Theoretical and numerical bimetric relativity

Francesco Torsello

Doctoral Thesis in Theoretical Physics at Stockholm University, Sweden 2020

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Francesco Torsello

Academic dissertation for the Degree of Doctor of Philosophy in Theoretical Physics at Stockholm University to be publicly defended on Wednesday 18 March 2020 at 13.15 in sal FB52, AlbaNova universitetscentrum, Roslagstullsbacken 21.

Abstract

General relativity (GR) is the standard physical theory describing gravitational interactions. All astrophysical and cosmological observations are compatible with its predictions, provided that unknown matter and energy components are included. These are called dark matter and dark energy.

In addition, GR describes the nonlinear self-interaction of a massless spin-2 field. In particle physics, there are both massless and massive fields having spin 0, 1 and 1/2. It is then well-justified to ask whether a mathematically consistent nonlinear theory describing a massive spin-2 field exists.

The Hassan–Rosen bimetric relativity (BR) is a mathematically consistent theory describing the nonlinear interaction between a massless and a massive spin-2 field. These fields are described by two metrics, out of which only one can be directly coupled to us and determines the geometry we probe.

Since it includes GR, BR is an extension of it and provides us with new astrophysical and cosmological solutions. These solutions, which may give hints about the nature of dark matter and dark energy, need to be tested against observations in order to support or falsify the theory. This requires predictions for realistic physical systems. One such system is the spherically symmetric gravitational collapse of a dust cloud, and its study is the overarching motivation behind the thesis.

Studying realistic physical systems in BR requires the solving of the nonlinear equations of motion of the theory. This can be done in two ways: (i) looking for methods that simplify the equations in order to solve them exactly, and (ii) solving the equations numerically.

The studies reviewed in the thesis provide results for both alternatives. In the first case, the results concern spacetime symmetries (e.g., spherical symmetry) and how they affect particular solutions in BR, especially those describing gravitational collapse. In the second case, inspired by the success of numerical relativity, the results initiate the field of numerical bimetric relativity. The simulations provide us with the first hints about how gravitational collapse works in BR.

Keywords: *spin-2 fields, extension of general relativity, ghost-free bimetric theory, Hassan–Rosen bimetric relativity, numerical relativity.*

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Stockholm University

Department of Physics

Stockholm University, 106 91 Stockholm

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Francesco Torsello

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Cover image: Vincent van Gogh, "Tree Roots", 1890, oil on canvas. Credits: Van Gogh Museum, Amsterdam (Vincent van Gogh Foundation).
Tree roots have several branches and allow trees to interact with one another (see https://www.smithsonianmag.com/science-nature/the-whispering-trees-180968084/), precisely as the square root in bimetric relativity, which has several branches and is responsible for the interaction between the metrics.

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To Science

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Supervisor

Edvard Mörtsell
Professor, The Oskar Klein Centre, Department of Physics, Stockholm University, AlbaNova, Stockholm, Sweden

Opponent

Eugeny Babichev
Chargé de recherche, Laboratoire de Physique Théorique (UMR 8627), CNRS, Univ. Paris-Sud, Université Paris-Saclay, Orsay, France

Committee

Marcus Berg
Professor, Department of Physics, Karlstad University, Karlstad, Sweden

Agnese Bissi
Associate senior lecturer, Department of Physics and Astronomy, Uppsala University, Uppsala, Sweden

Stephan Rosswog
Professor, The Oskar Klein Centre, Department of Astronomy, Stockholm University, AlbaNova, Stockholm, Sweden

Matthias Blennow — backup member
Associate professor, Department of Theoretical Physics, School of Engineering Sciences, KTH Royal Institute of Technology, AlbaNova, Stockholm, Sweden

Chairman

Ingemar Bengtsson
Professor, Department of Physics, Stockholm University, AlbaNova, Stockholm, Sweden

Abstract

General relativity (GR) is the standard physical theory describing gravitational interactions. All astrophysical and cosmological observations are compatible with its predictions, provided that unknown matter and energy components are included. These are called dark matter and dark energy.

In addition, GR describes the nonlinear self-interaction of a massless spin-2 field. In particle physics, there are both massless and massive fields having spin 0, 1 and 1/2. It is then well-justified to ask whether a mathematically consistent nonlinear theory describing a massive spin-2 field exists.

The Hassan–Rosen bimetric relativity (BR) is a mathematically consistent theory describing the nonlinear interaction between a massless and a massive spin-2 field. These fields are described by two metrics, out of which only one can be directly coupled to us and determines the geometry we probe.

Since it includes GR, BR is an extension of it and provides us with new astrophysical and cosmological solutions. These solutions, which may give hints about the nature of dark matter and dark energy, need to be tested against observations in order to support or falsify the theory. This requires predictions for realistic physical systems. One such system is the spherically symmetric gravitational collapse of a dust cloud, and its study is the overarching motivation behind the thesis.

Studying realistic physical systems in BR requires the solving of the nonlinear equations of motion of the theory. This can be done in two ways: (i) looking for methods that simplify the equations in order to solve them exactly, and (ii) solving the equations numerically.

The studies reviewed in the thesis provide results for both alternatives. In the first case, the results concern spacetime symmetries (e.g., spherical symmetry) and how they affect particular solutions in BR, especially those describing gravitational collapse. In the second case, inspired by the success of numerical relativity, the results initiate the field of numerical bimetric relativity. The simulations provide us with the first hints about how gravitational collapse works in BR.

Svensk sammanfattning

Den allmänna relativitetsteorin är den fysikaliska standardteorin som beskriver gravitation. Alla astrofysikaliska och kosmologiska observationer överensstämmer med teorins förutsägelser, förutsatt att okända komponenter av materia och energi ingår. De kallas för mörk materia och mörk energi.

Dessutom beskriver den allmänna relativitetsteorin den icke-linjära självinteraktionen av ett masslöst spinn-2 fält. Inom partikelfysik finns det både masslösa och massiva fält med spinn 0, 1 och 1/2. Det är alltså väl motiverat att fråga om det finns en matematisk konsistent och icke-linjär teori som kan beskriva ett massivt spin-2 fält.

Hassan–Rosens bimetriska relativitetsteori (BR) är en matematisk konsistent teori som beskriver den icke-linjära interaktionen mellan ett masslöst och ett massivt spin-2 fält. Båda dessa fält beskrivs av två metriker. Endast en av dem kan kopplas direkt till oss och den bestämmer vilken geometri vi kan undersöka.

Eftersom BR inkluderar den allmänna relativitetsteorin är den en utvidgad teori om gravitation och den ger oss nya astrofysikaliska och kosmologiska lösningar. Dessa lösningar, som kan ge oss ledtrådar om ursprunget av mörk materia och mörk energi, måste testas observationellt för att stödja eller förfalska teorin. Detta kräver förutsägelser för realistiska fysiska system. Ett sådant system är en sfäriskt symmetrisk gravitationskollaps av ett stoftmoln. Studien av sådana system är den övergripande motivationen bakom avhandlingen.

Att studera realistiska fysiska system i BR kräver lösningen av icke-linjära rörelseekvationer. Detta kan göras på två sätt: (i) genom att leta efter metoder som förenklar ekvationerna för att lösa dem exakt och (ii) att lösa ekvationerna numeriskt.

Studierna i avhandlingen ger resultat för båda alternativen. I bägge fallen är de inspirerade av allmän relativitetsteori. I det första fallet rör resultaten rymdtidssymmetrier (till exempel sfärisk symmetri) och hur de påverkar specifika lösningar i BR, särskilt de som beskriver gravitationskollaps. I det andra fallet, som är inspirerat av den numeriska relativitetens framgångar, inleder resultaten fältet för numerisk bimetrisk relativitet. Resultaten ger oss de första idéerna om hur gravitationskollaps fungerar i BR.

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List of Figures

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List of included papers

The following papers are included in this thesis, and are referred to by the Roman numbers assigned below.

I Francesco Torsello, Mikica Kocic, Marcus Högås, and Edvard Mörtsell. "Spacetime symmetries and topology in bimetric relativity". *Phys. Rev. D* 97 (8 Apr. 2018), p. 084022. DOI: 10.1103/PhysRevD.97.084022. URL: https://link.aps.org/doi/10.1103/PhysRevD.97.084022

II Mikica Kocic, Marcus Högås, Francesco Torsello, and Edvard Mörtsell. "On Birkhoff's theorem in ghost-free bimetric theory" (2017). arXiv: 1708.07833 [hep-th]. Submitted to *Phys. Rev. D*, under peer-review.

III Francesco Torsello, Mikica Kocic, and Edvard Mörtsell. "Classification and asymptotic structure of black holes in bimetric theory". *Phys. Rev. D* 96 (6 Sept. 2017), p. 064003. DOI: 10.1103/PhysRevD.96.064003. URL: https://link.aps.org/doi/10.1103/PhysRevD.96.064003

IV Mikica Kocic, Anders Lundkvist, and Francesco Torsello. "On the ratio of lapses in bimetric relativity". *Classical and Quantum Gravity* 36.22 (Oct. 2019), p. 225013. DOI: 10.1088/1361-6382/ab497a. URL: https://doi.org/10.1088%2F1361-6382%2Fab497a

V Francesco Torsello, Mikica Kocic, Marcus Högås, and Edvard Mörtsell. "Covariant BSSN formulation in bimetric relativity". *Classical and Quantum Gravity* 37.2 (Dec. 2019), p. 025013. DOI: 10.1088/1361-6382/ab56fc. URL: https://doi.org/10.1088%2F1361-6382%2Fab56fc

VI Francesco Torsello. "The mean gauges in bimetric relativity". *Classical and Quantum Gravity* 36.23 (Nov. 2019), p. 235010. DOI: 10.1088/1361-6382/ab4ccf. URL: https://doi.org/10.1088%2F1361-6382%2Fab4ccf

VII Francesco Torsello. "bimEX: A Mathematica package for exact computations in 3+1 bimetric relativity". *Computer Physics Communications* 247 (2020), p. 106948. ISSN: 0010-4655. DOI: https://doi.org/10.1016/j.cpc.2019.106948. URL: http://www.sciencedirect.com/science/article/pii/S0010465519303030

VIII Mikica Kocic, Francesco Torsello, Marcus Högås, and Edvard Mortsell. "Initial data and first evolutions of dust clouds in bimetric relativity" (2019). arXiv: `1904.08617 [gr-qc]`. Submitted to *Classical and Quantum Gravity*, under peer-review.

The following papers are not included in the thesis, and are cited as ordinary references in the main text.

[**Koc+19a**] Mikica Kocic, Marcus Högås, Francesco Torsello, and Edvard Mörtsell. "Algebraic properties of Einstein solutions in ghost-free bimetric theory". *Journal of Mathematical Physics* 60.10 (2019), p. 102501. DOI: `10.1063/1.5100027`. eprint: `https://doi.org/10.1063/1.5100027`. URL: `https://doi.org/10.1063/1.5100027`

[**Hög+19**] Marcus Högås, Mikica Kocic, Francesco Torsello, and Edvard Mörtsell. "Exact solutions for gravitational collapse in bimetric gravity" (2019). arXiv: `1905.09832 [gr-qc]`

[**HTM19**] Marcus Högås, Francesco Torsello, and Edvard Mörtsell. "On the stability of bimetric structure formation" (2019). arXiv: `1910.01651 [gr-qc]`

The chronological order of the papers is: **III, II, I, IV, V, VI, VII, VIII**.

Preface

This doctoral thesis collects the scientific results obtained during my Ph.D. studies, from September 2015 to February 2020, at the Oskar Klein Centre, Department of Physics, Stockholm University, in the Cosmology, Particle Astrophysics and Strings (CoPS) division. It is accompanied by the articles written during the same period, published or submitted to peer-reviewed scientific journals.

The thesis is divided in three parts. The last part contains the included papers. Part I and Part II constitute the body of the thesis. Specifically, Part I includes the results obtained during the first two years of the Ph.D. studies, whereas Part II includes the results obtained in the last two years and a half of the Ph.D. studies. The reason for keeping the two parts separated is that the results obtained in the first and last part of the Ph.D. studies, even though they have a shared motivation, use different methods. The Introduction, the Conclusion of Part I and the Conclusion and outlook connect the two parts and explain why they have both a standalone and a combined importance, in addition to clarifying what is the common motivation behind them and how the results tie together.

The thesis is written assuming the reader to be another Ph.D. student who starts to work in the field. Since there are no books describing the theory of extended gravity I have been working on, in writing the thesis I wanted both to explain my work and to provide the necessary background needed to understand my work in the clearest and simplest possible way. Also, I collected what I think are the relevant references that explain in a clear way both the physics and the mathematics needed. I hope to have succeeded in this task, and I leave the final judgment to the reader. Lastly, the thesis is not only a comprehensive summary of the included papers, but it includes complements to them and some original results obtained during the writing of the thesis.

Contributions to the papers

Paper I. I had the original idea for the paper while working on Paper III, and the work on [Koc+19a] motivated its development. All the authors contributed to the results. I guided the work and wrote most of the paper, with the help from the coauthors.

Paper II. This paper was born while looking for example solutions for Paper I. I contributed in analyzing the solution, in checking the results, in discussing the physical implications and in writing the final version of the manuscript.

Paper III. Edvard Mörtsell and I initiated the project. Mikica Kocic and I performed the analytical and numerical studies. All the authors had many discussions about the physical implications of the results. I wrote the majority of the paper, with the help from the coauthors.

Paper IV. The idea for the paper was proposed by Mikica Kocic. I made the computations included in Appendix E and checked part of the computations included in the main text. I contributed to writing the final version of the paper in all of its parts. All the authors discussed extensively about the meaning of the concepts introduced in the paper.

Paper V. The original idea for the work resulting in this paper was proposed by Mikica Kocic. I performed all the computations included in the paper and its appendix and wrote the majority of the paper. All the authors had several discussions about the meaning and physical implications of the results.

Paper VI. The original idea was proposed in [HK18]. I made all the computations and wrote the entire paper. I had many discussions with Marcus Högås, Mikica Kocic, Anders Lundkvist and Edvard Mörtsell.

Paper VII. I developed the package presented in this paper during the work on Paper V and Paper VI. I wrote the entirety of the code, with support from Mikica Kocic. I wrote the entire paper, with the help from Edvard Mörtsell and Mikica Kocic.

Paper VIII. I reproduced the numerical solutions and checked the final results, and contributed to the writing of the final version of the paper. All the authors had several discussions about both the numerical issues and the physical implications of the numerical solutions.

Material from the Licentiate thesis

The entirety of Part I and Appendix A of this thesis is based on my Licentiate thesis, "Symmetries and black holes in Hassan–Rosen bimetric theory," 2018 (unpublished). All chapters and sections of the Licentiate thesis were slightly modified, improved and harmonized with Part II during the work leading to the final form of this thesis.

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First and foremost, I express my gratitude to my supervisor Edvard Mörtsell, for being always willing to discuss and answer my questions each time I knock on the door of his office. I cannot think of a more encouraging, positive and inspiring supervisor. I am also grateful to Fawad Hassan, my assistant supervisor, whose answers to my questions guided me many times towards the right direction.

I am pleased to thank my collaborators Marcus Högås, Mikica Kocic and Anders Lundkvist. As I like to say, we are all meticulous in different ways, and this makes us enlighten each other very profitably. All our work together and our (sometimes endless) discussions have been invaluable.

My gratitude goes to Eddy Ardonne, Ingemar Bengtsson, Julius Engelsöy, Marcus Högås, Mikica Kocic and Axel Widmark for reading the thesis and providing valuable comments and remarks.

I am thankful to the Oskar Klein Centre and the CoPS division. Their dynamical and inspirational atmosphere made me enjoy the years of my Ph.D. in the best way. In particular, during these years the interaction with many people was beneficial to me. Many thanks go to Rahman Amanullah, Luis Apolo, Jorge Laraña Aragon, Sebastian Baum, Ingemar Bengtsson, Lars Bergström, Giovanni Camelio, Jonas Enander, Julius Engelsöy, Ariel Goobar, Christoffer Lundman, David Marsch, Kjell Rosquist, Stephan Rosswog, Angnis Schmidt-May, Christian Setzer, Stefan Sjörs, Mikael von Strauss, Bo Sundborg, Axel Widmark and Nico Wintergerst for all the pleasant and profitable discussions we had.

I thank my office mates Marcus Högås, Mikica Kocic and Anders Lundkvist for making the "bimetric office" a great place to spend the day. I thank them and my former office mates Sebastian Baum, Adri Duivenvoorden, Julius Engelsöy, Daniel Palm and Sunny Vagnozzi for always knowing when to start a new intriguing (or funny, or both) conversation.

In addition, I thank my mentor Stefano Bonetti, for being always supportive and for sharing with me his wisdom about the academic life and career.

I am thankful to Viktoria Larsson, the coach of AlbaNova Gym. She taught me how to increase all my personal best results in amateur powerlifting. I will miss both her professional coaching and the gym environment that my gym mates Alexander Agapow, Björn Ahlgren, Egor Babaev, Matthias Blennow, Emil Blomqvist, Julius Engelsöy, Raphael Kalender, Anders Källberg, Jakob Larsson, Mats Larsson, Per Moosavi, Patrick Mutter, Frida Navratil, Petros Papadogiannis, Marcus Pernow, Sergey Pershoguba, Jonas Persson, Dmitry Romashchenko, Num Vistbacka and Axel Widmark contributed to create and maintain during these years. I hope I will be able to join other AlbaNova Gym competitions as an external athlete.

Infine ringrazio Sonia, la mia compagna di vita da oltre dieci anni, con cui ho condiviso momenti bellissimi e superato momenti difficili. Ringrazio la mia famiglia, da sempre inesauribile fonte di fiducia e supporto, e i miei amici Andrea, Emanuele, Gigi, Marcello, Marco, Matteo e Piergiorgio, che da più di quindici anni mi aiutano a definire chi sono e mi ricordano da dove vengo.

Francesco Torsello
Stockholm, January 2020

List of Acronyms

ΛCDM	Λ cold dark matter (Standard Model of cosmology)
nubirel	numerical bimetric relativity
AdS	Anti–de Sitter
BD	Boulware–Deser
BFE	bimetric field equations
BH	black hole
BR	bimetric relativity
BSSN	Baumgarte–Shapiro–Shibata–Nakamura
BSSNOK	Baumgarte–Shapiro–Shibata–Nakamura–Oohara–Kojima
cBSSN	covariant Baumgarte–Shapiro–Shibata–Nakamura
CC	curvature collineation
CFL	Courant–Lax factor
CK	Cauchy–Kovalevskaya
DoF	degree of freedom
dRGT	de Rham–Gabadadze–Tolley
EC	Einstein–Christoffel
EFE	Einstein field equations
EH	Einstein–Hilbert
ESDIRK	diagonally implicit Runge–Kutta with an explicit first stage and a single eigenvalue
FOSH	first-order symmetric hyperbolic
FTCS	forward time–centered space
GHF	generalized harmonic formulation
GR	general relativity
ICN	iterative Crank–Nicholson
ISW	integrated Sachs–Wolfe effect
KST	Kidder–Scheel–Teukolsky

KVF	Killing vector field
LD	linear diffeomorphism
MC	matter collineation
MoL	method of lines
MOTS	marginally outer trapped surface
NOR	Nagy–Ortiz–Reula
NR	numerical relativity
NS-SS	nonstationary spherically symmetric
ODE	ordinary differential equation
PDE	partial differential equation
QFT	quantum field theory
RC	Ricci collineation
RK	Runge–Kutta
SAdS	Schwarzschild–anti–de Sitter
SdS	Schwarzschild–de Sitter
SM	Standard Model of particle physics

Conventions and notation

The metric signature is $(-1, 1, 1, 1)$. Geometric quantities are indicated both with indices and without depending on the context, if this does not create confusion. For example, a metric is written both g and $g_{\mu\nu}$. Analogously, a vector field can be written $\mathcal{V}$ or $\mathcal{V}^{\mu}$. Spacetime indices are denoted with Greek letters and run from 0 to 3; spatial indices are denoted with Latin letters and run from 1 to 3; boldface uppercase indices are Lorentz indices and run from 0 to 3; boldface lowercase indices are spatial Lorentz indices and run from 1 to 3.

4-dimensional geometric quantities related to g are indicated with the subscript g, for example G_g is the Einstein tensor for g. 4-dimensional geometric quantities related to f are indicated with the subscript f. In the expressions $\sqrt{-g}$ and $\sqrt{-f}$, the symbols g and f always refer to the determinants of the metrics g and f.

In the $3+1$ decomposition, the spatial parts of the 4-dimensional metrics g, f, h are γ, φ and χ, and 4-dimensional geometric quantities are denoted with the prescript 4, for example ${}^4R_{\mu\nu}$ is the Ricci tensor for the metric g. We denote the determinant of a spatial metric with the symbol Δ, and use the following notation taken from Paper V and Paper VI,

Δ ,	no accent :	quantity refers to the g-sector,
$\tilde{\Delta}$,	tilde :	quantity refers to the f-sector,
$\overset{\#}{\Delta}$,	hash :	quantity refers to the h-sector,
$\boldsymbol{\Delta}$,	boldface :	quantity refers to the Lorentz frame.
$\bar{\Delta}$,	overbar :	quantity refers to the g-sector in BSSN,
$\widehat{\Delta}$,	wide hat :	quantity refers to the f-sector in BSSN,
$\mathring{\Delta}$,	circle :	quantity refers to the h-sector in BSSN,
$\overset{*}{\boldsymbol{\Delta}}$,	boldface, asterisk :	quantity refers to the Lorentz frame in BSSN.

The symbol $:=$ is used to define objects; the symbol $\equiv$ is used for identities. *Italics* are used to emphasize and the quotation marks "" are used to introduce or define a new word or a new expression.

The notation "(metrics)-geometric property" means that an object has a geometric property with respect to the specified metrics. For example, f-spacelike means "spacelike with respect to f" and (g, h)-orthogonal means "orthogonal with respect to g and h."

In the plots, the color blue is associated with the metric g, the color red with the metric f and the color green with their geometric mean metric h.

Introduction

Presentation of the thesis. This thesis is about a theory of gravity, called Hassan–Rosen bimetric theory or bimetric relativity, which is an extension of general relativity. Bimetric relativity is a geometric theory of gravity, as general relativity, and as such it can be studied using the methods developed for general relativity. The studies described in this thesis extend several theoretical and numerical methods used in general relativity, to bimetric relativity.

An intuitive phenomenon caused by gravitational interaction is gravitational collapse, due to the innate attractiveness of gravity. Gravitational collapse is at the basis of our understanding of physical systems at very different length scales, for example the formation of stars, compact stars, black holes, solar systems, galaxies and galaxy clusters. In cosmology, structure formation relies on gravitational collapse in a homogeneous and isotropic expanding spacetime.

Therefore, it is fundamental to understand the dynamics of gravitational collapse in extended theories of gravity, since any such theory has to be able to provide sensible predictions about the physical processes mentioned above. This thesis discusses necessary steps undertaken in order to shed light on gravitational collapse in bimetric relativity. These studies clarify many aspects of the theory, and open new research paths. Understanding gravitational collapse in bimetric relativity is the overarching motivation behind the studies described in the thesis.

The main results concern different, but connected, theoretical areas of bimetric relativity. One is the formal theory, with results on: (i) how the symmetries of spacetime (e.g., spherical symmetry) can be used to constrain the solutions to the equations of motion of bimetric relativity, in particular those describing gravitational collapse and (ii) how the equations of motion can be integrated numerically. A symmetry of spacetime is a transformation of spacetime which leaves some of its geometrical properties unaltered. These results are obtained with methods closely following those used in general relativity, originating from geometry and the theory of ordinary and partial differential equations. The other area concerns the numerical integration of the equations of motion of the theory, the bimetric field equations, involving: (i) the study of static, spherically symmetric black hole solutions, which, under appropriate conditions, are assumed to be the end state of a spherically symmetric gravitational collapse and (ii) the study of spherically symmetric gravitational collapse of pressureless matter.

The thesis points out the great success of general relativity in describing Nature, but it also provides motivations for extending it, based on some of the most important contemporary open problems in physics, namely the dark matter and dark energy problems, and the cosmological constant problem. To be well-motivated, an extended theory of gravity as bimetric relativity should provide better explanations than general relativity to some of these problems, and at the same time provide predictions in equal or better agreement than general relativity with observations.

With this considerations in mind, the aim of this thesis is twofold. First, to concisely explain why and how bimetric relativity helps us in the understanding of the aforementioned open problems. Second, the thesis provides the background needed to understand the included research papers. The goal of these studies is to allow future research to obtain predictions that can be compared with observations. This constitutes the major part of the thesis. Besides, the thesis provides details about the research studies not contained in the research papers.

Motivation behind the thesis. In general relativity, spherically symmetric gravitational collapse is studied both in asymptotically flat spacetime and in homogeneous, isotropic spacetimes. In the first case, an exact solution to the Einstein field equations in the presence of pressureless dust, namely the Oppenheimer–Snyder one [OS39], can be found, making use of the Jebsen–Birkhoff theorem. This theorem states that, in general relativity, any region of a spherically symmetric vacuum spacetime must be equivalent to a region of the Schwarzschild spacetime describing a static and spherically symmetric black hole. The second case makes use of the linear perturbation theory established in [Lif46] (see also [Wei08, Ch. 5-6]). This consists in introducing linear scalar, vector and tensor perturbations and studying their linear dynamics. Vector perturbations do not play an important role in general relativity, since they decay quickly. Tensor perturbations describe gravitational waves and scalar perturbations describe gravitational collapse of matter. The study of scalar perturbations culminates in the explanation of structure formation in our Universe.

In bimetric relativity, the standard way to study gravitational collapse has been to introduce linear perturbations on homogeneous, isotropic background spacetimes. This is due to two reasons, namely the success of this method in general relativity and the fact that the equations of motion are harder to deal with in bimetric relativity compared to general relativity, hence linearization helps to simplify the problem. The study of scalar perturbations in bimetric relativity led to the result that either matter collapses too quickly (the so-called gradient instability) compared to general relativity, or that the theory is inconsistent [CCP12; KSS12; Ber+12; FT13; KA14; CCP14; Fel+14; SAK14; Koe+14; LF14; Ena+15; Kön15; KGK19].

This conclusion resulted in a skepticism towards bimetric relativity over the past few years, after a very fertile period of several years immediately subsequent to its birth in 2011. However, the conclusion that linear perturbation theory leads to instabilities, does not imply that bimetric relativity cannot explain gravitational collapse in a way consistent with observations. It only means that linear perturbation theory, whose validity is restricted to when higher order perturbations are much smaller than the linear ones, which have to be much smaller than the background as well, very quickly loses its validity and cannot be used to draw conclusions about the physical system under consideration.

This state of affairs is not new in the field of modified and extended theories of gravity. Already in the seventies, van Dam and Veltman [DV70], and Zakharov [Zak70] showed that predictions obtained in general relativity are not recovered in the limit when these theories are supposed to tend to general relativity. However, the conclusion was based on a linear analysis. In [Vai72], Vainshtein argued that the inclusion of nonlinearities provides a smooth limit to general relativity. The Vainshtein mechanism was then studied in more detail in bimetric relativity in [AMN15; EM15].

There are many modified and extended theories of gravity. The possible ways one can construct such theories are constrained and at the same time suggested by Lovelock's theorem [Lov71; Lov72]. Before stating it, we remind the reader that the dynamical field in general relativity is the metric, that is, a function that defines a measurement of distances between any two points in spacetime. Lovelock's theorem establishes that, starting from a Lagrangian density involving only the metric, the only possible second-order Euler–Lagrange field equations in four dimensions are the Einstein field equations with a nonzero cosmological constant. In [Cli+12] it is pointed out that, due to Lovelock's theorem, a modified or extended theory of gravity can be constructed by doing one or more of the following:

(i) Replace the metric, or add other fields interacting with it.
(ii) Consider field equations with derivatives higher than second.
(iii) Consider a space with dimension different than 4.
(iv) Consider field equations not arising from an action principle.
(v) Consider non-local field equations.

Reviews on theories constructed by using these pathways can be found in [Cli+12]. Here, we do not talk about higher-derivative, non-local and higher dimensional theories, since bimetric relativity does not belong to these categories. Rather, it is obtained by adding a new field interacting with the metric, namely another metric. Hence, in a bimetric spacetime there are two different distance measures but, for any given type of matter inhabiting the spacetime, only one of them is relevant. More details are provided in the thesis. For topics not treated in the thesis, we refer the reader to the review [SS16] on bimetric relativity and [Rha14] on the closely related massive gravity theory. Other types of theories obtainable by adding fields interacting with the metric tensor are, for example, scalar-tensor theories adding a scalar field, and tensor-vector-scalar theories, adding both a scalar and a vector field. The reason why bimetric relativity stands out among these theories is explained in the thesis.

Summary of the thesis. Since the results coming from linear analysis are not reliable in bimetric relativity, the studies presented in this thesis are fully nonlinear. That is, the bimetric field equations are never considered under limits that neglect some of their parts, or under analytical approximations. The nonlinear study is done in different ways. One way is to study symmetries of spacetime and how they affect the bimetric field equations and the relevant fields in the theory. Another is to look for exact solutions to the bimetric field equations. Yet another way is to numerically integrate the bimetric field equations; this requires a recasting of them in a suitable form.

Symmetries of spacetime have been widely studied in general relativity, with the purpose of simplifying the Einstein field equations in the aim of finding exact solutions to them. The first part of the thesis uses the same approach in bimetric relativity, as described in Paper I. The study of spacetime symmetries led to the discovery of a novel nonstationary spherically symmetric vacuum solution in bimetric relativity, whose relevance and importance are explained in Paper II. Symmetries of spacetime also play a major role in determining the properties of static and spherically symmetric black hole solutions, whose analytical and numerical properties are reported in Paper III.

The numerical integration of the bimetric field equations can be done in several ways. If the equations are reduced to ordinary differential equations through, for example, the assumptions of some spacetime symmetries, the numerical integration can be performed after rewriting the equations in so-called normal form. This form only involves first-order derivatives with respect to the unique independent variable considered, and is solved for them. This is done in Paper III when solving for static and spherically symmetric black hole solutions. When moving to spherically symmetric gravitational collapse, the bimetric field equations become partial differential equations. As such, they need to be treated in a specific way to ensure existence, uniqueness and, roughly speaking, stability of the solutions. The same is true in the case of general relativity, where the problem is faced by means of a rewriting of the Einstein field equations which differentiates between space and time, the so-called $3+1$ formalism. The same rewriting can be applied to the bimetric field equations, and the results of Paper IV, Paper V, Paper VI and Paper VII concern the development of this particular reformulation of the bimetric field equations. The first results on bimetric gravitational collapse obtained with the numerical integration of such a reformulation of the bimetric field equations are presented in Paper VIII, and constitute the first results in numerical bimetric relativity.

The importance of Paper I, Paper V, Paper VI and Paper VII is based on the fact that their results are fully general. Hence, they may be used not only to look for solutions describing gravitational collapse, but also to look for other type of solutions, for example rotating black holes and stars, binary systems of black holes and compact stars, and gravitational radiation. The importance of Paper II and Paper III resides in clarifying how the results of Paper I about spacetime symmetries affect particular solutions of the theory, and to clarify what types of spherically symmetric static and dynamical vacuum spacetimes are solutions to the bimetric field equations. Paper IV and Paper VIII are the first works which prepare and perform the numerical integration of the bimetric field equations, rewritten in the $3+1$ formalism.

The findings of Paper VIII do not show any sign of physical instabilities in the gravitational collapse of a spherically symmetric cloud of pressureless dust. The simulations described in Paper VIII are not numerically stable, hence the evolution can be followed for a rather short time. Yet, if an exponential growth was to happen, it should be evident early in the evolution, and this is not the case. Obtaining long-term numerically stable bimetric simulations is subject of ongoing work, initiated with the studies in Paper V and Paper VI.

Structure of the thesis. The thesis is structured in three main parts. Part I is independent from Part II. The reading of Part I before Part II is advised, though not

The reader familiar with	History of and motivations for bimetric relativity	may skip	Section 1.1
	Covariant formulation of bimetric relativity		Section 1.2
	Symmetries of spacetime		Section 2.2
	Theory of partial differential equations		Chapter 4
	$3+1$ decomposition in general relativity		Chapter 5
	$3+1$ decomposition in bimetric relativity		Section 6.1 Section 6.2

Table 1: Suggestions about how to read this thesis.

necessary. Suggestions about what parts of the thesis may be skipped by a reader familiar with some of the treated topics are provided in Table 1.

Part I contains the results obtained in the covariant formulation, that is, without rewriting the bimetric field equations in the $3+1$ formalism. Chapter 1 provides motivations in support of the study of bimetric relativity, and introduces some of its theoretical features. Chapter 2 reviews the concept of spacetime symmetry and describes the results of Paper I. Chapter 3 summarizes the results of Paper II and Paper III mainly from the viewpoint of Paper I, but also emphasizes their standalone importance.

Part II includes the results obtained in the $3+1$ formulation. Chapter 4 reviews aspects of the theory of partial differential equations, needed to understand the results of Paper V, Paper VI, Paper VII and Paper VIII. Chapter 5 reviews the $3+1$ decomposition in general relativity, needed to understand the results of Paper IV, Paper V, Paper VI, Paper VII and Paper VIII. Chapter 6 reviews the literature concerning the $3+1$ decomposition in bimetric relativity and summarizes the results of Paper IV, Paper V, Paper VI, Paper VII and Paper VIII.

Part III comprises the included papers.

Part I

COVARIANT FORMULATION

The connections between symmetry and beauty are a well-trodden area, with Hermann Weyl's Symmetry the classic reference. Ms. Cole sees invariance and symmetry as a way to get from truth to beauty, adding that "...deep truths can be defined as invariants—things that do not change no matter what; how invariants are defined by symmetries, which in turn define which properties of nature are conserved, no matter what. These are the selfsame symmetries that appeal to the senses in art and music and natural forms like snowflakes and galaxies. The fundamental truths are based on symmetry, and there's a deep kind of beauty in that."

Rockmore [Roc99] about Cole [Col99], found in Duggal and Sharma [DS99]

Chapter 1

Bimetric relativity

In this chapter, we introduce the Hassan–Rosen bimetric relativity (BR) and describe some of its theoretical features. We provide some motivations to consider it as a candidate theory describing the gravitational interaction in Section 1.1, and perform a more formal analysis of it in Section 1.2.

1.1 Motivations behind bimetric relativity

Our physical understanding of Nature relies on the two pillars of modern physics, quantum field theory (QFT) [PS95; Wei95; Wei96; Wei00] and general relativity (GR) [OR96; MTW73; Wal10]. These two theoretical frameworks describe how the four interactions of Nature govern the phenomena we experience and see. In particular, the strong, electromagnetic and weak interactions are understood in the realm of QFT, whereas the gravitational interaction is comprehended within the elegance of GR. Both QFT and GR provide astonishingly accurate and precise predictions [Wil14; Abb+16; Abb+17].

Some of the most compelling open questions in physics concern particle physics and cosmology [Han15], described by the Standard Model of particle physics (SM) [Nag13; Sch14], and the Standard Model of cosmology (ΛCDM) [Dod03; Spe15]. The acronym ΛCDM (for "Λ Cold Dark Matter") indicates that this model describes a universe whose main components are an unknown form of matter, "cold dark matter," and an unknown form of energy, denoted Λ. The latter is homogeneous in space and time and drives the accelerated expansion of the Universe. Both standard models have passed experimental tests with incredible success [CM15; Ade+16].

The SM is built within the framework of QFT, whereas the ΛCDM model makes use of GR. In spite of the many passed experimental tests, it is common to regard these theories as incomplete. In this section we will present some of the reasons for this. In particular, we focus on the motivations for exploring BR.

The history of BR. With "classical field theory," we refer to a theory describing the dynamics of continuous systems and fields by means of the Lagrangian formulation [GPS02, Chapter 13], when quantization is not taken into account. The dynamics is governed by the field equations, which are the Euler–Lagrange equations obtained by applying the Hamilton

	Massless	Massive
Spin-0	Klein–Gordon $\partial_\mu \partial^\mu \phi = 0$	Klein–Gordon with mass $\left(\partial_\mu \partial^\mu - m^2\right) \phi = 0$
Spin-1/2	Dirac $i\not{\partial}\, \psi = 0$	Dirac with mass $\left(i\not{\partial} - m\right) \psi = 0$
Spin-1	Maxwell $\partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = 0$	Proca $\partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) - m^2 A^\nu = 0$
Spin-3/2	Rarita–Schwinger $i\not{\partial}\, \psi^\mu = 0$	Rarita–Schwinger with mass $\left(i\not{\partial} - m\right) \psi^\mu = 0$
Spin-2	Einstein (nonlinear) $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\, R + \Lambda_{\mathrm{B}}\, g_{\mu\nu} = 0$	?

Table 1.1: Field equations for massless and massive spin-0, spin-1/2, spin-1 and spin-3/2 free fields in natural units where $c = \hbar = 1$, and for a massless spin-2 field in geometrized units where $c = G = 1$ [Wei00; Sre07; Sch14]. The Minkowski metric is $\mathrm{diag}(-1, 1, 1, 1)$.

principle to the action.

The Euler–Lagrange equations for a certain field are completely specified by its spin and mass, as summarized in Table 1.1. The linear field equations for massless and massive spin-0, spin-1/2, spin-1 and spin-3/2 fields have known formulations, as well as the nonlinear Einstein field equations (EFE) for a massless spin-2 field. On the other hand, the field equations for a massive spin-2 field are not included in Table 1.1, because they are not included in the two main theoretical frameworks of GR and QFT.

Table 1.1 shows that, once the spin is fixed, the field equations for massless and massive fields are very similar; the former can be obtained from the latter by taking the limit $m \to 0$. Therefore, it is natural to look for the field equations for a massive spin-2 field by trying to extend or modify GR.

An important difference between the Klein–Gordon, Dirac, Maxwell, Proca and Rarita–Schwinger equations on one side, and the EFE on the other side is that the latter are nonlinear. This rendered the search for a classical field theory of a massive spin-2 field to be long and hard. It started in 1939, when Fierz and Pauli formulated a linear theory for a massive spin-2 field [Fie39; FP39]. Only in recent times was a nonlinear completion finally found. In 2010, de Rham, Gabadadze and Tolley proposed a theory, the dRGT massive gravity [RG10; RGT11], which is free from the pathological Boulware–Deser (BD) ghost [BD72], as proven in [HR12c].[1] The dRGT massive gravity describes the interaction between

[1]A ghost is a nonphysical degree of freedom having a negative kinetic term in the action, thus rendering the Hamiltonian not bounded from below [HH02]. In a quantum theory, ghosts are associated with states having a negative norm [HH02; Bar+09], therefore their presence impede the quantization of the theory both mathematically, because a Hilbert space has a non-negative norm *by definition* [HS65, Section 16], and

a dynamical and a non-dynamical metric. The nonlinear Hamiltonian analysis in [HR12c] shows that the dRGT massive gravity propagates five degrees of freedom (DoF), which can be associated to the polarizations of a massive spin-2 field.

In 2012, soon after the dRGT massive gravity appeared in the literature, Hassan and Rosen generalized it [HRS12], by proving that dynamics could be given to the second spin-2 field, without reintroducing the BD ghost or destroying the consistency of the theory [HR12a; HR12b]. This led to the Hassan–Rosen bimetric relativity, where both the spin-2 fields possess a kinetic term in the action and interact through the same potential as in the dRGT massive gravity. The Hamiltonian analysis shows that BR propagates seven DoF. For linear perturbations around background solutions having proportional metrics, five DoF can be assigned to the polarizations of a massive spin-2 field, and two to the polarizations of a massless spin-2 field [Com+12b; HSS13b; HSS13a]. However, the distinction between massless and massive mode is blurred at the nonlinear level.

Reviews about the dRGT massive gravity and BR can be found in [Rha14] and [SS16], respectively.

It is noteworthy that a classical consistent nonlinear theory of interacting massless and massive spin-2 fields exists, since theories for massless fields with spin higher than 2 and interacting with any other fields, do not exist [Wei95, p. 538][Sch14, p. 138]. Similarly, interacting theories of only massless spin-2 fields do not exist [Bou+01]. There are, however, theories describing free massless fields with an arbitrary spin on Minkowski spacetime [Fro78; FF78] and on AdS spacetime [Vas80; Vas87; LV88].

The cosmological constant problems. In this paragraph, we discuss a fundamental problem in contemporary physics which was one of the first motivations prompting the search for modified or extended theories of gravity. We will closely follow the arguments in Martin [Mar12], which we refer the reader to for details. Another review on the cosmological constant problems can be found in Burgess [Bur15].

When deriving the EFE in GR, no physical requirement forbids the inclusion of the cosmological constant term $\Lambda_{\mathrm{B}}\, g_{\mu\nu}$ (see Table 1.1), where the subscript B stands for "bare." This term can be interpreted as the energy density of an empty universe. From a QFT perspective, this is the energy density of the ground state of the system, that is, the "vacuum" state $|0\rangle$. It can be shown that, for any arbitrary field, Lorentz *invariance* (not just covariance) forces such an energy density to be [Sak91; Vis18]

$$\langle 0|T_{\mu\nu}|0\rangle = -\rho_{\mathrm{vac}}\, g_{\mu\nu}. \tag{1.1}$$

If we substitute (1.1) in the EFE with a matter source,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\, R + \Lambda_{\mathrm{B}}\, g_{\mu\nu} = 8\pi \langle 0|T_{\mu\nu}|0\rangle\,, \tag{1.2}$$

we find

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\, R = -\left(\Lambda_{\mathrm{B}} + 8\pi\rho_{\mathrm{vac}}\right) g_{\mu\nu}. \tag{1.3}$$

physically, because they allow for negative probabilities.

We can define the effective cosmological constant to be

$$\Lambda_{\text{eff}} := \Lambda_{\text{B}} + 8\pi\rho_{\text{vac}}, \tag{1.4}$$

which is the quantity that can be probed with cosmological measurements. We can regard ρ_{vac} as a correction to the bare value Λ_{B}.

Since there are many interacting fields in Nature, corresponding to all the observed particles in the SM, and all of them are supposed to give a contribution to the vacuum energy, we may have many such corrections to Λ_{B}. These corrections can have a classical and a quantum origin. Accordingly, there is a classical and a quantum cosmological constant problem.

The classical cosmological constant problem originates from corrections to the vacuum energy density due to phase transitions in the early Universe. In particular, the contributions from electroweak and QCD phase transitions would be, respectively,

$$\rho_{\text{vac}}^{\text{EW}} \simeq -1.2 \times 10^{8}\,\text{GeV}^4, \qquad \rho_{\text{vac}}^{\text{QCD}} \simeq 10^{-2}\,\text{GeV}^4. \tag{1.5}$$

From (1.4), we can deduce that the vacuum energy must be corrected after every phase transition,

$$\rho_{\text{vac}}^{\text{classical}} = \rho_{\text{vac}}^{\text{B}} + \rho_{\text{vac}}^{\text{EW}} + \rho_{\text{vac}}^{\text{QCD}} + \ldots, \qquad \rho_{\text{vac}}^{\text{B}} := \frac{\Lambda_{\text{B}}}{8\pi}, \tag{1.6}$$

where the dots refer to other possible or hypothetical phase transitions. Observations imply,

$$\rho_{\text{vac}}^{\text{obs}} \simeq 10^{-47}\,\text{GeV}^4, \tag{1.7}$$

which is many order of magnitude smaller than the predicted corrections in (1.5). This would require Λ_{B}, or, equivalently, the associated $\rho_{\text{vac}}^{\text{B}}$, to be fine tuned with an enormous precision. This severe fine-tuning problem is the "classical cosmological constant problem."

The quantum cosmological problem arises when accounting for quantum corrections to the vacuum energy. These can be collectively written [KP11],

$$\rho_{\text{vac}}^{\text{quantum}} = \sum_i (\pm n_i)\frac{m_i^2}{64\pi^2}\ln\left(\frac{m_i^2}{\mu^2}\right) \simeq -2 \times 10^{8}\,\text{GeV}^4, \tag{1.8}$$

where the index i runs over all the elementary particles included in the SM, m_i is the mass of the particle i and n_i is the total number of DoF of the particle i. The $+$ sign holds for bosons and the $-$ sign for fermions. We can now write the expression for the vacuum energy as,

$$\begin{aligned}
\rho_{\text{vac}}^{\text{predicted}} &= \rho_{\text{vac}}^{\text{B}} + \rho_{\text{vac}}^{\text{quantum}} + \rho_{\text{vac}}^{\text{classical}} \\
&= \rho_{\text{vac}}^{\text{B}} + \rho_{\text{vac}}^{\text{quantum}} + \rho_{\text{vac}}^{\text{EW}} + \rho_{\text{vac}}^{\text{QCD}} + \ldots \\
&\simeq \rho_{\text{vac}}^{\text{B}} - 3.2 \times 10^{8}\,\text{GeV}^4,
\end{aligned} \tag{1.9}$$

Again, $\rho_{\text{vac}}^{\text{B}}$ needs to be fine tuned to an extraordinary precision, in order to cancel the contributions from the quantum corrections. This second fine-tuning problem is the "quantum cosmological constant problem."

These calculations rely on some important assumptions:

(i) The observational estimation of $\rho_{\text{vac}}^{\text{exp}}$ assumes isotropy and homogeneity.

(ii) Gravitation is correctly described by GR, and particle physics is correctly described by the SM.

(iii) The physical origin of the cosmological constant (i.e., of the accelerated expansion of the Universe) is vacuum energy.

Relaxing assumption (i) does not imply any modification to GR or the SM. Assumptions (ii) can be relaxed by assuming that particle physics is described by a model other than the SM; however, none of the extensions of the SM proposed so far is able to solve the two problems without spoiling some other experimental or theoretical constraints [Bur15]. Another approach would be to modify GR or to extend it; in this respect, one can look at the dRGT massive gravity and BR as candidate theories of gravity. Assumption (iii) is related to the fundamental question "does the vacuum energy gravitate?" Decoupling the vacuum energy from gravity is called "degravitation."

Within BR, solving the cosmological constant problems would require some form of degravitation. For spherically symmetric solutions, this is discussed in [PS17]. However, the cosmological constant problems are still present in BR.

The dark energy problem. The previous two cosmological constant problems can be collectively referred to as the "old cosmological constant problem." In addition, there is a "new cosmological constant problem," or "dark energy problem" [Bur15]. This problem arises when one *assumes* that the vacuum energy is degravitated. Hence, a new source for the accelerated expansion of the Universe has to be found, and it is called "dark energy."

BR solves the dark energy problem, and the resulting dark energy is also protected against quantum corrections, as we shall motivate. Suppose we have a field theory (classical or quantum), described by an action containing a parameter. Suppose further that setting the parameter to zero cancels some terms in the action and enhances its symmetry group to a broader one. In such a situation, the parameter is said to be "technically natural in the sense of 't Hooft" [t H80], that is, quantum corrections will not change its order of magnitude. Hence, if this parameter is small, quantum corrections will be small. Note that technical naturalness also holds for gauge symmetries, as it happens for the masses of the gauge bosons in electroweak theory [CT16, p. 3].

BR contains five energy scales in total, called $m_{(i)}$ with $i \in \{0, \ldots, 4\}$. Two of them, namely $m_{(0)}$ and $m_{(4)}$, act as cosmological constants for the two metrics g and f, respectively. Therefore, degravitating vacuum energy corresponds to degravitating $m_{(0)}$ and $m_{(4)}$ (or possibly only the one directly coupled to the metric coupled to us). In such a case, a dynamical accelerated expansion can still be obtained [Vol12; Str+12; Com+12a], thanks to the remaining terms in the interaction between the metrics [AKS13]. These involve three energy scales which are technically natural. As we will see in Section 1.2, if $m_{(1)}, m_{(2)}, m_{(3)} \neq 0$, the action is invariant under diffeomorphisms, as is GR. However, if $m_{(1)} = m_{(2)} = m_{(3)} = 0$, the action reduces to two decoupled EH actions. The latter are invariant under two independent groups of diffeomorphism, hence the symmetry group of the action is enhanced.

Dark matter. In the ΛCDM model, a theoretically well-motivated unknown matter component is needed in order to explain many astronomical and cosmological observations [Ber00; BHS05; Fen10]. To present day, only its gravitational interactions have been detected [Gib17], and therefore the possibility of a purely gravitational origin of it is still open. BR provides an interesting framework for investigating this possibility as shown, for example, by the works in [AM14; BH15; EM15; Bab+16].

In particular, in [Bab+16] it is stressed that a heavy massive spin-2 field has all the properties to be a dark matter particle candidate, since it would couple negligibly to ordinary matter.

Motivations to study bimetric relativity. The old cosmological constant problem was one of the first important motivations to look for modified and extended theory of gravity. However, once BR was formulated it was clear that it does not solve this problem. Hence, solving the old cosmological constant problem is not the primary motivation to study BR further.

A strong motivation to study BR nowadays is that it is the *minimal* theory of nonlinearly interacting massless and massive spin-2 fields. With "minimal" we mean that BR is obtained by the minimal requirements needed for a theory to be consistent, namely the absence of ghosts. The imposition of these requirements does not leave any freedom to specify arbitrary interaction potentials between the metrics. This makes the bimetric equations of motion (to be introduced in the next section) as fundamental as the Klein–Gordon, Dirac or EFE. The latter equations have described the dynamics of spin-1, spin-$1/2$ and spin-2 fields since their discovery without modifications, for they only depend on the spin and the mass of the particles whose dynamics they describe, and are obtained imposing minimal requirements only. This suggests that studying BR is worthwhile to understand both the structure of the interactions between (multiple) spin-2 fields and to study new features of gravity. Note also that dark matter and dark energy *have only been detected gravitationally,* hence it is natural to try to identify their origin by exploring spin-2 interactions and their phenomenology. In addition, the study of multiple metrics defined on the same differentiable manifold has been considered mostly for Riemannian metrics. Hence, studying BR has also the mathematical motivation to understand the structure of multiple Lorentzian metrics defined on the same differentiable manifold. Moreover, at present day, the sole physical theories that solve the dark energy problem are theories of modified or extended gravity.

Another motivation to study BR is that its cosmological solutions without an explicit cosmological constant term, simultaneously can explain the origin of dark energy as the interaction between the massless and massive spin-2 fields, and have a particle dark matter candidate in the heavy massive spin-2 field, as we shall describe at the end of the next section.

All of these reasons justify the study of BR both theoretically and phenomenologically. Indeed, since the bimetric cosmological solutions are compatible with the cosmological data, it is necessary to test other predictions, such as those arising from gravitational collapse. Due to the complexity of the theory, the study of realistic physical systems requires the development of both theoretical and numerical strategies, which is what this thesis is about.

1.2 Formulation of bimetric relativity

Hassan–Rosen action. The action of BR in four dimensions and SI units is [HR12a][2]

$$\mathcal{S} := \int \mathrm{d}^4x\sqrt{-g}\bigg[\left(\frac{c^4}{16\pi G_g}R_g + \mathcal{L}_g\right) - \frac{c^4}{G\ell^2}V(S) + \det(S)\left(\frac{c^4}{16\pi G_f}R_f + \mathcal{L}_f\right)\bigg], \quad (1.10)$$

where the symbol g refers to the determinant of the metric g when it appears in the expression $\sqrt{-g}$ (same for f), and $S := \left(g^{-1}f\right)^{1/2}$ is the principal square root of the matrix $g^{-1}f$.[3] We stress that the square root of a matrix is not unique. The choice of the square root branch that guarantees a spacetime interpretation of the theory, the existence of a common $3+1$ decomposition for both metrics (which is a necessary condition for the ghost-free proof), and is compatible with general covariance, is the principal square root [HK17]. We will come back to these topics in more detail in Chapter 6, since they are relevant for the $3+1$ decomposition in BR. In (1.10), G is Newton's gravitational constant, G_g and G_f are the modified gravitational constants of g and f, c is the speed of light and ℓ is the length scale of the bimetric interaction. In geometrized units where $c = G = 1$ and defining $G_g = \kappa_g\, G/(8\pi)$, $G_f = \kappa_f\, G/(8\pi)$, with κ_g, κ_f dimensionless constants, (1.10) reads,

$$\mathcal{S} = \int \mathrm{d}^4x\sqrt{-g}\bigg[\left(\frac{1}{2\kappa_g}R_g + \mathcal{L}_g\right) - \frac{1}{\ell^2}V(S) + \det(S)\left(\frac{1}{2\kappa_f}R_f + \mathcal{L}_f\right)\bigg] \quad (1.11a)$$

$$=: \mathcal{S}^g_{\mathrm{GR}} + \mathcal{S}_{\mathrm{int}} + \mathcal{S}^f_{\mathrm{GR}}. \quad (1.11b)$$

The kinetic terms for the two metrics are given by Einstein–Hilbert (EH) terms. Note that since $\det(S) = \sqrt{-f}/\sqrt{-g}$, the term multiplied by $\det(S)$ in (1.11) has integration measure $\mathrm{d}^4x\,\sqrt{-g}\,\det(S) = \mathrm{d}^4x\,\sqrt{-f}$, reproducing an EH term for f. The two matter lagrangians $\mathcal{L}_g$ and $\mathcal{L}_f$ are independent, and minimally coupled to only one metric, as prescribed in [RHR15]. The actions $\mathcal{S}^g_{\mathrm{GR}}$ and $\mathcal{S}^f_{\mathrm{GR}}$ include the EH terms and the matter terms for g and f, respectively.

The interaction action $\mathcal{S}_{\mathrm{int}}$ between the metrics is determined by the "bimetric potential"

$$V(S) := \sum_{n=0}^{4} \beta_{(n)}\, e_n(S), \quad (1.12)$$

where the $\beta_{(n)}$ are constant dimensionless real parameters and the $e_n(S)$ are the elementary symmetric polynomials of the eigenvalues λ_i of S,

$$e_0(S) := 1, \qquad e_1(S) := \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \quad (1.13a)$$

$$e_2(S) := \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4, \quad (1.13b)$$

$$e_3(S) := \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4, \qquad e_4(S) := \lambda_1\lambda_2\lambda_3\lambda_4. \quad (1.13c)$$

[2] We do not include explicitly the Gibbons–Hawking–York boundary terms in the action [Wal10, Appendix E.1], but it is understood that they are present.

[3] Note that $g^{-1}f$ is a linear operator, as $(g^{-1}f)^{1/2}$. In index notation, $S^\mu{}_\nu := \left(g^{-1}f\right)^{1/2\,\mu}{}_\nu$, $S^\mu{}_\rho S^\rho{}_\nu = g^{\mu\rho}f_{\rho\nu}$.

In dimension d, the elementary symmetric polynomials $e_m(S)$ with $m > d$ are zero. Also, they can be rewritten in terms of the traces of powers of S,

$$e_0(S) = 1, \qquad e_n(S) = S^{[\alpha_1}{}_{\alpha_1} \dots\dots S^{\alpha_n]}{}_{\alpha_n}, \tag{1.14}$$

that is,

$$e_0(S) = 1, \qquad e_1(S) = \mathrm{Tr}\,(S)\,, \qquad e_2(S) = \frac{1}{2}\Big[\mathrm{Tr}\,(S)^2 - \mathrm{Tr}\,\big(S^2\big)\Big]\,, \tag{1.15a}$$

$$e_3(S) = \frac{1}{6}\Big[\mathrm{Tr}\,(S)^3 - 3\,\mathrm{Tr}\,\big(S^2\big)\,\mathrm{Tr}\,(S) + 2\,\mathrm{Tr}\,\big(S^3\big)\Big]\,, \qquad e_4(S) = \det\,(S)\,. \tag{1.15b}$$

More properties of the elementary symmetric polynomials can be found in [Koc18, Sec. 1.3].

Variation of the action and bimetric field equations. We now vary the action (1.11) with respect to g and f, to get the bimetric field equations (BFE).

The variations of $\mathcal{S}^g_{\mathrm{GR}}$ and $\mathcal{S}^f_{\mathrm{GR}}$ are the same as in GR,[4,5]

$$\delta\,\mathcal{S}^g_{\mathrm{GR}} = \frac{1}{2}\int_\Omega \mathrm{d}^4x\sqrt{-g}\,\left(\frac{1}{\kappa_g}G_{g\mu\nu} - T_{g\mu\nu}\right)\delta g^{\mu\nu}, \tag{1.16a}$$

$$\delta\,\mathcal{S}^f_{\mathrm{GR}} = \frac{1}{2}\int_\Omega \mathrm{d}^4x\sqrt{-f}\,\left(\frac{1}{\kappa_f}G_{f\mu\nu} - T_{f\mu\nu}\right)\delta f^{\mu\nu}, \tag{1.16b}$$

with Einstein tensors for the metrics

$$G_{g\mu\nu} := R_{g\mu\nu} - \frac{1}{2}g_{\mu\nu}R_g, \qquad G_{f\mu\nu} := R_{f\mu\nu} - \frac{1}{2}f_{\mu\nu}R_f \tag{1.17}$$

and matter stress–energy tensors[6]

$$T_{g\mu\nu} := \frac{-2}{\sqrt{-g}}\frac{\partial}{\partial g^{\mu\nu}}\big[\sqrt{-g}\,\mathcal{L}_g\big], \qquad T_{f\mu\nu} := \frac{-2}{\sqrt{-f}}\frac{\partial}{\partial f^{\mu\nu}}\big[\sqrt{-f}\,\mathcal{L}_f\big]. \tag{1.18}$$

Following [HSS13a], the variation of the elementary symmetric polynomials in $\mathcal{S}_{\mathrm{int}}$, which makes use of Newton's identities, gives[7]

$$\delta e_n(S) = \frac{1}{2}\sum_{k=1}^{n}(-1)^{k-1}e_{n-k}(S)\,\mathrm{Tr}\left[S^{k-2}\delta\left(S^2\right)\right], \tag{1.19}$$

with $\delta\big(S^2\big) = \delta\big(g^{-1}f\big) = \big(\delta g^{-1}\big)f + g^{-1}\big(\delta f\big)$. The variations of $\mathcal{S}_{\mathrm{int}}$ with respect to g and f become

$$\delta_g\,\mathcal{S}_{\mathrm{int}} = -\frac{1}{2\ell^2}\int_\Omega \mathrm{d}^4x\sqrt{-g}\,V_g(S)_{\mu\nu}\,\delta g^{\mu\nu}, \tag{1.20a}$$

$$\delta_f\,\mathcal{S}_{\mathrm{int}} = -\frac{1}{2\ell^2}\int_\Omega \mathrm{d}^4x\sqrt{-f}\,V_f(S)_{\mu\nu}\,\delta f^{\mu\nu}, \tag{1.20b}$$

[4]The boundary terms of the variations are canceled by the Gibbons–Hawking–York boundary terms.

[5]These variations use Dirichlet boundary conditions [Bla17].

[6]For a definition of the stress–energy tensor not relying on the matter Lagrangian, but only on the properties of the matter itself, see [RZ13, Sec. 2.3].

[7]Concerning the Newton's identities, or Newton–Girard identities, see for example [OS12, Sec. 10.12].

with

$$V_g(S)_{\mu\nu} := \frac{-2}{\sqrt{-g}}\frac{\partial}{\partial g^{\mu\nu}}\big[\sqrt{-g}\,V(S)\big] \;= -g_{\mu\lambda}\sum_{n=0}^{3}(-1)^n\beta_{(n)}\,{Y_{(n)}}^{\lambda}{}_{\nu}(S), \tag{1.21a}$$

$$V_f(S)_{\mu\nu} := \frac{-2}{\sqrt{-f}}\frac{\partial}{\partial f^{\mu\nu}}\big[\sqrt{-f}\,V(S)\big] = -f_{\mu\lambda}\sum_{n=0}^{3}(-1)^n\beta_{(4-n)}\,{Y_{(n)}}^{\lambda}{}_{\nu}(S^{-1}). \tag{1.21b}$$

The operator ${Y_{(n)}}^{\mu}{}_{\nu}(X)$, with X being a linear operator, is defined by

$${Y_{(n)}}^{\mu}{}_{\nu}(X) := \sum_{k=0}^{n}(-1)^k e_k(X)(X^{n-k})^{\mu}{}_{\nu}. \tag{1.22}$$

We call $V_g(S)_{\mu\nu}$ and $V_f(S)_{\mu\nu}$ "bimetric stress–energy tensors."

