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Q-Form Field on a p-Brane with Codimension Two

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Abstract: This paper investigates gauge invariance in a bulk massless q-form field on a p-brane with codimension two, utilizing a general Kaluza–Klein (KK) decomposition. The KK decomposition analysis reveals four distinct KK modes: the conventional q-form, two $(q-1)$-forms and one $(q-2)$-form. These diverse modes are essential for maintaining gauge invariance. We also find eight Schrödinger-like equations for the four modes due to the two extra dimensions, and their mass spectra are closely related. The KK decomposition process gives rise to four dualities on the p-brane, originating from the inherent Hodge duality present in the bulk. Notably, these dual symmetries play a significant role in maintaining the equivalence of bulk dual fields during dimensional reduction.

Keywords: q-form field; p-brane; localization; Kaluza–Klein decomposition; Hodge duality

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1. Introduction

In extra-dimensional theories, the Kaluza–Klein (KK) theory provides the possibility of unification of electromagnetism and gravity by introducing one compact extra dimension. This idea has attracted attention and has been widely studied [1–18] since the Arkani-Hamed–Dimopoulos–Dvali (ADD) model [19] and the Randall–Sundrum (RS) model [20,21] provided new ways to solve the long-existing hierarchy problem [19,20,22–24] and the cosmology problem [25–28].

In the braneworld scenario, the KK modes of bulk fields are of particular importance [29–48]. Through a reduction mechanism, various bulk fields naturally generate a series of KK modes. The zero modes should coincide with particles in our observed 4D spacetime, while massive KK modes reveal the physics of the extra dimensions. In this work, we consider a massless q-form field in a $D = p+3$ dimensional spacetime bulk where a codimensional two brane resides, i.e., the brane has p spatial dimensions. The zero-form and one-form fields are, respectively, the well-known scalar and vector fields. The usual two-form field is the Kalb–Ramond field, which appears as the torsion of the spacetime in Einstein–Cartan theory and as a massless mode in string theory as well. Higher q-form fields ($q > 2$) only exist in high-dimensional spacetime with $D > q+1$ and represent new particles with $D > q+4$. To avoid a vacuous discussion, we do not restrict our p-brane to $p = 3$. This is plausible because, in principle, there can be compact small-scale dimensions besides the ordinary three infinite dimensions, and together they form the p-brane.

There has been some work on localization of q-form fields [35,49–59]. In order to obtain a series of KK modes of the bulk field, one can carry out a dimensional reduction with some localization mechanism. In the conventional approach, KK decomposition is performed after gauge fixing to simplify the derivation. This eventually causes a problem: it is known that a q-form field and its Hodge dual, a $(D-2-q)$-form field, are physically equivalent, while via this mechanism, only one of them is localizable. In order to eliminate this unreasonable contradiction, modifications have been proposed in some work, one of which is of particular interest: a general KK decomposition without gauge fixing, under which the Hodge duality in the bulk naturally transfers onto the brane, suggested by the authors of [60]. With this new idea, another work discussed the localization of a vector field on a codimension-two brane [61]. Here, we will adopt their approach to study the more generalized scenario of a q-form on a p-brane of codimension two. Revisiting the duality issue strongly suggests that the results are unaffected by the number of codimensions.

In more detail, we start by carrying the general KK decomposition to a bulk q-form X, yielding four distinct categories of KK modes: $X^{(n)}$, $\overline{X}^{(n)}$, $\underline{X}^{(n)}$, and $\overline{\underline{X}}^{(n)}$. Among these, $X^{(n)}$ represents the conventional q-form mode, while the remaining three, $\overline{X}^{(n)}$, $\underline{X}^{(n)}$, and $\overline{\underline{X}}^{(n)}$, belong to lower rank forms ($q-1$, $q-1$, and $q-2$, respectively), which typically exist in this new mechanism and are essential for the gauge invariance of the brane action and brane Hodge duality. By assuming orthogonality conditions and employing a technique of comparing two sets of equations, we can separate equations of the KK modes into brane parts and extra parts. For each type of extra dimensional functions, there are two Schrödinger-like equations that correspond to the two codimensions; thus, each type obtains two parts of masses from extra dimensions. In total, we have eight Schrödinger-like equations, and their mass spectra are closely related as the KK modes couple with each other. In contrast to the codimension one case, these equations are partial differential equations, and we do not analyze their solutions since our discussion maintains a focus on formalism.

Consistently, in the brane action obtained from KK reduction, three of the fields have mass terms, which are considered as obstacles to gauge invariance of the action. Fortunately, we are able to reformulate the brane action and find out that the four brane fields couple in a nice way so that under proper gauge transformations they compensate each other, which eventually make the action invariant. Despite its extra dimensional origin, this is similar to the Higgs mechanism, where, via coupling with a scalar, a massless vector gains mass without loss of gauge invariance.

By virtue of this novel mechanism, Hodge duality is also preserved under the codimension-two reduction. We will find two dual forms in the bulk of ranks q and $p+1-q$, i.e., of strength ranks $q+1$ and $p+2-q$, naturally lead to four pairs of coupled dualities between brane form fields of ranks $q-1 \sim p+2-q$, $q \sim p+1-q$, $q \sim p+1-q$, and $q+1 \sim p-q$, respectively. They are exactly the counterparts that appear in the effective brane action; therefore, the brane action equals its dual action. In this sense, the equivalent bulk fields reduce to equivalent brane fields. Moreover, the previous localizability contradiction will automatically disappear due to some relations between extra-dimensional functions.

This paper is organized as follows. In Section 2, a general KK decomposition is performed, and four types of KK modes are obtained. The gauge invariance of the brane action is discussed in Section 3. Section 4 shows the Hodge duality in the bulk and on the brane. Finally, the conclusion and a discussion are presented in Section 5.

2. A General Kaluza–Klein Decomposition

We specify the metric $ds^2 = e^{2A(y,z)}\left(\hat{g}_{\mu\nu}(x^\lambda)dx^\mu dx^\nu + dy^2 + dz^2\right)$ for our discussion, where $\hat{g}_{\mu\nu}$ is the induced metric on the (p+1)-dimensional brane world and $A(y,z)$ is the warp factor which only depends on the two extra dimensional coordinates. Consider a massless q-form field $X_{M_1\cdots M_q}$ in the bulk, and the action reads

$$\begin{aligned} S &= -\frac{1}{2(q+1)!}\int d^D x\sqrt{-g}Y^{M_1\cdots M_{q+1}}Y_{M_1\cdots M_{q+1}}, \\ &= -\frac{1}{2(q+1)!}\int d^D x\sqrt{-g}\Big[Y^{\mu_1\cdots\mu_{q+1}}Y_{\mu_1\cdots\mu_{q+1}} + (q+1)Y^{\mu_1\cdots\mu_q y}Y_{\mu_1\cdots\mu_q y} \\ &\quad + (q+1)Y^{\mu_1\cdots\mu_q z}Y_{\mu_1\cdots\mu_q z} + q(q+1)Y^{\mu_1\cdots\mu_{q-1}yz}Y_{\mu_1\cdots\mu_{q-1}yz}\Big], \end{aligned} \quad (1)$$

where $Y_{M_1\cdots M_{q+1}} = \partial_{[M_1}X_{M_2\cdots M_{q+1}]}$ is the field strength of $X_{M_1\cdots M_q}$. The equations of motions (EoMs) for the bulk field $\partial_{M_1}\left(\sqrt{-g}Y^{M_1\cdots M_{q+1}}\right) = 0$ are then

$$\begin{aligned}
&\partial_{\mu_1}\left(\sqrt{-g}Y^{\mu_1\cdots\mu_{q-1}yz}\right) = 0,\\
&\partial_{\mu_1}\left(\sqrt{-g}Y^{\mu_1\cdots\mu_q z}\right) + \partial_y\left(\sqrt{-g}Y^{y\mu_2\cdots\mu_q z}\right) = 0,\\
&\partial_{\mu_1}\left(\sqrt{-g}Y^{\mu_1\cdots\mu_q y}\right) + \partial_z\left(\sqrt{-g}Y^{z\mu_2\cdots\mu_q y}\right) = 0,\\
&\partial_{\mu_1}\left(\sqrt{-g}Y^{\mu_1\cdots\mu_{q+1}}\right) + \partial_y\left(\sqrt{-g}Y^{y\mu_2\cdots\mu_{q+1}}\right) + \partial_z\left(\sqrt{-g}Y^{z\mu_2\cdots\mu_{q+1}}\right) = 0.
\end{aligned} \tag{2}$$

