Noninertial symmetry group with invariant Minkowski line element consistent with Heisenberg quantum commutation relations

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Noninertial symmetry group with invariant Minkowski line element consistent with Heisenberg quantum commutation relations

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Abstract.
The maximal symmetry of a quantum system with Heisenberg commutation relations is given by the projective representations of the automorphism group of the Weyl-Heisenberg algebra. The automorphism group is the central extension of the inhomogeneous symplectic group with a conformal scaling that acts on extended phase space. We determine the subgroup that also leaves invariant a degenerate Minkowski orthogonal line element. This defines noninertial relativistic symmetry transformations that have the expected classical limit as $c \to \infty$.

1. Introduction

The inhomogeneous Lorentz group defines the relation between inertial states. Clocks locally at rest relative to an inertial state are related to the clocks of other inertial states through the Minkowski proper time line element. Quantum states are rays in a Hilbert space and therefore inertial quantum states are related through the projective representation of the inhomogeneous Lorentz group. Projective representations are equivalence classes of unitary representations of the central extension of the group. The central extension of the inhomogeneous Lorentz group is its cover, the Poincaré group, as this group does not admit an algebraic extension. [1],[2]

The equivalence principle of general relativity enables the noninertial frames of a particle accelerating under gravity to be understood as locally inertial states on a curved manifold. Particles under gravity follow geodesics that are locally inertial trajectories and neighboring locally inertial frames are related by the connection. The clock locally at inertial rest is related to the local clocks of other neighboring locally inertial states in the gravitating system through the Riemannian proper time line element.

Neither general relativity nor special relativity addresses the issue of noninertial states that are not due to gravity, but rather one of the other forces. A special case is a region in which gravity is negligible and the underlying manifold may be considered to be flat. Consider for example an electron in a region that gravity is negligible that encounters an electromagnetic field and therefore has a noninertial trajectory. How are the clocks of such noninertial states related?

We hypothesize that the noninertial relativistic symmetry group relating these states is the most general group consistent with the requirements that

1) the Heisenberg uncertainty principle holds in the noninertial as well as inertial states

2) the proper time given by the Minkowski line element that is invariant for noninertial as well as inertial states

Consider a quantum system in which the position, momentum, energy and time degrees of freedom are represented by the Hermitian representation of the algebra of the Weyl-Heisenberg group $\mathcal{H}(n+1)$ where the number of spacial dimensions is $n=3$. The requirement that the algebra transforms into itself under the action of the relativistic symmetry group means that the symmetry group is a subgroup of the automorphism group of the Weyl-Heisenberg algebra. [3],[4],[5] This automorphism group is

$$\mathcal{A}ut_{\mathcal{H}} \simeq \mathbb{Z}_2 \otimes_s \mathcal{D} \otimes_s \overline{\mathcal{S}p}(2n+2) \otimes_s \mathcal{H}(n+1). \tag{1}$$

$\mathbb{Z}_2$ is the 2 element discrete group, $\mathcal{D}$ is the abelian group isomorphic to the reals under multiplication, $\mathcal{S}p(2n+2)$ is the symplectic group with the over-bar denoting the universal cover and $\mathcal{H}(n+1)$ is the Weyl-Heisenberg group. At this point there is no concept of a relativistic symmetry.

The Minkowski line element is $d\tau^2 = dt^2 - \frac{1}{c^2}dq^2$. This is an invariant for states that are inertially related and the second assertion is that this continues to be true for general noninertial states.

We will show that the homogeneous relativistic symmetry group that is a subgroup of the automorphism group of the Weyl-Heisenberg group that leaves the Minkowski line element invariant is

$$\mathcal{O}a(1,n) \simeq \mathcal{O}(1,n) \otimes_s \mathcal{A}(m), \tag{2}$$

where $m=(n+1)(n+2)/2$ and $\mathcal{A}(m)$ is the abelian group isomorphic to $\mathbb{R}^m$ under addition. The additional generators of the abelian group behave as a power-force stress tensor that is the proper time derivative of the energy-momentum stress tensor. We show that this leads to expected relativistic results in transforming to noninertial states.