Finally, the variations of the full action with respect to g and f are,

$$\delta_g\,\mathcal{S} = \frac{1}{2}\int_\Omega \mathrm{d}^4x\sqrt{-g}\,\left(\frac{1}{\kappa_g}G_{g\mu\nu} - \frac{1}{\ell^2}V_{g\mu\nu} - T_{g\mu\nu}\right)\delta g^{\mu\nu}, \tag{1.23a}$$

$$\delta_f\,\mathcal{S} = \frac{1}{2}\int_\Omega \mathrm{d}^4x\sqrt{-f}\,\left(\frac{1}{\kappa_f}G_{f\mu\nu} - \frac{1}{\ell^2}V_{f\mu\nu} - T_{f\mu\nu}\right)\delta f^{\mu\nu}. \tag{1.23b}$$

Hamilton's principle of stationary action imposes that these two variations must be zero for every value of the variations $\delta g^{\mu\nu}$ and $\delta f^{\mu\nu}$ with Dirichlet boundary conditions, and for every integration domain Ω. From the continuity of the integrand, we deduce the BFE,

$$G_{g\mu\nu} = \kappa_g\big(\ell^{-2}\,V_{g\mu\nu} + T_{g\mu\nu}\big), \tag{1.24a}$$

$$G_{f\mu\nu} = \kappa_f\big(\ell^{-2}\,V_{f\mu\nu} + T_{f\mu\nu}\big). \tag{1.24b}$$

In vacuum, if the metrics are proportional, $f_{\mu\nu} = c^2\,g_{\mu\nu}$, $0 \neq c \in \mathbb{R}$, the bimetric stress–energy tensors reduce to two cosmological constant terms and the BFE become two *decoupled* EFE. Therefore, in this case the solutions to the BFE are GR solutions.

The algebraic and the differential identities. Two important identities relate the two bimetric stress–energy tensors and the bimetric potential.

The first, proven in [BMV13; HSS14], is the algebraic identity

$$V_g(S)^{\mu}{}_{\nu} + \det(S)\,V_f(S)^{\mu}{}_{\nu} = V(S)\,\delta^{\mu}{}_{\nu}, \tag{1.25}$$

where the indices of each tensor are raised and lowered with the corresponding metric, for example $V_f(S)^{\mu}{}_{\nu} = f^{\mu\alpha}V_f(S)_{\alpha\nu}$.

The second is the differential identity,

$$\nabla_\mu V_g(S)^{\mu}{}_{\nu} + \det(S)\,\tilde{\nabla}_\mu V_f(S)^{\mu}{}_{\nu} = 0, \tag{1.26}$$

proved in [DK02] by using diffeomorphism invariance. The operator ∇ is the covariant derivative for g and $\tilde{\nabla}$ for f.

Diffeomorphism invariance. The action (1.11) is invariant under diffeomorphisms. Consider a diffeomorphism φ generated by a smooth vector field $\mathcal{V}$. The variations of the metrics are equal to their Lie derivative along $\mathcal{V}$ (see, e.g., [Bla17]),

$$\delta_{\mathcal{V}} g_{\mu\nu} := \varphi^* g_{\mu\nu} - g_{\mu\nu} = \mathcal{L}_{\mathcal{V}} g_{\mu\nu} = \nabla_\mu \mathcal{V}_\nu + \nabla_\nu \mathcal{V}_\mu = 2\nabla_{(\mu}\mathcal{V}_{\nu)}, \tag{1.27a}$$

$$\delta_{\mathcal{V}} f_{\mu\nu} := \varphi^* f_{\mu\nu} - f_{\mu\nu} = \mathcal{L}_{\mathcal{V}} f_{\mu\nu} = \widetilde{\nabla}_\mu \mathcal{V}_\nu + \widetilde{\nabla}_\nu \mathcal{V}_\mu = 2\widetilde{\nabla}_{(\mu}\mathcal{V}_{\nu)}, \tag{1.27b}$$

where φ^* is the pullback induced by the diffeomorphism φ.[8] We do not assume the vector field $\mathcal{V}$ to vanish at the boundary Ω. The computations regarding the EH actions and the matter actions are well-known in GR and can be found, for example, in [Bla17]. Hence, we concentrate on $\mathcal{S}_{\text{int}}$.

It is possible to show that [Bla17] (see also Appendix A)

$$\delta_{\mathcal{V}} \left(\int_\Omega \mathrm{d}^4x \sqrt{-g}\mathcal{L} \right) = \int_\Omega \mathrm{d}^4x \sqrt{-g}\, \nabla_\mu \left(\mathcal{V}^\mu \mathcal{L}\right), \tag{1.28}$$

for every covariant action with scalar Lagrangian density $\mathcal{L}$. In our case,

$$\delta_{\mathcal{V}} \mathcal{S}_{\text{int}} = -\frac{1}{\ell^2} \int_\Omega \mathrm{d}^4x \sqrt{-g}\, \nabla_\mu \left[\mathcal{V}^\mu V(S)\right], \tag{1.29}$$

that is, the variation of $\mathcal{S}_{\text{int}}$ under a generic diffeomorphism is the integral of a total derivative. Therefore, if we assume the vector field to vanish at the boundary Ω, the variation is zero and the action is strictly invariant. If we do not assume this, the variation is a total derivative added to the Lagrangian. Hence, it does not affect the Euler–Lagrange equations in the bulk of the integration domain. In other words, two Lagrangians differing only by a total derivative can be considered equivalent [GPS02, Sec. 1.4].[9] In this sense, the interaction action is invariant under arbitrary diffeomorphisms. The same is true for the GR terms, and therefore the entire bimetric action is invariant.

As a last comment, we note that the differential identity follows from diffeomorphism invariance, as stated in [DK02], if one assumes the algebraic identity (1.25). A direct

[8] We remind the reader that the Lie derivative along $\mathcal{V}^\mu$ of a generic tensor $T^{\mu_1\dots\mu_n}{}_{\nu_1\dots\nu_m}$ is equal to

$$\mathcal{L}_{\mathcal{V}} T^{\mu_1\dots\mu_n}{}_{\nu_1\dots\nu_m} = \mathcal{V}^\rho \partial_\rho T^{\mu_1\dots\mu_n}{}_{\nu_1\dots\nu_m} - \sum_{i=1}^n T^{\mu_1..\rho..\mu_n}{}_{\nu_1\dots\nu_m} \partial_\rho \mathcal{V}^{\mu_i} + \sum_{j=1}^m T^{\mu_1\dots\mu_n}{}_{\nu_1..\rho..\nu_m} \partial_{\nu_i} \mathcal{V}^\rho,$$

independently on any metric. If a metric is defined, the Lie derivative can be written in terms of the compatible covariant derivative ∇_μ as [Wal10]

$$\mathcal{L}_{\mathcal{V}} T^{\mu_1\dots\mu_n}{}_{\nu_1\dots\nu_m} = \mathcal{V}^\rho \nabla_\rho T^{\mu_1\dots\mu_n}{}_{\nu_1\dots\nu_m} - \sum_{i=1}^n T^{\mu_1..\rho..\mu_n}{}_{\nu_1\dots\nu_m} \nabla_\rho \mathcal{V}^{\mu_i} + \sum_{j=1}^m T^{\mu_1\dots\mu_n}{}_{\nu_1..\rho..\nu_m} \nabla_{\nu_i} \mathcal{V}^\rho.$$

[9] Actually, one should be careful in dealing with total derivatives too. The Gibbons–Hawking–York boundary term in GR arises from a total derivative and it is non-zero, for Dirichlet boundary conditions [Bla17]. This renders the variational problem for the EH action ill-posed (see, e.g., [DH09]), and that is why the Gibbons–Hawking–York term must be subtracted, even if does not affect the EFE. Regarding diffeomorphism invariance of the bimetric interaction action, Dirichlet boundary conditions make (1.29) to vanish identically at the boundary, because the bimetric potential does not contain any derivative of the metrics.

computation of the variation of the action, included in Appendix A, leads to

$$\begin{aligned}\delta_{\mathcal{V}}\mathcal{S}_{\text{int}} &= \frac{1}{2}\ell^{-2}\int_\Omega \mathrm{d}^4x\sqrt{-g}\,\left(\nabla_\mu {V_g}^{\mu\nu} + \det(S)\,\tilde{\nabla}_\mu {V_f}^{\mu\nu}\right)\mathcal{V}_\nu \\ &\quad - \ell^{-2}\int_\Omega \mathrm{d}^4x\sqrt{-g}\,\nabla_\mu\left[\mathcal{V}^\mu V(S)\right].\end{aligned} \tag{1.30}$$

Equating (1.29) and (1.30), we find

$$\nabla_\mu {V_g}^{\mu\nu} + \det(S)\,\tilde{\nabla}_\mu {V_f}^{\mu\nu} = 0. \tag{1.31}$$

Technical naturalness of $m_{(1)}, m_{(2)}, m_{(3)}$ in the sense of 't Hooft. Diffeomorphism invariance of the bimetric action is crucial for the technical naturalness of the three energy scales $m_{(1)}, m_{(2)}, m_{(3)}$ in the bimetric potential. We already stressed the importance of this in the context of the dark energy problem, so here we expand on it. In natural units $c = \hbar = 1$ and the action (1.10) reads,

$$\mathcal{S} = \int \mathrm{d}^4x\sqrt{-g}\left[\left(\frac{M_g^2}{2}R_g + \mathcal{L}_g\right) - m^4 V(S) + \det(S)\left(\frac{M_f^2}{2}R_f + \mathcal{L}_f\right)\right], \tag{1.32}$$

where M_g, M_f and m have dimension of energy. Let us define the energy scales $m_{(i)}^4 := m^4\beta_{(i)}$ and write the bimetric potential as,

$$\begin{aligned}m^4 V(S) &= m_{(0)}^4 e_0(S) + m_{(1)}^4 e_1(S) + m_{(2)}^4 e_2(S) + m_{(3)}^3 e_1(S) + m_{(4)}^4 e_4(S) \\ &= -\Lambda_g + m_{(1)}^4 e_1(S) + m_{(2)}^4 e_2(S) + m_{(3)}^3 e_1(S) - \Lambda_f \det(S),\end{aligned} \tag{1.33}$$

where $\Lambda_g := -m_{(0)}^4$ and $\Lambda_f := -m_{(4)}^4$ are the cosmological constants for g and f. In the limit $m_{(1)}, m_{(2)}, m_{(3)} \to 0$,

$$\begin{aligned}\lim_{\substack{m_{(1)}\to 0\\ m_{(2)}\to 0\\ m_{(3)}\to 0}} \mathcal{S} &= \int \mathrm{d}^4x\sqrt{-g}\left[\left(\frac{M_g^2}{2}R_g + \mathcal{L}_g + \Lambda_g\right) + \det(S)\left(\frac{M_f^2}{2}R_f + \mathcal{L}_f + \Lambda_f\right)\right] \\ &= \int \mathrm{d}^4x\sqrt{-g}\left(\frac{M_g^2}{2}R_g + \mathcal{L}_g + \Lambda_g\right) + \int \mathrm{d}^4x\sqrt{-f}\left(\frac{M_f^2}{2}R_f + \mathcal{L}_f + \Lambda_f\right),\end{aligned} \tag{1.34}$$

we obtain the sum of two independent EH actions with non-zero cosmological constants, separately invariant under independent diffeomorphism groups.

The action (1.32) including $\mathcal{S}_{\text{int}}$ is not invariant under separate diffeomorphisms. We call the vector fields generating the diffeomorphisms $\mathcal{U}$ and $\mathcal{V}$. A direct computation, analogous to that reported in Appendix A, gives

$$\delta\mathcal{S}_{\text{int}} = -m^4\left(\int_\Omega \mathrm{d}^4x\sqrt{-g}\,{V_g}^{\mu\nu}\nabla_{(\mu}\mathcal{U}_{\nu)} + \int_\Omega \mathrm{d}^4x\sqrt{-f}\,{V_f}^{\mu\nu}\tilde{\nabla}_{(\mu}\mathcal{V}_{\nu)}\right), \tag{1.35}$$

for arbitrary vector fields $\mathcal{U}$ and $\mathcal{V}$. From (1.21),

$$\begin{aligned}-m^4 V_g(S)_{\mu\nu} &= \Lambda_g\, g_{\mu\nu} - m^4_{(1)}\; g_{\mu\lambda} {Y_{(1)}}^{\lambda}{}_{\nu}(S) \\ &\quad + m^4_{(2)}\, g_{\mu\lambda} {Y_{(2)}}^{\lambda}{}_{\nu}(S) - m^4_{(3)}\, g_{\mu\lambda} {Y_{(3)}}^{\lambda}{}_{\nu}(S) = \Lambda_g\, g_{\mu\nu} + {V_g}'(S)_{\mu\nu}, \qquad (1.36a)\\ -m^4 V_f(S)_{\mu\nu} &= m^4_{(4)}\, f_{\mu\nu} - m^4_{(3)}\; f_{\mu\lambda} {Y_{(1)}}^{\lambda}{}_{\nu}(S^{-1}) \\ &\quad + m^4_{(2)}\, f_{\mu\lambda} {Y_{(2)}}^{\lambda}{}_{\nu}(S^{-1}) - m^4_{(1)}\, f_{\mu\lambda} {Y_{(3)}}^{\lambda}{}_{\nu}(S^{-1}) = \Lambda_f\, f_{\mu\nu} + {V_f}'(S)_{\mu\nu}, \qquad (1.36b)\end{aligned}$$

hence,

$$\begin{aligned}\delta\mathcal{S}_{\text{int}} &= \int_\Omega \mathrm{d}^4x\sqrt{-g}\,\Lambda_g\, g^{\mu\nu}\nabla_\mu \mathcal{U}_\nu + \int_\Omega \mathrm{d}^4x\sqrt{-f}\,\Lambda_f\, f^{\mu\nu}\tilde{\nabla}_\mu \mathcal{V}_\nu \\ &\quad \int_\Omega \mathrm{d}^4x\sqrt{-g}\,{V_g}'(S)^{\mu\nu}\nabla_\mu \mathcal{U}_\nu + \int_\Omega \mathrm{d}^4x\sqrt{-f}\,{V_f}'(S)^{\mu\nu}\tilde{\nabla}_\mu \mathcal{V}_\nu \qquad (1.37a)\\ &= \int_\Omega \mathrm{d}^4x\sqrt{-g}\,\nabla_\mu\Big(\Lambda_g \mathcal{U}^\mu + \Lambda_f \mathcal{V}^\mu\Big) \\ &\quad \int_\Omega \mathrm{d}^4x\sqrt{-g}\,{V_g}'(S)^{\mu\nu}\nabla_\mu \mathcal{U}_\nu + \int_\Omega \mathrm{d}^4x\sqrt{-f}\,{V_f}'(S)^{\mu\nu}\tilde{\nabla}_\mu \mathcal{V}_\nu. \qquad (1.37b)\end{aligned}$$

The expression in (1.37b) is neither zero nor a total derivative, unless $m_{(1)} = m_{(2)} = m_{(3)} = 0$. Each of the interaction terms proportional to $m_{(1)}, m_{(2)}, m_{(3)}$ reduces the gauge group of the action, making the energy scales $m_{(1)}, m_{(2)}, m_{(3)}$ technically natural in the sense of 't Hooft. Quantum corrections *to all of them* are proportional *to all of them*. Hence, if they are small, quantum corrections are also small.

The same does not apply to Λ_g and Λ_f, which is why the old cosmological constant problem is not solved in BR.

The bimetric conservation laws. The divergence of the field equations (1.24) with respect to g and f respectively is

$$\begin{aligned}\nabla_\mu {G_g}^{\mu\nu} &= \kappa_g \ell^{-2}\nabla_\mu {V_g}^{\mu\nu} + \kappa_g \nabla_\mu {T_g}^{\mu\nu}, \qquad (1.38a)\\ \tilde{\nabla}_\mu {G_f}^{\mu\nu} &= \kappa_f \ell^{-2}\tilde{\nabla}_\mu {V_f}^{\mu\nu} + \kappa_f \tilde{\nabla}_\mu {T_f}^{\mu\nu}. \qquad (1.38b)\end{aligned}$$

Using the contracted Bianchi identities and the diffeomorphism invariance of the matter actions, we are left with,

$$\nabla_\mu {V_g}^{\mu\nu} = 0, \qquad \tilde{\nabla}_\mu {V_f}^{\mu\nu} = 0, \qquad (1.39)$$

which are called "Bianchi constraints" or "bimetric conservation laws." They are useful tools for finding solutions to the BFE (1.24), by replacing some of them. We used this approach in the included papers. Note that the bimetric conservation laws cannot be used to make (1.35) vanish, since they hold on-shell, that is, they are obtained using the BFE, see (1.38). The variation (1.35), instead, has to be evaluated off-shell, since it is the variation of interaction action under a *gauge* symmetry and should be zero without the imposition of any other condition.

Topological constraint. BR is invariant under diffeomorphisms applied to both sectors simultaneously. This means that the atlas (i.e., the set of charts covering the entire manifold) used to cover the spacetime, once chosen, has to be the same for both the metric sectors. If we introduce an atlas in a set, in addition to introducing a differential structure, we also uniquely determine the topology of the set [Lan95, p. 20]. In BR, this means that the two metrics must be compatible with the topology induced by the chosen atlas, when considering global solutions. This restricts the consistent metric combinations that one can have in particular solutions. This is one of the results of Paper I.

Cosmological solutions. A short review of the bimetric cosmological solutions is now given, to show what are commonly used values for the β parameters. This will be used in the last chapter of the thesis to compare the β parameters used in the numerical simulations of a gravitationally collapsing dust cloud with values obtained by testing cosmological solutions against the observations. We follow the treatment in [LMS18]. Under the assumption of metrics sharing the same homogeneity and isotropy (a concept discussed in Chapter 2), the BFE reduce to two Friedmann equations [Akr+15],

$$3H^2 = \kappa_g\Big[\rho + m^4\Big(\beta_{(0)} + 3\beta_{(1)}y + 3\beta_{(2)}y^2 + \beta_{(3)}y^3\Big)\Big], \tag{1.40}$$

$$3\kappa H^2 = \kappa_g m^4\Big(\frac{\beta_{(1)}}{y} + 3\beta_{(2)} + 3\beta_{(3)}y + \beta_{(4)}y^2\Big), \tag{1.41}$$

where $H = \dot{a}/a$ is the Hubble function for the metric $g_{\mu\nu}$, $y := Y/a$ with Y the scale factor for $f_{\mu\nu}$, ρ is the matter energy density for a perfect fluid and $\kappa := \kappa_g/\kappa_f$. We assume $m = \mathcal{O}(1)$, since we can always rescale the β parameters to account for different values for m. Usually, $\beta_{(0)}$ and $\beta_{(4)}$ are set to 0 since it is desirable to obtain self-acceleration without explicit cosmological constant terms in the action.

The Fierz–Pauli (FP) mass is[10]

$$m_{\mathrm{FP}} = \big(1 + \kappa^{-2}c^{-2}\big)\big(c\beta_{(1)} + 2c^2\beta_{(2)} + c^3\beta_{(3)}\big)\kappa_g m^4, \tag{1.42}$$

with $c \in \mathbb{R}$ given by [LMS18]

$$\kappa^2\Big(c\beta_{(0)} + 3c^2\beta_{(1)} + 3c^3\beta_{(2)} + c^4\beta_{(3)}\Big) = \beta_{(1)} + 3c\beta_{(2)} + 3c^2\beta_{(3)} + c^3\beta_{(4)}. \tag{1.43}$$

There are two types of cosmological solutions which are compatible with the cosmological data. The first type considers a large value for the FP mass. In order to discuss it, we now note that in BR there is a redundancy in the parameter space since the action is invariant under the following rescaling (see, e.g., [Akr+15] and references therein),

$$\big(f_{\mu\nu}, \kappa_f, \beta_{(n)}\big) \to \big(\omega f_{\mu\nu}, \omega\kappa_f, \omega^{-n/2}\beta_{(n)}\big), \quad \omega \in \mathbb{R}. \tag{1.44}$$

The BR model obtained by doing this rescaling is called here the "dual model." For a BR model with generic β parameters, the GR limit is given by taking the limit $\kappa_f \to \infty$ with κ_g

[10] The Fierz–Pauli mass is defined only in those regions of spacetime where the metrics are close to being proportional, that is, where there are small deviations from GR.

and the β parameters fixed [SS16, Sec. 5.4.2]. Suppose we start with a value for κ_f and we send it to infinity by rescaling it with a constant $\alpha \to \infty$. Due to (1.44), a model with such a large $\alpha\kappa_f$ is dual to a model with a fixed κ_f if we apply the rescaling

$$\big(f_{\mu\nu}, \alpha\kappa_f, \beta_{(n)}\big) \to \big(f_{\mu\nu}/\alpha, \kappa_f, \alpha^{n/2}\beta_{(n)}\big). \tag{1.45}$$

Therefore, the GR limit can be equivalently taken by keeping κ_f fixed and sending $f_{\mu\nu}$ to zero and the β parameters to infinity according to (1.45). For instance, we can set $\kappa_f = \kappa_g$, that is, $\kappa = 1$.

Setting $\kappa = 1$ in the GR limit, the FP mass in (1.42) becomes very large. In addition, the metrics will tend to be proportional with proportionality constant given by c, implying that a cosmological constant term appears in the Friedmann equations. In a model where only $\beta_{(1)}$ and $\beta_{(2)}$ are nonzero, then the cosmological constant is [LMS18]

$$\Lambda = \frac{2\kappa_g m^4 \beta_{(1)}}{|\beta_{(2)}|}. \tag{1.46}$$

Since it does not depend on $\beta_{(0)}$ and $\beta_{(4)}$, it is technically natural. Deviations from GR can be made arbitrarily small by increasing the β parameters and decreasing $f_{\mu\nu}$.

The bimetric cosmological solutions also provide a good fit to cosmological observational data [Dha+17] for small values of the FP mass, $m_{\text{FP}} \simeq H_0$ with H_0 Hubble constant. Setting $\kappa = 1$, the β parameters are of the order of $H_0^2 \simeq 10^{-52}\text{m}^{-2}$. This model provides testable deviations from ΛCDM and has a dynamical dark energy driving the accelerated expansion of the Universe. However, in [Dha+17] it is shown that there may be exceptions where the values of some of the β parameters can be much larger than H_0^2 but still be consistent with the cosmological data.

Chapter 2

Symmetries of spacetime

This chapter is focused on spacetime symmetries in BR. It constitutes both a complement and a review of Paper I.

We begin by motivating the study of spacetime symmetries in Section 2.1. Afterwards, we present the required mathematical concepts in Section 2.2, and summarize the main results of Paper I in Section 2.3.

2.1 Motivations

The study of spacetime symmetries has a great importance in GR (see [Zaf97] and references therein), the primary purpose being the simplification of the EFE and the finding of exact solutions.

BR, like GR, is a geometric theory of gravitation. Hence, in studying it, one can employ the same methods as in GR. Since the BFE (1.24) are more complicated than the EFE, it is wise to try to simplify them in order to find exact solutions. One way to do this is to study spacetime symmetries in BR. The most straightforward result of this study is the understanding of what types of ansatzes one can make for the metrics.

In BR, two metrics are interacting nonlinearly. Therefore, even without referring to particular solutions, it is of interest to study what are the relations between the spacetime symmetries of two Lorentzian metrics defined on the same differentiable manifold.

In principle, the metrics may be thought as part of the gravitational sector or not. However, there are reasons to believe that both of them are part of it. First, since the BFE are a system of *coupled* partial differential equations, the system cannot be easily separated into a gravitational and a nongravitational part—more details on the properties of the BFE as a system of partial differential equations are provided in Chapter 4 and Chapter 6. Second, the degrees of freedom of the massless and massive mode are distributed between the two metrics, supporting a geometric interpretation of the second metric. The study of the relations between the two metric sectors—for example, between their symmetries of spacetime—may clarify how the geometry of the second metric affects the gravitational interaction.

2.2 Geometrical concepts

The bimetric spacetime. When referring to "bimetric spacetime," we consider a triple $(\mathscr{M}, g, f)$, with $\mathscr{M}$ a differentiable manifold satisfying certain mathematical properties, and g and f being Lorentzian metrics defined on it such that $S := \left(g^{-1}f\right)^{1/2}$ is a positive definite (1,1) tensor.[1] Under these assumptions, g and f admit a common 3+1 decomposition [HK18], which guarantees the viability of this definition of spacetime, see Section 6.1. Therefore, when we discuss a "symmetry of spacetime" in BR, we may refer to a symmetry of g (or its related geometrical quantities) or to a symmetry of f (or its related geometrical quantities), or both.

Lie groups and Lie algebras. We refer the reader to [CDD82; Hal03] for a rigorous treatment of what follows in this paragraph.

Spacetime symmetries are usually described in terms of differentiable groups of transformations from the manifold to itself. For this reason, we now make a quick review about such differentiable groups.

A finite dimensional Lie group G is a finite dimensional group having the structure of a differentiable manifold. A finite dimensional Lie algebra $\mathfrak{g}$ is a finite dimensional vector space with a map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which is bilinear, antisymmetric and satisfies the Jacobi identity,

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0, \qquad \forall\, X,Y,Z \in \mathfrak{g}. \tag{2.1}$$

The elements of a basis of the Lie algebra are the "generators" of the Lie algebra. The tangent space T_eG at the identity element e of the Lie group is, by definition, the Lie algebra $\mathrm{Lie}(G) \equiv \mathfrak{g}$ of G.

A homomorphism $\Phi : (G, \star) \to (H, \times)$ between two Lie groups, with $\star$ and $\times$ being the group laws, is a continuous group homomorphism, that is, it is a continuous function such that,

$$\Phi(U \star V) = \Phi(U) \times \Phi(V), \tag{2.2}$$

for every $U, V \in G$. The property (2.2) simply means that the function Φ respects the operations of the two groups. A homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ between two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ is a continuous function satisfying

$$\phi(\alpha u, \beta v) = \alpha\phi(u) + \beta\phi(v), \tag{2.3a}$$

$$\phi([u,v]) = [\phi(u), \phi(v)], \tag{2.3b}$$

for every $u, v \in \mathfrak{g}$. The property (2.3a) tells us that ϕ is a bilinear map, whereas property (2.3b) tells us that ϕ respects the algebra product, that is, the Lie bracket, in our case. If the homomorphism is also bijective and its inverse is continuous, then it is an isomorphism

[1] For a more precise definition of spacetime in GR, see, e.g., [Wal10]. In our case, $\mathscr{M}$ is defined in the same way as in GR and we only need to add the f-sector and the principal square root S to the definition.

between the Lie algebras. Two isomorphic Lie algebras can be considered to be equivalent.

There is a theorem which states the following: Let G, H be two Lie groups with Lie algebras $\mathfrak{g} = \mathrm{Lie}(G), \mathfrak{h} = \mathrm{Lie}(H)$. Let $\phi : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. If G is simply connected, there exist a unique Lie group homomorphism $\Phi : G \to H$, with $\phi = \mathrm{d}\Phi$. An immediate corollary is that, if both G and H are simply connected and $\mathfrak{g}$ is isomorphic to $\mathfrak{h}$, then G and H are also isomorphic, which basically means that they are the same. If G is connected only, the theorem does not hold. In this case, another result can be stated. Suppose G is a connected Lie group and let $\widetilde{G}$ be its unique universal covering.[2] Then, if H is a Lie group with Lie algebra $\mathfrak{h}$ and $\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism, then there exist a unique homomorphism $\Phi : \widetilde{G} \to H$.

After this recapitulation about Lie groups and Lie algebras, we can come back to spacetime symmetries. Spacetime symmetries usually form a Lie group acting on the manifold $\mathscr{M}$. The Killing vector fields (KVFs), then, generate a Lie algebra which is isomorphic to the Lie algebra of the isometry group. The algebra operation is the Lie bracket between vector fields; for any two vector fields X and Y, the Lie bracket is a third vector field given by,

$$[X, Y](f) = X(Y(f)) - Y(X(f)), \qquad \forall f \in C^{\infty}(\mathscr{M}). \tag{2.4}$$

Note that $X(f) = X^{\mu}\partial_{\mu} f$ in a coordinate chart.

Spacetime symmetries. Following [Hal04], we define a symmetry of spacetime as a (local) diffeomorphism of $\mathscr{M}$ which preserves some of its geometric properties. Usually, instead of specifying the (local) diffeomorphism, one specifies a smooth vector field, called a "symmetry vector field," whose integral curves define the action of the diffeomorphism group on the spacetime. If the symmetry vector field is defined globally on $\mathscr{M}$, then the symmetry is said to be global.

Quite often, the set of spacetime symmetries preserving some geometrical properties forms a finite-dimensional Lie group, and the associated symmetry vector fields form a finite-dimensional Lie algebra. However, this is not always the case; for example, the set of homothetic vector fields forms a group, but not a Lie algebra [DS99, p. 55].

For a generic spacetime, some of the most important spacetime symmetries are reviewed in Table 2.1, where the terminology is introduced. The relations between them and other spacetime symmetries are summarized in Figure 2.2.

As a first example, for the sake of familiarity and simplicity, we consider some symmetries of Minkowski spacetime. In particular, those transformations keeping the metric η invariant. They belong to the Poincaré group and are the isometries of η: four translations, three rotations and three boosts. We could also consider the transformations that preserve η up to a constant, or up to a non-constant conformal factor.

In addition to pure geometric collineations, in physics is also important to consider "matter collineations" (MC), that is, the symmetries of the stress–energy tensor $T_{\mu\nu}$. Then, as an appendage to Figure 2.2, we draw the diagram in Figure 2.1.

Spacetime symmetries have been studied for a long time, both in mathematics and

[2] Up to canonical isomorphism.

Symmetry vector field ξ	Property	Definition	Group
Projective vector field	Preserves geodesics	$\exists\, \omega_\mu : \quad \nabla_\mu \nabla_\nu \xi_\rho = R_{\mu\nu\rho}{}^{\sigma} \xi_\sigma + 2 g_{\mu(\nu} \omega_{\rho)}$	$P(\mathscr{M})$
Conformal vector field	Preserves the metric up to a conformal factor	$\mathscr{L}_\xi\, g(x) = \phi(x)\, g(x)$	$C(\mathscr{M})$
Affine vector field	Preserves geodesics together with their affine parameters	$\mathscr{L}_\xi \nabla = 0$ or $\nabla_\mu \nabla_\nu \xi_\rho = R_{\mu\nu\rho}{}^{\sigma} \xi_\sigma$	$A(\mathscr{M}) \subset P(\mathscr{M})$
Homothetic vector field	Preserves the metric up to a constant conformal factor	$\mathscr{L}_\xi\, g(x) = c\, g(x)$	$H(\mathscr{M}) \subset A(\mathscr{M})$ $H(\mathscr{M}) \subset C(\mathscr{M})$
Killing vector field (metric collineation)	Preserves the metric	$\mathscr{L}_\xi\, g(x) = 0$	$K(\mathscr{M}) \subset H(\mathscr{M})$ $K(\mathscr{M}) \subset C(\mathscr{M})$
Collineation of tensor X	Preserves the tensor X	$\mathscr{L}_\xi\, X(x) = 0$	

Comments.

(i) All groups but $H(\mathscr{M})$ are Lie groups [DS99].

(ii) Some important collineations are those of the Riemann curvature tensor, called "curvature collineation" (CC), and those of the Ricci tensor, called "Ricci collineation"(RC). Every CC is a RC, but the converse is not true [Hal04; DS99].

Table 2.1: Some of the most important symmetries of spacetime [DS99; Hal04]. $\mathscr{L}_\xi$ is the Lie derivative along ξ.

Figure 2.1: The relation between KVFs and the CC, RC and MC. The last implication uses the EFE, hence it is valid only in GR.

physics. Essentially, the definitions and the results contained in Table 2.1 and Figure 2.2 are geometric, and can be applied to every theory founded on (pseudo-)Riemannian geometry.

In BR, all of these results hold for each of the metrics separately, but the last implication in Figure 2.1 does not, that is, a KVF of one metric is not necessarily a MC. We will treat spacetime symmetries in BR in the next section, but we can already motivate the previous statement by just looking at the BFE in (1.24). For instance, taking their Lie derivative of (1.24a) along a KVF ξ of g does not imply $\mathscr{L}_\xi T_g{}^{\mu\nu} = 0$, but rather implies $\mathscr{L}_\xi T_g{}^{\mu\nu} = -\ell^{-2}\, \mathscr{L}_\xi V_g{}^{\mu\nu}$. We will clarify this issue in the next section.

At this point, we have all the ingredients needed to treat spacetimes symmetries in BR.

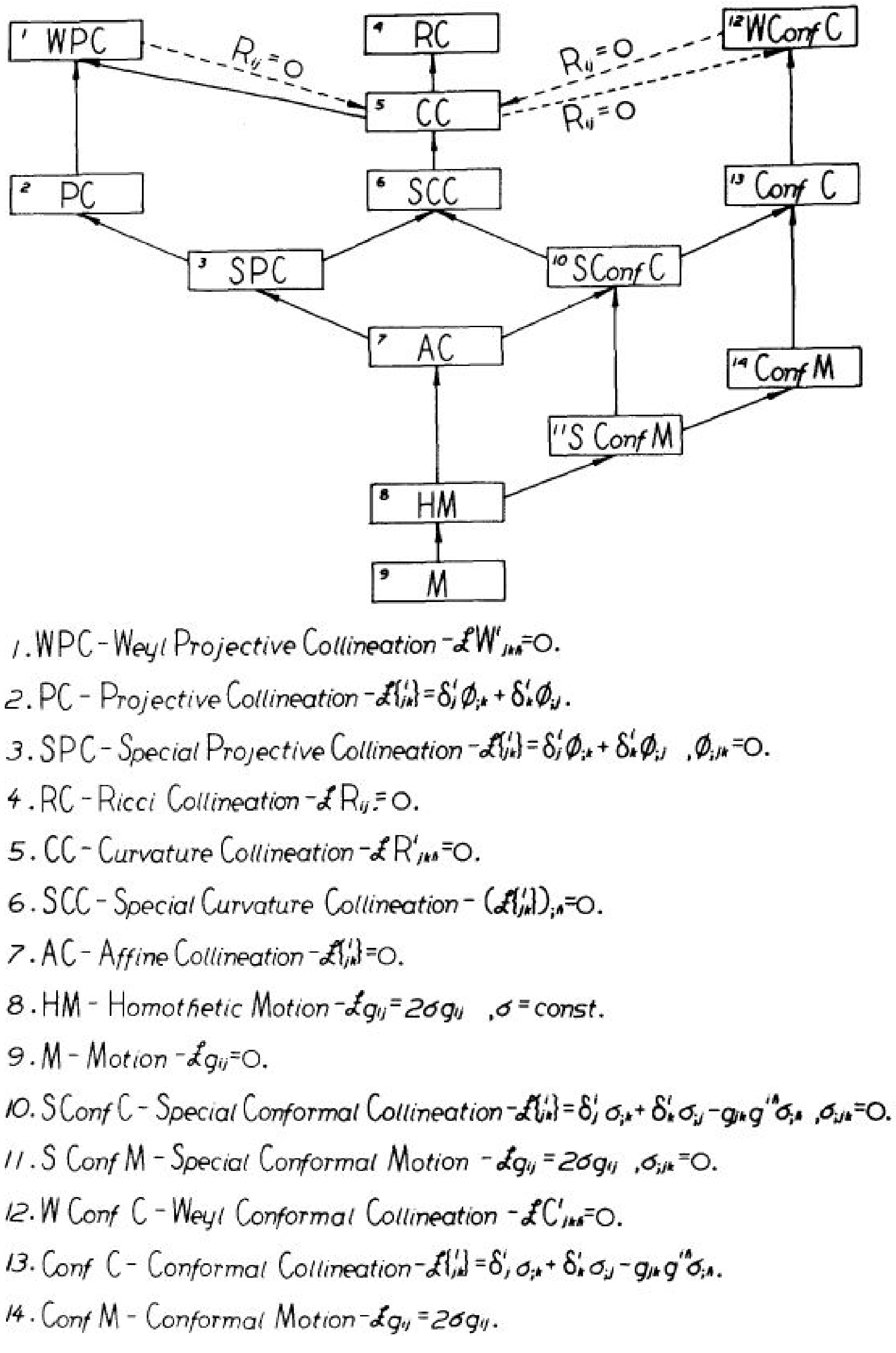


Figure 2.2: The diagram shows the relations between various symmetries of spacetime. In modern language, "motion" would be similar to "metric collineation." Reproduced from Gerald H. Katzin, Jack Levine, and William R. Davis. "Curvature Collineations: A Fundamental Symmetry Property of the Space-Times of General Relativity Defined by the Vanishing Lie Derivative of the Riemann Curvature Tensor". *Journal of Mathematical Physics* 10.4 (1969), pp. 617–629. DOI: `10.1063/1.1664886`. eprint: `http://dx.doi.org/10.1063/1.1664886`. URL: `http://dx.doi.org/10.1063/1.1664886`, with the permission of AIP Publishing.

2.3 Symmetries of spacetime in bimetric relativity

In Paper I we study spacetime symmetries in BR, with major focus on the isometries of the two metrics and the relation between them, the collineations of the stress–energy tensors and those of tensors not defined in GR, namely the square root (1,1) tensor field $S = (g^{-1}f)^{1/2}$ and the bimetric stress–energy tensors V_g and V_f defined in (1.21).

As previously noted, the results summarized in Figure 2.2 hold also in BR, because they do not rely on the EFE. Therefore, the main motivations for the analysis of Paper I are:

(i) Are there any relations between the isometry groups of g and f?

(ii) Are there any relations between the KVFs of g and f?

(iii) What is the equivalent of Figure 2.1 in BR? More explicitly, what are the relations between the isometries of each metric and the collineations of the associated bimetric and matter stress–energy tensors?

We saw in the previous section that an isometry group is a Lie group, with an associated Lie algebra. Loosely speaking, the KVFs generate such Lie algebra. Hence, question (i) and question (ii) are closely related, although not equivalent, as we will explicitly see.

The technicalities regarding the answers to these questions can be found in Paper I. Here, we present the answers by means of example solutions to the BFE (1.24), described in Paper I.

The relation between the isometry groups of g and f. In Paper I, it is shown that an Anti–de Sitter (AdS) g and a Minkowski f satisfy the BFE. This answers question (i). In general, there are no relations between the isometry groups of the two metrics. However, let us study this solution in detail.

The two metrics have the following form in outgoing Eddington–Finkelstein coordinates (v, r, θ, ϕ),

$$\mathrm{d}{s_g}^2 = -\left(1 - \frac{\Lambda_g(R_0;\{\beta_{(i)}\})\, r^2}{3}\right)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2\mathrm{d}\Omega^2,$$
$$\mathrm{d}{s_f}^2 = R_0^2\left[-\left(1 - \frac{\Lambda_f(R_0;\{\beta_{(i)}\})\, r^2}{3}\right)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2\mathrm{d}\Omega^2\right], \tag{2.5a}$$

where $\mathrm{d}\Omega^2 := \mathrm{d}\theta^2 + \sin(\theta)^2\mathrm{d}\phi^2$, $R_0 = \left(-\beta_{(2)} + \sqrt{\beta_{(2)}^2 - \beta_{(1)}\beta_{(3)}}\right)/\beta_{(3)}$, and $\Lambda_g(R_0;\{\beta_{(i)}\})$ and $\Lambda_f(R_0;\{\beta_{(i)}\})$ are the cosmological constants for the g and f metrics, whose explicit expressions are not needed here. For the following set of β parameters,

$$\left\{\beta_{(0)}, \beta_{(1)}, \beta_{(2)}, \beta_{(3)}, \beta_{(4)}\right\} = \{1, 1, -1, 1, -1\}, \tag{2.6}$$

we have $R_0 = 1$ and[3]

$$\Lambda_g(R_0;\{\beta_{(i)}\}) = -2\ell^{-2} < 0, \qquad \Lambda_f(R_0;\{\beta_{(i)}\}) = 0. \tag{2.7}$$

[3]In Paper I, eqs. (13)-(14) include the formulas for $\Lambda_g(R_0;\{\beta_{(i)}\})$ and $\Lambda_f(R_0;\{\beta_{(i)}\})$ in natural units with a typo: they should be proportional to m^4, not m^2.

Here, ℓ is the length scale for the interactions between the metrics, introduced in (1.10).

The two metrics cannot be simultaneously diagonalized, as can be seen from the form of the square root,

$$S = \partial_v \otimes \mathrm{d}v + \frac{r^2}{3\ell^2}\,\partial_v \otimes \mathrm{d}r + \partial_r \otimes \mathrm{d}r + \partial_\theta \otimes \mathrm{d}\theta + \partial_\phi \otimes \mathrm{d}\phi, \tag{2.8}$$

which cannot be diagonalized. This configuration belongs to Type IIa in the algebraic classification presented in [HK17]. Note that the square root becomes diagonal if the two cosmological constants are the same, that is, for proportional metrics.

Thus, we have found a solution of the BFE in which both g and f are maximally symmetric, but have different isometry groups and, hence, different KVFs.

Before proceeding further, we make one last important remark about this solution, already present in Paper I. AdS has topology $\mathbb{S} \times \mathbb{R}^3$, whereas Minkowski has topology $\mathbb{R}^4$. Since the two topologies do not agree, this solution cannot cover the whole spacetime. On the other hand, if we take the universal covering of AdS in its timelike direction, we can "unwrap" $\mathbb{S}$ onto $\mathbb{R}$, and have the topology $\mathbb{R}^4$ for AdS too.[4] In this case, compatible with the topological constraint discussed in Section 1.2, the solution is global, that is, is defined over the whole manifold.

We now state a more general answer to question (i), from Paper I: a necessary condition for the two metrics to share their isometry groups is that the Lie algebras generated by their respective KVFs are isomorphic. This means that the KVFs of one metric can be obtained from the KVFs of the other by applying a Lie algebra isomorphism. This is not a sufficient condition, since two different Lie groups can have the same Lie algebra. As described in Section 2.2, in this case one Lie group would be the covering of the other (e.g., SU(2) and SO(3) share the Lie algebra, and the first is the universal covering of the second).[5]

The relation between the KVFs of g and f. The answer to question (i) partially answers question (ii). If the two isometry groups are different, then the KVFs are in general different.

However, this is not the only possibility, as pointed out in Paper I. Suppose that the metrics share a timelike KVF, that is, they are stationary [Wal10]. If the KVF is orthogonal to a family of spacelike hypersurfaces with respect to g, but not with respect to f, then g would be static and f would not. Hence, the metrics would have different isometries, even with the same KVF.

Let us now consider the case when the metrics share their isometry groups, by showing another example solution from Paper I. The solution comprises two Minkowski metrics written in the usual spherical polar chart (t, r, θ, ϕ) for g,

$$\begin{aligned}
\mathrm{d}{s_g}^2 &= -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin(\theta)^2\mathrm{d}\phi^2\right)\,, \\
\mathrm{d}{s_f}^2 &= -\tau'(t)^2\mathrm{d}t^2 + \rho'(r)^2\mathrm{d}r^2 + \rho(r)^2\left(\mathrm{d}\theta^2 + \sin(\theta)^2\mathrm{d}\phi^2\right)\,,
\end{aligned} \tag{2.9a}$$

[4]Note, however, that even the universal covering of AdS does not forget the periodicity in time, since geodesics starting from the same event meet an infinite number of times in the covering [Mos06, p. 8].

[5]To be universal, a covering has to be simply connected.

with $\tau(t)$ and $\rho(r)$ arbitrary functions. As in Paper I, we choose $\tau(t) := t + \operatorname{arcsinh}(t)$ and $\rho(r) := r + \operatorname{arcsinh}(r)$ for convenience.

The Einstein tensors for the metrics are zero. Hence, in order to solve the bimetric field equations, we have to introduce two independent stress–energy tensors to cancel the contribution from the bimetric stress–energy tensors. In other words, we define

$$T_g{}^{\mu\nu} := -\ell^{-2}\, V_g{}^{\mu\nu}, \qquad T_f{}^{\mu\nu} := -\ell^{-2}\, V_f{}^{\mu\nu}. \tag{2.10}$$

With our choice of $\tau(t)$ and $\rho(r)$, the bimetric and matter stress-energy tensors do not diverge at any (t, r).

Being the Minkowski metric written in two different ways, g and f share the Poincaré group of isometries. In addition, they share the null–cones. This can be seen if we consider only the (t, r) parts of the line element of f (since the metrics are spherically symmetric, we can neglect the angular part). The null–cone of f is defined by the equation

$$-\tau'(t)^2 t^2 + \rho'(r)^2 r^2 = 0 \quad \Longleftrightarrow \quad \left(1 + \frac{1}{\sqrt{1+t^2}}\right)^2 t^2 = \left(1 + \frac{1}{\sqrt{1+r^2}}\right)^2 r^2. \tag{2.11}$$

This equation can be satisfied only if $t = \pm r$, which is exactly the same relation defining the null–cone of g. However, they do not share the same KVFs. Indeed, as shown in Paper I, we can obtain f from g by applying the diffeomorphism

$$\mathrm{d}t = \frac{\mathrm{d}\tau}{\tau'(t)}, \quad \mathrm{d}r = \frac{\mathrm{d}\rho}{\rho'(r)}. \tag{2.12}$$

Therefore, the KVFs of g and f are related by the Jacobian matrix

$$\left(J^{-1}\right)^{\mu}{}_{\nu} = \operatorname{diag}\left(\tau'(t)^{-1},\, \rho'(r)^{-1},\, 1,\, 1\right), \tag{2.13}$$

in the following way

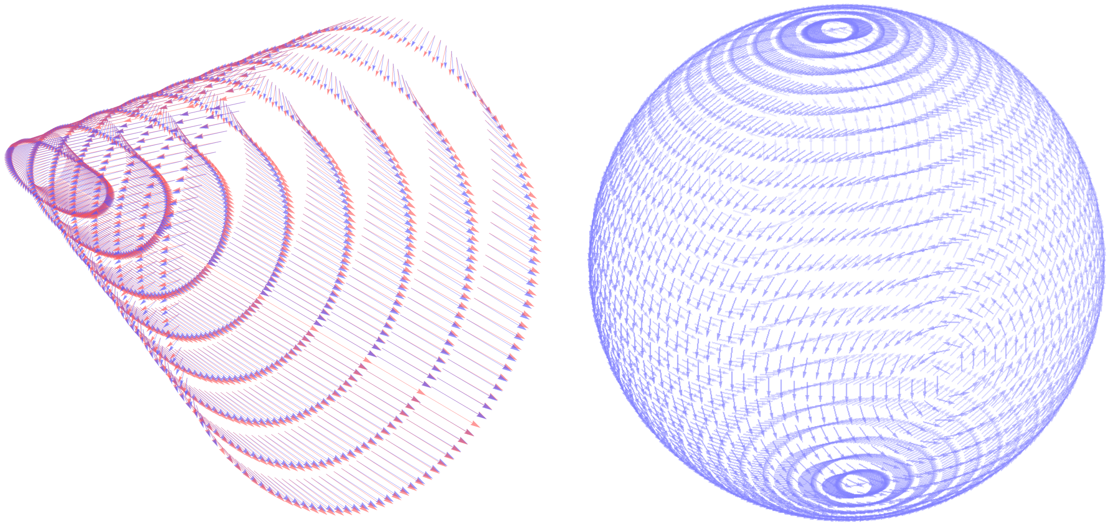
$$\xi_f{}^{\mu}\left(\tau(t), \rho(r), \theta, \phi\right) = \left(J^{-1}\right)^{\mu}{}_{\nu}\, \xi_g{}^{\nu}\left(t, r, \theta, \phi\right), \tag{2.14}$$

where ξ_g is a generic KVF of g and ξ_f is the related KVF of f. Figure 2.3a represents the boost

$$b^{\mu} = \left(r \sin(\theta)\cos(\phi),\, t\sin(\theta)\cos(\phi),\, \frac{t}{r}\cos(\theta)\cos(\phi),\, -\frac{t\cos(\phi)}{r\sin(\theta)}\right), \tag{2.15}$$

equal to $(x, t, 0, 0)$ in Cartesian coordinates, which is a KVF for g (in blue), together with the related boost for f (in red), obtained by applying (2.14) to b^{μ}. Both KVFs are plotted on the null-cone shared by the metrics. The explicit form of the KVF of f is not needed here. From Figure 2.3a, we can confirm that the two metrics have the same null-cone, since their boosts, though different, lie on it.

We conclude this section by noticing that the Jacobian in (2.13) leaves the angular part unchanged. Therefore, the KVFs corresponding to SO(3) (spherical symmetry) are the same for both metrics. The KVF $-\big(0, 0, \sin(\phi), \cos(\phi)\cot(\theta)\big)$ is plotted in Figure 2.3b, and lies



(a) One boost KVF of g (blue), and the related boost KVF of f (red) obtained through (2.14), are plotted on a null–cone shared by the metrics. They are different, though they lie on the null–cone, meaning that they map null directions to null directions.

(b) The KVF $-\big(0, 0, \sin(\phi), \cos(\phi)\cot(\theta)\big)$, shared by g and f, is plotted on the sphere S^2. It lies on S^2, which is then preserved by its flow.

Figure 2.3: Selected KVFs for the solution in (2.9).

on the sphere S^2.

For this solution, the KVFs of g are related to those of f by the differential map of a diffeomorphism. This is consistent with the fact that the two metrics, being the same, share the isometry group. Indeed, the differential map of a diffeomorphism (represented by the Jacobian) is a Lie algebra isomorphism, because it is linear, differentiable and preserves the Lie bracket, as pointed out in Paper I.

We can now answer question (ii): in general, there is no relation between the KVFs of g and f.

The relation between the isometries of each metric and the collineations of the associated bimetric and matter stress–energy tensors. Consider the last example solution again. To satisfy the BFE, we had to define two stress–energy tensors in (2.10), each of them coupled to one metric. We need not write their explicit form, but only state that the three KVFs generating spherical symmetry,

$$\mathcal{R}_{xy} = (0, 0, 0, 1), \tag{2.16a}$$

$$\mathcal{R}_{xz} = (0, 0, \cos(\phi), -\cot(\theta)\sin(\phi)), \tag{2.16b}$$

$$\mathcal{R}_{yz} = (0, 0, -\sin(\phi), -\cot(\theta)\cos(\phi)), \tag{2.16c}$$

are collineations for T_g and T_f,

$$\mathscr{L}_X {T_g}^{\mu\nu} = \mathscr{L}_X {T_f}^{\mu\nu} = 0, \quad \forall X \in \{\mathcal{R}_{xy}, \mathcal{R}_{xz}, \mathcal{R}_{yz}\}. \tag{2.17}$$

Therefore, the matter stress–energy tensors describe an isotropic (i.e., spherically symmetric) non-homogeneous and non-static fluid. This means that also the bimetric stress–energy tensors V_g and V_f represent the same type of fluid. The matter stress–energy tensors have the form,

$$T_{g\mu\nu} = \mathrm{diag}\,[\rho(r),\, p_r(r,t),\, p_t(r,t),\, p_t(r,t)]\,, \tag{2.18a}$$
$$T_{f\mu\nu} = \mathrm{diag}\,[\widetilde{\rho}(r),\, \widetilde{p}_r(r,t),\, \widetilde{p}_t(r,t),\, \widetilde{p}_t(r,t)]\,. \tag{2.18b}$$

This type of stress–energy tensor is referred to as an "anisotropic perfect fluid," since it includes a radial pressure p_r and a different tangential pressure p_t [RZ13, p. 141]. However, in this thesis, we use the word "isotropic" as a synonym of spherical symmetry. Given the result in (2.17), we refer to the stress–energy tensors in (2.18a) as an isotropic, non-homogeneous fluid. The features of the fluid would be the same after an arbitrary rotation, which is our definition of isotropy (or, equivalently, of spherical symmetry), formalized by (2.17).[6] Nevertheless, the pressure does change along the chosen direction, that is, when varying r, showing that the fluid is not homogeneous (for the definition of homogeneity, we refer to [Cho09]).

After this brief digression on the interpretation of the stress–energy tensors, we come back to symmetries. In the example, both sectors have a maximally symmetric metric coupled to a stress–energy tensor having only three KVFs. This would be impossible in GR; we now show why this is allowed in BR. We recall the BFE in presence of two independent matter sources,

$$G_{g\mu\nu} = \kappa_g\big(\ell^{-2}\, V_{g\mu\nu} + T_{g\mu\nu}\big), \tag{2.19a}$$
$$G_{f\mu\nu} = \kappa_f\big(\ell^{-2}\, V_{f\mu\nu} + T_{f\mu\nu}\big). \tag{2.19b}$$

Consider the g-sector (analogous computations can be carried out for the f-sector). Suppose ξ is a KVF for g, η is a collineation for V_g and ζ is a collineation for T_g. Taking the Lie derivatives of (2.19a) along the three vector fields ξ, η, ζ,

$$\ell^{-2}\mathscr{L}_\xi\, V_{g\mu\nu} = -\mathscr{L}_\xi\, T_{g\mu\nu}, \tag{2.20a}$$
$$\mathscr{L}_\eta\, G_{g\mu\nu} = \kappa_g \mathscr{L}_\eta\, T_{g\mu\nu}, \tag{2.20b}$$
$$\mathscr{L}_\zeta\, G_{g\mu\nu} = \ell^{-2}\kappa_g\, \mathscr{L}_\zeta\, V_{g\mu\nu}. \tag{2.20c}$$

If all equations in (2.20) are satisfied, $G_{g\mu\nu}, V_{g\mu\nu}$ and $T_{g\mu\nu}$ can have completely independent collineations (which, in principle, can also be different from those in the f-sector).

We now repeat the same argument without matter stress–energy tensors. Suppose ξ is a KVF for g. Then, as shown in Paper I,

$$0 = \mathscr{L}_\xi\, G_{g\mu\nu} = \ell^{-2}\kappa_g \mathscr{L}_\xi\, V_{g\mu\nu}. \tag{2.21}$$

In "bimetric vacuum" (a concept introduced in Paper II and discussed in Chapter 3), that

[6] These rotations are precisely generated by the SO(3) KVFs in (2.16).

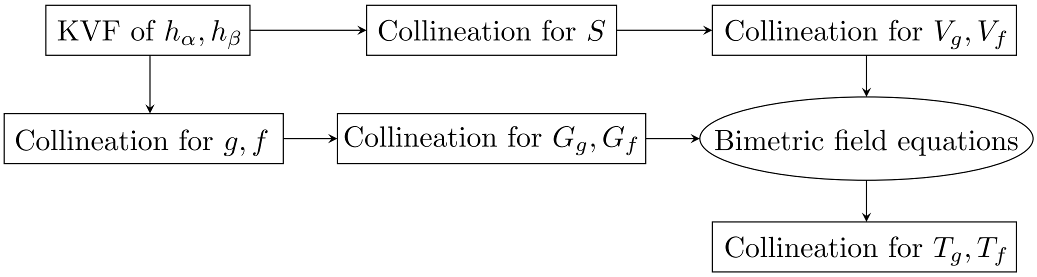


Figure 2.4: The relations between the collineations of h_α and h_β, and those of the square root $S = \left(g^{-1}f\right)^{1/2}$, the bimetric stress–energy tensors V_g, V_f and the stress–energy tensors T_g, T_f. This result is stated in Proposition 2 in Paper I.

is, in absence of external matter sources, a KVF for one metric is also a collineation for the associated bimetric stress–energy tensor.

We now report one of the main result of Paper I, starting with the following definition.

Definition 1. Consider Lorentzian metric fields g and f. We define the set $h_\alpha := g\,(g^{-1}f)^\alpha$, $\alpha \in \mathbb{R}$, where the matrix power function is defined through $X^\alpha := \exp\left[\alpha \log(X)\right]$. Note that $g = h_0$ and $f = h_1$.

In Paper I, the following proposition is proved,

Proposition. *Consider a vector field ξ and arbitrary h_α and h_β such that $\mathscr{L}_\xi h_\alpha = 0$ and $\mathscr{L}_\xi h_\beta = 0$. Then $\mathscr{L}_\xi h_\gamma = 0$ for any $\gamma \in \mathbb{R}$.*

As a corollary, if g and f have the same KVF ξ, then the latter is a collineation for every tensor in the theory, as explained in Paper I. Indeed, in this case ξ is a collineation for S, and therefore also for the bimetric stress–energy tensors. In presence of matter stress–energy tensors, (2.20) implies that ξ is a collineation also for them. We summarize the result of the Proposition in Figure 2.4.

We can now understand why the stress–energy tensors in (1.18) are only spherically symmetric: the SO(3) KVFs are the only ones shared by the metrics (equal to h_0 and h_1), and therefore they are collineations for every tensor in the theory.

The definition of a spacetime symmetry in BR. One of the unanswered questions in Paper I is what the best definition for a spacetime symmetry in BR is. The two metric sectors can either share or have unrelated collineations. In the former case, it is natural to talk about a spacetime symmetry of the system. In the latter case the definition of a spacetime symmetry is more ambiguous. Should our example solution be considered as a maximally symmetric system, or only a spherically symmetric one? The question remains open.

The impact of spacetime symmetries on gravitational wave emission. The conclusions of Paper I have interesting consequences concerning gravitational wave emission.

In GR, the leading term sourcing helicity-2 gravitational waves is determined by the "quadrupole moment" of the matter source, which, assuming a Minkowski background, is defined as

$$Q^{ij}(t,\vec{x}) := \int \mathrm{d}^3x\, T^{00}(t,\vec{x})\left(x^i x^j - \frac{1}{3}|\vec{x}|^2\,\eta^{ij}\right). \tag{2.22}$$

The leading term of the gravitational wave $h^{\mathrm{TT}}_{\mu\nu}$, where TT stands for "transverse–traceless," obeys the equation,

$$\left[h^{\mathrm{TT}}_{ij}\right]_{\text{leading}} = \frac{2}{|\vec{x}|}\partial_t^2 Q^{\mathrm{TT}}_{ij}(t,\vec{x}). \tag{2.23}$$

We refer the reader to [Mag08, Ch. 3] for more details. If the stress–energy tensor is spherically symmetric, then the quadrupole moment is zero and there is no gravitational wave emission.

In BR, however, a spherically symmetric stress–energy tensor does not imply a spherically symmetric Einstein tensor nor a spherically symmetric bimetric stress–energy tensor. Therefore, in presence of spherically symmetric matter sources, in BR the helicity-2 tensor perturbations in both metrics can be sourced by the "bimetric quadrupole moments,"

$$Q^{\mathrm{b}ij}_g(t,\vec{x}) := \int \mathrm{d}^3x\, {V_g}^{00}(t,\vec{x})\left(x^i x^j - \frac{1}{3}|\vec{x}|^2\,\eta^{ij}\right), \tag{2.24a}$$

$$Q^{\mathrm{b}ij}_f(t,\vec{x}) := \int \mathrm{d}^3x\, {V_f}^{00}(t,\vec{x})\left(x^i x^j - \frac{1}{3}|\vec{x}|^2\,\eta^{ij}\right). \tag{2.24b}$$

Since this phenomenon is not excluded at this level, one should solve the BFE with a spherically symmetric stress–energy tensor and bimetric stress-energy tensors with nonzero quadrupole moment in order to extract these "bimetric gravitational waves."

Chapter 3

Symmetries of spacetime in particular solutions

In this chapter, we discuss two classes of solutions having different spacetime symmetries. Contrary to the examples of the previous chapter, these solutions have metrics that share their KVFs and therefore, following the discussion at the end of Section 2.3, there is no ambiguity in the spacetime symmetries of the system.

In the following, Section 3.1 and Section 3.2 summarize selected topics from Paper II and Paper III. They also contain complementary information to the two papers.

3.1 Nonstationary spherically symmetric solutions

The Jebsen–Birkhoff theorem. In GR, the Jebsen–Birkhoff theorem [Jeb21; BL23; Eie25; DF05; Jeb05; VR06; CCH12][HE11, Appendix B] states that, in vacuum, that is, whenever $R_{g\mu\nu} = 0$, a spherically symmetric solution in an open set $\mathscr{U} \subset \mathscr{M}$ has to be locally isometric (i.e., equivalent) to part of the maximally extended Schwarzschild solution in $\mathscr{U}$.[1,2]

This means that a spherically symmetric vacuum solution has to have at least four KVFs: the three generators of SO(3), expressed in spherical polar coordinates in (2.16), and another KVF—orthogonal to a congruence of hypersurfaces [Wal10]—which can be timelike, spacelike or null. Where it is timelike, the metric is static; where it is spacelike, the metric is spatially homogeneous; where it is null, there is a Killing horizon [Deb72][CCH12, Section 2.5]. This theorem is of great importance in GR. Together with the no-hair theorems [Isr67; Isr68; Car71][CCH12, Section 3.3][MTW73, p. 876], it implies that the only spherically symmetric black hole (BH) solution is the Schwarzschild solution [Sch16].