Instead of using the usual KK decomposition with the specific gauge condition, we follow the method developed in ref. [60], where a general decomposition is assumed to preserve the full information of the extra dimension. We start with the general gauge-free KK decomposition for the bulk q-form field

$$\begin{aligned}
X_{\mu_1\cdots\mu_q}(x_\mu,y,z) &= \sum_n X^{(n)}_{\mu_1\cdots\mu_q}(x^\mu)W_1^{(n)}(y,z)e^{aA(y,z)},\\
X_{\mu_1\cdots\mu_{q-1}y}(x_\mu,y,z) &= \sum_n \overline{X}^{(n)}_{\mu_1\cdots\mu_{q-1}}(x^\mu)W_2^{(n)}(y,z)e^{aA(y,z)},\\
X_{\mu_1\cdots\mu_{q-1}z}(x_\mu,y,z) &= \sum_n \underline{X}^{(n)}_{\mu_1\cdots\mu_{q-1}}(x^\mu)W_3^{(n)}(y,z)e^{aA(y,z)},\\
X_{\mu_1\cdots\mu_{q-2}yz}(x_\mu,y,z) &= \sum_n \underline{\overline{X}}^{(n)}_{\mu_1\cdots\mu_{q-2}}(x^\mu)W_4^{(n)}(y,z)e^{aA(y,z)},
\end{aligned} \tag{3}$$

where W_i^ns $(i=1,2,3,4)$ are functions related to the extra dimensional coordinates. Thus, we have decomposed the bulk field potential X into four types of KK mode fields, $X^{(n)}$, $\overline{X}^{(n)}$, $\underline{X}^{(n)}$, and $\underline{\overline{X}}^{(n)}$, on the brane, which are, respectively, q, $q-1$, $q-1$, and $q-2$ forms. Together with their field strengths $Y^{(n)}$, $\overline{Y}^{(n)}$, $\underline{Y}^{(n)}$, and $\underline{\overline{Y}}^{(n)}$, their indices are raised or lowered by the induced brane metric $\hat{g}_{\mu\nu}$. Here, a is an arbitrary parameter, we choose it to be $q-(p+1)/2$ for convenience. The KK decomposition of X yields the decomposition of the field strength Y as

$$\begin{aligned}
Y^{\mu_1\cdots\mu_{q+1}} &= \sum_n Y^{\mu_1\cdots\mu_{q+1}}_{(n)}W_1^{(n)}e^{(a-2q-2)A},\\
Y^{\mu_1\cdots\mu_q y} &= \sum_n\left[\frac{(-1)^q e^{-2(q+1)A}}{q+1}X^{\mu_1\cdots\mu_q}_{(n)}\partial_y\left(W_1^{(n)}e^{aA}\right) + \frac{qe^{-2(q+1)A}}{q+1}\overline{Y}^{\mu_1\cdots\mu_q}_{(n)}W_2^{(n)}e^{aA}\right],\\
Y^{\mu_1\cdots\mu_q z} &= \sum_n\left[\frac{(-1)^q e^{-2(q+1)A}}{q+1}X^{\mu_1\cdots\mu_q}_{(n)}\partial_z\left(W_1^{(n)}e^{aA}\right) + \frac{qe^{-2(q+1)A}}{q+1}\underline{Y}^{\mu_1\cdots\mu_q}_{(n)}W_3^{(n)}e^{aA}\right],\\
Y^{\mu_1\cdots\mu_{q-1}yz} &= \sum_n\Bigg[\frac{(-1)^{q-1}e^{-2(q+1)A}}{q+1}\underline{X}^{\mu_1\cdots\mu_{q-1}}_{(n)}\partial_y\left(W_3^{(n)}e^{aA}\right)\\
&\quad+\frac{(-1)^q e^{-2(q+1)A}}{q+1}\overline{X}^{\mu_1\cdots\mu_{q-1}}_{(n)}\partial_z\left(W_2^{(n)}e^{aA}\right) + \frac{(q-1)e^{-2(q+1)A}}{(q+1)}\underline{\overline{Y}}^{\mu_1\cdots\mu_{q-1}}_{(n)}W_4^{(n)}e^{aA}\Bigg].
\end{aligned} \tag{4}$$

In order to obtain the $(p+1)$-dimensional effective brane action, we place the decomposition of field strength Y into Equation (1), and the action reads,

$$\begin{aligned}
S = &-\frac{1}{2(q+1)!}\sum_n\sum_{n'}\int d^{p+1}x\sqrt{-\hat{g}}\quad\Big[I_1^{nn'}Y^{\mu_1\cdots\mu_{q+1}}_{(n)}Y^{(n')}_{\mu_1\cdots\mu_{q+1}} + \left(I_2^{nn'}+I_4^{nn'}\right)X^{\mu_1\cdots\mu_q}_{(n)}X^{(n')}_{\mu_1'\cdots\mu_q}\\
&+I_3^{nn'}\overline{Y}^{\mu_1\cdots\mu_q}_{(n)}\overline{Y}^{(n')}_{\mu_1\cdots\mu_q} + I_5^{nn'}\underline{Y}^{\mu_1\cdots\mu_q}_{(n)}\underline{Y}^{(n')}_{\mu_1\cdots\mu_q} + 2I_6^{nn'}X^{\mu_1\cdots\mu_q}_{(n)}\overline{Y}^{(n')}_{\mu_1\cdots\mu_q} + 2I_8^{nn'}X^{\mu_1\cdots\mu_q}_{(n)}\underline{Y}^{(n')}_{\mu_1\cdots\mu_q}\\
&+I_7^{nn'}\overline{X}^{\mu_1\cdots\mu_{q-1}}_{(n)}\overline{X}^{(n')}_{\mu_1\cdots\mu_{q-1}} + I_9^{nn'}\underline{X}^{\mu_1\cdots\mu_{q-1}}_{(n)}\underline{X}^{(n')}_{\mu_1\cdots\mu_{q-1}} + 2I_{10}^{nn'}\overline{X}^{\mu_1\cdots\mu_{q-1}}_{(n)}\underline{X}^{(n')}_{\mu_1\cdots\mu_{q-1}} + I_{11}^{nn'}\underline{\overline{Y}}^{\mu_1\cdots\mu_{q-1}}_{(n)}\underline{\overline{Y}}^{(n')}_{\mu_1\cdots\mu_{q-1}}\\
&+2I_{12}^{nn'}\overline{X}^{\mu_1\cdots\mu_{q-1}}_{(n)}\underline{\overline{Y}}^{(n')}_{\mu_1\cdots\mu_{q-1}} + 2I_{13}^{nn'}\underline{X}^{\mu_1\cdots\mu_{q-1}}_{(n)}\underline{\overline{Y}}^{(n')}_{\mu_1\cdots\mu_{q-1}}\Big],
\end{aligned} \tag{5}$$

where the extra-dimensional parts are separately integrated:

$$
\begin{aligned}
I_1^{nn'} &= \int dydz W_1^{(n)} W_1^{(n')}, \qquad I_2^{nn'} = \frac{1}{q+1}\int dydz \partial_y\left(W_1^{(n)}e^{aA}\right)\partial_y\left(W_1^{(n')}e^{aA}\right)e^{-2aA},\\
I_3^{nn'} &= \frac{q^2}{q+1}\int dydz W_2^{(n)} W_2^{(n')}, \qquad I_4^{nn'} = \frac{1}{q+1}\int dydz \partial_z\left(W_1^{(n)}e^{aA}\right)\partial_z\left(W_1^{(n')}e^{aA}\right)e^{-2aA},\\
I_5^{nn'} &= \frac{q^2}{q+1}\int dydz W_3^{(n)} W_3^{(n')}, \qquad I_6^{nn'} = \frac{(-1)^q q}{q+1}\int dydz W_2^{(n)}\partial_y\left(W_1^{(n')}e^{aA}\right)e^{-aA},\\
I_7^{nn'} &= \frac{q}{q+1}\int dydz \partial_z\left(W_2^{(n)}e^{aA}\right)\partial_z\left(W_2^{(n')}e^{aA}\right)e^{-2aA}, \quad I_8^{nn'} = \frac{(-1)^q q}{q+1}\int dydz W_3^{(n)}\partial_z\left(W_1^{(n')}e^{aA}\right)e^{-aA},\\
I_9^{nn'} &= \frac{q}{q+1}\int dydz \partial_y\left(W_3^{(n)}e^{aA}\right)\partial_y\left(W_3^{(n')}e^{aA}\right)e^{-2aA},\\
I_{10}^{nn'} &= \frac{-q}{q+1}\int dydz \partial_y\left(W_3^{(n)}e^{aA}\right)\partial_z\left(W_2^{(n')}e^{aA}\right)e^{-2aA},\\
I_{11}^{nn'} &= \frac{q(q-1)^2}{q+1}\int dydz W_4^{(n)} W_4^{(n')}, \quad I_{12}^{nn'} = \frac{(-1)^q q(q-1)}{q+1}\int dydz \partial_z\left(W_2^{(n)}e^{aA}\right)W_4^{(n')}e^{-aA},\\
I_{13}^{nn'} &= \frac{(-1)^{q-1} q(q-1)}{q+1}\int dydz \partial_y\left(W_3^{(n)}e^{aA}\right)W_4^{(n')}e^{-aA}.
\end{aligned} \tag{6}
$$

Here, these parameters correspond to mass terms or the coupling strength of fields on the brane. We impose the following orthogonality and finiteness conditions for a localizable case:

$$
\begin{aligned}
I_1^{nn'} = \delta_{nn'}, \quad I_3^{nn'} = I_5^{nn'} = (q+1)\delta_{nn'}, \quad I_{11}^{nn'} = q(q+1)\delta_{nn'},\\
I_k^{nn'} < \infty \quad \text{otherwise.}
\end{aligned} \tag{7}
$$

In order to separate the EoMs into brane parts and extra-dimensional parts, we will derive the EoMs in two different ways by exchanging the order of variation and KK decomposition. Since these two sets of EoMs should basically be consistent, we can obtain useful results by comparing them. Firstly, we derive the EoMs by varying the effective action (5) with respect to the four types of fields $X_{(n)}$, $\overline{X}_{(n)}$, $\underline{X}_{(n)}$, and $\underline{\overline{X}}_{(n)}$:

$$
\begin{aligned}
&\sum_{n'}\left[\frac{1}{\sqrt{-\hat{g}}}\partial_\mu\left(\sqrt{-\hat{g}}I_1^{nn'}Y_{(n')}^{\mu\mu_1\cdots\mu_q}\right) - \left(I_2^{nn'}+I_4^{nn'}\right)X_{(n')}^{\mu_1\cdots\mu_q} - I_6^{nn'}\overline{Y}_{(n')}^{\mu_1\cdots\mu_q} - I_8^{nn'}\underline{Y}_{(n')}^{\mu_1\cdots\mu_q}\right] = 0,\\
&\sum_{n'}\left[\frac{1}{\sqrt{-\hat{g}}}\partial_\mu\left(\sqrt{-\hat{g}}I_3^{nn'}\overline{Y}_{(n')}^{\mu\mu_1\cdots\mu_{q-1}} + \sqrt{-\hat{g}}I_6^{nn'}X_{(n')}^{\mu\mu_1\cdots\mu_{q-1}}\right) - I_7^{nn'}\overline{X}_{(n')}^{\mu_1\cdots\mu_{q-1}} - I_{10}^{nn'}\underline{X}_{(n')}^{\mu_1\cdots\mu_{q-1}} - I_{12}^{nn'}\underline{\overline{Y}}_{(n')}^{\mu_1\cdots\mu_{q-1}}\right] = 0,\\
&\sum_{n'}\left[\frac{1}{\sqrt{-\hat{g}}}\partial_\mu\left(\sqrt{-\hat{g}}I_5^{nn'}\underline{Y}_{(n')}^{\mu\mu_1\cdots\mu_{q-1}} + \sqrt{-\hat{g}}I_8^{nn'}X_{(n')}^{\mu\mu_1\cdots\mu_{q-1}}\right) - I_{10}^{nn'}\overline{X}_{(n')}^{\mu_1\cdots\mu_{q-1}} - I_9^{nn'}\underline{X}_{(n')}^{\mu_1\cdots\mu_{q-1}} - I_{13}^{nn'}\underline{\overline{Y}}_{(n')}^{\mu_1\cdots\mu_{q-1}}\right] = 0,\\
&\sum_{n'}\frac{1}{\sqrt{-\hat{g}}}\partial_\mu\left[\sqrt{-\hat{g}}\left(I_{11}^{nn'}\underline{\overline{Y}}_{(n')}^{\mu\mu_1\cdots\mu_{q-2}} + I_{12}^{nn'}\overline{X}_{(n')}^{\mu\mu_1\cdots\mu_{q-2}} + I_{13}^{nn'}\underline{X}_{(n')}^{\mu\mu_1\cdots\mu_{q-2}}\right)\right] = 0.
\end{aligned} \tag{8}
$$

Alternatively, we can obtain EoMs by directly inserting the decomposition (4) into the undecomposed EoMs (2):

$$\frac{1}{\sqrt{-\hat{g}}}\partial_{\mu_1}\left(\sqrt{-\hat{g}}Y^{\mu_1\cdots\mu_{q+1}}_{(n)}\right)+\left(\lambda_1^{(n)}+\lambda_2^{(n)}\right)X^{\mu_2\cdots\mu_{q+1}}_{(n)}+\lambda_3^{(n)}\overline{Y}^{\mu_2\cdots\mu_{q+1}}_{(n)}+\lambda_4^{(n)}\underline{Y}^{\mu_2\cdots\mu_{q+1}}_{(n)}=0,$$
$$\frac{1}{\sqrt{-\hat{g}}}\partial_{\mu_1}\left(\sqrt{-\hat{g}}\overline{Y}^{\mu_1\cdots\mu_q}_{(n)}\right)+\frac{\lambda_5^{(n)}}{\sqrt{-\hat{g}}}\partial_{\mu_1}\left(\sqrt{-\hat{g}}X^{\mu_1\cdots\mu_q}_{(n)}\right)+\lambda_6^{(n)}\underline{X}^{\mu_2\cdots\mu_q}_{(n)}+\lambda_7^{(n)}\overline{X}^{\mu_2\cdots\mu_q}_{(n)}+\lambda_8^{(n)}\underline{Y}^{\mu_2\cdots\mu_q}_{(n)}=0,$$
$$\frac{1}{\sqrt{-\hat{g}}}\partial_{\mu_1}\left(\sqrt{-\hat{g}}\underline{Y}^{\mu_1\cdots\mu_q}_{(n)}\right)+\frac{\lambda_9^{(n)}}{\sqrt{-\hat{g}}}\partial_{\mu_1}\left(\sqrt{-\hat{g}}X^{\mu_1\cdots\mu_q}_{(n)}\right)+\lambda_{10}^{(n)}\underline{X}^{\mu_2\cdots\mu_q}_{(n)}+\lambda_{11}^{(n)}\overline{X}^{\mu_2\cdots\mu_q}_{(n)}+\lambda_{12}^{(n)}\overline{Y}^{\mu_2\cdots\mu_q}_{(n)}=0,$$
$$\frac{1}{\sqrt{-\hat{g}}}\partial_{\mu_1}\left[\sqrt{-\hat{g}}\left(\underline{\overline{Y}}^{\mu_1\cdots\mu_{q-1}}_{(n)}+\lambda_{13}^{(n)}\overline{X}^{\mu_1\cdots\mu_{q-1}}_{(n)}+\lambda_{14}^{(n)}\underline{X}^{\mu_1\cdots\mu_{q-1}}_{(n)}\right)\right]=0, \tag{9}$$