This relativistic theory must lead to expected classical results in the limit $c\to\infty$ where the Minkowski line element reduces to the invariant Newtonian time line element dt^2. We have previously studied the most general group that leaves invariant the Newtonian time line element dt^2 that is a subgroup of the automorphisms of the Weyl-Heisenberg group. This results in a group that leads directly to Hamilton's equations and, with the additional requirement of orthonormal position frames, describes the Hamilton symmetry group for noninertial transformations in a classical context. [4]

2. Consistency between a relativistic symmetry group and quantum mechanics

States in quantum mechanics are represented by rays Ψ in a Hilbert space $\mathbf{H}$ that are the equivalence class of states $|\psi\rangle$ in the Hilbert space that are related by a phase

$$\Psi \simeq \left\{ e^{i\omega} \left|\psi\right\rangle \middle| \omega\in\mathbb{R}\right\}, \qquad |\psi\rangle \in H. \tag{3}$$

A relativistic symmetry group $g\in\mathcal{G}$ acts on the states through a projective representation π, $\tilde{\Psi} = \pi(g)\Psi$, with the property that it also has a phase,

$$\pi(\tilde{g}\cdot g) = e^{i\omega(\tilde{g},g)}\pi(\tilde{g})\pi(g), \omega(\tilde{g},g)\in\mathbb{R}. \tag{4}$$

Projective representations are equivalence classes of the unitary representations ϱ of the central extension $\check{\mathcal{G}}$ (denoted by the inverted hat) of the group $\mathcal{G}$ that act on the states, [6],[7]

$$|\tilde{\psi}\rangle = \varrho(g)\left|\psi\right\rangle, \quad g\in\check{\mathcal{G}}, \qquad |\psi\rangle \in \mathbf{H}^{\varrho}. \tag{5}$$

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The Hilbert space is determined by the unitary representation ϱ and so we label it as $\mathbf{H}^{\varrho}$. Observables corresponding to a the relativistic symmetry group $\mathcal{G}$ are represented by the Hermitian representations ϱ' of elements Z of the algebra of the centrally extended group$\check{\mathcal{G}}$, $\hat{Z} = \varrho'(Z)$. The action of the projective representation of the group element $g \in \check{\mathcal{G}}$ on these observables is

$$\hat{\tilde{Z}}\,|\tilde{\psi}\rangle = \varrho(g)\hat{Z}\,|\psi\rangle = \varrho(g)\hat{Z}\varrho(g)^{-1}\varrho(g)\,|\psi\rangle = \varrho(g)\hat{Z}\varrho(g)^{-1}\,|\tilde{\psi}\rangle \tag{6}$$

and so $\hat{\tilde{Z}} = \varrho'(\tilde{Z}) = \varrho'(gZg^{-1})$. Therefore, if the representation ϱ is faithful, we have $\tilde{Z} = gZg^{-1}$.

Position, momentum, energy and time observables are the Hermitian representation of the algebra of the Weyl-Heisenberg group $\mathcal{H}(n+1)$ with a general element given by $Z = z^{\alpha}Z_{\alpha}$, $\alpha = 1, ..2n{+}2$ where $\{z^{\alpha}\} \in \mathbb{P} \simeq \mathbb{R}^{2n+2}$ and Z_{α} are a dimensionless basis for the Weyl-Heisenberg algebra that satisfy the commutation relations

$$[Z_{\alpha}, Z_{\beta}] = \zeta_{\alpha,\beta} I, \tag{7}$$

where $\zeta_{\alpha,\beta}$ are the components of a symplectic metric. The Hermitian representation of the algebra satisfies $[\hat{Z}_{\alpha}, \hat{Z}_{\beta}] = i\zeta_{\alpha,\beta}\hat{I}$, where $\hat{Z}_{\alpha} = \varrho'(Z_{\alpha})$ and $\hat{I} = \varrho'(I)$ is the unit operator on the Hilbert space. The position-time and momentum-energy Hermitian operators are $\{\hat{Z_{\alpha}}\} = \{\hat{X}_a, \hat{P}_a\}$ with $a = 0, ..n$. Note that there is no relativistic symmetry yet and hence no concept of a mass shell.

The basic physical assumption is that the Heisenberg commutation relations are satisfied by any state related by a relativistic symmetry group $\mathcal{G}$. That is, position, momentum, energy and time observables satisfying the Heisenberg quantum commutation relations will also satisfy the Heisenberg quantum commutation relations for any states related by the projective representations of the symmetry group (6). This implies that if $\{X_a, P_a, I\}$ are a basis of the Weyl-Heisenberg algebra, then $\{\tilde{X}_a, \tilde{P}_a, \tilde{I}\}$ are also a basis of the Weyl-Heisenberg algebra where

$$\tilde{X}_a = gX_ag^{-1}, \tilde{P}_a = gP_ag^{-1}, \tilde{I} = gIg^{-1} = I. \tag{8}$$

and $g \in \check{\mathcal{G}}$ and ϱ is a faithful representation. The maximal group for which this property is true is the automorphism group of the Weyl-Heisenberg group. This results in basic consistency condition that the central extension $\check{\mathcal{G}}$ of the relativistic symmetry group $\mathcal{G}$ must be a subgroup of the automorphism group of the Weyl-Heisenberg group and algebra, $\check{\mathcal{G}} \subseteq \mathcal{A}ut_{\mathcal{H}(n+1)}$.

