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The QES sextic and Morse potentials: exact WKB condition and supersymmetry

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Abstract. In this paper, as a continuation of [Contreras-Astorga A., Escobar-Ruiz A. M. and Linares R., *Phys. Scr.* **99** 025223 (2024)] the one-dimensional quasi-exactly solvable (QES) sextic potential $V^{(\mathrm{qes})}(x) = \frac{1}{2}(\nu\, x^6 + 2\,\nu\,\mu\, x^4 + \left[\mu^2 - (4N+3)\nu\right]\, x^2)$ is considered. In the cases $N = 0, \frac{1}{4}, \frac{1}{2}, \frac{7}{10}$ the WKB correction $\gamma = \gamma(N,n)$ is calculated for the first lowest 50 states $n \in [0, 50]$ using highly accurate data obtained by the Lagrange Mesh Method. Closed analytical approximations for both γ and the energy $E = E(N,n)$ of the system are constructed. They provide a reasonably relative accuracy $|\Delta\gamma|$ and $|\Delta\epsilon|$, respectively, with upper bound $\lesssim 10^{-3}$ for all the values of (N,n) studied. Also, it is shown that the QES Morse potential is shape invariant characterized by a hidden $\mathfrak{sl}_2(\mathbb{R})$ Lie algebra and vanishing WKB correction $\gamma = 0$.

1 Introduction

Quasi-exactly solvable (QES) systems in quantum mechanics represent a class of Hamiltonian operators for which solely a finite sector of the whole spectra can be determined exactly (closed analytical expressions for spectra and wavefunctions) using algebraic methods [1]. Unlike exactly solvable systems (such as hydrogen atom, the harmonic oscillator and the Morse potentials), in QES systems only a subset of eigenstates and eigenvalues can be found exactly, while the rest remains unknown.

QES systems play an intermediate role between exactly solvable models and non-integrable systems. They are instrumental in understanding the boundaries of solvability in quantum mechanics and have applications in various fields, such as mathematical physics, quantum field theory, and condensed matter physics.

In a recent publication [2], the present authors studied the SUSY partner Hamiltonians of the QES sextic potential[1] $V^{(\mathrm{qes})} = \frac{1}{2}(\nu\, x^6 + 2\,\nu\,\mu\, x^4 + \left[\mu^2 - (4N+3)\nu\right]\, x^2)$ from an algebraic perspective, here N, $\nu > 0$, μ are real parameters. If $N \in \mathbb{Z}^+$, the potential $V^{(\mathrm{qes})}$ admits $(N+1)$ exact solutions (including the ground state) described by the underlying hidden $\mathfrak{sl}_2(\mathbb{R})$ Lie algebra. Hence, for each $N \in \mathbb{Z}^+$ the SUSY partner potential $V_1(x,N)$ can be constructed [3]. Interestingly, in the variable x or $z = x^2$, it was found that the corresponding SUSY Hamiltonian with potential $V_1(x,N)$ does not possess a hidden $\mathfrak{sl}_2(\mathbb{R})$ algebraic structure.

In this work, we aim to construct closed compact analytical approximations for the energy and for the so called WKB correction γ of the QES sextic potential. The SUSY partner Hamiltonians of the QES Morse potential are also investigated at the level of the corresponding algebraic operators governing the polynomial part of the exact QES eigenfunctions.

[1]There is a global factor of $\frac{1}{2}$ with respect to the definition given in [2]

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2 Exact WKB condition: analytical interpolations

Let us consider the one-dimensional Schrödinger equation for the QES sextic potential ($\nu = 1, \mu = 1$)

$$\mathcal{H}_{\rm qes}\,\psi(x) \;=\; \left[-\frac{1}{2}\frac{d^2}{dx^2} \;+\; \frac{1}{2}\left(x^6 \;+\; 2\,x^4 \;-\; 2\,(2N+1)\,x^2\right)\right]\psi(x) \;=\; E\,\psi(x)\;, \tag{1}$$

($\hbar = 1,\ m = 1$) where N is a real parameter. The problem is defined in the domain $x \in (-\infty, \infty)$ with $\psi \in \mathcal{L}^2$. For any N, using the Lagrange Mesh Mathematica Package [4] (LMMP), the numerical eigenfunctions $\psi_n(x)$ and eigenvalues E_n can be computed in a straightforward manner. Since the LMMP allows us to achieve very accurate numerical results with high precision (not less than 15 significant figures in accuracy) we can denote them, in practice, as the *exact* ones. These exact values serve as the starting point to study the so called exact WKB condition (see below).

The Bohr-Sommerfeld quantization condition combined with the *WKB correction*

$$\gamma \;=\; \gamma(N,n)\;,$$

(arising from the sum of higher-order WKB terms taken at the exact energies) define the exact WKB condition [5] which, by construction, must reproduce the *exact* energies. The above relation just means that γ is a function of both N and n. Explicitly, the correction γ is defined by the exact WKB condition

$$\int_{x_1(E)}^{x_2(E)} \sqrt{2\,(\,E \;-\; V^{(\rm qes)}(x,N)\,)}\,dx \;=\; \pi\left(n \;+\; \frac{1}{2} \;+\; \gamma\right)\;, \tag{2}$$

where $V^{(\rm qes)}(x,N)$ is the sextic potential function appearing in (1), $x_1(E)$ and $x_2(E)$ are the corresponding (symmetric) turning points obeying $V(x) = E$, $n \in \mathbb{Z}^+$ is the principal quantum number labeling the state (the number of nodes in ψ), and $E = E_n$ is taken as the *exact* energy obtained from the LMMP. Therefore, by fixing (N,n), one can consider (2) as the defining equation for the WKB correction $\gamma = \gamma(N,n)$.

For $N < N_{\rm critical} \approx 0.73295$, the ground state energy $E_0 = E_0(N)$ is always greater than the maximum of the potential $V^{(\rm qes)}$, thus, no instantons (tunneling) effects would be present, see Figs. 1, 2. The instantons effects are beyond the scope of this consideration.