The theorem can be extended to Einstein spacetimes, that is, spacetimes for which $R_{g\mu\nu} = \Lambda\, g_{\mu\nu}$, with Λ (cosmological) constant (see, e.g., [SW09; SW10]), and to the Einstein–Maxwell equations [dIn92, Section 18.1]. It follows that, for $\Lambda > 0$, the only spherically

[1] The reader might reply that the Minkowski solution is also a vacuum solution to the EFE; indeed, the latter is the Schwarzschild solution with zero integration constant m.

[2] References [Jeb21; BL23; Eie25; DF05; Jeb05; VR06; CCH12] follow the interesting story of this theorem, universally attributed to Birkhoff since 1923, but proved before him by Jebsen in 1921.

symmetric BH solution is the Schwarzschild–de Sitter (SdS); for $\Lambda < 0$, it is the Schwarzschild–anti–de Sitter (SAdS); for the Einstein–Maxwell equations, it is the Reissner–Nordström solution.

In Paper II, we show that the Jebsen–Birkhoff theorem is not valid in BR by explicitly solving the BFE for a class of nonstationary, spherically symmetric (NS-SS) solutions.

Nonstationary spherically symmetric class of solutions. The class of NS-SS solutions is obtained, in natural units, by imposing the conditions,

$$\beta_{(0)} = 3\beta_{(2)}\frac{M_g^2}{M_f^2}, \quad \beta_{(4)} = 3\beta_{(2)}\frac{M_f^2}{M_g^2}, \quad \beta_{(1)} = \beta_{(3)} = 0, \tag{3.1}$$

which are also imposed on the dS background in the attempt to look for partially massless bimetric gravity [HSS13c]. This suggests the interesting possibility of a generic solution of this class to be a background geometry for candidate partially massless theories [DN84; DW01a; DW01c; DW01d; DW01b; DW04; DW06].

In ingoing Eddington–Finkelstein coordinates (v, r, θ, ϕ), the class of NS-SS solutions is

$$\begin{aligned} \mathrm{d}s_g{}^2 &= -\left(1 - \frac{1}{3}\,\Lambda(v)\,r^2\right)\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r + r^2\mathrm{d}\Omega^2, \\ \mathrm{d}s_f{}^2 &= \lambda(v)^2\left[-\left(1 - \frac{1}{3}\,\Lambda(v)\,r^2 - 2\frac{\lambda'(v)}{\lambda(v)}r\right)\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r + r^2\mathrm{d}\Omega^2\right], \end{aligned} \tag{3.2a}$$

with $\mathrm{d}\Omega^2 \coloneqq \mathrm{d}\theta^2 + \sin(\theta)^2\,\mathrm{d}\phi^2$. The class is parameterized by the arbitrary function $\lambda(v)$, which is not determined by the BFE. The function $\Lambda(v)$ is defined in terms of $\lambda(v)$ as

$$\Lambda(v) \coloneqq 3\beta_{(2)}m^4 M_g{}^{-2}\left(\alpha^{-2} + \lambda^2(v)\right), \tag{3.3}$$

with $\alpha \coloneqq M_f/M_g$. We refer to $\Lambda(v)$ as the "curvature field", because the Ricci and Kretschmann scalars of the two metrics are

$$R_g = 4\,\Lambda(v), \quad R_f = R_g\,\lambda^{-2}(v), \quad K_g = \frac{8}{3}\,\Lambda^2(v), \quad K_f = K_g\,\lambda^{-4}(v). \tag{3.4}$$

The Weyl tensors of g and f are both identically zero, therefore both geometries are of Petrov Type O [Pet16].

When $\lambda'(v) = 0$, $\Lambda'(v) = 0$, g and f become proportional, maximally symmetric and equal to Minkowski, dS or AdS depending on the constant value of $\Lambda(v)$.

We now focus on the spacetime symmetries of this solution. As already mentioned, both metrics have three generators of SO(3) as KVFs, therefore there is no ambiguity in the spacetime symmetries of the system. The metrics do not have any timelike KVF, as evident from Figure 3.1. The plotted geodesics are not invariant under translations along the v direction (the advanced time), unless $\lambda'(v) = 0$, that is, unless we have GR solutions. The metrics also possess conformal vector fields, with three of them explicitly computed in Paper II. They are given in terms of the vector field

$$\xi = V(v)\partial_v + r\,\chi(v)\partial_r \tag{3.5}$$

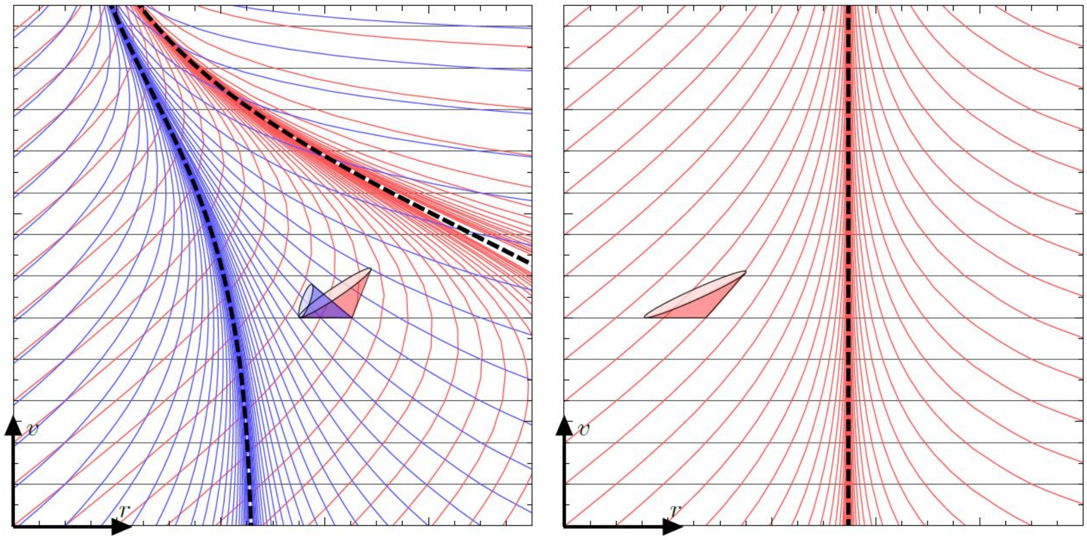


Figure 3.1: Figure 1 in Paper II. Null radial geodesics for the NS-SS solutions defined by $\lambda(v) = 2e^v$ (left panel) and $\lambda(v) = 3$ (right panel), both with $\alpha = 1$, $\beta_{(2)} = 1/3$. The horizontal black lines are ingoing null radial geodesics, shared by g and f. The outgoing null radial geodesics of g are in red, and those of f are in blue. The dashed lines are the cosmological horizons. In the right panel, all geodesics coincide because $\lambda(v)$ is constant and the two metrics are proportional, i.e. they are both GR solutions. Selected null-cones are plotted, for clarity.

with $\chi(v) = V'(v) - r\,V''(v)$. The function $V(v)$ is found by solving the conformal equation

$$\mathscr{L}_\xi\, g = 2\chi\, g, \tag{3.6}$$

which, under the simplifying assumption of ξ not having components along ∂_θ and ∂_ϕ, reduces to a third-order linear homogeneous ordinary differential equation (ODE),

$$V'''(v) - \frac{1}{3}\Lambda(v)V'(v) - \frac{1}{6}\Lambda'(v)V(v) = 0. \tag{3.7}$$

For $\Lambda(v) \neq 0$, it has three linearly independent solutions, giving the three conformal vector fields. For $\lambda(v) = 2\,e^v$, one solution is given by,

$$V(v) = {}_1F_2\left(\frac{1}{2};\, 1 - \frac{\sqrt{\beta_{(2)}}}{2\alpha}, 1 - \frac{\sqrt{\beta_{(2)}}}{2\alpha};\, \beta_2\, e^{2v}\right), \tag{3.8}$$

where the generalized hypergeometric function ${}_pF_q$ is defined by [Nat, eq. 16.2.1],

$${}_pF_q\,(a_1, ..., a_p; b_1, ..., b_q; z) := \sum_{n=0}^{\infty} \frac{(a_1)_n ... (a_p)_n}{(b_1)_n ... (b_q)_n} \frac{z^n}{n!}, \tag{3.9}$$

and $(\cdot)_n$ denotes the Pochhammer symbol or "rising factorial" [Nat, eqs. 5.2.4, 5.2.5],

$$\begin{aligned} (x)_0 &:= 1, \\ (x)_n &:= x(x+1)\cdots(a+n-1), \quad n \geqslant 1. \end{aligned} \tag{3.10}$$

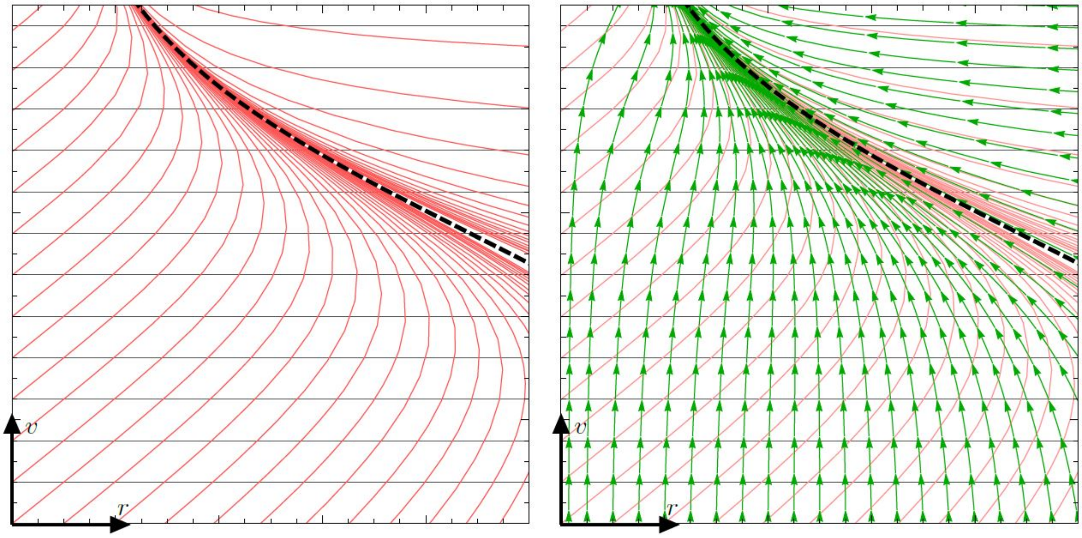


Figure 3.2: Figure 2 in Paper II. Integral curves of the conformal vector field ξ (green) on top of the null radial geodesics of g for $\lambda(v) = 2\mathrm{e}^v$. The ingoing null radial geodesics are horizontal and black, the outgoing are in red. The dashed line represents the cosmological horizon.

For the same parameters α and $\beta_{(2)}$ as in Figure 3.1 and for $V(v)$ given by (3.8), the integral curves of ξ are plotted in Figure 3.2.

In the limit $\lambda'(v) \to 0$, also $\Lambda'(v) \to 0$, and the equation (3.7) reduces to,

$$V'''(v) - \frac{1}{3}\Lambda\, V'(v) = 0. \tag{3.11}$$

The three independent solutions of (3.11) are,

$$V_1(v) = \sqrt{\frac{3}{\Lambda}}\mathrm{e}^{v\sqrt{\Lambda/3}}, \quad V_2(v) = -\sqrt{\frac{3}{\Lambda}}\mathrm{e}^{-v\sqrt{\Lambda/3}}, \quad V_3(v) = 1. \tag{3.12}$$

The solution $V_3(v) = 1$ corresponds to the vector field ∂_v, which is a timelike KVF for both metrics when $\lambda'(v) = 0$.[3] The reason for this is that, from Table 2.1, the Killing algebra is a Lie subalgebra of the conformal algebra. Hence, solving the conformal equations, we also get the KVFs, if they exist.

Interpretation of the NS-SS solution. The existence of this NS-SS solution and the one in [HSS13c], provides two counterexamples to a statement analogous to the Jebsen–Birkhoff theorem in BR. A closer look at this NS-SS solution provides a physical interpretation of the absence of such a statement, as discussed in detail in Paper II.

In summary, the Jebsen–Birkhoff theorem is stated for systems satisfying the vacuum EFE, that is, $R_{g\mu\nu} = 0$, and it is extended to Einstein spacetimes, satisfying $R_{g\mu\nu} = \Lambda\, g_{\mu\nu}$, with Λ (cosmological) constant. In BR, the employed definition of vacuum is different, because $R_{g\mu\nu} \neq 0$ and $R_{g\mu\nu} \neq \Lambda\, g_{\mu\nu}$ even in absence of matter stress–energy tensors, since BR includes another spin-2 field. The "bimetric vacuum" is different from the GR vacuum.

[3] This can be easily seen by a direct computation of $\mathscr{L}_{\partial_v} g$ and $\mathscr{L}_{\partial_v} f$.

Indeed, when the curvature field $\Lambda(v)$ is constant, the NS-SS solution turns into a maximally symmetric GR solution, respecting the Jebsen–Birkhoff theorem.

For the NS-SS solution, the bimetric stress–energy tensors can be recognized as matter stress–energy tensors of Type II [HE11], and their components as energy fluxes, energy densities and tangential pressures. In Paper II, we show that ${V_g}^{\mu\nu}$ can satisfy the null and weak energy conditions, but not the strong one (for the energy conditions, see [HE11]). However, ${V_f}^{\mu\nu}$ cannot satisfy the null energy condition, if ${V_g}^{\mu\nu}$ does; the anticorrelation of the null energy conditions for the bimetric stress–energy tensors is a known result [BMV12]. Hence, the NS-SS solution can be thought as a non-vacuum solution of the EFE (of the generalized Vaidya type [WW99]; see Paper II for more details).

3.2 Static and spherically symmetric black holes

The ansatz and its spacetime symmetries. In BR, the "bidiagonal" ansatz is an ansatz for which there exist a coordinate chart in which both metrics are diagonal.

In Paper III, static and spherically symmetric bidiagonal BHs are studied. Nonbidiagonal solutions were studied in [Com+12b], and are equivalent to GR solutions. Therefore, in looking for static and spherically symmetric BH solutions specific to BR, the bidiagonal ansatz is more promising.

Paper III was the first one chronologically, therefore the results about spacetime symmetries were not known yet and we did not consider different KVFs in the two sectors in Paper III.

The ansatz used in Paper III, in Eddington–Finkelstein coordinates (v, r, θ, ϕ) adapted to g, is

$$
\begin{aligned}
\mathrm{d}{s_g}^2 &= -\mathrm{e}^{q(r)} F(r)\, \mathrm{d}v^2 + 2\mathrm{e}^{q(r)/2}\, \mathrm{d}v\mathrm{d}r + r^2\, \mathrm{d}\Omega^2 \\
\mathrm{d}{s_f}^2 &= -\mathrm{e}^{q(r)} F(r)\, \tau(r)^2\, \mathrm{d}v^2 + 2\mathrm{e}^{q(r)/2}\tau(r)^2\, \mathrm{d}v\mathrm{d}r + \frac{\Sigma(r)^2 - \tau(r)^2}{F(r)}\, \mathrm{d}r^2 + R(r)^2 r^2\, \mathrm{d}\Omega^2, \quad (3.13)
\end{aligned}
$$

with $\mathrm{d}\Omega^2 \coloneqq \mathrm{d}\theta^2 + \sin(\theta)^2\, \mathrm{d}\phi^2$. The square root $S = \left(g^{-1} f\right)^{1/2}$ is

$$
\begin{aligned}
S = {} & \tau(r)\, \partial_v \otimes \mathrm{d}v + \Sigma(r)\, \partial_r \otimes \mathrm{d}r + R(r)\, (\partial_\theta \otimes \mathrm{d}\theta + \partial_\phi \otimes \mathrm{d}\phi) + \\
& + \mathrm{e}^{-q(r)/2}\, \Phi(r)\, \partial_r \otimes \mathrm{d}v, \qquad (3.14)
\end{aligned}
$$

with $\Phi(r) \coloneqq (\Sigma(r) - \tau(r))\, / F(r)$ defined as the "crossing function." One can recover the bidiagonal ansatz by changing chart to the usual spherical polar (t, r, θ, ϕ), with $\mathrm{d}t = \mathrm{d}v - \mathrm{e}^{-q(r)/2} F(r)^{-1}\, \mathrm{d}r$.[4] The eigenvalues of S are τ, Σ, R. We assume that they are strictly positive, since this corresponds to the principal branch of the square root matrix function. Choosing the principal branch guarantees that the theory is well-defined, as explained in [HK17], and that the two sectors share the curvature singularities at $r = 0$, by means of Proposition 1 in Paper III.

[4] The nonbidiagonal solutions in [Com+12b] involve a nonzero (t, r) component for f. The assumption of metrics sharing the KVF rules out the (r, θ) and (r, ϕ) components, if they depend of r, θ, ϕ.

In Eddington–Finkelstein coordinates, the KVFs of the two metrics are

$$
\begin{aligned}
\mathcal{K} &= (1,0,0,0),\\
\mathcal{R}_{xy} &= (0,0,0,1),\\
\mathcal{R}_{xz} &= (0,0,\cos(\phi),\, -\cot(\theta)\sin(\phi)),\\
\mathcal{R}_{yz} &= (0,0,-\sin(\phi),\, -\cot(\theta)\cos(\phi)).
\end{aligned} \tag{3.15}
$$

The norms of $\mathcal{K}$ with respect to g and f are,

$$
\begin{aligned}
|\mathcal{K}|_g &= g_{\mu\nu}\mathcal{K}^\mu\mathcal{K}^\nu = g_{00} = -\mathrm{e}^q F,\\
|\mathcal{K}|_f &= f_{\mu\nu}\mathcal{K}^\mu\mathcal{K}^\nu = f_{00} = -\mathrm{e}^q F\tau^2 = |\mathcal{K}|_g\tau^2.
\end{aligned} \tag{3.16}
$$

Both norms are timelike when $F > 0$, null when $F = 0$ and spacelike when $F < 0$. This implies that $F(r) = 0$ defines the Killing horizon for both metrics, in accordance with the Proposition in [DJ12], which was generalized in Appendix C of Paper III to include the new category of BHs introduced in Paper III and reviewed in the next paragraph. For stationary spacetimes, the strong rigidity theorem tells us that an event horizon must be a Killing horizon [Haw72, p. 161] (see also [Heu96, Theorem 6.7] and [CCH12, Section 3.3.1]). Therefore, in looking for static BHs, it is reasonable to impose the existence of at least one Killing horizon. As shown in Paper III, the Killing horizon defined by $F(r) = 0$ turns out to be an event horizon, since the geodesics of g and f entering it do not exit it, but converge to the shared curvature singularity at $r = 0$.

The new classification for bidiagonal BHs. $\mathcal{K}$ plays a key role in defining the classification for bidiagonal BHs presented in Paper III. At the Killing horizon it becomes a null vector for both metrics, showing that the null-cones of the metrics necessarily have at least one common null direction.

If the metrics, at the Killing horizon, share only one null direction, then the solution is called "improper bidiagonal." If they share two or four null directions, the solution is called "proper bidiagonal." This is the geometric interpretation of the algebraic types I and IIa in [HK17], which are the ones allowed in the chosen ansatz. We will come back to the algebraic types of the square root in Chapter 6.

Definition. A BH solution is said to be "improper bidiagonal" if and only if the crossing function is finite and nonzero at the Killing horizon r_{H}, that is,

$$0 \neq \Phi(r_{\mathrm{H}}) < \infty.$$

A BH solution is said to be "proper bidiagonal" if and only if the crossing function is zero at the Killing horizon r_{H}, that is,

$$\Phi(r_{\mathrm{H}}) = 0.$$

Let us define the algebraic types A and B,

$$A = \begin{pmatrix} 0 & \bullet & 0 & 0 \\ \bullet & 0 & 0 & 0 \\ 0 & 0 & \bullet & 0 \\ 0 & 0 & 0 & \bullet \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & \bullet & 0 & 0 \\ \bullet & \bullet & 0 & 0 \\ 0 & 0 & \bullet & 0 \\ 0 & 0 & 0 & \bullet \end{pmatrix}. \tag{3.17}$$

We can see from (3.13) that the proper bidiagonal BHs have both metrics of type A at the Killing horizon, whereas the improper bidiagonal have g and f of algebraic types A and B, respectively. Two matrices, both having algebraic type A, can be simultaneously diagonalized, but two matrices having algebraic types A and B can not [Uhl73; HK17]. Therefore, improper bidiabonal BHs have metrics which cannot be simultaneously diagonalized at the Killing horizon, hence their name.

Proportional BH solutions, that is, GR solutions, constitute a subset of the set of proper bidiagonal BHs; they have coinciding null cones, therefore they share all of their null directions.

Lyapunov instability of the GR solutions. One of the main results of Paper III concerns the behavior of static spherically symmetric BHs when $r \to \infty$. The reason for exploring this is to understand if there can be asymptotically flat, dS or AdS solutions, different from Schwarzschild, SdS or SAdS, that can constitute end states of spherically symmetric gravitational collapse. In addition, if these non-rotating and non-electrically charged BH solutions would depend on some measurable parameters other than their mass, this would disprove statements similar to the no-hair theorems in BR.

For this system, the BFE can be considered both as a boundary value problem and as an initial value problem. In looking for asymptotically flat solutions, the boundary conditions are fixed by imposing reasonable values for some of the fields at the Killing horizon, and values for other fields when $r \to \infty$ by imposing asymptotic flatness. The Schwarzschild solution was the only found solution to this boundary value problem. Notice that a boundary value problem involving ODEs may have one, several, or no solutions at all. This is in contrast with the general results concerning an initial value problem involving ODEs, which can be found, for example, in [Gra14, Ch. 1][NDG17, Ch. 4]. For the initial value problem consisting of a nonlinear system of first-order ODEs,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y), \quad y \in U \subset \mathbb{R}^n, \quad f : [t_0, t_1] \times U \to \mathbb{R}^n, \tag{3.18a}$$

$$y(t_0) = y_0, \tag{3.18b}$$

the Picard–Lindelöf theorem establishes that Lipschitz continuity of f with respect to y is sufficient to guarantee existence and uniqueness of the solution for a given specification of the initial condition, and also the continuous dependence of the solutions on the initial conditions.[5] Hence, the number of solutions to a boundary value problem can be determined

[5] Such systems are called "well-posed." The situation is radically different if the initial value problem involves partial differential equations, as we shall see in Chapter 4.

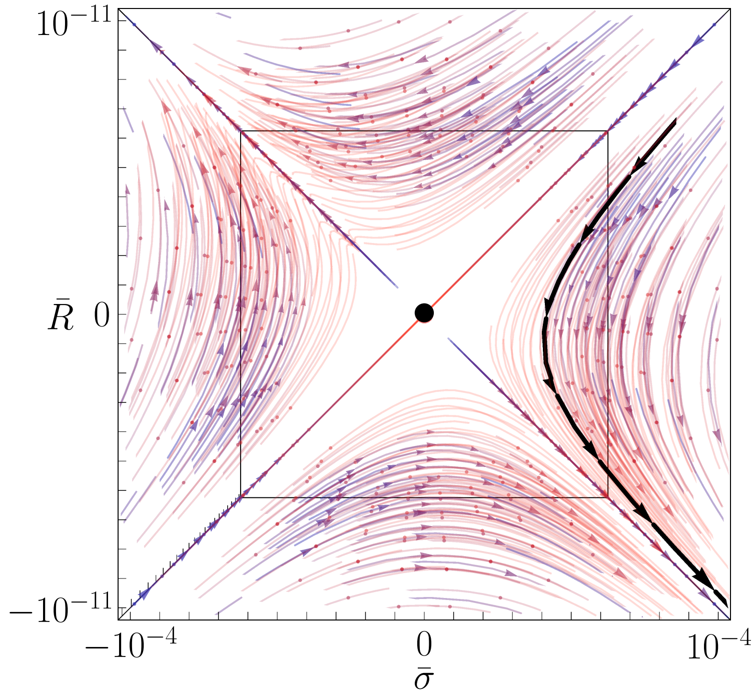


Figure 3.3: Figure adapted from Figure 7a in Paper III. Phase space plot around the Schwarzschild solution [black dot at $(0, 0)$]. The variables are $\bar{R} = R - 1, \bar{\sigma} = \sigma - 1$, with $\sigma(r) := \Sigma(r)/\left(R(r)\,r\right)'$. The black thicker trajectory is emphasized as an example. Every trajectory is a BH solution to the BFE. The redder the trajectory, the larger r; the bluer the trajectory, the smaller r. However, these solutions around Schwarzschild have $\det(S) = 0$ at some finite r, therefore they are not acceptable. This is a typical saddle point diagram, which shows the "repulsive" character of the Schwarzschild solution when $r \to \infty$.

through the shooting method [NDG17, Subsec. 9.3]. The boundary value problem is turned into an initial value problem by the replacement of some boundary conditions with initial conditions. Then, the number of solutions to the boundary value problem is equal to the number of solutions to the initial value problem that converge to the replaced boundary conditions. This is the approach used in Paper III, where the initial conditions are specified at the shared Killing horizon, and solutions converging to Minkowski, dS or AdS spacetimes when $r \to \infty$ are searched for.

Paper III shows that theoretically consistent solutions, other than Schwarzschild, SdS and SAdS, exist, but diverge from the GR backgrounds. This conclusion was reached by way of a phase space analysis of the BFE. We refer to Paper III for the details. In Figure 3.3, the phase space around the Schwarzschild solution is shown (see Paper III for the explicit parameter values). Every trajectory is a BH solution. We can clearly see the saddle point structure of the trajectories around the central black dot representing the Schwarzschild solution. From this, we can argue that the Schwarzschild solution is unstable in the sense of Lyapunov, that is, all other solutions diverge from it when $r \to \infty$. Similar conclusions are drawn for dS and AdS, see Figure 7 in Paper III. We remark that we have not formulated an exact proof for these claims, but numerical solutions, the phase plots and analytical considerations strongly suggest their validity.

Paper III shows that, in addition to GR solutions, in BR there are also BHs whose leading term is AdS, but that do not converge to it when $r \to \infty$. They oscillate around it with an increasing amplitude. Since they do not show any singular behavior, that is, all their scalar invariants are finite and $\det(S) \neq 0$ everywhere, they are theoretically valid solutions which do not have an analog in GR. When $r \to \infty$, their subleading term, which introduces the growing amplitude oscillations, appears multiplied by a parameter. If this can be considered as a hair, thus disproving analogous statements to the no-hair theorems, depends on the definition of "hair" employed. Since these solutions do not have GR asymptotics, and in particular are not asymptotically flat, it is not clear if and how this parameter can be measured and if and how it can be related to a physical observable.

We also conjectured that the instability properties of GR solutions in the bidiagonal ansatz is due to the sharing of the KVFs. A possibility, that should be investigated further, is that, if they do not share the KVFs, the ansatz is not bidiagonal and the solutions equivalent to those in GR, as shown in [Com+12b].

Conclusion of Part I

Summary of the scientific results in Part I. The first part of this thesis reports the scientific results obtained during the first two years of the Ph.D. studies. It is based primarily on Paper I. Paper II and Paper III are reviewed from the viewpoint of Paper I, but their standalone importance is also pointed out.

Paper I concludes that in BR the isometries of the two metrics are unrelated generically; nonetheless, we proved a proposition stating when the metrics can share the isometries. We analyzed both the relations between the KVFs of the two metrics and the relation between their isometry groups, and found that, generically, the two relations are uncorrelated. In other words, one could have different isometry groups and different KVFs, or the same isometry group but different KVFs, or the same KVFs and different isometry groups (e.g., staticity and stationarity). We also studied the relation between collineations of the tensors within the same metric sector, and found that three possible situations can occur:

(i) The two metrics share KVFs, which also define the collineations for the bimetric and matter stress–energy tensors.

(ii) The two metrics do not share the KVFs.

   (a) In vacuum, the KVFs of each metric define the collineations for the corresponding bimetric stress-energy tensor.

   (b) When matter stress–energy tensors are present, the KVFs of each metric do not define the collineations for the corresponding bimetric and matter stress–energy tensors.

The main result of Paper II is the presentation of a new class of NS-SS solutions, parameterized by an arbitrary function of the advanced time coordinate, disproving an analogous statement to the Jebsen–Birkhoff theorem in BR. Interestingly, this solution is obtained by imposing the same β parameters used in the quest for partially massless bimetric gravity [HSS13c]. This suggests the possibility to use it as a background solution in that context.

Bidiagonal static and spherically symmetric black hole solutions sharing their KVFs where studied in Paper III. Again, KVFs are of great importance in determining the properties of the solutions. When the static KVF crosses the Killing horizon, the two metrics can be either simultaneously diagonalizable or not. This defines two categories of solutions, denoted "proper bidiagonal" and "improper bidiagonal," respectively. GR solutions are always proper bidiagonal, but the converse is not true. Among the large variety of BH solutions, only GR solutions converge to Minkowski, dS or AdS when $r \to \infty$. We conjecture that this instability property of the bidiagonal solutions may be due to imposing the same

KVFs for both metrics.

Discussion of the scientific results in Part I. The results summarized above do not give us a conclusive answer about how we can simplify the BFE in order to study spherically symmetric gravitational collapse of matter in BR.

More in detail, the results of Paper I do not restrict the possible ansatzes one can choose in order to get analytic solutions describing the gravitational collapse and its possible end states, since the metrics can have different KVFs even if they share the isometry group.

The NS-SS class of solutions presented in Paper II is the first exact vacuum solution in BR without an analogous vacuum solution in GR. However, it is not a BH solution, that is, the end state of spherically symmetric gravitational collapse in GR. In addition, the solution only holds for specific values of the parameters of the theory, hence it cannot be considered as the end state of gravitational collapse for generic configurations. Moreover, disproving an analogous statement to the Jebsen–Birkhoff theorem in BR, it tells us that we cannot assume a static spacetime outside a spherically symmetric collapsing dust cloud.

The results of Paper III tell us that, if we are looking for asymptotically flat static end states of spherically symmetric gravitational collapse, then the only possible alternative is the Schwarzschild solution.

In conclusion, these findings show how difficult it is to obtain exact results in BR. There are two possible ways to overcome this problem. First, one can still look for exact solutions by, for example, relaxing the assumption of staticity to get dynamical solutions in the presence of matter sources. Outcomes of this approach are presented in [Hög+19; HTM19]. However, these works also show that analytical methods are not enough to obtain more general solutions. Second, one can focus on finding numerical dynamical solutions. This is also a nontrivial task, and the subject of the second part of the thesis. Part II provides some of the mathematical background needed to understand how the BFE can be treated as a system of partial differential equations, and reviews the results of the remaining included papers. The topic unifies the theory of partial differential equations and pseudo-Riemannian geometry, and it is inspired by the success of numerical relativity.

Part II

$3 + 1$ FORMULATION

We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes.

Pierre Simon Laplace [LD98]

I think that when we know that we actually do live in uncertainty, then we ought to admit it; it is of great value to realize that we do not know the answers to different questions. This attitude of mind—this attitude of uncertainty—is vital to the scientist, and it is this attitude of mind which the student must first acquire. It becomes a habit of thought. Once acquired, one cannot retreat from it any more.

Richard P. Feynman, talk given at the Caltech YMCA Lunch Forum on May 2, 1956

Chapter 4

Partial differential equations

The BFE constitute a system of partial differential equations (PDEs) and, aside from specific cases as in [Hög+19; HTM19], their numerical integration becomes a necessity when solutions describing realistic physical systems are searched for. In this chapter, we discuss aspects of the theory of PDEs, needed to understand the recasting of the BFE in a form suitable for the numerical integration. We start by introducing the notion of "Cauchy problem" and "well-posedness" in Section 4.1, and continue by reviewing the classification of PDEs in Section 4.2.

Multi-index notation. Following [Tay10a, p.5] [Eva10, Appendix A], we introduce a useful notation to improve readability. A "multi-index" α is a vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ with non-negative integer components; the "order" of a multi-index is $|\alpha| := \alpha_1 + \cdots + \alpha_n$. Given a multi-index α, we introduce the following notation for the partial derivatives of the real function u of n real variables $x = \left(x^1, \ldots, x^n\right)$,

$$\mathscr{D}^\alpha u(x) := \frac{\partial^{|\alpha|} u(x)}{(\partial x^1)^{\alpha_1} \cdots (\partial x^n)^{\alpha_n}} \equiv \partial_{x^1}^{\alpha_1} \cdots \partial_{x^n}^{\alpha_n} u(x). \tag{4.1}$$

For k being a non-negative integer, we define,

$$\mathscr{D}^k u(x) := \{\mathscr{D}^\alpha u(x) \mid |\alpha| = k\}, \tag{4.2}$$

that is, the set of all partial derivatives of order k. We refer to $\mathscr{D}^\alpha$ as a "differential operator" of order α. If $k = 1$, we consider the elements of $\mathscr{D}u(x) \equiv \mathscr{D}^1 u(x)$ as arranged in the gradient vector $\mathscr{D}u(x) = (\partial_{x^1} u(x), \ldots, \partial_{x^n} u(x))$. We also use a subscript to indicate which variables the differentiation refers to, $\mathscr{D}_x u(x)$. Finally, we introduce the following notation for monomials constructed from the components of a vector $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, $v^\alpha = v_1^{\alpha_1} \cdots v_n^{\alpha_n}$.

In this chapter, we will consider only C^∞, that is smooth, real functions of real arguments.

4.1 The well-posedness of the Cauchy problem

The first treatise about the Cauchy problem was written by Hadamard in 1923 [Had23]. Here, we mainly follow the more systematic exposition in Petrovsky [PS64], to which we refer the reader for technical details.

The Cauchy problem. Consider the following system of k PDEs,

$$\partial_t^{n^i} u^i = F^i\left(t, x, u, \left\{\partial_t^{\alpha_0}\mathscr{D}^\alpha u^j\right\}_{|\alpha|+\alpha_0 \leqslant n^j}\right), \quad \forall i, j \in \{1, \ldots, k\}\,, \tag{4.3a}$$

where $x = \left(x^1, \ldots, x^n\right)$ is the vector of all the variables except t, $u = \left(u^1, \ldots, u^k\right)$ is the vector of all the unknown functions, $\left\{\partial_t^{\alpha_0}\mathscr{D}^\alpha u^j\right\}$, with $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_0 < n^j$, is the set of all possible partial derivatives of all the functions in u and F^i is a real-valued function of its arguments. The variable t is singled-out among the others since the derivatives $\partial_t^{n^i} u^i$ must appear among the derivatives of the highest order n^i of the function u^i and the system is solved for them. In physical systems, the independent variable t is usually the time coordinate. For some value $t = t_0$ and $\forall x \in G$, with G subspace of the space of independent variables $\left(x^1, \ldots, x^n\right)$, we prescribe values for the unknown functions and their derivatives with respect to t up to order $n^i - 1$,

$$\partial_t^m u^i(t, x)\Big|_{t=t_0} = \phi^i_{(m)}(x)\,, \qquad \forall m \in \left\{1, \ldots, n^i - 1\right\}, \quad \forall i \in \{1, \ldots, k\}\,, \tag{4.3b}$$

where the functions $\phi^i_{(m)}(x)$ are all defined (at least) on G. The prescriptions (4.3b) are referred to as the "Cauchy conditions" or as the "initial data." The Cauchy problem consists in determining the solution of the differential system (4.3a) which satisfies (4.3b).

The system of PDEs (4.3a) can always be reduced to another system which is first-order in t with respect to all the unknown functions. Here, we refer to the system obtained from (4.3a) by means of this first-order reduction as the "first-order Cauchy problem."

The generalized Cauchy problem. Consider the more general system of PDEs,

$$\Phi^i\left(x, u, \left\{\mathscr{D}^\alpha u^j\right\}_{|\alpha| \leqslant n^j}\right) = 0, \quad \forall i, j \in \{1, \ldots, k\}\,, \tag{4.4}$$

with the same notation as in (4.3a), but now all the coordinate are included in $x = \left(t, x^1, \ldots, x^n\right)$ and the multi-index notation runs over all the variables, $\alpha = (\alpha_0, \ldots, \alpha_n)$. Hence, α_0 can be equal to n^j for each j. Suppose that, in the space of the independent variables $\left(t, x^1, \ldots, x^n\right)$, a surface $\mathscr{S}$ is defined together with a congruence of curves $\mathcal{C}$ such that every curve belonging to $\mathcal{C}$ passes through at most one point of $\mathscr{S}$ and it is not tangent to it at that point. On the surface $\mathscr{S}$, we can prescribe values for the unknown functions u^i and their derivatives along the directions defined by the curves in $\mathcal{C}$. This prescription is the generalization of the Cauchy conditions, and together with (4.4) constitutes the "generalized Cauchy problem." It can be shown that the generalized Cauchy problem can be reduced to a Cauchy problem locally, that is, in a neighborhood U of $\mathscr{S}$, provided that the surface $\mathscr{S}$ is nowhere "characteristic"—a concept defined below.

Given the system (4.4), let us construct a set of smooth surfaces in the space of independent variables,

$$\psi^{\ell}(t,x) = y^{\ell}, \quad \forall \ell \in \{0,\ldots,n\}, \tag{4.5}$$

with y^{ℓ} real parameters, such that:

(i) The surface $\mathcal{S}$ has equation $y^0 = 0$.

(ii) There exists a sufficiently small $\epsilon > 0$ such that the curves in the congruence $\mathcal{C}$ are parametrized by

$$-\epsilon \leqslant y^0 \leqslant \epsilon, \quad y^1 = c_1, \cdots, y^n = c_n, \tag{4.6}$$

with the c_i real constants. The formulas (4.6) define the neighborhood U of $\mathcal{S}$.

(iii) The determinant of the Jacobian matrix with elements $J^{\ell}{}_i = \partial\psi^{\ell}/\partial x^i$, $\forall \ell, i \in \{0,\ldots,n\}$, is non-zero in the neighborhood U of the surface $\mathcal{S}$.

When these conditions are satisfied, we can use the y^{ℓ} as a set of new coordinates. We denote them with $y = \left(y^0,\ldots,y^n\right)$. Intuitively, we can choose $y^1,\ldots,y^n$ to be smooth coordinates on the surface $\mathcal{S}$, and y^0 to be the coordinate varying along the curves in $\mathcal{C}$. This is enough to guarantee that y can be used as a new system of coordinates in place of x. At this point, we change coordinates from x to y in (4.4). We are interested in rewriting the transformed system as a Cauchy problem (4.3a), that is, in solving this system for $\partial^{n^i}_{y^0} u^i$. According to the implicit function theorem [PM96, Ch. 7], this is possible if and only if the determinant of the following Jacobian matrix is non-zero in U,

$$P^i{}_j(u;y) := \frac{\partial \Phi^i}{\partial\left(\partial^{n^j}_{y^0} u^j\right)}, \qquad \det P\,(u;y) \neq 0. \tag{4.7}$$

Under the coordinate transformation from x to y,

$$\mathcal{D}^{\alpha} u^i = \partial^{\alpha}_{y^0} u^i \left(\partial_{x^0} y^0\right)^{\alpha_0} \cdots \left(\partial_{x^n} y^0\right)^{\alpha_n} + \ldots, \tag{4.8}$$

where the terms denoted with $\ldots$ include derivatives of u^i with lower order with respect to y^0, which are not important for our purpose. Hence, using the chain rule, each element in the Jacobian in (4.7) can be rewritten as,

$$\frac{\partial \Phi^i}{\partial\left(\partial^{n^j}_{y^0} u^j\right)} = \sum_{|\alpha|=n^j} \frac{\partial \Phi^i}{\partial\left(\mathcal{D}^{\alpha} u^j\right)} \frac{\partial\left(\mathcal{D}^{\alpha} u^j\right)}{\partial\left(\partial^{n^j}_{y^0} u^j\right)}, \tag{4.9}$$

and from (4.7, 4.8) follows that,

$$P^i{}_j(u;x) = \sum_{|\alpha|=n^j} \frac{\partial \Phi^i}{\partial\left(\mathcal{D}^{\alpha} u^j\right)} \left(\mathcal{D}_x y^0\right)^{\alpha}. \tag{4.10}$$

For a fixed solution $\bar{u} = (\bar{u}_1,\ldots,\bar{u}_k)$, we define the "principal symbol" of (4.4) as a map (see, e.g., [Ren08, eq. (7.4)][Tay10a, Sec. 2.9]),

$$P\left(\bar{u};\cdot,\cdot\right):\mathbb{R}^{2\,(n+1)}\to\mathbb{R}^{k\times k} \qquad (4.11)$$
$$(x,\xi)\mapsto P(\bar{u};x,\xi),\quad \xi\coloneqq(\xi_0,\ldots,\xi_n)\in\mathbb{R}^{n+1},$$

with components,

$$P^i{}_j(\bar{u};x,\xi)=\sum_{|\alpha|=n^j}\frac{\partial\Phi^i}{\partial\left(\mathscr{D}^\alpha u^j\right)}\bigg|_{u=\bar{u}}\xi^\alpha. \qquad (4.12)$$

The determinant of the principal symbol is called the "characteristic polynomial" of (4.4). Finally, the system (4.4) can be written as a Cauchy problem in the neighborhood U of $\mathscr{S}$ if and only if,

$$\det P^i{}_j\left(\bar{u};x,\mathscr{D}_x y^0\right)\neq 0, \qquad (4.13)$$

at every point on $\mathscr{S}$. The "characteristic equation" is then,

$$\det P^i{}_j\left(\bar{u};x,\xi\right)=0. \qquad (4.14)$$

The "characteristic set" is the set of all the solution (x,ξ) of the characteristic equation such that ξ is nonzero. A surface $\Psi\left(x\right)=0$ is called a "characteristic surface" for the system (4.4) if it belongs to the characteristic set, that is, if at each of its points,

$$\det P^i{}_j\left(\bar{u};x,\mathscr{D}_x\Psi\right)=0. \qquad (4.15)$$

Therefore, if $\mathscr{S}$ is nowhere characteristic and (4.13) holds everywhere in U, the system (4.4) can be reduced to a Cauchy problem in U.

We remark that, when the derivatives $\partial\Phi^i/\partial\left(\mathscr{D}^\alpha u^j\right)$ in (4.12) do not depend on u, the principal symbol itself does not depend on it, but only on the independent variables and on the function Φ^i. Since in this case the principal symbol is the same for all the solutions, it only depends on the system of PDEs itself. We will come back to this point in the next section, when talking about the classification of PDEs based on the principal symbol.

The Cauchy–Kovalevskaya theorem. It is of great importance to determine if a given system of PDEs, once the appropriate boundary conditions are specified, has a unique solution.[1] For the Cauchy problem (4.3), the Cauchy–Kovalevskaya (CK) theorem [Cau42; Kow75] (see also [PS64, p. 15][Tay10b, Theorem 4.1]) states the following. If all the functions F^i in (4.3a) are analytic in some neighborhood of the point (t_0,x_0,ϕ_0) and if all the initial data $\phi^j_{(k)}$ are analytic in some neighborhood of the point x_0, then the Cauchy problem has a

[1]Here, we use the expression "boundary conditions" in a broad sense. We refer to the conditions that are to be imposed on the unknown functions and possibly their derivatives, depending on the mathematical and physical setup under consideration. They can be boundary conditions in the strict sense, setting values at the boundary of some region, or initial conditions.

unique analytic solution in some neighborhood of the point (t_0, x_0).[2] When the system (4.4) can be reduced to a Cauchy problem, that is, when the surface $\mathcal{S}$ is nowhere characteristic, and in addition $\mathcal{S}$ is specified implicitly by an analytic function, the CK theorem also holds.

Well-posedness. The CK theorem guarantees the existence and uniqueness of the solution of a Cauchy problem, provided that the *analytic* surface $\mathcal{S}$ where the Cauchy conditions are prescribed is nowhere characteristic, and that all the functions involved in the definition of the problem are *analytic*. In physics, generalized Cauchy problems arise in many systems involving analytic equations, but non-analytic initial data. Then the CK theorem does not apply. An interesting remark is made in [PS64]. Thanks to the Stone–Weierstrass theorem [Wei85; Sto37; Sto48] (see also [Haz19]), one can choose polynomial functions that uniformly approximate the Cauchy conditions as closely as desired, on a compact surface. Upon doing that, the resulting Cauchy problem has a unique solution due to the CK theorem. It is reasonable to guess that the solution to the original system, if it exists, differs by little from the solution of the approximated problem with analytic initial data. However, Hadamard constructed a counterexample where this is not the case [Had23, p. 33], described in Appendix B.

The most general conditions under which a Cauchy problem involving smooth functions admits a unique solution have not been found yet. What has been done, though, is to define a class of Cauchy problems that satisfy the following requirements:

(i) For any smooth functions $\phi^i_{(k)}(x)$ given on the region G constituting the Cauchy conditions, there exists a unique solution to the Cauchy problem (4.3).

(ii) For an arbitrary $\epsilon > 0$ it is possible to find an $\eta > 0$ such that, if the functions $\phi^i_{(k)}$ and all of their derivatives change by less than η on G, then the solution to the Cauchy problem changes by less than ϵ in $t_0 \leqslant t \leqslant \bar{t}$, with $\bar{t}$ real constant.

Condition (ii) means that the solution depends continuously on the initial data and all of its derivatives. This definition of well-posedness is taken from [PS64] (see also [Eva10, p. 7] and [Str07, p. 25] for modern references). The more recent reference [KL89, p. 39] explains in detail how, assuming that condition (i) holds, different formulations of condition (ii) may lead to different behaviors of the system. In particular, systems featuring a continuous dependence of the solution on the derivatives of the initial data only, are problematic when dealing with equations with non-constant coefficients; such systems are called "weakly well-posed." Systems showing a continuous dependence of the solution on the initial data (and not necessarily on its derivatives) do not show the same problem, and are then called "well-posed." All other systems are called "ill-posed." The concept of well-posedness does not refer only to the Cauchy problem, but can be applied to any system of PDEs with the appropriate boundary conditions [CH62, p. 226]. It was first introduced by Hadamard in [Had02]. We remark that being able to write a systems of PDEs as a Cauchy problem is a problem separate from that of proving its well-posedness. For the first problem concerns the existence of a non-characteristic surface, whereas the second problem concerns the fulfillment

[2] An real "analytic" function is a function which can locally be expressed as a convergent power series. A function is analytic if and only if its Taylor series about a given point converges to the function in some neighborhood of the point in its domain.

of conditions (i) and (ii) above.

Since well-posed problems "behave well" by definition, it is usually desirable to write physical problems in a well-posed way. It is important to note that the same physical problem can, in principle, be written both in a well-posed and in an ill-posed way. This may happen, for example, if the system is constrained in the Hamiltonian sense (for Hamiltonian constrained systems, see, e.g., [GPS02]). Since the constraints are always satisfied, that is, they are always equal to 0, one can freely add them to the dynamical equations. This may spoil the well-posedness of the problem, or, vice versa, recast it in a well-posed way. The choice of coordinates may affect the well-posedness as well. As we will see in Chapter 5 and Chapter 6, both these factors alter the well-posedness of the EFE and the BFE.

In the next section, we will review the possible expressions that the functions F^i and Φ^i in (4.3a, 4.4) can have, and how they affect the well-posedness of the Cauchy problem.

4.2 Classification of partial differential equations

This section provides an overview of the different types and classes of PDEs, which affect the well-posedness of the differential problem. Some conditions under which a certain type of PDEs leads to a well-posed Cauchy problem are described. The EFE can be recast in different forms to satisfy these conditions, as reviewed in Appendix C. First steps toward the recasting of the BFE in a form that satisfy these conditions are made in Paper V, Paper VI and reviewed in Section 6.3.

Linear and nonlinear PDEs. A PDE of order s, with n independent variables $x = \left(x^1, \dots, x^n\right)$ and k unknown functions $u = (u^1, \dots, u^k)$ is called [Eva10]:

(i) "linear" if it is of the following form,

$$\sum_{|\alpha| \leqslant s} a_\alpha(x) \mathscr{D}^\alpha u = f(x), \tag{4.16}$$

where a_α, f are given functions. It is called "homogeneous" if $f(x) = 0$.

(ii) "semilinear" if it is of the following form,

$$\sum_{|\alpha| = s} a_\alpha(x) \mathscr{D}^\alpha u + a_0 \left(\mathscr{D}^{s-1} u, \dots, \mathscr{D} u, u, x\right) = 0, \tag{4.17}$$

(iii) "quasilinear" if it is of the following form,

$$\sum_{|\alpha| = s} a_\alpha(\mathscr{D}^{s-1} u, \dots, \mathscr{D} u, u, x) \mathscr{D}^\alpha u + a_0 \left(\mathscr{D}^{s-1} u, \dots, \mathscr{D} u, u, x\right) = 0, \tag{4.18}$$

(iv) "(fully) nonlinear" if its dependence on the derivatives of the highest order is nonlinear.

The principal symbols (4.12) of the linear equation (4.16) and the semilinear equation (4.17) are both equal to,

$$P(x, \xi) = \sum_{|\alpha| = s} a_\alpha(x) \xi^\alpha, \tag{4.19}$$

with $\xi \in \mathbb{R}^n$. Therefore, they do not depend on the solution u. This is not the case for quasilinear and fully nonlinear PDEs (and systems of PDEs).

Notions of ellipticity, parabolicity and hyperbolicity. The mathematical field of study of PDEs is rich. A variety of possible PDEs arise when modeling different physical phenomena. Their study spans more than a century, and has led to a classification of PDEs based on the algebraic properties of their principal symbol. The classification includes three main classes: "elliptic," "parabolic" and "hyperbolic," discussed by Hadamard in [Had23] for linear PDEs. The prototypes of these classes are three linear second-order PDEs; we now introduce them as concrete, simple representatives of the three classes. Suppose $u(x)$ is a smooth function of the variables $x = \left(x^1, \ldots, x^n\right)$, with $x \in U$ and $U \subset \mathbb{R}^n$ neighborhood of the origin. In the elliptic class we have the Laplace equation,

$$\triangle u = 0. \tag{4.20a}$$

Suppose that u also depends on a $(n+1)^{\text{th}}$ variable, called t and such that $t_0 \leqslant t \leqslant T$, t_0, T constants. Then, in the parabolic class there is the heat equation,

$$\partial_t u = \triangle u. \tag{4.20b}$$

In the hyperbolic class, we find the wave equation,

$$\partial_{tt} u = \triangle u. \tag{4.20c}$$

In the parabolic and hyperbolic case, the coordinate t has a special role, such that the equations are already written as a Cauchy problem. Not surprisingly, t is the time coordinate in physical systems, whereas the other coordinates are spatial. The elliptic equations describe equilibrium states, which do not depend on time by definition. Parabolic and hyperbolic equations involve time derivatives and therefore are "evolution equations," that is, they describe the evolution of a physical system in time. However, they model different physical systems. Making the substitution $t \to -t$, the heat equation (4.20b) is not invariant, but the wave equation (4.20c) is. This tells us that parabolic equation arise when describing processes whose evolution is not the same forward and backward in time, such as diffusion, transport, and creation processes like, indeed, heat propagation [Eva10, p. 372]. On the other hand, hyperbolic equations arise when describing wave-like propagation processes [Eva10, p. 399].

We now give a review of the definitions of linear and nonlinear elliptic, parabolic and hyperbolic PDEs of any order. Let us start with linear PDEs (the argument straightforwardly generalizes to semilinear PDEs, since the principal symbol is the same as for linear PDEs). Consider a differential operator of order m on a manifold $\mathscr{M}$ of dimension n,

$$P(x, \partial) u = \sum_{|\alpha| \leqslant m} a_\alpha(x) \mathscr{D}^\alpha u, \tag{4.21}$$

with $x \in U \subset \mathscr{M}$ and U open set, $a_\alpha(x)$ smooth coefficients and u vector of k unknown smooth functions. The operator $P(x, \partial)$ is said to be "elliptic" if its principal symbol

$$P(x, \xi) = \sum_{|\alpha|=m} a_\alpha(x) \xi^\alpha \tag{4.22}$$

is invertible for any non-zero $\xi \in \mathbb{R}^n$ [Tay10a]. This implies that an elliptic system has an empty characteristic set and therefore, can always be written as a Cauchy problem. We remind the reader, though, that this does not mean that such a Cauchy problem is well-posed.

There exists a stronger concept of ellipticity, called "strong ellipticity," having different definitions in the literature. We refer to [Nar85, p. 201] and define a differential operator $P(x, \partial)$ of even order $m = 2\mu$, $\mu \in \mathbb{N}$, as a "strongly elliptic" operator if its principal symbol satisfies

$$Q(\xi) := (P(x, \xi)u, u) \geqslant C|\xi|^m (u, u), \tag{4.23}$$

with $C > 0$, for all $x \in U$, $\xi \in \mathbb{R}^n$, $u \in \mathbb{R}^k$ and $(\cdot, \cdot)$ some appropriate positive definite inner product in the space of solutions. This means that the principal symbol is a positive definite operator.[3] The second-order Laplace operator $\triangle$ (or its negative, depending on conventions) is strongly elliptic [Eva10, p. 313]. A strongly elliptic operator must be of even order if $n > 1$ [Nar85, p. 201].[4]

We now turn to the concept of "strong parabolicity." A linear system of PDEs of the form

$$\partial_t u = P(x, \partial) u, \tag{4.24}$$

where $P(x, \partial)$ is a strongly elliptic differential operator of any even order, is called "strongly parabolic" [Lad85, p. 191].[5] The heat equation (4.20b) is an example of a strongly parabolic PDE.

[3]The definitions of strong ellipticity given in [Tay10a, pp. 455, 461] for even order operators, and in [KL89, p. 178] for second-order operators are compatible with the one in (4.23); however, in [KL89] strong ellipticity is defined indirectly through the definition of "strong parabolicity", which we will define soon. Moreover, the definition of strong ellipticity in [Lad85, p. 191] is slightly different than the one in (4.23). However, in all the cases, strong ellipticity requires the principal symbol not be only invertible, but also positive definite for all x in the ξ-space [Ren08, Sec. 7.1]. In [Eva10, p. 312] the concept of uniform ellipticity is defined, which is stronger than ellipticity but weaker than strong ellipticity (see also [Mai18]).

[4]Indeed, one can always choose ξ such that the real function $Q(\xi)$ is a polynomial of order m. Suppose to parametrize $\xi = a + \lambda b$, $a, b \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$. Then, if m was odd, the polynomial $Q(a + \lambda b)$ in λ would have a real zero. If this zero was trivial for $Q(a + \lambda b)$, (4.23) would still hold. If $n = 1$, it is not possible to find a, b such that $a + \lambda b \neq 0 \ \forall \lambda \in \mathbb{R}$, since there will always be $\lambda = -a/b$. Hence for $n = 1$, that is, one independent variable only, $Q(a + \lambda b)$ only has a trivial zero meaning that $Q(a + \lambda b) \iff \xi = a + \lambda b = 0$. However, for $n > 1$, one can find $a, b \in \mathbb{R}^n$ such that $a + \lambda b \neq 0 \ \forall \lambda \in \mathbb{R}$. As an example, consider $a = (1, 0, \ldots, 0)$ and $b = (1/2, 0, 1/2, 0, \ldots, 0)$. Then we have $a + \lambda b = (1 + \lambda/2, 0, \lambda/2, 0, \ldots, 0)$ and there is no λ such that $a + \lambda b$ is the zero vector. Hence, $Q(a + \lambda b)$, which must have a zero because m is odd, has actually a nontrivial zero, that is, $Q(a + \lambda b) = 0$ with $\xi = a + \lambda b \neq 0 \ \forall \lambda \in \mathbb{R}$. In this case, (4.23) does not hold. Note that this counterexample holds in any odd m.

[5]See also [Ren08, Sec. 8.10]. The definition is compatible with [KL89, p. 178, 197] and [Eva10, p. 372] for second-order strongly parabolic systems (called only "parabolic" in [Eva10]). Compare with the definition of second-order parabolic systems in [KL89, p. 180], which reads $\lambda_i \geqslant C|\xi|^2$ for all $x \in U$, $\xi \in \mathbb{R}^n$, with λ_i eigenvalues of the principal symbol $P(x, \xi)$.

Lastly, a linear system of PDEs of the form

$$\partial_{tt} u = P(x, \partial)u, \tag{4.25}$$

where $P(x, \partial)$ is a strongly elliptic differential operator of second order, is called "strongly hyperbolic" [Lad85, p. 192][Eva10, p. 399]. These systems can be interpreted as a generalization of the wave equation (4.20c).

It is possible to encounter hyperbolic systems where the time derivatives are first-order, not second-order as in (4.25). Actually, in the rest of the thesis, we are going to encounter such first-order systems of equations when dealing with the EFE and BFE, and to consider the well-posedness of the resulting Cauchy problem. Hence, in the next paragraph, we define the concept of strong hyperbolicity for linear systems with first-order derivatives both in the time and spatial variables.

As a final remark, we note that the classification built for linear systems of PDEs can be generalized to nonlinear systems by means of the linearization around a given solution [Tay10b, p. 127], which leads to the principal symbol in (4.12). The linearized system depends on the chosen background solution for quasilinear and fully nonlinear systems; hence, in these cases the class of the system of PDEs depends on the background solution at the same point x [PM06; Ren08; Tay10b].

Hyperbolicity of first-order systems and well-posedness of the Cauchy problem. The parabolic and hyperbolic PDEs in (4.24, 4.25) have the form required in the definition of the Cauchy problem (4.3a). Hence, we can consider the Cauchy problems arising from these PDEs and the desired initial data at some initial time $t = t_0$. It is natural to ask about the well-posedness of such a Cauchy problem. The Cauchy problem involving a strongly parabolic system is well-posed for $t \geqslant t_0$ [Lad85, p. 191][Ren08, Sec. 8.10], providing formal mathematical support to the intuition about parabolic PDEs as describing systems whose evolution is different forward and backward in time.

In the previous paragraph, we defined strong hyperbolicity for second-order systems. Now we define the same concept for first-order systems. Consider the first-order, linear system of PDEs in n spatial variables plus the time variable,

$$L(t, x, \partial)u := \partial_t u - A(t, x, \partial)u - B(t, x)u = 0 \tag{4.26a}$$

$$A(t, x, \partial)u := \sum_{j=1}^{n} A_j(t, x)\partial_j u, \tag{4.26b}$$

where $u = \left(u^1, \ldots, u^k\right)$, $x = (x^1, \ldots, x^n) \in U \subset \mathbb{R}^n$, and $A_j(t, x)$, $B(t, x)$ are $k \times k$ matrix valued functions defined in a neighborhood U of the origin in $\mathbb{R}^{n+1}$. We define the following two differential operators,

$$L_1(t, x, \partial) := \partial_t - A(t, x, \partial), \qquad L_0(t, x) := B(t, x), \tag{4.27}$$

where $L_0(t, x)$ includes all the lower order terms, that is, non-derivative terms. We say that

$L(t, x, \partial)$ is "hyperbolic" in U if the Cauchy problem,

$$L\,(t, x, \partial)\, u = f(t, x), \quad f(t < t_0) = 0, \tag{4.28a}$$

$$u(t < t_0) = 0 \tag{4.28b}$$

is well-posed [Nis03]. We say that $L_1(t, x, \partial)$ is "strongly hyperbolic" if the Cauchy problem (4.28) is well-posed for arbitrary lower order term $B(t, x)$ [Nis03]. In the case of constant coefficients, the authors of [KY60] proved that $L_1(t, x, \partial)$ is strongly hyperbolic if and only if the matrix $A(\xi)$ is uniformly diagonalizable, that is, if all of the roots of its determinant are real for any $\xi \in \mathbb{R}^n$ and there exists a matrix $N(\xi)$ with nonvanishing determinant and unit row vectors (with respect to some appropriate norm) and such that $N(\xi)A(\xi)N(\xi)^{-1}$ is diagonal. Many authors during the last century studied the conditions under which a first-order linear system with non-constant coefficients is strongly hyperbolic—see [Nis03] and references therein for a complete historical account on this study. These conditions were established by Kreiss and Lorenz [KL89, p. 186]. For (4.28) to be well-posed or, equivalently, for $L_1(t, x, \partial)$ to be strongly hyperbolic, the eigenvalues of $A(t, x, \xi)$ need to be all real and the eigenvectors must be linearly independent and smooth functions of (t, x, ξ) (see also [Ren08, Sec. 8.4] and [Eva10, p. 421]).

