

The incompressible non-relativistic Navier-Stokes equation from gravity

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ABSTRACT: We note that the equations of relativistic hydrodynamics reduce to the incompressible Navier-Stokes equations in a particular scaling limit. In this limit boundary metric fluctuations of the underlying relativistic system turn into a forcing function identical to the action of a background electromagnetic field on the effectively charged fluid. We demonstrate that special conformal symmetries of the parent relativistic theory descend to ‘accelerated boost’ symmetries of the Navier-Stokes equations, uncovering a conformal symmetry structure of these equations. Applying our scaling limit to holographically induced fluid dynamics, we find gravity dual descriptions of an arbitrary solution of the forced non-relativistic incompressible Navier-Stokes equations. In the holographic context we also find a simple forced steady state shear solution to the Navier-Stokes equations, and demonstrate that this solution turns unstable at high enough Reynolds numbers, indicating a possible eventual transition to turbulence.

KEYWORDS: Gauge-gravity correspondence, Black Holes in String Theory, AdS-CFT Correspondence

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Contents

1	Introduction and discussion	1
2	Scaling the Navier-Stokes equation	6
2.1	The Navier-Stokes equation as a universal limit of fluid dynamics	6
2.2	A scaling limit	6
2.3	Charged fluids	9
2.4	Reduction of Cauchy data	9
3	Symmetries	9
3.1	Dilatations	10
3.2	Special conformal transformations	10
3.3	The full symmetry group	11
4	The gravity solution dual to fluid dynamics in the scaling regime	12
4.1	The dual bulk metric	12
4.2	Comments on the bulk metric	14
5	Simple steady state shear flows at arbitrary Reynolds number	15
A	Stability analysis	17
A.1	Setting up the eigenvalue equations	17
A.2	Qualitative nature of the solution	19
A.3	Numerical evaluation of the frequency	20
A.4	Perturbation theory at small Reynolds numbers	21
B	A simple flow on a sphere	22

1 Introduction and discussion

The flow of non-relativistic fluids is described by the Navier-Stokes equations

$$\begin{aligned} \dot{\vec{v}} + \vec{v} \cdot \nabla \vec{v} &= -\vec{\nabla} P + \nu \nabla^2 \vec{v} + \vec{f} \\ \vec{\nabla} \cdot \vec{v} &= 0 \end{aligned} \tag{1.1}$$

where \vec{v} is the fluid velocity, P the fluid pressure, ν the shear viscosity and \vec{f} an externally specified forcing function. Although these equations describe a wide variety of natural phenomena (see e.g. [1]) and have been intensively studied for almost two centuries, their extremely rich phenomenology remains very poorly understood. In particular, most fluid flows go turbulent at high Reynolds number, i.e. in the regime in which the viscous fluid

term is negligible compared to the nonlinear convective term in (1.1). Although turbulent flows appear complicated and statistical in nature, it has been suggested (see e.g. [2]) that these flows are in fact governed by a new and simple universal mathematical structure analogous to a fixed point of the renormalization group flow equations. A completely new angle on fluid dynamics could well be needed in order to uncover such a structure.

Recent investigations [3–13], within the framework of the AdS/CFT correspondence of string theory [14] have revealed an initially surprising relationship between the vacuum equations of Einstein gravity in an asymptotically locally AdS_{d+1} space and the equations of hydrodynamics in d dimensions. More concretely these papers study a class of regular, long wavelength locally asymptotically AdS_{d+1} solutions to the vacuum Einstein equations with a negative cosmological constant. These solutions are shown to be in one to one correspondence with solutions of the d dimensional hydrodynamical equations $\nabla_\mu T^{\mu\nu} = 0$. In the last equation the stress tensor $T^{\mu\nu}$ is a holographically determined functional of a d dimensional fluid velocity u^μ and temperature T . In the long distance limit under consideration it is appropriate to expand the stress tensor in a power series in the boundary derivatives of the velocity and temperature fields. Schematically

$$T^{\mu\nu} = \sum_{n=0}^{\infty} T^{d-n} T_n^{\mu\nu} \tag{1.2}$$

where T is the local fluid temperature and $T_n^{\mu\nu}$ is a local function of the fluid velocity and temperature of n^{th} order in spacetime derivatives. The expressions for $T_n^{\mu\nu}$ for $n \leq 2$ have been explicitly determined in the references cited above (see [13] for the most general result) and constitute a relativistic generalization of the incompressible Navier-Stokes stress tensor. In summary, classical asymptotically AdS_{d+1} gravity is ‘dual’ to relativistic generalizations of the Navier-Stokes equations at long distance and time scales.

Many theoretical and experimental investigations of fluid dynamics study the actual incompressible Navier-Stokes equations (1.1). It is consequently of interest to find a dual description of the Navier-Stokes equations (1.1) themselves rather than their relativistic generalizations. This may be simply achieved by taking the appropriate limit of the results of [4–13], and this limit is the topic of our note.

In order to find the gravitational description dual to (1.1), we adopt a two step procedure. First, purely at the level of fluid dynamics we make a straightforward and possibly well known observation. We note that the non-relativistic incompressible Navier-Stokes equations (1.1) are the precise and universal outcome of a particular combined scaling limit (one in which we scale to long distances, long times, low speeds and low amplitudes in a coordinated fashion) applied to any reasonable relativistic equations of hydrodynamics, i.e. the hydrodynamics a relativistic fluid with any reasonable equation of state.¹ The equations of fluid dynamics become non-relativistic at low speeds for the usual reason, and also become effectively incompressible, as we go to velocities much lower than the speed of sound (see for instance [1]).

¹For instance the fluid that is dual to gravity studied below whose equation of state is dictated by conformal invariance.

To be more specific we show that the equations of hydrodynamics reduce, in a precise fashion, to the incompressible non-relativistic Navier-Stokes equations (1.1) under the limit

$$\begin{aligned}
 \delta x &\sim \frac{1}{T\epsilon} \\
 \delta t &\sim \frac{1}{T\epsilon^2} \\
 v^i &\sim \epsilon \\
 \delta P &\sim T^d \epsilon^2 \\
 \epsilon &\rightarrow 0
 \end{aligned}
 \tag{1.3}$$

where δx is the spatial length scale, δt the temporal scale while v^i and δP represent estimates of the magnitude of velocity and pressure fluctuations about an ambient configuration of equilibrium fluid at rest. The rough contours of this choice of scaling are quite intuitive. It is clear we have to scale to long distances to be in the fluid dynamical regime. The dispersion relation for shear waves, $\omega = i\nu k^2$ suggests that time intervals should scale like spatial intervals squared.² Scalings of distances and time intervals determine the scaling law for velocities. Finally the pressure variations are scaled appropriately to ensure that they cannot accelerate the fluid into velocities outside this scaling limit.

As we have explained, the scaling (1.3) of the equations of relativistic fluid mechanics leads to the Navier-Stokes equations (1.1). It follows that this scaling operation is a symmetry of the same equations. This is easily directly verified. In particular, if the fields $v^i(x, t)$ and $p(x, t)$ obey the unforced Navier-Stokes equations, then the rescaled fields $\epsilon v^i(\epsilon x, \epsilon^2 t)$ and $\epsilon^2 p(\epsilon x, \epsilon^2 t)$ also obey the same equations. Consequently, the scaling operation described above is a symmetry of the unforced Navier-Stokes equations.

As we have described above, the Navier-Stokes equations may be obtained as the scaling limit of any reasonable relativistic equations of fluid dynamics. In describing the connection with gravity, in most of the rest of this note, we will take the parent fluid dynamical theory to be conformal. It is natural to wonder how much of the full relativistic conformal group descends to a symmetry of the Navier-Stokes equations, and in what form it does so. It is obvious that the relativistic Poincare group descends to the Galilean symmetry group of the Navier-Stokes equations. In the next section we explain that the scaling symmetry operation described in the previous paragraph is loosely related to the dilatation operator of the parent relativistic conformal theory.³ Further we demonstrate that all spatial special conformal transformations also descend to exact symmetries of the Navier-Stokes equations. After our scaling these transformations effectively turn out to be the ‘boost’ to a uniformly accelerated frame; the inertial forces one has to deal with when working in a non-inertial frame are compensated for by a shift of the pressure. These spatial special conformal transformations, together with the Galilean group and the scaling symmetry described above, form the $(d+2)(d+1)/2 - 1$ dimensional symmetry algebra of

²The scaling to arbitrarily low velocities projects out sound waves with dispersion relation $\omega \propto k$.