where

$$\lambda_1^{(n)}=\frac{\partial_y\left[\partial_y\left(W_1^{(n)}e^{aA}\right)e^{-2aA}\right]}{e^{-aA}(q+1)W_1^{(n)}},\ \lambda_2^{(n)}=\frac{\partial_z\left[\partial_z\left(W_1^{(n)}e^{aA}\right)e^{-2aA}\right]}{e^{-aA}(q+1)W_1^{(n)}},\ \lambda_3^{(n)}=\frac{(-1)^q q\partial_y\left(W_2^{(n)}e^{-aA}\right)}{e^{-aA}(q+1)W_1^{(n)}},$$
$$\lambda_4^{(n)}=\frac{(-1)^q q\partial_z\left(W_3^{(n)}e^{-aA}\right)}{e^{-aA}(q+1)W_1^{(n)}},\quad \lambda_5^{(n)}=\frac{(-1)^q\partial_y\left(W_1^{(n)}e^{aA}\right)}{e^{aA}qW_2^{(n)}},\quad \lambda_6^{(n)}=\frac{\partial_z\left[\partial_y\left(W_3^{(n)}e^{aA}\right)e^{-2aA}\right]}{-qe^{-aA}W_2^{(n)}},$$
$$\lambda_7^{(n)}=\frac{\partial_z\left[\partial_z\left(W_2^{(n)}e^{aA}\right)e^{-2aA}\right]}{e^{-aA}qW_2^{(n)}},\ \lambda_8^{(n)}=\frac{(q-1)\partial_z\left(W_4^{(n)}e^{-aA}\right)}{(-1)^q e^{-aA}qW_2^{(n)}},\quad \lambda_9^{(n)}=\frac{\partial_z\left(W_1^{(n)}e^{aA}\right)}{(-1)^q qe^{aA}W_3^{(n)}},$$
$$\lambda_{10}^{(n)}=\frac{\partial_y\left[\partial_y\left(W_3^{(n)}e^{aA}\right)e^{-2aA}\right]}{e^{-aA}qW_3^{(n)}},\ \lambda_{11}^{(n)}=\frac{\partial_y\left[\partial_z\left(W_2^{(n)}e^{aA}\right)e^{-2aA}\right]}{-qe^{-aA}W_3^{(n)}},\ \lambda_{12}^{(n)}=\frac{(q-1)\partial_y\left(W_4^{(n)}e^{-aA}\right)}{(-1)^{q-1}qe^{-aA}W_3^{(n)}},$$
$$\lambda_{13}^{(n)}=\frac{(-1)^q\partial_z\left(W_2^{(n)}e^{aA}\right)}{(q-1)e^{aA}W_4^{(n)}},\qquad \lambda_{14}^{(n)}=\frac{(-1)^{q-1}\partial_y\left(W_3^{(n)}e^{aA}\right)}{(q-1)e^{aA}W_4^{(n)}}. \tag{10}$$

By comparing Equation (8) with Equation (9), we find the $\lambda_i^{(n)}$s are necessarily constants under our former assumption, so we introduce the following mass parameters

$$\lambda_1^{(n)}=\frac{-m_{1y}^{(n)2}}{q+1},\qquad \lambda_2^{(n)}=\frac{-m_{1z}^{(n)2}}{q+1},\qquad \lambda_7^{(n)}=\frac{-m_{2z}^{(n)2}}{q},\qquad \lambda_{10}^{(n)}=\frac{-m_{3y}^{(n)2}}{q}, \tag{11}$$

which give a set of Schrödinger-like equations for the extra-dimensional functions:

$$\left[-\partial_z^2+(a\partial_z A)^2-a\partial_z^2 A\right]W_1^{(n)}=m_{1z}^{(n)2}W_1^{(n)},\quad \left[-\partial_y^2+(a\partial_y A)^2-a\partial_y^2 A\right]W_1^{(n)}=m_{1y}^{(n)2}W_1^{(n)},$$
$$\left[-\partial_z^2+(a\partial_z A)^2-a\partial_z^2 A\right]W_2^{(n)}=m_{2z}^{(n)2}W_2^{(n)},\quad \left[-\partial_y^2+(a\partial_y A)^2-a\partial_y^2 A\right]W_3^{(n)}=m_{3y}^{(n)2}W_3^{(n)}. \tag{12}$$

Note Equation (7) and compare the EoMs again to obtain the relations

$$\lambda_3^{(n)}=-(q+1)\lambda_5^{(n)},\quad \lambda_4^{(n)}=-(q+1)\lambda_9^{(n)},\quad \lambda_6^{(n)}=\lambda_{11}^{(n)},\quad \lambda_{12}^{(n)}=-q\lambda_{14}^{(n)},\quad \lambda_8^{(n)}=-q\lambda_{13}^{(n)}. \tag{13}$$

Substitution of $\lambda_3^{(n)}$ into $\lambda_5^{(n)}$, $\lambda_4^{(n)}$ into $\lambda_9^{(n)}$, $\lambda_{14}^{(n)}$ into $\lambda_{12}^{(n)}$, and $\lambda_{13}^{(n)}$ into $\lambda_8^{(n)}$ yields

$$
\begin{aligned}
&\lambda_3^{(n)}\lambda_5^{(n)} = \lambda_1^{(n)}, \qquad \lambda_3^{(n)} = m_{1y}^{(n)}, \qquad \lambda_5^{(n)} = \frac{-m_{1y}^{(n)}}{q+1}, \\
&\lambda_4^{(n)}\lambda_9^{(n)} = \lambda_2^{(n)}, \qquad \lambda_4^{(n)} = m_{1z}^{(n)}, \qquad \lambda_9^{(n)} = \frac{-m_{1z}^{(n)}}{q+1}, \\
&\lambda_{12}^{(n)}\lambda_{14}^{(n)} = \lambda_{10}^{(n)}, \qquad \lambda_{12}^{(n)} = m_{3y}^{(n)}, \qquad \lambda_{14}^{(n)} = \frac{-m_{3y}^{(n)}}{q}, \\
&\lambda_8^{(n)}\lambda_{13}^{(n)} = \lambda_7^{(n)}, \qquad \lambda_8^{(n)} = -m_{2z}^{(n)}, \qquad \lambda_{13}^{(n)} = \frac{m_{2z}^{(n)}}{q},
\end{aligned} \tag{14}
$$

where we have chosen the signs for consistency.