The automorphism group of the Weyl-Heisenberg group is [3],[4],[5]

$$\mathcal{A}ut_{\mathcal{H}(n+1)} = \overline{\mathcal{DS}p}(2n+2) \otimes_s \mathcal{H}(n+1). \tag{9}$$

The Heisenberg group itself are the inner automorphisms. The outer automorphisms are the cover of the homogeneous group

$$\mathcal{DS}p(2n+2) \simeq \mathbb{Z}_2 \otimes \mathcal{D} \otimes \mathcal{S}p(2n+2). \tag{10}$$

The matrix realization of this group and the group properties are given in [4],[5]. The inhomogeneous group includes translations on extended phase space $\mathbb{P}$,

$$\begin{aligned} \mathcal{IDS}p(2n+2) &= \mathcal{DS}p(2n+2) \otimes_s \mathcal{A}(2n+2) \\ &= \mathbb{Z}_2 \otimes_s \mathcal{D} \otimes_s \mathcal{S}p(2n+2) \otimes_s \mathcal{A}(2n+2) \end{aligned}. \tag{11}$$

The automorphism group is the central extension of this inhomogeneous group, $\mathcal{A}ut_{\mathcal{H}(n+1)} \simeq \mathcal{I}\check{\mathcal{D}}\mathcal{S}p(2n+2)$.

Projective representations of the classical inhomogeneous group are equivalence classes of the unitary representations of the automorphism group that is its central extension. The unitary representation determines the Hilbert space of quantum states. These representations are the largest symmetry of a quantum system that preserves the Weyl-Heisenberg commutation relations.

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3. Homogeneous relativistic symmetry group

We determine in this section the homogeneous relativistic symmetry group for noninertial frames that satisfies two conditions

1) It leaves invariant the Minkowski proper time line element.

2) It is a subgroup of the automorphism group of the Weyl-Heisenberg group.

The group $\mathcal{O}a(1,n)$ is dependent on the scale c. We show a homomorphism parameterized by c satisfies the conditions to define a Inönü-Wigner contraction. This contraction results in the Hamilton group that we have previously shown is the relativistic symmetry group for noninertial states in the nonrelativistic ($c\to\infty$) context.

3.1. The group $\mathcal{O}a(1,n)$ and its algebra

The postulates of special relativity requires the invariance of the Minkowski proper time line element

$$d\tau^2=\eta_{a,b}dx^a dx^b \tag{12}$$

with $a,b..=0,..n$ and η is the diagonal matrix $\eta=[\eta_{a,b}]=\mathrm{diag}\{-1,1,...1\}$ with units where $c=1$.

Consider the $2n+2$ dimensional time, position, energy, momentum extended phase space $\mathbb{P}\simeq\mathbb{R}^{2n+2}$ with coordinates $\{z^\alpha\}=\{x^a,p^a\}$ where $\alpha,\beta=1,...2n+2$, $a,b=0,1..n$. The Minkowski metric is a degenerate line element on the cotangent space $T_z^*\mathbb{P}$

$$d\tau^2=\tilde{\eta}_{\alpha,\beta}dz^\alpha dz^\beta \tag{13}$$

where $\tilde{\eta}_{\alpha,\beta}$ are the components of the $(2n+2)\times(2n+2)$ dimensional matrix $\tilde{\eta}$

$$\tilde{\eta}=[\tilde{\eta}_{\alpha,\beta}]=\begin{pmatrix}[\eta_{a,b}] & 0\\ 0 & 0\end{pmatrix}.$$