³This loose ‘descent’ is analogous to the relation of the dilatation operator of the Schroedinger group to the generator of scale transformations of the massless Klein Gordon equation.

the Navier-Stokes equations. We list the commutation relations of this algebra,⁴ and the action of its generators on the velocity fields, in detail in the next section.

The conformal symmetry algebra described above is just subset of the full infinite dimensional symmetry algebra of the Navier Stokes equations [16] (see also [17–19] for other related work) . The additional generators of the full symmetry algebra are very easy to describe; they consist of boosts to a reference frame whose velocity is homogeneous in space but an arbitrary function of time. Just as in our discussion in the paragraph above, the pseudo force from such a frame change may be cancelled by an appropriate shift in pressure, and so is a symmetry of the Navier Stokes equations.⁵

The scaling limit described above admits an interesting generalization. Consider the equations of fluid dynamics on a base manifold $G_{\mu\nu} = g_{\mu\nu} + H_{\mu\nu}$ where $H_{\mu\nu}$ is small. By taking all terms that depend on $H_{\mu\nu}$ to the r.h.s., $\nabla_\mu T^{\mu\nu} = 0$ reduces effectively to the equations fluid dynamics on the base space $g_{\mu\nu}$ forced by an $H_{\mu\nu}$ dependent forcing function. If we combine the scaling described in the paragraph above with the H_{00} , $H_{ij} = \mathcal{O}(\epsilon^2)$ and $H_{i0} = \epsilon A_i(x^i, t)$, the effective resultant forcing function survives and is finite in the $\epsilon \rightarrow 0$ limit. It turns out that the effective scaled forcing function depends only on $A_i(x^i, t)$ and has a very simple form. It is precisely the force applied on a charged fluid by effectively a background electromagnetic potential $A_0 = 0$, $A_i = A_i(x^i, t)$. Consequently the ‘magnetohydrodynamical’ Navier-Stokes equations (i.e. the Navier-Stokes equations with a forcing function from an arbitrary background electromagnetic field) follows as a universal result of a scaling of the equations of relativistic hydrodynamics with small metric fluctuations.

We now return to the duality between gravity and fluid dynamics. We apply the scaling limit described in the previous paragraphs to the equations of fluid dynamics that are holographically dual to gravity. This procedure gives us an asymptotically locally AdS_{d+1} gravity dual to any solutions of the magnetohydrodynamical Navier-Stokes equations. The resultant metric describes small - but non linearly propagating fluctuations about a uniform black brane. We present the explicit form of the resultant bulk metric in section 4 below.

It follows from our discussion that every solution to the Navier-Stokes equations (1.1) is also a scaling limit of a solution to Einstein’s equations with a negative cosmological constant with one important caveat. The proviso is that many actual solutions of the Navier-Stokes equations describe fluids subject to hard wall type boundary conditions, and we do not (yet?) understand how to generate gravitational duals of these boundary conditions. However the qualitative effects of boundary conditions may easily be mimicked by appropriate forcing functions which are completely in our hands. In particular, while several experiments that study fully developed steady state turbulence do so in fluids with hard wall boundary conditions, there should be no barrier to setting up the same

⁴The nonrelativistic conformal symmetry group of the Navier-Stokes equations includes the contraction of spatial conformal generators K_i but does not include a generator that descends from the temporal conformal generator K_0 . Our algebra is distinct from the Schroedinger group studied for example in [15]

⁵We thank J. Maldacena and R. Gopakumar for suggesting a symmetry enhancement along these lines, and thank J. Maldacena for drwaing our attention to [16], see especially the top of pg 68. R. Gopakumar and collaborators are currently further studying this algebra and its extensions.

phenomenon for a fluid with no boundaries (e.g. on R^d or on a compact manifold) with the appropriate forcing function. In order to make this expectation concrete we have identified one forcing function (of a likely infinite plethora of possibilities) applied to a fluid on R^d , whose steady state end flow we expect to be turbulent at asymptotically high Reynolds numbers. In the rest of this introduction we describe this forcing function, and the reason we expect the flow it generates to be turbulent.

Consider the forcing function $A_x = \alpha e^{i\omega_0 t + ik_0 y} + cc$, together with $A_i = 0$ for $i \neq x$ acting on (for concreteness) a fluid, on $R^{2,1}$ whose spatial sections are parameterized by the Cartesian coordinates x and y . This gauge fields sets up a time and y dependent electric field in the x direction, together with a magnetic field in the plane. It is very easy to find one exact solution to the equations of fluid dynamics subject to this forcing function. On this solution v_x is proportional to A_x and $v_y = 0$. This solutions describes a fluid driven into motion in the direction of an applied electric field (the x direction), while Lorentz forces from the magnetic field, in the y direction, are balanced by pressure gradients. All nonlinear terms in the Navier-Stokes equations vanish when evaluated on our solution. However nonlinear terms have an important effect on the dynamics of small fluctuations about our solution at high Reynolds number. In particular, in appendix A below we demonstrate that nonlinear terms drive some fluctuation modes (with momentum in the x direction) unstable at high enough Reynolds numbers. In order to explain the possible significance of this instability, it is useful to recall the usual situation with fluids at high Reynolds numbers.

As we have described earlier in our introduction, fluid flows at high Reynolds numbers are very rich, and of potential theoretical interest. Nonetheless, in every highly symmetrical situation, there exists a simple ‘laminar’ solution, that preserves all the symmetries of the problem, at every value of the Reynolds number. This solution is often simple to determine analytically, and certainly shows none of the fascinating phenomenology of turbulence. However in interesting situations this solution becomes ‘tachyonic’ i.e. goes unstable to linear fluctuations that break some of the symmetries of the problem, above a critical Reynolds numbers. The ‘end point’ of this ‘tachyon condensation’ typically has richer dynamical behavior than the original solution itself. As the Reynolds number is further increased, further instabilities are usually triggered, and at arbitrarily high Reynolds numbers flow is turbulent.

We suspect that the dynamical pattern described in the previous paragraph applies to the exact solution described in this paper. As we currently have no theoretical tools to predict the onset of turbulence in any fluid flow this is necessarily a guess, but one that we believe is natural, given the results of our stability analysis. This guess suggests that the stable steady state solution to AdS₄ gravity, with the effective gauge field A_x described above, is dual to a turbulent fluid flow at high Reynolds numbers. Of course the particular situation described above is only one of a plethora of possibilities. We describe this solution in detail in section 5 and appendix A below only in order to have one concrete example of a gravitational set up that is likely to be dual to a turbulent fluid flow; not because we think that our particular is distinguished in any way.

We believe it is likely that the fluid-gravity map will lead to interesting new insights on the nature of solutions of Einstein gravity in the presence of a horizon.⁶ It also does not seem impossible that gravitational techniques and methods will prove useful in bringing new insights into the investigation of fascinating fluid phenomena like turbulence. In particular, it would be fruitful to understand the Kolmogorov laws on well-developed turbulence and their modification within the gravity framework. The symmetry algebra (3.5) may also throw new light on these issues. We leave further investigation of these issues to future work.

Note added. While we were completing this paper we received the preprint [20] which has substantial overlap with section 2 of this paper.

2 Scaling the Navier-Stokes equation

2.1 The Navier-Stokes equation as a universal limit of fluid dynamics

In this section we will display a scaling limit⁷ that reduces fluid dynamical equations of the form $\nabla_\mu T^{\mu\nu} = 0$, to the incompressible non-relativistic Navier-Stokes equations usually studied in fluid dynamics text books (see e.g. [1]).

The equations of relativistic fluid dynamics are

$$\begin{aligned} \nabla_\mu T^{\mu\nu} &= 0 \\ T^{\mu\nu} &= \rho u^\mu u^\nu + P \mathcal{P}^{\mu\nu} - 2\eta \sigma^{\mu\nu} - \zeta \theta \mathcal{P}^{\mu\nu} + \dots \\ \mathcal{P}^{\mu\nu} &= g^{\mu\nu} + u^\mu u^\nu, \\ \sigma^{\mu\nu} &= \mathcal{P}^{\mu\alpha} \mathcal{P}^{\nu\beta} \left(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - \frac{g_{\alpha\beta}}{d-1} \partial \cdot u \right), \quad \theta = \nabla^\beta u_\beta \end{aligned} \tag{2.1}$$

Here P is the pressure, ρ the energy density, η the shear viscosity, ζ the bulk viscosity of the fluid, and $g_{\alpha\beta}$ the metric of the space on which the fluid propagates. Each of the fluid quantities listed above may be regarded as a function of the fluid temperature - and also of chemical potentials if the fluid is charged. The ... in the equation above refer to terms of second or higher order in spacetime derivatives.