In addition to strong hyperbolicity, there are weaker and stronger concepts of hyperbolicity for linear first-order systems of PDEs, given below: $\forall x \in U$, $\forall t \geqslant 0$, $\forall \xi \in \mathbb{R}^n, \xi \neq 0$, the system (4.26a) is called [KL89, p. 57][Eva10, p. 422]:

(i) "Weakly hyperbolic" in U if the eigenvalues of $A(t, x, \xi)$ are real.
(ii) "Symmetric hyperbolic" in U if $A_j(t, x) = A_j(t, x)^\top$ $\forall j = 1, \ldots, s$.
(iii) "Strictly hyperbolic" in U if the eigenvalues of $A(t, x, \xi)$ are real and distinct.

Weakly hyperbolic systems lead to ill-posed Cauchy problems [KL89, p. 30]. Symmetric and strictly hyperbolic systems are special cases of strongly hyperbolic systems. Indeed, symmetric hyperbolic systems have a symmetric principal symbol, which can then be uniformly diagonalized by an orthogonal transformation and has real eigenvalues; strictly hyperbolic systems have a principal symbol with a complete set of eigenvectors, which makes it uniformly diagonalizable.

The concept of strict hyperbolicity is a bit restrictive. As an example, let us consider the wave equation. It is a strictly hyperbolic equation. However, a system of two decoupled wave equations is not strictly hyperbolic [Ren08, Sec. 8.3]. On the contrary, two symmetric hyperbolic systems taken together form a symmetric hyperbolic system. An example of a symmetric hyperbolic system is constituted by the Maxwell equations written in terms of the electric and magnetic fields [Ren08, Sec. 8.3]. Also, it can be shown that a strictly hyperbolic system in 3 space dimensions must have at least a 7 equations [KL89]. In conclusion, strictly hyperbolic systems are unlikely to appear in realistic physical systems.

The strong and symmetric hyperbolicity of systems with first-order time derivatives and second-order spatial derivatives was studied in detail in [GM06]. Possible definitions of hyperbolicities for such systems refer to their first-order reduction: if the latter has a certain type of hyperbolicity, then the original system shares it as well. In [GM06], criteria are given to determine the type of hyperbolicity of the original system without the need to reduce it to a first-order system also in the spatial derivatives.

Chapter 5

The Einstein equations as a Cauchy problem

After having discussed the relevant aspects of the theory of PDEs, we now describe a formulation of the EFE resulting in a system of PDEs determining a Cauchy problem. That is, the $3+1$ decomposition of the spacetime and the EFE, which is the topic of this chapter.

5.1 The foliation of spacetime

In this section we review some elements of differential geometry needed to write down the $3+1$ decomposition of the EFE. We closely follow the reference [Gou12].

Hypersurfaces and induced metric. Consider a differentiable manifold $\mathscr{M}$ of dimension 4 with a metric g of signature $(-,+,+,+)$. The couple $(\mathscr{M}, g)$ is called a Lorentzian manifold. We assume that $\mathscr{M}$ is time orientable according to [Wal10, Sec. 8.1]. Consider a 3-dimensional manifold $\hat{\Sigma}$ and an embedding $\Phi : \hat{\Sigma} \to \mathscr{M}$.[1] The image $\Sigma = \Phi(\hat{\Sigma}) \subset \mathscr{M}$ is called a hypersurface of $\mathscr{M}$.

$\hat{\Sigma}$ being a 3-dimensional manifold, we can consider its tangent space $T_p(\hat{\Sigma})$ at a certain arbitrary point $p \in \hat{\Sigma}$, and ask how an arbitrary tangent vector $v \in T_p(\hat{\Sigma})$ is mapped to the tangent space of $\mathscr{M}$ at $\Phi(p)$. This mapping induced by Φ is called "pushforward," is denoted by Φ_* and, for any smooth function $f : \mathscr{M} \to \mathbb{R}$, acts as follows,

$$\Phi_*(v)(f) := v(f \circ \Phi), \tag{5.1}$$

where we remind the reader that $v(f)$ gives the directional derivative of f in the direction of v, and $\circ$ denotes the composition of functions. The image $\Phi_*(v)$ of v is a tangent vector belonging to $T_{\Phi(p)}(\mathscr{M})$. In addition to the pushforward, there exists a linear map between linear forms induced by Φ, called "pullback." It is denoted Φ^* and defined by,

$$\Phi^*(\omega)(v) := \omega\big(\Phi_*(v)\big) \tag{5.2}$$

[1] An embedding is a homeomorphism onto its image. A homeomorphism is a continuous function whose inverse is also continuous.

The definition of pull-back can be extended to an arbitrary multilinear form T as,

$$\forall(v_1,\ldots,v_n)\in T_p(\hat{\Sigma})^n,\quad \Phi^*T(v_1,\ldots,v_n)=T(\Phi_*v_1,\ldots,\Phi_*v_n). \tag{5.3}$$

It is not necessary to consider a generic embedding Φ in what follows, hence we assume it to be the identity, $\Sigma=\hat{\Sigma}$. Therefore, the pushforward and the pullback become maps acting as follows,

$$\Phi_*:T_p(\Sigma)\to T_p(\mathscr{M}),\quad \Phi^*:T_p^*(\mathscr{M})^{\otimes n}\to T_p^*(\Sigma)^{\otimes n}, \tag{5.4}$$

with $\otimes n$ the n^{th} tensor power.

The pullback of the metric g is called the "first fundamental form" or "induced metric" of Σ,

$$\gamma\equiv\Phi^*(g):T_p(\Sigma)\times T_p(\Sigma)\to\mathbb{R}. \tag{5.5}$$

The induced metric γ is also called the 3-metric, since it is a metric on the 3-dimensional hypersurface Σ. Depending on the signature of γ, Σ is called:

(i) "spacelike" if γ is Riemannian with signature $(+,+,+)$;
(ii) "timelike" if γ is Lorentzian with signature $(-,+,+)$;
(iii) "null" if γ is degenerate.

The induced metric has a unique Levi–Civita connection D, which allows to compute covariant derivatives of tensors defined in $T(\Sigma)^{\otimes n}\otimes T^*(\Sigma)^{\otimes m}$.

Normal vector field. A hypersurface can be locally defined as the set of points for which a scalar field defined on $\mathscr{M}$ is constant. Hence, denoting this scalar field $\mathcal{T}$,

$$\forall p\in\mathscr{M},\quad p\in\Sigma\Longleftrightarrow\mathcal{T}(p)=0. \tag{5.6}$$

The scalar field $\mathcal{T}$ is also called the "time function." The gradient 1-form $\nabla\mathcal{T}$ is normal to Σ, that is, $\nabla\mathcal{T}(v)=0\ \forall v$ tangent to Σ, as follows from multivariate calculus. The vector $\overrightarrow{\nabla\mathcal{T}}$ dual to $\nabla\mathcal{T}$, having components $\overrightarrow{\nabla\mathcal{T}}^\mu=g^{\mu\nu}(\nabla\mathcal{T})_\mu$, is a vector normal to Σ with character:

(i) timelike if and only if Σ is spacelike;
(ii) spacelike if and only if Σ is timelike;
(iii) null if and only if Σ is null.

When $\overrightarrow{\nabla\mathcal{T}}$ is not null, we can normalize it,

$$\nu:=-\alpha\,\overrightarrow{\nabla\mathcal{T}},\quad \alpha:=\left(\pm\overrightarrow{\nabla\mathcal{T}}\cdot\overrightarrow{\nabla\mathcal{T}}\right)^{-1/2},\quad \nu\cdot\nu=\pm1, \tag{5.7}$$

where the $+(-)$ sign holds for a timelike(spacelike) hypersurface.[2] We will come back to the meaning of the scalar field α later.

The extrinsic curvature. The normal vector field ν has a unit norm by definition. Hence, its derivative along a vector field v tangent to Σ must be orthogonal to ν. This can

[2]When $\overrightarrow{\nabla\mathcal{T}}$ is null, there is no privileged normalization.

be easily verified,

$$\nu \cdot \nabla_v \nu = \frac{1}{2}\big(\nu \cdot \nabla_v \nu + \nabla_v \nu \cdot \nu\big) = \frac{1}{2}\nabla_v\big(\nu \cdot \nu\big) = 0, \tag{5.8}$$

since $\nu \cdot \nu$ is constant. This means that the vector field $\nabla_v \nu$ belongs to $T(\Sigma)$. Hence, we define the rank-2 tensor K as,

$$\begin{aligned} \forall p \in \Sigma, \quad K : T_p(\Sigma) \times T_p(\Sigma) &\to \mathbb{R} \\ (u, v) &\mapsto K(u, v) := -u \cdot \nabla_v \nu. \end{aligned} \tag{5.9}$$

The tensor K is symmetric and is called the "second fundamental form" or the "extrinsic curvature" of the hypersurface Σ.

Since ν is a unit vector field, its covariant derivative along v tells us how much it bends in the direction defined by v, or equivalently, how much the hypersurface bends in that direction. The adjective "extrinsic" comes from the fact that it is defined in terms of ν, which is a vector field defined in the spacetime and not on $T(\Sigma)$; in addition, the definition of ν requires $\overrightarrow{\nabla \mathcal{T}}$, which is the dual to $\nabla \mathcal{T}$ and requires the spacetime metric g to be defined. The "intrinsic curvature" of the hypersurface Σ is the Ricci curvature scalar R obtained from the Riemann curvature tensor $R^{\mu}{}_{\nu\rho\sigma}$ specified by the Levi–Civita connection D of γ.

The importance of the induced metric γ and the extrinsic curvature K resides in them being the dynamical variables in the $3+1$ decomposition of the EFE, as we shall see soon.

Projector on a spacelike hypersurface. In the 3+1 formalism of GR, only spacelike hypersurfaces are considered. From now on, we limit our analysis to them. Hence, the induced metric γ is Riemannian and called the "spatial metric," $\nu \cdot \nu = -1$ and the vector and tensor fields tangent to Σ are called "spatial vectors/tensors."

Since ν is normal to Σ, the tangent space $T_p(\mathcal{M})$ at each $p \in \Sigma$ can be orthogonally decomposed as,

$$T_p(\mathcal{M}) = T_p(\Sigma) \oplus \mathrm{span}(\nu_p). \tag{5.10}$$

This means we can define the "orthogonal projector" $\perp$ onto $T_p(\Sigma)$ as,

$$\begin{aligned} \perp : T_p(\mathcal{M}) &\to T_p(\Sigma) \\ v &\mapsto v + (\nu \cdot v)\nu. \end{aligned} \tag{5.11}$$

The components of the orthogonal projector can be readily written,

$$\perp^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \nu^{\mu}\nu^{\rho} g_{\rho\nu}. \tag{5.12}$$

The orthogonal projector annihilates every vector normal to Σ,

$$\perp(\nu) = \nu + (\nu \cdot \nu)\nu = \nu - \nu = 0, \tag{5.13}$$

and acts as the identity map on an arbitrary spatial vector u orthogonal to ν,

$$\perp(u) = u + (u \cdot \nu)\nu = u. \tag{5.14}$$

The orthogonal projector can be seen as the map reverse to the pushforward in (5.4). It is also possible to define the reverse map to the pullback in (5.4),

$$\begin{aligned} \perp^* : T_p^*(\Sigma) &\to T_p^*(\mathscr{M}) \\ \omega &\mapsto \omega\big(\perp(\cdot)\big), \end{aligned} \tag{5.15}$$

which gives a 1-form in $T_p^*(\mathscr{M})$. This map can be extended to multilinear forms,

$$\begin{aligned} \perp^* : T_p^*(\Sigma)^{\otimes n} &\to T_p^*(\mathscr{M})^{\otimes n} \\ A &\mapsto A\big(\perp(\cdot), \dots, \perp(\cdot)\big). \end{aligned} \tag{5.16}$$

In particular, we can apply $\perp^*$ to the induced metric γ, to extend it to the tangent vectors in $T_p(\mathscr{M})$,

$$\begin{aligned} \perp^* \gamma : T_p(\mathscr{M}) \times T_p(\mathscr{M}) &\to \mathbb{R} \\ (u, v) &\mapsto \gamma\big(\perp(u), \perp(v)\big). \end{aligned} \tag{5.17}$$

We can compute the components of $\perp^* \gamma$ in a basis (ν, v_1, v_2, v_3) with v_1, v_2, v_3 linearly independent spatial vectors,

$$(\perp^* \gamma)(\nu, \nu) = \gamma\big(\perp(\nu), \perp(\nu)\big) = \gamma\big(\vec{0}, \vec{0}\big) = 0, \tag{5.18a}$$

$$(\perp^* \gamma)(\nu, v_i) = \gamma\big(\perp(\nu), \perp(v_i)\big) = \gamma\big(\vec{0}, v_i\big) = 0, \tag{5.18b}$$

$$(\perp^* \gamma)(v_i, v_j) = \gamma\big(\perp(v_i), \perp(v_j)\big) = \gamma\big(v_i, v_j\big) = \gamma_{ij}. \tag{5.18c}$$

Now, if we lower the index μ in (5.12),

$$g_{\alpha\rho} \perp^\rho{}_\beta = g_{\alpha\rho}\delta^\rho{}_\beta + g_{\alpha\rho}\nu^\rho\nu_\beta = g_{\alpha\beta} + \nu_\alpha\nu_\beta. \tag{5.19}$$

and contract the indices in (5.19) with the basis vectors (ν, v_1, v_2, v_3), we get,

$$\nu^\alpha g_{\alpha\beta}\nu^\beta + \nu^\alpha\nu_\alpha\nu_\beta\nu^\beta = \nu_\beta\nu^\beta - \nu_\beta\nu^\beta = -1 + 1 = 0, \tag{5.20a}$$

$$\nu^\alpha g_{\alpha\beta}v_i^\beta + \nu^\alpha\nu_\alpha\nu_\beta v_i^\beta = \nu_\beta v_i^\beta - \nu_\beta v_i^\beta = 0 - 0 = 0, \tag{5.20b}$$

$$v_i^\alpha g_{\alpha\beta}v_j^\beta + v_i^\alpha\nu_\alpha\nu_\beta v_j^\beta = (v_i)_\beta(v_j)^\beta = \gamma_{ij}. \tag{5.20c}$$

The comparison between (5.18) and (5.20) tell us that,

$$g_{\alpha\rho} \perp^\rho{}_\beta = g_{\alpha\beta} + \nu_\alpha\nu_\beta = (\perp^* \gamma)_{\alpha\beta}. \tag{5.21}$$

In index-free notation we have,

$$\perp^* \gamma = g + \nu \otimes \nu = g \perp . \tag{5.22}$$

It is common practice to use the same symbol for γ and $\perp^* \gamma$; from now on, we will denote both of them with γ. The distinction will be made by the indices: Greek letters denote spacetime indices, whereas Latin letters denote spatial indices. Consistently, γ_{ij} refers to the induced metric in (5.5), whereas $\gamma_{\mu\nu}$ refers to its extension in (5.17). The same convention is used for all other spatial tensors and their extensions.

From (5.21) it follows that $\gamma_{\mu\nu}$ and $g_{\mu\nu}$ act in the same way on spacetime vectors u, v tangent to Σ,

$$\gamma_{\mu\nu} u^\mu v^\nu = g_{\mu\nu} u^\mu v^\nu + \nu_\mu \nu_\nu u^\mu v^\nu = g_{\mu\nu} u^\mu v^\nu. \tag{5.23}$$

As the last step in this paragraph, we compute a quantity needed later, namely,

$$\perp^\mu{}_\alpha \perp^\nu{}_\beta g_{\mu\nu} = \perp^\mu{}_\alpha \gamma_{\mu\beta} = \delta^\mu{}_\alpha \gamma_{\mu\beta} + \nu^\mu \nu_\alpha \gamma_{\mu\beta} = \gamma_{\alpha\beta}, \tag{5.24}$$

where we used (5.12, 5.21).

The Gauss and Codazzi relations. The $3+1$ decomposition of the EFE makes use of three mathematical relations concerning the decomposition of the 4-dimensional Riemann curvature tensor ${}^4R^\alpha{}_{\beta\gamma\delta}$ in terms of the induced metric $\gamma_{\mu\nu}$, the 3-dimensional Riemann tensor $R^\alpha{}_{\beta\gamma\delta}$ and the extrinsic curvature $K_{\mu\nu}$. We will state these relations only, and refer the reader to [Gou12, Subsec. 3.5.1] for their proofs. In this paragraph we state two of them, since the third one makes use of concepts introduced later.

The first relation is called "Gauss relation" and reads,

$${}^4R^\rho{}_{\sigma\mu\nu} \perp^\mu{}_\alpha \perp^\nu{}_\beta \perp^\gamma{}_\rho \perp^\sigma{}_\delta = R^\gamma{}_{\delta\alpha\beta} + K^\gamma{}_\alpha K_{\delta\beta} - K^\gamma{}_\beta K_{\alpha\delta}. \tag{5.25}$$

The contraction of the indices γ and α in (5.25) results in,

$${}^4R_{\mu\nu} \perp^\mu{}_\alpha \perp^\nu{}_\beta + {}^4R^\mu{}_{\nu\rho\sigma} \gamma_{\alpha\mu} \nu^\nu \perp^\rho{}_\beta \nu^\sigma = R_{\alpha\beta} + K K_{\alpha\beta} - K^\mu{}_\beta K_{\alpha\mu}, \tag{5.26}$$

called the "contracted Gauss relation." Taking the trace of (5.26) with respect to γ results in,

$${}^4R + 2\, {}^4R_{\mu\nu} \nu^\mu \nu^\nu = R + K^2 - K_{ij} K^{ij}, \tag{5.27}$$

which is called the "scalar Gauss relation."

The second relation is called "Codazzi relation" and reads,

$${}^4R^\rho{}_{\sigma\mu\nu} \perp^\gamma{}_\rho \nu^\sigma \perp^\mu{}_\alpha \perp^\nu{}_\beta = D_\beta K^\gamma{}_\alpha - D_\alpha K^\gamma{}_\beta. \tag{5.28}$$

The contraction of α and γ in (5.28) gives,

$$^4R_{\mu\nu} \perp^{\mu}{}_{\alpha} \nu^{\nu} = D_\alpha K - D_\mu K^{\mu}{}_{\alpha}, \tag{5.29}$$

referred to as the "contracted Codazzi relation."

We are going to use these relations in Section 5.2 when projecting the EFE onto Σ and span(ν).

Foliation of a manifold. A set of hypersurfaces $\{\Sigma_t\}_{t\in\mathbb{R}}$ such that there is a smooth scalar field $\mathcal{T}$ on $\mathcal{M}$, with nonvanishing gradient, such that,

$$\forall t \in \mathbb{R}, \quad \Sigma_t := \{p \in M, \mathcal{T}(p) = t\}, \tag{5.30}$$

is called a "foliation" or "slicing." Each hypersurface Σ_t is called a "leaf" or "slice" of the foliation. The assumption of nonvanishing gradient for the scalar field $\mathcal{T}$ implies that the slices do not intersect,

$$t \neq t' \Longrightarrow \Sigma_t \cap \Sigma_{t'} = \varnothing. \tag{5.31}$$

Global hyperbolicity. Consider a spacelike hypersurface $\Sigma \subset \mathcal{M}$. If every timelike or null curve without end intersects Σ in one and only one point, then the spacelike hypersurface is called a "Cauchy surface." A spacetime admitting a Cauchy surface is said to be "globally hyperbolic."[3]

The definition of globally hyperbolic spacetime implies that its topology has to be $\mathbb{R} \times \Sigma$, where Σ is the Cauchy surface and $\mathbb{R}$ is the topology of span(ν), that is, of the timelike direction. An example of a non globally hyperbolic spacetime is AdS, which has topology $S \times \mathbb{R}^3$. Given its topology, a globally hyperbolic spacetime can be foliated by a family of spacelike hypersurface $\{\Sigma_t\}_{t\in\mathbb{R}}$, and in particular,

$$\mathcal{M} = \bigcup_{t\in\mathbb{R}} \Sigma_t. \tag{5.32}$$

This means that, if a spacetime admits a global $3+1$ decomposition, then it is globally hyperbolic, and vice versa.

The lapse function and the normal evolution vector. In (5.7), we introduced the scalar field α, denoted "lapse function" by Wheeler [Whe64]. The definition of the lapse function in (5.7) in the case of a spacelike (and timelike) hypersurface, tells us that

$$\alpha > 0, \tag{5.33}$$

unless $\nabla\mathcal{T} = 0$, that is, unless the foliation is not regular.

[3] The name "globally hyperbolic" follows from the fact that the scalar wave equation has a well-posed Cauchy problem in these spacetimes [Cho09].

We now define the "normal evolution vector" as,

$$\mu \coloneqq \alpha\, \nu = -\alpha^2\, \overrightarrow{\nabla \mathcal{T}}. \tag{5.34}$$

The normal evolution vector is timelike, normal to Σ_t $\forall t \in \mathbb{R}$ and its square norm is equal to $-\alpha^2$. Consider a point $p \in \Sigma_t$ and its infinitesimal displacement $p' = p + \mu\, \delta t$, $\delta t \ll 1$. The value of $\mathcal{T}(p')$ is then,

$$\mathcal{T}(p') = \mathcal{T}(p + \mu\delta t) \simeq \mathcal{T}(p) + \delta t\, \mu^\mu\, \nabla_\mu \mathcal{T}. \tag{5.35}$$

Since

$$\mu^\mu\, \nabla_\mu \mathcal{T} = -\alpha^2\, \overrightarrow{\nabla \mathcal{T}}^{\,\mu} \nabla_\mu \mathcal{T} = \frac{-\alpha^2}{-\alpha^2} = 1, \tag{5.36}$$

where we used (5.7), it follows

$$\mathcal{T}(p') \simeq \mathcal{T}(p) + \delta t = t + \delta t. \tag{5.37}$$

This means that the displaced point p' belongs to $\Sigma_{t+\delta t}$, that is, the vector $\mu\, \delta t$ drags the point p to the neighboring hypersurface. If we take the scalar field $\mathcal{T}$ as the time coordinate (as we shall do later), we say that the point p has evolved from t to $t + \delta t$.

We now expand on the physical meaning of the lapse function. Since ν is a unit vector, it can be promoted to be the 4-velocity of an observer. Such an observer is called "Eulerian" and its wordlines are orthogonal to Σ_t $\forall t \in \mathbb{R}$, by definition. Consider two events p and p' belonging to the wordline of an Eulerian observer, and such that $p \in \Sigma_t$, $p' \in \Sigma_{t+\delta t}$. We saw above that $p' = p + \mu\, \delta t$. Then, the proper time $\delta\tau$ separating p and p' is

$$\delta\tau = \sqrt{-g_{\rho\sigma}\, \delta t\, \mu^\rho\, \delta t\, \mu^\sigma} = \delta t \sqrt{-\mu^\rho \mu_\rho} = \delta t\, \sqrt{\alpha^2} = \alpha\, \delta t, \tag{5.38}$$

since $\alpha > 0$ by definition, for a regular foliation. Now we can better understand the name "lapse function" given to α; it quantifies the ratio between the coordinate time and the proper time measured by an Eulerian observer between two neighboring events.

The evolution of the spatial metric. We saw that a displacement in the direction of μ drags a point p to the neighboring slice. Mathematically, this can be extended to any tensor by means of the Lie derivative. The variation of any spatial tensor T between neighboring hypersurfaces of the foliation is equal to its Lie derivative along μ, which is a spatial tensor itself [Gou12, eq. (4.12)],

$$\forall t \in \mathbb{R}, \quad \forall p \in \Sigma_t, \quad \forall T \in T_p(\Sigma_t)^{\otimes n} \otimes T^*_p(\Sigma_t)^{\otimes m}, \quad \mathscr{L}_{\mu} T \in T_p(\Sigma_t)^{\otimes n} \otimes T^*_p(\Sigma_t)^{\otimes m}. \tag{5.39}$$

In particular, the Lie derivative of the spatial metric $\gamma_{\mu\nu}$ along μ turns out to be equal to [Gou12, eq. (4.30)]

$$\mathscr{L}_{\mu} \gamma_{\mu\nu} = -2\alpha K_{\mu\nu}. \tag{5.40}$$

We refer to this variation as the "evolution" of the spatial metric.

Since $\mu = \alpha\,\nu$, (5.40) is equivalent to

$$K_{\mu\nu} = -\frac{1}{2}\mathscr{L}_{\nu}\gamma_{\mu\nu}. \tag{5.41}$$

This provides us with an intuitive understanding of the extrinsic curvature, that is, it quantifies the change of the spatial metric along the direction normal to the spacelike hypersurface.

The Ricci relation. The third mathematical equation we need to establish the $3+1$ decomposition of the EFE is the "Ricci relation," which gives the expression of another projection of the 4-dimensional Riemann tensor. It reads [Gou12, Subsec. 4.4.1],

$${}^4R^{\mu}{}_{\rho\nu\sigma}\gamma_{\alpha\mu}\nu^{\rho}\perp^{\nu}{}_{\beta}\nu^{\sigma} = \frac{1}{\alpha}\mathscr{L}_{\mu}K_{\alpha\beta} + \frac{1}{\alpha}D_{\alpha}D_{\beta}\alpha + K_{\alpha\mu}K^{\mu}{}_{\beta}. \tag{5.42}$$

The left-hand side of (5.42) can be replaced by making use of (5.26),

$${}^4R_{\mu\nu}\perp^{\mu}{}_{\alpha}\perp^{\nu}{}_{\beta} = -\frac{1}{\alpha}\mathscr{L}_{\mu}K_{\alpha\beta} - \frac{1}{\alpha}D_{\alpha}D_{\beta}\alpha + R_{\alpha\beta} + KK_{\alpha\beta} - 2K_{\alpha\mu}K^{\mu}{}_{\beta}. \tag{5.43}$$

5.2 The $3+1$ decomposition of the Einstein equations

After having introduced the geometrical background in Section 5.1, in this section we review the $3+1$ decomposition of the EFE and discuss some of its features as a system of PDEs.

The projections of the stress–energy tensor. The unit normal ν is the 4-velocity of an Eulerian observer. Hence, given a stress–energy tensor $T_{\mu\nu}$, the matter energy density measured by such an observer is (see, e.g., [RZ13, Sec. 3.2]),

$$\rho^{\mathrm{m}} := T_{\mu\nu}\nu^{\mu}\nu^{\nu}. \tag{5.44}$$

The matter momentum density measured by the Eulerian observer is,

$$j^{\mathrm{m}}{}_{\alpha} := -T_{\mu\nu}\perp^{\mu}{}_{\alpha}\nu^{\nu}, \tag{5.45}$$

and is tangent to Σ_t. Due to the symmetry of the stress–energy tensor, the matter momentum density is equal to the energy flux,

$$\phi_{\alpha} := -T_{\mu\nu}\nu^{\mu}\perp^{\nu}{}_{\alpha} = -T_{\mu\nu}\perp^{\mu}{}_{\alpha}\nu^{\nu} = j^{\mathrm{m}}{}_{\alpha}. \tag{5.46}$$

Finally, the stress tensor measured by the Eulerian observer is,

$$J^{\mathrm{m}}{}_{\alpha\beta} := T_{\mu\nu}\perp^{\mu}{}_{\alpha}\perp^{\nu}{}_{\beta}, \tag{5.47}$$

tangent to Σ_t. Knowing $\rho^{\mathrm{m}}, j^{\mathrm{m}}{}_\alpha$ and $J^{\mathrm{m}}{}_{\alpha\beta}$, it is possible to reconstruct $T_{\alpha\beta}$ as follows,

$$T_{\alpha\beta} = \rho^{\mathrm{m}}\, \nu_\alpha \nu_\beta + \nu_\alpha\, j^{\mathrm{m}}{}_\beta + j^{\mathrm{m}}{}_\alpha\, \nu_\beta + J^{\mathrm{m}}{}_{\alpha\beta}. \tag{5.48}$$

Equation (5.48) is the $3+1$ decomposition of the stress–energy tensor. The trace of $T_{\alpha\beta}$ becomes,

$$\begin{aligned} T &= g^{\alpha\beta}\rho^{\mathrm{m}}\nu_\alpha\nu_\beta + g^{\alpha\beta}\nu_\alpha j^{\mathrm{m}}{}_\beta + g^{\alpha\beta} j^{\mathrm{m}}{}_\alpha\nu_\beta + g^{\alpha\beta} J^{\mathrm{m}}{}_{\alpha\beta} \\ &= -\rho^{\mathrm{m}} + 2 j^{\mathrm{m}}{}_\mu \nu^\mu + J^{\mathrm{m}} = J^{\mathrm{m}} - \rho^{\mathrm{m}}. \end{aligned} \tag{5.49}$$

The projections of the Einstein equations. The EFE in Table 1.1, after setting $\Lambda_{\mathrm{B}} = 0$ for the sake of simplicity (it can be easily restored at the end of the computation, or it can be thought as a part of the stress–energy tensor), can be rewritten as,

$${}^4R_{\mu\nu} = 8\pi\Big(T_{\mu\nu} - \frac{1}{2}\,T^\alpha{}_\alpha\, g_{\mu\nu}\Big). \tag{5.50}$$

Consider a spacelike hypersurface Σ_t, with $t \in \mathbb{R}$, and assume that the spacetime can be foliated by a family of spacelike hypersurfaces in a neighborhood of Σ_t. Notice that we are not assuming global hyperbolicity here, that is, we are assuming that the spacetime admits a foliation in a neighborhood of Σ_t only. This does not restrict the space of solutions that we can obtain *locally* using the $3+1$ formulation.

Consider the projection of the EFE (5.50) onto Σ_t,

$${}^4R_{\mu\nu} \perp^\mu{}_\alpha \perp^\nu{}_\beta = 8\pi\Big(\perp^\mu{}_\alpha \perp^\nu{}_\beta\, T_{\mu\nu} - \frac{1}{2}\,T^\alpha{}_\alpha \perp^\mu{}_\alpha \perp^\nu{}_\beta\, g_{\mu\nu}\Big). \tag{5.51}$$

These projections are all known—see (5.24, 5.43, 5.47, 5.49)—hence we can replace them and obtain,

$$\begin{aligned} \mathcal{L}_{\mu} K_{\alpha\beta} &= -D_\alpha D_\beta \alpha \\ &\quad + \alpha\Big\{R_{\alpha\beta} + K K_{\alpha\beta} - 2K_{\alpha\mu}K^\mu{}_\beta + 4\pi\big[(J^{\mathrm{m}} - \rho^{\mathrm{m}})\,\gamma_{\alpha\beta} - 2J^{\mathrm{m}}{}_{\alpha\beta}\big]\Big\}. \end{aligned} \tag{5.52}$$

All tensors in (5.52) are tangent to Σ_t and therefore are the extensions of spatial tensors defined on $T(\Sigma)$. Therefore, we can replace the spacetime indices with the spatial indices without losing any information,

$$\begin{aligned} \mathcal{L}_{\mu} K_{ij} &= -D_i D_j \alpha \\ &\quad + \alpha\Big\{R_{ij} + K K_{ij} - 2K_{ik}K^k{}_j + 4\pi\big[(J^{\mathrm{m}} - \rho^{\mathrm{m}})\,\gamma_{ij} - 2J^{\mathrm{m}}{}_{ij}\big]\Big\}. \end{aligned} \tag{5.53}$$

Equation (5.53) is the evolution equation for the extrinsic curvature K_{ij}, which tells us how the extrinsic curvature changes when dragged along μ from one slice to the neighboring one. Equations (5.40, 5.53) are the "evolution equations" in the $3+1$ formalism.

Now consider the projection purely orthogonal to Σ_t, along ν,

$${}^4R_{\mu\nu}\nu^\mu\nu^\nu - \frac{1}{2}\,{}^4R\, g_{\mu\nu}\nu^\mu\nu^\nu = 8\pi\, \nu^\mu\nu^\nu T_{\mu\nu}, \tag{5.54a}$$

$$ {}^4R_{\mu\nu}\nu^\mu\nu^\nu + \frac{1}{2}\,{}^4R = 8\pi\,\nu^\mu\nu^\nu T_{\mu\nu}. \tag{5.54b} $$

We can use the scalar Gauss equation (5.27) and (5.44) to write

$$ R + K^2 - K_{ij}K^{ij} = 16\pi\,\rho^{\mathrm{m}}. \tag{5.55} $$

This equation is the "Hamiltonian constraint." It does not involve derivatives along the normal evolution vector; it relates objects on the same slice and it must be satisfied on each slice separately. Therefore, it constitutes a constraint on the evolution.

The last projection to be considered is the mixed one, that is, once along ν and once onto Σ_t. We successively get,

$$ {}^4R_{\mu\nu}\,\nu^\mu \perp^\nu{}_\beta - \frac{1}{2}\,{}^4R\,g_{\mu\nu}\,\nu^\mu \perp^\nu{}_\beta = 8\pi\,\nu^\mu \perp^\nu{}_\beta\,T_{\mu\nu}, \tag{5.56a} $$

$$ {}^4R_{\mu\nu}\,\nu^\mu \perp^\nu{}_\beta = 8\pi\,\nu^\mu \perp^\nu{}_\beta\,T_{\mu\nu}. \tag{5.56b} $$

Using the contracted Codazzi relation (5.29) and (5.45) we can write,

$$ D_\alpha K - D_\mu K^\mu{}_\alpha = -8\pi\,j^{\mathrm{m}}{}_\alpha. \tag{5.57} $$

As for the evolution equation (5.53), all tensors in (5.57) are spatial, hence

$$ D_j K^j{}_i - D_i K = 8\pi\,j^{\mathrm{m}}{}_i. \tag{5.58} $$

The equation (5.58) is called "momentum constraint." As the Hamiltonian constraint, it must hold on each slice separately and does not contain any derivative along μ.

The equations (5.40, 5.53, 5.55, 5.58) are called the "standard $3+1$ equations." One can look at the evolution equations as those dictating the dynamics of GR, and at the constraint equations as those establishing its kinematics. These equations were obtained by Darmois in [Dar27] in Gaussian normal coordinates, which we will consider later. Lichnerowicz generalized them using a more general set of coordinates [Lic39; Lic44; Lic52] and, finally, Choquet–Bruhat wrote them in arbitrary coordinates [Fou52; Fou56]. The $3+1$ equations were used by Dirac [Dir58; Dir59] and Arnowitt, Deser and Misner [ADM62; ADM08] to develop the Hamiltonian formulation of GR. The $3+1$ equations were later used in the development of numerical relativity (NR), starting with the works by York [Yor73; Yor78], whose approach is restated in [Yor04]. For a historical account on the Cauchy problem in GR, we refer the reader to [Cho14; Rin15].

The choice of coordinates and the shift vector. Equations (5.53, 5.55) and (5.58), equivalent to the EFE, plus the mathematical relation (5.40) dictating the evolution of the spatial metric, are tensorial. Introducing coordinates, they become a system of coupled PDEs, and can be studied using the concepts introduced in Chapter 4.

As we will see in Section 5.3, it is useful to choose basis vectors and coordinates adapted to the foliation. A frame having three basis vectors tangent to Σ_t and the last one orthogonal to Σ_t is adapted to the foliation and is called "Cauchy adapted frame" [Cho15, Subsec.VIII.7.1].

To adopt it, one chooses some coordinates (x^1, x^2, x^3) on each hypersurface Σ_t. If they vary smoothly from one hypersurface to the neighboring one, the coordinate system $(\mathcal{T}, x^1, x^2, x^3)$ is a well-defined coordinate system for the spacetime $\mathscr{M}$. In that case, (x^1, x^2, x^3) are called "spatial coordinates." Once we choose such coordinates, we consider the natural basis on $T_p(\mathscr{M})$ $\forall p \in \mathscr{M}$ given by,

$$\partial_{\mathcal{T}} := \frac{\partial}{\partial \mathcal{T}}, \quad \partial_i := \frac{\partial}{\partial x^i}. \tag{5.59}$$

The basis vector $\partial_{\mathcal{T}}$ is tangent to the lines of constant spatial coordinates and called the "time vector." It is the algebraic dual to $\nabla \mathcal{T} \equiv \mathrm{d}\mathcal{T}$,

$$(\mathrm{d}\mathcal{T})_\alpha (\partial_{\mathcal{T}})^\alpha = 1. \tag{5.60}$$

Note that, since $(\mathrm{d}\mathcal{T})_\alpha \overrightarrow{\nabla \mathcal{T}}^\alpha = -\alpha^{-2}$, $\mathrm{d}\mathcal{T}$ and $\overrightarrow{\nabla \mathcal{T}}$ are not algebraic duals, and we cannot use $\overrightarrow{\nabla \mathcal{T}}$ as our basis vector in the coordinate basis. However, we know that $(\mathrm{d}\mathcal{T})_\alpha \mu^\alpha = 1$, see (5.36). The relations (5.36, 5.60) imply that $\partial_{\mathcal{T}}$ and μ can differ only by a spatial vector, denoted "shift vector" by Wheeler [Whe64],

$$\beta := \partial_{\mathcal{T}} - \mu = \partial_{\mathcal{T}} - \alpha\, \nu, \tag{5.61a}$$

$$1 = (\mathrm{d}\mathcal{T})_\alpha (\partial_{\mathcal{T}})^\alpha = (\mathrm{d}\mathcal{T})_\alpha \beta^\alpha + (\mathrm{d}\mathcal{T})_\alpha \mu^\alpha = (\mathrm{d}\mathcal{T})_\alpha \beta^\alpha + 1 \Longrightarrow (\mathrm{d}\mathcal{T})_\alpha \beta^\alpha = 0. \tag{5.61b}$$

The lapse function and the shift vector were first introduced by Choquet–Bruhat [Fou56]. The square norm of $\partial_{\mathcal{T}}$ is given by

$$\partial_{\mathcal{T}} \cdot \partial_{\mathcal{T}} = -\alpha^2 + \beta \cdot \beta, \tag{5.62}$$

hence it can be timelike, null or spacelike depending on the lapse and the square norm of the shift.

At this point, we choose the Cauchy adapted frame to be,

$$\theta_{(0)} = \mu = \partial_{\mathcal{T}} - \beta, \quad \theta_{(i)} = \partial_i. \tag{5.63}$$

Its dual coframe reads,

$$\theta^{(0)} = \mathrm{d}\mathcal{T}, \quad \theta^{(i)} = \mathrm{d}x^i + \beta^i \mathrm{d}\mathcal{T}. \tag{5.64}$$

Since the Lie derivative along μ drags geometric objects from one slice to the neighboring one, it is clear that the choice of this Cauchy adapted frame is particularly suitable for the $3+1$ formulation.

We stress that, if one fixes the scalar field $\mathcal{T}$ which defines the slicing, then the lapse function α is uniquely determined according to (5.7), and if one chooses $\mathcal{T}$ to be the time coordinate, the shift vector is uniquely determined according to (5.61a). Conversely, the specification of α, β, and a spatial coordinate system (x^1, x^2, x^3) on a spacelike hypersurface, uniquely determines a coordinate system $(\mathcal{T}, x^1, x^2, x^3)$ in a neighborhood of that hypersurface. Indeed, if the lapse is specified, we can construct the normal evolution vector and we know how to drag geometrical objects from one slice to the other; if the shift is specified, we know how to propagate the spatial coordinate system from one slice to the other. To show this, consider again two events $p \in \Sigma_t$ and $p' = p + \mu\, \delta t \in \Sigma_{t+\delta t}$, with $\delta t \ll 1$. The difference in their time coordinate is δt. To compute the difference between their spatial coordinates, choose a 4-dimensional chart $(\mathcal{T}, x^1, x^2, x^3)$ covering a neighborhood of Σ_t which includes $\Sigma_{t+\delta t}$. The difference between the i^{th} spatial coordinate of p and p' is given by,

$$\begin{aligned} x^i(p') - x^i(p) &= x^i(p + \mu\delta t) - x^i(p) = (\nabla x^i)_\alpha \mu^\alpha\, \delta t \\ &= \Big[(\nabla x^i)_\alpha (\partial_t)^\alpha - (\nabla x^i)_\alpha \beta^\alpha\Big]\delta t = -\beta^i \delta t, \end{aligned} \tag{5.65}$$

following from the fact that ∇x^i is the 1-form dual to the canonical basis vector ∂_i. Hence, the freedom to choose the lapse function on each slice corresponds to the freedom to change the time coordinate on each slice, and the freedom to choose the shift vector on each slice reflects the freedom to change spatial coordinates on each slice. We refer to the lapse function and the shift vector collectively as "gauge variables."

Having introduced the shift vector, the components of the normal vector read,

$$\nu^\alpha = \frac{1}{\alpha}\Big[\big(1,0,0,0\big) - \big(\beta^1, \beta^2, \beta^3\big)\Big] = \frac{1}{\alpha}\big(1, -\beta^1, -\beta^2, -\beta^3\big), \tag{5.66}$$

where we used (5.61a).

We note here that the free choice of the gauge variables means that not all functions in the evolution equations are analytic.[4] Even if we choose the gauge variables to be analytic, the initial data are not constrained to be such, since all the geometric objects are defined on a differentiable manifold which requires them to be only smooth, not analytic. Therefore, if we write the 3+1 equations as a (generalized) Cauchy problem, we will not be able to invoke the CK theorem to guarantee existence and uniqueness of the solution, and the rewriting of the Cauchy problem in a well-posed form will be necessary.

The metric components. Since we have chosen a basis, we can compute the components of the spacetime metric $g_{\mu\nu}$ by evaluating the scalar product between the basis vectors. We have,

$$g_{00} = \partial_\mathcal{T} \cdot \partial_\mathcal{T} = -\alpha^2 + \beta^i \beta_i, \tag{5.67a}$$

$$g_{0i} = \partial_\mathcal{T} \cdot \partial_i = \big(\mu + \beta\big)\cdot \partial_i = \beta \cdot \partial_i = \beta_i, \tag{5.67b}$$

$$g_{ij} = \partial_i \cdot \partial_j = g_{\mu\nu}(\partial_i)^\mu (\partial_j)^\nu = \gamma_{k\ell}(\partial_i)^k(\partial_j)^\ell = \gamma_{ij}, \tag{5.67c}$$

where we used the fact that ∂_i is spatial and μ is normal to Σ_t. In matrix notation,

$$g_{\alpha\beta} = \begin{pmatrix} -\alpha^2 + \gamma_{k\ell}\beta^k\beta^\ell & \gamma_{jk}\beta^k \\ \gamma_{ik}\beta^k & \gamma_{ij} \end{pmatrix}, \quad g^{\alpha\beta} = \alpha^{-2}\begin{pmatrix} -1 & \beta^j \\ \beta^i & \alpha^2\gamma^{ij} - \beta^i\beta^j \end{pmatrix}, \tag{5.68}$$

[4] The gauge variables do not appear in the constraint equations.

with $g_{\alpha\rho}g^{\rho\beta} = \delta_\alpha{}^\beta$ as it can be verified with a direct computation. In terms of the Cauchy coframe, the line element reads,

$$\mathrm{d}s^2 = -\alpha^2\big(\theta^{(0)}\big)^2 + \gamma_{ij}\theta^{(i)}\theta^{(j)} = -\alpha^2\mathrm{d}\mathcal{T}^2 + \gamma_{ij}\big(\mathrm{d}x^i + \beta^i\mathrm{d}\mathcal{T}\big)\big(\mathrm{d}x^j + \beta^j\mathrm{d}\mathcal{T}\big). \tag{5.69}$$

From (5.66, 5.68) it follows,

$$\nu_\alpha = \big(-\alpha, 0, 0, 0\big). \tag{5.70}$$

The counting of propagating degrees of freedom. A confirmation that the standard $3+1$ equations are equivalent to the EFE is that both sets of equations propagate the same number of degrees of freedom. Since the EFE are second-order in time, in order to solve their Cauchy problem (which may or may not be formulated in terms of the $3+1$ decomposition, as pointed out in Appendix C), we need to specify the initial values for the 10 components of the metrics and their 10 time derivatives. Then, we determine the second time derivatives of the metric components by means of the EFE. However, not all of the EFE contain second-order time derivatives. Consider the Bianchi identity,

$$\nabla_\rho G^{\mu\rho} \equiv 0 \Longrightarrow \partial_t G^{\mu 0} \equiv -\partial_i G^{\mu i} - G^{\rho\sigma}\Gamma^\mu_{\rho\sigma} - G^{\mu\rho}\Gamma^\sigma_{\rho\sigma}. \tag{5.71}$$

Since (5.71) is an identity and since the right-hand side does not contain third-order time derivatives, $G^{\mu 0}$ does not contain second-order time derivatives of the components of the metric, and

$$G^{\mu 0} = 8\pi\, T^{\mu 0} \tag{5.72}$$

do not determine any second-order time derivative of the components of the metric. The equations (5.72) correspond to the 4 constraints. Therefore, we have 6 dynamical equations for 10 metric components. Since 4 of the latter can be fixed by diffeomorphism invariance, we have 6 dynamical equations for 6 metric components. However, not all of these components are independent, because of the constraint equations. They can be solved for 4 of them in terms of the remaining 2, which constitute the independent dynamical modes. Each of the 2 modes needs its first time derivative to be specified, hence there are 4 independent degrees of freedom.[5]

In the $3+1$ decomposition, we start with 12 degrees of freedom, which are the 6 components of the spatial metric and the 6 components of the extrinsic curvature, which must be specified in order to solve the Cauchy problem.[6] The constraints determine 4 of them in terms of the remaining 8. Then, 3 among the components of the spatial metric or the extrinsic curvature are fixed on each slice by spatial diffeomorphism invariance, that is,

[5]In the Hamiltonian formalism, one would need to specify a dynamical variable and its conjugate momentum. In our approach, the conjugate momentum is replaced by the time derivative of the metric components.

[6]The extrinsic curvature basically represents the time derivative of the metric according to (5.40), and it is in direct relation with the conjugate momentum to the spatial metric in the Hamiltonian formalism of GR, $\pi^{ij} = \sqrt{\Delta}\left(K\gamma^{ij} - K^{ij}\right)$ [Gou12, Sec. 5.5].

by specifying the shift vector, and 1 more component is fixed by specifying the lapse function. The latter is possible since we can always pick one of the evolution equations for a dynamical variable and specify the lapse function to set it equal to 0, thus making that degree of freedom nondynamical. Finally, the independent degrees of freedom are $12 - 4 - 3 - 1 = 4$, corresponding to 2 propagating modes.

Both the covariant EFE and the standard $3+1$ equations describe the propagation of 2 modes, associated to a massless spin-2 field. For more details about the counting of degrees of freedom, see [Wal10, Sec. 10.2] and [BS10, Ch. 2].

The standard $3+1$ equations as a system of PDEs. We summarize the $3+1$ decomposition of the EFE,

$$\begin{aligned}
\mathscr{L}_{\mu}\, \gamma_{ij} &= -2\alpha K_{ij}, && (5.73\text{a})\\
\mathscr{L}_{\mu}\, K_{ij} &= -D_i D_j \alpha \\
&\quad + \alpha\Big\{R_{ij} + K K_{ij} - 2K_{ik}K^k{}_j + 4\pi\big[(J^{\mathrm{m}} - \rho^{\mathrm{m}})\,\gamma_{ij} - 2J^{\mathrm{m}}{}_{ij}\big]\Big\}, && (5.73\text{b})\\
16\pi\,\rho^{\mathrm{m}} &= R + K^2 - K_{ij}K^{ij}, && (5.73\text{c})\\
8\pi j^{\mathrm{m}}{}_i &= D_j K^j{}_i - D_i K. && (5.73\text{d})
\end{aligned}$$

In order to write it as a system of PDEs, we need to expand the covariant and Lie derivatives. The Lie derivative along the normal evolution vector μ is equal to,

$$\mathscr{L}_{\mu} = \mathscr{L}_{\partial_{\mathfrak{T}}} - \mathscr{L}_{\beta}, \tag{5.74}$$

and since $\partial_{\mathfrak{T}}$ is the canonical basis vector, the Lie derivative along μ of a tensor field T tangent to Σ_t with components $T^{i\ldots}{}_{j\ldots}$ is,

$$\mathscr{L}_{\mu} T^{i\ldots}{}_{j\ldots} = \left(\frac{\partial}{\partial \mathfrak{T}} - \mathscr{L}_{\beta}\right) T^{i\ldots}{}_{j\ldots}. \tag{5.75}$$

The Lie derivative of a tensor field along the shift vector can be expressed directly in terms of spatial partial derivatives using the definition of Lie derivative, and the spatial covariant derivatives of a tensor field can also be expanded in terms of spatial partial derivatives and Christoffel symbols of the spatial metric. Finally, the Christoffel symbols can also be written in terms of the spatial partial derivatives of the spatial metric.

After expanding all these terms, the equations (5.73) constitute a systems of coupled PDEs. At this point, we are almost ready to study them with the methods reviewed in Chapter 4. Before doing that, the last thing to consider is that GR is diffeomorphism invariant. Hence, the choice of coordinates modifies (5.73) and alters its property as a PDE system. In order to gain some insights on the structure of the equations, we make the simplest possible choice of coordinates, called “Gaussian normal coordinates” or “geodesic slicing”,[7]

$$\alpha = 1, \quad \beta = 0. \tag{5.76}$$

In these coordinates, the spacetime line element becomes

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}\mathcal{T}^2 + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j. \tag{5.77}$$

By substituting (5.76) into (5.73), expanding the Ricci tensor and Ricci scalar in terms of the spatial partial derivatives of the spatial metric, and replacing K_{ij} by means of (5.73a), one gets the following system of PDEs for the spatial metric (see Section 5.4.2 of [Gou12] for the details),

$$-\frac{\partial^2\gamma_{ij}}{\partial\mathcal{T}^2} + \gamma^{k\ell}\left(\frac{\partial^2\gamma_{ij}}{\partial x^k\partial x^\ell} + \frac{\partial^2\gamma_{k\ell}}{\partial x^i\partial x^j} - \frac{\partial^2\gamma_{\ell j}}{\partial x^i\partial x^k} - \frac{\partial^2\gamma_{\ell i}}{\partial x^j\partial x^k}\right) = 8\pi\big[(J^{\mathrm{m}} - \rho^{\mathrm{m}})\,\gamma_{ij} - 2S_{ij}\big] + \mathcal{Q}_{ij}\left(\gamma_{k\ell}, \frac{\partial\gamma_{k\ell}}{\partial x^m}, \frac{\partial\gamma_{k\ell}}{\partial\mathcal{T}}\right), \tag{5.78a}$$

$$\gamma^{ik}\gamma^{j\ell}\frac{\partial^2\gamma_{ij}}{\partial x^k\partial x^\ell} - \gamma^{ij}\gamma^{k\ell}\frac{\partial^2\gamma_{ij}}{\partial x^k\partial x^\ell} = 16\pi\rho^{\mathrm{m}} + \mathcal{Q}\left(\gamma_{k\ell}, \frac{\partial\gamma_{k\ell}}{\partial x^m}, \frac{\partial\gamma_{k\ell}}{\partial\mathcal{T}}\right), \tag{5.78b}$$

$$\gamma^{jk}\frac{\partial^2\gamma_{ki}}{\partial x^j\partial\mathcal{T}} - \gamma^{k\ell}\frac{\partial^2\gamma_{k\ell}}{\partial x^i\partial\mathcal{T}} = -16\pi {j^{\mathrm{m}}}_i + \mathcal{Q}_i\left(\gamma_{k\ell}, \frac{\partial\gamma_{k\ell}}{\partial x^m}, \frac{\partial\gamma_{k\ell}}{\partial\mathcal{T}}\right), \tag{5.78c}$$

where all the first- and zeroth-order derivative terms are included in the *nonlinear* $\mathcal{Q}$ terms. The system (5.78) is second-order and quasilinear, according to the classification in Section 4.2, that is, it is linear with respect to the highest order (second) derivatives, but the coefficients of these derivatives depend on the spatial metric. The system of 6 equations (5.78a) for 6 unknowns γ_{ij} can be written,

$$\frac{\partial^2\gamma_{ij}}{\partial\mathcal{T}^2} = F_{ij}\left(\gamma_{k\ell}, \frac{\partial\gamma_{k\ell}}{\partial x^m}, \frac{\partial\gamma_{k\ell}}{\partial\mathcal{T}}, \frac{\partial^2\gamma_{k\ell}}{\partial x^m\partial x^n}\right). \tag{5.79}$$

This is a system of PDEs in the form (4.3a), therefore it can be treated as a Cauchy problem. However, this is not the entire system, since (5.78b, 5.78c) must be satisfied as well. The latter are not treatable as a Cauchy problem; rather, as we already said, they constitute constraints on the Cauchy problem.

The generalized Cauchy problem for the EFE. The EFE are equivalent to (5.73) assuming that the spacetime admits a $3+1$ decomposition, that is, assuming that in the neighborhood of the initial hypersurface Σ_{t_0} it is possible to define a regular foliation. Note that the spacetime itself is not known before solving the standard $3+1$ equations, hence there is no way to know, before solving the equations, in what region the spacetime admits a regular foliation. Therefore, the procedure of solving the equations can be continued until the foliation stays regular. If it is always regular, then the solution is a globally hyperbolic spacetime, Σ_{t_0} is a Cauchy surface, and the topology is $\mathbb{R}\times\Sigma_{t_0}$. However, if this does not

[7] Notice that not all the spacetimes admit a global geodesic slicing.

happen, it doesn't necessarily mean that the spacetime is not globally hyperbolic, since the failure could be due to an unfortunate choice of the slicing. Indeed, we remind the reader that the (free) choice of the lapse determines the foliation; if the spacetime admits a regular foliation, but we choose another one through the specification of the lapse, which becomes non-regular at some time $\widetilde{t}$ during the evolution (and at the initial time t_0 we do not know that this will happen), the latter cannot continue after $\widetilde{t}$. Hence, in GR (and BR, as we will see in the Chapter 6) we are always dealing with a local generalized Cauchy problem (see Section 4.1), rather than a Cauchy problem.

The free evolution scheme. In order to find a solution to the constrained Cauchy problem (5.73), we need to specify the initial data. If the Cauchy problem was not constrained, the choice of the initial data would be free. Since it is constrained, the choice of the initial data is partially free, in the sense that we can choose freely some of the variables, but not all of them, since the constraints must be satisfied. In principle, this freedom in choosing some of the variables on the initial hypersurfaces can be used to simplify the constraint equations. In practice, the freely specifiable part of the initial data will be determined by the physical properties of the system under consideration, for example the matter density profile for spherically symmetric gravitational collapse. The constraints must be valid at all times and therefore have to be solved at all times, together with the evolution equations. Solving evolution and constraint equations altogether is referred to as the "constrained scheme." If there was reason to believe that it is not needed to solve the constraints at all times, but only on the initial hypersurface to determine the initial data, this would simplify the problem considerably.

Fortunately, there is reason to avoid solving the constraints at all times. Indeed, it is possible to compute evolution equations for the constraints themselves, which we denote as,

$$H = 0, \quad H := \frac{1}{2}\big(R + K^2 - K_{ij}K^{ij}\big) - 8\pi\rho^{\mathrm{m}}, \tag{5.80a}$$

$$M_i = 0, \quad M_i := D_j K^j{}_i - D_i K - 8\pi j^{\mathrm{m}}{}_i. \tag{5.80b}$$

For convenience, we also denote the two evolution equations (5.73a, 5.73b) collectively as,

$$\mathcal{E}_{\mu\nu} = 0, \tag{5.81}$$

where the symbol $\mathcal{E}_{\mu\nu}$ includes the nonzero terms in (5.73a, 5.73b), moved to the left-hand side. The evolution equations for the constraints are obtained by projecting the equation

$$\nabla_\mu\big(G_g{}^\mu{}_\nu - 8\pi\, T^\mu{}_\nu\big) = 0, \tag{5.82}$$

which follows from the contracted Bianchi identity $\nabla_\mu G_g{}^\mu{}_\nu \equiv 0$ that in turn follows from the symmetries of the Riemann curvature tensor, and the *independent assumption* $\nabla_\mu T^\mu{}_\nu = 0$. We remark that the latter must be assumed, since at this stage the EFE are not solved yet. On the other hand, this is a very reasonable assumption, since it is the energy–momentum conservation law for the matter source [RZ13, Subsec. 2.3.5]. It is possible to rewrite (5.82) as [Gou12, eq. (11.15)],

$$\nabla_\mu\big(\mathcal{E}^\mu{}_\nu + (H-\mathcal{E})\perp^\mu{}_\nu + \nu^\mu M_\nu + M^\mu \nu_\nu + H\,\nu^\mu\nu_\nu\big) = 0, \tag{5.83}$$

with $\mathcal{E} = g^{\mu\nu}\mathcal{E}_{\mu\nu}$. The projections of (5.83) onto span(ν) and Σ_t give [Gou12, eqs. (11.22), (11.26)],

$$\mathscr{L}_\mu H = -D_i(\alpha\, M^i) - M^i D_i\alpha + \alpha K(2H-\mathcal{E}) + \alpha K^{ij}\mathcal{E}_{ij}, \tag{5.84a}$$

$$\begin{aligned}\mathscr{L}_\mu M^i &= -D^j(\alpha\mathcal{E}^{ij}) + 2\alpha K^i{}_j M^j + \alpha K M^i \\ &\quad + \alpha D^i(\mathcal{E}-H) + (\mathcal{E}-2H)D^i\alpha.\end{aligned} \tag{5.84b}$$

At this point, we apply the assumptions at the base of the free evolution scheme, that is, the constraints are solved only to obtain the initial data,

$$H\big|_{\mathcal{T}=0} = M^i\big|_{\mathcal{T}=0} = 0, \tag{5.85}$$

and the evolution equations (5.81) are satisfied at all times. This implies,

$$\mathscr{L}_\mu H = -D_i(\alpha\, M^i) - M^i D_i\alpha + 2\alpha K H, \tag{5.86a}$$

$$\mathscr{L}_\mu M^i = -D^i(\alpha H) + 2\alpha K^i{}_j M^j + \alpha K M^i - H D^i\alpha. \tag{5.86b}$$

At the initial time $\mathcal{T}=0$, due to (5.85), the system (5.86) reads,

$$\left.\frac{\partial H}{\partial \mathcal{T}}\right|_{\mathcal{T}=0} = \left.\frac{\partial M^i}{\partial \mathcal{T}}\right|_{\mathcal{T}=0} = 0. \tag{5.87}$$

One solution to (5.86, 5.87) is the trivial one, $H = M^i = 0$. Since the system (5.86) is symmetric hyperbolic, its Cauchy problem is well-posed and the trivial solution is unique.

Therefore, if the constraints are zero initially and the evolution equations are satisfied, the constraints are satisfied also for $t > 0$ without solving them explicitly. This is referred to as the "stable propagation of the constraints" and was first shown in [Fri97]. Solving the constraints to determine the initial data, and evolving these data by solving the evolution equations only, is called "free evolution scheme."

The free evolution scheme is very beneficial, since solving the constraints numerically at each time step during a simulation is demanding. Indeed, in most of the cases the constraints constitute a boundary value problem which has to be solved over the entire spatial domain at each time step. In addition, employing the free evolution scheme means that it is enough to consider the well-posedness of the Cauchy problem for the dynamical variables (γ_{ij}, K_{ij}) defined by the evolution equations (5.73a, 5.73b), without considering the constraints. It is natural to ask if this Cauchy problem is well-posed. As we saw in Section 4.2, this is equivalent to ask about the hyperbolicity of the system (5.73a, 5.73b). It turns out that this system of PDEs, after reducing it to a first-order system in the spatial derivatives, is only weakly hyperbolic [Alc08, Sec. 5.4], [BS10, p. 377] and, as such, does not lead to a well-posed Cauchy problem. It is then desirable to recast the standard $3+1$ equations in a way such that the Cauchy problem is well-posed.

From ill-posedness to well-posedness. Due to its definition in Section 4.1, well-posedness is a property referring to the PDEs one is working with, neither to the physics nor to the theory behind them. Indeed, it is possible to write the same tensorial equations as different Cauchy problems, well- or ill-posed, keeping the physics described by the equations the same.

Generically, if a mathematical boundary value problem is ill-posed, a possible strategy to try to rewrite it in a well-posed form is to define auxiliary variables, compute the differential equations which determine them, and study the well-posedness of the new problem. Note that, if a new independent dynamical variable is introduced and evolved, a new constraint, solvable for it, must be provided. Otherwise, the theory is extended and propagates more degrees of freedom than the original one.

In the case of GR there are two more possibilities one can use to rewrite the equations in a well-posed form: diffeomorphism invariance and the fact that GR is a constrained theory. The first possibility, already discussed in Section 5.2, allows us to change the evolution equations by freely specifying the gauge variables (lapse function and shift vector). This may change the principal symbol of the equations, hence their hyperbolicity. The second possibility allows us to add multiples of the constraints H and M_i defined in (5.80) to the evolution equations, since they are both equal to 0 at all times. Since H contains second-order spatial derivatives, it does alter the principal symbol and the hyperbolicity of the equations; regarding M_i, which contain first-order spatial derivatives of $K^i{}_j$ and its trace K, the hyperbolicity is changed if one considers a first-order reduction of the system.