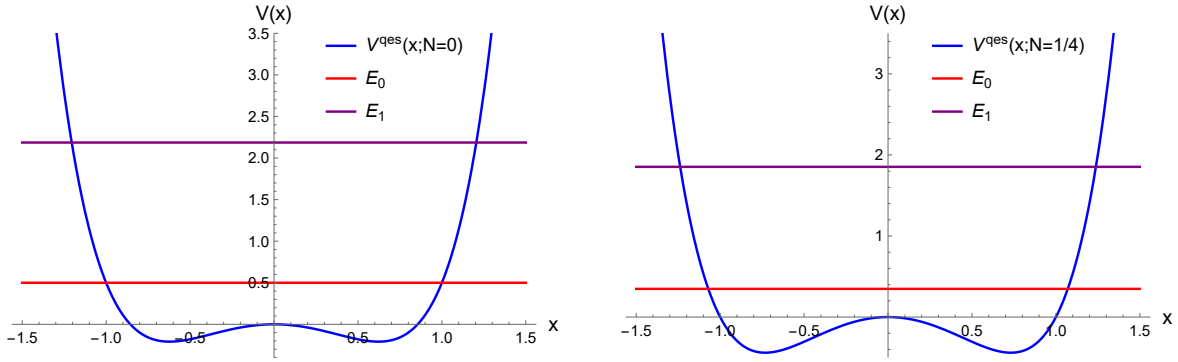

Figure 1: Cases $N = 0$ (left) and $N = \frac{1}{4}$ (right). The QES sextic potential $V^{(\rm qes)} = \frac{1}{2}\left(x^6 + 2\,x^4 - 2\,(2N+1)\,x^2\right)$ is shown (blue line). The ground state energy (in red) and the first excited level (in purple) are displayed as well. No instanton/tunneling effects occur. For $N = 0$, the exact analytical ground state is known only, for other levels the numerical results were calculated using the LMMP. For $N = \frac{1}{4}$, no exact analytical solutions exist.

In the cases $N = 0,\ \frac{1}{4},\ \frac{1}{2},\ \frac{7}{10}$, we calculate numerically the WKB correction $\gamma = \gamma(N,n)$ for the first 50 states, $n = 0, 1, 2, \ldots, 50$. The results are displayed in Fig. 3. At fixed N, the parameter $\gamma \sim 10^{-2}$ turns out to be a slowly decreasing, smooth function as n increases. Based on [5] and numerical experiments, it can be fitted by

$$\gamma_{\rm fit} \;=\; \frac{a_0 \;+\; a_1\,\tilde{n}}{\sqrt{1 \;+\; b_1^2\,\tilde{n} \;+\; b_2^2\,\tilde{n}^2 \;+\; b_3^2\,\tilde{n}^3 \;+\; b_4^2\,\tilde{n}^4}}\;, \tag{3}$$

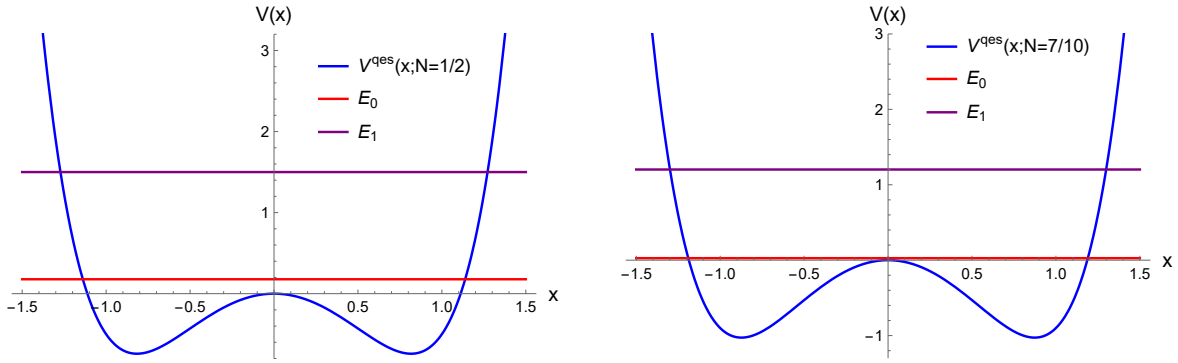

Figure 2: Cases $N = \frac{1}{2}$ (left) and $N = \frac{7}{10}$ (right). The QES sextic potential $V^{(\mathrm{qes})} = \frac{1}{2}\left(x^6 + 2\,x^4 - 2\,(2N+1)\,x^2\right)$ (blue line). The ground state energy (in red) and the first excited level (in purple) are displayed. No instanton/tunneling effects occur. For $N = \frac{1}{2}$, the exact analytical first excited state is known only, for other levels the numerical results were calculated using the LMMP. For $N = \frac{7}{10}$, no exact analytical solutions exist.

here $\tilde{n} \equiv n - 2 > 0$, $a_i = a_i(N)$ and $b_i = b_i(N)$ are interpolating parameters. In the particular case $N = 0$, we have

$$\begin{aligned} a_0 &\approx 0.019202\,, \qquad a_1 \approx 0.0038722\,, \\ b_1 &\approx 1.09402\,, \quad b_2 \approx 0.672026\,, \quad b_3 \approx 0.291328\,, \quad b_4 \approx 0.068666\,, \end{aligned} \tag{4}$$

(see Fig. 4). For the cases $N = \frac{1}{4}$, $\frac{1}{2}$, $\frac{7}{10}$, in Table 1 we report the corresponding values of the interpolating parameters in γ_{fit} (3), respectively. The fit (3), in comparison with the exact numerical values γ_{exact} obtained from (2), exhibits a small relative error $|\Delta\gamma| \lesssim 10^{-3}$ in the domain $2 < n < 50$ for all values of the parameter N considered, see Fig. 5.