If the fluid is charged, then we must supplement the equation (2.1) with an equation of charge conservation for every conserved charge. Below, we comment briefly on the scaling limit of this equation.

2.2 A scaling limit

Now let us study the motion of a fluid on a metric of the form $G_{\mu\nu} = g_{\mu\nu} + H_{\mu\nu}$. We assume that the background metric $g_{\mu\nu}$ has the form

$$g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + g_{ij} dx^i dx^j. \tag{2.2}$$

⁶For instance, consider a black brane in asymptotically flat space. At sufficiently low energies, a beam of gravitons shot at this brane perturbs the non-normalizable boundary conditions of the effectively AdS near horizon region of the brane. It may be possible to choose this perturbation to drive the gravity in the near horizon AdS region turbulent.

⁷This section was worked out in collaboration with J. Maldacena.

(this is simply a choice of coordinate system, for a large class of metrics) while the small fluctuation $H_{\mu\nu}$ is completely arbitrary. As we have explained in the introduction, we intend to view the fluid flow on the space with metric $G_{\mu\nu}$ as an effectively forced flow on the space with metric $g_{\mu\nu}$. In order to do this we need to map the velocity field \tilde{u}^μ on the space $G_{\mu\nu}$ to a velocity field on $g_{\mu\nu}$. This map, which must respect the requirement that $u^2 = \tilde{u}^2 = -1$, may be chosen in a natural fashion if one thinks of the velocity field as generated by the path of ‘particles’ through spacetime. We now describe this in more detail.

Let the fluid velocity on the space $G_{\mu\nu}$ be given by \tilde{u}^μ where

$$\tilde{u}^\mu = \frac{1}{\sqrt{V^2}} (1, \vec{V}) \tag{2.3}$$

where \vec{V} is a $d - 1$ spatial vector with components V^i , V^α is a d component object with components $(1, \vec{V})$ and V^2 is $G_{\alpha\beta}V^\alpha V^\beta$ (the indices α, β run over d spacetime indices while the indices i, j run over the $d - 1$ spatial indices). Expanding \tilde{u}^μ to first order in $H_{\alpha\beta}$ we have

$$\begin{aligned} \tilde{u}^\mu &= u^\mu + \delta u^\mu + \dots \\ u^\mu &= \frac{1}{\sqrt{1 - g_{ij}V^iV^j}} (1, \vec{V}) \\ \delta u^\mu &= -u^\mu \frac{u^\alpha u^\beta H_{\alpha\beta}}{2} \end{aligned} \tag{2.4}$$

where u^μ is the d velocity of the fluid referred to the metric $g_{\mu\nu}$. All terms in δu^μ above depend on the fluctuation metric $H_{\alpha\beta}$; below we will take all these terms to the r.h.s. of (1.1) and view them as contributions to the effective forcing function.

One obvious solution to the equations of fluid dynamics on the space with metric $g_{\mu\nu}$ is simply the fluid at rest with constant pressure P_0 and density ρ_0 . We now turn to a particular kind of small amplitude and long distance fluctuation about this uniform fluid at rest and with pressure P_0 and energy density ρ_0 on a manifold that is ‘close’ to $g_{\mu\nu}$. More specifically we set

$$\begin{aligned} H_{00} &= \epsilon^2 h_{00}(\epsilon x^i, \epsilon^2 t) \\ H_{0i} &= \epsilon A_i(\epsilon x^i, \epsilon^2 t) \\ H_{ij} &= \epsilon^2 h_{ij}(\epsilon x^i, \epsilon^2 t) \\ V^i &= \epsilon v^i(\epsilon x^i, \epsilon^2 t) \\ \frac{P - P_0}{\rho_0 + P_0} &= \epsilon^2 p(\epsilon x^i, \epsilon^2 t) \end{aligned} \tag{2.5}$$

and take ϵ to be arbitrarily small. Although we have explicitly listed only the scaling of the pressure P above, the energy density ρ and viscosity ν also scale in a similar fashion (this is consistent with the fact that they are all functions of the same underlying variables). We have normalized pressure fluctuations by $\rho_0 + P_0$ rather than P_0 for future convenience.

We will now examine what happens to the Navier-Stokes equations under this scaling. Let us first start with the 0 or temporal component of these equations. It is easy to check

that

$$\begin{aligned}\nabla_\mu T^{\mu 0} &= \epsilon^2 [\rho_e (\nabla_i v^i)] + \mathcal{O}(\epsilon^4) \\ \rho_e &= \rho_0 + P_0\end{aligned}\tag{2.6}$$

Consequently, in the limit of small ϵ this equation reduces simply to $\nabla_i v^i = 0$, where ∇_i is the covariant derivative with respect to the purely spatial metric g_{ij} .

Let us now turn to the spatial Navier-Stokes equations. After some calculation we find

$$\begin{aligned}\nabla_\mu T^{\mu i} &= \epsilon^3 \left[\rho_e \nabla^i p + \rho_e \nabla_\mu (v^i v^\mu) \right. \\ &\quad \left. - 2\eta \nabla_j \left(\frac{\nabla^j v^i + \nabla^i v^j}{2} - g^{ij} \frac{\vec{\nabla} \cdot \vec{v}}{d-1} \right) - \zeta \nabla_i \vec{\nabla} \cdot \vec{v} - f^i \right] + \mathcal{O}(\epsilon^5) \\ f^i &= \rho_e \left(\frac{\partial_i h_{00}}{2} - \partial_0 A_i - \frac{\partial_j (\sqrt{g} A_i v^j)}{\sqrt{g}} + v^j \partial^i A_j \right)\end{aligned}\tag{2.7}$$

where $v^\mu = (1, v^i)$. Using the equation $\nabla_i v^i = \frac{\partial_i (\sqrt{g} v^i)}{\sqrt{g}} = 0$, the coefficient of ϵ^3 in the equation above may be simplified to

$$\nabla^i p + \partial_0 v^i + \vec{v} \cdot \nabla v^i - \nu (\nabla^2 v^i + R_j^i v^j) = \frac{\partial^i h_{00}}{2} - \partial_0 A^i + F_j^i v^j\tag{2.8}$$

where $F_{ij} = \partial_i A_j - \partial_j A_i$ is the field strength for the vector field A_i and $\nu = \eta/\rho_e$ is the ‘kinematical viscosity’ of the fluid.⁸ (2.8) takes a somewhat simpler form in terms of slightly redefined variables. Let

$$A_i = a_i + \nabla_i \chi$$

where χ is chosen to ensure that $\nabla_i a^i = 0$. This is the usual split of a gauge field into its pure curl and pure divergence parts. Note that

$$f_{ij} \equiv \partial_i a_j - \partial_j a_i = F_{ij}.$$

We also define the effective pressure

$$p_e = p - \frac{1}{2} h_{00} + \dot{\chi}$$

in terms of which (2.8) reduces to

$$\nabla_i p_e + \partial_0 v_i + \vec{v} \cdot \nabla v_i - \nu (\nabla^2 v_i + R_{ij} v^j) = -\partial_0 a^i - v^j f_{ji}\tag{2.9}$$

(2.9) is precisely the Navier-Stokes equation with forcing function generated by an effective background electromagnetic field (with $a^0 = 0$ and spatial vector a^i) on the effectively charged fluid.

⁸ Note also that we have used conventions in which

$$[\nabla_\rho, \nabla_\sigma] A_\nu = A_\beta R_{\nu\rho\sigma}^\beta, \quad R_{\alpha\theta}^\beta = R_{\alpha\theta}$$

With these conventions the scalar curvature of a unit d sphere is $d(d-1)$ and the Ricci tensor of a unit sphere is given by $R_{ij} = (d-1)g_{ij}$.

2.3 Charged fluids

If the fluid under study carries an extra set of conserved charges, it obeys an extra set of conservation equations of the form $\nabla_\mu J^\mu = 0$. It is easy to verify that these equations all reduce to the condition that the fluid velocity is divergence free in the scaling limit under study in this section. Consequently charged fluids obey the same equations as uncharged fluids in the scaling limit under study in this subsection.

2.4 Reduction of Cauchy data

Returning to the study of uncharged fluids, the Cauchy data⁹ of the parent relativistic fluid dynamical equations consists of d real functions of space; the value of the pressure field and the value of the $d - 1$ independent velocity fields on an initial time slice. As the fluid dynamical equations are of first order in time, the Cauchy data of the problem does not include the time derivatives of all these fields.