Similarly, by substituting $\lambda_5^{(n)}$ into $\lambda_3^{(n)}$, $\lambda_9^{(n)}$ into $\lambda_4^{(n)}$, $\lambda_{12}^{(n)}$ into $\lambda_{14}^{(n)}$, and $\lambda_8^{(n)}$ into $\lambda_{13}^{(n)}$, we obtain another four Schrödinger-like equations with mass terms

$$
\begin{aligned}
&\Big[-\partial_y^2 + (a\partial_y A)^2 + a\partial_y^2 A\Big] W_2^{(n)} = m_{1y}^{(n)2} W_2^{(n)}, \quad \Big[-\partial_z^2 + (a\partial_z A)^2 + a\partial_z^2 A\Big] W_3^{(n)} = m_{1z}^{(n)2} W_3^{(n)}, \\
&\Big[-\partial_y^2 + (a\partial_y A)^2 + a\partial_y^2 A\Big] W_4^{(n)} = m_{3y}^{(n)2} W_4^{(n)}, \quad \Big[-\partial_z^2 + (a\partial_z A)^2 + a\partial_z^2 A\Big] W_4^{(n)} = m_{2z}^{(n)2} W_4^{(n)}.
\end{aligned} \tag{15}
$$

Comparing the expressions of λ_5, λ_9, λ_6, and those of λ_5, λ_9, λ_{11}, we have

$$
\lambda_5^{(n)}\lambda_7^{(n)} = -\lambda_6^{(n)}\lambda_9^{(n)}, \qquad \lambda_9^{(n)}\lambda_{10}^{(n)} = -\lambda_5^{(n)}\lambda_{11}^{(n)}, \qquad \lambda_6^{(n)} = \frac{m_{1y}^{(n)} m_{2z}^{(n)2}}{q m_{1z}^{(n)}}, \tag{16}
$$

which leads to $\frac{m_{3y}^{(n)2}}{m_{2z}^{(n)2}} = \frac{m_{1y}^{(n)2}}{m_{1z}^{(n)2}}$; therefore, we can introduce constant parameters $\eta^{(n)}$s such that $m_{2z}^{(n)} = \eta^{(n)} m_{1z}^{(n)}$, $m_{3y}^{(n)} = \eta^{(n)} m_{1y}^{(n)}$, and rewrite $m_{1y}^{(n)}$, $m_{1z}^{(n)}$ as $m_y^{(n)}$, $m_z^{(n)}$. Consequently, we can summarize the eight Schrödinger-like equations as

$$
\begin{aligned}
&\Big[-\partial_y^2 + (a\partial_y A)^2 - a\partial_y^2 A\Big] W_1^{(n)} = m_y^{(n)2} W_1^{(n)}, \qquad \Big[-\partial_z^2 + (a\partial_z A)^2 - a\partial_z^2 A\Big] W_1^{(n)} = m_z^{(n)2} W_1^{(n)}, \\
&\Big[-\partial_y^2 + (a\partial_y A)^2 + a\partial_y^2 A\Big] W_2^{(n)} = m_y^{(n)2} W_2^{(n)}, \qquad \Big[-\partial_z^2 + (a\partial_z A)^2 - a\partial_z^2 A\Big] W_2^{(n)} = \eta_{(n)}^2 m_z^{(n)2} W_2^{(n)}, \\
&\Big[-\partial_y^2 + (a\partial_y A)^2 - a\partial_y^2 A\Big] W_3^{(n)} = \eta_{(n)}^2 m_y^{(n)2} W_3^{(n)}, \quad \Big[-\partial_z^2 + (a\partial_z A)^2 + a\partial_z^2 A\Big] W_3^{(n)} = m_z^{(n)2} W_3^{(n)}, \\
&\Big[-\partial_y^2 + (a\partial_y A)^2 + a\partial_y^2 A\Big] W_4^{(n)} = \eta_{(n)}^2 m_y^{(n)2} W_4^{(n)}, \quad \Big[-\partial_z^2 + (a\partial_z A)^2 + a\partial_z^2 A\Big] W_4^{(n)} = \eta_{(n)}^2 m_z^{(n)2} W_4^{(n)}.
\end{aligned} \tag{17}
$$

From the above equations, we obtain two series of mass parameters from the two codimensions, and the mass spectra of each codimension are quite similar in spite of the common factors $\eta_{(n)}$s. Under certain background solutions such as $\partial_y \partial_z A = 0$, or the condition that one of $W_2^{(n)}$ or $W_3^{(n)}$ is zero, we will have $\eta_{(n)} = 1$. As usual, the operator on the left side of each equation can be written as a product of an operator and its conjugate, $PP^{\dagger}$; therefore, mass spectra of the KK modes are non-negative definite. With an explicit solution of the space-time background, one can solve these equations to get their mass spectra and wave functions. Since the subsequent discussion is independent of the solution of the Schrödinger-like equations, we are not going to solve these equations specifically. Under certain conditions, the q-form field can be localized on p-branes with two extra dimensions, so we reserve localizablility as an assumption.

3. Gauge Invariance of the Brane Action

With the previous results, one can continue compare the two sets of EoMs to exhaust relations between $I^{nn\prime}$s and $m^{(n)}$s

$$
\begin{aligned}
& I_1^{nn'} = \delta_{nn'}, \qquad I_3^{nn'} = I_5^{nn'} = (q+1)\delta_{nn'}, \qquad I_{11}^{nn'} = q(q+1)\delta_{nn'}, \\
& I_2^{nn'} + I_4^{nn'} = -\left(\lambda_1^{(n)} + \lambda_2^{(n)}\right)\delta_{nn'} = \frac{\delta_{nn'}}{q+1}\left(m_y^{(n)2} + m_z^{(n)2}\right), \qquad I_6^{nn'} = -\lambda_3^{(n)}\delta_{nn'} = -m_y^{(n)}\delta_{nn'}, \\
& I_7^{nn'} = -(q+1)\lambda_7^{(n)}\delta_{nn'} = \frac{q+1}{q}\eta_{(n)}^2 m_z^{(n)2}\delta_{nn'}, \qquad I_8^{nn'} = -\lambda_4^{(n)}\delta_{nn'} = -m_z^{(n)}\delta_{nn'}, \\
& I_9^{nn'} = -(q+1)\lambda_{10}^{(n)}\delta_{nn'} = \frac{q+1}{q}\eta_{(n)}^2 m_y^{(n)2}\delta_{nn'}, \qquad I_{10}^{nn'} = -(q+1)\lambda_6^{(n)}\delta_{nn'} = \frac{q+1}{-q}\eta_{(n)}^2 m_y^{(n)} m_z^{(n)}\delta_{nn'}, \\
& I_{12}^{nn'} = -(q+1)\lambda_8^{(n)}\delta_{nn'} = (q+1)\eta_{(n)} m_z^{(n)}\delta_{nn'}, \qquad I_{13}^{nn'} = -(q+1)\lambda_{12}^{(n)}\delta_{nn'} = -(q+1)\eta_{(n)} m_y^{(n)}\delta_{nn'}.
\end{aligned} \tag{18}
$$