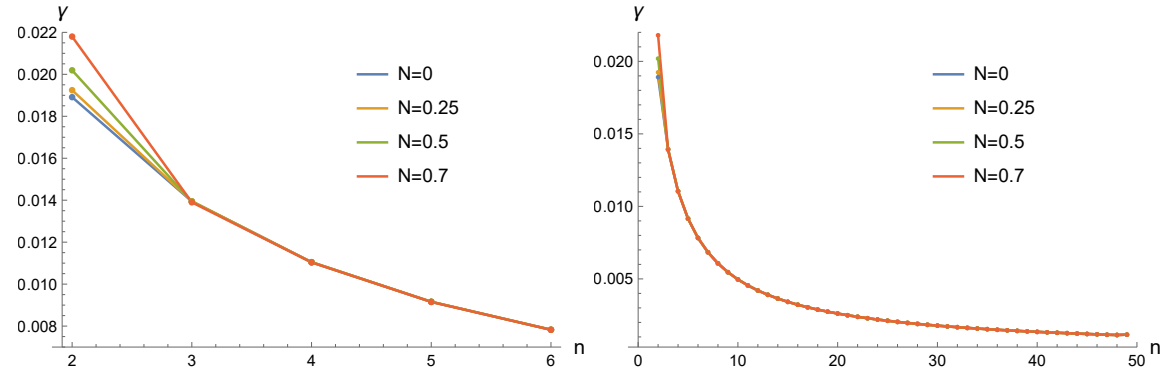

Figure 3: Behaviour of the WKB correction $\gamma = \gamma(n)$ at different values of N. Already for $n > 2$, the function $\gamma = \gamma(n)$ remains almost unaffected as the parameter N increases from 0 up to $\frac{7}{10}$.

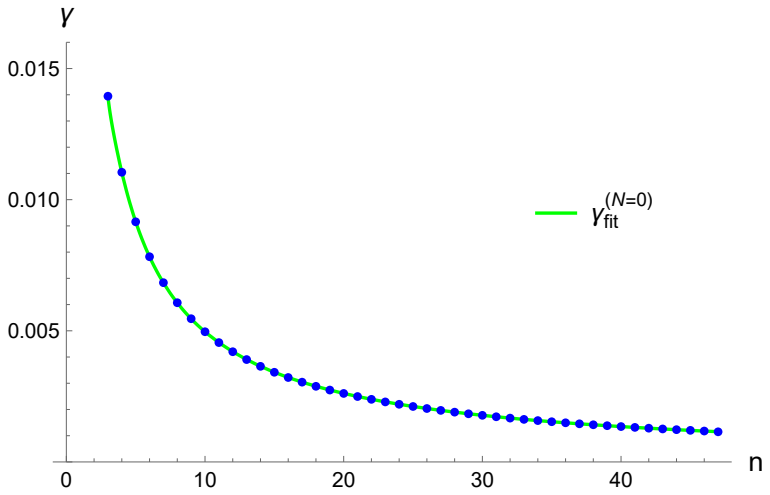

Figure 4: Case $N = 0$. The analytical interpolation (3) with parameters (4) corresponds to the green line whilst the bullets (in blue) mark the numerical values $\gamma_{\rm exact}$ of the WKB correction $\gamma = \gamma(n)$.

Table 1: The parameters of the analytical interpolation $\gamma_{\rm fit}$ (3) for the cases $N = \frac{1}{4},\ \frac{1}{2},\ \frac{7}{10}$.

parameter	$N = \frac{1}{4}$	$N = \frac{1}{2}$	$N = \frac{7}{10}$
a_0	0.0332512	0.0469388	0.041918
a_1	0.0419986	0.0656	0.0622568
b_1	3.70908	5.57123	5.08444
b_2	2.96296	4.45283	4.17168
b_3	2.23779	3.46547	3.26907
b_4	0.749137	1.17011	1.11086

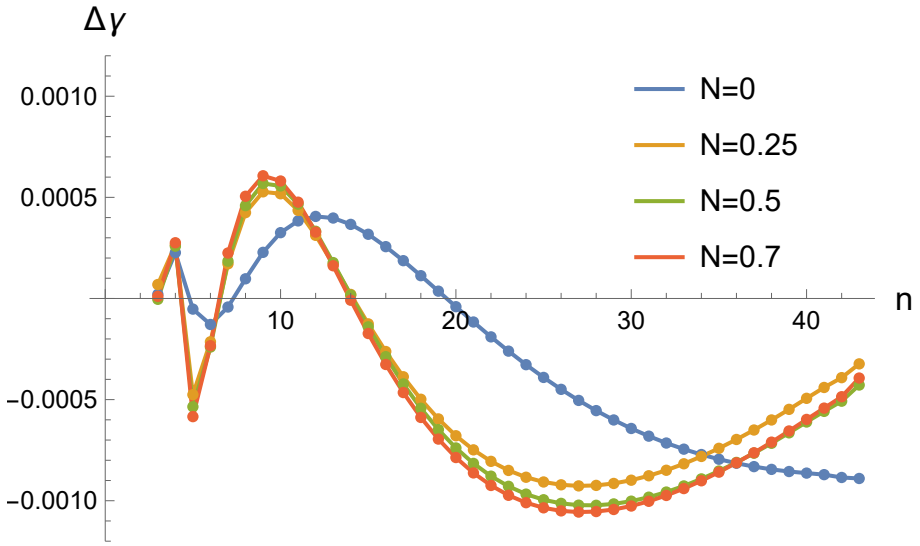

Figure 5: Relative error $\Delta\gamma \equiv \frac{\gamma_{\rm exact}-\gamma_{\rm fit}}{\gamma_{\rm exact}}$ *vs* n at different values of N. The function $\gamma_{\rm fit}$ is taken as in (3), the values of the interpolation parameters are presented in (4) and Table 1, respectively.

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3 Energies: analytical interpolations

As mentioned in Section 2, for the QES sextic potential (1) using the LMMP we computed at $N = 0, \frac{1}{4}, \frac{1}{2}, \frac{7}{10}$ the energy E_n for the first 50 states. At fixed N, it is a slowly increasing, smooth function as n grows, see Fig. 6. In the domain $n \in [0, 50]$ the energy varies from ~ 1 up to ~ 410. At fixed n, the energy decreases monotonously as the parameter N changes from 0 up to $N = 0.7$.