Now let us examine the Cauchy data of the incompressible Navier-Stokes equations. Note that the spatial divergence of (2.8)

$$\nabla^2 p_e = -\nabla_i v^j \nabla_j v^i - v^i v^j R_{ij} + \nabla_i [(-\nu R_j^i + f_j^i) v^j] \quad (2.10)$$

determines the pressure of the fluid as a function of the fluid velocity (only the velocity - not its time derivatives). In other words the independent data in the fluid is simply given by $d - 2$ real functions that parameterize an arbitrary divergence free velocity field.

It follows that two of the degrees of freedom of the equations of relativistic fluid dynamics are lost on taking the scaling limit of the previous subsection. These two degrees of freedom are simply the fluctuations of the pressure and the divergence of the velocity. At the linearized level these two degrees of freedom combine together in sound mode fluctuations. (Note that the relativistic dispersion of sound implies that a sound mode has twice as much data as a shear mode.). Consequently the reduction of Cauchy data in the scaling limit of this paper follows simply a nonlinear restatement of the observation that sound waves are projected out in our scaling limit.

3 Symmetries

As we have noted above, the Navier-Stokes equations may be obtained by applying the appropriate scalings to the equations of relativistic hydrodynamics with an arbitrary equation of state. The parent hydrodynamical system may in particular be chosen to be conformally invariant. It is natural to wonder whether either the dilatation or special conformal transformations of the parent theory descend to a symmetry of the the Navier-Stokes equations. In this section we answer this question in the affirmative.¹⁰

⁹We thank S. Trivedi for very useful discussions on the topic of this subsection.

¹⁰This section was worked out in collaboration with R. Loganayagam.

3.1 Dilatations

Let us first explain how this works for dilatations. Dilatations consist of a diffeomorphism $(x')^\mu = \frac{x^\mu}{\lambda}$, $T' = T$, $(u')^\mu = \frac{u^\mu}{\lambda}$ $(g')_{\mu\nu} = \lambda^2 g_{\mu\nu}$ compounded with the Weyl transformation $\tilde{x}^\mu = (x')^\mu$, $\tilde{g}_{\mu\nu} = \frac{(g')_{\mu\nu}}{\lambda^2}$, $\tilde{u}^\mu = \lambda(u')^\mu$, $\tilde{T} = \lambda T$. In sum

$$\tilde{x}^\mu = \frac{x^\mu}{\lambda}, \quad \tilde{g}_{\mu\nu} = g_{\mu\nu}, \quad \tilde{u}^\mu = u^\mu, \quad \tilde{T} = \lambda T.$$

This action on coordinates, temperatures and fields, is a symmetry of the equations of conformal relativistic fluid dynamics. While this operation commutes with our scaling limit, it is not by itself a symmetry of the Navier-Stokes equations. This is because the kinematical viscosity ν , regarded as a parameter in the Navier-Stokes equations, is proportional to $\frac{1}{T}$ and so changes under the variable transformation described above. The Navier-Stokes equations are however simple enough to allow dilatation transformations listed above to be modified into a true symmetry of the Navier-Stokes equations by ‘absorbing’ the transformation of ν into ‘anomalous’ transformations of time and velocity. The result of this procedure is simply (1.3) with which we started this note.

3.2 Special conformal transformations

The situation is more straightforward for special conformal transformations. The scaling law for the velocity and temperature fields, under a special conformal transformation, may also be obtained by compounding a diffeomorphism with the appropriate Weyl transformation. Restricting to infinitesimal conformal transformations, we find^{11, 12}

$$\begin{aligned} \delta x^\mu &= -2c_\nu x^\mu + x^2 c^\mu \\ \delta u^\mu &= -2[x^\mu c^\nu - x^\nu c^\mu] u_\nu - \delta x^\nu \partial_\nu u^\mu \\ \delta T &= 2c_\nu x T - \delta x^\nu \partial_\nu T \end{aligned} \tag{3.2}$$

Now note that special conformal transformations induce an additive shift on the temperature fluctuation, δT , proportional to $x \cdot c T_0$ where T_0 is the temperature of the background. In order that this shift respect the ϵ^2 scaling of δT we are required to scale $c_0 \propto \epsilon^4$ and $c_i \propto \epsilon^3$. Imposing this scaling and retaining terms only to leading order in the velocity

¹¹In order to verify the covariance of local equations under these symmetry transformations, it is sufficient to omit the terms proportional to $\delta x^\mu \partial_\mu$ but instead to transform all derivatives according to the rule

$$\delta(\partial_\beta) = 2[c_\beta x \cdot \partial - x_\beta c \cdot \partial + x \cdot c \partial_\beta]$$

¹²Under this transformation, the shift in a conformal stress tensor is given by

$$\delta T^{\mu\nu} = 2d(c \cdot x) T^{\mu\nu} + 2(x^\lambda c^\mu - x^\mu c^\lambda) T_\lambda^\nu + 2(x^\lambda c^\nu - x^\nu c^\lambda) T_\lambda^\mu - \delta x^\lambda \partial_\lambda T^{\mu\nu} \tag{3.1}$$

Upon accounting also for the shift in the derivatives, it is easy to convince oneself that the equation of energy momentum conservation, for any identically traceless stress tensor, is invariant under special conformal transformations.

expansion (3.3) reduces to¹³

$$\begin{aligned}
 \delta t &= 0 \\
 \delta x^i &= -t^2 c^i \\
 \delta v^i &= -2c^i t + t^2 c_j \partial_j v^i \\
 \delta T &= 2(-c^0 t + c^i x^i) T + t^2 c_j \partial_j T
 \end{aligned}
 \tag{3.3}$$

It is clear that the symmetry generated by c^0 acts trivially (it does not act on coordinates or velocities, but merely generates a shift, linear in time, of the pressure; more about this below). However the symmetries generated by c^i act nontrivially, and are directly verified to be symmetries of the Navier-Stokes equations.¹⁴

3.3 The full symmetry group

We have thus discovered that the Navier-Stokes equations enjoy invariance under a conformal symmetry group. The generators of this group are the dilatation D , special conformal symmetries K_i Gallilian boosts B_i , the generator of time translations (energy) H , momenta P_i and spatial rotations M_{ij} . The action of these generators on velocity fields is given by

$$\begin{aligned}
 Dv^j &= (-2t\partial_t - x^m \partial_m - 1) v^j \\
 K_i v^j &= -2t\delta_{ij} + t^2 \partial_i v^j \\
 B_i v^j &= \delta_{ij} - t\partial_i v^j \\
 H v^j &= -\partial_t v^j \\
 P_i v^j &= -\partial_i v^j \\
 M_{ik} v^j &= \delta_{ij} v^k - \delta_{kj} v^i - (x^k \partial_i - x^i \partial_k) v^j
 \end{aligned}
 \tag{3.4}$$

The commutation relations between these various generators is given by

$$\begin{aligned}
 [D, K_i] &= -3K_i \\
 [D, B_i] &= -B_i \\
 [D, H] &= 2H \\
 [D, P_i] &= P_i \\
 [D, M_{ij}] &= 0 \\
 [M_{ij}, P_k] &= -\delta_{ik} P_j + \delta_{jk} P_i \\
 [M_{ij}, K_k] &= -\delta_{ik} K_j + \delta_{jk} K_i
 \end{aligned}
 \tag{3.5}$$

¹³As above, in verifying the covariance of equations under these transformation, it is sufficient to omit the terms proportional to $t^2 c_j \partial_j$ above, but instead to transform derivatives according to

$$\delta(\partial_t) = 2tc^i \partial_i \quad \delta(\partial_i) = 0.$$

¹⁴Under the transformations listed in (3.3) we have $\delta p_e = \frac{\delta p}{\delta P_0} = 2c \cdot x$ so that $\delta \partial_i p_e = 2c^i$. Further $\delta(\dot{v}^i + v \cdot \nabla v^i) = -2c^i$ and the viscous term is unchanged. Adding all terms together we have a symmetry of the equations.

$$\begin{aligned}
[M_{ij}, B_k] &= -\delta_{ik}B_j + \delta_{jk}B_i \\
[M_{ij}, H] &= 0 \\
[K_i, P_j] &= 0 \\
[K_i, B_j] &= 0 \\
[K_i, H] &= -2B_i \\
[H, B_j] &= -P_j
\end{aligned}$$

The symmetry algebra listed in (3.5) may presumably be obtained from the appropriate contraction of the parent symmetry algebra $SO(d, 2)$. A related study is currently under progress in [21].