Appendix C shortly reviews some formulations of the EFE which result in a well-posed Cauchy problem, and points out how to generalize one of them—namely, the "generalized harmonic formulation"—to BR. All of them rewrite the EFE in a form which is, at least, strongly hyperbolic. The reader is pointed to the cited references for more details about the formulation of their interest. Below, we introduce the strongly hyperbolic formulation relevant in Paper V and Paper VI.

The covariant BSSN formulation. The Baumgarte–Shapiro–Shibata–Nakamura–Oohara–Kojima (BSSNOK) formulation [NOK87; SN95; BS98], more commonly known as just the BSSN formulation, is a recasting of the standard $3+1$ equations. Together with the generalized harmonic formulation (described in Appendix C), it is the most widely used in the NR community, since it has proven to result in very robust numerical simulations of different spacetimes, both in vacuum and in the presence of matter—see, for example, [Alc08, p. 82], [BS10, Sec. 11.4], [Gou12, Sec. 11.4.5] and references therein, [LP14]. Two of the most renowned numerical codes, namely the Einstein Toolkit [Löf+12] and SENR/NRPy+ [REB18], are using the BSSN formulation. For all of these reasons, we chose to recast the $3+1$ formulation of the BFE in the BSSN formalism in Paper V.

We follow [BS10, Sec. 11.5]. The key feature of the BSSN formulation is the separation of the longitudinal mode from the transverse ones, by making use of the conformal decomposition,

$$\bar{\gamma}_{ij} := \mathrm{e}^{-4\phi}\gamma_{ij}, \tag{5.88}$$

where $\bar{\gamma}_{ij}$ is called the "conformal spatial metric" or just the "conformal metric." By assumption, the determinant $\bar{\Delta}$ of the conformal metric is equal to 1 during the evolution, hence (5.88) implies,

$$\phi = \frac{1}{12}\log(\Delta). \tag{5.89}$$

Since the determinant Δ of γ is a scalar density of weight 2, ϕ is a scalar density of weight 1/6. This implies that $\bar{\gamma}_{ij}$ is a tensor density of weight $-2/3$. The extrinsic curvature is also separated into its conformal traceless part and its trace, according to,

$$\bar{A}_{ij} := \mathrm{e}^{-4\phi}\left(K_{ij} - \frac{1}{3}\gamma_{ij}K\right). \tag{5.90}$$

The conformal traceless extrinsic curvature $\bar{A}_{ij}$ is a tensor density of weight $-2/3$. The indices of the conformal tensors are raised and lowered with the conformal metric. The quantities $\phi, K, \bar{\gamma}_{ij}, \bar{A}_{ij}$ are promoted to be the new dynamical variables. In addition, other new dynamical variables, the "conformal connections," are introduced,

$$\bar{\Gamma}^i := \bar{\gamma}^{jk}\bar{\Gamma}^i_{jk} = -\partial_j\bar{\gamma}^{ij}, \tag{5.91}$$

where $\bar{\Gamma}^i_{jk}$ are the Christoffel symbol of the conformal metric, and the last equality follows because $\bar{\Delta} = 1$. The definitions (5.91) are used as new constraints on the conformal connections.

In [Sar+02; BS04], the BSSN formulation is proven to be strongly hyperbolic if the standard gauge (introduced in the next section) is imposed on the gauge variables. In addition, the BSSN formulation can be made even symmetric hyperbolic by choosing appropriate values of the free parameters introduced in [Sar+02].

The BSSN formulation was rewritten in a form which involves spatial tensors, not spatial tensor densities, in [Bro09]. This rewriting is referred to as the "covariant BSSN formulation" (cBSSN), and makes it possible to compare the same physical solution evolved in different coordinate systems (i.e., with different choices of lapse, shift or spatial coordinates on the initial hypersurface). For more details about the BSSN formulation, we refer the reader firstly to the cited references, and secondly to Paper V for a brief review. Here, it is enough to say that in the cBSSN formulation, an external arbitrary connection $\bar{\Gamma}_{\mathtt{B}}$ is introduced in order to make the conformal connection (5.91) a spatial vector (the difference between connections is a tensor [Lee97, p. 63]). This may be the Levi–Civita connection of an arbitrary background metric, and is usually chosen to be the connection associated with the chosen spatial curvilinear coordinate system, hence it is nondynamical, $\partial_t\bar{\Gamma}_{\mathtt{B}} = 0$. As mentioned in Paper V and proved in Paper VI, there is a natural candidate for the background metric in BR, as we shall review in Section 6.3.

5.3 Two important choices of coordinates

In the last section of this chapter, we introduce two of the most commonly used gauge choices in NR, which will also be relevant in Chapter 6 when discussing the results of Paper V and especially Paper VI and Paper VIII. They are the "maximal slicing" and the "standard gauge," and both, in different ways, describe how the choices of the lapse function and the shift vector can be used to impose desirable properties to the geometry of the spacetime and to the system of PDEs.

Maximal slicing. We follow [BS10, Sec. 4.2] and refer the reader to it for a more detailed exposition.

Taking the trace of the evolution equation (5.73b) results in,

$$D^2\alpha = -\partial_t K + \alpha\Big[K_{ij}K^{ij} + 4\pi\Big(\rho^{\mathrm{m}} + J^{\mathrm{m}}\Big)\Big] + \beta^i D_i K. \tag{5.92}$$

We want to find coordinates such that the trace K of the extrinsic curvature has a specific value. We can do this by solving (5.92) for the lapse α. The specification of K, together with (5.73b), completely determines K_{ij}. Hence, (5.92) turns into an elliptic PDE for the lapse α, which is the only unknown function assuming that the shift is fixed by the remaining gauge choice. Solving for the lapse in this way is equivalent to find a time coordinate such that K has the desired value.

Maximal slicing, suggested in [Lic44] and introduced in [SY78; Yor78], corresponds to setting the trace of the extrinsic curvature identically to zero. This is done by setting $K = 0$ as part of the initial data, and imposing $\partial_t K = 0$ as a gauge choice. The PDE (5.92) then reads,

$$D^2\alpha = \alpha\Big[K_{ij}K^{ij} + 4\pi\Big(\rho^{\mathrm{m}} + J^{\mathrm{m}}\Big)\Big]. \tag{5.93}$$

In practice, this elliptic PDE has to be solved numerically at each time t to determine the lapse on each hypersurface Σ_t. This slows down the integration, since evolving a quantity at each spatial grid point is less demanding than solving for it over the entire grid.[8] The benefit is that maximal slicing is "singularity avoiding," that is, during the process of black hole formation, the slicing enters the event horizon, but never reaches the central curvature singularity. The slices tend to a limiting surface at $r = 3M/2$ in Schwarzschild coordinates, not covering the entire inner region of the black hole. This is useful since the numerical integration cannot handle the presence of a curvature singularity.

The name "maximal" has the following origin. Consider a foliation $\{\Sigma_t\}_{t\in(t_0,t_1)}$ in an open subset U of spacetime, and the open set $V \in \Sigma_t$, with Σ_t a hypersurface in the foliation. Every point in V can be dragged to the neighboring slices by the normal evolution vector, hence we can consider the family of open sets $V(t)$, $t \in (t_0, t_1)$, each one lying on Σ_t. If the area of $V(t)$, $A(t)$, has a critical point at $\bar{t} \in (t_0, t_1)$, the hypersurface $\Sigma_{\bar{t}}$ is called "extremal." This happens if and only if $K = 0$ on $V(t)$. In Euclidean geometry, hypersurfaces with $K = 0$

[8] Note, however, that it is possible to rewrite the maximal slicing condition as an evolution equation for the lapse, see [BS10, Sec. 4.2] for more details.

are minimal, whereas in pseudo-Riemannian geometry they are maximal [BS10, p. 104].

The singularity avoidance of the maximal slicing can be understood as follows. Since [Gou12, eq. (3.66)],

$$\nabla_\mu \nu^\mu = -K, \tag{5.94}$$

with ν being the vector field normal to the spacelike hypersurfaces, setting $K = 0$ means that the congruence $\mathcal{N}$ of integral curves of ν is not-expanding (see [Poi04, Secs. 2.2, 2.3]). In addition, this congruence is also irrotational [BS10, Exercise 2.5]. Therefore, maximal slicing makes Eulerian observers move as an incompressible fluid. Since also [BS10, eq. (2.136)],

$$\mathscr{L}_\nu \log\bigl(\sqrt{\Delta}\bigr) = -K, \tag{5.95}$$

we can equivalently say that the spatial coordinate volume element $\sqrt{\Delta}\, \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3$ is conserved along the normal congruence $\mathcal{N}$. This is enough to prevent the focusing of Eulerian geodesics, which would result in a coordinate singularity, since a coordinate system is not one-to-one at the point of intersection of multiple geodesics.

Also, with the appropriate inner boundary condition, maximal slicing makes the lapse "collapse," that is, to tend to zero for late times in the black hole interior, in such a way that the proper time $\delta\tau = \alpha\, \delta t$ of an Eulerian observer is slowed down when the coordinate time $t \to \infty$. In this way, an Eulerian observer (hence the numerical integration) never reaches the curvature singularity.

The standard gauge. The standard gauge is a set of evolution equations for the gauge variables. Namely, the lapse is determined by the "1+log" slicing α [Bon+95], and the shift by the "Γ-driver" condition β [Alc+03],

$$\partial_t \alpha = \beta^j \partial_j \alpha - 2\alpha K, \tag{5.96a}$$

$$\partial_t \beta^i = \beta^j \partial_j \beta^i + \frac{3}{4} B^i, \tag{5.96b}$$

$$\partial_t B^i = \beta^j \partial_j B^i + \partial_t \bar{\Gamma}^i - \beta^j \partial_j \bar{\Gamma}^i - \eta B^i, \tag{5.96c}$$

where B^i is an auxiliary variable. The standard gauge (5.96) is adapted to the cBSSN formalism in [Bro09]. With this gauge, the (c)BSSN formulation is strongly hyperbolic [BS04; Bro09].

The 1+log slicing condition was shown to be stable for various systems with strong gravitational fields [Arb+99; Alc+01; Alc+03], and has therefore been the standard choice for fixing the lapse in many simulations involving black holes and neutron stars, as pointed out in [Alc08]. The 1+log slicing basically reproduces the singularity avoidance properties of maximal slicing [Alc03; Bon+97], but has the advantage to be an evolution equation, not an elliptic one. It is actually embedded in the Bona–Masso family of slicing conditions (C.2). The Γ-driver shift condition has also been proved to be stable, particularly together with the BSSN formulation, see [Alc08, Subsec. 4.3.2] and [Gou12, Subsec. 10.3.5] for collections of references using this gauge condition.

Chapter 6

Numerical bimetric relativity

Provided with the necessary background from the theory of PDEs in Chapter 4 and from the $3+1$ decomposition in GR in Chapter 5, in this chapter we review the $3+1$ decomposition in BR and the results of Paper IV, Paper V, Paper VI, Paper VII and Paper VIII, which, together with the work in [Koc18], pave the way toward the field of numerical bimetric relativity (nubirel).

6.1 The foliation of a bimetric spacetime

This section is devoted to the discussion on how to establish the $3+1$ decomposition in BR from the geometric viewpoint. The BFE are not taken into account, hence all the presented concepts originate from the geometric and algebraic structure of BR.

The existence of a $3+1$ decomposition in BR. In order to formulate the BFE as a generalized Cauchy problem (see Section 4.1), it is necessary to define a surface to specify the initial data on. In GR, this is a spacelike hypersurface which is defined by the equation $\mathcal{T}=0$, with the scalar field $\mathcal{T}$, called the "time function," having a timelike gradient $\nabla\mathcal{T}$. In BR, one can proceed in the same way, but has to face a fundamental problem already at this stage: Is the chosen hypersurface spacelike with respect to both metrics, or, equivalently, is the gradient $\nabla\mathcal{T}$ timelike with respect to both metrics? The answer to this question turns out to be fundamental, and was stated in [HK18]. We mentioned this in Section 2.2; here we expand on it, since it is of crucial importance in the $3+1$ decomposition of BR.

Consider two Lorentzian metric tensors defined on the same manifold $\mathscr{M}$, and their null cones at an arbitrary point $p\in\mathscr{M}$. We call the metrics and their null cones "causally coupled" at p if and only if a common timelike vector *and* a common spacelike surface element with respect to both metrics exist at p. We call the metrics "null coupled" at p if and only if neither a common timelike vector nor a common spacelike surface element exist at p. Two arbitrary Lorentzian metrics can be causally coupled, null coupled, or none of the two. More intuitively, two causally coupled null cones intersect, and it is possible to find an inner cone inside the intersection and an outer cone containing both null cones. The intersection of two null coupled null cones reduces to two null directions. Causally and null coupled null cones are shown in Figure 6.1 on p. 86.

In BR, the two metrics are not arbitrary. Even off-shell, that is, before solving the BFE, the two metrics are related. This relation originates from the existence of the square root matrix $S = \sqrt{g^{-1}f}$ which, as we saw in Section 1.2, is necessary to formulate the theory. It is easy to provide an example of two metrics which do not admit a real square root. Their matrix representations read [Koc14, Example 5.1],[1]

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}, \quad C = \sqrt{A^{-1}B} \notin \mathbb{R}. \tag{6.1}$$

Note that A and B have Lorentzian signature, as can be seen by computing their eigenvalues. The main theorem proved in [HK18] establishes that a real square root of $A = g^{-1}f$, with g and f two Lorentzian metrics, exists if and only if the metrics are causally or null coupled. A common timelike vector and a common spacelike surface element are explicitly computed in Appendix B of [HK18], for causally coupled metrics.[2] The existence of the square root is also equivalent to the existence of the geometric mean Lorentzian metric $h = g\,S$ [HK18], as we will review in more detail later.

The proof of the theorem in [HK18] follows the reasoning below. Given two Lorentzian metrics g and f, the conditions for the existence of a real square root of A, proved in [Hig08, Th. 1.23, 1.26, 1.29] and summarized in [HK18, Subsec. 2.3.1], involve its eigenvalues. Therefore, it is convenient to express A in terms of its eigenvalues. Consider an arbitrary point $p \in \mathscr{M}$. In general, A cannot be diagonalized at p, but it can always be put in *real* Jordan normal form A_{J} by an appropriate nonunique similarity transformation Z, that is, $A = Z\,A_{\mathrm{J}}\,Z^{-1}$, with A_{J} containing the eigenvalues of A (see, e.g., [Uhl73, Theorem 0.3]).[3] All the possible ways in which this can be done, depending on the properties of g and f, are provided by the theorem on canonical pair forms [Uhl73]. This theorem states that there always exists a nonsingular Z such that $A_{\mathrm{J}} = Z^{-1}\,A\,Z$, and simultaneously $Z^{\top}\,g\,Z$ and $Z^{\top}\,f\,Z$ have specific forms. The matrix representations of A_{J}, $Z^{\top}\,g\,Z$ and $Z^{\top}\,f\,Z$ at p, depend on the eigenvalues of A at p, hence it is possible to impose the conditions of existence of a real square root of A on them. The result is that a real square root exists if and only if $Z^{\top}\,g\,Z$ and $Z^{\top}\,f\,Z$ fall into one out of five possible algebraic types. These algebraic types correspond to causally or null coupled null cones for g and f, and are listed in Table 6.1. The algebraic type can change from point to point due, for example, to the dynamics or to the geometry of the system. For instance, we saw in Section 3.2 that improper bidiagonal BHs are of Type I everywhere except at the Killing horizon, where they become of Type IIa.

It follows that it is possible to find a hypersurface which is spacelike with respect to both g and f *and* a direction which is timelike with respect to both g and f—hence, it is possible to establish the $3+1$ decomposition in BR—if and only if the metrics are causally

[1] To be more precise, the metrics in (6.1) do not admit a real square root isometry. For more details about the mathematical structure at the basis of BR, see [Koc14].

[2] We assume that there exist distributions of common spacelike surface elements which are integrable to common spacelike hypersurfaces forming a foliation. For a distribution to be integrable, the condition established by Frobenius theorem must be satisfied [Wal10, App. B.3]. In our specific case, we know that the set of such integrable distributions is non-empty, since we do pose the initial data and find numerical solutions, as we shall describe in Section 6.4.

[3] We refer the reader to [Bre06] for a historical account about the Jordan normal form.

Algebraic type	$\mathrm{diag}(Z^\top g\, Z)$	$\mathrm{diag}(Z^\top f\, Z)$	$\mathrm{diag}(Z^{-1} g^{-1} f\, Z)$
I	$(-1,1,1,1)$	$(-\lambda_1,\lambda_2,\lambda_3,\lambda_4)$	$(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$
IIa	$(\pm\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},1,1)$	$(\pm\begin{pmatrix}0 & \lambda\\ \lambda & 1\end{pmatrix},\lambda_2,\lambda_3)$	$(\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix},\lambda_2,\lambda_3)$
IIb	$(\pm\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},1,1)$	$(\pm\begin{pmatrix}b & a\\ a & -b\end{pmatrix},\lambda_2,\lambda_3)$	$(\begin{pmatrix}a & -b\\ b & a\end{pmatrix},\lambda_2,\lambda_3)$
III	$(\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix},1)$	$(\begin{pmatrix}0 & 0 & \lambda\\ 0 & \lambda & 1\\ \lambda & 1 & 0\end{pmatrix},\lambda_2)$	$(\begin{pmatrix}\lambda & 1 & 0\\ 0 & \lambda & 1\\ 0 & 0 & \lambda\end{pmatrix},\lambda_2)$
IV	$(-1,1,1,1)$	$(\lambda,-\lambda,\lambda_2,\lambda_3)$	$(-\lambda,-\lambda,\lambda_2,\lambda_3)$

Table 6.1: Algebraic types for which a real square root of $g^{-1}f$ exists [HK18]. It holds $\lambda,\lambda_i>0\ \forall i$, $a\in\mathbb{R}$, $b\neq 0$.

coupled, that is, of algebraic types I, IIa, IIb and III, as can be seen in Figure 6.1. Note that the matrix square root function is multivalued, and the different square roots are classified in nonprimary, primary and principal [Hig08, Sec. 1.4, 1.5, 1.6, 1.7]. Therefore, there is an ambiguity in the choice of the square root. In [HK18, Sec. 3] it is argued that the only choice which is compatible with general covariance, in the sense that the resulting square root transforms as a $(1,1)$ tensor field under a generic diffeomorphism, is the choice of the principal square root. The principal square root is *the one* with all eigenvalues having a nonnegative real part, hence it can never be of Type IV (see Table 6.1). Finally, the choice of the principal square root guarantees that the metrics are causally coupled and the $3+1$ decomposition can be established.

Bimetric geometry of a spacelike hypersurface. Once we are assured that we can define a common foliation of spacelike hypersurfaces $\{\Sigma_t\}_{t\in\mathbb{R}}$ in an open set $U\in\mathscr{M}$, defined as the level sets of a scalar field $\mathcal{T}$, we can proceed as in Section 5.1 for the metrics g,f,h. We define the common gradient 1-form $\nabla\mathcal{T}=\tilde{\nabla}\mathcal{T}=\overset{\#}{\nabla}\mathcal{T}=\partial\mathcal{T}\equiv\nabla\mathcal{T}$ normal to Σ_t. Its geometric dual vectors with respect to g, f and h have components,

$$\mathtt{t}^\mu := g^{\mu\nu}\big(\nabla\mathcal{T}\big)_\nu,\quad \tilde{\mathtt{t}}^\mu := f^{\mu\nu}\big(\nabla\mathcal{T}\big)_\nu,\quad \overset{\#}{\mathtt{t}}{}^\mu := h^{\mu\nu}\big(\nabla\mathcal{T}\big)_\nu. \tag{6.2}$$

The vectors normal to Σ_t with respect to g, f and h are,

$$\nu := -\alpha\,\mathtt{t},\quad \alpha := \big[-g\big(\mathtt{t},\mathtt{t}\big)\big]^{-1/2},\quad g(\nu,\nu) = -1, \tag{6.3a}$$

$$\tilde{\nu} := -\tilde{\alpha}\,\tilde{\mathtt{t}},\quad \tilde{\alpha} := \big[-f\big(\tilde{\mathtt{t}},\tilde{\mathtt{t}}\big)\big]^{-1/2},\quad f(\tilde{\nu},\tilde{\nu}) = -1, \tag{6.3b}$$

$$\overset{\#}{\nu} := -H\,\overset{\#}{\mathtt{t}},\quad H := \big[-h\big(\overset{\#}{\mathtt{t}},\overset{\#}{\mathtt{t}}\big)\big]^{-1/2},\quad h(\overset{\#}{\nu},\overset{\#}{\nu}) = -1, \tag{6.3c}$$

where $g\big(\mathtt{t},\mathtt{t}\big)=g_{\mu\nu}\mathtt{t}^\mu\mathtt{t}^\nu$, analogously for the other metrics. The spatial metrics

$$\gamma := \Phi^*(g),\quad \varphi := \Phi^*(f),\quad \chi := \Phi^*(h), \tag{6.4}$$

are defined again as the pullback of the spacetime metrics onto $T_p^*(\Sigma_t) \otimes T_p^*(\Sigma_t)$ for each $p \in \mathscr{M}$, induced by the embedding of the common spacelike hypersurface Σ_t into $\mathscr{M}$. The extrinsic curvatures read,

$$\begin{aligned}
\forall p \in \Sigma_t, \quad & K : T_p(\Sigma_t) \times T_p(\Sigma_t) \to \mathbb{R} \\
& (u, v) \mapsto K(u, v) = -g\big(u, \nabla_v \nu\big), && (6.5a) \\
\forall p \in \Sigma_t, \quad & \widetilde{K} : T_p(\Sigma_t) \times T_p(\Sigma_t) \to \mathbb{R} \\
& (u, v) \mapsto \widetilde{K}(u, v) = -f\big(u, \widetilde{\nabla}_v \widetilde{\nu}\big), && (6.5b) \\
\forall p \in \Sigma_t, \quad & \overset{\#}{K} : T_p(\Sigma_t) \times T_p(\Sigma_t) \to \mathbb{R} \\
& (u, v) \mapsto \overset{\#}{K}(u, v) = -h\big(u, \overset{\#}{\nabla}_v \overset{\#}{\nu}\big), && (6.5c)
\end{aligned}$$

and the projectors, for each $p \in \mathscr{M}$, are defined as,

$$\begin{aligned}
\perp : T_p(\mathscr{M}) \to T_p(\Sigma_t) \qquad & \widetilde{\perp} : T_p(\mathscr{M}) \to T_p(\Sigma_t) \qquad & \overset{\#}{\perp} : T_p(\mathscr{M}) \to T_p(\Sigma_t) \\
v \mapsto v + g(\nu, v)\nu, \qquad & v \mapsto v + f(\widetilde{\nu}, v)\,\widetilde{\nu}, \qquad & v \mapsto v + h(\overset{\#}{\nu}, v)\,\overset{\#}{\nu}. \qquad (6.6)
\end{aligned}$$

The normal evolution vectors are defined analogously,

$$\mu := \alpha\,\nu = -\alpha^2\,\mathtt{t}, \quad \widetilde{\mu} := \widetilde{\alpha}\,\widetilde{\nu} = -\widetilde{\alpha}^2\,\widetilde{\mathtt{t}}, \quad \overset{\#}{\mu} := H\,\overset{\#}{\nu} = -H^2\,\overset{\#}{\mathtt{t}}. \tag{6.7}$$

Consider a point $p \in \Sigma_t$, and displace it along μ, $\widetilde{\mu}$ and $\overset{\#}{\mu}$ to obtain $p' = p + \mu\,\delta t$, $\widetilde{p} = p + \widetilde{\mu}\,\delta t$ and $\overset{\#}{p} = p + \overset{\#}{\mu}\,\delta t$. Since

$$\begin{aligned}
\mu^\mu\,\nabla_\mu \mathfrak{T} &= \mu^\mu\,\mathtt{t}^\rho g_{\rho\mu} = -\alpha^2\,\mathtt{t}^\mu \mathtt{t}^\rho g_{\rho\mu} = \frac{-\alpha^2}{-\alpha^2} = 1, && (6.8a) \\
\widetilde{\mu}^\mu\,\nabla_\mu \mathfrak{T} &= \widetilde{\mu}^\mu\,\widetilde{\mathtt{t}}^\rho f_{\rho\mu} = -\widetilde{\alpha}^2\,\widetilde{\mathtt{t}}^\mu \widetilde{\mathtt{t}}^\rho f_{\rho\mu} = \frac{-\widetilde{\alpha}^2}{-\widetilde{\alpha}^2} = 1, && (6.8b) \\
\overset{\#}{\mu}{}^\mu\,\nabla_\mu \mathfrak{T} &= \overset{\#}{\mu}{}^\mu\,\overset{\#}{\mathtt{t}}{}^\rho h_{\rho\mu} = -H^2\,\overset{\#}{\mathtt{t}}{}^\mu \overset{\#}{\mathtt{t}}{}^\rho h_{\rho\mu} = \frac{-H^2}{-H^2} = 1, && (6.8c)
\end{aligned}$$

it follows that $p', \widetilde{p}, \overset{\#}{p} \in \Sigma_{t+\delta t}$, as in GR. Hence, the Lie derivative of a geometric object defined in a metric sector, along the normal evolution vector in that metric sector, equals the variation of the geometric object between neighboring hypersurfaces. The meaning of the lapses is also the same, for each metric,

$$\begin{aligned}
\delta\tau &= \sqrt{-g_{\rho\sigma}\,\delta t\,\mu^\rho\,\delta t\,\mu^\sigma} = \delta t\sqrt{\alpha^2} = \alpha\,\delta t, && (6.9a) \\
\delta\widetilde{\tau} &= \widetilde{\alpha}\,\delta t, \qquad \delta\overset{\#}{\tau} = H\,\delta t, && (6.9b)
\end{aligned}$$

where τ is the proper time measured by an observer with 4-velocity ν between the events p and p', $\widetilde{\tau}$ is the proper time measured by an observer with 4-velocity $\widetilde{\nu}$ between the events p and $\widetilde{p}$, and $\overset{\#}{\tau}$ is the proper time measured by an observer with 4-velocity $\overset{\#}{\nu}$ between the events p and $\overset{\#}{p}$. Note that the three observers measure the same coordinate time δt. These observers are Eulerian observers with respect to g, f and h; we call them g-Eulerian, f-Eulerian and

h-Eulerian. Now,

$$\mathscr{L}_{\mu}\,\gamma_{\mu\nu} = -2\alpha K_{\mu\nu}, \quad \mathscr{L}_{\widetilde{\mu}}\,\varphi_{\mu\nu} = -2\widetilde{\alpha}\widetilde{K}_{\mu\nu}, \quad \mathscr{L}_{\overset{\#}{\mu}}\,\chi_{\mu\nu} = -2H\overset{\#}{K}_{\mu\nu}, \tag{6.10}$$

and the Gauss, Codazzi and Ricci relations are valid separately for each metric.

One common timelike direction. Since we only consider the principal square root, the null cones of the metrics are causally coupled. This implies that many common timelike directions exist; namely, those lying within the intersection of the null cones of g and f. Here, we compute explicitly one special common timelike direction.

Since $f = g\,S^2$, $h = g\,S$, we can compute the relation between the timelike vectors $\mathtt{t}, \widetilde{\mathtt{t}}, \overset{\#}{\mathtt{t}}$,

$$\widetilde{\mathtt{t}} = S^{-1}\,\overset{\#}{\mathtt{t}} = S^{-2}\,\mathtt{t}, \quad \overset{\#}{\mathtt{t}} = S\,\widetilde{\mathtt{t}} = S^{-1}\,\mathtt{t}, \quad \mathtt{t} = S^2\,\widetilde{\mathtt{t}} = S\,\overset{\#}{\mathtt{t}}. \tag{6.11}$$

We now compute the square norm of $\overset{\#}{\mathtt{t}}$ with respect to g and f, using (6.11),

$$g_{\mu\nu}\overset{\#}{\mathtt{t}}{}^{\mu}\overset{\#}{\mathtt{t}}{}^{\nu} = g_{\mu\nu}S^{\mu}{}_{\rho}S^{\nu}{}_{\sigma}\widetilde{\mathtt{t}}^{\rho}\widetilde{\mathtt{t}}^{\sigma} = f_{\rho\sigma}\widetilde{\mathtt{t}}^{\rho}\widetilde{\mathtt{t}}^{\sigma} = -\frac{1}{\widetilde{\alpha}^2} < 0, \tag{6.12a}$$

$$f_{\mu\nu}\overset{\#}{\mathtt{t}}{}^{\mu}\overset{\#}{\mathtt{t}}{}^{\nu} = f_{\mu\nu}\big(S^{-1}\big)^{\mu}{}_{\rho}\big(S^{-1}\big)^{\nu}{}_{\sigma}\mathtt{t}^{\rho}\mathtt{t}^{\sigma} = g_{\rho\sigma}\mathtt{t}^{\rho}\mathtt{t}^{\sigma} = -\frac{1}{\alpha^2} < 0. \tag{6.12b}$$

This means that $\overset{\#}{\mathtt{t}}$ is a timelike vector with respect to g, f and h [Koc14, eq. (4.39)] and it provides a common timelike direction.[4] The statement generalizes straightforwardly to the normal $\overset{\#}{\mathtt{v}}$ and the normal evolution vector $\overset{\#}{\mu}$, both collinear to $\overset{\#}{\mathtt{t}}$. The timelike vectors $\mathtt{t}, \widetilde{\mathtt{t}}$ and $\overset{\#}{\mathtt{t}}$ are plotted together with the null cones in Figure 6.1.

The choice of coordinates and the shift vectors. As in GR, we now need to choose coordinates and a basis adapted to the foliation. Analogously to (5.59) we choose

$$\partial_{\mathfrak{T}} := \frac{\partial}{\partial \mathfrak{T}}, \quad \partial_i := \frac{\partial}{\partial x^i}. \tag{6.13}$$

Since (5.60) and (6.8) hold, the differences between the basis vector $\partial_{\mathfrak{T}}$ and each of the normal evolution vectors $\mu, \widetilde{\mu}, \overset{\#}{\mu}$, can only be given by spatial vectors. These are the shift vectors of g, f, h,

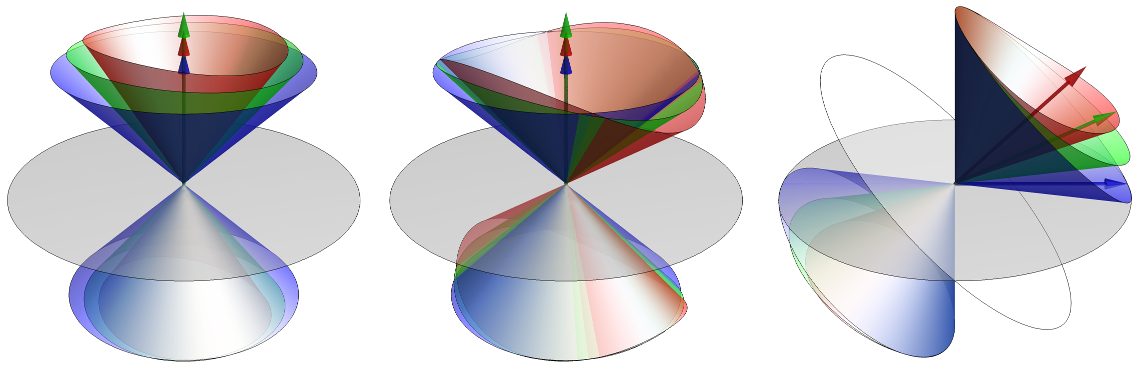
$$\beta := \partial_{\mathfrak{T}} - \mu = \partial_{\mathfrak{T}} - \alpha\,\mathtt{v}, \quad (\mathrm{d}\mathfrak{T})_{\alpha}\beta^{\alpha} = 0, \tag{6.14a}$$

$$\widetilde{\beta} := \partial_{\mathfrak{T}} - \widetilde{\mu} = \partial_{\mathfrak{T}} - \widetilde{\alpha}\,\widetilde{\mathtt{v}}, \quad (\mathrm{d}\mathfrak{T})_{\alpha}\widetilde{\beta}^{\alpha} = 0, \tag{6.14b}$$

$$q := \partial_{\mathfrak{T}} - \overset{\#}{\mu} = \partial_{\mathfrak{T}} - H\,\overset{\#}{\mathtt{v}}, \quad (\mathrm{d}\mathfrak{T})_{\alpha}q^{\alpha} = 0. \tag{6.14c}$$

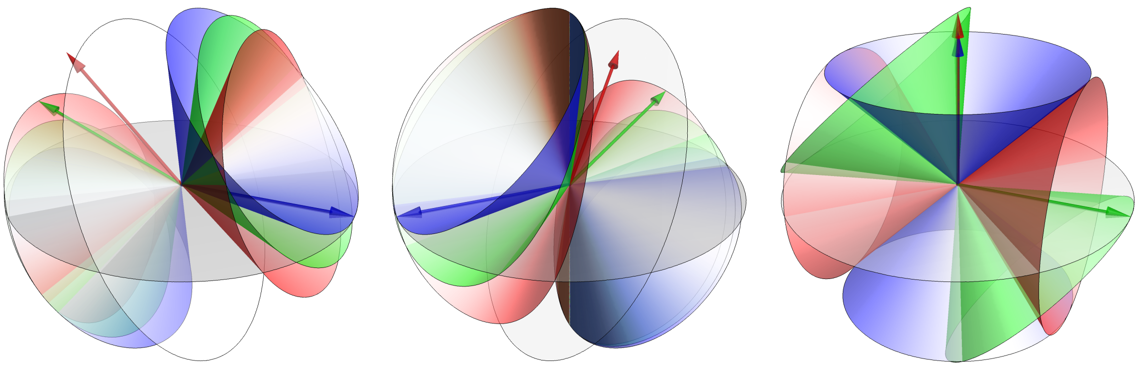
Since there are more than one lapse function and shift vector, it is important to clarify how a spacetime coordinate system is specified in $3+1$ BR. As we saw in Section 5.2, the specification of spatial coordinates on the initial hypersurface Σ_t, of a lapse function used in the definition of the normal evolution vector, and of a shift vector which establishes how the spatial coordinates propagate to the neighboring hypersurface $\Sigma_{t+\delta t}$, $\delta t \ll 1$, is enough

[4] This simple proof that $\overset{\#}{\mathtt{t}}$ is a common timelike direction, provided that the principal S exists and that the hypersurface is spacelike with respect to g and f, was derived independently from [Koc14].



(a) Type I. Σ is (g, f, h)-spacelike. The null cone of g may be contained in the null cone of f, or vice versa. In these cases, the three null cones share none (left figure) or two null directions. If one of the g or f null cones is not contained into the other, then they share four null directions, and each of them shares other four null directions with the h null cone (right figure).

(b) Type IIa. Σ is g-null and (f, h)-spacelike. The three null cones share one null direction.



(c) Type IIb. Σ is g-null and (f, h)-timelike. The null cones share two null directions pairwise.

(d) Type III. Σ is g-null and (f, h)-timelike. The null cones share two null directions pairwise, but one of them is the same for the three null cones.

(e) Type IV. Σ is g-spacelike, f-timelike and h-null. The g and f null cones share *only* two null directions, and each of them shares other two null directions with the h null cone.

Figure 6.1: Null cones associated with the tridimensional metrics of different algebraic types, in the chart where the metrics have the form specified in Table 6.1. The algebraic types encode all the possible ways the null cones can intersect and all the possible configurations of the (appropriately normalized) vectors $\mathtt{t}$ (blue), $\widetilde{\mathtt{t}}$ (red), $\overset{\#}{\mathtt{t}}$ (green) normal to the light grey horizontal plane Σ with respect to g, f, h, respectively. Note that $\mathtt{t}, \widetilde{\mathtt{t}}, \overset{\#}{\mathtt{t}}$ are not necessarily timelike, since here we are not considering a (g, f)-spacelike hypersurface. However, it is clear that we could do that for the types I, IIa, IIb and III by choosing, for example, the white oblique planes [Σ is already (g, f, h)-spacelike for Type I]. The shared timelike directions lie within the intersections of the null cones.

to determine a 4-dimensional coordinate system in a neighborhood of Σ_t. Consider a 4-dimensional coordinate system $x^\mu = (x^0, x^1, x^2, x^3)$ in a neighborhood of Σ_t; then, with $p' = p + \mu\,\delta t$, $\widetilde{p} = p + \widetilde{\mu}\,\delta t = \hat{p} + \mu\,\delta t$, $p, \hat{p} \in \Sigma_t$, $p', \widetilde{p} \in \Sigma_{t+\delta t}$, and for a fixed i,

$$x^i(p') - x^i(p) = -\beta^i\,\delta t, \tag{6.15a}$$

$$x^i(\widetilde{p}) - x^i(p) = -\widetilde{\beta}^i\,\delta t, \tag{6.15b}$$

$$x^i(\widetilde{p}) - x^i(\hat{p}) = -\beta^i\,\delta t. \tag{6.15c}$$

These conditions are all consistent with each other. Indeed, (6.15b, 6.15c) imply

$$x^i(p) = x^i(\hat{p}) + \big(\widetilde{\beta}^i - \beta^i\big)\delta t, \tag{6.15d}$$

and from (6.15a, 6.15b) follows that

$$x^i(\widetilde{p}) = x^i(p') - \big(\widetilde{\beta}^i - \beta^i\big)\delta t. \tag{6.15e}$$

Plugging $x^i(p')$ from (6.15a) into (6.15e), we get (6.15b); then, plugging $x^i(p)$ from (6.15d) into (6.15b), we get (6.15c). Hence, the presence of two normal evolution vectors which map events from Σ_t to $\Sigma_{t+\delta t}$ is compatible with the existence of one spacetime coordinate system in a neighborhood of Σ_t. The same reasoning applies if we consider $\overset{\#}{p} = p + \overset{\#}{\mu}\,\delta t$ together with $p', \widetilde{p}$. Also, the relations in (6.15) show us that it is enough to specify one shift vector on Σ_t to specify the coordinate system in its neighborhood. Compare, for example, (6.15b) with (6.15c). In the same way, the specification of one lapse function determines the coordinate time in terms of the proper time of an Eulerian observer, according to (6.9), and allows us to define one normal evolution vector to drag geometric objects from one hypersurface to neighboring one.

Once we fix one lapse and one shift, we do not have any residual gauge freedom to fix the remaining lapses and shifts. Hence, it seems that we cannot construct the spacetime metrics whose lapses and shifts we do not gauge fix. In the remainder of this section and the next one, we shall see that this is, in fact, not an issue, since the remaining lapse functions and shift vectors are uniquely fixed by the requirement of existence of the square root S and by the BFE, as discussed later at the end of Section 6.2.

The symmetrization condition. The existence of the square root is equivalent to a condition on the tetrads of the metrics, called “symmetrization condition” [DMZ13; HK18]. It is also called (with an abuse of terminology, since it is not a gauge) the “Deser–van Nieuwenhuizen gauge,” and was first introduced in [Zum70].

Given two Lorentzian metrics g and f, it is possible to write them as,

$$g_{\mu\nu} = E^{\top}{}_{\mu}{}^{\mathbf{A}}\,\boldsymbol{\eta}_{\mathbf{AB}}\,E^{\mathbf{B}}{}_{\nu}, \quad f_{\mu\nu} = M_{\circ}{}^{\top}{}_{\mu}{}^{\mathbf{A}}\,\mathbf{L}^{\top}{}_{\mathbf{A}}{}^{\mathbf{C}}\boldsymbol{\eta}_{\mathbf{CD}}\mathbf{L}^{\mathbf{D}}{}_{\mathbf{B}}\,M_{\circ}{}^{\mathbf{B}}{}_{\nu}, \tag{6.16}$$

where the Lorentz transformation $\mathbf{L}$ accounts for the freedom of choosing different Lorentz frames for the metrics. For the ease of notation, we now move to matrix notation and compute $g^{-1}f$,

$$g^{-1}f = E^{-1}\boldsymbol{\eta}^{-1}E^{\top,-1}M_{\circ}{}^{\top}\mathbf{L}^{\top}\boldsymbol{\eta}\mathbf{L}M_{\circ}. \tag{6.17}$$

Let us insert the identity in the Lorentz frame $\mathbb{1}^{\top} = \big(\boldsymbol{\eta}EE^{-1}\boldsymbol{\eta}^{-1}\big)\big(E^{\top,-1}E^{\top}\big)$ in (6.17),

$$g^{-1}f = E^{-1}\boldsymbol{\eta}^{-1}E^{\top,-1}M_{\circ}{}^{\top}\mathbf{L}^{\top}\big(\boldsymbol{\eta}EE^{-1}\boldsymbol{\eta}^{-1}\big)\big(E^{\top,-1}E^{\top}\big)\boldsymbol{\eta}\mathbf{L}M_{\circ}. \tag{6.18}$$

In (6.18) we can recognize

$$g^{-1}f = E^{-1}\boldsymbol{\eta}^{-1}E^{\top,-1}\big(M_\circ{}^{\top}\mathbf{L}^{\top}\boldsymbol{\eta}E\big)E^{-1}\boldsymbol{\eta}^{-1}E^{\top,-1}\big(E^{\top}\boldsymbol{\eta}\mathbf{L}M_\circ\big), \tag{6.19}$$

and notice that the terms in parenthesis are the transpose of each other. Suppose that these two transposes are equal,

$$E^{\top}\boldsymbol{\eta}\mathbf{L}M_\circ = \big(E^{\top}\boldsymbol{\eta}\mathbf{L}M_\circ\big)^{\top}, \tag{6.20}$$

which is the symmetrization condition. Then, (6.18) becomes,

$$\begin{aligned} g^{-1}f &= E^{-1}\boldsymbol{\eta}^{-1}E^{\top,-1}\big(E^{\top}\boldsymbol{\eta}\mathbf{L}M_\circ\big)E^{-1}\boldsymbol{\eta}^{-1}E^{\top,-1}\big(E^{\top}\boldsymbol{\eta}\mathbf{L}M_\circ\big) \\ &= \big(E^{-1}\mathbf{L}M_\circ\big)\big(E^{-1}\mathbf{L}M_\circ\big). \end{aligned} \tag{6.21}$$

It immediately follows,

$$S = \sqrt{g^{-1}f} = E^{-1}\mathbf{L}M_\circ. \tag{6.22}$$

We just showed that, if the symmetrization condition holds, a real square root of $g^{-1}f$ exists. The converse is also true; consider (6.19) and suppose that a real square root of $g^{-1}f$ exists,

$$g^{-1}f = \Big[E^{-1}\boldsymbol{\eta}^{-1}E^{\top,-1}\big(M_\circ{}^{\top}\mathbf{L}^{\top}\boldsymbol{\eta}E\big)\Big]\Big[E^{-1}\boldsymbol{\eta}^{-1}E^{\top,-1}\big(E^{\top}\boldsymbol{\eta}\mathbf{L}M_\circ\big)\Big] = XX. \tag{6.23}$$

This equation implies,

$$X = E^{-1}\boldsymbol{\eta}^{-1}E^{\top,-1}\big(M_\circ{}^{\top}\mathbf{L}^{\top}\boldsymbol{\eta}E\big) = E^{-1}\boldsymbol{\eta}^{-1}E^{\top,-1}\big(E^{\top}\boldsymbol{\eta}\mathbf{L}M_\circ\big), \tag{6.24}$$

that is, the symmetrization condition (6.20). We proved that a real square root of $g^{-1}f$ exists if and only if there exist $\mathbf{L}$ such that the object $E^{\top}\boldsymbol{\eta}\mathbf{L}M_\circ$ is symmetric.

We stress that, even if such $\mathbf{L}$ exists, we do not need to use it in (6.16).[5] Given any two tetrads E_1 and M_1, if a Lorentz transformation $\mathbf{L}$ exists such that the symmetrization condition holds, then a real square root $\sqrt{g^{-1}f}$ exists for *any* choice of the tetrads of g and f. Indeed, given any two tetrads E_1 and E_2 for the same metric g, there always exists a Lorentz transformation $O \coloneqq E_1E_2^{-1}$ such that $E_1 = OE_2$,

$$O^{\top}\boldsymbol{\eta}O = E_2^{\top,-1}E_1^{\top}\boldsymbol{\eta}E_1E_2^{-1} = E_2^{\top,-1}E_1^{\top}\big(E_1^{\top,-1}gE_1^{-1}\big)E_1E_2^{-1} \tag{6.25a}$$

$$= E_2^{\top,-1}gE_2^{-1} = \boldsymbol{\eta}, \tag{6.25b}$$

that is, O is a Lorentz transformation. It follows,

$$S = \sqrt{g^{-1}f} = \sqrt{E_1^{-1}\boldsymbol{\eta}^{-1}E_1^{\top,-1}M_1^{\top}\boldsymbol{\eta}M_1} \tag{6.26a}$$

$$= \sqrt{E^{-1}O_1^{-1}\boldsymbol{\eta}^{-1}O_1^{\top,-1}E^{\top,-1}M_\circ{}^{\top}O_2^{\top}\boldsymbol{\eta}O_2M_\circ} \tag{6.26b}$$

$$= \sqrt{E^{-1}\boldsymbol{\eta}^{-1}E^{\top,-1}M_\circ{}^{\top}\boldsymbol{\eta}M_\circ} \tag{6.26c}$$

[5]However, the symmetrizing $\mathbf{L}$ must be used in (6.22).

$$= \sqrt{E^{-1}\boldsymbol{\eta}^{-1}E^{\top,-1}M_\circ{}^\top\mathbf{L}^\top\boldsymbol{\eta}\mathbf{L}M_\circ} \tag{6.26d}$$
$$= E^{-1}\mathbf{L}M_\circ, \tag{6.26e}$$

with E_1 and M_1 arbitrary tetrads, $O_1 \coloneqq E_1E^{-1}$, $O_2 \coloneqq M_1M_\circ{}^{-1}$ Lorentz transformations, and $\mathbf{L}$ the symmetrizing Lorentz transformation for E and $M_\circ$. In other words, given the metrics, the *choice* of their tetrads does not affect the existence of the square root; what affects it is the *possibility to symmetrize any two arbitrary tetrads g* and f with a Lorentz transformation. Indeed, if it is possible to symmetrize two tetrads E and $M_\circ$, then it is possible to symmetrize all other tetrads of the same metrics with a different symmetrizing Lorentz transformation,

$$E^\top\boldsymbol{\eta}\mathbf{L}M_\circ = \big(E^\top\boldsymbol{\eta}\mathbf{L}M_\circ\big)^\top, \tag{6.27a}$$
$$E_1^\top O_1^{\top,-1}\boldsymbol{\eta}\mathbf{L}O_2^{-1}M_1 = M^\top O_2^{\top,-1}\mathbf{L}^\top\boldsymbol{\eta}O_1^{-1}E_1, \tag{6.27b}$$
$$E_1^\top\big(O_1^{\top,-1}\boldsymbol{\eta}O_1^{-1}\big)\big(O_1\mathbf{L}O_2^{-1}\big)M_1 = M^\top\big(O_2^{\top,-1}\mathbf{L}^\top O_1^\top\big)\big(O_1^{\top,-1}\boldsymbol{\eta}O_1^{-1}\big)E_1, \tag{6.27c}$$
$$E_1^\top\boldsymbol{\eta}\mathbf{L}_1M_1 = M^\top\mathbf{L}_1^\top\boldsymbol{\eta}E_1, \tag{6.27d}$$

with $\mathbf{L}_1 \coloneqq O_1\mathbf{L}O_2^{-1}$. As a last remark, we note that since, given the metrics, the choice of their tetrads does not affect the existence of the square root, it does not affect the causal coupling of the metrics either. Indeed, given a metric, its null cone is insensitive to the choice of the metric's tetrad.

We already mentioned that the existence of the square root S is equivalent to the existence of the geometric mean metric h of g and f. We now review this equivalence in more detail, since it is relevant in the formulation of the $3+1$ decomposition in BR. The geometric mean of two sesquilinear positive definite forms was first defined in [PW75].[6] We limit ourselves to symmetric positive definite quadratic forms (i.e., Riemannian metrics), and consider their matrix representation. Then, the matrix representation of the geometric mean $A \# B$ of two symmetric positive definite quadratic forms A and B is defined to be the unique solution to the matrix equation $XA^{-1}X = B$, given by [Hig08, Subsec. 2.10]

$$A \# B \coloneqq X = B^{1/2}\big(B^{-1/2}AB^{-1/2}\big)^{1/2}B^{1/2} = B\big(B^{-1}A\big)^{1/2} = \big(AB^{-1}\big)^{1/2}B. \tag{6.28}$$

This definition can be extended to Lorentzian symmetric quadratic forms if and only if the square root S exists [HK18]. Indeed, if we replace $B \to g$ and $A \to f$,

$$h \coloneqq f \# g = g\big(g^{-1}f\big)^{1/2} = gS = fS^{-1} = f\big(f^{-1}g\big)^{1/2} = g \# f. \tag{6.29}$$

Hence, as for Riemannian metrics, also for two arbitrary Lorentzian metrics $g \# f = f \# g$. The geometric mean h exists if and only if the square root S exists. If it exists, then, given

[6] A sesquilinear form over a complex vector space V is a map $\phi : V \times V \to \mathbb{C}$ such that

$$\phi(x+y, z+w) = \phi(x,z) + \phi(x,w) + \phi(y,z) + \phi(y,w),$$
$$\phi(ax, by) = a^* b\,\phi(x,y),$$

for all $x, y, z, w \in V$ and all $a, c \in \mathbb{C}$; a^* is the complex conjugate of a [Lan05, p. 531].

any two tetrads E_1 and M_1 and their symmetrizing Lorentz transformation $\mathbf{L}_1$,

$$h = gS = E_1^\top \boldsymbol{\eta} E_1 E_1^{-1} \mathbf{L}_1 M_1 = E_1^\top \boldsymbol{\eta} \mathbf{L}_1 M_1 = M_1^\top \mathbf{L}_1^\top \boldsymbol{\eta} E_1. \tag{6.30}$$

Hence, solving the symmetrization condition for the symmetrizing Lorentz transformation is equivalent to compute the geometric mean h in terms of the symmetrizing Lorentz transformation and the tetrads of the metrics. For a formal and rigorous treatment of these topics, we refer the reader to [Koc14].

The symmetrized principal tetrads. In this paragraph we expand on the meaning of the symmetrization condition. In [Wat04, Theorem 1.7.30] it is proved that a symmetric matrix g having nonsingular leading principal submatrices, has a unique LDLT decomposition, that is, it can be uniquely written as,[7]

$$g = LDL^\top, \tag{6.31}$$

with L lower triangular matrix and D diagonal matrix. Considering, for simplicity, the 2-dimensional case, the matrix D can be further decomposed as,

$$D = \begin{pmatrix} D_{00} & 0 \\ 0 & D_{11} \end{pmatrix}, \quad C = \begin{pmatrix} c_1\sqrt{|D_{00}|} & 0 \\ 0 & c_2\sqrt{|D_{11}|} \end{pmatrix} \Longrightarrow D = C\ \mathrm{Sig}(g)\, C, \tag{6.32}$$

where $\mathrm{Sig}(g)$ is the signature matrix of g, and the c_i are free to be arbitrarily set to ± 1, which corresponds to the freedom of choosing the sign in front of the square roots. For the matrix representation (in a given chart) of a Lorentzian metric,

$$g = L\, C\, \boldsymbol{\eta}\, C\, L^\top. \tag{6.33}$$

We see from (6.33) that we have computed a tetrad for g, equal to

$$\Pi \coloneqq C\, L^\top. \tag{6.34}$$

This tetrad is not uniquely defined, since we can choose arbitrarily the signs in C in (6.32). However, since Π is upper triangular, the i^{th} column of the inverse Π^{-1} is multiplied by c_i only. Hence, the directions defined by the columns of Π^{-1} are uniquely specified. In index notation, ${\Pi^{-1}}^{\mu}{}_{\mathbf{A}}$, hence the columns of Π^{-1} are spacetime vectors and define the orthonormal frame of the tetrad Π. We call this tetrad "principal tetrad" and the directions defined by the associated orthonormal frame "principal directions," since they are analogous to the principal axes of rotation for a rigid body obtained by diagonalizing the symmetric matrix representing the inertia tensor [GPS02, Sec. 5.4].

[7] Having nonsingular leading principal submatrices is a stronger condition than being invertible. For example, the Schwarzschild metric in null coordinates does not have nonsingular leading principal submatrices, even though it is invertible. However, there may exist a coordinate transformation such that the matrix representation of the metric has nonsingular leading submatrices.

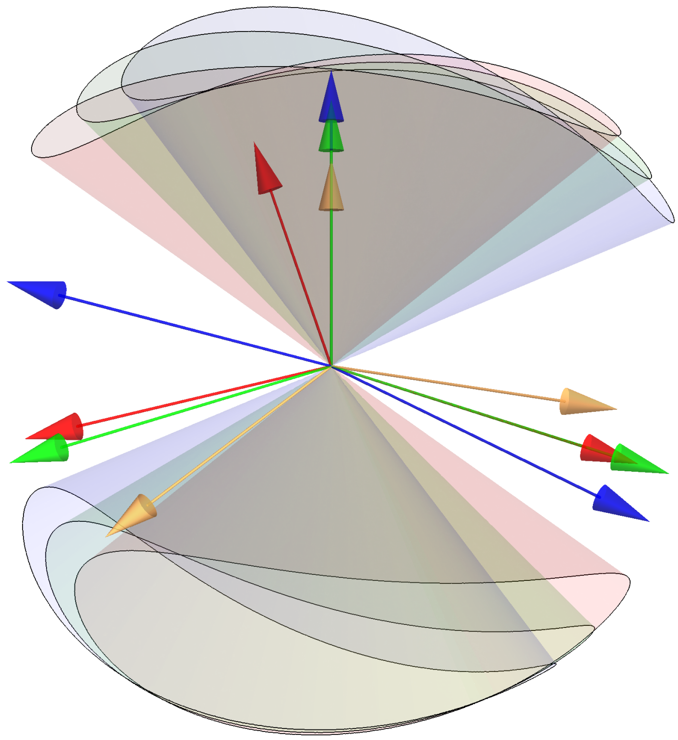


Figure 6.2: The (appropriately normalized) symmetrized principal directions Π^{-1} (blue), ${\tilde{\Pi}_{\circ}}^{-1}$ (orange), $\tilde{\Pi}^{-1}$ (red) and $\overset{\#}{\Pi}{}^{-1}$ (green) are plotted together with the null cones, for tridimensional Lorentzian metrics. The symmetrization condition maps the orange directions into the red ones, such that one spacelike principal direction in $\tilde{\Pi}^{-1}$ coincides with one spacelike principal direction in $\overset{\#}{\Pi}{}^{-1}$.

Consider the principal tetrads for the spacetime metrics g and f,

$$g = \Pi^{\top}\boldsymbol{\eta}\,\Pi, \quad {\tilde{\Pi}_{\circ}}^{\top}\boldsymbol{\eta}\,\tilde{\Pi}_{\circ}, \tag{6.35}$$

assuming that the square root S exists. Then, we can compute the principal symmetrizing Lorentz transformation $\boldsymbol{\Omega} = \Pi\, S\, {\tilde{\Pi}_{\circ}}^{-1}$, the geometric mean $h = gS$ and its principal tetrad $h = \overset{\#}{\Pi}{}^{\top}\boldsymbol{\eta}\,\overset{\#}{\Pi}$. With these objects defined, we can introduce an intuitive meaning for the symmetrization condition involving the principal tetrads. Consider the symmetrized tetrad for f, $\tilde{\Pi} := \boldsymbol{\Omega}\,\tilde{\Pi}_{\circ}$. The principal directions associated with Π (blue), $\tilde{\Pi}_{\circ}$ (orange), $\tilde{\Pi}$ (red) and $\overset{\#}{\Pi}$ (green) are plotted in Figure 6.2 for generic Lorentzian 3-dimensional metrics g, f and their geometric mean. The principal directions of f are mapped to new principal directions such that one of them coincides with one principal direction of h. In Figure 6.2, the coinciding principal directions for f and h are the spacelike directions at the right of the picture. This means that the third (or the fourth, in 4 dimensions) column of $\tilde{\Pi}, \overset{\#}{\Pi}$ are proportional. Note that the timelike principal directions of $\Pi, \tilde{\Pi}_{\circ}, \overset{\#}{\Pi}$ are collinear. This is the case since the principal tetrads are upper triangular, that is, the only nonzero component of the first column of $\Pi^{-1}, {\tilde{\Pi}_{\circ}}^{-1}, \overset{\#}{\Pi}{}^{-1}$ is the timelike one.

Suppose there exist a chart where the matrix representation of a metric has nonsingular leading submatrices. Then the decomposition (6.31) is unique, as are the principal directions. Even in a chart where the matrix representation of the metric does not have nonsingular leading submatrices, we can still consider the principal directions by writing the inverse of the principal tetrad, which transforms as a spacetime vector under coordinate transformations, in the considered chart.

We remark that the considerations in this paragraph about the principal tetrads are not formally proved, but have been verified numerically in a large variety of cases, one of them being shown in Figure 6.2.

The $3+1$ decomposition of the symmetrization condition. A generic Lorentz transformation, in the canonical basis of the Lorentz frame, can be parametrized as [Mei13, Sec. 1.6],

$$\mathbf{L} = \mathbf{\Lambda P} = \begin{pmatrix} \lambda & \mathbf{p}^\top \boldsymbol{\delta} \\ \mathbf{p} & \mathbf{\Lambda}_\mathrm{s} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{R} \end{pmatrix}, \tag{6.36}$$

that is, as the product of a generic boost $\mathbf{\Lambda}$ and a generic spatial rotation $\mathbf{P}$. The quantities appearing in (6.36) are,

$$\begin{aligned}
&\boldsymbol{\delta} && : \text{spatial part of the Minkowski metric, i.e.,} \\
& && \quad \text{the spatial Euclidean metric,} && (6.37\text{a}) \\
&\mathbf{p} \;\; := \sinh(\mathbf{w}) && : \text{spatial real vector in the Lorentz frame,} \\
& && \quad \text{called ``separation parameter,''} && (6.37\text{b}) \\
&\mathbf{w} && : \text{rapidities of the Lorentz boost } \mathbf{\Lambda}, && (6.37\text{c}) \\
&\lambda \;\; = \sqrt{1 + \mathbf{p}^\top \boldsymbol{\delta} \mathbf{p}} && : \text{Lorentz ``}\gamma\text{'' factor of the Lorentz boost } \mathbf{\Lambda}, && (6.37\text{d}) \\
&\mathbf{\Lambda}_\mathrm{s} = \big(\mathbf{I} + \mathbf{p}\mathbf{p}^\top \boldsymbol{\delta}\big)^{1/2} \\
&\quad\;\; = \mathbf{I} + \frac{\mathbf{p}\mathbf{p}^\top \boldsymbol{\delta}}{\lambda + 1} && : \text{spatial part of the Lorentz boost } \mathbf{\Lambda}, && (6.37\text{e}) \\
&\mathbf{I} && : \text{spatial part of the identity } \mathtt{1} \text{ in the Lorentz frame,} && (6.37\text{f}) \\
&\mathbf{R}\,, \quad \boldsymbol{\delta}^{-1} \mathbf{R}^\top \boldsymbol{\delta} = \mathbf{R}^{-1} && : \text{Euclidean spatial rotation in the Lorentz frame.} && (6.37\text{g})
\end{aligned}$$

Note that the Lorentz boost vector of $\mathbf{\Lambda}$, usually denoted with "β", is $\mathbf{v} = \lambda^{-1}\mathbf{p}$. The relation (6.36) can be seen as the $3+1$ decomposition of a Lorentz transformation.