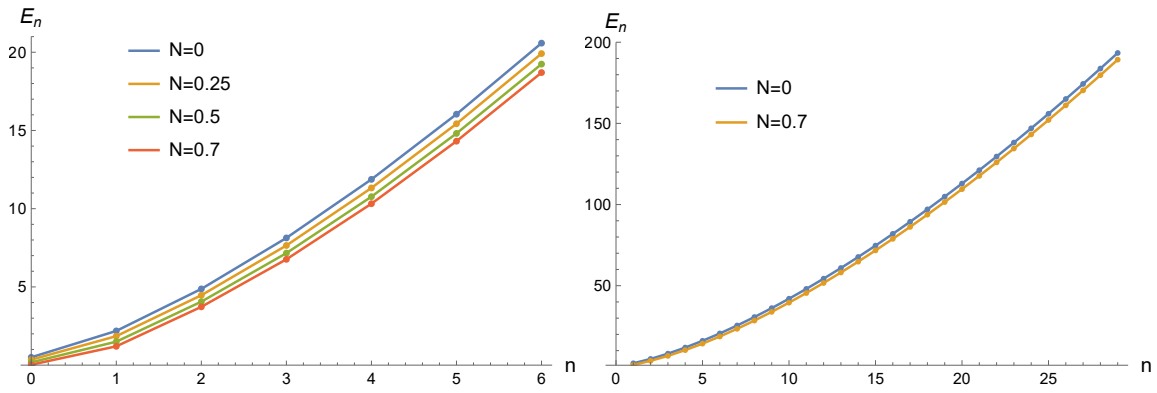

Figure 6: The energy E_n *vs* n for the cases $N = 0, \frac{1}{4}, \frac{1}{2}, \frac{7}{10}$. Results are displayed in atomic units.

At fixed N, the energy $E_n = E(n)$ can be interpolated by

$$E_{\rm fit}(n) \;=\; E_0\,\tilde{n} \;+\; \sqrt{\tilde{n}-1}\,\frac{(a_0 \,+\, A_1\,\tilde{n} \,+\, A_2\,\tilde{n}^2 \,+\, A_3\,\tilde{n}^3 \,+\, A_4\,\tilde{n}^4 \,+\, A_5\,\tilde{n}^5 \,+\, A_6\,\tilde{n}^6)}{1 \,+\, B_1^2\,\tilde{n} \,+\, B_2^2\,\tilde{n}^2 \,+\, B_3^2\,\tilde{n}^3 \,+\, B_4^2\,\tilde{n}^4 \,+\, B_5^2\,\tilde{n}^5}\;, \tag{5}$$

where $\tilde{n} \equiv n + 1 > 0$, $A_i = A_i(N)$ and $B_i = B_i(N)$ are interpolating parameters. At $n = 0$ ($\tilde{n} = 1$) the analytical interpolation $E_{\rm fit}$ reproduces the exact numerical value of the ground state energy $E_{\rm fit} \to E_0$ whereas at large $n \to \infty$ (thus, $\tilde{n} \to \infty$) we reproduce the asymptotic behaviour $E \propto n^{3/2}$. If $n \to \infty$, in the semi-classical limit we consider large distances $x \to \infty$ where the sextic potential behaves as $V^{(\rm qes)} \sim \frac{x^6}{2}$. Standard Bohr-Sommerfeld quantization condition provides the dominant term

$$E_n \;\sim\; \frac{1}{2}\pi^{3/4}\left(\frac{\Gamma\left(\frac{5}{3}\right)}{\Gamma\left(\frac{7}{6}\right)}\right)^{3/2} n^{3/2} \;=\; 1.13254\,n^{3/2}\;,$$

in the limit $n \to \infty$. Specifically, for the interpolation (5) with $N = 0$ we obtain

$$\begin{aligned}
&A_0 \;=\; -303.678, \quad A_1 \;=\; -13.8988, \quad A_2 \;=\; 185.841, \quad A_3 \;=\; 22.0053, \quad A_4 \;=\; 13.4821,\\
&A_5 \;=\; 0.777246, \quad A_6 \;=\; 1.06539,\\
&B_1 \;=\; 16.0303, \quad B_2 \;=\; 4.22639, \quad B_3 \;=\; 3.82635, \quad B_4 \;=\; 1.17858, \quad B_5 \;=\; 0.969174,
\end{aligned} \tag{6}$$

see Fig. 7. In this case, for the asymptotic behaviour at $n \to \infty$ we found that $E_{\rm fit} \sim 1.13424\,n^{3/2}$.

In Table 2 the corresponding values of the interpolating parameters in $E_{\rm fit}$ (5) are shown as a function of N. The fit (5), in comparison with the exact numerical values $E_{\rm exact}$ obtained using the LMMP, exhibits a small relative error $|\Delta\epsilon| \lesssim 10^{-3}$ in the whole domain $2 < n < 50$ for all values of the parameter N considered, see Fig. 8. In the particular case $N = 0$, this error reduces up to $|\Delta\epsilon| \lesssim 10^{-5}$.

Table 2: The parameters of the analytical interpolation $E_{\rm fit}$ (5) for the cases $N = \frac{1}{4}, \frac{1}{2}, \frac{7}{10}$. The ratio A_6/B_5^2 defines the asymptotic behaviour at $n \to \infty$ in $E_{\rm fit}$ (see text).

parameter	$N = \frac{1}{4}$	$N = \frac{1}{2}$	$N = \frac{7}{10}$
A_0	-2961.78	-4024.34	-6563.79
A_1	407.572	1776.87	3617.77
A_2	687.568	553.412	25.2448
A_3	684.966	210.095	443.209
A_4	-1.90414	13.202	20.7142
A_5	0.755926	1.7131	2.55102
A_6	1.06975	1.07253	1.07404
B_1	40.1137	37.9358	40.0919
B_2	22.2057	5.64541	11.9183
B_3	3.02188	5.36947	6.51999
B_4	0.936421	1.09358	1.18494
B_5	0.967749	0.965024	0.962401
A_6/B_5^2	1.14224	1.15169	1.1596