The group $\mathcal{GL}(2n+2,\mathbb{R})$ of nonsingular $(2n+2)\times(2n+2)$ matrices acts naturally on the cotangent space $T_z^*\mathbb{P}$ with basis $\{dz^\alpha|_z\}$. Elements Γ of the subgroup $\mathcal{OGL}a(1,n)\subset\mathcal{GL}(2n+2,\mathbb{R})$ that is defined as the subgroup that leaves invariant the degenerate line element (13) satisfies $\Gamma^{\mathrm{t}}\tilde{\eta}\Gamma=\tilde{\eta}$. Γ may be written in terms of $(n+1)\times(n+1)$ submatrices as $\Gamma=\left(\begin{smallmatrix}\Lambda & B\\ \Xi & A\end{smallmatrix}\right)$ and therefore

$$\begin{pmatrix}\eta & 0\\ 0 & 0\end{pmatrix}=\begin{pmatrix}\Lambda^{\mathrm{t}} & \Xi^{\mathrm{t}}\\ B^{\mathrm{t}} & A^{\mathrm{t}}\end{pmatrix}\begin{pmatrix}\eta & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\Lambda & B\\ \Xi & A\end{pmatrix}=\begin{pmatrix}\Lambda^{\mathrm{t}}\eta\Lambda & \Lambda^{\mathrm{t}}\eta B\\ B^{\mathrm{t}}\eta\Lambda & B^{\mathrm{t}}\eta B\end{pmatrix}. \tag{14}$$

It follows immediately that $B=0$ and $\Lambda\in\mathcal{O}(1,n)$. As $\det\Gamma=\det\Lambda\det A$, $\det A\neq0$ and $A\in\mathcal{GL}(n+1,\mathbb{R})$. Therefore,

$$\mathcal{OGL}a(1,n)\simeq(\mathcal{O}(1,n)\otimes\mathcal{GL}(n+1,\mathbb{R}))\otimes_s\mathcal{A}((n+1)^2) \tag{15}$$

that has elements Γ and Γ^{-1} of the form

$$\Gamma=\begin{pmatrix}\Lambda & 0\\ \Xi & A\end{pmatrix},\quad \Gamma^{-1}=\begin{pmatrix}\Lambda^{-1} & 0\\ -A^{-1}\Xi\Lambda^{-1} & A^{-1}\end{pmatrix},\qquad \begin{matrix}\Lambda\in\mathcal{O}(1,n)\\ A\in\mathcal{GL}(n+1,\mathbb{R})\end{matrix}\qquad \Xi\in\mathcal{A}((n+1)^2) \tag{16}$$

It can be verified that this has the semidirect product structure as claimed following exactly the same steps as given explicitly below for the $\mathcal{O}a(1,n)$ group. The homogeneous relativistic symmetry group, that is called $\mathcal{O}a(1,n)$, is the homogenous subgroup of the automorphism group that leaves invariant the degenerate line element,

$$\mathcal{O}a(1,n)=\mathcal{DS}p(2n+2)\cap\mathcal{OGL}a(1,n). \tag{17}$$

The elements of the $\mathcal{D}\mathcal{S}p(2n+2)$ group are of the form $\Delta\Sigma$ where $\Delta \in \mathbb{Z}_2 \otimes \mathcal{D}$ and $\Sigma \in \mathcal{S}p(2n+2)$. The symplectic matrices satisfy the condition $\Sigma^{\rm t}\zeta\Sigma = \zeta$ and so $\Sigma^{-1} = -\zeta\Sigma^{\rm t}\zeta$ with $\zeta = \left(\begin{array}{cc} 0 & \eta \\ -\eta & 0 \end{array}\right)$. This may also be written in terms of $(n+1)\times(n+1)$ submatrices $\Sigma_{\mu,\nu}$ $\mu,\nu = 1,2$ with the matrix and inverse having the form

$$\Sigma = \left(\begin{array}{cc} \Sigma_{1,1} & \Sigma_{1,2} \\ \Sigma_{2,1} & \Sigma_{2,2} \end{array}\right), \quad \Sigma^{-1} = \left(\begin{array}{cc} \eta\Sigma_{2,2}^{\rm t}\eta & -\eta\Sigma_{1,2}^{\rm t}\eta \\ -\eta\Sigma_{2,1}^{\rm t}\eta & \eta\Sigma_{1,1}^{\rm t}\eta \end{array}\right). \tag{18}$$

If Γ in (16) is a subgroup of the outer automorphism group, we have $\Sigma = \Delta^{-1}\Gamma$ or expanding,

$$\left(\begin{array}{cc} \Sigma_{1,1} & \Sigma_{1,2} \\ \Sigma_{2,1} & \Sigma_{2,2} \end{array}\right) = \Delta^{-1}\left(\begin{array}{cc} \Lambda & 0 \\ \Xi & A \end{array}\right). \tag{19}$$

