$ is evaluated at a point in the moduli space, associated to the black hole charges. Then it is shown in [144] that, for a large black hole charge N, the black hole partition function is given by a sum of chiral blocks where each block corresponds to a D-brane amplitude. The leading block is associated to the closed string amplitudes. They claim that these give a nonperturbative description for topological strings for finite N.

With the relation in [143], the black hole entropy was calculated in [145] by using matrix models. They conclude that the black hole entropy at zero temperature is equivalent to the matrix model free energy.

APPENDIX A

UNIVERSAL KERNEL FROM HERMITE POLYNOMIALS

The universal kernel and its endpoint form can also be computed using orthogonal polynomials

$$\int \mathrm{d}\lambda\, \mathrm{e}^{-\frac{V(\lambda)}{g_s}} P_n(\lambda)\, P_m(\lambda) = \delta_{nm}.$$

Defining the wave functions as

$$\Psi_n(\lambda_i) = P_n(\lambda_i)\, \mathrm{e}^{-V(\lambda) 2 g_s},$$

and using the Darboux-Christoffel formula, the kernel (up to a constant) can be expressed as, Eq.7.1,

$$K(\lambda_i,\lambda_j)=\frac{1}{N}\sum_{n=0}^{N-1}\Psi_n(\lambda_i)\,\Psi_n(\lambda_j)=\frac{\Psi_N(\lambda_1)\,\Psi_{N-1}(\lambda_2)-\Psi_{N-1}(\lambda_1)\,\Psi_N(\lambda_2)}{\lambda_1-\lambda_2}.$$

For the Gaussian potential we have

$$\Psi_n(x)=\frac{1}{(2g_s\pi)^{\frac{1}{4}}2^{\frac{n}{2}}\sqrt{n!}}\,H_n\left(\frac{x}{\sqrt{2g_s}}\right)\mathrm{e}^{-\frac{x^2}{4g_s}},$$

where H_n are the Hermite polynomials. These are the wave functions of the harmonic oscillator with $\hbar=2g_s$. Using $H_n'(x)=2nH_{n-1}(x)$ we can rewrite Eq.7.1 as

$$\kappa(\lambda_i,\lambda_j)=\frac{A_NA_{N-1}}{2N}\mathrm{e}^{-\frac{\lambda_1^2}{4g_s}-\frac{\lambda_2^2}{4g_s}}\left(\frac{H_N(\frac{\lambda_1}{\sqrt{2g_s}})\,H_N'(\frac{\lambda_2}{\sqrt{2g_s}})-H_N(\frac{\lambda_2}{\sqrt{2g_s}})\,H_N'(\frac{\lambda_1}{\sqrt{2g_s}})}{\lambda_1-\lambda_2}\right) \tag{7.7}$$

where A_n is the normalization factor of the Gaussian wave functions

$$A_n=\frac{1}{(2g_s\pi)^{\frac{1}{4}}2^{\frac{n}{2}}\sqrt{n!}}.$$

We need $\kappa(\lambda_i,\lambda_j)$ in the limit $N\to\infty$, $g_s\to 0$, $S=g_sN$ fixed and large. To find the asymptotic expansion of the Hermite polynomials, it is useful to rewrite them in terms of parabolic cylinder functions $U(a,x)$ as

$$H_n(x)=2^{n/2}\,\mathrm{e}^{\frac{1}{2}x^2}\,U(-n-\frac{1}{2},\sqrt{2}x).$$

The asymptotic expansion of the parabolic cylinder function $U(a,x)$ is given in terms of the quantity $Y=\sqrt{4|a|-x^2}$. Since $a=-N-1/2$, we expand in the region where a is large, negative. In this region the argument $\lambda_i/\sqrt{g_s}\sim\sqrt{\frac{N}{S}}\lambda_i\sim\sqrt{N}$, so x is moderately large. Using the asymptotic expansion in this regime, and neglecting lower order terms in the $1/N$ expansion, we find

$$U(-N-1/2,x)=\frac{2\sqrt{N!}}{(2\pi)^{1/4}}\frac{1}{Y(x)}\,\cos\left(\frac{1}{2}\int_0^x Y(x')dx'-\frac{N\pi}{2}\right).$$