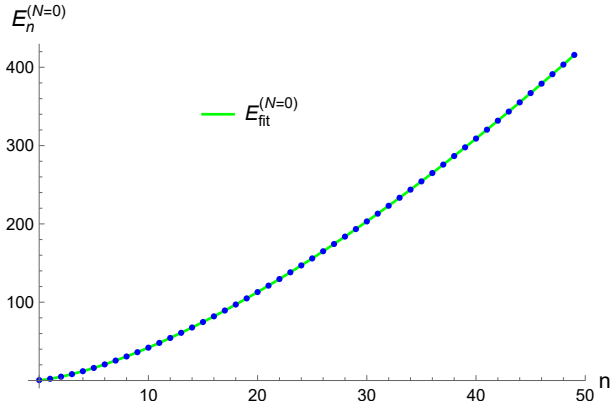

Figure 7: Case $N = 0$. The analytical interpolation (5) with parameters (6) corresponds to the green line whilst the bullets (in blue) mark the numerical values obtained using the LMMP.

4 Supersymmetric quantum mechanics and intertwining relations

Supersymmetric quantum mechanics (SUSY) is a technique used to expand the family of solvable potentials in quantum mechanics [6]. It relates two different Hamiltonian quantum systems, H_0 and H_1, by means of an intertwining operator A_1^+ as follows:

$$H_1 A_1^+ = A_1^+ H_0, \quad \text{where} \quad H_i = -\frac{1}{2}\frac{d^2}{dx^2} + V_i(x), \quad i = 0, 1\ . \tag{7}$$

This intertwining relation allows us to map solutions of the eigenvalue equation $H_0\,\psi = E\,\psi$ to solutions of the isospectral problem $H_1\,\phi = E\,\phi$ (except possibly for certain energy eigenstates). Naturally, the relation (7) imposes certain restrictions on H_1. To see this, let us consider the simplest case when A_1^+ is a first-order differential operator

$$A_1^+ = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + W(x)\right) = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + (\ln u(x))'\right), \tag{8}$$

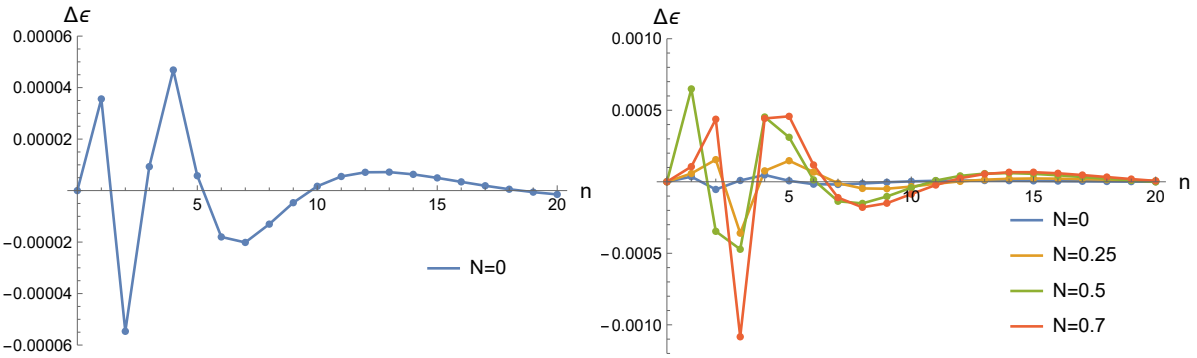

Figure 8: Relative error $\Delta\epsilon \equiv \frac{E_{\rm exact}-E_{\rm fit}}{E_{\rm exact}}$ *vs* n. The function $E_{\rm fit}$ is taken as in (5) whilst the values of the interpolation parameters are displayed in (6) and Table 2.

where $W(x)$ or, equivalently, $u(x)$ are functions to be found. We denote $u'(x)$ as $\frac{du(x)}{dx}$. In the literature, $W(x)$ is known as superpotential and $u(x)$ as the seed function. By substituting the explicit form of A_1^+ (8) in (7), we obtain the conditions

$$H_1 = H_0 - (\ln u(x))'' \ , \quad H_0\, u(x) = \epsilon\, u(x) \ , \quad u(x) \neq 0, \tag{9}$$

here ϵ is a constant; i.e., $u(x)$ must be a nodeless solution for the initial Schrödinger equation with Hamiltonian H_0. Once $u(x)$ is chosen, H_1 is fixed.

5 SUSY partners of the sextic potentials

Recently, in the work [2], the present authors studied the algebraic structure of the SUSY partners $V_1(x)$ of the sextic potential $V_0 = V^{\rm qes}(x) = \frac{1}{2}(\nu^2\, x^6 + 2\,\nu\,\mu\, x^4 + (\mu^2 - (4N+3)\nu)\ x^2)$ as a function of N. Accordingly, the central object was the Hamiltonian $H_0 = -\frac{1}{2}\frac{d^2}{dx^2} + V^{\rm qes}(x)$. Here, in the $\mathbb{Z}_2$-invariant variable $z = x^2$, we further elaborate on the relevant intertwining relation between H_0 and H_1 at the level of the algebraic gauge rotated operators, h_0 and h_1, which govern the polynomial part of the exact QES eigenfunctions, respectively. Explicitly, using the gauge factor

$$\Gamma(z) = \exp\left(-\frac{\nu}{4}z^2 - \frac{\mu}{2}z\right) \ , \tag{10}$$

the gauge-rotated Hamiltonian $h_0 = \Gamma^{-1}\, H_0\, \Gamma$ takes the form:

$$h_0 \ = \ -2\,z\,\frac{d^2}{dz^2} \ + \ \left(2\,\nu\,z^2 + 2\,\mu\,z - 1\right)\frac{d}{dz} \ - \ 2\,N\,\nu\,z \ + \ \frac{\mu}{2} \ . \tag{11}$$

If the parameter N takes positive integer values, the spectral problem $h_0\, P(z) = E\, P(z)$ has $(N+1)$ polynomial eigenfunctions $P_j^{(N)}(z)$, $j = 0, 1, 2, N$. Accordingly, the QES solutions of the original Schrödinger equation $H_0\,\psi = E\,\psi$ become $\psi_j(x) = \Gamma(z = x^2) P_j^{(N)}(z = x^2)$. The above operator h_0 possesses a hidden $\mathfrak{sl}_2(\mathbb{R})$ algebra.