$\Gamma^{-1} = \Delta^{-1}\Sigma^{-1}$ that we can compute from (18) and equate to the inverse calculated in (16)

$$\Gamma^{-1} = \left(\begin{array}{cc} \Lambda^{-1} & 0 \\ -\Lambda^{-1}\Xi A^{-1} & A^{-1} \end{array}\right) = \Delta^{-2}\left(\begin{array}{cc} \eta A^{\rm t}\eta & 0 \\ -\eta\Xi^{\rm t}\eta & \eta\Lambda^{\rm t}\eta \end{array}\right) \tag{20}$$

from which it follows that $\Lambda^{-1} = \Delta^{-2}\eta A^{\rm t}\eta$ and $A^{-1} = \Delta^{-2}\eta\Lambda^{\rm t}\eta$. This has a solution if and only if $\Delta = \pm 1_n \in \mathbb{Z}_2 \subset \mathcal{D}$ and $A^{\rm t} = \eta\Lambda^{-1}\eta$. 1_n is the $n\times n$ unit matrix. Noting that $\Lambda^{-1} = \eta\Lambda^{\rm t}\eta$ this gives $A^{\rm t} = \Lambda^{\rm t}$ and therefore $A = \Lambda$. Finally,

$$\Xi^{\rm t} = \eta\Lambda^{-1}\Xi\Lambda^{-1}\eta = \Lambda^{\rm t}\eta\Xi\eta\Lambda^{\rm t}. \tag{21}$$

Thus elements of $\mathcal{O}a(1,n)$ have the form

$$\Gamma(\Lambda,\Xi) = \left(\begin{array}{cc} \Lambda & 0 \\ \Xi & \Lambda \end{array}\right), \ \Lambda \in \mathcal{O}(1,n), \quad \Xi \in \mathcal{A}(m) \ . \tag{22}$$

with $m = (n+1)(n+2)/2$. The group multiplication and inverse of $\mathcal{O}a(1,n)$ are

$$\begin{array}{ll} \Gamma(\Lambda,\Xi) & = \Gamma(\Lambda',\Xi')\Gamma(\Lambda'',\Xi'') \\ & = \Gamma(\Lambda'\Lambda'',\Xi'\Lambda''+\Lambda'\Xi''), \end{array} \qquad \Gamma(\Lambda,\Xi)^{-1} = \Gamma(\Lambda^{-1},-\Lambda^{-1}\Xi\Lambda^{-1}). \tag{23}$$

The Lorentz group is the subgroup $\Gamma(\Lambda,0)$. The matrix components of the Lorentz matrices may be given as the usual expressions in regular and hyperbolic trigonometry terms of the rotation angles and boost angles.

The elements $\Gamma(1_n,\Xi)$ define an abelian normal subgroup with group multiplication, inverse and automorphisms given by

$$\Gamma(1_n,\Xi')\Gamma(1_n,\Xi'') = \Gamma(1_n,\Xi'+\Xi''), \qquad \Gamma(1_n,\Xi)^{-1} = \Gamma(1_n,-\Xi) \tag{24}$$

$$\Gamma(\Lambda',\Xi')\Gamma(1_n,\Xi)\Gamma(\Lambda',\Xi')^{-1} = \Gamma(1_n,\Lambda'\Xi\Lambda'^{-1}) \ . \tag{25}$$

The intersection $\Gamma(\Lambda,0)\cap\Gamma(1_n,\Xi) = 1_n$ and $\Gamma(\Lambda,\Xi) = \Gamma(1_n,\Xi)\Gamma(\Lambda,0)$. Therefore it is the semidirect product

$$\mathcal{O}a(1,n) \simeq \mathcal{O}(1,n)\otimes_s \mathcal{A}((n+1)(n+2)/2). \tag{26}$$

It can be shown that it does not admit an algebraic central extension and therefore the central extension of this group is simply its cover.

A general element of the algebra of $\mathcal{O}a(1,n)$ is $Z = \lambda^{a,b}L_{a,b} + \xi^{a,b}M_{a,b}$. Note that as $\xi^{a,b} = \xi^{b,a}$, that $M_{a,b} = M_{b,a}$. The Lie algebra relations may be directly computed to be

$$\begin{array}{l} [L_{a,b}, L_{c,d}] = -L_{b,d}\eta_{a,c} + L_{b,c}\eta_{a,d} + L_{a,d}\eta_{b,c} - L_{a,c}\eta_{b,d}, \\ [L_{a,b}, M_{c,d}] = -M_{b,d}\eta_{a,c} - M_{b,c}\eta_{a,d} + M_{a,d}\eta_{b,c} + M_{a,c}\eta_{b,d}, \quad [M_{a,b}, M_{c,d}] = 0. \end{array} \tag{27}$$

The $M_{a,b}$ abelian generators transform as a symmetric $(0,2)$ tensor under the Lorentz generators $L_{a,b}$.