Substituting in Eq.7.7, and expanding when $\lambda_i-\lambda_j\sim O(1/N)$ we arrive at

$$K(\lambda_i,\lambda_j)\sim\frac{\sin N\pi\rho(\bar\lambda)(\lambda_i-\lambda_j)}{\pi(\lambda_i-\lambda_j)},$$

where $\bar\lambda=(\lambda_i+\lambda_j)/2$, and $\rho(\bar\lambda)$ is the normalized eigenvalue density

$$\rho(\bar\lambda)=\frac{1}{2S\pi}\sqrt{4S-\bar\lambda}.$$

The overall normalization factor can be fixed from the definitions, using that $\kappa(\lambda,\lambda) = \rho(\lambda)$. Finally

$$\kappa(\lambda_i,\lambda_j) = \frac{\sin N\pi\rho(\bar{\lambda})(\lambda_i-\lambda_j)}{N\pi(\lambda_i-\lambda_j)},$$

in agreement with Eq.7.2.

ENDPOINT ASYMPTOTICS

When scaling close to one of the branch points, $\lambda \to \pm 2\sqrt{S}$, the kernel has a different asymptotic expansion in terms of Airy functions. Let us scale to the positive branch point $2\sqrt{S}$. At large a the parabolic cylinder functions can be expressed in terms of Airy functions as

$$U(a,x) \sim 2^{-\frac{1}{4}-\frac{1}{2}a}\Gamma\left(\frac{1}{4}-\frac{1}{2}a\right)\left(\frac{t}{\xi^2-1}\right)^{\frac{1}{4}}\mathrm{Ai}(t),$$

where

$$x = 2\sqrt{|a|}\xi, \quad t = (4|a|)^{2/3}\tau,$$

and

$$\tau = \left(\frac{3}{8}(\xi\sqrt{\xi^2-1}-\cosh^{-1}\xi)\right)^{\frac{2}{3}}.$$

We need the scaling limit when $x \to 2\sqrt{|a|}$, that is when $\xi \to 1$. In this limit,

$$U(a,x) \sim 2^{-\frac{1}{4}-\frac{1}{2}a}\Gamma\left(\frac{1}{4}-\frac{1}{2}a\right)|a|^{\frac{1}{6}}\,\mathrm{Ai}(t),$$

where $t = 2|a|^{\frac{2}{3}}(\xi-1)$ measures the distance from the branch point. Close to the branch point then the Hermite polynomials have the asymptotics

$$H_N(x) \sim 2^{\frac{N}{2}}\mathrm{e}^{\frac{x^2}{2}}\left(N+\frac{1}{2}\right)^{\frac{1}{6}}\Gamma\left(\frac{1+N}{2}\right)\mathrm{Ai}(u),$$

where

$$u = \left(N+\frac{1}{2}\right)^{\frac{1}{6}}\left(x-\sqrt{4N+2}\right).$$

Substituting in Eq.7.7 we find

$$K(u_1,u_2) \sim \frac{\mathrm{Ai}(u_1)\,\mathrm{Ai}'(u_2)-\mathrm{Ai}(u_2)\,\mathrm{Ai}'(u_1)}{u_1-u_2}.$$

Here we rescaled u by a $1/\sqrt{g_s}$ in the substitution.

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SAMENVATTING

Random matrices worden veel gebruikt in vakgebieden waar men te kampen heeft met enorme hoeveelheden data waardoor de verwerking daarvan een moeizaam proces wordt. Het grootste voordeel van random matrices is dat het helemaal niet nodig blijkt te zijn om alle data afzonderlijk te verwerken, maar dat men kan volstaan met een benadering door gebruik te maken van zeer grote matrices, en wel *random* matrices. Het is inderdaad wonderbaarlijk dat men hiermee correcte resultaten kan vinden, maar het is keer op keer gebleken, gelukkig, dat de methode werkt.

In de natuurkunde werden random matrices voor het eerst gebruikt door Wigner in 1951. Hij hield zich bezig met het probleem dat, in de kernfysica, het zeer moeilijk is om de de energieniveau's van zware atoomkernen exact te bepalen omdat ze zeer dicht op elkaar zitten. De oplossing is dan om de data statistisch te benaderen en te kijken naar de gemiddelde waarden. Wigner vond dat de gemiddelde verdeling van de energieniveau's voor zware atomen gelijk is aan de gemiddelde verdeling van de eigenwaardeniveau's van een random matrix. Dit werd later bevestigd door verschillende experimenten.