In order to write the $3+1$ decomposition of the vielbeins E and $M_\circ$, we first need to define the Cauchy frames for the metrics, in the basis (6.13). The Cauchy frames read,

$$\theta_g{}^0 = \mathrm{d}\mathcal{T}, \qquad \theta_g{}^i = \mathrm{d}x^i + \beta^i \mathrm{d}\mathcal{T}, \qquad \theta_{g0} = \partial_\mathcal{T} - \beta^i \partial_i, \qquad \theta_{gi} = \partial_i, \tag{6.38a}$$

$$\theta_f{}^0 = \mathrm{d}\mathcal{T}, \qquad \theta_f{}^i = \mathrm{d}x^i + \widetilde{\beta}^i \mathrm{d}\mathcal{T}, \qquad \theta_{f0} = \partial_\mathcal{T} - \widetilde{\beta}^i \partial_i, \qquad \theta_{fi} = \partial_i, \tag{6.38b}$$

and the vielbeins of g and f are [Koc18],

$$E^\mathbf{0} = \alpha\, \theta_g{}^0 = \alpha\, \mathrm{d}\mathcal{T}, \qquad E^\mathbf{a} = e^\mathbf{a}{}_i \theta_g{}^i = e^\mathbf{a}{}_i \big(\mathrm{d}x^i + \beta^i \mathrm{d}\mathcal{T}\big), \tag{6.39a}$$

$$E_\mathbf{0} = \alpha^{-1} \theta_{g0} = \alpha^{-1} \big(\partial_\mathcal{T} - \beta^i \partial_i\big), \qquad E_\mathbf{a} = e^i{}_\mathbf{a} \theta_{gi} = e^i{}_\mathbf{a} \partial_i, \tag{6.39b}$$

$$M_\circ{}^\mathbf{0} = \widetilde{\alpha}\, \theta_f{}^0 = \widetilde{\alpha}\, \mathrm{d}\mathcal{T}, \qquad M_\circ{}^\mathbf{a} = m_\circ{}^\mathbf{a}{}_i \theta_f{}^i = m_\circ{}^\mathbf{a}{}_i \big(\mathrm{d}x^i + \widetilde{\beta}^i \mathrm{d}\mathcal{T}\big), \tag{6.39c}$$

$$M_{\circ\mathbf{0}} = \widetilde{\alpha}^{-1}\theta_{f0} = \widetilde{\alpha}^{-1}\big(\partial_{\mathfrak{T}} - \widetilde{\beta}^i\partial_i\big), \qquad M_{\circ\mathbf{a}} = m_\circ{}^i{}_{\mathbf{a}}\theta_{fi} = m_\circ{}^i{}_{\mathbf{a}}\partial_i, \tag{6.39d}$$

where e, m are the spatial tetrads of the spatial metrics,

$$\gamma_{ij} = e^\top{}_i{}^{\mathbf{a}}\,\boldsymbol{\delta}_{\mathbf{ab}}\,e^{\mathbf{b}}{}_j, \quad \varphi_{ij} = m_\circ{}^\top{}_i{}^{\mathbf{a}}\,\boldsymbol{\delta}_{\mathbf{ab}}\,m_\circ{}^{\mathbf{b}}{}_j. \tag{6.40}$$

The expressions in (6.39) provide the $3+1$ decomposition for the tetrads. With these, we can now write the $3+1$ decomposition for the geometric mean h,

$$h = E^\top\boldsymbol{\eta}\mathbf{L}M_\circ = \begin{pmatrix}\alpha & \beta^\top e^\top\\ 0 & e^\top\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & \boldsymbol{\delta}\end{pmatrix}\begin{pmatrix}\lambda & \mathbf{p}^\top\boldsymbol{\delta}\\ \mathbf{p} & \boldsymbol{\Lambda}_{\rm s}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \mathbf{R}\end{pmatrix}\begin{pmatrix}\widetilde{\alpha} & 0\\ m_\circ\widetilde{\beta} & m_\circ\end{pmatrix}, \tag{6.41}$$

which in [Koc14, p. 79], using (6.37), is proved to be equal to

$$h_{00} = -\frac{\alpha\widetilde{\alpha}}{\lambda} + \Big(\beta - \alpha e^{-1}\mathbf{v}\Big)^\top e^\top\boldsymbol{\delta}\boldsymbol{\Lambda}_{\rm s}\mathbf{R}m_\circ\Big[\widetilde{\beta} + \widetilde{\alpha}\big(\mathbf{R}m_\circ\big)^{-1}\mathbf{v}\Big], \tag{6.42a}$$

$$h_{0i} = \Big[\Big(\beta - \alpha e^{-1}\mathbf{v}\Big)^\top\Big(e^\top\boldsymbol{\delta}\boldsymbol{\Lambda}_{\rm s}\mathbf{R}m_\circ\Big)^\top\Big]_i, \tag{6.42b}$$

$$h_{i0} = \Big\{e^\top\boldsymbol{\delta}\boldsymbol{\Lambda}_{\rm s}\mathbf{R}m_\circ\Big[\widetilde{\beta} + \widetilde{\alpha}\big(\mathbf{R}m_\circ\big)^{-1}\mathbf{v}\Big]\Big\}_i, \tag{6.42c}$$

$$h_{ij} = \Big(e^\top\boldsymbol{\delta}\boldsymbol{\Lambda}_{\rm s}\mathbf{R}m_\circ\Big)_{ij}. \tag{6.42d}$$

Hence, the $3+1$ decomposition of the symmetrization condition $h = h^\top$ reads [Koc14, Theorem 4.1],

$$\widetilde{\beta} - \beta = -\Big[\alpha e^{-1} + \widetilde{\alpha}\big(\mathbf{R}m_\circ\big)^{-1}\Big]\mathbf{v}, \tag{6.43a}$$

$$e^\top\boldsymbol{\delta}\boldsymbol{\Lambda}_{\rm s}\mathbf{R}m_\circ = \big(e^\top\boldsymbol{\delta}\boldsymbol{\Lambda}_{\rm s}\mathbf{R}m_\circ\big)^\top, \tag{6.43b}$$

that is, the spatial part χ of the geometric mean is symmetric and the shift vectors of g and f are dependent. In (6.42) we can identify the $3+1$ variables of h as

$$H = \sqrt{\frac{\alpha\widetilde{\alpha}}{\lambda}}, \tag{6.44a}$$

$$q = \beta - \alpha e^{-1}\mathbf{v} = \widetilde{\beta} + \widetilde{\alpha}\big(\mathbf{R}m_\circ\big)^{-1}\mathbf{v}, \tag{6.44b}$$

$$\chi = e^\top\boldsymbol{\delta}\boldsymbol{\Lambda}_{\rm s}\mathbf{R}m_\circ. \tag{6.44c}$$

Note that the relation between the shifts implies that the normal evolution vectors are also related,

$$\mu - \widetilde{\mu} = -\Big[\alpha e^{-1} + \widetilde{\alpha}\big(\mathbf{R}m_\circ\big)^{-1}\Big]\mathbf{v}, \tag{6.45a}$$

$$\overset{\#}{\mu} = \mu + \alpha e^{-1}\mathbf{v} = \widetilde{\mu} - \widetilde{\alpha}\big(\mathbf{R}m_\circ\big)^{-1}\mathbf{v}. \tag{6.45b}$$

The conditions (6.43) state the casual coupling between the metrics. This is easily seen in (6.45a), where the difference between the normal directions is bounded since $\mathbf{v}$ is a boost vector, which means that $\mathbf{v}^\top\boldsymbol{\delta}\mathbf{v} < 1$.

In (6.43), the spatial rotation $\mathbf{R}$ and the separation parameter $\mathbf{p}$ are not determined yet. Their expressions are needed to be able to write down the 3+1 decomposition of the bimetric interactions. The computation of $\mathbf{R}$ is reported in [HKS14, Appendix A] and performed in more detail in [Koc14, eqs.(A.81)–(A.87)], and consists in solving (6.43b) for $\mathbf{R}$; the solution is,

$$\mathbf{R} = \left(\boldsymbol{\delta}^{-1}{\mathbf{R}_{\mathrm{o}}}^{\top}\boldsymbol{\delta}\mathbf{R}_{\mathrm{o}}\right)^{1/2}{\mathbf{R}_{\mathrm{o}}}^{-1}, \quad \mathbf{R}_{\mathrm{o}} \coloneqq \boldsymbol{\delta}^{-1}\left(e\, {m_{\circ}}^{-1}\right)^{\top}\boldsymbol{\delta}\boldsymbol{\Lambda}_{\mathrm{s}}. \tag{6.46}$$

This solution always exists, since $\boldsymbol{\delta}^{-1}{\mathbf{R}_{\mathrm{o}}}^{\top}\boldsymbol{\delta}\mathbf{R}_{\mathrm{o}}$ is strictly positive definite, and it is unique.[8] Existence and uniqueness follow from the fact that this solution corresponds to the polar decomposition for the invertible matrix ${\mathbf{R}_{\mathrm{o}}}^{-1}$, which always exists and is unique (regarding the polar decomposition of a matrix, we refer the reader to [Hal15, Sec. 2.5]),[9]

$${\mathbf{R}_{\mathrm{o}}}^{-1} = \left(\boldsymbol{\delta}^{-1}{\mathbf{R}_{\mathrm{o}}}^{\top}\boldsymbol{\delta}\mathbf{R}_{\mathrm{o}}\right)^{-1/2}\mathbf{R} = \left({\mathbf{R}_{\mathrm{o}}}^{-1}\boldsymbol{\delta}^{-1}{\mathbf{R}_{\mathrm{o}}}^{-1,\top}\boldsymbol{\delta}\right)^{1/2}\mathbf{R}. \tag{6.47}$$

In general, the matrix $\mathbf{R}_{\mathrm{o}}$ does not have any other property than being invertible. The solution for $\mathbf{R}$ always exists, implying that the spatial part of the symmetrization condition can always be satisfied. The solution for $\mathbf{R}$ depends on $e, m_{\circ}$ and $\boldsymbol{\Lambda}_{\mathrm{s}}$. We stress that $\boldsymbol{\Lambda}_{\mathrm{s}}$ is not free; it is uniquely specified by $\mathbf{p}$, which can be determined by solving the 3+1 equations, as pointed out in [HL18; Koc18]. Now we can write the $3+1$ decomposition of the square root matrix [Koc18],

$$S = \begin{pmatrix} 1 & 0 \\ -q & \mathbf{I} \end{pmatrix} \begin{pmatrix} \dfrac{\widetilde{\alpha}}{\alpha\lambda} & \dfrac{\lambda}{\alpha}\mathbf{v}^{\top}\boldsymbol{\delta}\mathbf{R}m_{\circ} \\ -\dfrac{\widetilde{\alpha}}{\lambda}e^{-1}\mathbf{v} & e^{-1}{\boldsymbol{\Lambda}_{\mathrm{s}}}^{-1}\mathbf{R}m_{\circ} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q & \mathbf{I} \end{pmatrix}. \tag{6.48}$$

The construction of a bimetric foliation. When the metrics are causally coupled, it is possible to formulate the $3+1$ decomposition in BR, and the geometric mean and the square root exist and can be computed using the $3+1$ decomposition of g, f and the symmetrizing Lorentz transformation $\mathbf{L}$.

In practice, when trying to solve the BFE for a specific solution—e.g., a spherically symmetric solution—an ansatz is chosen for the two metrics. However, as we saw in (6.1), we cannot choose any ansatz that we want, since the square root may not exist. Therefore, in BR, it is preferable to specify an ansatz for the tetrads, not the metrics, and check if the symmetrization condition holds.

In the $3+1$ formulation, we would like to construct a 4-dimensional bimetric spacetime

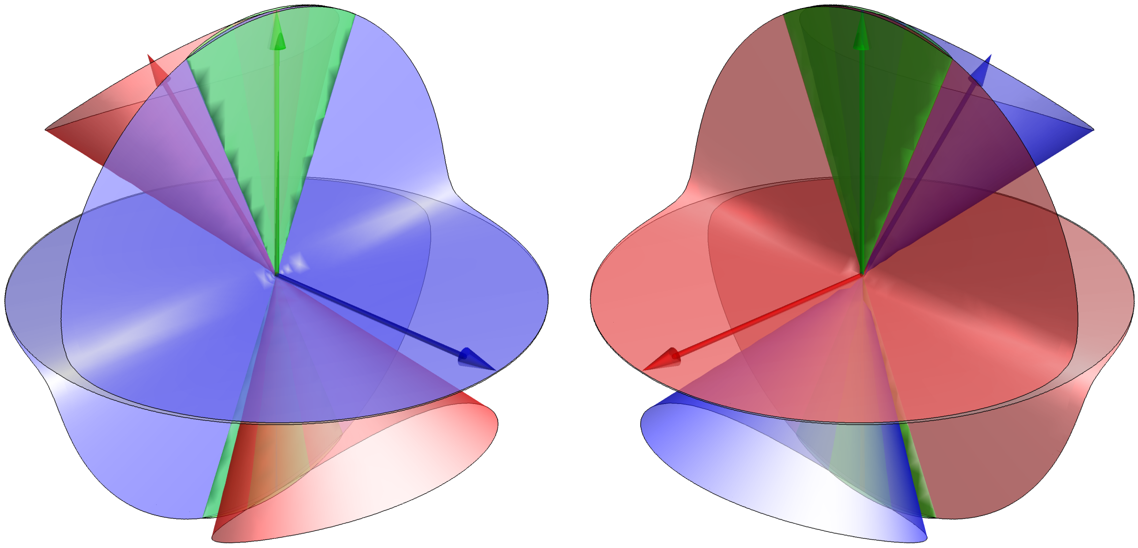
[8] Note that, if an indefinite metric was involved, instead of $\boldsymbol{\delta}$, the positive definiteness would not be guaranteed and the real square root in (6.46) would not always exist. Consider the invertible matrix $M = \begin{pmatrix} 0 & 1 \\ 1/2 & 0 \end{pmatrix}$. The product $MM' = M\boldsymbol{\eta}^{-1}M^{\top}\boldsymbol{\eta}$, with $\boldsymbol{\eta}$ being the 2-dimensional Minkowski metric, is equal to $\operatorname{diag}(-1, -1/4)$, which clearly does not have a real square root. That is why we did not solve the 4-dimensional symmetrization condition (6.20) with the same strategy.

[9] Note that in [HKS14, Appendix A], the inverse of both sides of (6.43b) is solved for $\mathbf{R}$, whereas in [Koc14, eqs.(A.81)–(A.87)], (6.43b) itself is solved for $\mathbf{R}$. This corresponds to consider, respectively, the right or left polar decomposition of ${\mathbf{R}_{\mathrm{o}}}^{-1}$ and yields equivalent results. We use the same solution as in [HKS14, Appendix A], which is also used in [Koc18].

by specifying the initial data on a hypersurface which is spacelike with respect to both metrics and then evolve them by solving the Cauchy problem for the BFE (discussed in the next section) along a common timelike direction. In order to do this, we must guarantee that the specified initial data admit the existence of a real square root. In other words, in the $3+1$ formulation, we must set up the spatial quantities on the initial hypersurface Σ_t such that Σ_t is spacelike with respect to both spacetime metrics and there is a common timelike direction for the metrics. This is done in the following way. Consider a 4-dimensional differentiable manifold $\mathscr{M}$, with no metrics (hence no square root) defined on it. Define a hypersurface Σ_t on it, as the set determined by the equation $\mathcal{T} = t$ for the scalar field $\mathcal{T}$. Define *any* two Riemannian metrics γ, φ in $T^*(\Sigma_t) \otimes T^*(\Sigma_t)$ by specifying their spatial tetrads; it is always possible to do that, see for example the theory of the space of Riemannian metrics defined on a smooth open manifold, also called "superspace" or "space of geometries" [Whe64] (see also [MTW73, Ch. 43], [Sch01] and [KM97, Sec. 45]).[10] Consider three vector fields $\beta, \widetilde{\beta}, q$ on $T(\Sigma_t)$ and impose the relations (6.43a, 6.44b) between them, with $\mathbf{p}$ real vector in the Lorentz frame. Solve the spatial symmetrization condition (6.43b) for $\mathbf{R}$; since the two metrics are Riemannian and $\mathbf{p}$ is a *real* vector, it is always possible to do that and obtain (6.46). At this point we also have the Riemannian metric χ according to (6.44c). Consider the metric fields g, f, h on $T^*(\mathscr{M}) \otimes T^*(\mathscr{M})$ in a neighborhood of Σ_t, which are constructed with the Riemannian metrics as their spatial parts, the shifts, and the lapses, according to (5.68, 6.42, 6.44). Then, the square root tensor field $S = \left(g^{-1}f\right)^{1/2}$ exists *by construction* in a neighborhood of Σ_t, and is computed by using the spatial tetrads, the shifts, the lapses, $\mathbf{R}$ and $\mathbf{p}$, according to (6.48). Equivalently, the hypersurface Σ_t is spacelike with respect to the metrics g, f, h, since their projections onto $T^*(\Sigma_t) \otimes T^*(\Sigma_t)$ are the Riemannian metrics γ, φ, χ, and the direction defined by $\overset{\#}{\mathfrak{t}}{}^\mu := h^{\mu\nu}\nabla_\nu \mathcal{T}$ is timelike with respect to g, f, h, as shown in (6.12). Proposition 1 in [Koc18] proves that this construction does not miss any square root realization *in the considered foliation,* meaning that, if the foliation stays regular and spacelike during the dynamics, this parametrization is the most general one. The next paragraph describes how the foliation can become ill-suited for the $3+1$ decomposition during the dynamics.

Bimetric gauge pathologies. In GR, if the dynamics of the foliation develops a null hypersurface $\Sigma_{\hat{t}}$ at some time $\hat{t}$, then the $3+1$ formalism is not defined anymore at $\hat{t}$, and the generalized Cauchy problem has to be stated in a different way [dIn80, p. 98]. This can happen due to the choice of the slicing, as described in [AM98, p. 3] in the context of the "gauge pathologies." On a null hypersurface, the metric is degenerate by definition, that is, not invertible. In addition, as we mentioned in Section 5.1, the normal vector is itself null and cannot be normalized. The inability to normalize the normal vector leads to complications in the decomposition of the tangent space of the ambient space into the direct sum of a subspace of vectors orthogonal to the hypersurface and a subspace of vectors tangent to it. Due to these reasons, it is not possible to extend the definition (5.12) of the projector to the null case [Spe19, Sec. 7.1]. Note, however, that it is possible to pose the generalized Cauchy problem using null hypersurfaces by means of the $2+2$ decomposition,

[10] The superspace can also be defined on smooth compact manifolds.



(a) The gray horizontal surface is g-null. The null cone of g is extremely wide and the intersection with the null cone of f shrinks to a shared null direction in the limit. The configuration has: $\alpha = 100$, $\tilde{\alpha} = 1$, $\mathbf{p}^1 = 500$, the spatial tetrads for γ and φ are the identity, the rest is set to zero.

(b) The gray horizontal surface is f-null. The null cone of f is extremely wide and the intersection with the null cone of g shrinks to a shared null direction in the limit. The configuration has: $\alpha = 1$, $\tilde{\alpha} = 100$, $\mathbf{p}^1 = 500$, the spatial tetrads for γ and φ are the identity, the rest is set to zero.

Figure 6.3: Two bimetric gauge pathologies. The figures show the null cones and $\mathtt{t}$ (blue), $\tilde{\mathtt{t}}$ (red), $\overset{\#}{\mathtt{t}}$ (green) close enough to the gauge pathologies (the closer to the limit, i.e., to the pathology, the less readable the figures are). These configurations are off-shell, that is, they arise from the causal coupling between the (three-dimensional) metrics only, not from the dynamics. In these two cases, a (g, f)-spacelike hypersurface exists, but a (g, f)-timelike direction does not, hence the metrics are neither causally nor null coupled and the limit of the square root does not exist (see how the null cone of h is shrinking). These configurations constitute real singularities in BR.

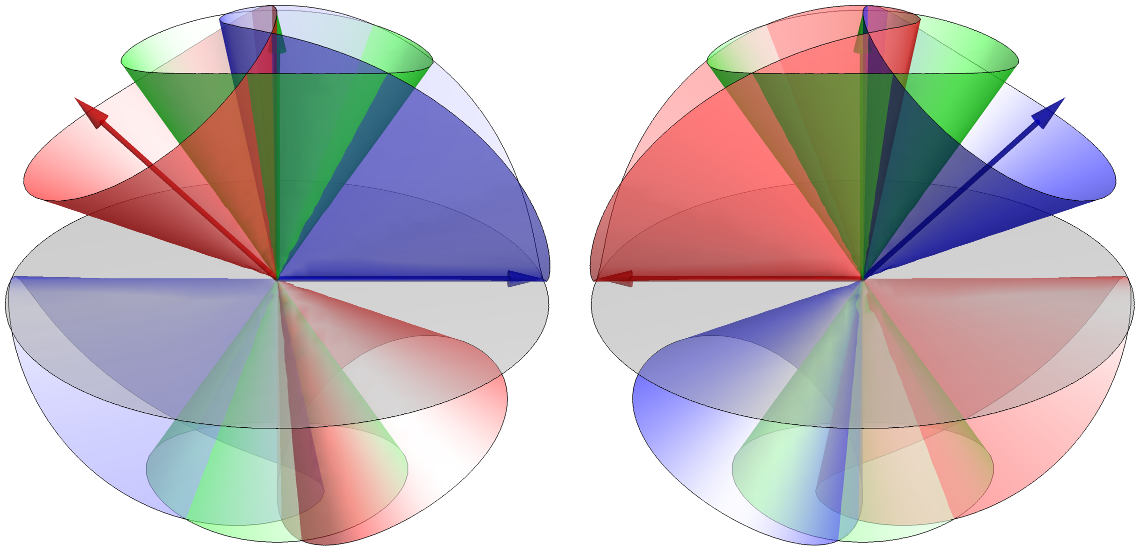
which decomposes the spacetime into two foliations of spacelike 2-dimensional surfaces, each of them foliating a hypersurface which can be null, timelike or spacelike. We will not deepen this formalism here, and refer the reader to [dIn80, p. 98][dIn84, p. 221] and references therein for more details.

In GR, the foliation becomes null if and only if the normal vector field becomes null. Since the normal vector field is normalized, consider the timelike vector in (5.7),

$$\overrightarrow{\nabla\mathfrak{T}}^{\alpha} = \frac{1}{\alpha^2}\big(-1, \beta^i\big), \quad \overrightarrow{\nabla\mathfrak{T}} \cdot \overrightarrow{\nabla\mathfrak{T}} = -\frac{1}{\alpha^2}. \tag{6.49}$$

The necessary condition for the timelike vector $\overrightarrow{\nabla\mathfrak{T}}$ to become null is that the lapse diverges. However, this condition is not sufficient since a diverging lapse may make $\overrightarrow{\nabla\mathfrak{T}}$ the zero vector, which would mean that the foliation becomes nonregular. In order for $\overrightarrow{\nabla\mathfrak{T}}$ to become a nonzero null vector, it is also necessary that the shift diverges in such a way that β^i/α^2 stays finite for at least one of the components of β^i.

In BR, the slicing becomes null with respect to one metric under the same conditions as in GR. However, given the increased number of metrics in the theory, there are several



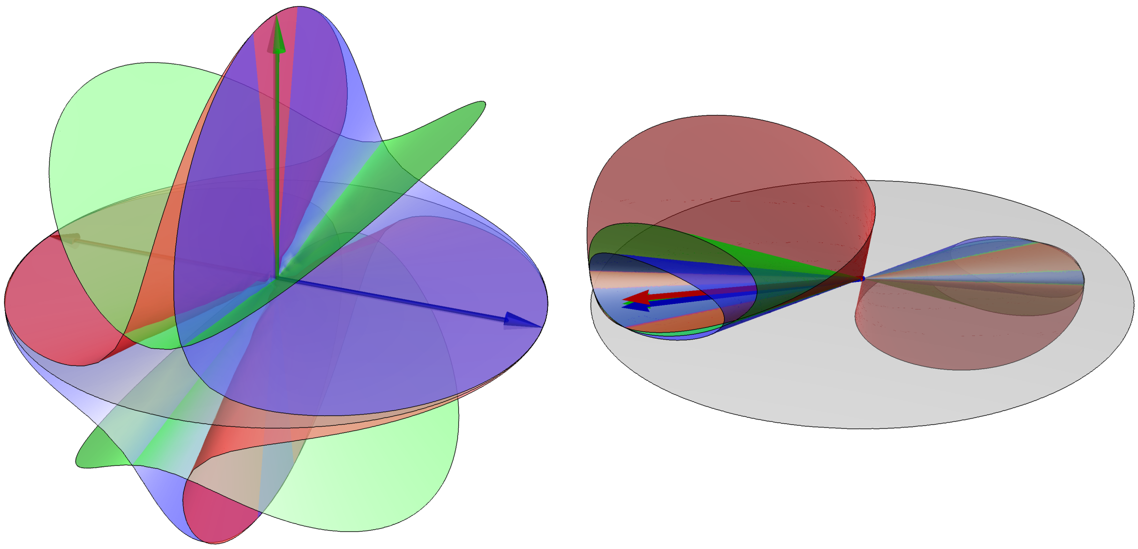
(a) The gray horizontal surface is g-null. The intersection between the null cones of g and f does not shrink to a shared null direction in the limit. The configuration is obtained with the same numeric values as in Figure 6.3a, but ${e^0}_0 = 10^{-3}$.

(b) The gray horizontal surface is f-null. The intersection between the null cones of g and f does not shrink to a shared null direction in the limit. The configuration is obtained with the same numeric values as in Figure 6.3b, but ${m_\circ{}^0}_0 = 10^{-3}$.

Figure 6.4: Two more bimetric gauge pathologies. The figures show the null cones and $\mathtt{t}$ (blue), $\widetilde{\mathtt{t}}$ (red), $\overset{\#}{\mathtt{t}}$ (green) close enough to the gauge pathologies (the closer to the limit, i.e., to the pathology, the less readable the figures are). These configurations arise from the causal coupling between the (three-dimensional) metrics only, not from the dynamics. In these two cases the metrics are casually coupled of Type IIb, hence the square root exists and these pathologies are only due to an unfortunate choice of the foliation.

possibilities. For example, there is the possibility of the slicing becoming null with respect to only g or only f, with an existing square root, as pointed out in [HK18, Sec. 3.2]. Indeed, the algebraic types in Table 6.1 encode all the possible configurations the metrics can develop during the dynamics and, in particular, all the possible null configurations. In Appendix D we describe all the possible null configurations that can be realized starting from a $3+1$ configuration. Some of them are shown in Figure 6.3, Figure 6.4, Figure 6.5.

Note that when one of these configurations takes place, the bimetric $3+1$ decomposition fails. However, in some of the cases the metrics are still causally or null coupled, that is, the square root exists (see the captions of Figure 6.3, Figure 6.4, Figure 6.5 for more details). Hence, the failure of the $3+1$ decomposition is only due to a bad choice of the slicing; such situations can happen dynamically as in GR, and are called "bimetric gauge pathologies." In Appendix D it is shown that the cases shown in Figure 6.3, Figure 6.4, Figure 6.5a can happen only if $\mathbf{p}$ diverges together with the affected lapses, whereas the last case happens only if $\mathbf{p}$ is finite. As a consequence, if a gauge condition keeping $\mathbf{p}$ always finite was known, then the gauge pathologies in Figure 6.3, Figure 6.4, Figure 6.5a would be excluded by construction. No such gauge condition is known at the time of writing this thesis, and this certainly motivates further work on this topic.



(a) The gray surface is (g, f)-null. The null cones of g and f are extremely wide and their intersection shrinks to two shared null directions in the limit. The configurations has: $\alpha = 100$, $\widetilde{\alpha} = 100$, $\mathbf{p}^1 = 500$, ${e^0}_0 = {m_\circ{}^0}_0 = 0.2$, ${e^1}_1 = {m_\circ{}^1}_1 = 1$, the rest is set to zero.

(b) The gray surface is (g, f, h)-null. The null cones stay always causally coupled. The three normal directions will eventually coincide in the limit and become (g, f, h)-null. The configuration has: $\alpha = \widetilde{\alpha} = q^1 = q^2 = 100$, ${e^0}_0 = 2$, ${e^1}_0 = -1$, ${e^1}_1 = 3/2$, ${m_\circ{}^0}_0 = 8/5$, ${m_\circ{}^1}_0 = -9/5$, ${m_\circ{}^1}_1 = 3/2$, $\mathbf{p}^1 = 1/10$, $\mathbf{p}^2 = 0$.

Figure 6.5: Two more bimetric gauge pathologies. The figures show the null cones and $\mathtt{t}$ (blue), $\widetilde{\mathtt{t}}$ (red), $\overset{\#}{\mathtt{t}}$ (green) close enough to the gauge pathologies (the closer to the limit, i.e., to the pathology, the less readable the figures are). These configurations arise from the causal coupling between the (three-dimensional) metrics only, not from the dynamics. In case (a), the metrics become null coupled, that is, they tend to Type IV from Type IIb. Therefore the limit of the square root exists but, as stressed in [HK18, Subsec. 3.3], it is on a branch cut of the square root matrix function. Since it is not possible to change branch with continuous operations only (neither with differentiable operations, hence with dynamics), case (a) may be a real singularity (i.e., not coordinate) of the theory if, after the limit, the null cones become neither causally nor null coupled. It is not a real singularity if they become of Type IIb again. However, a square root of Type IV is not a tensor, as argued in [HK18, Subsec. 3.1], hence this case may be regarded as a real singularity irrespectively of the subsequent dynamical evolution. In case (b), the metrics are casually coupled of Type IIa, hence the square root also exists. Therefore, this pathology is only due to an unfortunate choice of the foliation.

An example of the gauge pathology in Figure 6.5b occurs at the Killing horizon of the static BHs studied in Paper III. There, since g and f share the timelike KVF, h also share it, as explained in Paper I. Hence, on the Killing horizon, the KVF becomes null with respect to all the metrics since it is orthogonal to the foliation whose leaves are $S^2 \times \mathbb{R}$, which becomes null at the Killing horizon, as we described in Section 3.2. It is clear that, even if the $3+1$ decomposition fails if we choose this specific spacelike slicing, nothing pathological is happening in the covariant formalism. Actually, we described in Section 5.3 the maximal slicing condition, which allows the slicing to be horizon penetrating and singularity avoiding, meaning that there exists a spacelike foliation that can be used to set up the $3+1$ decomposition at and inside the Killing horizon.

g-sector	f-sector
$n := e^{-1}\mathbf{v}$,	$\tilde{n} := m^{-1}\mathbf{v}$,
$\mathcal{Q} := e^{-1}{\mathbf{\Lambda}_\mathrm{s}}^2 e$,	$\tilde{\mathcal{Q}} := m^{-1}{\mathbf{\Lambda}_\mathrm{s}}^2 m$,
$\mathcal{D} := m^{-1}{\mathbf{\Lambda}_\mathrm{s}}^{-1} e$,	$\tilde{\mathcal{D}} := e^{-1}{\mathbf{\Lambda}_\mathrm{s}}^{-1} m$,
$\mathcal{B} := \mathcal{D}^{-1} = e^{-1}\mathbf{\Lambda}_\mathrm{s} m$,	$\tilde{\mathcal{B}} := \tilde{\mathcal{D}}^{-1} = m^{-1}\mathbf{\Lambda}_\mathrm{s} e$,
$\mathcal{V} := -\,\ell^{-d/2}\sum_{n=0}^{d}\beta_{(n)}e_n(\tilde{\mathcal{D}})$,	$\tilde{\mathcal{V}} := -\lambda^{-1}\,\ell^{-d/2}\sum_{n=0}^{d}\beta_{(n)}e_{n-1}(\mathcal{B})$,
$\mathcal{U} := -\lambda^{-1}\,\ell^{-d/2}\sum_{n=0}^{d}\beta_{(n)}Y_{n-1}(\mathcal{B})$,	$\tilde{\mathcal{U}} := -\tilde{\mathcal{D}}\,\ell^{-d/2}\sum_{n=0}^{d}\beta_{(n)}Y_{n-1}(\tilde{\mathcal{D}})$,
$(\mathcal{Q}\tilde{\mathcal{U}}) := \mathcal{Q}\tilde{\mathcal{U}} = -\mathcal{B}\,\ell^{-d/2}\sum_{n=0}^{d}\beta_{(n)}Y_{n-1}(\tilde{\mathcal{D}})$,	$(\tilde{\mathcal{Q}}\mathcal{U}) := \tilde{\mathcal{Q}}\mathcal{U} = -\lambda^{-1}\tilde{\mathcal{Q}}\,\ell^{-d/2}\sum_{n=0}^{d}\beta_{(n)}Y_{n-1}(\mathcal{B})$,
$e_n(X) = X^{[a_1}{}_{a_1}X^{a_2}{}_{a_2}\dots X^{a_n]}{}_{a_n}$,	$Y_n(X) := \sum_{k=0}^{n}(-1)^{n+k}e_k(X)X^{n-k}$.

Table 6.2: Definitions of the spatial bimetric interactions in $d = N + 1$ dimensions, based on the quantities (6.37) in the Lorentz frame [Koc18]. The $e_n(X)$ are the elementary symmetric polynomials of the linear operator X.

We remark that all the configurations in Figure 6.3, Figure 6.4 and Figure 6.5 are obtained off-shell, that is, varying the $3+1$ parameters without imposing the BFE. To understand which of them can actually happen during the dynamical evolution, it is needed to solve the BFE in the $3+1$ formalism, which we are going to introduce in the next section.

6.2 The $3+1$ decomposition of the bimetric equations

The $3+1$ decomposition of the BFE is obtained in three steps: the projection of the Einstein (or Ricci) tensor, the projection of the matter stress–energy tensors, and the projection of the bimetric stress–energy tensors. The first two steps are the same as in GR, and the computations described in Section 5.2 can be directly applied to the BFE. The projection of the bimetric stress–energy tensors is specific to BR, and it is described below. In this section, we review it and emphasize the most relevant steps; we refer the reader to [Koc18] for more details.

The projections of the bimetric stress–energy tensors. We define the spatial bimetric interactions in Table 6.2, with the identification $m := \mathbf{R}m_\circ$. Note that m is again a tetrad for the spatial metric φ. The $3+1$ decomposition of the bimetric stress–energy tensors V_g, V_f is written in terms of the spatial bimetric interactions. We start with the decomposition of the bimetric potential V in (1.12) [Koc18],

$$\sqrt{-g}\,V = \alpha\sqrt{\Delta}e_n(S) = \alpha\det(e)\,e_n(S)$$

$$= \alpha \det(e)\, e_n(\widetilde{\mathcal{D}}) + \widetilde{\alpha} \det(m)\, e_{4-n}(\mathcal{D}). \tag{6.51}$$

Since it holds $\lambda^{-1} e_{n-1}(\widetilde{\mathcal{B}}) = e_{4-n}(\mathcal{D}) \det\left(m e^{-1}\right)$, (6.51) is equivalent to,

$$\alpha V = \alpha \mathcal{V} + \widetilde{\alpha}\widetilde{\mathcal{V}}. \tag{6.52}$$

The projections of the bimetric stress–energy tensors V_g, V_f, called the "bimetric sources," are

$$\rho^{\mathrm{b}} := \nu^\mu \, V_g{}^\rho{}_\nu \, \nu^\nu \qquad = -e_n(\mathcal{B}), \tag{6.53a}$$
$$j^{\mathrm{b}i} := \perp^\mu{}_\rho \, V_g{}^\rho{}_\nu \, \nu^\nu \qquad = (\mathcal{Q}\widetilde{\mathfrak{u}})^i{}_j n^j, \tag{6.53b}$$
$$J^{\mathrm{b}i}{}_j := \perp^\mu{}_\rho \, V_g{}^\rho{}_\sigma \perp^\sigma{}_\nu = \mathcal{V}\,\mathbf{I}^i{}_j - (\mathcal{Q}\widetilde{\mathfrak{u}})^i{}_j + \frac{\widetilde{\alpha}}{\alpha}\,\mathcal{U}^i{}_j, \tag{6.53c}$$
$$\widetilde{\rho}^{\mathrm{b}} := \widetilde{\nu}^\mu \, V_f{}^\rho{}_\nu \, \widetilde{\nu}^\nu \qquad = -\det\left(e m^{-1}\right) \lambda e_n(\widetilde{\mathcal{B}}), \tag{6.53d}$$
$$\widetilde{j}^{\mathrm{b}i} := \widetilde{\perp}^\mu{}_\rho \, V_f{}^\rho{}_\nu \, \widetilde{\nu}^\nu \qquad = -\det\left(e m^{-1}\right) j^{\mathrm{b}i}, \tag{6.53e}$$
$$\widetilde{J}^{\mathrm{b}i}{}_j := \widetilde{\perp}^\mu{}_\rho \, V_f{}^\rho{}_\sigma \, \widetilde{\perp}^\sigma{}_\nu = \det\left(e m^{-1}\right)\left(\widetilde{\mathcal{V}}\,\mathbf{I}^i{}_j - (\widetilde{\mathcal{Q}}\mathfrak{u})^i{}_j + \frac{\alpha}{\widetilde{\alpha}}\,\widetilde{\mathcal{U}}^i{}_j\right). \tag{6.53f}$$

The detailed computations of the bimetric sources can be found in the ancillary files of [Koc18].

The $3+1$ decomposition of the BFE. With the bimetric sources and interactions at hand, we can now consider the projection of the BFE (1.24). Since in BR we have (at least) the three projectors in (6.6), there are many ways to project the BFE onto the initial hypersurface Σ_t. Since the following completeness relation holds for any symmetric tensor T of rank 2 [Gou12, p. 75],

$$\begin{aligned} T_{\mu\nu} = {} & (\mathrm{Proj}^\top)_\mu{}^\rho T_{\rho\sigma} \, \mathrm{Proj}^\sigma{}_\nu - \mathrm{nor}_{(\mu} (\mathrm{Proj}^\top)_{\nu)}{}^\rho T_{\rho\sigma} \, \mathrm{nor}^\sigma \\ & + \mathrm{nor}^\rho \, T_{\rho\sigma} \, \mathrm{nor}^\sigma \, \mathrm{nor}_\mu \, \mathrm{nor}_\nu, \end{aligned} \tag{6.54}$$

where the couple $\mathrm{Proj}, \mathrm{nor}$ can be any of $\perp, \nu$, $\widetilde{\perp}, \widetilde{\nu}$ and $\overset{\#}{\perp}, \overset{\#}{\nu}$, we can choose to project each of the BFE using the most convenient projector and normal vector. The most convenient choice, which allows to use known results from GR, is to project the equations in the g-sector with using $\perp$ and ν, and the equations in the f-sector using $\widetilde{\perp}$ and $\widetilde{\nu}$. With this choice, spatial tensors in the $g(f)$-sector are dragged to the neighboring hypersurface by μ $(\widetilde{\mu})$.

In this way, the projected equations look formally the same as the standard 3+1 equations in GR, but with the bimetric sources included. The bimetric evolution equations are,

$$\mathscr{L}_{\mu} \gamma_{ij} = -2\alpha K_{ij}, \tag{6.55a}$$
$$\begin{aligned} \mathscr{L}_{\mu} K_{ij} = {} & -D_i D_j \alpha + \alpha \left[R_{ij} - 2K_{ik}K^k{}_j + K K_{ij}\right] \\ & - \alpha \kappa_g \left[\gamma_{ik} J^{\mathrm{eff}\,k}{}_j - \frac{1}{2}\gamma_{ij}\left(J^{\mathrm{eff}\,i}{}_i - \rho^{\mathrm{eff}}\right)\right], \end{aligned} \tag{6.55b}$$
$$\mathscr{L}_{\widetilde{\mu}} \varphi_{ij} = -2\widetilde{\alpha}\widetilde{K}_{ij}, \tag{6.55c}$$

$$\mathscr{L}_{\widetilde{\mu}}\widetilde{K}_{ij} = -\widetilde{D}_i\widetilde{D}_j\widetilde{\alpha} + \widetilde{\alpha}\left[\widetilde{R}_{ij} - 2\widetilde{K}_{ik}\widetilde{K}^k{}_j + \widetilde{K}\widetilde{K}_{ij}\right] - \widetilde{\alpha}\kappa_f\left[\varphi_{ik}\widetilde{J}^{\mathrm{eff}\,k}{}_j - \frac{1}{2}\varphi_{ij}\left(\widetilde{J}^{\mathrm{eff}\,i}{}_i - \widetilde{\rho}^{\mathrm{eff}}\right)\right], \tag{6.55d}$$

with the "effective sources" being,

$$\rho^{\mathrm{eff}} = \rho^{\mathrm{b}} + \rho^{\mathrm{m}}, \qquad j^{\mathrm{eff}}{}_i = j^{\mathrm{b}}{}_i + j^{\mathrm{m}}{}_i, \qquad J^{\mathrm{eff}}{}_{ij} = J^{\mathrm{b}}{}_{ij} + J^{\mathrm{m}}{}_{ij}, \tag{6.56a}$$

$$\widetilde{\rho}^{\mathrm{eff}} = \widetilde{\rho}^{\mathrm{b}} + \widetilde{\rho}^{\mathrm{m}}, \qquad \widetilde{j}^{\mathrm{eff}}{}_i = \widetilde{j}^{\mathrm{b}}{}_i + \widetilde{j}^{\mathrm{m}}{}_i, \qquad \widetilde{J}^{\mathrm{eff}}{}_{ij} = \widetilde{J}^{\mathrm{b}}{}_{ij} + \widetilde{J}^{\mathrm{m}}{}_{ij}, \tag{6.56b}$$

where the superscript "m" stands for "matter." The bimetric constraint equations read,

$$2\mathcal{C} := R + K^2 - K_{ij}K^{ij} - 2\kappa_g\rho^{\mathrm{eff}} = 0, \tag{6.57a}$$

$$\mathcal{C}_i := D_kK^k{}_i - D_iK - \kappa_g j^{\mathrm{eff}}{}_i = 0, \tag{6.57b}$$

$$2\widetilde{\mathcal{C}} := \widetilde{R} + \widetilde{K}^2 - \widetilde{K}_{ij}\widetilde{K}^{ij} - 2\kappa_f\widetilde{\rho}^{\mathrm{eff}} = 0, \tag{6.57c}$$

$$\widetilde{\mathcal{C}}_i := \widetilde{D}_k\widetilde{K}^k{}_i - \widetilde{D}_i\widetilde{K} - \kappa_f\widetilde{j}^{\mathrm{eff}}{}_i = 0. \tag{6.57d}$$

Since the two bimetric currents j^{b} and $\widetilde{j}^{\mathrm{b}}$ are related by (6.53e), one of the momentum constraints becomes,

$$\sqrt{\gamma}\left\{\kappa_g{}^{-1}\left(D_kK^k{}_i - D_iK\right) - j^{\mathrm{m}}{}_i\right\} + \sqrt{\varphi}\left\{\kappa_f{}^{-1}\left(\widetilde{D}_k\widetilde{K}^k{}_i - \widetilde{D}_i\widetilde{K}\right) - \widetilde{j}^{\mathrm{m}}{}_i\right\} = 0. \tag{6.58}$$

In principle, one of the momentum constraints can be solved for the separation parameter $\mathbf{p}$. A generic solution is still lacking; more details about how to determine $\mathbf{p}$ can be found in Appendix E of Paper IV.

Analogously to GR, the constraints (6.57) are stably propagated in the free evolution scheme, as proved in [Koc19a].

The projection of the bimetric conservation law. Here we project the Bianchi constraint (1.39), which is also called "bimetric conservation law" due to its similarity to the conservation law for the matter stress–energy tensor $\nabla_\mu T^{\mu\nu} = 0$. We remark that the bimetric conservation law is not an identity as the Bianchi identity, but it has to be solved, as stressed at the end of Section 1.2. Here we assume that it is valid. Using the analogous result in [Gou12, Subsec. 6.2.1], we write,

$$\begin{aligned} 0 &= \nabla_\mu V_g{}^\mu{}_\nu = \nabla_\mu\big(J^{\mathrm{b}\mu}{}_\nu + \mathrm{v}^\mu\, j^{\mathrm{m}}{}_\nu + j^{\mathrm{m}\mu}\,\mathrm{v}_\nu + \rho^{\mathrm{b}}\,\mathrm{v}^\mu\mathrm{v}^\nu\big), \\ &= \nabla_\mu J^{\mathrm{m}\mu}{}_\nu - K\, j^{\mathrm{m}}{}_\nu + \mathrm{v}^\mu\nabla_\mu j^{\mathrm{m}}{}_\nu + \nabla_\mu j^{\mathrm{m}\mu}\, j^{\mathrm{m}}{}_\nu - j^{\mathrm{m}\mu}K_{\mu\nu} \\ &\quad - K\,\rho^{\mathrm{m}}\,\mathrm{v}_\nu + \rho^{\mathrm{m}}\, D_\nu\ln(\alpha) + \mathrm{v}_\nu\mathrm{v}^\mu\nabla_\mu\rho^{\mathrm{m}}. \end{aligned} \tag{6.59}$$

The projections of (6.59) read,

$$\mathscr{L}_{\mu}\,\rho^{\mathrm{b}} = -\alpha D_i j^{\mathrm{b}i} - 2j^{\mathrm{b}i}D_i\alpha + \alpha\, K\,\rho^{\mathrm{m}} + \alpha K^j{}_i J^{\mathrm{b}i}{}_j, \tag{6.60a}$$

$$\mathscr{L}_{\mu}\, j^{\mathrm{b}}{}_i = -D_j\big(\alpha\, J^{\mathrm{b}j}{}_i\big) - \rho^{\mathrm{m}}\, D_i\alpha + \alpha\, K\, j^{\mathrm{b}}{}_i. \tag{6.60b}$$

In [Koc18, Appendix D], it is proven that

$$\partial_{\mathcal{T}}\rho^{\mathrm{b}} - n^i \partial_{\mathcal{T}} j^{\mathrm{b}}{}_i = \frac{1}{2}(\mathcal{Q}\widetilde{\mathcal{U}})^{ij}\partial_{\mathcal{T}}\gamma_{ij} - \frac{1}{2}\widetilde{\mathcal{U}}^{ij}\partial_{\mathcal{T}}\varphi_{ij}. \tag{6.61}$$

Substituting (6.60) (after having converted the Lie derivatives along μ into partial derivatives in time and Lie derivatives along the shift vector β) and the evolution equations (6.55a, 6.55c) into (6.61), we get [Koc18, Appendix D]

$$\mathcal{C}_{\mathrm{b}} := \mathcal{U}^i{}_j\Big(D_i n^j - K^j{}_i\Big) + \widetilde{\mathcal{U}}^i{}_j\Big(\widetilde{D}_i \widetilde{n}^j + \widetilde{K}^j{}_i\Big) - D_i\Big(\mathcal{U}^i{}_j n^j\Big) \tag{6.62a}$$

$$\begin{aligned} &= -\mathcal{U}^i{}_j\left(D_i n^j - \frac{1}{2}n^j\partial_i \ln\Big(\sqrt{\Delta}\Big) - K^j{}_i\right) + \frac{1}{2}D_i\Big(\mathcal{U}^i{}_j n^j\Big) \\ &\quad - \widetilde{\mathcal{U}}^i{}_j\left(\widetilde{D}_i \widetilde{n}^j - \frac{1}{2}\widetilde{n}^j\partial_i \ln\Big(\sqrt{\widetilde{\Delta}}\Big) + \widetilde{K}^j{}_i\right) + \frac{1}{2}\widetilde{D}_i\Big(\widetilde{\mathcal{U}}^i{}_j \widetilde{n}^j\Big) = 0, \end{aligned} \tag{6.62b}$$

where the expression (6.62b) for $\mathcal{C}_{\mathrm{b}}$ is obtained by using

$$D_i\big(\widetilde{\mathcal{U}}^i{}_j\widetilde{n}^j\big) - \widetilde{D}_i\big(\widetilde{\mathcal{U}}^i{}_j\widetilde{n}^j\big) = \left\{\partial_i\left[\ln\Big(\sqrt{\Delta}\Big) - \ln\Big(\sqrt{\widetilde{\Delta}}\Big)\right]\right\}\widetilde{\mathcal{U}}^i{}_j\widetilde{n}^j, \tag{6.63}$$

which follows from the differential identity (1.26) [Koc18, Appendix D].

The equation (6.62) provides another constraint specific to BR, with no analog in GR. This constraint is equivalent to the additional constraint computed in [HL18] in the Hamiltonian formalism, as it is made clear in Paper IV. Since, in the $3+1$ formalism, it is obtained by projecting the bimetric conservation law (1.39), it is also called bimetric conservation law in Paper V. In this thesis, we call it “bimetric constraint.”

The preservation of the bimetric constraint. The bimetric constraint $\mathcal{C}_{\mathrm{b}} = 0$ is preserved in time, that is, $\partial_t\mathcal{C}_{\mathrm{b}} = 0$. In [HL18], in the Hamiltonian formalism, the preservation of the bimetric constraint is computed explicitly and it is shown that it is equivalent to,

$$\mathcal{C}_{\mathrm{lap}} := \alpha - W\,\widetilde{\alpha} = 0, \tag{6.64}$$

which we call “lapse constraint.” In (6.64), W is a spacetime scalar field (as the lapses are) which depends only on the dynamical variables of BR, that is, the spatial metrics, their extrinsic curvatures, and their spatial derivatives. However, the most general explicit expression of W was not computed in [HL18].

In Paper IV, we clarify the equivalence between the Hamiltonian formalism and the $3+1$ formalism in BR, showing how the bimetric constraints computed in [HL18] and [Koc18] are equivalent. In addition, in the same paper we compute W explicitly assuming a spherically symmetric ansatz. We also notice that, for a regular foliation the lapses are nonzero, and for a spacelike foliation they are finite, hence the relation (6.64) is well-defined as long as the foliation is regular and (g, f)-spacelike.

The preservation of the lapse constraint. The lapse constraint is also preserved in time as, that is, $\partial_t \mathcal{C}_{\text{lap}} = 0$. This derivative is equal to the evolution equation

$$\partial_t \alpha - \widetilde{\alpha}\, \partial_t W - W\, \partial_t \widetilde{\alpha} = 0. \tag{6.65}$$

Since W is specified by the dynamical variables, its time derivative is specified by their evolution equations, which contain the lapses and their radial derivatives. Hence, (6.65) is a PDE with respect to the lapses, and can be used to evolve one of them, namely, the one that we do not gauge fix. We will return to gauge fixing in BR in the next paragraph.

In [Koc19a], the stable propagation of the constraints (6.57, 6.62) is established assuming that the lapse constraint is satisfied during the evolution. There are two ways to ensure that: (i) imposing the lapse constraint explicitly during the evolution, and (ii) adding (6.65) to the set of evolution equations to be solved (i.e., adding it to the Cauchy problem) and obtaining the initial data for one of the lapses by solving the lapse constraint of the initial hypersurface. The two approaches are theoretically equivalent. In the first case, the evolution is partially constrained; the second case defines the free evolution scheme in BR. Since in the first case the imposed constraint does not fix a dynamical variable, but one of the lapses, we call it "partially free," to distinguish it from other (partially) constrained evolution schemes where the constraints determine some of the dynamical variables.

Finally, we remark that, since the preservation of the lapse constraint yields an evolution equation and not an additional constraint, there are no other constraints whose preservation in time has to be considered.

Gauge fixing in bimetric relativity. As pointed out in Section 6.1, the specification of one lapse function uniquely determines the coordinate time. This is consistent with the lapse constraint (6.64), since, once one lapse is specified, the remaining ones are determined by,

$$\alpha^2 = H^2 \lambda W, \qquad \widetilde{\alpha}^2 = \frac{H^2 \lambda}{W}, \qquad \alpha = W \widetilde{\alpha}, \tag{6.66a}$$

which have to be imposed explicitly during the evolution, or by

$$\partial_t \log\big(\alpha\big) + \partial_t \log\big(W_g\big) = \partial_t \log\big(\widetilde{\alpha}\big) + \partial_t \log\big(W_f\big), \tag{6.66b}$$

$$\partial_t \log\big(\alpha\big) = \partial_t \log\big(H\big) + \frac{1}{2}\Big[\partial_t \log\big(\lambda\big) + \partial_t \log\big(W\big)\Big], \tag{6.66c}$$

$$\partial_t \log\big(\widetilde{\alpha}\big) = \partial_t \log\big(H\big) + \frac{1}{2}\Big[\partial_t \log\big(\lambda\big) - \partial_t \log\big(W\big)\Big], \tag{6.66d}$$

which have to be integrated in time with initial values given by (6.66a). The evolution equations (6.66b, 6.66c, 6.66d) are obtained using (6.65), the time derivative of (6.44a) and the expression $W =: -W_f/W_g$. A similar argument holds for the shift vectors, since, after one of them is specified, the spatial coordinates are uniquely propagated to the neighboring slice and the remaining shifts are determined by,

$$\beta = q + \frac{\alpha}{\lambda} e^{-1} \mathbf{p}, \qquad \widetilde{\beta} = q - \frac{\widetilde{\alpha}}{\lambda} m^{-1} \mathbf{p}, \qquad \widetilde{\beta} = \beta - \frac{\widetilde{\alpha}}{\lambda} \left(W\, e^{-1} + m^{-1}\right) \mathbf{p}. \tag{6.66e}$$

Finally, there are no undetermined variables and the bimetric standard $3+1$ equations (6.55, 6.57, 6.62) are closed. In summary, the diffeomorphism invariance of BR allows us to specify one lapse, one shift and a spatial coordinate system on the initial hypersurface Σ_t, as in GR. This is enough to determine a spacetime coordinate system in a neighborhood of Σ_t. From now on, we will refer to the three lapses and the three shifts collectively as "the gauge variables."

As remarked in Paper V and Paper VI, in BR it is convenient to distinguish between the chosen gauge condition and the actual gauge fixing, the latter being the equation that fixes the gauge variable. The same gauge condition can be expressed in 9 *geometrically* equivalent, but *analytically* inequivalent gauge fixings. For example, suppose we choose the geodesic slicing with respect to h as our gauge condition; this can be expressed in the following geometrically equivalent ways,

$$H = 1, \qquad q = 0, \tag{6.67a}$$

$$\alpha = \sqrt{\lambda W}, \qquad \beta = \frac{\alpha}{\lambda} e^{-1} \mathbf{p}, \tag{6.67b}$$

$$\widetilde{\alpha} = \sqrt{\lambda W^{-1}}, \qquad \widetilde{\beta} = -\frac{\widetilde{\alpha}}{\lambda} m^{-1} \mathbf{p}, \tag{6.67c}$$

$$H = 1, \qquad \widetilde{\beta} = -\frac{\widetilde{\alpha}}{\lambda} m^{-1} \mathbf{p}. \tag{6.67d}$$

One can combine these gauge fixings in 9 ways in total (3 lapses times 3 shifts).[11] When we choose a gauge condition tied to the geometry of one metric, we say that we choose a gauge condition "with respect to" that metric.

In the simple case (6.67), all the gauge fixings are analytically equivalent, since no derivatives are involved. However, if we consider for example the standard gauge with respect to h—to be introduced in the next section—then the gauge fixings corresponding to it are analytically inequivalent. Indeed, $\mathbf{p}$ can be expressed in terms of the dynamical variables by solving one constraint, and its derivatives become derivatives of the dynamical variables, thus affecting the principal symbol of the bimetric $3+1$ equations.

Finally, we arrived at the topic of the hyperbolicity of the bimetric $3+1$ equations. Therefore, it is time to discuss the main topics of Paper V and Paper VI.

The counting of propagating degrees of freedom in BR. As we did for GR, we conclude the section by counting the propagating degrees of freedom in BR.

In the $3+1$ formulation, we start with 6 independent components for each spatial metric and each extrinsic curvature, that is, 24 independent components. We fix 4 of them by diffeomorphism invariance, that is, by specifying one lapse function and one shift vector. In addition, there are 2 Hamiltonian constraint, 2 vector momentum constraints and the bimetric constraint, which can be used to fix 9 independent components. However, 3 of them have to be used to solve for the separation parameter $\mathbf{p}$, which is not a dynamical variable, in terms of the dynamical variables. Hence, with the constraints we can fix $9 - 3 = 6$

[11] In principle, it is possible to gauge fix any lapse function and any shift vector of the metrics defined in Definition 1 on p. 30. Hence, there are actually infinitely many equivalent gauge fixings for the same gauge condition.

components of the spatial metrics or the extrinsic curvatures. Finally, we end up with $24 - 4 - 6 = 14$, corresponding to 7 propagating modes. In the partially free evolution scheme, the lapse constraint fixes one lapse function in terms of the other, hence it does not constrain a dynamical variable. In the free evolution scheme, we start with 25 independent components—24 as in the partially free scheme plus the evolved lapse. Then, the lapse constraint does constrain a dynamical variable and the counting of degrees of freedom gives again 14. The 7 degrees of freedom, around proportional backgrounds, are associated to the 2 degrees of freedom for a massless spin-2 field and the 5 degrees of freedom for a massive spin-2 field [Com+12b; HSS13b; HSS13a].

We remark that solving one momentum constraint for $\mathbf{p}$ is not straightforward in the most general case. It can be done assuming the same spherical symmetry in both sectors, as it is shown in [Koc18] for the standard $3+1$ equations and in Paper V for the cBSSN equations. Appendix E of Paper IV shows what are the difficulties in the most general case.

6.3 The bimetric covariant BSSN formulation and the mean gauges

The cBSSN formulation of the BFE is established in Paper V. Paper VI can be seen as the natural continuation of Paper V, since some technical issues related to the bimetric cBSSN formulation are discussed, and the two gauge conditions discussed in Section 5.3 are computed with respect to the geometric mean metric in BR. In this section, we are going to summarize these topics without focusing on the technicalities, which can be found in the papers.

The BSSN formulation of the bimetric interaction and sources. The main goal of Paper V is to clarify that the hyperbolic structure of the bimetric evolution equations is not just two copies of the hyperbolic structure of the evolution equations in GR, and to establish if it is possible to extend the results in GR about the strong hyperbolicity of cBSSN, to BR. In other words, given that the cBSSN formulation of the EFE is strongly hyperbolic, and given that the standard $3+1$ formulation of the BFE (6.55, 6.57) looks formally the same to the standard $3+1$ formulation in GR, what can we conclude about the cBSSN formulation of the BFE?

The results in Paper V show that the strong hyperbolicity of the cBSSN equations cannot be *directly* extended to the bimetric case, assuming the standard gauge with respect to g or f. The reason is that the lapse functions and the shift vectors of g and f are related by (6.66). Aside from the spatial tetrads, the quantities linking the gauge variables are W and $\mathbf{p}$. The explicit expressions for these two quantities are not known in the most general case. However, due to the Hamiltonian analysis in [HL18], we know that they can be expressed in terms of the dynamical variables and their spatial derivatives. Assuming a spherically symmetric ansatz for the spatial metrics, with spherical polar coordinates on the initial hypersurface and the same SO(3) KVFs, it is possible to compute both W and $\mathbf{p}$ explicitly. Their expressions are given in [Koc18] and Paper IV in the standard $3+1$

formalism and in Paper V and its supplementary material in the cBSSN formalism. The spherically symmetric expressions of W and $\mathbf{p}$ do depend only on the dynamical fields and their spatial derivatives. While the expression for $\mathbf{p}$ is manageable, the expression for W is not. It is instead nonlinear in the dynamical fields and their derivatives, and it is very lengthy. As the reader at this point may have guessed, since W and $\mathbf{p}$ involve spatial derivatives, they do affect the principal symbol of the system of PDEs (6.55). If we impose the lapse constraint explicitly during the evolution, then W will appear wherever the non gauge fixed lapse appears. Then, the evolution equations will contain its spatial derivatives, which contain the spatial derivatives of the dynamical fields. Instead, if we determine the non gauge fixed lapse with (6.65), we have one more evolution equation to include in the Cauchy problem. This evolution equation contains both W and $\partial_t W$, hence it contains the spatial derivatives of the dynamical fields. In both cases, the principal symbol is affected.

Hence, the analysis of the hyperbolicity of the bimetric cBSSN equations requires much more work than the one in GR, and it is still an open problem. Accordingly, in Paper V we say that we cannot claim the bimetric cBSSN formulation to be strongly hyperbolic yet. In addition to clarifying that the hyperbolic structure of the bimetric evolution equations in cBSSN and the standard $3+1$ form is not just equal to two copies of the hyperbolic structure of the corresponding formulations of the EFE, the merit of Paper V is to show that it is possible to rewrite the bimetric interactions in the cBSSN formalism and to point out what are the steps to take in order to establish the field of nubirel. Since BR is a theory of gravity alternative to GR, it has to be tested against the observational data. In order to get sensible prediction for realistic physical systems, it is necessary to face the numerical integration of the BFE, as argued in [Shi99] already in the case of GR. To do that, one has to recast the BFE as a well-posed Cauchy problem, following the same path outlined by NR. With Paper V, we show that the BFE can be tackled in the same way as one would do with the EFE, but that the specific bimetric features require additional strategies to be found. One possible new strategy is to impose a gauge specific to BR such that the system of PDEs simplifies and allows to use known results in GR. Two such possibilities are explored in Paper VI, as we shall see.