In addition to the symmetries listed above, the Navier-Stokes equations have an infinite dimensional group of trivial symmetries, under which the pressure is simply shifted by an arbitrary function of time. These are symmetries of the equation because only gradients of the pressure enter the Navier-Stokes equations, and they are trivial because the pressure is not really an independent variable of the Navier-Stokes equations, which may in fact be eliminated by taking the curl of those equations. For this reason one need not keep track of the action of symmetry generators on the pressure. However it is not difficult to do so and we find

$$\begin{aligned}
Dp_e &= (-2t\partial_t - x^m\partial_m - 2)p_e \\
K_i p_e &= 2x^i + t^2\partial_i p_e \\
B_i p_e &= -t\partial_i p_e \\
H p_e &= -\partial_t p_e \\
P_i p_e &= -\partial_i p_e \\
M_{ik} p_e &= -(x^k\partial_i - x^i\partial_k)p_e
\end{aligned} \tag{3.6}$$

This action of the symmetry generators on the pressure field do not quite yield the commutation relations (3.5), but instead have additional terms on the r.h.s. corresponding to generators of the trivial symmetries referred to above (i.e. generators that shift the pressure field by a function of time). Spatial derivatives of the pressure field, however, honestly transform according to the algebra (3.5). Consequently, the symmetry algebra (3.5) is not represented on the pressure field itself, but on its spatial derivatives. This is vaguely reminiscent of the fact that the two dimensional conformal group has well defined action on all derivatives of massless scalar fields, but not on the field itself.

4 The gravity solution dual to fluid dynamics in the scaling regime

4.1 The dual bulk metric

As we have described above, any solution of the incompressible non-relativistic Navier-Stokes equations solves the equations of fluid dynamics dual to gravity up to $\mathcal{O}(\epsilon^3)$, under the scaling listed in the previous section. Now in [9, 10, 13] the equations of fluid dynamics are obtained as the Einstein constraint equation of a bulk asymptotically locally AdS_{d+1}

space. It is thus natural to define the gravitational dual to a solution of the Navier-Stokes equations as any small fluctuation about a black brane background that solves all of Einstein's equations (constraint as well as dynamical) to cubic order in ϵ . Adopting this definition, it is easy to read off the bulk metric dual to the Navier-Stokes equations from an appropriate scaling of the bulk metric of [9, 10, 13].

In computing the bulk metric upto $\mathcal{O}(\epsilon^3)$ in the sense described above, it turns out that terms from only the zeroth order and the first order in the derivative expansion of the gravitational solutions of [9, 10, 13] are relevant. The metric up to first order in derivative has the following form

$$\begin{aligned}
 ds^2 &= ds_0^2 + ds_1^2 \\
 \text{where} \\
 ds_0^2 &= -2u_\mu dx^\mu dr + \frac{1}{b^d r^{d-2}} u_\mu u_\nu dx^\mu dx^\nu + r^2 g_{\mu\nu} dx^\mu dx^\nu \\
 ds_1^2 &= -2r u_\nu (u^\alpha \bar{\nabla}_\alpha) u_\mu dx^\mu dx^\nu + \frac{2}{d-1} r (\bar{\nabla}_\alpha u^\alpha) u_\mu u_\nu dx^\mu dx^\nu \\
 &\quad + 2br^2 F(br) \sigma_{\mu\nu} dx^\mu dx^\nu \\
 \sigma_{\mu\nu} &= \frac{1}{2} (\bar{\nabla}_\mu u_\nu + \bar{\nabla}_\nu u_\mu) + \frac{1}{2} (u_\nu (u^\alpha \bar{\nabla}_\alpha) u_\mu + u_\mu (u^\alpha \bar{\nabla}_\alpha) u_\nu) \\
 &\quad - \frac{1}{d-1} (\bar{\nabla}_\alpha u^\alpha) (u_\mu u_\nu + g_{\mu\nu}) \\
 F(x) &= \int_x^\infty dy \left(\frac{y^{d-1} - 1}{y(y^d - 1)} \right) \\
 b &= \frac{d}{4\pi T} = b_0 + \delta b \\
 T &= T_0 + \delta T
 \end{aligned} \tag{4.1}$$

Here $\bar{\nabla}$ denotes the covariant derivative with respect to the full boundary metric which is equal to a background $g_{\mu\nu}$ plus perturbation $H_{\mu\nu}$. T_0 is the temperature of the background blackbrane. The terms that appear in the metric involve covariant derivatives of the d velocity u_μ with respect to the full boundary metric. These terms can be expressed as covariant derivatives of the $d-1$ velocity v_i and the metric perturbation $A_i = H_{0i}$ with respect to spatial part of the background metric g_{ij} .

$$\begin{aligned}
 \bar{\nabla}_i u_j &= \nabla_i v_j + \mathcal{O}(\epsilon^4) \\
 \bar{\nabla}_i u_0 + \bar{\nabla}_0 u_i &= \partial_0(v_i + A_i) - \frac{1}{2} \partial_i h_{00} - \frac{1}{2} \partial_i (v_j v^j) - v^j F_{ij} + \mathcal{O}(\epsilon^4) \\
 \bar{\nabla}_\mu u^\mu &= \nabla_j v^j + \mathcal{O}(\epsilon^4) \\
 u^\mu \bar{\nabla}_\mu u_0 &= \mathcal{O}(\epsilon^4) \\
 u^\mu \bar{\nabla}_\mu u_i &= \partial_0(v_i + A_i) - \frac{1}{2} \partial_i h_{00} + (v^j \nabla_j) v_i - v^j F_{ij} + \mathcal{O}(\epsilon^4) \\
 F_{ij} &= \partial_i A_j - \partial_j A_i
 \end{aligned} \tag{4.2}$$

Here ∇ denotes the covariant derivative with respect to g_{ij} . The raising and lowering of the i, j indices are also with respect to the metric g_{ij} . To simplify the expression of $\sigma_{\mu\nu}$

in (4.2) the constraint $\nabla_i v^i = 0$ has been used. Using these expressions the derivative part of the metric can be written as

$$\begin{aligned}
 ds_1^2 = & b_0 r^2 F(b_0 r) (\nabla_i v_j + \nabla_j v_i) dx^i dx^j - 2b_0 r^2 F(b_0 r) v^j (\nabla_i v_j + \nabla_j v_i) dt dx^i \\
 & + 2r \left(\partial_0(v_i + A_i) - \frac{1}{2} \partial_i h_{00} - v^j F_{ij} + (v^j \nabla_j) v_i \right) dt dx^i + \mathcal{O}(\epsilon^4)
 \end{aligned} \tag{4.3}$$

Here the first term is of order ϵ^2 and the last two terms are of order ϵ^3 . Since from the constraint equations $\nabla_i v^i = 0$, there is no contribution from the scalar sector. The zeroth order metric can also be expanded in powers of ϵ . It turns out that to solve Einstein equation up to order ϵ^3 , it is sufficient to expand the zeroth order metric up to order ϵ^2 in the fluctuations.

$$\begin{aligned}
 ds_0^2 = & \frac{1}{b_0^d r^{d-2}} dt^2 + r^2 (-dt^2 + g_{ij} dx^i dx^j) + 2dt dr \\
 & - \frac{2}{b_0^d r^{d-2}} (A_i + v_i) dt dx^i - 2(A_i + v_i) dx^i dr \\
 & + \frac{1}{b_0^{d+1} r^{d-2}} (-d \delta b + v_j v^j - h_{00}) dt^2 \\
 & + \frac{1}{r^{d-2}} (A_i + v_i) (A_j + v_j) dx^i dx^j - (-v_j v^j + h_{00}) dt dr
 \end{aligned} \tag{4.4}$$

Here the first line is of order ϵ^0 , the second line is of order ϵ^1 and the third is of order ϵ^2 . The full metric is given by the sum of (4.4) and (4.3); and solves Einsteins equations to $\mathcal{O}(\epsilon^3)$ provided the velocity and temperature fields above obey the incompressible Navier-Stokes equations (1.1). These equations imply in particular that

$$\frac{\nabla^2 T}{T_0} = -\nabla_i v^j \nabla_j v^i - v^i v^j R_{ij} + \nabla_i [(-\nu R_j^i + F_j^i) v^j] + \frac{1}{2} \nabla^2 h_{00} - \partial_0(\nabla \cdot A) \tag{4.5}$$

an equation that determines δT (and hence δb in (4.4)) as a spatially nonlocal but temporally ultralocal functional of the velocity fields v^i . It follows that though the bulk metric at x^μ is determined locally as a function of temperatures and velocities at x^μ , it is not determined locally as a function of velocities at x^μ . This non locality is a consequence of the infinite speed of sound (and consequent action at a distance) in our scaling limit.

4.2 Comments on the bulk metric

Several comments on the bulk metric (4.3) and (4.4) are in order. Let us first spell out some terminology. We refer to the parts of the bulk metric that are proportional to a linear combination of dt^2 or $\sum_i (dx^i)^2$ as its scalar components. Terms in the metric proportional to $dt dx^i$ are called its vector components, while terms proportional to $dx^i dx^j$ are referred to as its tensor components.