By inserting these relations into Equation (5), we can convert the effective action on the brane into

$$
\begin{aligned}
S_{eff} = \; & \frac{-1}{2(q+1)!}\sum_n \int d^{p+1}x\sqrt{-\hat{g}}\Big[\left(Y^{(n)}_{\mu_1\cdots\mu_{q+1}}\right)^2 + (q+1)\left(\overline{Y}^{(n)}_{\mu_1\cdots\mu_q}\right)^2 + (q+1)\left(\underline{Y}^{(n)}_{\mu_1\cdots\mu_q}\right)^2 \\
& + q(q+1)\left(\overline{\underline{Y}}^{(n)}_{\mu_1\cdots\mu_{q-1}}\right)^2 + q(q+1)\left(\overline{\underline{Y}}^{(n)}_{\mu_1\cdots\mu_{q-1}}\right)^2 + \frac{1}{q+1}\left(m_y^{(n)2} + m_z^{(n)2}\right)\left(X^{(n)}_{\mu_1\cdots\mu_q}\right)^2 \\
& - 2m_y^{(n)} X^{\mu_1\cdots\mu_q}_{(n)}\overline{Y}^{(n)}_{\mu_1\cdots\mu_q} - 2m_z^{(n)} X^{\mu_1\cdots\mu_q}_{(n)}\underline{Y}^{(n)}_{\mu_1\cdots\mu_q} + \frac{q+1}{q}\eta_{(n)}^2 m_z^{(n)2}\left(\overline{X}^{(n)}_{\mu_1\cdots\mu_{q-1}}\right)^2 \\
& + \frac{q+1}{q}\eta_{(n)}^2 m_y^{(n)2}\left(\underline{X}^{(n)}_{\mu_1\cdots\mu_{q-1}}\right)^2 - \frac{2(q+1)}{q}\eta_{(n)}^2 m_y^{(n)} m_z^{(n)}\overline{X}^{(n)}_{\mu_1\cdots\mu_{q-1}}\underline{X}^{\mu_1\cdots\mu_{q-1}}_{(n)} \\
& + 2(q+1)\eta_{(n)} m_z^{(n)}\overline{X}^{(n)}_{\mu_1\cdots\mu_{q-1}}\overline{\underline{Y}}^{\mu_1\cdots\mu_{q-1}}_{(n)} - 2(q+1)\eta_{(n)} m_Y^{(n)}\overline{X}^{(n)}_{\mu_1\cdots\mu_{q-1}}\overline{\underline{Y}}^{\mu_1\cdots\mu_{q-1}}_{(n)}\Big] \\
= \; & \frac{-1}{2(q+1)!}\sum_n \int d^{p+1}x\sqrt{-\hat{g}}\left(Y^{(n)}_{\mu_1\cdots\mu_{q+1}}\right)^2 + \\
& \frac{-1}{2q!}\sum_n \int d^{p+1}x\sqrt{-\hat{g}}\left[\left(\overline{Y}^{(n)}_{\mu_1\cdots\mu_q} - \frac{m_y^{(n)}}{q+1}X^{(n)}_{\mu_1\cdots\mu_q}\right)^2 + \left(\underline{Y}^{(n)}_{\mu_1\cdots\mu_q} - \frac{m_z^{(n)}}{q+1}X^{(n)}_{\mu_1\cdots\mu_q}\right)^2\right] + \\
& \frac{-1}{2(q-1)!}\sum_n \int d^{p+1}x\sqrt{-\hat{g}}\left(\overline{\underline{Y}}^{(n)}_{\mu_1\cdots\mu_{q-1}} + \frac{\eta_{(n)} m_z^{(n)}}{q}\overline{X}^{(n)}_{\mu_1\cdots\mu_{q-1}} - \frac{\eta_{(n)} m_y^{(n)}}{q}\underline{X}^{(n)}_{\mu_1\cdots\mu_{q-1}}\right)^2.
\end{aligned} \tag{19}
$$

One can observe that through coupling of the four fields, the effective action appears exactly as a sum of four squares, which is then apparently invariant under the following gauge transformation

$$
\begin{aligned}
X^{(n)}_{\mu_1\cdots\mu_q} &\to X^{(n)}_{\mu_1\cdots\mu_q} + \partial_{[\mu_1}\Lambda^{(n)}_{\mu_2\cdots\mu_q]}, \\
\overline{X}^{(n)}_{\mu_1\cdots\mu_{q-1}} &\to \overline{X}^{(n)}_{\mu_1\cdots\mu_{q-1}} + \tfrac{m_y^{(n)}}{q+1}\Lambda^{(n)}_{\mu_1\cdots\mu_{q-1}}, \\
\underline{X}^{(n)}_{\mu_1\cdots\mu_{q-1}} &\to \underline{X}^{(n)}_{\mu_1\cdots\mu_{q-1}} + \tfrac{m_z^{(n)}}{q+1}\Lambda^{(n)}_{\mu_1\cdots\mu_{q-1}}, \\
\overline{\underline{X}}^{(n)}_{\mu_1\cdots\mu_{q-2}} &\to \overline{\underline{X}}^{(n)}_{\mu_1\cdots\mu_{q-2}} + \partial_{[\mu_1}\Gamma^{(n)}_{\mu_2\cdots\mu_{q-2}]},
\end{aligned} \tag{20}
$$

where $\Lambda^{(n)}$ and $\Gamma^{(n)}$ are an arbitrary $(q-1)$-form and $(q-2)$-form, respectively.

4. Induced Hodge Dualities on The Brane

In the bulk, a q-form potential X is said to be dual to a $(D-2-q)$-form potential X^* (though not uniquely determined) via Hodge duality between their strength fields Y and Y^*

$$
Y^{*M_1\cdots M_{p+2-q}} = \frac{\epsilon^{M_1\cdots M_{p+2-q}N_1\cdots N_{q+1}}}{(q-1)!\sqrt{-g}} Y_{N_1\cdots N_{q+1}}, \tag{21}
$$

i.e., a $q+1 \sim p+2-q$ duality. Direct substitution into Equation (1) shows

$$\begin{aligned} S &= -\frac{1}{2(q+1)!}\int d^D x\sqrt{-g}Y^{N_1\cdots N_{q+1}}Y_{N_1\cdots N_{q+1}} \\ &= -\frac{1}{2(p+2-q)!}\int d^D x\sqrt{-g}Y^{*N_1\cdots N_{p+2-q}}Y^*_{N_1\cdots N_{p+2-q}} = S^*, \end{aligned} \tag{22}$$

which indicates the two massless dual potentials X and X^* are physically equivalent.

However, when performing localization for X and X^* through the usual KK decomposition, we obtain two brane fields of the same order as their bulk ones, which are therefore impossible to be Hodge-dual on the brane for dimensional reasons. In contrast, the novel decomposition mechanism yields a series of brane forms of different orders, and the dimensional requirement is satisfied when we match them reversely. Therefore, it is very likely that the bulk duality will naturally reduce to brane dualities, which is what we are about to derive.

Subsequently, we denote $X^*_{(n)}$ and $Y^*_{(n)}$ as the brane components of X^* and Y^* rather than the duals of some fields. From Equation (21), one can obtain

$$\begin{aligned} &\sqrt{-g}Y^{*\mu_1\cdots\mu_{p+2-q}} = \frac{\epsilon^{\mu_1\cdots\mu_{p+2-q}\nu_1\cdots\nu_{q-1}yz}}{(q-1)!}Y_{\nu_1\cdots\nu_{q-1}yz}, \quad \sqrt{-g}Y^{*\mu_1\cdots\mu_{p+1-q}y} = \frac{\epsilon^{\mu_1\cdots\mu_{p+1-q}y\nu_1\cdots\nu_q z}}{q!}Y_{\nu_1\cdots\nu_q z}, \\ &\sqrt{-g}Y^{*\mu_1\cdots\mu_{p+1-q}z} = \frac{\epsilon^{\mu_1\cdots\mu_{p+1-q}z\nu_1\cdots\nu_q y}}{q!}Y_{\nu_1\cdots\nu_q y}, \quad \sqrt{-g}Y^{*\mu_1\cdots\mu_{p-q}yz} = \frac{\epsilon^{\mu_1\cdots\mu_{p-q}yz\nu_1\cdots\nu_{q+1}}}{(q+1)!}Y_{\nu_1\cdots\nu_{q+1}}. \end{aligned} \tag{23}$$