Returning to the group, the transformation equations are $d\tilde{z} = \Gamma dz$. Using the definition of Γ in (26) results in

$$d\tilde{x} = \Gamma dx, \quad d\tilde{p} = \Gamma dp + \Xi dx, \tag{28}$$

that in component form are (with units where $c = 1$)

$$d\tilde{x}^a = \lambda^a_b dx^b, \ \ d\tilde{p}^a = \lambda^a_b dp^b + \xi^a_b dx^b. \tag{29}$$

Then, the proper time line element is invariant as required,

$$d\tau^2 \ = \eta_{a,b} d\tilde{x}^a d\tilde{x}^b = \eta_{a,b}\lambda^a_c dx^c \lambda^b_d dx^d = \eta_{a,b} dx^a dx^b. \tag{30}$$

The λ^a_c are the components of the Lorentz transformation that, as usual, depend on the relative rotation angle and hyperbolic boost angle. The mass μ satisfies

$$\begin{array}{ll} c^2 d\tilde{\mu}^2 & = \eta_{a,b} d\tilde{p}^a d\tilde{p}^b = \eta_{a,b}(\lambda^a_c dp^c + \xi^a_c dx^c)\left(\lambda^c_d dp^d + \xi^b_d dx^d\right) \\ & = c^2 d\mu^2 + \eta_{a,b}\xi^a_c \xi^b_d dx^c dx^d + 2\eta_{a,b}\xi^a_c \lambda^b_d dx^c dp^d. \end{array} \tag{31}$$

From basic dimensional analysis, the ξ^a_c have the dimensions of force or power (in units with $c = 1$, these are the same). It is a symmetric tensor satisfying $\xi^a_b = \eta^{a,c}\eta_{b,d}\xi^d_c$ that transforms as an (1,1) tensor under the Lorentz transformation

$$\tilde{\xi}^a_b = \lambda^a_c \lambda^d_b \xi^c_d. \tag{32}$$

These are the properties of a power-force stress tensor that is the proper time derivative of the energy-momentum stress tensor.

The rate of change of the mass squared with respect to the proper time is given by

$$\frac{d\tilde{\mu}^2}{d\tau^2} = \frac{d\mu^2}{d\tau^2} + \frac{1}{c^2}\eta_{a,b}\xi^a_c V^c(\xi^b_d V^d + 2 \ \ \lambda^b_d F^d) \tag{33}$$

where $V^a = \frac{dx^a}{d\tau}$ is the *four* velocity and $F^a = \frac{dp^a}{d\tau}$ is the *four* force for the case $n = 3$.

3.2. Three notation

Further insight into the physical meaning of the group may be obtained by converting to $n+1$ notation that for $n = 3$ is the familiar *three* notation $\{x^a\} = \{t, \frac{1}{c}q^i\}$, $\{p^a\} = \{\frac{1}{c}e, p^i\}$, $i, j = 1, ..n$. The Lorentz matrix $\Lambda(\alpha, \beta)$ parameterized by rotation angles $\alpha^{i,j} = -\alpha^{j,i}$ and hyperbolic boost rotations β^i that have the usual form. As usual, we identify velocity as $v^i = c\frac{\beta^i}{|\beta|}\tanh(\beta)$ and define $\gamma(\beta) = \cosh(\beta) = \lambda^0_0$ or equivalently $\gamma(v) = (1 - (\frac{v}{c})^2)^{-\frac{1}{2}}$.

The velocity *four* vectors are given as usual by $\{V^0, V^i\} = \{\gamma, \gamma v^i\} = \gamma\{1, \frac{dx^i}{dt}\}$ where $\gamma = \frac{dt}{d\tau}$. The *four* force likewise is $\{F^0, F^i\} = \{\gamma r, \gamma f^i\}$ where $f^i = \frac{dp^i}{dt}$ and $r = \frac{de}{dt}$ and f^i has the dimensions of force and r has the dimensions of power. The power-force-stress components are

$$\Xi = \begin{pmatrix} \xi^0_0 & \xi^0_i \\ \xi^j_0 & \xi^j_i \end{pmatrix} = \gamma \begin{pmatrix} \frac{1}{c}r & -f_i \\ f^j & \frac{1}{c}m^j_i \end{pmatrix}. \tag{34}$$