Er zijn vele toepassingen van random matrices en het is onmogelijk om ze hier allemaal op te noemen. Een van de meest relevante toepassingen in de context van dit proefschrift is in de quantumchromodynamica (QCD). QCD beschrijft de sterke wisselwerking tussen quarks en gluonen in het binnenste van deeltjes als protonen en neutronen. De gluonen zijn "lijm-deeltjes" die de quarks bij elkaar houden. In de natuur komen de quarks namelijk nooit vrij voor, maar altijd in *confinement*, ze zijn als het ware opgesloten. In QCD speelt het aantal "kleuren" van de quarks een belangrijke rol. De "kleur" wordt gebruikt om intrinsieke eigenschappen van deze deeltjes te benoemen. De quarks maken altijd zulke combinaties dat het geheel geen "kleur" heeft.

In 1973 heeft 't Hooft laten zien dat de beschrijving van de quark-interacties veel simpeler wordt wanneer men in de perturbatieve expansie een limiet neemt waarbij

het aantal kleuren naar oneindig gaat, terwijl de koppelingsconstante van de theorie naar nul gaat, en hun produkt constant blijft. Dit wordt de 't Hooft-limiet genoemd. De perturbatieve expansie stelt een oneindige reeks voor, in toenemende machten van de genoemde koppelingsconstante. In deze limiet bleek het mogelijk om de Feynman-diagrammen, die de interacties weergeven, op zo'n manier te tekenen dat er alleen maar "planaire" diagrammen overblijven. Dat zijn diagrammen die op een plat vlak kunnen worden getekend zonder dat ze zichzelf kruisen. In plaats van de enkelvoudige lijnen in een Feynman-diagram, gebruikt men nu dubbele lijnen met een bepaalde oriëntatie, aangegeven door pijltjes. Door deze georiënteerde lijnen te markeren met indices, kan men dit alles ook opschrijven in termen van matrices. Op deze manier kan de partitiefunctie van de theorie, die de informatie, of het structuur, van de theorie bevat, geschreven worden als een matrix-integraal met random matrices. Het gaat hier om Hermitische, en niet om reële matrices, zodat het mogelijk blijft om de oriëntatie van de dubbele lijnen te onderscheiden. Voor simpele matrix modellen, bijvoorbeeld een Gaussisch model dat kwadratisch is in de matrices, kan de matrix-integraal exact worden opgelost in termen van de constanten van het model en kan de partitiefunctie uitgerekend worden. Deze "vette" Feynmandiagrammen kunnen ook worden beschreven door het tellen van het aantal vertices, zijden en lussen in het diagram en die op de bijbehorende manier te combineren. Hiermee wordt er ook een link gemaakt naar een geometrische beschrijving, namelijk naar Riemann-oppervlakken met gaten.

Deze beschrijving is ook gerelateerd aan de snaartheorie. De snaartheorie is een theorie met een rijke structuur die uitgaat van het idee dat het universum is obgebouwd uit zeer kleine objecten, namelijk "snaren". Deze theorie heeft als belangrijkste resultaat dat zij de quantumveldentheorie (QFT) en de algemene relativiteitstheorie (ART) combineert. QFT beschrijft, met succes, de kleine deeltjes in het universum en hun wisselwerking onderling. Aan de andere kant van de meetlat beschrijft ART de werking van de zwaartekracht op grote schaal. Het is een fundamenteel probleem in de hoge-energie fysica dat deze afzonderlijk succesvolle theorieën niet met elkaar zijn te verenigen en dat er tot zover bekend nog geen complete theorie van quantum-zwaartekracht is. De snaartheorie, die een theorie-in-wording en dus nog verre van compleet is, doet hier een poging toe met toenemende succes.

De bovengenoemde matrix-integraal is equivalent aan een nul-dimensionale veldentheorie. In de snaartheorie vindt men dezelfde matrix-integraal, geschreven in termen van de eigenwaarden van de matrix, wanneer men kijkt naar een supersymmetrische ijktheorie met een superpotentiaal en deze superpotentiaal minimaliseert ten opzichte van de supervelden van deze ijktheorie. In deze vergelijking blijkt de superpotentiaal in de snaartheorie gelijk te zijn aan de potentiaal van het matrix model en de rangorde van de supersymmetrische ijkgroep is dan gelijk aan de grootte van de matrix. Men beschouwt dan weer de 't Hooft-limiet van de perturbatieve

expansie met alleen planaire diagrammen als gevolg.