Now the derivative expansion of [4–13] solves the Einstein equations in a multi step process. In the first step the dynamical Einstein equations are used to determine the r dependence of the bulk metric as a function of boundary data, in each of the sectors described above. The requirement of regularity of the future horizon is then used to

determine the boundary data of the tensor sector in terms of the boundary data in the vector and scalar sectors (roughly the fluid temperature and velocity).

In the scaling limit described in the previous subsection, the tensor sector of the bulk metric is particularly simple. It is given by

$$b_0 r^2 F(b_0 r) (\nabla_i v_j + \nabla_j v_i) dx^i dx^j + \frac{1}{r^{d-2}} (A_i + v_i) (A_j + v_j) dx^i dx^j. \quad (4.6)$$

Note that even though the velocity v_i is scaled to zero in the scaling limit under study in this paper, the leading order in ϵ metric (4.6) includes a quadratically nonlinear in the velocities. This is a consequence of the fact that we scale distances to infinity at the same rate as velocities to zero. Had we simply scaled amplitudes to be small, while keeping distance scales finite, we would have ended up with only the first of these two terms in (4.6) and the resultant geometry would simply have been the dual of linearized fluid dynamics.

The Navier-Stokes equations are obtained out of the Einstein constraint equations acting on the bulk metric described above. The first term in (4.6) gives rise to the viscous term in the Navier-Stokes equations, while the second term in (4.6) is the origin of the nonlinear convective term in those equations. It is consequently not surprising that ratio of the first to the second term in (4.6) is proportional to the Reynolds number Re of the flow: in fact it is of order $Re \times \frac{1}{r^d F(r)}$. Now $F(r) \sim \frac{1}{r}$ at large r ; so that the viscous term in (4.6) appears to dominate the nonlinear term in that expression when $r > (Re)^{1/d-1}$. This appearance is at least partly a coordinate artifact, as is evidenced by the fact that the two terms in (4.6) contribute equally to the Einstein constraint equations at every r (this follows as these equations are independent of r).¹⁵

5 Simple steady state shear flows at arbitrary Reynolds number

Consider a fluid on $R^{d-1,1}$, subject to an effective forcing

$$\begin{aligned} a_x &= \int d\omega dk a(k, \omega) \exp[iky + i\omega t] \\ a_i &= 0, \quad (i \neq x) \end{aligned} \quad (5.1)$$

Here x and y parameterize orthogonal displacements in the $d - 1$ spatial directions, and $a(k, \omega)$ is any function.

In the presence of this forcing function, it is easy to verify that the velocity field

$$\begin{aligned} v_x &= - \int dk d\omega \frac{a(k, \omega) e^{iky + i\omega t}}{1 - i \frac{\nu k^2}{\omega}} \\ v_i &= 0 \quad i \neq x \end{aligned} \quad (5.2)$$

¹⁵The ratio computed above naively suggests that nonlinearities at different effective Reynolds numbers are separated in the bulk radial coordinate; a result reminiscent of the approximate locality in scale of high Reynolds number fluid flows. While this interpretation is too quick for the purposes of evaluating the Einstein constraint equation, perhaps the naive ratio of the two terms in (4.6) is physically relevant for some physically interesting questions. We feel this deserves further contemplation.

(together with the pressure field which may be obtained by integrating (2.10)), is an exact solution of (2.9).

In order to be concrete let us make the simple choice $a(k, \omega) = a(-k, -\omega) = \alpha \delta(k - k_0) \delta(\omega - \omega_0)$. In this special case the forcing function is given by

$$\begin{aligned} a_x &= \alpha \exp [ik_0 y + i\omega_0 t] + cc \\ a_i &= 0, \quad (i \neq x) \end{aligned} \tag{5.3}$$

and the velocity field is given by¹⁶

$$\begin{aligned} v_x &= -\frac{\alpha e^{ik_0 y + i\omega_0 t}}{1 - i \frac{\nu k_0^2}{\omega_0}} + cc \\ v_i &= 0 \quad i \neq x \end{aligned} \tag{5.4}$$

This solution is characterized by two dimensionless numbers, $a = \frac{\alpha}{k_0 \nu}$ and $b = \frac{\nu k_0^2}{\omega}$. The Reynolds number for this solution is given by

$$R = \frac{vL}{\nu} = \frac{a}{\sqrt{1 + b^2}}. \tag{5.5}$$

and can be made large by taking a to infinity at fixed b . This may be achieved, for instance, by increasing the strength of the forcing amplitude α keeping everything else fixed.

In appendix A we outline a linear stability analysis of the special flow described above. We find that in $d = 2$ this flow is unstable at large enough Reynolds numbers for a range of the parameter b . As we have described in the introduction, this suggests that the flow is dynamically interesting, and likely turbulent, at asymptotically high Reynolds numbers. Assuming this is the case, we have identified at least one gravitational system that is dual to steady state turbulence. In some detail, consider gravity with a negative cosmological constant, subject to the small z boundary condition on the metric

$$\lim_{z \rightarrow 0} ds^2 = \frac{1}{z^2} (dz^2 - dt^2 + \epsilon dt dx^i a_i + dx^i dx_i)$$

(where $i = 1 \dots d - 1$ and a_i is the gauge field (5.1)). The steady state solution of this gravitational system is dual to a turbulent flow of d dimensional fluid dynamics whenever the parameters in the function a_i above are chosen to satisfy the inequality $a \gg \sqrt{1 + b^2}$.

In appendix B we have presented a generalization of the solution described in this section to the forced flow of a fluid on a sphere. In that situation one should obtain the dual of a turbulent fluid in an spacetime that is asymptotically global AdS.

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¹⁶This special solution may be thought of as living either on a spatial R^{d-1} or on a spatial torus, whose y edge is of size $\frac{2\pi}{k_0}$.

and useful discussions and comments throughout its execution. We would also especially like to thank R. Loganayagam for collaboration on the calculations in section 3, for useful discussions throughout the execution of this project and for several detailed comments on the manuscript. We have profited greatly from discussions with S. Trivedi and have also had fruitful discussions with J.Bhattacharya, R. Gopakumar, G. Mandal, H. Ooguri and M. Van Raamsdonk. We would like to acknowledge useful discussion with the students in the TIFR theory room. The work of S.M. was supported in part by Swarnajayanti Fellowship. The work of S.R.W was supported by J.C.Bose fellowship. We would all also like to acknowledge our debt to the people of India for their generous and steady support to research in the basic sciences.

A Stability analysis

In this appendix we study linear fluctuations about the solution (5.4) for the special case $d = 3$. In two spatial dimensions the dual of a divergenceless velocity is the gradient of a scalar, and the field strength of a gauge field is itself dual to a scalar. In other words we can write

$$\begin{aligned} v_i &= \epsilon_{ij} \partial_j \chi \\ f_{ij} &= \epsilon_{ij} f \end{aligned} \tag{A.1}$$

The Navier-Stokes equations (2.9) may be rewritten (after eliminating the pressure by taking the curl) in terms of these scalars as

$$-\nabla^2 \dot{\chi} + \nu \nabla^4 \chi - \epsilon^{jm} \nabla_m \chi \nabla_j \nabla^2 \chi + \dot{f} + \epsilon^{iq} \nabla_i \chi \nabla_q f = 0 \tag{A.2}$$

In terms of these variables the solution (5.4) takes the form

$$\begin{aligned} f_0 &= -ik_0 \alpha \exp [ik_0 y + i\omega_0 t] + \text{cc} \\ \chi_0 &= -\frac{1}{ik_0} \times \frac{\alpha e^{ik_0 y + i\omega_0 t}}{1 - i \frac{\nu k_0^2}{\omega_0}} + \text{cc} \end{aligned} \tag{A.3}$$