Therefore, the transformation equations for the position, time, momentum, energy basis is

$$\begin{array}{ll} d\tilde{t} = \gamma dt + \frac{1}{c}\lambda_i^0 \; dq^i, & d\tilde{p}^i = \lambda_j^i dp^j + \lambda_i^0 \; dt, \\ d\tilde{q}^i = \lambda_j^i dq^j + c\lambda_0^i \; dt, & d\tilde{e} = \gamma de + c\lambda_i^0 \; dp^i - \gamma f_i dq^i + c\gamma r dt \; . \end{array} \tag{35}$$

For $n = 1$ these are simply

$$\begin{array}{ll} d\tilde{t} = \gamma(v)\left(\; dt + \frac{1}{c^2}v dq\right), & d\tilde{p} = \gamma(v)\left(dp^j + \frac{1}{c^2}v de + f dt + \frac{1}{c^2} \; m \; dq\right), \\ d\tilde{q} = \gamma(v)\left(dq + v dt\right), & d\tilde{e} = \gamma(v)\left(de + v dp - f dq + r dt\right). \end{array} \tag{36}$$

and the corresponding group parameter transformations using (23) are

$$\gamma(v)\begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix} = \gamma(v'')\gamma(v')\begin{pmatrix} 1 & v' \\ v' & 1 \end{pmatrix}\begin{pmatrix} 1 & v'' \\ v'' & 1 \end{pmatrix} \tag{37}$$

$$\gamma(v)\begin{pmatrix} r & f \\ f & m \end{pmatrix} = \gamma(v'')\gamma(v')\left(\begin{pmatrix} r' & f' \\ f' & m' \end{pmatrix}\begin{pmatrix} 1 & v'' \\ v'' & 1 \end{pmatrix} + \begin{pmatrix} 1 & v' \\ v' & 1 \end{pmatrix}\begin{pmatrix} r'' & f'' \\ f'' & m'' \end{pmatrix}\right) \tag{38}$$

and so

$$\begin{array}{l} v = (v'' + v')\,/\left(1 + \frac{v'v''}{c^2}\right), \quad f = \left(f'' + f' + \frac{1}{c^2}\left(r'v'' - v'r''\right)\right)/\left(1 + \frac{v'v''}{c^2}\right), \\ r = (r'' + r' - f'v'' + v'f'')\,/\left(1 + \frac{v'v''}{c^2}\right), m = (m'' + m' + f'v'' - v'f'')\,/\left(1 + \frac{v'v''}{c^2}\right). \end{array} \tag{39}$$

Consider next the algebra. We have the infinitesimal parameter correspondence

$$\lambda^{0,i} = \frac{1}{c}\beta^i, \lambda^{j,i} = \alpha^{i,j}, \xi^{0,0} = \frac{1}{c}r, \xi^{j,0} = f^j, \xi^{j,i} = \frac{1}{c}m^{j,i} \tag{40}$$

where $\alpha^{i,j} = -\alpha^{i,j}$ and $m^{i,j} = m^{j,i}$ with the corresponding generators

$$L_{0,j} = cK_j, L_{i,j} = J_{i,j}, M_{0,0} = cR, M_{i,0} = N_i, M_{i,j} = cM^{\circ}_{i,j}. \tag{41}$$

A general element of the algebra is $Z = \alpha^{i,j}J_{i,j} + \beta^i K_i + f^i N_i + rR + m^{i,j}M^{\circ}_{i,j}$. The nonzero commutators of the Lie algebra (27) written in terms of these generators.

$$\begin{array}{ll} [J_{i,j}, J_{k,l}] = -J_{j,l}\delta_{i,k} + J_{j,k}\delta_{i,l} + J_{i,l}\delta_{j,k} - J_{i,k}\delta_{j,l}, & [K_i, K_k] = \frac{1}{c^2}J_{i,k}, \\ [J_{i,j}, K_k] = -K_j\delta_{i,k} + K_i\delta_{j,k}, & [K_i, N_k] = -M^{\circ}_{i,k} - R\delta_{i,k}, \\ [J_{i,j}, N_k] = -N_j\delta_{i,k} + N_i\delta_{j,k}, & [K_i, R] = -\frac{2}{c^2}N_i, \\ \left[J_{i,j}, M^{\circ}_{k,l}\right] = M^{\circ}_{j,l}\delta_{i,k} - M^{\circ}_{j,k}\delta_{i,l} + M^{\circ}_{i,l}\delta_{j,k} + M^{\circ}_{i,k}\delta_{j,l} & \\ \left[K_i, M^{\circ}_{k,l}\right] = \frac{-1}{c^2}\left(N_l\delta_{i,k} + N_k\delta_{i,l}\right), & \end{array} \tag{42}$$