Ook wanneer we vanuit een ander oogpunt kijken komen we deze matrix-integraal tegen in de snaartheorie. De snaartheorie heeft namelijk ook een geometrische beschrijving, die niet afhangt van de metriek, dat wil zeggen de intrinsieke details, van de achtergrondruimte. Deze theorie wordt de topologische snaartheorie genoemd en een bepaald topologisch snaartheorie-model, namelijk het topologische B-model op een Calabi-Yau variëteit, reduceert tot een random matrix model. In deze vergelijking wordt de matrix-model potentiaal weergegeven door de polynoom in de geometrie van het topologisch model. Daarnaast is de som van de eigenwaarden van de matrix gelijk aan de som over de D-branen in het topologische model. D-branen zijn twee-dimensionale objecten in de snaartheorie. Ook hier wordt de 't Hooft-limiet in beschouwing genomen.

De relatie tussen matrix modellen en supersymmetrische ijktheorieën aan de ene kant, en tussen matrix modellen en de topologische snarentheorie aan de andere kant werd voor het eerst geformuleerd door Dijkgraaf en Vafa in 2002.

Tot nu toe hadden we het alleen over de planaire Feynman-diagrammen. Wanneer men niet alleen naar de leidende term in de perturbatieve expansie kijkt, maar ook de volgende term in beschouwing neemt, komt dit overeen met een bijdrage aan de zwaartekrachtsterm in de snaartheorie. Door deze term uit te rekenen in èn de snaartheorie, èn door gebruik te maken van de vergelijking met de matrix modellen, kan men laten zien dat de resultaten exact overeen komen. Dit is dan een bevestiging van de formulering van Dijkgraaf en Vafa.

Een ander thema in dit proefschrift is de universaliteit van de matrix-model kernel. Dat houdt in dat deze kernel, een functie waarvan de determinant gelijk is aan de eigenwaardedichtheid van het matrix model, in de limiet waarbij de grootte van de matrix naar oneindig gaat, niet afhangt van de details van de matrix-model potentiaal. Voor iedere potentiaal neemt de kernel in deze limiet dus dezelfde vorm aan. Deze limiet komt overeen met een korte-afstandsschaal en met deze kernel kan men het korte-afstand-gedrag van systemen bestuderen.

Dezelfde kernel komt ook voor in de theorie van partities. Een partitie is een rij van positieve gehele getallen waarvan de waarde niet groter wordt wanneer men deze rij van voor naar achter doorloopt. Zo'n partitie wordt weergegeven door een twee-dimensionaal diagram. Wanneer het aantal getallen in de partitie groot wordt, vindt men de bovengenoemde kernel. In de snaartheorie blijkt dit equivalent te zijn aan het plaatsen van D-branen in de achterliggende geometrie.

In de literatuur is het plaatsen van D-branen ook gerelateerd aan zwarte gaten. Het bleek dat het mogelijk is om de entropie van zwarte gaten te berekenen door een matrix model te gebruiken.

Dit is maar een kleine greep uit de vele toepassingen van random matrices. Er valt nog zeker veel te leren en te voorspellen door het bestuderen van random matrices.

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CURRICULUM VITAE

Mine Temürhan was born in Istanbul, Turkey on December 15th, 1975. Moving to the Netherlands in 1991 with her mother and sister, she went to college and studied physics at the Free University Amsterdam. She graduated in theoretical physics in 2000 under supervision of prof. Piet Mulders and prof. Robbert Dijkgraaf. Her master's thesis was on holography, a subject in string theory.

In 2001 she started her Ph.D. at the Institute for Theoretical Physics, part of the University of Amsterdam, with prof. Robbert Dijkgraaf as her supervisor. During these four years she studied, among other things, the subject of random matrix models in connection to the Dijkgraaf-Vafa theory. This resulted in the papers [1, 2], in colloboration with Robbert Dijkgraaf and Annamaria Sinkovics, and in this thesis.

As part of her education she tutored several classes, including string theory, electrodynamics, gravitation and cosmology. She has also visited many Ph.D. schools, workshops and conferences in Europe.

After her Ph.D., she will continue her studies as a postdoc at the Free University Brussels in Belgium.