Note that when working with dual fields, we just need to replace $q+1$ by $p+2-q$. Thus, the substitution of (4) into the first equation of Equation (23) gives

$$\begin{aligned} \sqrt{-g}\ &\sum_n Y^{*\mu_1\cdots\mu_{p+2-q}}_{(n)} W_1^{*(n)} e^{-(a+(p+2-q))A} \\ &= \sum_n \frac{\epsilon^{\mu_1\cdots\mu_{p+2-q}\nu_1\cdots\nu_{q-1}yz}}{(q-1)!}\Bigg[\frac{(-1)^{q-1}}{q+1}\underline{X}^{(n)}_{\mu_1\cdots\mu_{q-1}}\partial_y\Big(W_3^{(n)}e^{aA}\Big) + \frac{(-1)^q}{q+1}\overline{X}^{(n)}_{\mu_1\cdots\mu_{q-1}}\partial_z\Big(W_2^{(n)}e^{aA}\Big) \\ &\quad + \frac{q-1}{q+1}\underline{\overline{Y}}^{(n)}_{\mu_1\cdots\mu_{q-1}}W_4^{(n)}e^{aA}\Bigg] \\ &= \sum_n \frac{\epsilon^{\mu_1\cdots\mu_{p+2-q}\nu_1\cdots\nu_{q-1}yz}}{(q-1)!}\Bigg[\frac{q-1}{q+1}\lambda_{14}^{(n)}W_4^{(n)}e^{aA}\underline{X}^{(n)}_{\mu_1\cdots\mu_{q-1}} + \frac{q-1}{q+1}\lambda_{13}^{(n)}W_4^{(n)}e^{aA}\overline{X}^{(n)}_{\mu_1\cdots\mu_{q-1}} \\ &\quad + \frac{q-1}{q+1}\underline{\overline{Y}}^{(n)}_{\mu_1\cdots\mu_{q-1}}W_4^{(n)}e^{aA}\Bigg] \\ &= \frac{\epsilon^{\mu_1\cdots\mu_{p+2-q}\nu_1\cdots\nu_{q-1}yz}W_4^{(n)}e^{aA}}{(q-2)!(q+1)}\left(\underline{\overline{Y}}^{(n)}_{\mu_1\cdots\mu_{q-1}} + \frac{\eta_{(n)}m_z^{(n)}}{q}\overline{X}^{(n)}_{\mu_1\cdots\mu_{q-1}} - \frac{\eta_{(n)}m_y^{(n)}}{q}\underline{X}^{(n)}_{\mu_1\cdots\mu_{q-1}}\right), \end{aligned} \tag{24}$$

which can be simplified to

$$\sqrt{-\hat{g}}\sum_n Y^{*\mu_1\cdots\mu_{p+2-q}}_{(n)}W_1^{*(n)} = \sum_n \frac{\epsilon^{\mu_1\cdots\mu_{p+2-q}\nu_1\cdots\nu_{q-1}}W_4^{(n)}}{(q-2)!(q+1)}\left(\underline{\overline{Y}}^{(n)}_{\mu_1\cdots\mu_{q-1}} + \frac{\eta_{(n)}m_z^{(n)}}{q}\overline{X}^{(n)}_{\mu_1\cdots\mu_{q-1}} - \frac{\eta_{(n)}m_y^{(n)}}{q}\underline{X}^{(n)}_{\mu_1\cdots\mu_{q-1}}\right). \tag{25}$$

Here, we find that $W_1^{*(n)}$ is proportional to $W_4^{(n)}$, so according to the relation $\frac{q-1}{q+1}\int dydz W_4^{(n)}W_4^{(n')} = \delta_{nn'} = \int dydz W_1^{*(n)}W_1^{*(n')}$, we can let $W_4^{(n)} = \frac{q+1}{q-1}W_1^{*(n)}$, then Equation (25) is further simplified to

$$\sqrt{-\hat{g}}Y^{*\mu_1\cdots\mu_{p+2-q}}_{(n)} = \frac{\epsilon^{\mu_1\cdots\mu_{p+2-q}\nu_1\cdots\nu_{q-1}}}{(q-1)!}\left(\underline{\overline{Y}}^{(n)}_{\mu_1\cdots\mu_{q-1}} + \frac{\eta_{(n)}m_z^{(n)}}{q}\overline{X}^{(n)}_{\mu_1\cdots\mu_{q-1}} - \frac{\eta_{(n)}m_y^{(n)}}{q}\underline{X}^{(n)}_{\mu_1\cdots\mu_{q-1}}\right). \tag{26}$$

This is exactly a pair of Hodge duality on the brane.

Similarly, the remaining three equations of Equation (23) will lead to

$$
\begin{aligned}
&W_3^{(n)} = \frac{q+1}{q}\frac{p+1-q}{p+2-q}W_2^{*(n)},\\
&\sqrt{-\hat{g}}\left(\overline{Y}^{*\mu_1\cdots\mu_{p+1-q}}_{(n)} - \frac{m_y^{*(n)}}{p+2-q}X^{*\mu_1\cdots\mu_{p+1-q}}_{(n)}\right) = \frac{\epsilon^{\mu_1\cdots\mu_{p+1-q}\nu_1\cdots\nu_q}}{(-1)^q q!}\left(\underline{Y}^{(n)}_{*\nu_1\cdots\nu_q} - \frac{m_z^{(n)}}{q+1}X^{(n)}_{*\nu_1\cdots\nu_q}\right);\\
&W_2^{(n)} = \frac{q+1}{q}\frac{p+1-q}{p+2-q}W_3^{*(n)},\\
&\sqrt{-\hat{g}}\left(\underline{Y}^{*\mu_1\cdots\mu_{p+1-q}}_{(n)} - \frac{m_z^{*(n)}}{p+2-q}X^{*\mu_1\cdots\mu_{p+1-q}}_{(n)}\right) = \frac{\epsilon^{\mu_1\cdots\mu_{p+1-q}\nu_1\cdots\nu_q}}{(-1)^{q+1} q!}\left(\overline{Y}^{(n)}_{*\nu_1\cdots\nu_q} - \frac{m_y^{(n)}}{q+1}X^{(n)}_{*\nu_1\cdots\nu_q}\right);\\
&W_1^{(n)} = \frac{p-q}{p+2-q}W_4^{*(n)},\\
&\sqrt{-\hat{g}}\left(\underline{\overline{Y}}^{*\mu_1\cdots\mu_{p-q}}_{(n)} + \frac{\eta^*_{(n)}m_z^{*(n)}}{p+1-q}\overline{X}^{*\mu_1\cdots\mu_{p-q}}_{(n)} - \frac{\eta^*_{(n)}m_y^{*(n)}}{p+1-q}\underline{X}^{*\mu_1\cdots\mu_{p-q}}_{(n)}\right) = \frac{\epsilon^{\mu_1\cdots\mu_{p-q}\nu_1\cdots\nu_{q+1}}}{(q+1)!}Y^{(n)}_{\nu_1\cdots\nu_{q+1}}.
\end{aligned}
\tag{27}
$$

These are another three pairs of dualities and relations between extra-dimensional functions, where, because of the correspondence between the extra-dimensional functions, the mass spectra for the dual modes are altered via $m_y^{*(n)} = \eta_{(n)}m_y^{(n)}$, $m_z^{*(n)} = \eta_{(n)}m_z^{(n)}$, and $\eta^*_{(n)} = 1/\eta_{(n)}$. Incidentally, our $\pm$ sign choices are determined by the identity $** = (-1)^{sgn(g)+(n-k)k}$.