3.3. Contraction in the limit $c \to \infty$

The scaling with c given in (42) satisfies the conditions for an Inönü-Wigner [8] contraction $c \to \infty$ for which the nonzero contracted commutators are

$$\begin{array}{ll} [J_{i,j}, J_{k,l}] = -J_{j,l}\delta_{i,k} + J_{j,k}\delta_{i,l} + J_{i,l}\delta_{j,k} - J_{i,k}\delta_{j,l}, & [K_i, N_k] = -M^{\circ}_{i,k} - R\delta_{i,k}, \\ [J_{i,j}, K_k] = -K_j\delta_{i,k} + K_i\delta_{j,k}, & [J_{i,j}, N_k] = -N_j\delta_{i,k} + N_i\delta_{j,k}, \\ \left[J_{i,j}, M^{\circ}_{k,l}\right] = -M^{\circ}_{j,l}\delta_{i,k} - M^{\circ}_{j,k}\delta_{i,l} + M^{\circ}_{i,l}\delta_{j,k} + M^{\circ}_{i,k}\delta_{j,l}. & \end{array} \tag{43}$$

The subgroup spanned by $\{J_{i,j}, K_i, N_i, R\}$ is the algebra of the Hamilton group $\mathcal{H}a(n)$. The full algebra of the group defined by

$$\mathcal{O}(n) \otimes_s \mathcal{A}(n(n+1)/2) \otimes_s \mathcal{H}(n) = \hat{\mathcal{H}a}(n) \otimes_s \mathcal{A}(n(n+1)/2) \tag{44}$$

where $\hat{\mathcal{H}a}(n)$ is the extended Hamilton group

$$\hat{\mathcal{H}a}(n) = \mathbb{Z}_2 \otimes_s \mathcal{H}a(n), \quad \mathcal{H}a(n) = \mathcal{SO}(n) \otimes_s \mathcal{H}(n) \tag{45}$$

where $\{J_{i,j}\}$ are the generators of $\mathcal{O}(n)$, $\{M_{i,j}\}$ are the generators of $\mathcal{A}(n(n+1)/2)$ and $\{K_i, N_i, R\}$ are the generators of the Weyl-Heisenberg group $\mathcal{H}(n)$.

The basis transformation equations (35) contract to the expected transformation equations in the limit [5]

$$\begin{array}{ll} d\tilde{t} = \ dt, & d\tilde{p}^i = \lambda(\alpha)^i_j dp^j + f^i dt, \\ d\tilde{q}^i = \lambda(\alpha)^i_j dq^j + v^i \ \ dt, & d\tilde{e} = de + v_i dp^i - f_i dq^i + r dt. \end{array} \tag{46}$$

where the $\lambda(\alpha)^i_j$ are now the components of a rotation matrix, $\mathcal{O}(n)$.

4. Summary

We started by noting that neither special relativity nor general relativity address the problem of how clocks of noninertial states due to forces other than gravity are related.

The hypothesis that the Minkowski proper time line element is invariant in these states that are not necessarily inertial and requiring that the Heisenberg commutation relations hold in all noninertial states results in the noninertial relativistic symmetry group $\mathcal{O}a(1,n)$. This group gives the expected transformations to noninertial states in terms of a power-force stress tensor that is the proper time derivative of the energy-momentum stress tensor.

The $\mathcal{O}a(1,n)$ group is also the $b \to \infty$ of the $\mathcal{U}(1,n)$ group of reciprocal relativity described in [9]. This gives an understanding of the behavior of reciprocal relativity in the small interaction limit (that is, small forces relative to b) that is analogous to the manner in which the Euclidean group that is the homogeneous group of the Galilei group gives the small velocity limit, relative to c, of the Lorentz group.

Spacetime is an invariant subspace under the actions of the $\mathcal{O}a(1,n)$ group and therefore is observer independent or *absolute*. In this limit, there is an apparent global inertial frame that all observers agree on. Forces appear to be relative to this frame rather than being strictly relative to particle states. Forces and the power-force-stress energy tensor are simply additive and unbounded. Velocities are bounded by c and strictly relative to particle states.

In the $c \to \infty$ limit yields the classical *nonrelativistic* Hamilton theory that describes particles undergoing general noninertial motion. In this case, there is an apparent global inertial rest frame that all observers agree on. Forces and velocities appear to be relative to this frame rather than being strictly relative to particle states. Forces and velocities are simply additive and unbounded.

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