A.1 Setting up the eigenvalue equations

Let us now set

$$\chi = \chi_0 + \theta \left(e^{iq_0 x} \int d\omega dk \chi_{q_0}(\omega, k) e^{i\omega y + ikx} + e^{-iq_0 x} \int d\omega dk \chi_{q_0}^*(\omega, k) e^{-i\omega y - ikx} \right)$$

where θ is a small parameter, and solve (A.2) to linear order in θ . Note that while our fluctuation has a specific frequency (namely q_0) in the x direction, it is a sum over frequencies in the y direction and in time. This is necessary as our background breaks translational invariance in the y and t directions, but preserves this invariance in the x direction. It is easy to work out the linear equations that our fluctuation coefficients $\chi_{q_0}(\omega, k)$ obey. We find

$$\begin{aligned} &[i\omega(k^2 + q_0^2) + \nu(k^2 + q_0^2)^2] \chi_{q_0}(\omega, q) \\ &+ [\beta(iq_0) ((k - k_0)^2 + q_0^2 - k_0^2) + \alpha iq_0 k_0^2] \chi_{q_0}(\omega - \omega_0, k - k_0) \\ &+ [\beta^*(iq_0) ((k + k_0)^2 + q_0^2 - k_0^2) + \alpha^* iq_0 k_0^2] \chi_{q_0}(\omega + \omega_0, k + k_0) = 0 \end{aligned} \tag{A.4}$$

where

$$\beta = -\frac{\alpha}{1 + \frac{\nu k_0^2}{i\omega_0}}.$$

Let us define

$$f_k(\gamma, \kappa) = \chi_{q_0} [\omega_0(\gamma + k), k_0(\kappa + k)].$$

The fluctuation equation (A.4) may be recast in terms of f_k as (we will usually omit to write the functional dependence of f_k in the equations that follow)

$$\begin{aligned} & \left[\frac{i}{b}(\gamma + k)((\kappa + k)^2 + c^2) + ((\kappa + k)^2 + c^2)^2 \right] f_k \\ & + ica \left(-\frac{(\kappa + k - 1)^2 + c^2 - 1}{1 - ib} + 1 \right) f_{k-1} \\ & + ica^* \left(-\frac{(\kappa + k + 1)^2 + c^2 - 1}{1 + ib} + 1 \right) f_{k+1} \end{aligned} \quad (\text{A.5})$$

The dimensionless quantities a, b and c are parameters in (A.5) (a and b were defined in section 4 while $c = \frac{q_0}{k_0}$). γ and κ are variables in this equation; (A.5) is the condition that a matrix acting on the columns $\{f_k\}$ has a zero eigenvalue. Together with the physical requirement that f_k decay at large $|k|$; this eigenvalue equation yields an expression for unknown temporal frequency γ as a function of κ .¹⁷ That is, the equation (A.5) should yield a dispersion relation of the form

$$\gamma = \gamma(\kappa, a, b, c, n) \quad (\text{A.6})$$

where the integer n labels which of the infinitely many solutions to the zero eigenvalue condition we have chosen to study. A result for γ with a negative imaginary part represents an instability of the system.

Let us study the large $|k|$ asymptotics of the variables f_k in a little more detail. According to our boundary conditions, at large positive k f_{k+1} is much smaller than f_{k-1} . As a consequence (A.5) implies that

$$\frac{f_k}{f_{k-1}} = \frac{ica}{1 - ib} \times \frac{k + 2(\kappa - 1)}{k^2(k + 4\kappa + \frac{i}{b})} \times \left(1 + \mathcal{O}\left(\frac{1}{k^2}\right) \right) \quad (\text{A.7})$$

It follows that, at large k ,

$$f_k \approx D(a, b, c\kappa) \left(\frac{ica}{i - ib} \right)^k \frac{\Gamma(k + 2\kappa - 1)}{\Gamma(k + 1)^2 \Gamma(k + 4\kappa + \frac{i}{b} + 1)}. \quad (\text{A.8})$$

where D is a constant. This estimate is valid provided that $k \gg 1$ and that $k^2 \gg \frac{ca}{\sqrt{1+b^2}}$. Similarly, the behavior of f_k at large negative values of k is given by

$$f_k \approx D'(a^*, b, c, \kappa) \left(\frac{ica^*}{i + ib} \right)^{-k} \frac{\Gamma(-k - 2\kappa - 1)}{\Gamma(-k + 1)^2 \Gamma(-k - 4\kappa + \frac{i}{b} + 1)} \quad (\text{A.9})$$

¹⁷Recall that κ is a real number (in order that the mode in question is well defined at all y) of unit periodicity.

where D' is an independent constant. Note χ is real provided that $D(a, b, c, \kappa) = D'(a^*, b, -c, -\kappa)$; a condition that we consequently demand on physical grounds.

We are interested in determining the dispersion relation (A.6) as the condition for the existence of a solution of (A.5) that achieves both asymptotic conditions (A.8) and (A.9) (this leads to an equation as, generic solutions of (A.5) that obey (A.8) blow up at large negative k).

A.2 Qualitative nature of the solution

Let us first note that the off diagonal terms in (A.5) are both proportional to $a \times c$. The proportionality to a reflects the fact that the problem gets strongly coupled at high Reynolds numbers; we will have a lot more to say about this below. The fact that these terms are proportional to c , however, implies that fluctuations with no x momentum are governed by the linear Navier-Stokes equations, and so are always stable. An instability, if it occurs, must necessarily break a symmetry (in this case of x translations) that the background solution preserves.

In much of the rest of this section we specialize our analysis to $\kappa = 0$ and $c = 1$ (i.e. $k_0 = q_0$) but for a reasonably wide range of the parameters a and b . We perform this specialization for convenience in the numerical analysis that we will describe below; we expect qualitatively similar results at all values of κ and reasonable nonzero values of c , though we have not explicitly tested this expectation.

Let us start by painting a qualitative picture of physically relevant solutions of (A.5). We start with the simple limit $a \rightarrow 0$. Solutions are given exactly by $f_k = \delta_{k,K}$ with $\gamma = -K + ib(K^2 + 1)$ for K ranging over integers. The time dependence of this solution is given by $f_k(t) = f_k(0)e^{-iK\omega_0 t - b\omega_0(K^2+1)t}$; it follows that all these modes decay in time, so that our solution is stable against small fluctuations.

At small but nonzero values of Reynolds number $Re = \frac{a}{\sqrt{1+b^2}}$ the solutions to (A.5) are small perturbations of the solutions described in the paragraph above. On the K^{th} such solution f_k is nonzero at $k = K$, but decays rapidly as k moves away from K . In particular $f_{k-K} \propto (Re)^{|k-K|}$.¹⁸ The dispersion relation for γ , described in the previous paragraph, is corrected in a power series in $(Re)^2$. We will compute the first term in this series below.

As the Reynolds number is increased, the width in the distribution of f_k (as a function of k centered around $k = K$) increases. For $Re \gg 1$ and $Re \gg \sqrt{n}$ this width is of order $k \sim (Re)^{\frac{1}{2}}$. (see (A.8) and (A.9)).

In summary, the fluctuation mode we study is highly localized about a particular y momentum at small Reynolds number, but consists of a ‘cascade’ or roughly equal superpositions of order $(Re)^{\frac{1}{2}}$ modes at large Reynolds numbers. The fluctuation modes oscillate rather than growing exponentially in time at small Reynolds numbers; however the stability properties of these modes must be elucidated by explicit calculation at large Reynolds numbers.

¹⁸This decay is visible, for instance, in the asymptotic forms (A.8) and (A.9).

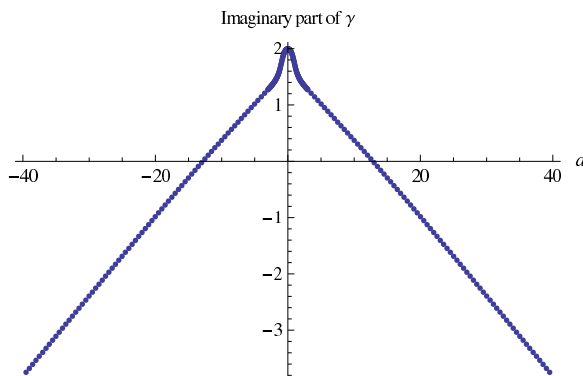


Figure 1. The imaginary part of the frequency γ plotted against a , for $\kappa = 0$, $c = 1$ and the root with $K = 1$. Note that $Im(\gamma) = 2$ at $b = 0$ and decreases monotonically as $|b|$ is increased.

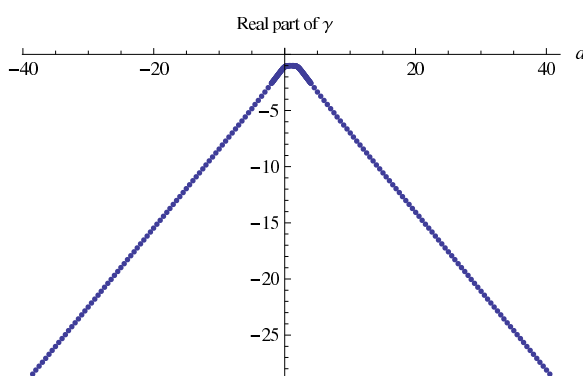


Figure 2. The real part of the frequency γ plotted against a , for $\kappa = 0$, $c = 1$ and the root with $K = 1$. Note that $Re(\gamma) = -1$ at $b = 0$ and decreases monotonically as $|b|$ is increased.