Now, we can perform a first-order SUSY transformation of H_0, described above in section 4, using as a seed function the exact (analytical) ground state $\psi_0(x) = \Gamma(x^2) P_0^{(N)}(x^2)$. That way, we obtain the SUSY partner Hamiltonian H_1. For this operator H_1, the N exact QES solutions can be written as $\phi_i(x) \ = \ \Gamma_N(x)\,\mathcal{P}_i(x)$, $i = 0, 1, 2, (N-1)$, where the factor $\mathcal{P}_i(x)$ is a polynomial odd-function in $x-$variable whilst Γ_N is the non-polynomial part. Further details can be found in [2].

Similarly to H_0, one can introduce the operator $h_1 = \Gamma_N^{-1}\, H_1\, \Gamma_N$ which describes the polynomial QES solutions $\mathcal{P}_i(x)$ of H_1. From the intertwining relation (7), we can immediately derive the SUSY partner of h_0, namely

$$h_1\, \mathcal{A}_1^+ \ = \ \mathcal{A}_1^+\, h_0 \ , \tag{12}$$

with intertwining operator $\mathcal{A}_1^+ = \Gamma_N^{-1} A_1^+\, \Gamma$ and A_1^+ defined in (8).

Eventually, from the equation $h_0\,P(z) = E\,P(z)$ and (12) we obtain that $\mathcal{P}(\sqrt{z}) = \mathcal{A}_1^+ P(z)$ solves $h_1\mathcal{P}(\sqrt{z}) = E\,\mathcal{P}(\sqrt{z})$. In variable z, the operators $\mathcal{A}_1^+$ and h_1 read

$$\begin{aligned}\mathcal{A}_1^+ &= -\sqrt{2\,z}\,p(z)\,\left(\frac{d}{dz}\;+\;2\,\frac{\dot{p}(z)}{p(z)}\right),\\ h_1 &= h_0\;+\;4\,z\,\frac{\dot{p}(z)}{p(z)}\frac{d}{dz}\;-\;2\,z\frac{\ddot{p}(z)}{p(z)} - (2\,\nu\,z^2+2\,\mu\,z+1)\frac{\dot{p}(z)}{p(z)}\;+\;3\,\nu\,z\;+\;\mu\;,\end{aligned} \tag{13}$$

where we have used the notation $p(z) = P_0^{(N)}(z)$ and $\dot{p}(z) = \frac{dp(z)}{dz}$. Note that the global factor $p(z)$ in $\mathcal{A}_1^+$ will avoid rational expressions in the functions $\mathcal{P}(\sqrt{z})$. Except for the case $N = 0$, no Lie algebraic structure is known in h_1.

6 SUSY partners of the quasi-exactly solvable Morse potential

In this Section, we consider the QES Morse potential [7]

$$V_0(x;N) \;=\; \frac{1}{2}\big(a^2\,e^{-2\,\alpha\,x}\;-\;a\,e^{-\alpha\,x}(2\,b+\alpha\,(2\,N+1)\,)\;+\;(N\alpha+b)^2\big)\;, \tag{14}$$

a QES system of the first type, $x \in (-\infty,\infty)$ where $a > 0$, b, $\alpha > 0$ and N are parameters. Let us take the *fermionic* Hamiltonian

$$H_0 \;=\; -\frac{1}{2}\frac{d^2}{dx^2}\;+\;V_0(x;N)\;. \tag{15}$$

In the variable

$$z \;=\; e^{-\alpha\,x}\;, \tag{16}$$

the operator (15) can be transformed to a quantum "top" [7]; namely, it can be rewritten as a constant coefficient quadratic combination in the first order differential operators

$$\mathcal{J}^+(N) \;\equiv\; z^2\,\partial_z\;-\;N\,z\;,\qquad \mathcal{J}^0(N) \;\equiv\; z\,\partial_z\;-\;\frac{N}{2}\;,\qquad \mathcal{J}^-\;\equiv\;\partial_z\;, \tag{17}$$

which span, for any value of the parameter N, the $\mathfrak{sl}_2(\mathbb{R})$ algebra. Explicitly, using the gauge factor

$$\Gamma(z) \;=\; e^{-\frac{a}{\alpha}\,e^{-\alpha x}-b\,x} \;=\; e^{-\frac{a}{\alpha}\,z+\frac{b}{\alpha}\,\log(z)}\;, \tag{18}$$

one obtains the gauge-rotated algebraic operator

$$\begin{aligned}h_0^{(N)} &\equiv \Gamma^{-1}\,H_0\,\Gamma\\ &= -\frac{1}{2}\left(2\,a\,\alpha\,N z\;+\;b^2-2c\right)\;-\;\frac{1}{2}\alpha\,z\,(-2\,az+\alpha+2\,b)\,\partial_z\;-\;\frac{1}{2}\,\alpha^2\,z^2\,\partial_z^2\;,\end{aligned} \tag{19}$$

or, equivalently, in Lie algebraic form

$$h_0^{(N)} \;=\; -\frac{\alpha^2}{2}\,\mathcal{J}^+\,\mathcal{J}^-\;+\;a\,\alpha\,\mathcal{J}^+\;-\;\frac{1}{2}\,\alpha\,(2b+\alpha\,(N\;+\;1)\,)\,\mathcal{J}^0\;+\;\frac{1}{2}\left(2\,c-b^2\right)\;. \tag{20}$$

At fixed N, the spectral problem

$$h_0^{(N)}\,P(z) \;=\; E\,P(z)\;, \tag{21}$$

admits $(N+1)$ polynomial solutions $P_i(z)$ in the $z-$variable. Accordingly, the original Hamiltonian operator H_0 (15) possesses $(N+1)$ eigenfunctions in the form

$$\psi_i \;=\; \Gamma(z)\,P_i(z)\;,\quad i = 1,2,\ldots,N\;. \tag{22}$$

In $z-$variable, the original Hamiltonian operator H_0 (15) reads

$$H_0 \;=\; -\frac{1}{2}\alpha^2\,(z^2\,\partial_z^2\;+\;z\,\partial_z)\;+\;\frac{1}{2}\left(a^2\,z^2\;-\;a\,(\alpha\,(2N+1)\;+\;2b)\,z+2\,c\right)\;. \tag{23}$$

Below, we present concrete examples.