We now give the essential information needed to understand how to compute the BSSN formulation of the bimetric interactions. The appendix of Paper V is devoted to present the computations in much more detail. The conformal metrics are defined as,

$$\bar{\gamma}_{ij} \coloneqq \mathrm{e}^{-4\phi}\,\gamma_{ij}, \qquad \bar{\gamma}^{ij} = \mathrm{e}^{4\phi}\,\gamma^{ij}, \tag{6.68a}$$

$$\widehat{\varphi}_{ij} \coloneqq \mathrm{e}^{-4\psi}\,\varphi_{ij}, \qquad \widehat{\varphi}^{ij} = \mathrm{e}^{4\psi}\,\varphi^{ij}, \tag{6.68b}$$

$$\mathring{\chi}_{ij} = \mathrm{e}^{-2(\phi+\psi)}\,\chi_{ij}, \qquad \mathring{\chi}^{ij} = \mathrm{e}^{2(\phi+\psi)}\,\chi^{ij}. \tag{6.68c}$$

These definitions imply that the conformal decomposition of the tetrads reads,

$$\bar{\gamma} = \mathrm{e}^{-4\phi}\,\gamma \Longrightarrow \bar{e}^{\intercal}\,\boldsymbol{\delta}\,\bar{e} \;\;= \left(\mathrm{e}^{-2\phi}e^{\intercal}\right)\boldsymbol{\delta}\left(\mathrm{e}^{-2\phi}e\right), \tag{6.69a}$$

$$\widehat{\varphi} = \mathrm{e}^{-4\psi}\,\varphi \Longrightarrow \widehat{m}^{\intercal}\,\boldsymbol{\delta}\,\widehat{m} = \left(\mathrm{e}^{-2\psi}m^{\intercal}\right)\boldsymbol{\delta}\left(\mathrm{e}^{-2\psi}m\right). \tag{6.69b}$$

It follows,

$$\bar{e} = \mathrm{e}^{-2\phi} e, \qquad \widehat{m} = \mathrm{e}^{-2\psi} m, \tag{6.70a}$$

$$\bar{e}^{-1} = \mathrm{e}^{2\phi} e^{-1}, \qquad \widehat{m}^{-1} = \mathrm{e}^{2\psi} m^{-1}. \tag{6.70b}$$

The bimetric interactions in Table 6.2 depend on the tetrads and on the spatial parts of the Lorentz boost $\mathbf{\Lambda}_\mathrm{s}$ and Euclidean rotation $\mathbf{R}$. Both $\mathbf{\Lambda}_\mathrm{s}$ and $\mathbf{R}$ are determined by $\mathbf{p}$, hence if we compute the conformal decomposition of $\mathbf{p}$, we can use it together with (6.70) to compute the conformal decomposition of all the bimetric interactions. Since the bimetric sources depend on the bimetric interactions, this allows us to compute their conformal decomposition as well. Thus, the bimetric part of the equations (6.55) is rewritten in the BSSN formalism; the remaining part is the same as in GR, hence it is not discussed here.

In Appendix A.1 of Paper V it is shown that it is always possible to keep $\mathbf{p}$ unchanged in recasting the BFE in the (c)BSSN formulation, hence $\mathbf{\Lambda}_\mathrm{s}$ and $\mathbf{R}$ also stay the same.

The mean gauges. In Paper VI, the maximal slicing and the standard gauge are computed with respect to h. The motivation that lies behind this work is a conjecture which arose during the work on Paper III and Paper V. For instance, Proposition 1 in Paper III states that, assuming one of the metrics g and f is regular at a point—that is, its curvature scalars are finite—the vanishing of $\det(S)$ induces a curvature singularity in the other metric. Since $h = gS$, this suggests that at least two of the metrics g, f, h have to share a curvature singularity; more support to this conjecture is given by the exact solution in [Hög+19], where g does not have a curvature singularity at the origin, but f and h do (see Sec. 2.4 in the paper). Hence, imposing a gauge with respect to h seems promising in order to detect curvature and coordinate singularities happening in any metric sector. In addition, it does not single out any of the metrics g and f, hence keeping the symmetry of the theory under their exchange (in vacuum). In the following we review the results of Paper VI by describing how to obtain the mean gauges.

The "mean maximal slicing" is defined by choosing $\overset{\#}{K} = 0$ on the initial hypersurface and demanding $\partial_t \overset{\#}{K} = 0$. Defining the auxiliary variables

$$\zeta \coloneqq \log(H), \quad \xi \coloneqq \partial_t \zeta, \tag{6.71}$$

the condition $\partial_t \overset{\#}{K} = 0$ can be rewritten as,

$$\partial_t \zeta = \xi, \tag{6.72a}$$

$$0 = \mathpzc{n}^i \partial_i \xi + \xi \left(\mathpzc{n}^i \partial_i \zeta + \mathpzc{m} \right) + \left(\partial_t \mathpzc{n}^i \right) \partial_i \zeta + \partial_t \mathpzc{m}, \tag{6.72b}$$

where $\mathpzc{n}^i$ and $\mathpzc{m}$ are somewhat complicated functions of the dynamical fields that can be found in Paper VI. Contrary to the maximal slicing with respect to g or f, which would result in a boundary value problem as in GR, (6.72) is an evolution equation for ζ coupled to a boundary value problem for its time derivative ξ. Numerically, the first step is to solve (6.72b) for ξ using the initial data for ζ, provided by the initial data for H. Then, the value of ξ is used to evolve ζ to the next slice. The procedure can then be iterated.

The mean maximal slicing imposes that the foliation is h-maximal, which implies that it is singularity avoiding for all the singularities involving h and any of the other metrics. More in detail, the proper volume of a h-Eulerian observer is conserved during the evolution, according to the reasoning in Section 5.3.

The other gauge condition with respect to h computed in Paper VI is the standard gauge, which is called "mean standard gauge" and reads,

$$\begin{aligned}
\partial_t H &= q^j \mathring{D}_{\mathrm{B}j} H - 2H\overset{\#}{K}, && (6.73\text{a})\\
\partial_t q^i &= q^j \mathring{D}_{\mathrm{B}j} q^i + \frac{3}{4}\mathring{B}^i, && (6.73\text{b})\\
\partial_t \mathring{B}^i &= q^j \mathring{D}_{\mathrm{B}j} \mathring{B}^i + \big(\partial_t \mathring{\Lambda}^i - q^i \mathring{D}_{\mathrm{B}j}\mathring{\Lambda}^i\big) - \eta \mathring{B}^i. && (6.73\text{c})
\end{aligned}$$

The computation of both (6.72) and (6.73) relies on the computation of the trace $\overset{\#}{K}$ of the extrinsic curvature of h. This can be determined using the relation between the determinants of the spatial metrics,

$$\overset{\#}{\Delta} = \lambda\sqrt{\Delta\widetilde{\Delta}}, \tag{6.74}$$

and the fact that $\overset{\#}{K}$ determines the evolution of $\overset{\#}{\Delta}$,

$$\mathscr{L}_{\overset{\#}{\mu}} \log\big(\overset{\#}{\Delta}\big) = -2H\overset{\#}{K}. \tag{6.75}$$

Hence, by taking the Lie derivative along $\overset{\#}{\mu}$ of (6.74), in Paper VI we get an expression for $\overset{\#}{K}$ in terms of the dynamical fields and H,

$$\begin{aligned}
\overset{\#}{K} &= \frac{1}{2H}\left[\alpha K - \frac{\mathscr{L}_{\alpha\lambda^{-1}n}\Delta}{2\Delta} + \tilde{\alpha}\widetilde{K} - \frac{\mathscr{L}_{\tilde{\alpha}\lambda^{-1}\tilde{n}}\widetilde{\Delta}}{2\widetilde{\Delta}} - \frac{\mathbf{p_a}\left(\partial_t \mathbf{p^a} - q^i\partial_i \mathbf{p^a}\right)}{\lambda^2}\right]\\
&= \frac{1}{2}\Bigg\{\sqrt{\lambda W}\left[K - \frac{1}{2\lambda}n^i\partial_i \log\big(\Delta\big)\right] - \frac{1}{H}\partial_i\left(\alpha\lambda^{-1}n^i\right)\\
&\quad + \sqrt{\lambda W^{-1}}\left[\widetilde{K} + \frac{1}{2\lambda}\tilde{n}^i\partial_i\log\big(\widetilde{\Delta}\big)\right] + \frac{1}{H}\partial_i\left(\tilde{\alpha}\lambda^{-1}\tilde{n}^i\right) - \frac{\mathbf{p_a}\left(\partial_t\mathbf{p^a} - q^i\partial_i\mathbf{p^a}\right)}{H\lambda^2}\Bigg\},
\end{aligned} \tag{6.76}$$

where we have used (6.64) and explicitly computed the Lie derivatives of the determinants of the metrics, which are scalar densities of weight 2.

Concerning the mean standard gauge in particular, the evolution equation for χ is required in order to obtain the evolution equations for $\mathring{\chi}$ and $\mathring{\Lambda}$ in (6.73), and it must be given in terms of the dynamical fields. Since $\chi = e^{\top}\boldsymbol{\delta}\boldsymbol{\Lambda}_{\mathrm{s}}\mathbf{R}m_{\circ}$, the computation of the evolution equation for χ requires the computation of the evolution equations for the tetrads and for the spatial part of the Lorentz boost $\boldsymbol{\Lambda}_{\mathrm{s}}$ and the spatial rotation $\mathbf{R}$. The latter reduces to the computation of the evolution equation for $\mathbf{p}$, since

$$\mathscr{L}_{\underset{\mu}{\#}} \mathbf{\Lambda}_{\mathrm{s}} = -\frac{1}{\lambda+1}\left[\frac{\mathbf{p}^\top \boldsymbol{\delta}\, \mathscr{L}_{\underset{\mu}{\#}} \mathbf{p}}{(\lambda+1)\,\lambda}\mathbf{p}\mathbf{p}^\top\boldsymbol{\delta} - \big(\partial_{\#}\mathscr{L}_{\underset{\mu}{\#}}\mathbf{p}\big)\mathbf{p}^\top\boldsymbol{\delta} - \mathbf{p}\,\mathscr{L}_{\underset{\mu}{\#}}(\mathbf{p}^\top\boldsymbol{\delta})\right], \tag{6.77}$$

and $\mathscr{L}_{\underset{\mu}{\#}} \mathbf{R}$ can be written in terms of $\mathscr{L}_{\underset{\mu}{\#}} \mathbf{\Lambda}_{\mathrm{s}}$, see Appendix B.3 of Paper VI.

As pointed out in Paper VI, the effective usefulness of the mean gauges for stable long-term bimetric simulations has to be tested numerically. Also, the mean standard gauge does alter the hyperbolicity of the system in a nontrivial way, hence another possible research path is to study the hyperbolicity of the system with this gauge choice.

The geometric mean in the bimetric covariant BSSN formulation. In Paper VI, the role of the geometric mean in the bimetric cBSSN formulation is studied. The possibility to consider it as the background metric for both γ and φ is discussed, and since the evolution equation for χ is computed, the evolution equation for the conformal metric $\mathring{\chi}$ can be obtained. Therefore, the evolution equation for the Levi–Civita connection $\mathring{\Gamma}$ of $\mathring{\chi}$ can also be computed in terms of that of $\mathring{\chi}$. Hence, in BR there is no need to choose an external metric as the background; choosing $\mathring{\chi}$ is natural since it is an object already defined in the theory, and its dynamics is determined by that of $\bar{\gamma}$ and $\hat{\varphi}$.

A strong motivation for choosing different background metrics compared to GR is that the strong hyperbolicity of the bimetric cBSSN formulation is still an open question, and this strategy allow us to recast the equations and alter their principal symbol. Indeed, the evolution equation for $\mathring{\Gamma}$ contains the spatial derivatives of the dynamical fields. The study of the hyperbolicity of the bimetric cBSSN equations with $\mathring{\Gamma}$ as the background connection, is left for future work.

The package `bimEX`. During the work to obtain the results of Paper V and Paper VI, the need to write the bimetric cBSSN equations in spherically symmetric form motivated the writing and extensive testing of a `Mathematica` [Inc] package able to write the cBSSN equations in any ansatz. The package is based on the `xAct` bundle [xAc] and called `bimEX`, for "**bim**etric **e**xact **c**omputation**s**." It implements the procedure described at p. 94-95.

The package is described in Paper VII, and is available at the public GitHub repository named `nubirel/bimEX` and at [Tor19a]. It is made to be as simple and user-friendly as possible, and can compute the bimetric cBSSN equations in any desired ansatz, with the exception of the lapse constraint (which is unknown in the most general case). It is easily extendable to handle different sets of equations, for instance different formulations of the BFE (and the EFE).

6.4 Gravitational collapse: first numerical results

The first numerical results in nubirel, obtained by evolving the bimetric standard $3+1$ evolution equations (6.55) in the partially free evolution scheme where we impose the lapse constraint (6.64) during the evolution, are reported in Paper VIII. This section reviews its most important conclusions and presents original results, not contained in Paper VIII, about the extraction of the longitudinal gravitational wave emitted by a spherically symmetric collapsing dust cloud.

The $3+1$ Valencia formulation of relativistic hydrodynamics. Here we review the recasting of the equations of relativistic hydrodynamics in the "Valencia formulation," introduced in [MIM91]. This formulation is used in the numerical integration of the bimetric standard 3+1 equations in Paper VIII. We will follow closely the reference [RZ13, Sec. 7.3.3], to which we refer the reader for more details. Note that the discussion here regards GR, but it directly generalizes to BR since independent matter sources couple to different metrics.

The equations of relativistic hydrodynamics are [RZ13, Sec. 3.3],

$$\nabla_\mu \big(\rho_0 \, u^\mu\big) = 0, \tag{6.78a}$$

$$\nabla_\mu T^{\mu\nu} = 0, \tag{6.78b}$$

with u^μ 4-velocity vector field of the fluid with rest mass density ρ_0. These equations describe the general relativistic conservation of rest mass of the fluid and the conservation of energy and momentum of the fluid, respectively. In [RZ13, Sec. 7.3.2] it is explained how it is desirable to recast (6.78) into "conservative" form, defined to be

$$\partial_t U + \nabla \cdot F(U) = 0, \tag{6.79}$$

where U is a vector whose components are the unknown functions, F is a vector-valued function of U and $\nabla = (\partial_x, \partial_y, \partial_z)$. This is the case since in hydrodynamics shocks are admitted, defined as points in phase space where the derivatives of the unknown functions are not defined. The interested reader is referred to [TA77]. What is of interest here is a theorem stating that, if the equations are not recasted in conservative form and shocks are present, then the numerical solution does not converge to the correct solution [HF94]. For this reason, the Valencia formulation, being conservative, has become the most widely used in numerical relativity, as pointed out in [RZ13, Sec. 3.3].

We do not anticipate that the system described in this section, the spherically symmetric bimetric gravitational collapse, will develop shocks during the evolution, but we implemented the Valencia formulation to enable simulations of more complex systems in the future.

The first step to recast (6.78) in the Valencia formulation is to define the variable,

$$\mathcal{D} := \rho_0 \, \alpha u^t. \tag{6.80}$$

The Lorentz factor w of the relative spatial velocity between the fluid and the Eulerian observer with 4-velocity ν^μ is [Far13, p. 142],

$$w = -\nu_\mu u^\mu = \alpha u^t, \tag{6.81}$$

where we used (5.66). Hence,

$$\mathscr{D} = \rho_0\, w. \tag{6.82}$$

It is possible to define a spacelike vector v such that [Far13, p. 142],

$$w = \left(1 - v^\mu v_\mu\right)^{-1/2}, \quad u^\mu = w(\nu^\mu + v^\mu), \quad \nu_\mu v^\mu = 0. \tag{6.83}$$

It follows that,

$$v^\mu = \frac{\perp^\mu{}_\nu u^\nu}{w} = \frac{\perp^\mu{}_\nu u^\nu}{-\nu_\mu u^\mu} \quad \Longrightarrow \quad v^t = 0, \quad v^i = \frac{1}{\alpha}\left(\frac{u^i}{u^t} + \beta^i\right). \tag{6.84}$$

Starting from (6.78a) and using (6.83, 6.84), it is possible to obtain the evolution equation for $\mathscr{D}$ as,

$$\partial_t\left(\sqrt{\Delta}\mathscr{D}\right) + \partial_i\left[\sqrt{\Delta}\,\mathscr{D}\left(\alpha v^i - \beta^i\right)\right] = 0. \tag{6.85}$$

The stress–energy tensor of a perfect fluid with 4-velocity u^μ is,

$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + p g^{\mu\nu}, \tag{6.86}$$

with ϵ, p energy density and pressure measured by an observer for which the fluid has 4-velocity u^μ. After some rewriting, its $3+1$ decomposition results in,

$$T^{\mu\nu} = \rho^{\mathrm{m}}\, \nu^\mu \nu^\nu + j^{\mathrm{m}\mu}\nu^\nu + \nu^\mu j^{\mathrm{m}\nu} + J^{\mathrm{m}\mu\nu}, \tag{6.87a}$$

$$J^{\mathrm{m}\mu\nu} = \rho_0\, h\, w^2\, v^\mu v^\nu + p\,\gamma^{\mu\nu}, \tag{6.87b}$$

$$j^{\mathrm{m}\mu} = \rho_0\, h\, w^2\, v^\mu, \tag{6.87c}$$

$$\rho^{\mathrm{m}} = \rho_0\, h\, w^2 - p, \tag{6.87d}$$

with $h := (\epsilon + p)/\rho_0$ being the specific enthalpy.[12]

The 4-divergence of a symmetric rank-2 tensor can be written as,

$$\nabla_\mu T^{\mu\nu} = g^{\nu\lambda}\left[\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\, T^\mu{}_\lambda\right) - \frac{1}{2}T^{\alpha\beta}\partial_\lambda g_{\alpha\beta}\right]. \tag{6.88}$$

Setting (6.88) to zero due to (6.78b), and substituting (6.87) into it, gives the momentum conservation equation for $j^{\mathrm{m}}{}_i$,

$$\partial_t\left(\sqrt{\Delta}\, j^{\mathrm{m}}{}_j\right) + \partial_i\left[\sqrt{\Delta}\left(\alpha J^{\mathrm{m}i}{}_j - \beta^i\, j^{\mathrm{m}}{}_j\right)\right] = \frac{1}{2}\sqrt{-g}\, T^{\alpha\beta}\partial_j g_{\alpha\beta}. \tag{6.89}$$

[12] The enthalpy of a thermodynamic system is defined as $H := U + pV$, with U internal energy, p pressure and V volume [ZD97, Secs. 10.1-10.2]. The specific enthalpy h is the enthalpy per unit of rest mass, hence $h = (U + pV)/M = (U + pV)/(\rho_0 V)$, with ρ_0 rest mass density. It follows $h = (U/V + p)/\rho_0 = (\epsilon + p)/\rho_0$.

The energy conservation equation is obtained by projecting (6.78b) along ν^μ,

$$\nabla_\mu\big(T^{\mu\nu}\nu_\nu\big) - T^{\mu\nu}\nabla_\mu\nu_\nu = 0, \tag{6.90}$$

and again rewriting it by making use of (6.87). This results in,

$$\partial_t\big(\sqrt{\Delta}\,\rho^{\mathrm{m}}\big) + \partial_i\Big[\sqrt{\Delta}\Big(\alpha\, j^{\mathrm{m}i} - \beta^i\,\rho^{\mathrm{m}}\Big)\Big] = -\sqrt{-g}\,T^{\mu\nu}\nabla_\mu\nu_\nu. \tag{6.91}$$

The last step is to rewrite the right-hand sides of (6.89, 6.91) in the $3+1$ split,

$$\frac{1}{2}\sqrt{-g}\,T^{\alpha\beta}\partial_j g_{\alpha\beta} = \sqrt{-g}\Big[\frac{1}{2}J^{\mathrm{m}ik}\partial_j\gamma_{ik} + \frac{1}{\alpha}j^{\mathrm{m}}{}_i\partial_j\beta^i - \rho^{\mathrm{m}}\partial_j\ln(\alpha)\Big], \tag{6.92a}$$

$$\sqrt{-g}\,T^{\mu\nu}\nabla_\mu\nu_\nu = \sqrt{-g}\Big[K_{ij}J^{\mathrm{m}ij} - j^{\mathrm{m}i}\partial_i\ln(\alpha)\Big], \tag{6.92b}$$

$$\sqrt{-g} = \alpha\sqrt{\Delta}. \tag{6.92c}$$

Finally, the Valencia formulation reads,

$$\partial_t\big(\sqrt{\Delta}\,U\big) + \partial_i\big(\sqrt{\Delta}\,F^i\big) = S, \tag{6.93}$$

with

$$U = \begin{pmatrix}\mathscr{D}\\ j^{\mathrm{m}}{}_j\\ \rho^{\mathrm{m}}\end{pmatrix} := \begin{pmatrix}\rho_0\,w\\ \rho_0\,h\,w^2 v_j\\ \rho_0\,h\,w^2 - p\end{pmatrix}, \qquad F^i := \begin{pmatrix}\alpha\,v^i\,\mathscr{D} - \beta^i\mathscr{D}\\ \alpha J^{\mathrm{m}i}{}_j - \beta^i j^{\mathrm{m}}{}_j\\ \alpha\,j^{\mathrm{m}i} - \beta^i\rho^{\mathrm{m}}\end{pmatrix}, \tag{6.94a}$$

$$S := \sqrt{\Delta}\begin{pmatrix}0\\ \frac{1}{2}\alpha\,J^{\mathrm{m}ik}\partial_j\gamma_{ik} + j^{\mathrm{m}}{}_i\partial_j\beta^i - \rho^{\mathrm{m}}\partial_j\alpha\\ \alpha J^{\mathrm{m}ij}K_{ij} - j^{\mathrm{m}j}\partial_j\alpha\end{pmatrix}. \tag{6.94b}$$

The system (6.93) is in conservative form. It is numerically more accurate to evolve the quantity $\tau := \rho^{\mathrm{m}} - \mathscr{D}$ rather than ρ^{m} [RZ13, Sec. 3.3]. This changes the last component of F^i into $(F^i)_3 = \alpha\big(j^{\mathrm{m}i} - \mathscr{D}\,v^i\big) - \beta^i\,\tau$. Note also that, for simplicity, we also define the quantities,

$$\hat{\mathscr{D}} := \sqrt{\Delta}\,\mathscr{D}, \quad \hat{S} := \sqrt{\Delta}\,j^{\mathrm{m}}, \quad \hat{\tau} := \sqrt{\Delta}\,\tau. \tag{6.95}$$

Even if the conserved variables $\mathscr{D}, j^{\mathrm{m}i}, \tau$ are easily expressed in terms of the primitive variables $\rho_0, v^i, \rho^{\mathrm{m}}$, the converse is not true. In the general case, a root-finding algorithm is necessary to compute the primitive variables, which serve to physically interpret the results and to be compatible with the equations of state, usually given in terms of the primitive variables. As shown in Paper VIII, for our system the inverse relation can be computed exactly, hence the root-finding procedure is not necessary.

In the present version of our solver, discussed in the remainder of the section, the Valencia formulation is used to evolve the gravitational collapse of pressureless dust minimally coupled to g.

The reference GR solution. The first step in Paper VIII is to consider the gravitational collapse in GR described by the solution found in [Nak+80]. This solution constitutes the reference to compare the bimetric gravitational collapse with.

The solution describes a spherically symmetric gravitational collapse, and the line element reads,

$$\mathrm{d}s^2 = -\alpha^2 \mathrm{d}t^2 + a^2\big(\mathrm{d}r + \beta\,\mathrm{d}t\big)^2 + b^2 r^2\big(\mathrm{d}\theta^2 + \sin(\theta)^2 \mathrm{d}\phi^2\big), \tag{6.96}$$

where (r, θ, ϕ) are the spatial coordinates on the initial hypersurface. The nonzero components of the extrinsic curvature are,

$$K_1 \coloneqq K^r{}_r, \quad K_2 \coloneqq K^\theta{}_\theta = K^\phi{}_\phi. \tag{6.97}$$

The only nonzero component of the shift is the radial one β^r.

On the initial hypersurface defined by $t = 0$ the system is assumed to be time symmetric, that is, the first derivatives of the metric components are zero. Since the shift vector is assumed to be zero, the components of the extrinsic curvatures are also such,

$$K_1 = K_2 = 0, \tag{6.98}$$

and the spatial metric is assumed to be conformally flat,

$$\mathrm{d}\ell^2 = \psi(r)^4\Big(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin(\theta)^2\mathrm{d}\phi^2\Big). \tag{6.99}$$

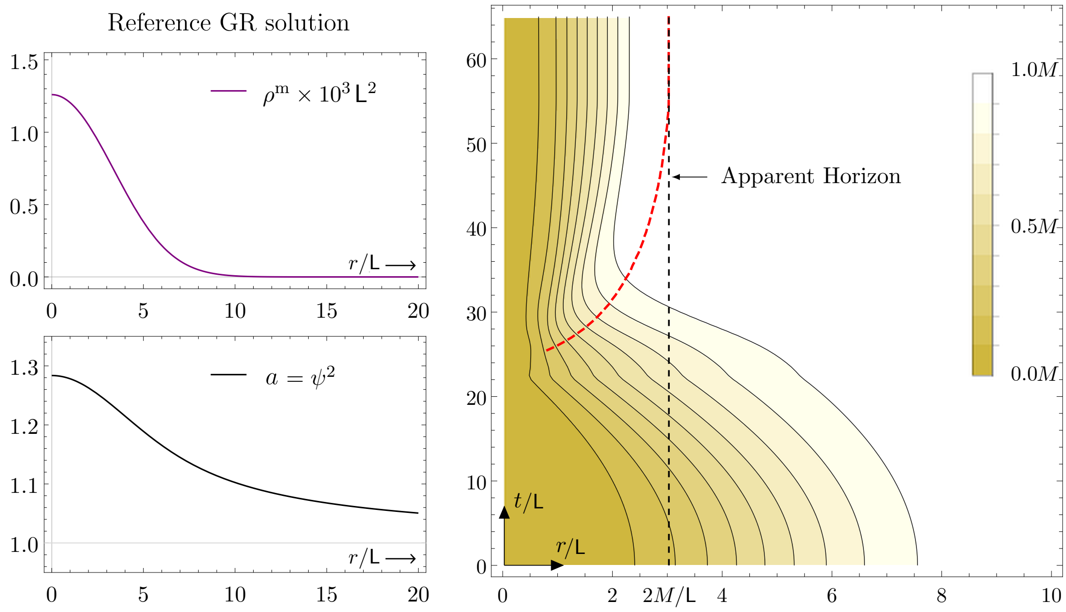
With these assumptions, the momentum constraint is satisfied. To solve the Hamiltonian constraint, we assume the following initial profile for $\hat{\mathscr{D}}$,

$$\hat{\mathscr{D}}(t=0,r) = c_0^3\Big[1 + \frac{1}{2r}\,\mathrm{erf}\big(c_0 r\sqrt{\pi}\big)\Big]\mathrm{e}^{-c_0^2 r^2 \pi}, \quad \mathrm{erf}(z) \coloneqq \frac{2}{\sqrt{\pi}}\int_0^z \mathrm{e}^{-x^2}\mathrm{d}x, \tag{6.100}$$

whose corresponding energy density initial profile is shown in Figure 6.6a and has a Gaussian shape. The Hamiltonian constraint is satisfied by the following conformal factor,

$$\psi(r) = 1 + \frac{1}{2r}\,\mathrm{erf}\left(c_0 r\sqrt{\pi}\right), \quad \lim_{r\to 0}\psi = 1, \quad \lim_{r\to\infty}\psi = 1, \tag{6.101}$$

hence the metric is flat at $r = 0$ and in the limit $r \to \infty$. The evolution of these initial data, obtained by choosing a zero shift and a lapse determined by maximal slicing, is shown in Figure 6.6b, where the collapse of the Lagrange shells is plotted. A “Lagrange shell” is defined as containing always the same fraction of the total rest mass of the cloud during the evolution. The solution to the constraints and the numerical evolution were obtained with a Mathematica/C++ toolkit called `bim-solver`, whose first version was written by Mikica Kocic. The author of this thesis worked on the later versions of `bim-solver`, on both Mathematica and C++, to adapt it to the bimetric cBSSN formulation and to make the bimetric simulations stable. This is ongoing work.



(a) Initial data

(b) Collapse of the Lagrange shells

Figure 6.6: Figure adapted from Figure 5 in Paper VIII. The values of (t, r) are given in geometrized units and in units of the length scale L, determined by the requirement that the event horizon of the final black hole is at $r = 2M$ with M mass of the black hole, as in [Nak+80]. In this case, the final location of the horizon is at $r/\mathsf{L} \simeq 3$, hence $2M/\mathsf{L} \simeq 3$ implying $\mathsf{L} \simeq 2M/3$. Note that the mass M of the initial dust cloud is arbitrary. (a) Initial profile for the matter density $\rho^{\mathrm{m}}(t = 0, r)$ (multiplied by $10^3\mathsf{L}^2$ for visibility) and the metric function $a(t = 0, r) = b(t = 0, r) = \psi(r)^2$, for $c_0 = (3\sqrt{2\pi})^{-1}$ (see main text for more details). (b) Development of the initial data obtained with `bim-solver`, a Mathematica/C++ toolkit written by Mikica Kocic and the author of this thesis. The Lagrange shells, whose corresponding fraction of rest mass is shown with the color code, collapse and form a black hole. The red dashed line shows the evolution of the radius of the apparent horizon, and the black dashed line indicates its final location at $r = 2M$. The evolution is obtained by using a zero shift and a lapse determined by maximal slicing. This can be seen by the fact that the Lagrange shells stop collapsing after entering the apparent horizon, since their proper time slows down more and more. The wiggles in the paths of the Lagrange shells are due to numerical errors. Improving the quality of the simulations is part of an ongoing project.

Some remarks about the implementation of `bim-solver`. The initial data are solved by `bim-solver` in `Mathematica`, and exported to a file imported by the C++ code which performs the numerical integration of the evolution equations.

The numerical methods employed in `bim-solver` consist in the finite difference scheme on a uniform grid, with the FTCS (forward time–centered space) method [Pre+07, p. 1032]. The finite difference scheme is implemented up to the sixth order. At the very first and last points of the grid, it is not possible to use a centered finite difference scheme, since, depending on the order of the scheme, a given point lacks the minimum number of neighboring points to be used to estimate the derivatives of the fields. For this reason, "ghost" points must be introduced, where only the values of the fields are populated, not of their derivatives [Goo+03, `Cactus` User's Guide, Subsec. C1.3.5]. The values of the fields at the ghost points are not populated through dynamical evolution; rather, by means of the inner and outer boundary conditions, briefly discussed below. At each grid point, the dynamical fields are evolved using an implementation of the method of lines (MoL) based on [Goo+03, Documentation of `CactusBase`], employing one explicit method among the Runge–Kutta (RK) explicit methods [Run95; Kut01][But08, Sec. 23, Ch. 3], the iterative Crank–Nicholson (ICN) method [CN47][BS10, p. 207], the explicit Runge–Kutta–Dormand–Prince methods with adaptive step size and error estimate [DP80][But08, p. 211]. In the version including the bimetric cBSSN evolution equations, there is also the possibility to choose two diagonally implicit Runge–Kutta methods [But65; Nør; Ale77][KNO96; KC16; JKT18][But08, p. 261] of third and fifth order, with an explicit first stage and a single eigenvalue (ESDIRK32a and ESDIRK54a) [Kvæ04]. The ESDIRK methods have been implemented in order to improve the stability of the numerical integration of the bimetric cBSSN equations, but so far we have been unlucky and the simulations are still unstable. The Kreiss–Oliger dissipation [KOC73][RZ13, Subsec. 8.3.7] is included in the evolution algorithm, to avoid spurious oscillating modes due to the finite difference approximation of the first and second radial derivatives.

Another central problem that has to be faced is the finite extent of the radial grid. This forces us to impose boundary conditions both at the inner boundary $r = 0$ and at the outer boundary $r = r_{\max}$. In principle, one could compactify the radial domain and impose the boundary conditions at $r = \infty$. However, this would imply using a non-uniform grid and `bim-solver` does not support this feature yet. In addition, it has been shown empirically that, for hyperbolic PDEs, imposing a physically well-motivated boundary condition at the outer boundary at finite $r_{\max}$ results in better-behaved simulations, see for example [GN09, Subsec. 3.1.2] and references therein. An example of physically well-motivated boundary conditions in NR is the Sommerfeld's "radiation boundary condition" [Som12; Sch92], which assumes asymptotic flatness and limits the reflection of unphysical (to be read, numerical) oscillating modes at the outer boundary. However, in nubirel we do not know if we can assume asymptotic flatness at infinity, since the bimetric sources may not tend to zero asymptotically and the assumption may be too restrictive. Hence, we implemented open boundary conditions, that is, we extrapolate the values of the dynamical fields at the outer boundary by Taylor expansion. The inner boundary conditions are more easily handled, since at $r = 0$ there is a coordinate singularity and we need to remove it, otherwise the numerical integration will develop instabilities there. This is done by imposing regularity of the radial derivatives of the fields at $r = 0$ by means of the parity conditions mentioned above, that is, components of even-rank tensors are even in r and components of odd-rank tensors are odd in r. This fixes the values of the fields on the ghost-cells.

The formation of the event horizon during the collapse is tracked by monitoring the formation of the apparent horizon. Since the event horizon is a global concept (see, e.g., [Wal10, Sec. 12.1] and [CCH12, Section 2.4]), it is not possible to establish if a surface is an event horizon during the numerical integration, since the global structure of the spacetime is not known yet (the integration is building the spacetime).

The formation of the apparent horizon is detected by tracking the marginally outer trapped surfaces (MOTS)—we refer the reader to [Wal10, Ch. 9-12] for the definitions of trapped surface, outer trapped surface, marginally trapped surface, apparent horizon, and their relation to the event horizon. Here, we only state that the apparent horizon is the outermost MOTS, and that it is always contained in, or coinciding with, the event horizon, if the null energy condition and some technical conditions about the spacetime structure at future null infinity hold [Wal10, Sec. 12.1]. The apparent horizon is a local concept, and can be easily detected in spherical symmetry. Indeed, a spherically symmetric MOTS is a surface such that [Bau+96; Tho07][Mas15, Ch. 7, App. G.5],

$$\zeta(r) \coloneqq \partial_r \log(b) - aK_2 = 0. \tag{6.102}$$

In `bim-solver`, the equation (6.102) is solved by a root-finding algorithm.

Solving for the bimetric initial data. The goal of Paper VIII is to describe a class of spherically symmetric gravitational collapse numerical solutions in BR. These are obtained in a very similar way as for the GR case. First, we assume that the two metrics share the same SO(3) KVFs. Second, we choose spherical coordinates on the initial hypersurface and write the line elements as,

$$\mathrm{d}s_g{}^2 = -\alpha^2 \mathrm{d}t^2 + a^2\big(\mathrm{d}r + \beta\, \mathrm{d}t\big)^2 + b^2 r^2\big(\mathrm{d}\theta^2 + \sin(\theta)^2 \mathrm{d}\phi^2\big), \tag{6.103}$$
$$\mathrm{d}s_f{}^2 = -\tilde{\alpha}^2 \mathrm{d}t^2 + \tilde{a}^2\big(\mathrm{d}r + \tilde{\beta}\, \mathrm{d}t\big)^2 + \tilde{b}^2 r^2\big(\mathrm{d}\theta^2 + \sin(\theta)^2 \mathrm{d}\phi^2\big). \tag{6.104}$$

We again assume that at $t = 0$, on the initial hypersurface, we have time symmetry,

$$K_1 = K_2 = \tilde{K}_1 = \tilde{K}_2 = 0. \tag{6.105}$$

This implies that $\mathbf{p}$ is zero on the initial hypersurface, that is, the radial components of the shifts β, $\tilde{\beta}$ and q (the only non-zero ones) coincide. We assume that both spatial metrics are conformally flat at $t = 0$,

$$\mathrm{d}\ell_\gamma{}^2 = \psi_\gamma(r)^4\Big(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin(\theta)^2\mathrm{d}\phi^2\Big), \tag{6.106}$$
$$\mathrm{d}\ell_\varphi{}^2 = \psi_\varphi(r)^4\Big(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin(\theta)^2\mathrm{d}\phi^2\Big). \tag{6.107}$$

With these assumptions, the momentum constraints and the bimetric constraint are identically satisfied. The Hamiltonian constraints read,

$$4\psi_\gamma{}^4\Delta\psi_\gamma + \kappa_g\ell^{-2}\Big(\beta_{(0)}\psi_\gamma{}^6 + 3\beta_{(1)}\psi_\varphi{}^2\psi_\gamma{}^4 + 3\beta_{(2)}\psi_\varphi{}^4\psi_\gamma{}^2 + 6\beta_{(3)}\psi_\varphi{}^6\Big) + \kappa_g\hat{\mathscr{D}} = 0, \tag{6.108a}$$
$$4\psi_\varphi{}^4\Delta\psi_\varphi + \kappa_f\ell^{-2}\Big(\beta_{(1)}\psi_\varphi{}^6 + 3\beta_{(2)}\psi_\gamma{}^2\psi_\varphi{}^4 + 3\beta_{(3)}\psi_\gamma{}^4\psi_\varphi{}^2 + 6\beta_{(4)}\psi_\gamma{}^6\Big) = 0. \tag{6.108b}$$

The initial profile for the conserved variable $\hat{\mathscr{D}}$ is the same as in (6.100). Note that, since $\hat{\mathscr{D}}$ and ρ_0 are related by (6.80, 6.95) which involve the metric components, even though

$\hat{\mathscr{D}}$ is the same for the reference GR initial data and the bimetric initial data, ρ_0 is not. However, if the initial data for the metric functions are close to the GR initial data, the density profiles are also close. Also, note that $w = 1$ on the initial hypersurface.

We remark that, in spherical coordinates, the metric components must be even function of r [AG05; RAN08; AM11]. Hence,

$$\partial_r \psi_\gamma = \partial_r \psi_\varphi = 0. \tag{6.109}$$

Therefore, we can solve (6.108), which are ODEs, as an initial value problem by using (6.109) and the specification of the values of the conformal factors at $r = 0$ as the initial conditions. This is the same strategy used when solving for static and spherically symmetric BHs in Paper III (see Section 3.2). In the present version of `bim-solver`, (6.108) are solved in `Mathematica` as an initial value problem. An improvement would be to solve them in C++ as a boundary value problem. This would result in a faster and more efficient implementation, and is left for future work.

We note that the Hamiltonian constraints (6.108) may be regarded as a (nontrivial) generalization of the Lane–Emden equation,

$$\Delta\theta + \theta^n = 0. \tag{6.110}$$

The Lane–Emden equation describes the matter density profile for polytropic fluids [Cha57, p. 88]. We refer to (6.108) as "Lane–Emden-like" equations.

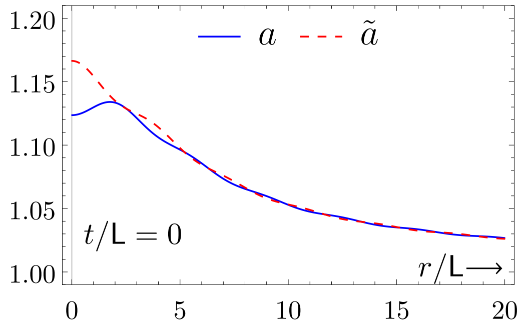
During the work leading to Paper VIII, thousands of simulations were performed varying the initial data by choosing different values of the $\beta_{(n)}$ parameters, κ_g, κ_f and the inner boundary conditions. Since the standard $3+1$ bimetric equations are ill-posed (as the GR standard $3+1$ equations are), the best simulations we have obtained evolve until $t/\mathsf{L} \simeq 7$, where we are measuring time and radial coordinate in geometrized units and in units of the same L as for the reference GR solution. In this paragraph we select one simulation, already discussed in Paper VIII and shown in Figure 6.7, with parameters

$$\kappa_g = \kappa_f = 8\pi, \quad \beta_{(1)} = -1, \quad \beta_{(n\neq 1)} = 0, \quad \ell = 1\,\mathrm{m}, \tag{6.111a}$$

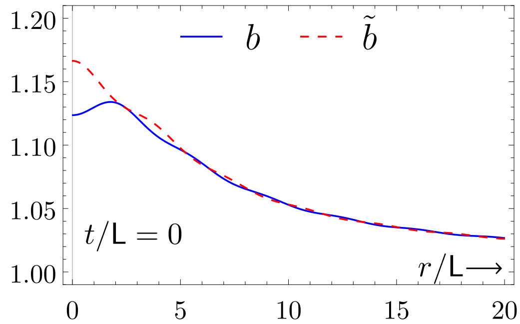
$$\psi_\gamma(r=0) = 1.06, \quad \psi_\varphi(r=0) = 1.08. \tag{6.111b}$$

The corresponding initial data for the metric functions are shown in Figures 6.7a, 6.7b. The physical origin of the oscillations is still unknown. The panels (c)–(f) of Figure 6.7 show the evolved metric functions at later times; more details about the numerical evolution are given below and in Paper VIII.

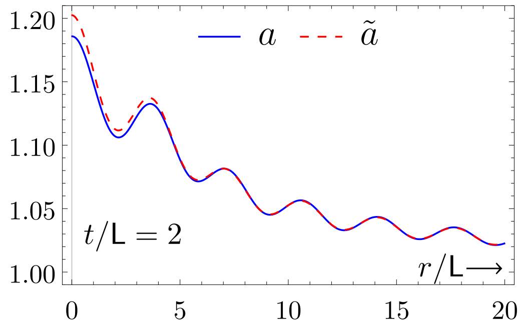
Short-term bimetric development. Regarding the numerical solutions presented in Paper VIII and reviewed here, the reference GR solution was obtained with a grid spacing $\Delta r = 0.01$, with $r_{\max} = 80$ and a fourth-order FTCS method. The MoL used a third-order RK with the Courant–Lax (CFL) factor [Pre+07, p. 1034] $\Delta t/\Delta r = 0.5$. The Kreiss–Oliger dissipation was of fourth-order with its coefficient being 0.03. The gauge condition for the lapse was maximal slicing with an identically zero shift. The resulting evolution reaches



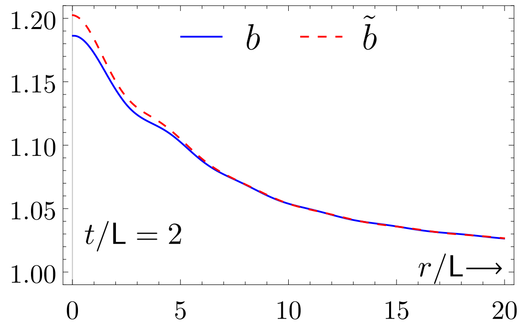
(a) Initial data for the metric functions a, $\tilde{a}$.



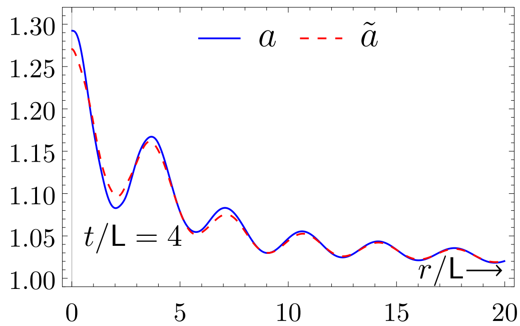
(b) Initial data for the metric functions b, $\tilde{b}$.



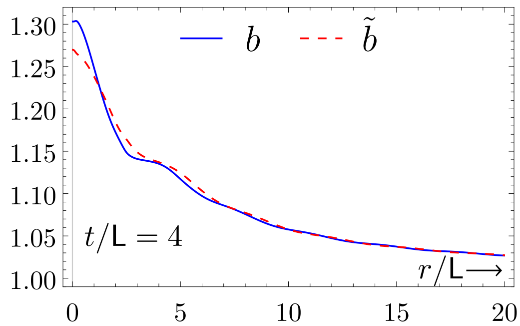
(c) Metric functions a, $\tilde{a}$ at $t/\mathsf{L} = 2$.



(d) Metric functions b, $\tilde{b}$ at $t/\mathsf{L} = 2$.



(e) Metric functions a, $\tilde{a}$ at $t/\mathsf{L} = 4$.



(f) Metric functions b, $\tilde{b}$ at $t/\mathsf{L} = 4$.

Figure 6.7: Figure adapted from Figure 10 in Paper VIII. Initial data and their evolution at $t/\mathsf{L} = 2, 4$ for the bimetric solution corresponding to the parameters (6.111). The oscillations of the metric fields in space and time are the most distinctive feature of the bimetric solutions compared with GR. The origin of these oscillations can be understood by looking at the linearized solutions, see p. 124.

beyond $t = 60$.

For the bimetric development of the initial data in Figures 6.7a, 6.7b, shown in Figures 6.7c, 6.7d, 6.7e, 6.7f, the grid spacing was increased to $\Delta r = 0.04$ with the outer boundary at $r_{\max} = 300$ and the CFL factor reduced to 0.25. All other methods are the same. The gauge condition is the maximal slicing with respect to g and $q = 0$. The evolution of β and $\tilde{\beta}$ is shown in Figure 6.8. The oscillations of the shifts (hence of $\mathbf{p}$), the lapses (see Figures 8 and 9 in Paper VIII) and the metric functions, make the causal coupling of the metrics change during the evolution. The null cones (hence the causal coupling) oscillate in time and space in a way depicted in the casual diagrams included in Paper VIII and

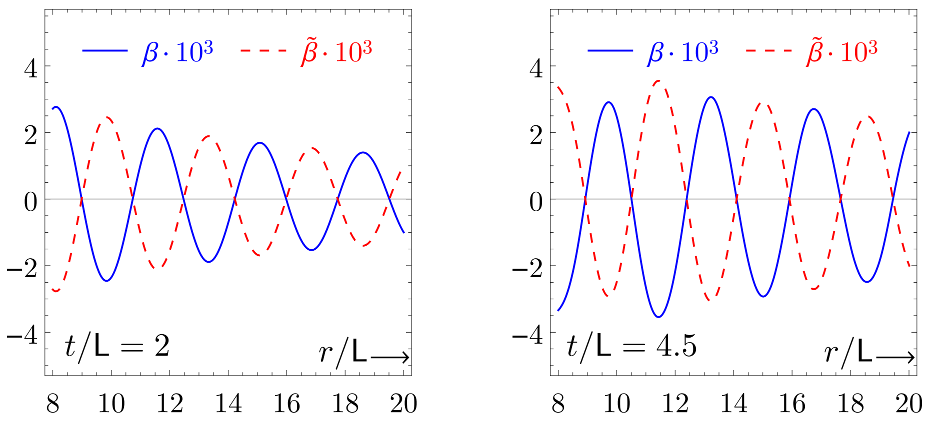


Figure 6.8: Figure adapted from Figure 11 in Paper VIII. The evolution of the shift vectors of g and f. They oscillate around the mean shift $q = 0$. This means that the separation parameter $\mathbf{p}$ is also oscillating in time and radial distance.

explained in more detail in [Koc19b]. The null cones oscillate alternating Type I and Type IIb configurations in different regions of spacetime. At the boundary of these regions, they assume a Type IIa configuration. We have never encountered Type III configurations so far. We conjectured that, since spherical symmetry makes the metrics to assume a 2-block form, and the metrics in Type III configurations have a 3-block form (see Table 6.1), it is not possible to obtain Type III configurations in spherical symmetry.

In Paper VIII, we also show one example of a bimetric vacuum numerical solution which is spherically symmetric and nonstationary, which confirms that the Jebsen–Birkhoff theorem is not valid in BR, compatibly with Paper II.

During the evolution, the constraints are evaluated to monitor their violation. This provides a measure of the quality of the solution, since large constraints' violations imply that the solution cannot be close to the exact solution, for which the constraints would be identically zero since they are stably propagated in the free evolution scheme. As reported in Paper VIII, the presented solutions are trustworthy for $t < 5$, where the constraints start to be too large compared with the energy density at each r. This comparison is meaningful since the constraints have the dimension of an energy density.

The development of the initial data presented in Paper VIII lasts too short to be able to draw conclusions about the end point of bimetric gravitational collapse of a spherical cloud of dust. However, as shown in Figure 6.9, the short-term bimetric evolution of initial data close to the initial data used for the reference GR solution, closely follows the GR evolution. Figure 6.9 shows the comparison between the evolutions of the Lagrange shells in GR and BR. There are no hints of physical—i.e, non-numerical—instabilities, as the bimetric collapse is slower than for the reference GR solution, for this model. This is certainly promising and further motivates the work to finally obtain stable long-term bimetric simulations, which is the subject of an ongoing project and would constitute the real birth of nubirel.

Extraction of gravitational waves. Even though the simulations are not stable enough to give us conclusive information about the end state of bimetric gravitational

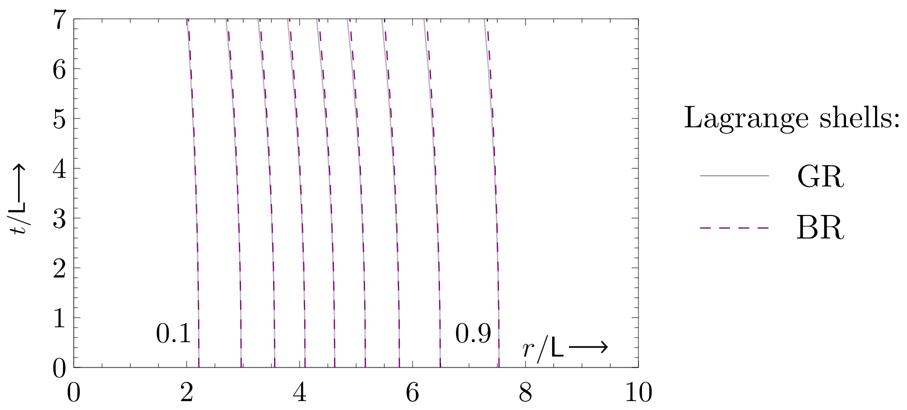


Figure 6.9: Figure adapted from Figure 3 in Paper VIII. Comparison between the collapse of the Lagrange shells for the bimetric solution described in the main text (dashed lines) and for the reference GR solution (solid lines). The bimetric collapse happens more slowly and there are no hints of physical instabilities. Long-term evolution is needed to be able to draw more meaningful physical conclusions about gravitational collapse in BR.

collapse, it is possible to extract other types of physical information from them. For instance, if we find a solution which is asymptotically flat, asymptotically dS or asymptotically AdS when $r \to \infty$, we can use linear perturbation theory to extract the propagating modes out from the nonlinear solution. We focus on the asymptotically flat case. We stress that we do not solve the linearized equations, since they are not reliable in BR as we stated in the Introduction. Rather, we will only *compute* the quantities of physical interests in a region where the *fully nonlinear* solution allows us to define them. This work is not included in any of the papers; it was done during the writing of this thesis. Our main references in this paragraph are [Mag08; Mag18].

We start by considering GR. Suppose that there exists a coordinate system in which the metric components satisfy,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll |\eta_{\mu\nu}|, \tag{6.112}$$

in a sufficiently large region of spacetime, where $\eta_{\mu\nu}$ is the flat metric in the specified coordinate system, and we call $h_{\mu\nu}$ the "metric perturbation."[13] In our specific case, the coordinate systems where (6.112) holds are the ones specified by the maximal slicing and $\beta = 0$ for GR, and by maximal slicing with respect to g and $q = 0$ for BR. It is possible to apply a coordinate transformation which preserves the form of (6.112). This is called "linear diffeomorphism" (LD) or "gauge transformation," and it has the following form,

$$x^\mu \to \qquad x'^\mu = x^\mu + \xi^\mu(x), \qquad\qquad |\partial_\mu \xi_\nu| = \mathcal{O}\big(|h_{\mu\nu}|\big), \tag{6.113a}$$

$$h_{\mu\nu}(x) \to h'_{\mu\nu}(x') = h_{\mu\nu}(x) - \big(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu\big). \tag{6.113b}$$

More details on this can be found in [Mag08, Sec. 1.1], [Mag18, Sec. 18.2.1] and [Ber01].

It is possible to uniquely decompose the metric perturbation $h_{\mu\nu}$ as [Mag18, Secs. 18.1.1,

[13]If some components of $\eta_{\mu\nu}$ are 0, then the corresponding components of the metric perturbations $h_{\mu\nu}$ should be much smaller than the nonzero components of $\eta_{\mu\nu}$.

18.2.1],

$$h_{00} = -2\varphi, \tag{6.114a}$$

$$h_{0i} = \delta_i + \overset{\mathrm{F}}{\nabla}_i\gamma = \delta_i + \partial_i\gamma, \tag{6.114b}$$

$$h_{ij} = -2\psi\,\eta_{ij} + \left(\overset{\mathrm{F}}{\nabla}_i\overset{\mathrm{F}}{\nabla}_j - \frac{1}{3}\eta_{ij}\overset{\mathrm{F}}{\nabla}^2\right)\lambda + \frac{1}{2}\left(\overset{\mathrm{F}}{\nabla}_i\epsilon_j + \overset{\mathrm{F}}{\nabla}_j\epsilon_i\right) + h^{\mathrm{TT}}_{ij}, \tag{6.114c}$$

where $\overset{\mathrm{F}}{\nabla}$ is the covariant derivative compatible with the flat metric in the specified coordinate system. We refer to the fields introduced in (6.114) as,

$\varphi, \psi, \gamma, \lambda$: scalar perturbations,

δ_i, ϵ_i : vector perturbations,

h^{TT}_{ij} : tensor perturbations; TT stands for "transverse–traceless."

These perturbations are subject to some constraints. Namely, the vectors δ^i and ϵ^i are divergenceless (transverse),

$$\overset{\mathrm{F}}{\nabla}_i\delta^i = \overset{\mathrm{F}}{\nabla}_i\epsilon^i = 0, \tag{6.116a}$$

and the tensor h^{TT}_{ij} is transverse and traceless,

$$\overset{\mathrm{F}}{\nabla}^i h^{\mathrm{TT}}_{ij} = \eta^{ij}h^{\mathrm{TT}}_{ij} = 0. \tag{6.116b}$$

The relations (6.116) and (6.114a) imply that,

$$\varphi = -\frac{1}{2}h_{00}, \quad \psi = -\frac{1}{6}\eta^{ij}h_{ij}. \tag{6.117}$$

The relations that determine the other perturbations in terms of $h_{\mu\nu}$ are more complicated and nonlocal, in the sense that a boundary value problem should be solved to determine them. For example, γ satisfies the boundary value problem $\overset{\mathrm{F}}{\nabla}^2\gamma = \eta^{ij}\overset{\mathrm{F}}{\nabla}_i h_{0j}$. The PDEs to be solved for the other perturbations can be found in [Mag18, Secs. 18.1.1].

In order to study how scalar, vector and tensor perturbations behave, it is possible to follow two routes. The first is to construct variables which do not change when performing LD; these are called "Bardeen's variables" [Bar80] and the scalar ones are equal to[14]

$$\Phi = \varphi + \partial_t\gamma - \frac{1}{2}\partial_t^2\lambda, \quad \Psi = -\psi - \frac{1}{6}\overset{\mathrm{F}}{\nabla}^2\lambda. \tag{6.118}$$

The second is to specify a LD ("to fix a gauge") and work in the resulting coordinates. In the latter case, the computed quantities depend on the coordinates specified by the LD. As pointed out in [Mag18, Secs. 18.1.1], it is possible to choose a LD such that $\lambda = \gamma = \delta_i = 0$.

[14]Note that the Bardeen's variables do not depend on the coordinates chosen for the metric perturbation and specified by a LD, but they do depend on the coordinates in which (6.112) holds in the first place. In our specific case, the Bardeen's variables depend on the choice of the spatial coordinates on the initial hypersurface, on the lapse and the shift, as we shall see explicitly in (6.121).

This choice completely determines the LD, thus exhausting the linear coordinate freedom, and is called "Newtonian gauge" or "Poisson gauge" or "longitudinal gauge." If only scalar perturbations are considered, then the line element in Newtonian gauge reads

$$\mathrm{d}s^2 = -\big(1+2\varphi\big)\mathrm{d}t^2 + \big(1-2\psi\big)\eta_{ij}\mathrm{d}x^i\mathrm{d}x^j. \tag{6.119}$$

Also, (6.118) implies that in the Newtonian gauge $\Phi = \varphi$ and $\Psi = -\psi$. Therefore, the Newtonian gauge simplifies the form of both the metric and the Bardeen's variables; in addition, since it exhaust the freedom to specify a LD, it excludes the presence of unphysical propagating modes, that is, propagating modes that can be eliminated or made nondynamical by applying a LD.

The two Bardeen's variables Φ and Ψ provide us with physical information. In particular in linearized GR, Φ satisfies the Poisson equation and defines the Newtonian potential, and Ψ satisfies the Poisson equation if the anisotropic part of the stress–energy tensor is equal to zero. In this case, Ψ tends to the Newtonian potential changed by sign [Mag18, Secs. 18.1.2]. Since the Poisson equation does not involve time derivatives, these Bardeen's variables are not dynamical. It turns out that the only LD-invariant dynamical variables are the components of h^{TT}_{ij} describing the two polarization of the massless spin-2 field. They describe gravitational waves in GR.

In linear perturbation theory, all the perturbations are determined by solving the linearized EFE, see [Mag18, Secs. 18.1.2]. However, if we know the value of the metric functions in $g_{\mu\nu}$ by other means, we can deduce $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$, and we can extract the scalar, vector and tensor perturbations from it.

We now move our attention to BR. If there exists a coordinate system such that,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{g\mu\nu}, \quad |h_{g\mu\nu}| \ll |\eta_{\mu\nu}|, \tag{6.120a}$$

$$f_{\mu\nu} = \eta_{\mu\nu} + h_{f\mu\nu}, \quad |h_{f\mu\nu}| \ll |\eta_{\mu\nu}|, \tag{6.120b}$$

then we can decompose both metrics as in (6.114). Note that, if only one of the metrics satisfies (6.120), it is still possible to decompose that metric only. In particular, we may be interested only in the metric directly coupled to us. Our purpose is to show one possible way to extract physics from the nonlinear numerical solutions in BR, and that, when the simulations will be good enough, these physical predictions can be compared directly with the astrophysical and cosmological observations. For this reason and for the sake of simplicity, here we assume to have only scalar perturbations. This is a strong assumption, but it is justified for the purposes of this paragraph.

Since we couple matter minimally to g, we choose to move to the Newtonian gauge with respect to g. Alternatively, we could also compute the Bardeen's variables for both g and f without specifying a gauge, and they could be compared with each other since they are invariant under LD. We compute φ and ψ by using (6.117) and the form of our spherically symmetric metric (6.103),

$$\varphi = -\frac{1}{2}h_{00} \quad = -\frac{1}{2}\big(g_{00} - \eta_{00}\big) \quad = \frac{1}{2}\big(\alpha^2 - a^2\beta^2 - 1\big), \tag{6.121a}$$

Model	$\beta_{(n)}$ parameters	$\left(\beta_{(1)} + 2\beta_{(2)} + \beta_{(3)}\right)^{1/2}$
BR1	$\beta_{(1)} = -\beta_{(2)} = 5 \cdot 10^{-5}$ $\beta_{(3)} = 4 \cdot 10^{-5}$	$\mathrm{i} \cdot 3.162 \cdot 10^{-3}$
BR2	$\beta_{(1)} = 5 \cdot 10^{-4}$ $\beta_{(2)} = -3 \cdot 10^{-4}$ $\beta_{(3)} = 0$	$\mathrm{i} \cdot 10^{-2}$
BR3	$\beta_{(1)} = 5 \cdot 10^{-4}$ $\beta_{(2)} = -2 \cdot 10^{-2}$ $\beta_{(3)} = 3 \cdot 10^{-2}$	$\mathrm{i} \cdot 9.747 \cdot 10^{-2}$
BR4	$\beta_{(1)} = 5 \cdot 10^{-4}$ $\beta_{(2)} = -2 \cdot 10^{-2}$ $\beta_{(3)} = 0$	$\mathrm{i} \cdot 1.987 \cdot 10^{-1}$
GR	$\beta_{(1)} = \beta_{(2)} = \beta_{(3)} = 0$	0

Table 6.3: BR models with $\kappa_g = \kappa_f = 8\pi$ and $\ell = 1\mathrm{m}$ used to find asymptotically flat solutions.

$$\psi = -\frac{1}{6}\eta^{ij}h_{ij} = -\frac{1}{6}\eta^{ij}\left(g_{ij} - \eta_{ij}\right) = -\frac{1}{6}\left(a^2 + 2b^2\right) + \frac{1}{2}. \tag{6.121b}$$

In BR, we do not want to linearize the BFE. Rather, we compute $\Phi = \varphi$ and $\Psi = \psi$ in the Newtonian gauge through (6.121). In GR, they do not depend on time and differ by a sign [Ber11].[15] Also, they are called "Newtonian potential" and "curvature potential," (often "spatial curvature") respectively. They were studied in the Fierz–Pauli linear theory of a massive spin-2 field on a flat background in [JMM13], where it is shown that, under appropriate conditions, Φ satisfies a Klein–Gordon equation at the linear level. It is therefore dynamical in that linear theory, contrary to GR, and represents the propagating 0-helicity longitudinal mode of the massive spin-2 field—a longitudinal gravitational wave. In our case, we are extracting Φ from the nonlinear numerical solutions, hence we cannot say what kind of equations Φ and Ψ satisfy, but we do expect them to change in time, that is, to be dynamical and to partly represent the propagating longitudinal gravitational wave.