In summary, the bulk duality of a $(q+1)$-form strength and a $(p+2-q)$-form strength (in other words, of potential ranks q and $p+1-q$) generates four coupled dualities on the brane, which is illustrated as follows (see Table 1):

Table 1. Duality/Dualities and their ranks.

Bulk	$Y \sim Y^*$	$q+1 \sim p+2-q$
Brane	$Y \sim \underline{\overline{Y}}^* + \overline{X}^* + \underline{X}^*$	$q+1 \sim p-q$
	$\underline{Y} + X \sim \overline{Y}^* + X^*$	$q \sim p+1-q$
	$\overline{Y} + X \sim \underline{Y}^* + X^*$	$q \sim p+1-q$
	$\underline{\overline{Y}} + \overline{X} + \underline{X} \sim Y^*$	$q-1 \sim p+2-q$

Interestingly, the coupled fields in the brane dualities are exactly the blocks appearing in the invariant brane action; thus, if we take the four dualities (26) and (27) into our brane action $S_q^{(n)}$(19) derived for a q-form potential, it will automatically equal its dual counterpart $S^{*(n)}_{p+1-q}$:

$$
\begin{aligned}
S_q^{(n)} &= \frac{-1}{2(q+1)!}\int d^{p+1}x\sqrt{-\hat{g}}\left(Y^{(n)}_{\mu_1\cdots\mu_{q+1}}\right)^2 + \\
&\frac{-1}{2q!}\int d^{p+1}x\sqrt{-\hat{g}}\left[\left(\overline{Y}^{(n)}_{\mu_1\cdots\mu_q} - \frac{m_y^{(n)}}{q+1}X^{(n)}_{\mu_1\cdots\mu_q}\right)^2 + \left(\underline{Y}^{(n)}_{\mu_1\cdots\mu_q} - \frac{m_z^{(n)}}{q+1}X^{(n)}_{\mu_1\cdots\mu_q}\right)^2\right] + \\
&\frac{-1}{2(q-1)!}\int d^{p+1}x\sqrt{-\hat{g}}\left(\underline{Y}^{(n)}_{\mu_1\cdots\mu_{q-1}} + \frac{\eta_{(n)}m_z^{(n)}}{q}\overline{X}^{(n)}_{\mu_1\cdots\mu_{q-1}} - \frac{\eta_{(n)}m_y^{(n)}}{q}\underline{X}^{(n)}_{\mu_1\cdots\mu_{q-1}}\right)^2 \\
&= \frac{-1}{2(p-q)!}\int d^{p+1}x\sqrt{-\hat{g}}\left(\underline{Y}^{*(n)}_{\mu_1\cdots\mu_{p-q}} + \frac{\eta^*_{(n)}m_z^{*(n)}}{p+1-q}\overline{X}^{*(n)}_{\mu_1\cdots\mu_{p-q}} - \frac{\eta^*_{(n)}m_y^{*(n)}}{p+1-q}\underline{X}^{*(n)}_{\mu_1\cdots\mu_{p-q}}\right)^2 + \\
&\frac{-1}{2(p+1-q)!}\int d^{p+1}x\sqrt{-\hat{g}}\left[\left(\overline{Y}^{*(n)}_{\mu_1\cdots\mu_{p+1-q}} - \frac{m_z^{*(n)}X^{*(n)}_{\mu_1\cdots\mu_{p+1-q}}}{p+2-q}\right)^2 + \left(\underline{Y}^{*(n)}_{\mu_1\cdots\mu_{p+1-q}} - \frac{m_y^{*(n)}X^{*(n)}_{\mu_1\cdots\mu_{p+1-q}}}{p+2-q}\right)^2\right] + \\
&\frac{-1}{2(p+2-q)!}\int d^{p+1}x\sqrt{-\hat{g}}\left(Y^{*(n)}_{\mu_1\cdots\mu_{p+2-q}}\right)^2 \quad = \quad S^{*(n)}_{p+1-q}.
\end{aligned} \tag{28}
$$

In this sense, one may conclude that equivalent bulk fields yield equivalent brane fields.

An equally important requirement is that the duality should be compatible with localizability; according to (26) and (27), the extra-dimensional parts of the four modes are (reversely) proportional to their duals, i.e., W_i to W^*_{4-i}, and thus two corresponding modes are simultaneously localizable or not. Therefore, the contradiction mentioned in this section does not appear.

5. Conclusions and Discussion

In this work, we investigated the localization of a q-form field in a bulk on a codimension-two p-brane using the gauge-free localization mechanism. There turns out to be four types of KK modes: one q-form that appears in the usual localization, and in addition, two $(q-1)$-forms and one $(q-2)$-form. Each type of mode was found to satisfy two Schrödinger equations due to the two codimensions; thus, in total, we have eight equations for all modes. From these equations and the brane action, we observed how the KK modes obtain masses from the codimensions: (i) the highest mode is free of a codimensional index and therefore obtains masses from both codimensions; (ii) the intermediate mode with index y gains mass from the z-dimension and the one with index z gains mass from the y-dimension; (iii) the lowest mode has index xy and so has no mass. The mass spectra for modes of different ranks are related by common parameters $\eta_{(n)}$s.

Then, we found the effective action on the brane is gauge invariant. Through multiple downward couplings, the mass terms in the original action are compensated by one-rank-lower forms, and the action can be rewritten as a sum of such compensated squares. Thus, under certain gauge transformations, the brane action is invariant.

The preservation of Hodge duality constitutes another aspect of our findings. We found the Hodge duality in the bulk naturally reduces to four coupled dualities on the brane, where the coupled brane fields in the brane action match with their duals reversely by order, which makes effective brane actions of the bulk dual fields equivalent. Incidentally, the codimensional functions of the four modes and their dual counterparts' are consistent with each other, so there is no localizability contradiction. It should be noted that the duality transformation possesses a generality regardless of the mass modes; dualities of zero modes or various massive modes are special aspects of one single entirety.

Furthermore, from the derivation, one can realize that there is nothing special about codimension two; generalizing the results to higher n-dimensional reduction is just routine. For example, there will be in total 2^n types of KK modes, ranking from q to $q-n$, and in the brane action each of them obtains masses from the codimensions whose index it does not contain. For example, the highest-rank KK mode has n mass terms while the lowest does not have mass. These KK modes couple rank by rank to guarantee action gauge invariance; Hodge duality is preserved in the same manner, by reverse-rank pairing. Even in the situation where the q-form fields do not have enough rank for the codimensions or have too much rank for the brane, i.e., $q < n$ or $q > p + 1$, everything still works well except that there will be fewer KK modes.

As a result of the emergence of those lower rank KK types, which are typical of the new mechanism, it is even harder to find solutions for the system and hence to determine localizability. Nevertheless, being free of physical restriction, this general KK decomposition enjoys a primitive mathematical nature; as a result, gauge invariance and Hodge duality are inheritable through dimensional reduction. One may understand this from another point of view [60] that choosing a gauge before the KK decomposition will implicitly eliminate parts of the localization information; thus, in order to see the whole view, it is reasonable to consider a gauge-free decomposition.

Although we have successfully shown the equivalence between the q-form and its Hodge dual within the classical framework, an important question arises regarding the behavior of this equivalence under certain gauge fixings. If the gauge fixing is carried out in a way that respects the relationship between these fields, then their equivalence may still hold. However, if the gauge fixing introduces changes that affect this equivalence, then the fields may no longer remain equivalent. Therefore, the maintenance of equivalence through gauge fixing depends on the implementation of the fixing procedure.

The transition from classical dynamics to quantum behavior is a non-trivial task, particularly in the context of gauge theories, which require maintaining the gauge invariance that has been established in the classical setting. The quantization of the brane fields would involve transitioning from the classical description to a quantum description, which typically involves the replacement of classical fields with operators and the introduction of commutation relations.

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