6.1 Case $N = 0$

For $N = 0$, the potential V_0 in (15) reads

$$V_0(x; N=0) \;=\; \frac{1}{2}\left(a^2 e^{-2\,\alpha\,x} \;-\; a\,(\alpha\,+\,2b)\,e^{-\alpha\,x} + b^2\right) \;=\; \frac{1}{2}\left(a^2\,z^2 \;-\; a\,(\alpha\,+\,2b)\,z + b^2\right)\;. \tag{24}$$

The ground state function of (15) is given by

$$\psi_0^{(N=0)} \;=\; \Gamma \;=\; e^{-\frac{a}{\alpha}\,e^{-\alpha x} - b\,x} \;=\; e^{-\frac{a}{\alpha}\,z + \frac{b}{\alpha}\,\log(z)}\;, \tag{25}$$

$P(z) = 1$, with zero energy

$$E_0^{(N=0)} \;=\; 0\;. \tag{26}$$

In this case,

$$h^{(N=0)} \;\equiv\; \left(\psi_0^{(N=0)}\right)^{-1} H_0\,\psi_0^{(N=0)} \;=\; \frac{1}{2}\,\alpha\left(\;-\,\alpha\,\mathcal{J}^+\,\mathcal{J}^- \;+\; 2\,a\,\mathcal{J}^+ \;-\; (\alpha+2\,b)\,\mathcal{J}^0\right)\;. \tag{27}$$

For instance, taking $a = 1,\ b = 8,\ \alpha = \sqrt{2}$, the next lowest energies levels are $E_1 = 10.313708498985$, $E_2 = 18.62741699797$, $E_3 = 24.94112549695$, $E_4 = 29.25483399594$, $E_5 = 31.56854249492$, see Fig. 9. Interestingly, for the QES Morse potential, it can be checked that the semiclassical WKB method reproduces the exact quantum mechanical results. Thus, the WKB correction vanishes $\gamma = 0$ for all the bound states. Explicitly,

$$\begin{aligned}\int_{x_1(E)}^{x_2(E)} \sqrt{2\left(\,E \;-\; V_0(x, N=0)\,\right)}\,dx \;&=\; \frac{\pi\left(\alpha - 2\sqrt{b^2 - 2\,E} + 2b\right)}{2\,\alpha} \\ &= \pi\,(n + 1/2)\;,\end{aligned} \tag{28}$$

($\gamma = 0$) which leads to the exact quantum spectra $E_n = -\frac{1}{2}\alpha\,n\,(\alpha\,n - 2\,b)$. For the treatment of the exactly solvable case $V_{ES} \propto (e^{-2\,\alpha\,x} - 2\,e^{-\alpha\,x})$, see [8].

The associated SUSY partner potential can be derived immediately

$$V_1(x; N=0) \;=\; \frac{1}{2}\,a^2\,e^{-2\alpha x} \;+\; a\,e^{-\alpha\,x}\left(\frac{\alpha}{2} - b\right) \;+\; \frac{b^2}{2} \;=\; \frac{1}{2}(\,a^2\,z^2 \;+\; a\,z\,(\alpha - 2\,b) \;+\; b^2\,)\;. \tag{29}$$

Also, at $N = 0$ the *bosonic* Hamiltonian

$$H_1^{(N=0)} \;=\; -\frac{1}{2}\frac{d^2}{dx^2} \;+\; V_1(x; N=0) \;=\; -\frac{1}{2}\frac{d^2}{dx^2} \;+\; V_0(x, N=-1)\;, \tag{30}$$

can be transformed into a quantum "top" of the form (20) but with a negative integer (cohomology) parameter $N = -1$. Therefore, similar to the QES sextic potential [2] the SUSY partner Hamiltonian $H_1^{(N=0)}$ with potential (29) still possesses a hidden $\mathfrak{sl}_2(\mathbb{R})$ Lie algebraic structure.

6.2 Case $N = 1$

For $N = 1$, we have

$$V_0 \;=\; \frac{1}{2}\left(a^2 e^{-2\alpha x} - a(3\alpha + 2b)e^{-\alpha\,x} + (\alpha + b)^2\,\right) \;=\; \frac{1}{2}\left(a^2\,z^2 - a\,z\,(3\alpha + 2b) + (\alpha + b)^2\,\right)\;, \tag{31}$$

see Fig. 10. The exact solutions are the ground state function given by

$$\psi_0^{(N=1)} \;=\; e^{-\frac{a}{\alpha}\,e^{-\alpha x} - b\,x}\,e^{-\alpha\,x} \;=\; e^{-\frac{a}{\alpha}\,z + \frac{b}{\alpha}\,\log(z)}\,z\;, \tag{32}$$

$P(z) = z$, again with zero energy

$$E_0^{(N=1)} \;=\; 0\;, \tag{33}$$

and the first excited state

$$\psi_1^{(N=1)} \;=\; e^{-\frac{a}{\alpha}\,e^{-\alpha x} - b\,x}(e^{-\alpha\,x} \;-\; \frac{\alpha + 2\,b}{2\,a}) \;=\; e^{-\frac{a}{\alpha}\,z + \frac{b}{\alpha}\,\log(z)}\,(z \;-\; \frac{\alpha + 2\,b}{2\,a})\;, \tag{34}$$

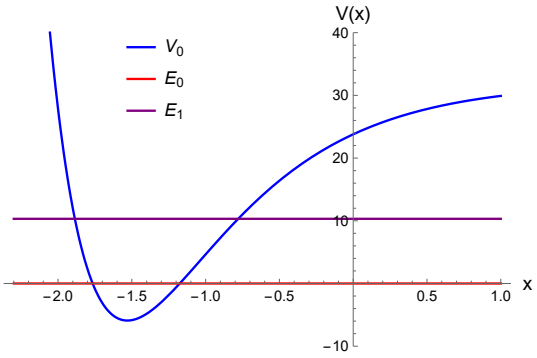

Figure 9: $a = 1, b = 8, \alpha = \sqrt{2}$ and $N = 0$, the potential V_0 (blue curve) in (15) and the first two energy levels E_0 and E_1.