A.3 Numerical evaluation of the frequency

In this subsection we present the results of our rough numerical evaluation of the frequency γ of the mode with $n = 1$ as a function of the Reynolds number, at various values of b . The method we use is very simple. We start with a value of k large enough for the asymptotic form (A.8) to be valid.¹⁹ We then use the recursion relations (A.5) to evaluate f_1/f_0 . We then independently evaluate the same ratio using (A.9) and recursion relations. Equating these expressions for the same ratio gives us an equation which we use to solve for γ as a function of all other parameters. Of course this equation has several solutions. At small Reynolds number we choose the solution that is near to $\gamma = -1 + 2ib$ (i.e. the mode with $K = 1$ in the language of the previous subsection) , and then ‘follow’ this root as we numerically increase the Reynolds number (in small steps) to large values. We have performed all our calculations using Mathematica.

Let us describe our results in detail at $b = 1$. In figure 1. below we present a plot of the imaginary part of γ as a function of Reynolds number. In figure 2 below we plot

¹⁹We have performed computations with values of this starting k ranging from 20 to 8. The plots presented below are generated using the starting value 10. We have verified that our starting value of k is large enough, by checking that our results are not significantly affected by increasing k .

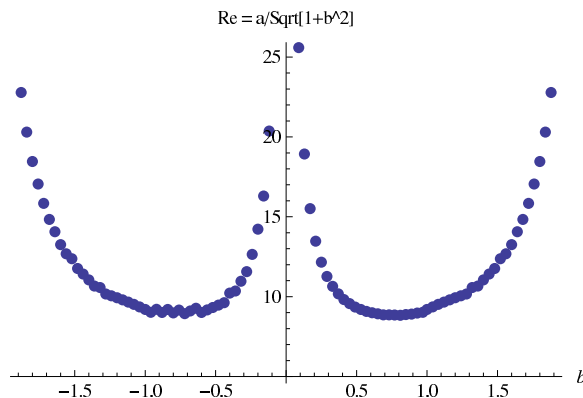


Figure 3. The curve that separates the region of stability (below) from instability (above) of the eigenvalue at $K = 1$, plotted as a function of the Reynolds number on the y axis versus b on the x axis. We have used $\kappa = 0$, $c = 1$ in the calculations that generated this plot.

the real part of the same frequency as a function of Reynolds number. Notice that the imaginary part changes sign (indicating a transition to instability) at a Reynolds number of order 13.

We have performed the same analysis at several different values of b . At each value of b the imaginary part of γ turns negative at a particular value of the Reynolds number. In figure 3 below we have plotted this critical Reynolds number as a function of b for a range of values of b .

As our principle aim in this subsection is to establish that the small fluctuations about our solutions are unstable at high enough Reynolds numbers we have not attempted to carefully estimate the errors in our numerical calculations. However we think it is unlikely that the errors in, for instance, figure 3 exceed a few percent.

As we have described above, at every value of b we start our calculations at small Reynolds numbers which are then slowly increased. As a check on our numerics, at small we have compared our numerical results for γ versus k against the predictions of perturbation theory (see the next subsection for details) at lowest order in $(Re)^2$; we find good agreement. In fact at $b = 1$ and $a = 1/100$, the difference between our numerical result and the zeroth order answer matched the prediction of perturbation theory upto one part in 10^5 ; a more accurate result than we had expected.

A.4 Perturbation theory at small Reynolds numbers

Let us formally identify the equation (A.5) with free index k with

$$\langle k|M|\psi\rangle = \sum_m \langle k|M|m\rangle \langle m|\psi\rangle$$

and simultaneously identify

$$\langle k|\psi\rangle = f_k$$

With these formal identifications, the set of equations (A.5) (for all k) are simply the equation for the operator M to have a zero eigenvalue. The corresponding eigenvector is specified by the values of f_k on this solution.

Let us write $M = M_0 + M_1$ where M_0 is a diagonal operator in the basis described above and M_1 is an off-diagonal operator, proportional to α . The elements of the matrix M_0 and M_1 are given by

$$\begin{aligned}
 \langle f_k | M_0 | f_k \rangle &= A(k) \\
 &= \left[\frac{i}{b} (\gamma + k) ((\kappa + k)^2 + c^2) + ((\kappa + k)^2 + c^2)^2 \right] \delta_{kj} \\
 \langle f_k | M_1 | f_{k-1} \rangle &= B(k) \\
 &= ica \left(-\frac{(\kappa + k - 1)^2 + c^2 - 1}{1 - ib} + 1 \right) \delta_{k,(j-1)} \\
 \langle f_k | M_1 | f_{k+1} \rangle &= C(k) \delta_{k,(j+1)} \\
 &= ica^* \left(-\frac{(\kappa + k + 1)^2 + c^2 - 1}{1 + ib} + 1 \right) \delta_{k,(j+1)}
 \end{aligned} \tag{A.10}$$

When $\alpha = 0$ the eigenvectors of the operator M are simply $|K\rangle$ with eigenvalue $A(K)$. This eigenvalue vanishes when

$$\gamma_0 = -K + i b [(\kappa + K)^2 + c^2].$$

We now wish to study how this eigenvector - and the corresponding solution for γ that keeps the eigenvalue zero - evolves in perturbation theory in α .

To lowest nontrivial order in perturbation theory, the shift in the eigenvalue $A(k)$ is given by

$$\begin{aligned}
 \tilde{A}(k) &= A(k, \omega) + \frac{\langle f_k | M_1 | f_{k-1} \rangle \langle f_{k-1} | M_1 | f_k \rangle}{A(k) - A(k-1)} + \frac{\langle f_k | M_1 | f_{k+1} \rangle \langle f_{k+1} | M_1 | f_k \rangle}{A(k) - A(k+1)} \\
 &= A(k) + \frac{B(k)C(k-1)}{A(k) - A(k-1)} + \frac{C(k)B(k+1)}{A(k) - A(k+1)}
 \end{aligned} \tag{A.11}$$

In order that the eigenvalue vanish at this order we must have

$$\gamma = -K + i b [(\kappa + K)^2 + c^2] - \frac{i b}{(K + \kappa)^2 + c^2} \left[\frac{B(K)C(K-1)}{A(K-1)} + \frac{C(K)B(K+1)}{A(K+1)} \right]$$

where the third term is evaluated at $\gamma = \gamma_0$ and we have used the fact that $A(K) = 0$ at $\gamma = \gamma_0$.

B A simple flow on a sphere

It is not difficult to generalize the simple laminar flow presented in section 4 above to a shear flow on a 2 sphere. The corresponding dual gravitational solution to this flow is asymptotic to (slightly perturbed) global AdS space.

For concreteness we choose $d = 3$ and choose the base metric of our space to be the two sphere, i.e.

$$ds^2 = -dt^2 + d\theta^2 + \sin^2 \theta d\phi^2 \tag{B.1}$$

Our velocity and forcing function fields will be vector fields on the two sphere, so be briefly pause to recall some necessary definitions. Recall that the spherical harmonics, Y_m^l form a basis for the expansion of an arbitrary scalar field on the sphere. On the other hand an arbitrary vector field on the sphere is given by linear combinations of $\partial_i Y_m^l$ and $\epsilon_i^j \partial_j Y_m^l$ where the ϵ symbol includes relevant factors of \sqrt{g} . Now, in our problem, both v^i and a^i are divergenceless. It follows that each of these fields may be expanded in a sum over only the vector (rather than also the derivative of scalar) spherical harmonics.

In order to obtain one solution (that can easily be generalized in many ways) to the Navier-Stokes equations, let

$$a_i = \alpha \epsilon_i^m \partial_m Y_0^l(\theta, \phi).$$

The velocity configuration²⁰

$$v_i = -\frac{\alpha}{1 - i \frac{l(l+1)-2}{\omega}} \times \epsilon_i^m \partial_m Y_0^l(\theta, \phi).$$

(together with the implied pressure field) yields a steady state, laminar, shear flow. The Reynolds number of this flow is given by (5.5) with a and b now given by $a = \frac{\alpha}{k_o \nu}$ and $b = \frac{\nu(l(l+1)-2}{\omega}$. As for the solution presented in section 4, we expect this flow to be unstable to small perturbations at fixed b in the limit of large a (we have not performed the linear stability analysis in this case).

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²⁰We use that

$$\nabla^m \nabla_m \nabla_i \phi = \nabla_i \nabla^m \nabla_m \phi + R_i^m \nabla_m \phi$$

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