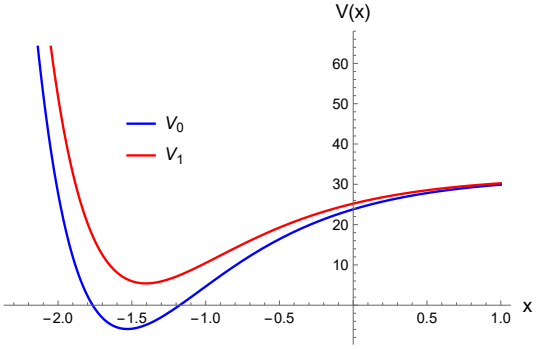

Figure 10: For fixed $a = 1, b = 8, \alpha = \sqrt{2}$ and $N = 0$, the confining potential V_0 (blue curve) in (15) and its susy partner $V_1(x)$.

thus, $P(z) = z - \frac{\alpha+2\,b}{2\,a}$, with energy

$$E_1^{(N=1)} \;=\; \frac{1}{2}\,\alpha\,(\alpha+2b)\;. \tag{35}$$

Eventually, the associated susy partner potential satisfies that

$$V_1(x; N=1) \;=\; V_0(x, N=0)\;. \tag{36}$$

6.3 Case arbitrary integer N

For arbitrary N, we have

$$\begin{aligned} V_0 \;&=\; \frac{1}{2}\left(a^2 e^{-2\alpha x} - a\big(\alpha(2N+1)+2b\big)e^{-\alpha\,x} + (N\,\alpha+b)^2\right) \\ &=\; \frac{1}{2}\left(a^2\,z^2 - a\,z\,\big(\alpha(2N+1)+2b\big) + (N\,\alpha+b)^2\right)\;. \end{aligned} \tag{37}$$

The ground state function given by

$$\psi_0^{(N)} \;=\; e^{-\frac{a}{\alpha}\,e^{-\alpha x} - b\,x}\,e^{-N\,\alpha\,x} \;=\; e^{-\frac{a}{\alpha}\,z + \frac{b}{\alpha}\,\log(z)}\,z^N\;, \tag{38}$$

$P(z) = z^N$, having zero energy

$$E_0^{(N)} \;=\; 0\;. \tag{39}$$

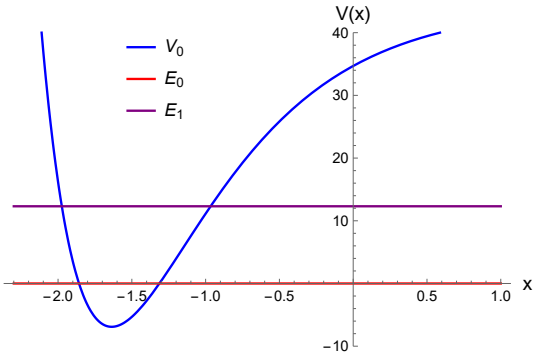

Figure 11: $a = 1, b = 8, \alpha = \sqrt{2}$ and $N = 1$, the potential V_0 (blue curve) in (15) and the first two energy levels E_0 and E_1.

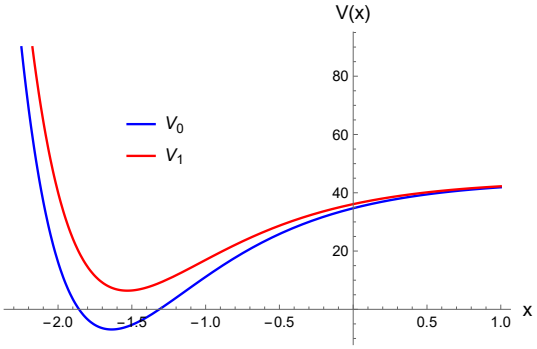

Figure 12: For fixed $a = 1$, $b = 8$, $\alpha = \sqrt{2}$ and $N = 1$, the confining potential V_0 (blue curve) in (15) and its susy partner $V_1(x)$.

The corresponding SUSY partner potential obeys the relation

$$V_1(x; N) \;=\; V_0(x, N-1)\;, \tag{40}$$

hence, akin to the case of the exactly solvable Morse potential, in the QES case, V_0 and V_1 are shape invariant potentials; see in [9] more details of the definition of shape invariant potential and the study of the exactly solvable Morse potential. In the case of the QES Morse potential and its SUSY partner, instantons effects are absent because no tunneling effects are present.

7 Summary

We studied two quasi-exactly solvable systems: the QES sextic and Morse potentials. For the sextic potential, approximate expressions for the WKB correction γ and the energy spectrum $E_n(N)$ were constructed. Explicit analytical fits were given in the cases $N = 0,\ 1/4,\ 1/2,\ 7/10$ where instanton effects are completely absent. At the level of algebraic operators which describe the polynomial QES solutions, the expression of the associated intertwining operator, in the variable $z = x^2$, were presented. Furthermore, a first-order SUSY transformation was performed to the QES Morse potential, showing that it is shape-invariant; as a consequence, the SUSY partner potential has the same hidden $\mathfrak{sl}_2(\mathbb{R})$ Lie algebraic structure with a shifted cohomology parameter $N \to N - 1$. Generalizations of the exact WKB correction as well as the relation shape invariance- intertwiners [10] for multidimensional systems (beyond separation of variables) is of great importance. As a future analysis, it would be interesting to study higher-order SUSY partners of the QES Morse potential using excited states as seed functions.

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Data availability
Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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