Studying the behavior of Newtonian potential and curvature potential provides important information since both these quantities enter the geodesic equations for massive and massless particles, see for example [Mag18, Sec. 20.3.1]. Hence, they can be used to get predictions about the dynamics of massive particles and about gravitational lensing of massless particles.

In addition, the time derivatives of the Newtonian and scalar potential determine the integrated Sachs–Wolfe effect (ISW), that is, the gravitational redshift or blueshift of the cosmic microwave background photons during their path towards us [Mag18, Sec. 20.3.2]. The nonlinear equations can be integrated assuming homogeneity and isotropy. In that case,

[15]Depending on sign conventions, they can either be the same or differ by a sign.

the solutions describe bimetric cosmologies and the potentials can be extracted with an analogous strategy, see for example [Mag18, Sec. 18.2]. Then, physical predictions about the bimetric ISW can be obtained using the same formalism, and tested against the observations.

The Newtonian and curvature potentials are computed for the five models in Table 6.3; there are four models in BR and one for GR. The initial data are determined by solving (6.108) for conformal factors tending to 1 when $r \to \infty$, that is, for asymptotically flat solutions. See Appendix E for more details.

The initial data obtained by solving the Hamiltonian constraints (6.108) with the initial conditions in Table E.1 are shown in Figure 6.10, where the radial coordinate is measured in terms of the Schwarzschild radius r_{H} of the black hole formed in the reference GR solution. The following features can be recognized in Table 6.3 and Figure 6.10:

(i) As noted in Paper VIII, regular initial data of this type could be found only for negative values of the combination $\beta_{(1)} + 2\beta_{(2)} + \beta_{(3)}$.

(ii) The amplitude and the wavelength of the oscillations over the entire radial range, depend inversely on the value of $\upbeta \equiv \mathrm{Im}\left[\left(\beta_{(1)} + 2\beta_{(2)} + \beta_{(3)}\right)^{1/2}\right]$.

These features can be understood by considering the linearized solutions far away from the source. In order to do that, we need to introduce the following concept. In the models defined in Table 6.3, we assume $\kappa_f = \kappa_g$, that is, $\kappa = \kappa_g/\kappa_f = 1$. This can be mapped to a dual model with arbitrary $\hat{\kappa} = \kappa_g/\hat{\kappa_f}$ as follows (see the end of Chapter 1 on bimetric cosmological solutions),

$$\left(f_{\mu\nu}, \kappa_g, \beta_{(n)}\right) \to \left(\hat{f}_{\mu\nu}, \hat{\kappa_f}, \hat{\beta}_{(n)}\right) = \left(f_{\mu\nu}/\hat{\kappa}, \kappa_g/\hat{\kappa} = \hat{\kappa_f}, \hat{\kappa}^{n/2}\beta_{(n)}\right). \tag{6.122}$$

In (6.122) we see that since we keep $\kappa = 1$, if we arbitrarily increase the β parameters of our models in Table 6.3, in general we are both decreasing $\hat{\kappa}$ and increasing the $\hat{\beta}$ parameters of the dual model. That is the case because the change in the β parameters does not necessarily respect the scaling in (6.122).

At this point, we introduce the linearized solution for a perturbation of the following form,

$$\alpha = 1 + \delta\alpha, \quad a = b = 1 - \delta a, \tag{6.123a}$$

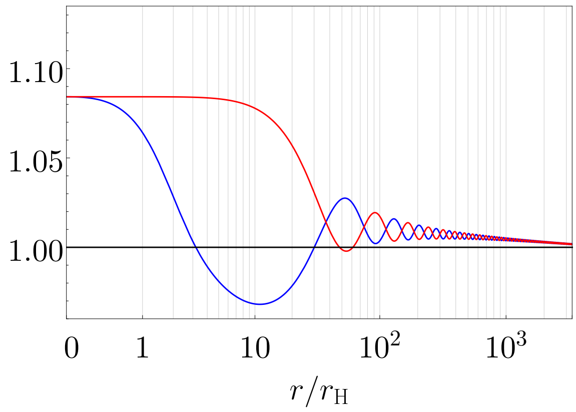
$$\widetilde{\alpha} = 1 + \delta\widetilde{\alpha}, \quad \widetilde{a} = \widetilde{b} = 1 - \delta\widetilde{a}, \tag{6.123b}$$

Usually, when the BFE are linearized around proportional backgrounds, $\beta_{(1)} + 2\beta_{(2)} + \beta_{(3)} > 0$ is assumed. In our case, though, this combination is negative, hence the linearized solutions in the g-sector for a model with arbitrary $\hat{k}$ read [Hög19],

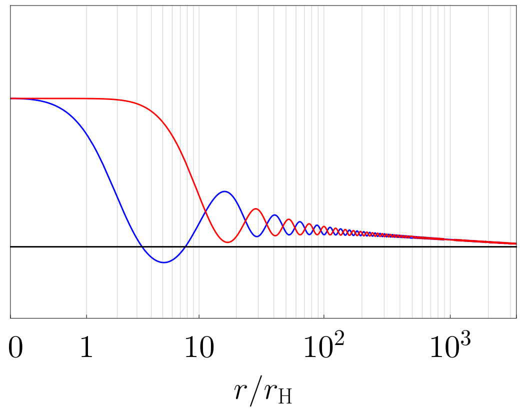
$$\delta\alpha = -\frac{\kappa_g}{1+\hat{\kappa}}\frac{M}{r}\left[1 + \frac{4}{3}\hat{\kappa}\cos\left(\frac{\hat{\upbeta}\, r}{\ell}\right)\right], \tag{6.124a}$$

$$\delta a = -\frac{\kappa_g}{1+\hat{\kappa}}\frac{M}{r}\left[1 + \frac{2}{3}\hat{\kappa}\cos\left(\frac{\hat{\upbeta}\, r}{\ell}\right)\right], \tag{6.124b}$$

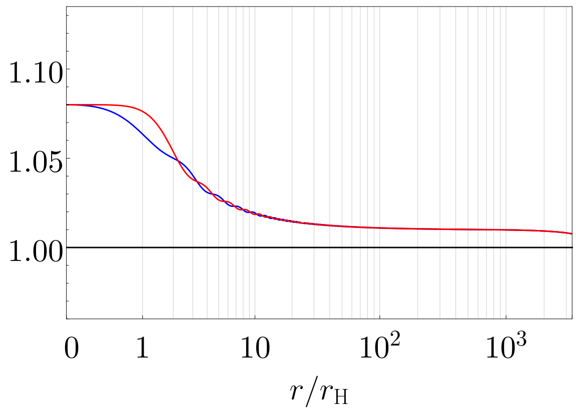
We see that if $\hat{\kappa}$ decreases the amplitude of the oscillations decreases, and if $\hat{\upbeta}$ increases, the



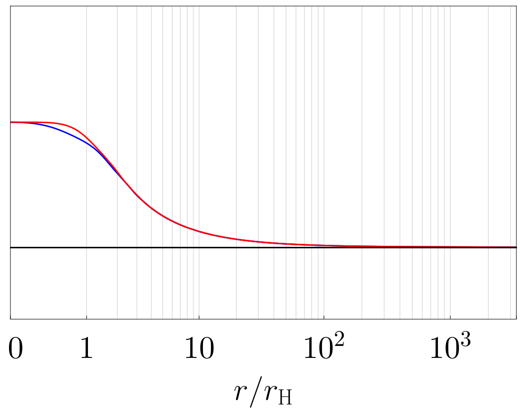
(a) BR1



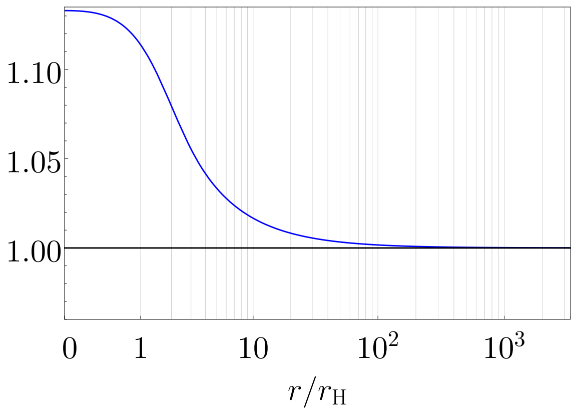
(b) BR2



(c) BR3



(d) BR4



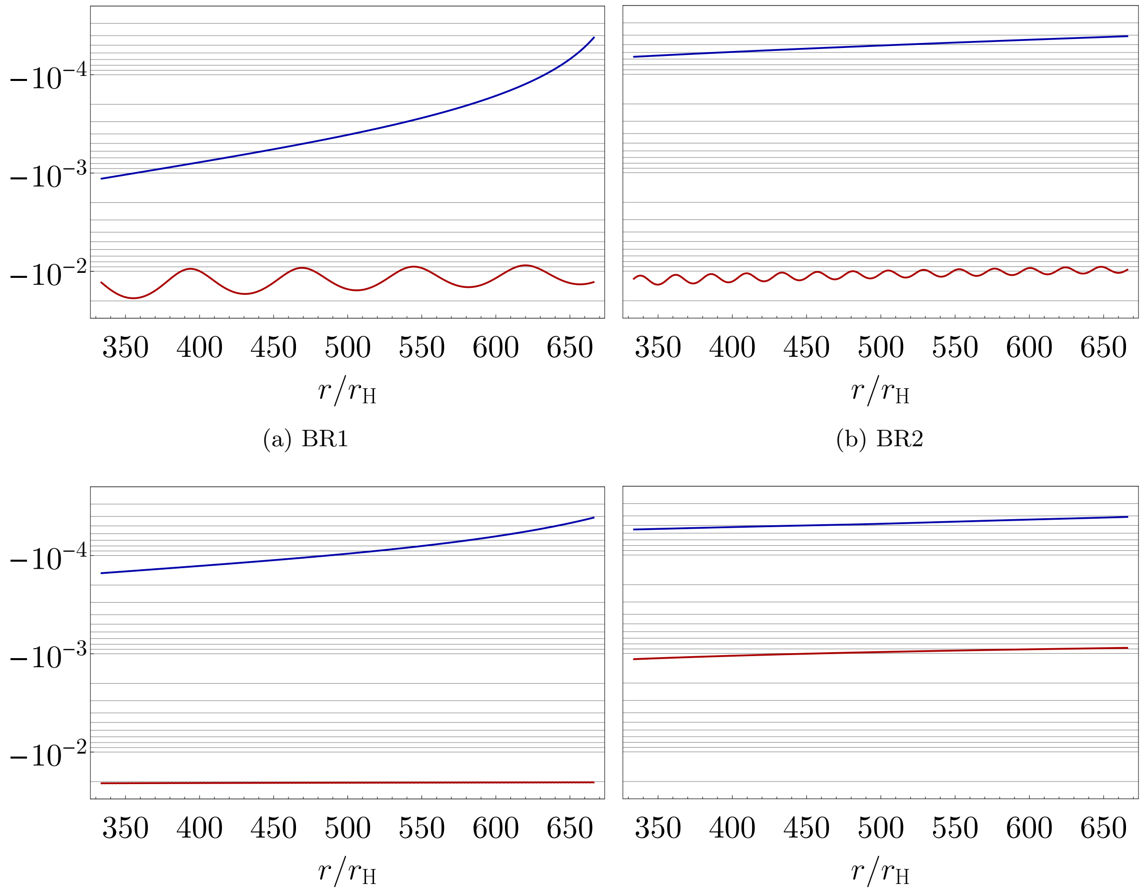
(e) GR

— ψ_γ — $\psi_\varphi - 1$

Figure 6.10: Asymptotically flat initial data for the conformal factors ψ_γ and ψ_φ, for the BR and GR models defined in Table 6.3. They are solution to (6.108). The black horizontal line indicates the value 1 to guide the eye.

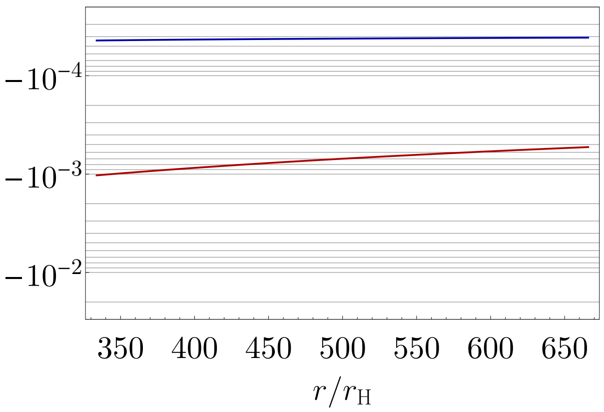
frequency increases (the wavelength decreases). This implies that, when we increase the β parameters (i.e., we *both* decrease $\hat{\kappa}$, and increase the $\hat{\beta}$ parameters and $\hat{\upbeta}$, as explained before), both the wavelength and the amplitude of the oscillations in (6.124a) decrease, which is exactly the behavior we see at the nonlinear level in Figure 6.10.

Once we have the initial data, we evolve them in `bim-solver` [see Appendix E for more details about the used numerical methods]. The BR1 and BR2 simulations stop at $t/\mathsf{L} \simeq 1.20$ and $t/\mathsf{L} \simeq 0.834$, respectively, but it is enough for the purposes of this section. The BR3, BR4 and GR simulations are integrated until $t/\mathsf{L} = 3$. Once we have the numerical solutions,



(a) BR1 (b) BR2

(c) BR3 (d) BR4



(e) GR

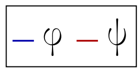


Figure 6.11: Newtonian and curvature potentials, φ and ψ, far away from the collapsing dust cloud, for the BR and GR models defined in Table 6.3 at $t/r_{\mathrm{H}} = 0.165$. The curvature potential oscillates in r in all bimetric models, but the amplitude of the oscillations is not visible in E.2c and E.2d. Figure 6.12 shows the oscillations for the BR3 model.

we compute the Newtonian and curvature potentials as described before. The results at a given instant of time, are plotted in Figure 6.11. We see that, in BR1 and BR2, the oscillations are present mostly in the scalar curvature, since the Newtonian potential is dominated by the lapse determined by maximal slicing (see (6.121a); the shift is extremely small at these radii, see Appendix E), which does not oscillates on these scales. The amplitude and wavelength of the oscillating curvature potential show the same dependence on the β parameters as for the conformal factors in the initial data. The potentials in the BR4 model are quite similar to GR, whose Newtonian and scalar potentials are not the same since we are considering the nonlinear solutions, but approach each other as the radius increases.

These results tell us that, in a generic bimetric model, an observer at $\sim 600 r_{\mathrm{H}}$ from a spherically symmetric collapsing cloud of dust, would measure potentials which can differ quite a lot from GR. However, there are also bimetric models as BR4, for which the observer would measure potentials similar to GR, even if we are not close to the GR limit of BR.

The oscillations we see in Figure 6.10 and Figure 6.11 are radial. However, as is shown in Paper VIII and reviewed previously, there are also oscillations in time. This is evident if we look at the evolution of the curvature potential, shown in Figure 6.12 for the BR3 model. The choice of this model rather than for example BR4, which is closer to GR, is due to a more accurate numerical integration, see Appendix E. The oscillations in time show that the scalar curvature is dynamical, contrary to GR. This extends the linear results in [JMM13] to the full nonlinear case and shows that spherically symmetric systems in BR with flat asymptotics produce a dynamical scalar curvature. The same is true for the Newtonian potential, not shown here since its evolution in time is slower then for the scalar curvature.

In all the plots, time and radial coordinates are given in units of the Schwarzschild radius, so if the mass of the final black hole is of the order of the mass of the Sun, the timescale of the evolution in SI units is of the order of,

$$1 \simeq t_{\mathrm{final}}/r_{\mathrm{H}} \Longrightarrow 1 \simeq (c\, t_{\mathrm{final}}^{\mathrm{SI}}) \left(\frac{2G\,\mathrm{M}_{\odot}}{c^2} \right)^{-1}, \tag{6.125a}$$

$$t_{\mathrm{final}}^{\mathrm{SI}} \simeq \frac{2G\,\mathrm{M}_{\odot}}{c^3} \simeq 9.848 \cdot 10^{-6}\,\mathrm{s} = 9.848\,\mu\mathrm{s}, \tag{6.125b}$$

where we used $G = 6.674 \cdot 10^{-11}\,\mathrm{m^3 kg^{-1} s^{-2}}$, $c = 2.998 \cdot 10^{8}\,\mathrm{m/s}$ and $\mathrm{M}_{\odot} = 1.988 \cdot 10^{30}\,\mathrm{kg}$. For a final supermassive black hole with a mass of the order $\mathrm{M} = 10^9 \mathrm{M}_{\odot}$, the timescale is $t_{\mathrm{final}}^{\mathrm{SI}} \simeq 9.848 \cdot 10^{3}\,\mathrm{s} \simeq 2\,\mathrm{h}\,44\,\mathrm{min}$.

Note that the trend of the Newtonian potential for $r \in \big(300\, r_{\mathrm{H}}, 700\, r_{\mathrm{H}}\big)$ depends on the boundary condition set at $r/\mathsf{L} = 2000$ when solving the boundary value problem of maximal slicing for the lapse of g. This boundary condition was determined following a strategy similar to the one used when solving the Hamiltonian constraints for the conformal factors, see Appendix E. Following this procedure, the Newtonian and curvature potentials are those in Figure 6.11 *in our coordinate system.*

In addition to computing the two potentials, it is also possible to extract gravitational waves in a coordinate-independent way, by computing the Moncrief variables or by using the Newman–Penrose formalism, see for example [BS10, Sec. 9.4]. However, since the Moncrief variables are related to tensor perturbations, which we do not expect to have in a spherically symmetric system where the two metrics share the same KVFs, their utility for our system is limited. Concerning the Newman–Penrose formalism and the computation of the Weyl scalars related to longitudinal and transverse gravitational waves, the difference in BR compared with GR is that the Ricci tensor is nonzero in bimetric vacuum, hence the physical information is not contained in the Weyl tensor only. Accordingly, the full Riemann tensor should be considered when computing the scalars, and their relation with longitudinal and transverse gravitational waves may change. The extension to BR of these interesting

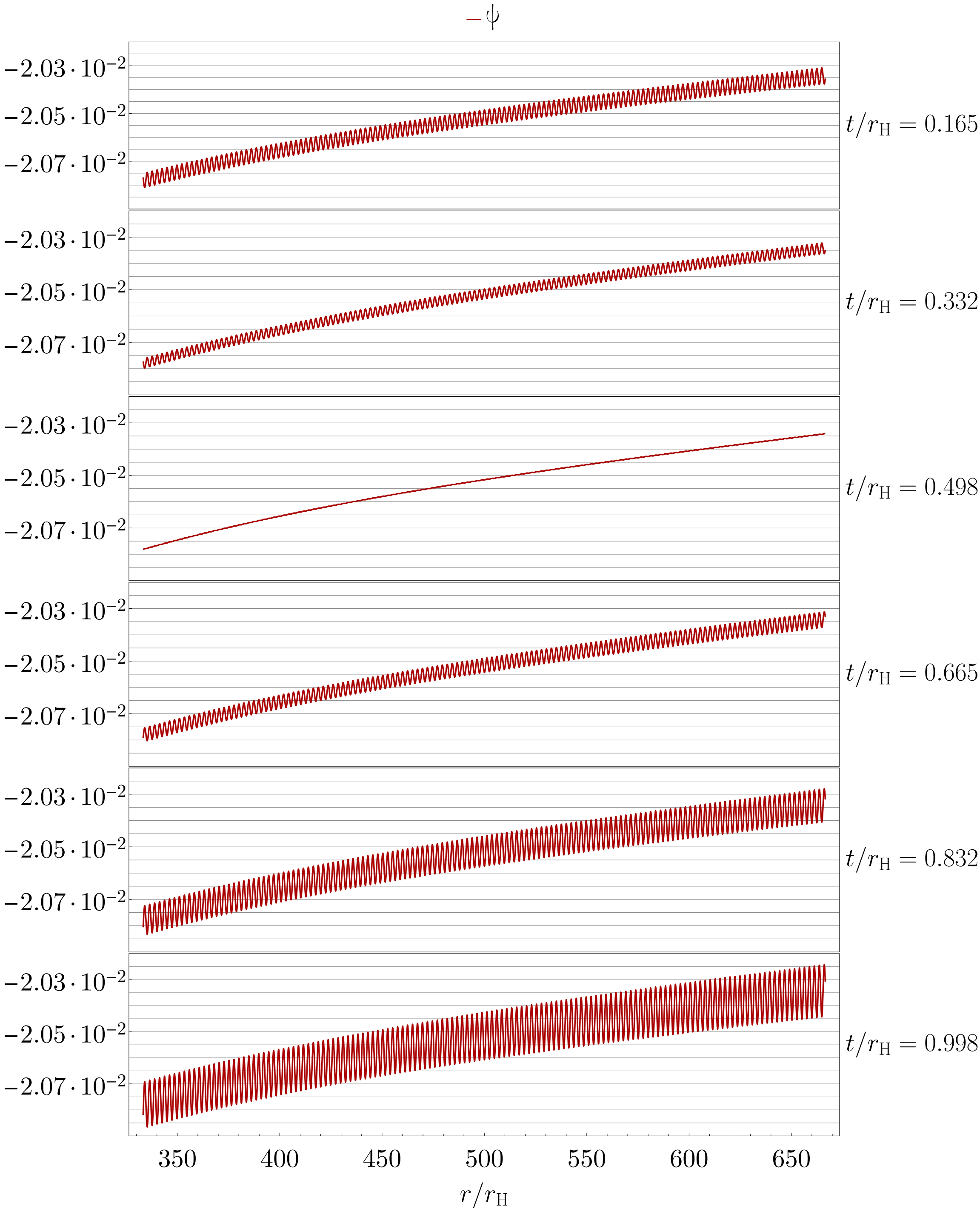


Figure 6.12: Time evolution of the curvature potential ψ for the BR3 model (see Table 6.3), which is the most accurate numerically (see Appendix E). The oscillations in time and space are clearly visible.

methods to extract gravitation waves in NR is left for future work.

Finally we note that, as we remarked at the end of Chapter 1, if $\kappa = 1$, the cosmological solutions are compatible with the cosmological observations both if the β parameters are very large and if they are very small. The bimetric models considered here all have β parameters of orders between 10^{-2} and 10^{-5}, hence they do not fit in any of the two cases for which the bimetric cosmologies are compatible with observations. This is because choosing very large or very small values for the β parameters would be challenging from the numerical viewpoint. We chose the values that made us obtain accurate asymptotically flat solutions. Exploring a more well-motivated region of the parameter space is left for future work.

Conclusion and outlook

Summary of results. The scientific results reviewed in this thesis and presented in the included papers do not fully answer the question of how gravitational collapse works in bimetric relativity. However, they do provide new methods to study the problem, inspired by similar studies in general relativity, and preliminary results with important clues on the physical properties of a spherically symmetric collapsing cloud of dust in bimetric relativity.

In Part I of the thesis, results obtained in the covariant formalism were discussed. In particular, the study of symmetries of spacetime is established in bimetric relativity, with the purpose of simplifying and solving exactly the bimetric field equations. This is the main topic of Paper I. The results clarify how symmetries of spacetime behave in bimetric relativity, but the configurations allowed by the theory do not straightforwardly single out a unique strategy that can be used to simplify the bimetric field equations. Nonetheless, they provide useful insights on how to find particular solutions with the desired symmetries, as pointed out by the discussed example solutions.

The study of spacetime symmetries brought us to the determination of the first exact vacuum solutions in bimetric relativity not equivalent to a vacuum solution in general relativity. This is presented in Paper II and is a nonstationary, spherically symmetric vacuum solution. Its existence provides a counter-example to a statement analogous to the Jebsen–Birkhoff theorem in bimetric relativity. Since this solution only exists for specific values of the parameters of the theory, the Jebsen–Birkhoff theorem may hold for other choices of the parameters. That this is not the case is shown in Paper VIII. Also, this vacuum solution cannot be regarded as an end state of gravitational collapse for generic spherically symmetric configurations.

In Paper III, studies concerning the possible static end states of bimetric gravitational collapse are presented. The imposition of one more spacetime symmetry, namely staticity, does not simplify the bimetric field equations enough to get exact solutions for generic configurations. Hence, the numerical integration of the bimetric field equations, reduced to ordinary differential equations, is necessary. The results suggest that the only possible static and spherically symmetric vacuum end states of spherically symmetric gravitational collapse are: (i) spacetimes equivalent to those in general relativity, namely the Schwarzschild, Schwarzschild–anti-de Sitter and Schwarzschild–de Sitter ones, and (ii) other spacetimes not equivalent to those in general relativity. The latter have a leading term equal to the one in the Schwarzschild–anti-de Sitter solution, but also a subleading term which grows when $r \to \infty$, hence they do not converge to the anti-de Sitter spacetime in that limit.

Some of the results summarized in Part I can be used to find other exact solutions to the bimetric field equations. Exact solutions for the gravitational collapse of pressureless radiation (so-called null dust) are found in [Hög+19]. They are analogous to the generalized Vaidya solutions in general relativity, and their static end states are spacetimes equivalent to those in general relativity. In addition, in [HTM19] it is proved that the most general solution to the bimetric field equations, describing the interior of a pressureless spherically symmetric dust cloud with homogeneous density, is the bimetric Friedmann–Lemaître–Robertson–Walker spacetime. This provides both physical information about structure formation on a homogeneous and spherically symmetric spacetime, and a strong argument against the claim that bimetric gravitational collapse happens too fast or the theory is inconsistent. Unfortunately, the solution describing the exterior of such a cloud is still unknown, and in [HTM19] it is pointed out that it cannot be any static solution, nor the nonstationary spherically symmetric solution from Paper II. This shows the need to tackle the problem numerically and thus brings us to the results discussed in Part II of this thesis.

Part II of the thesis concerns results obtained in the $3+1$ formulation of bimetric relativity. These results contribute to the new field of numerical bimetric relativity. The bimetric field equations are a system of coupled partial differential equations which can be rewritten as a constrained generalized Cauchy problem. One of the constraints has a very complicated form, which is not known explicitly in the most general case. In Paper IV, the explicit expression of this constraint is computed under the assumption of spherical symmetry, thus closing the system of equations and allowing for its numerical integration.

The first results in numerical bimetric relativity are presented in Paper VIII. The numerical simulations give us many interesting physical pieces of information. Nonstationary spherically symmetric vacuum solutions are found for generic values of the parameters of the theory, providing lots of counterexamples to a statement analogous to the Jebsen–Birkhoff theorem in bimetric relativity. One of them is shown in Paper VIII. In addition, no signs of physical instabilities can be seen in the solutions, though numerical instabilities are present which make the simulations break down before a possible bimetric black hole formation. The simulations were performed with a `Mathematica`/C++ code called `bim-solver`, whose construction constituted a sizable part of the Ph.D. studies.

In order to obtain long-term stable bimetric simulations, a necessary condition is to rewrite the bimetric field equations as a well-posed Cauchy problem. Paper V tackles this by recasting the bimetric field equations in a formulation which makes the Einstein field equations well-posed, the covariant BSSN formulation. Unfortunately, we cannot yet claim that this formulation makes the bimetric field equations well-posed.

In Paper VI, many possibilities to overcome this problem are proposed, both through additional recasting of the bimetric field equations and through the choice of suitable coordinates, which also affect the well-posedness of the system of partial differential equations. The choice of coordinates is indeed crucial both in general and bimetric relativity, since a bad choice can make the system evolve towards coordinate or curvature singularities, which are deleterious for the numerical simulations. These problems are also discussed in Paper VI, where gauge choices specific to bimetric relativity (i.e., with no analog in general relativity) are presented. The possibilities to recast the bimetric field equations discussed in Paper

VI are specific to bimetric relativity, though they are inspired by works done in general relativity.

The results of Paper V and Paper VI were obtained with a `Mathematica` package called `bimEX` which makes use of the `xAct` bundle. It is publicly available at the GitHub repository `nubirel/bimex` and at [Tor19a], where documentation and working examples can be found. The implementation of the package is described in Paper VII. The package allows the user to write down the covariant BSSN equations for any desired physical system. It has been tested in spherical symmetry, and was crucial for the work leading to Paper V and Paper VI, since it computed the spherically symmetric covariant BSSN equations to be solved by `bim-solver`. The package can be easily extended to handle other recastings of the bimetric field equations (and the Einstein field equations).

In this thesis, more details are provided about the topics discusses in the included papers. For example, a discussion about coordinate singularities in numerical bimetric relativity is provided. In addition, more details about the symmetries of spacetime for the nonstationary spherically symmetric solution of Paper II and for the static and spherically symmetric black hole solutions of Paper III are given.

Moreover, a preliminary analysis of gravitational wave extraction from the numerical solutions, not included in the papers, is performed. The result is that, very far from the collapsing body, an observer would measure an oscillating gravitational field both in space and time. The amplitude and wavelength of the oscillations depend on the parameters of the theory. For parameters compatible with the observational data, we expect the amplitude and wavelength to be very small. In general relativity, the same observer would measure the standard Newtonian gravitational force.

Outlook. The studies reported in the included papers open many avenues for future research. For example, it is possible to use the results of Paper I to deduce new ansatzes which are spherically symmetric, but have different Killing vector fields. This may lead to the discovery of new (exact) solutions. In addition, this method can be applied to static, nonstatic and nonstationary spherically symmetric solutions, and to less symmetric systems as, for example, stationary, axially symmetric ones (i.e., rotating black holes and stars). Also, it is possible to continue the study of spacetime symmetries in bimetric relativity along the line of [Hal04], that is, studying in more detail the mathematical structure of spacetime symmetries in bimetric relativity, with the purpose of understanding the properties of the solutions to the bimetric field equations. In addition, the symmetries of the bimetric field equations can be studied, and their relations with the symmetries of spacetime can be investigated, again with purpose of finding new exact solutions. For the symmetries of equations, we refer the reader to [Ste+03, Sec. 10.2, 10.3].

A possible research direction coming from Paper III would be to prove analytically that there are no static, spherically symmetric vacuum solutions, other than those equivalent to solutions from general relativity, which converge to the maximally symmetric solutions from general relativity, that is, Minkowski, anti-de Sitter and de Sitter spacetimes. Using the results from Paper I opens up the intriguing possibility to look for static and spherically symmetric black hole solutions with different timelike Killing vector fields.

The studies presented in Paper IV, Paper V, Paper VI, Paper VII and Paper VIII provide many connected possibilities for future research studies. The most compelling one is perhaps the recasting of the bimetric fields equations as a well-posed Cauchy problem. In Paper VI, possible strategies for doing this are presented. Another research direction is to compute the form of the complicated constraint of the theory in full generality, or to show that a close-form expression cannot be found in full generality. This would not constitute a severe problem in numerical bimetric relativity, since the constraint can be handled numerically. Yet another research path is to continue the work of numerically integrating the bimetric field equations in the recasting done in Paper V.

Another research path that can be undertaken is the study of spacetime symmetries in the bimetric $3+1$ formalism, inspired by the analogous study in general relativity [Chr91, Sec. 2.1]. This study would unify the two formalisms constituting the two parts of this thesis.

We highlight that the results in the papers reviewed in Part II open a new field of research in bimetric relativity, namely numerical bimetric relativity. Once this field will become fully established, that is, when stable long-term bimetric simulations will be available, there are exceedingly many possibilities about what physical systems to study, as in numerical relativity. It would be possible to obtain solutions describing the physics of realistic physical systems such as neutron star mergers, black hole-neutron star mergers or black hole-black hole mergers. In addition, since the Jebsen–Birkhoff theorem is not valid in bimetric relativity, spherically symmetric systems as a pulsating star or a cloud of pressureless dust emit gravitational waves. Analogously, transverse traceless gravitational waves in bimetric relativity can be emitted in the presence of spherically symmetric matter sources, as follows from the results of Paper I. All of this would provide different predictions compared to the ones from general relativity, where a spherically symmetric solution does not emit gravitational waves. Therefore, bimetric relativity should provide distinctive predictions compared with general relativity. As an example, in the thesis, a dynamical gravitational potential measured by an observer far away from the source is identified. Its relation to the longitudinal gravitational wave is not explored in full detail and is left for future work.

As pointed out in the thesis, both general relativity and the standard model of particle physics, though in excellent agreement with all the experiments and observations at present date, are incomplete. This is due, for example, to the dark matter, dark energy and the cosmological constant problems. Either general relativity or the standard model of particle physics, or both of them, have to be modified or extended in order to solve (some of) these problems. Exploring such extensions in order to test their value as physical theories is an important task of contemporary physics. This thesis is made with this spirit in mind.

Finally, nobody knows if bimetric relativity is the correct theory of gravity. The studies leading to this thesis certainly provide methods which can be used in future research projects with the aim of finding this out. The author is looking forward with maximum interest to these future studies and their results, irrespectively of what the result will be.

Appendix A

Variation of the bimetric potential under a diffeomorphism

In this appendix, we explicitly compute the variation of the action $\mathcal{S}_{\rm int}$, defined in (1.11), in natural units, under a generic diffeomorphism generated by the generic vector field $\mathcal{V}$.

The variation is,

$$\begin{aligned}
\delta_{\mathcal{V}}\mathcal{S}_{\rm int} &= \frac{\delta\mathcal{S}_{\rm int}}{\delta g^{\mu\nu}}\delta_{\mathcal{V}} g^{\mu\nu} + \frac{\delta\mathcal{S}_{\rm int}}{\delta f^{\mu\nu}}\delta_{\mathcal{V}} f^{\mu\nu} \\
&= -\frac{\delta\mathcal{S}_{\rm int}}{\delta g^{\mu\nu}} g^{\mu\alpha}\left(\delta_{\mathcal{V}} g_{\alpha\beta}\right) g^{\beta\nu} - \frac{\delta\mathcal{S}_{\rm int}}{\delta f^{\mu\nu}} f^{\mu\alpha}\left(\delta_{\mathcal{V}} f_{\alpha\beta}\right) f^{\beta\nu} \\
&= -m^4\left(\int_\Omega \mathrm{d}^4x\sqrt{-g}\, V_g{}^{\mu\nu}\nabla_{(\mu}\mathcal{V}_{\nu)} + \int_\Omega \mathrm{d}^4x\sqrt{-f}\, V_f{}^{\mu\nu}\tilde{\nabla}_{(\mu}\mathcal{V}_{\nu)}\right).
\end{aligned} \tag{A.1}$$

We apply the Leibniz rule and obtain,

$$\begin{aligned}
\delta_{\mathcal{V}}\mathcal{S}_{\rm int} &= m^4\left[\int_\Omega \mathrm{d}^4x\sqrt{-g}\,\left(\nabla_\mu V_g{}^{\mu\nu}\right)\mathcal{V}_\nu + \int_\Omega \mathrm{d}^4x\sqrt{-f}\,\left(\tilde{\nabla}_\mu V_f{}^{\mu\nu}\right)\mathcal{V}_\nu\right] \\
&\quad - m^4\left[\int_\Omega \mathrm{d}^4x\sqrt{-g}\,\nabla_\mu\left(V_g{}^{\mu\nu}\mathcal{V}_\nu\right) + \int_\Omega \mathrm{d}^4x\sqrt{-f}\,\tilde{\nabla}_\mu\left(V_f{}^{\mu\nu}\mathcal{V}_\nu\right)\right].
\end{aligned} \tag{A.2}$$

The last line in the previous equation can be written as,

$$\begin{aligned}
&m^4\int_\Omega \mathrm{d}^4x\,\partial_\mu\left[\sqrt{-g}\,\mathcal{V}^\nu\left(-V_g{}^\mu{}_\nu - \det(S)\,V_f{}^\mu{}_\nu\right)\right] \\
&= -m^4\int_\Omega \mathrm{d}^4x\,\partial_\mu\left(\sqrt{-g}\,\mathcal{V}^\mu V(S)\right) = -m^4\int_\Omega \mathrm{d}^4x\sqrt{-g}\,\nabla_\mu\left(\mathcal{V}^\mu V(S)\right),
\end{aligned} \tag{A.3}$$

where we have used the algebraic identity (1.25). We are left with

$$\begin{aligned}
\delta_{\mathcal{V}}\mathcal{S}_{\rm int} &= \frac{1}{2}m^4\left[\int_\Omega \mathrm{d}^4x\sqrt{-g}\,\left(\nabla_\mu V_g{}^{\mu\nu}\right)\mathcal{V}_\nu + \int_\Omega \mathrm{d}^4x\sqrt{-f}\,\left(\tilde{\nabla}_\mu V_f{}^{\mu\nu}\right)\mathcal{V}_\nu\right] \\
&\quad - m^4\int_\Omega \mathrm{d}^4x\sqrt{-g}\,\nabla_\mu\left(\mathcal{V}^\mu V(S)\right).
\end{aligned} \tag{A.4}$$

which can be rewritten as

$$\delta_{\mathcal{V}}\mathcal{S}_{\text{int}} = \frac{1}{2}m^4\int_\Omega \mathrm{d}^4x\sqrt{-g}\,\left(\nabla_\mu {V_g}^{\mu\nu} + \det(S)\,\tilde{\nabla}_\mu {V_f}^{\mu\nu}\right)\mathcal{V}_\nu$$
$$- m^4\int_\Omega \mathrm{d}^4x\sqrt{-g}\,\nabla_\mu\left(\mathcal{V}^\mu V(S)\right). \tag{A.5}$$

The differential identity (1.26) tells us that the first line is identically zero, so we find the result,

$$\delta_{\mathcal{V}}\mathcal{S}_{\text{int}} = -m^4\int_\Omega \mathrm{d}^4x\sqrt{-g}\,\nabla_\mu\left(\mathcal{V}^\mu V(S)\right) = \int_\Omega \mathrm{d}^4x\sqrt{-g}\,\nabla_\mu\left(\mathcal{V}^\mu \mathcal{L}_{\text{int}}\right), \tag{A.6}$$

with $\mathcal{L}_{\text{int}} \coloneqq -m^4 V(S)$. This is compatible with (1.29), which is obtained in an independent way for every covariant action written in terms of a scalar Lagrangian density $\mathcal{L}$ [Bla17],

$$\begin{aligned}\delta_{\mathcal{V}}\mathcal{S} &= \int_\Omega \mathrm{d}^4x\,\delta_{\mathcal{V}}\left(\sqrt{-g}\,\mathcal{L}\right) = \int_\Omega \mathrm{d}^4x\left[\left(\delta_{\mathcal{V}}\sqrt{-g}\right)\mathcal{L} + \sqrt{-g}\,(\delta_{\mathcal{V}}\mathcal{L})\right] \\ &= \int_\Omega \mathrm{d}^4x\left[\frac{1}{2}\sqrt{-g}\,g^{\mu\nu}\mathscr{L}_{\mathcal{V}}g_{\mu\nu} + \sqrt{-g}\,(\delta_{\mathcal{V}}\mathcal{L})\right] \\ &= \int_\Omega \mathrm{d}^4x\left[\sqrt{-g}\,(\nabla_\mu\mathcal{V}^\mu)\,\mathcal{L} + \sqrt{-g}\,\mathcal{V}^\mu\partial_\mu\mathcal{L}\right] \\ &= \int_\Omega \mathrm{d}^4x\sqrt{-g}\,\nabla_\mu\left(\mathcal{V}^\mu\mathcal{L}\right).\end{aligned} \tag{A.7}$$

The last equality uses

$$\sqrt{-g}\,\nabla_\mu\mathcal{V}^\mu \equiv \partial_\mu\left(\sqrt{-g}\,\mathcal{V}^\mu\right). \tag{A.8}$$

Appendix B

An ill-posed Cauchy problem

Consider the Cauchy problem given by the Laplace equation in one spatial dimension,

$$\Delta u = 0, \tag{B.1}$$

with initial data $u(0, x) = 0$, $u_t(0, x) = g(x) \coloneqq 0$. The solution to this initial value problem is $u(t, x) = 0$. Now consider the sequence of Cauchy problems expressed by the Laplace equation with the following sequence of initial data

$$u(0, x) = 0, \quad u_t(0, x) = g_n(x) \coloneqq \frac{\sin(nx)}{n^2}. \tag{B.2}$$

Note that the sequence of initial data converge to $g(x)$, that is, $\lim_{n\to\infty} g_n(x) = g(x) = 0$. If this problem was well-posed, then the sequence of solutions of these Cauchy problems would also converge to the solution corresponding to the initial data $g(x)$, namely $u(t, x) = 0$. The sequence of solutions is given by

$$u(t, x) = \frac{\sinh(nt)\sin(nx)}{n^3}, \tag{B.3}$$

and its limit when $n \to \infty$ does not exist. Therefore, the solutions to this Cauchy problem do not depend continuously on the initial data and the system is ill-posed [Had23]. The same Cauchy problem violates also the existence of the solution for non-analytic initial data at $t = 0$ [CH62, p. 228].

In order to clarify what kind of mathematical properties are involved here, we consider two examples of smooth non-analytic real functions (in the complex plane, a holomorphic function is necessarily analytic). The first is

$$f(x) \coloneqq \begin{cases} \mathrm{e}^{-1/x} & x > 0 \\ 0 & x \leqslant 0 \end{cases}, \tag{B.4}$$

which is smooth (C^∞) everywhere but not analytic at $x = 0$. Indeed, its Taylor series close to $x = 0$ converges to the zero function and not to $f(x)$. The second is defined as a Fourier series

$$F(x) := \sum_{k \in A} \mathrm{e}^{-\sqrt{k}} \cos(kx), \quad A = \{2^n, n \in \mathbb{N}\}, \tag{B.5}$$

and it is smooth everywhere, but nowhere analytic. A counterexample to the Cauchy–Kovalevskaya theorem in the case of smooth functions is given by “Lewy’s example” [Lew57]. It states that, on $\mathbb{R} \times \mathbb{C}$, there exists a smooth complex-valued function $F(t, z)$ such that the PDE

$$\frac{\partial u}{\partial \bar{z}} - \mathrm{i} z \frac{\partial u}{\partial t} = F(t, z) \tag{B.6}$$

has no solutions on any open set. If $F(t, z)$ is analytic, the Cauchy–Kovalevskaya applies and the equation has unique solution for any such $F(t, z)$.

Appendix C

Strongly hyperbolic formulations of the Einstein equations

In this appendix, we review some strongly hyperbolic formulations of the EFE which were used before the advent of the BSSN formulation, and the generalized harmonic formulation (GHF) which is still used nowadays. All these formulations with the exception of the GHF are obtained by using some (or all) of the methods described in Section 5.2 to recast the standard $3+1$ equations. The main references for this appendix are [Alc08; BS10]. We also show how to extend the GHF to BR.

The Bona–Masso and NOR formulations. We start with introducing the Bona–Masso formulation [BM89; BM92; BM93; Bon+95; Bon+97]. Its starting point is the definition of new dynamical variables,

$$a_i := \partial_i \ln \alpha, \quad d_{ijk} := \frac{1}{2}\partial_i \gamma_{jk}. \tag{C.1}$$

The shift is assumed to be a fixed function of space and time, whereas the lapse is assumed to evolve according to a Bona–Masso slicing [Bon+95],

$$\partial_t \alpha = -\alpha^2 f(\alpha) K + \beta^i \partial_i \alpha, \tag{C.2}$$

with $f(\alpha)$ arbitrary but positive function of α. The recasting procedure continues with the definition of three new dynamical variables,

$$V_i := d_{im}{}^m - d^m{}_{mi}, \tag{C.3}$$

that is also used as a constraint on V_i. From now on, we will consider only the principal part of the equations, and define the symbol $\asymp$ to mean "equal in its principal part." In terms of the V_i, the Ricci tensor is equal to

$$R_{ij} \asymp -\partial_m d^m{}_{ij} - \partial_i\left(V_j - \frac{1}{2}d_{jm}{}^m\right) - \partial_j\left(V_i - \frac{1}{2}d_{im}{}^m\right). \tag{C.4}$$

Using the expression in (C.4), the standard $3+1$ equations (5.73) read,

$$\partial_0 a_i \asymp -\alpha^2 Q, \tag{C.5a}$$

$$\partial_0 d_{ijk} \asymp -\alpha \partial_i K_{jk}, \tag{C.5b}$$

$$\partial_0 K_{ij} \asymp -\alpha \partial_k \Lambda^k_{ij}, \tag{C.5c}$$

with $\Lambda^k_{ij} := d^k{}_{ij} + \delta^k{}_{(i}\big[a_{j)} + 2V_{j)} - d_{j)m}{}^m\big]$ and $\partial_0 := \partial_t - \beta^k \partial_k$. Now we need to add the evolution equation for V_i to the system (C.5). This can be computed by noting that,

$$V^i = \frac{1}{2}\big(d^{im}{}_m - \Gamma^i\big), \quad \Gamma^i := \gamma^{lm}\Gamma^i_{lm}. \tag{C.6}$$

It holds,

$$\partial_t \Gamma^i = \beta^k D_k \Gamma^i - \Gamma^k D_k \beta^i + \gamma^{lm}\partial_l \partial_m \beta^i - D_l\big[\alpha\big(2K^{il} - \gamma^{il}K\big)\big] + 2\alpha K^{lm}\Gamma^i_{lm}. \tag{C.7}$$

The flat Laplacian of the shift can be written,

$$\gamma^{lm}\partial_l \partial_m \beta^i = D^k D_k \beta^2 - \beta^k D_k \Gamma^i + \Gamma^k D_k \beta^i - 2\Gamma^i_{lm} D^m \beta^n + R^{im}\beta_m. \tag{C.8}$$

Substituting (C.8) in (C.7) and then (C.7) in the time derivative of (C.6), one gets

$$\partial_0 V_i \asymp \alpha\big(\partial_j K^j{}_i - \partial_i K\big). \tag{C.9}$$

The above equation (C.9) has the same principal part as the momentum constraint. Therefore, the addition of $2\alpha M^i (= 0)$ to the right-hand side of (C.7) changes the evolution equation for V_i to,

$$\partial_0 V_i \asymp 0. \tag{C.10}$$

At this point, one can study the hyperbolicity of the system (C.5, C.10), and we refer the reader to [Alc08] for the details. It turns out that the first-order system in time and space is strongly hyperbolic, hence its Cauchy problem is well-posed.

The NOR (Nagy–Ortiz–Reula) formulation [NOR04] is related to the Bona–Masso one. It basically adds a multiple of the Hamiltonian constraint $\alpha\,\eta\,\gamma_{ij}\,H$ to the evolution equation (C.5c) for K_{ij}, and a multiple of the momentum constraint $\alpha\,\xi\,M^i$ to the evolution equation (C.7) for Γ^i or, equivalently, to the evolution equation for V^i. Here $\eta, \xi \in \mathbb{R}$ are constant parameters. The analysis of the hyperbolicity, which again can be found in [Alc08], shows that the NOR system is strongly hyperbolic in these three cases:

(i) $\eta = 0$, $\xi = 2$ (the Bona–Masso formulation);
(ii) $\eta = 0$, $\xi > 0$, $f(\alpha) \neq 1$;
(iii) $\eta \neq 0$, $\xi > 0$, $\eta\,(2-\xi) > -1/2$.

Note that the standard $3+1$ system, corresponding to $\eta = \xi = 0$, is not strongly hyperbolic, but only weakly hyperbolic.

The Einstein–Christoffel formulation. We now introduce a first-order symmetric hyperbolic (FOSH) formulation of the EE. It was presented in [AY99] and called "Einstein–Christoffel" (EC) formulation. The first step is again the definition of new dynamical variables,

$$f_{kij} := \Gamma_{(ij)k} + \gamma_{ki}\gamma^{lm}\Gamma_{[lj]m} + \gamma_{kj}\gamma^{lm}\Gamma_{[li]m}, \tag{C.11}$$

which are also new constraints on the f_{kij}. The standard $3+1$ evolution equations are then rewritten as,

$$\mathscr{L}_{\mu}\gamma_{ij} = -2\alpha K_{ij}, \tag{C.12a}$$

$$\mathscr{L}_{\mu}K_{ij} = \alpha\big(M_{ij} - \gamma^{kl}\partial_l f_{kij}\big), \tag{C.12b}$$

$$\mathscr{L}_{\mu}f_{kij} = \alpha\big(N_{kij} - \partial_k K_{ij}\big), \tag{C.12c}$$

where M_{ij} and N_{kij} are complicated functions of the dynamical variables only, not of their derivatives.[1] Hence, they do not contribute in the analysis of the hyperbolicity of the system. The principal part is,

$$\partial_t\gamma_{ij} \asymp \beta^k\,\partial_k\gamma_{ij}, \tag{C.13a}$$

$$\partial_t K_{ij} \asymp \beta^k\,\partial_k K_{ij} - \alpha\gamma^{mk}\,\partial_k f_{mij}, \tag{C.13b}$$

$$\partial_t f_{1ij} \asymp -\alpha\,\partial_1 K_{ij} + \beta^k\,\partial_k f_{1ij}, \tag{C.13c}$$

$$\partial_t f_{2ij} \asymp -\alpha\,\partial_2 K_{ij} + \beta^k\,\partial_k f_{2ij}, \tag{C.13d}$$

$$\partial_t f_{3ij} \asymp -\alpha\,\partial_3 K_{ij} + \beta^k\,\partial_k f_{3ij}. \tag{C.13e}$$

Being symmetric hyperbolic [AY99], its Cauchy problem is well-posed. The EC formulation can be embedded in the Kidder–Scheel–Teukolsky (KST) formulation, which constitutes a family of recasted versions of the standard $3+1$ equations, whose hyperbolicity depends on the 12 free parameters [KST01]. In the next paragraph we will discuss another subfamily of the KST family.

A subfamily of the Kidder–Scheel–Teukolsky formulation. This subfamily of the KST formulation depends on 3 free parameters. Again, the derivatives of the spatial metric are defined as independent quantities,

$$d_{ijk} := \frac{1}{2}\partial_i\gamma_{jk}. \tag{C.14}$$

The standard $3+1$ equations are rewritten in terms of d_{ijk} and constraints are added to them in the following way,

$$\partial_t d_{ijk} = \{3+1\} + \alpha\big[\xi\,\gamma_{i(j}M_{k)} + \chi\,\gamma_{jk}M_i\big], \tag{C.15a}$$

$$\partial_t K_{ij} = \{3+1\} + \alpha\,\eta\,\gamma_{ij}\,H, \tag{C.15b}$$

[1] The explicit expressions can be found in [BS10, Sec. 11.4]

where $\{3+1\}$ denotes the standard $3+1$ terms, and ξ, χ, η are the 3 free parameters. The lapse is evolved through the Bona–Masso slicing condition (C.2), and the principal part of the system reads

$$\partial_0 a_i \asymp -\alpha f(\alpha)\partial_i K, \tag{C.16a}$$

$$\partial_0 d_{ijk} \asymp -\alpha\, \partial_i K_{jk} + \alpha\big[\xi\, \gamma_{i(j} M_{k)} + \chi\, \gamma_{jk} M_i\big], \tag{C.16b}$$

$$\partial_0 K_{ij} \asymp -\alpha\, \partial_m \Lambda^m_{ij}, \tag{C.16c}$$

with $\Lambda^k_{ij} := d^k{}_{ij} + \delta_{(i}\big[a_{j)} + 2\big(d_{j)n}{}^n - d^n{}_{n|j)}\big) - d_{j)m}{}^m\big] + 2\eta\, \gamma_{ij}\gamma^{kl}\big(d_{ln}{}^n - d^n{}_{nl}\big)$. This system is strongly hyperbolic if all the relations below are satisfied (see [Alc08, Sec. 5.7] for the details),

$$\xi - \chi/2 > 0, \tag{C.17a}$$

$$1 + 2\chi + 4\eta\big(1 - \xi + 2\chi\big) > 0, \tag{C.17b}$$

$$1 - \xi + 2\chi \neq 0. \tag{C.17c}$$

The advantage of this formulation consists in providing a strongly hyperbolic system without introducing new dynamical variables and the related new constraints. The price to pay is that the modification of the evolution equation for d_{ijk} relies on the reduction of the system to a fully first-order one, hence the number of variables is larger compared with, for example, the BSSN formulation.

The generalized harmonic formulation and its extension to BR. This formulation is not derived from the standard $3+1$ equations, but from the covariant equations (5.50). The GHF and the BSSN are the most used nowadays in the field of NR; see, for example, the recent review [LP14]. The GHF was firstly used in [Pre05a; Pre05b].

Starting from the EFE,

$${}^4R_{\mu\nu} = 8\pi\Big(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\Big), \tag{C.18}$$

the 4-dimensional Ricci tensor is rewritten to obtain,

$$\begin{aligned} &\frac{1}{2} g^{\rho\sigma}\partial_\rho\partial_\sigma g_{\mu\nu} - g_{\rho(\mu}\partial_{\nu)}\,{}^4R^\rho - {}^4\Gamma^\rho\,{}^4\Gamma_{(\mu\nu)\rho} \\ &\quad -2g^{\rho\sigma}\,{}^4\Gamma^\beta_{\rho(\mu}\,{}^4\Gamma_{\nu)\beta\sigma} - g^{\rho\sigma}\,{}^4\Gamma^\beta_{\mu\sigma}\,{}^4\Gamma_{\beta\rho\nu} = -8\pi\Big(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\Big), \end{aligned} \tag{C.19}$$

where ${}^4\Gamma^\mu := g^{\rho\sigma}\,{}^4\Gamma^\mu_{\rho\sigma}$. In this formulation, the gauge is specified by choosing 4 coordinates such that the following 4 relations are satisfied,

$${}^4\Gamma^\mu - H^\mu(x) = 0, \tag{C.20}$$

with $H^\mu(x^\nu)$ arbitrary "gauge source functions." The substitution of (C.20) into (C.19) yields a nonlinear wave equation for the metric $g_{\mu\nu}$, which can be written,

$$g^{\rho\sigma}\partial_\rho\partial_\sigma g_{\mu\nu} + 2\,\partial_{(\mu}g^{\rho\sigma}\partial_\rho g_{\nu)\sigma} + 2\,\partial_{(\mu}H_{\nu)} \\ -2\,H_\rho\,{}^4\Gamma^\rho_{\mu\nu} + 2\,{}^4\Gamma^\rho_{\nu\sigma}\,{}^4\Gamma^\sigma_{\mu\rho} = -8\pi\Big(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\Big). \tag{C.21}$$

In order to stabilize the numerical evolution obtained with (C.21), the constraint (C.20) can be added to it. Setting $H^\mu = 0$ yields the harmonic formulation.

A key difference between the GHF and the formulations based on the 3+1 decomposition is that the GHF is second-order in time, and this affects both the method adopted to find the initial data and the numerical implementation. The nonlinear wave equation (C.21) needs initial data for $g_{\mu\nu}$ and its first time derivative $\partial_t g_{\mu\nu}$, whereas the $3+1$ equations need initial data for the spatial metric γ_{ij} and the extrinsic curvature K_{ij}. One approach to evaluate the initial data for the GHF is to solve the $3+1$ constraint equations in the "conformal thin-sandwich formalism," to obtain $\gamma_{ij}, K_{ij}, \alpha, \beta^i$ on the initial hypersurface.[2] Note that the lapse and the shift are not freely specifiable in the GHF formulation, since we are fixing the gauge through (C.20). Knowing the initial data for $\gamma_{ij}, \alpha, \beta^i$ allows to reconstruct the initial data for $g_{\mu\nu}$ according to (5.68). Then, the evaluation of the evolution equation (5.73a) at the initial time gives us the initial data for $\partial_t\gamma_{ij}$, and the evaluation of the constraint (C.20) at the initial time gives the initial data $\partial_t\alpha$ and $\partial_t\beta$. This determines $\partial_t g_{\mu\nu}$ on the initial hypersurface.

Lastly, we note that, since (C.21) is second-order in the time derivatives and also includes mixed second-order derivatives in time and space, the numerical methods to integrate them are different compared with the ones used to integrate the equations in any of the formulations based on the $3+1$ equations. One approach is to recast (C.21) as a system of coupled first-order PDEs [Lin+06], which can then be treated as the $3+1$ equations.

An advantage of the (c)BSSN formulation (and of any formulation based on the $3+1$ decomposition) over the GHF formulation is that the coordinates are chosen by fixing the lapse and the shift. These are connected with the choice of the slicing and therefore have a direct geometrical meaning. This can be used to choose a slicing with the desired geometrical properties, as we saw in Section 5.3. In the GHF formulation, where the gauge is chosen by specifying the gauge source functions H^μ, how to impose desirable geometrical properties to the system through the gauge choice is not as straightforward.

The generalized harmonic formulation of the BFE can be obtained by considering two copies of (C.21), one per each metric sector, including the bimetric stress-energy tensors and defining a second gauge source function $\widetilde{H}(x)$ in the f-sector. Since only one gauge source function can be specified, the second one has to be written in its terms. This can be done by noticing that [Wal10, Sec. 3.1] (see also Sec. III.A in Paper I),

$$\Gamma^\alpha{}_{\mu\nu} - \widetilde{\Gamma}^\alpha{}_{\mu\nu} = C^\alpha{}_{\mu\nu}, \tag{C.22a}$$

$$C^\alpha{}_{\mu\nu} := \frac{1}{2}g^{\alpha\beta}\Big(\widetilde{\nabla}_\mu g_{\beta\nu} + \widetilde{\nabla}_\nu g_{\mu\beta} - \widetilde{\nabla}_\beta g_{\mu\nu}\Big) \tag{C.22b}$$

[2] For more details about the conformal thin-sandwich formalism, we refer the reader to [BS10, Sec. 3.3].

$$= -\frac{1}{2} f^{\alpha\beta}\Big(\nabla_\mu f_{\beta\nu} + \nabla_\nu f_{\mu\beta} - \nabla_\beta f_{\mu\nu}\Big). \tag{C.22c}$$

We can now contract the μ, ν indices in (C.22a) first with g and then with f, and sum the resulting two equations. The result is,

$$g^{\mu\nu}\Gamma^\alpha{}_{\mu\nu} - g^{\mu\nu}\widetilde{\Gamma}^\alpha{}_{\mu\nu} + f^{\mu\nu}\Gamma^\alpha{}_{\mu\nu} - f^{\mu\nu}\widetilde{\Gamma}^\alpha{}_{\mu\nu} = \Big(g^{\mu\nu} + f^{\mu\nu}\Big)C^\alpha{}_{\mu\nu}, \tag{C.23}$$

and using $g^{\mu\nu}\Gamma^\alpha{}_{\mu\nu} = \Gamma^\alpha = H^\alpha(x)$, $f^{\mu\nu}\widetilde{\Gamma}^\alpha{}_{\mu\nu} = \widetilde{\Gamma}^\alpha = \widetilde{H}^\alpha(x)$,

$$\widetilde{H}^\alpha(x) = H^\alpha(x) - g^{\mu\nu}\widetilde{\Gamma}^\alpha{}_{\mu\nu} + f^{\mu\nu}\Gamma^\alpha{}_{\mu\nu} - \Big(g^{\mu\nu} + f^{\mu\nu}\Big)C^\alpha{}_{\mu\nu}. \tag{C.24}$$

Therefore, once coordinates are chosen by specifying *one* gauge source function, the other one is determined by (C.24). Note that this follows from geometry only, and can be analogously applied to obtain the relation between $\Gamma^\alpha{}_{\mu\nu}$ and $\overset{\#}{\Gamma}{}^\alpha{}_{\mu\nu}$, and between $\widetilde{\Gamma}^\alpha{}_{\mu\nu}$ and $\overset{\#}{\Gamma}{}^\alpha{}_{\mu\nu}$,

$$\Gamma^\alpha{}_{\mu\nu} - \overset{\#}{\Gamma}{}^\alpha{}_{\mu\nu} = \overset{\#}{C}{}^\alpha{}_{\mu\nu}, \qquad \widetilde{\Gamma}^\alpha{}_{\mu\nu} - \overset{\#}{\Gamma}{}^\alpha{}_{\mu\nu} = \widetilde{C}^\alpha{}_{\mu\nu}, \tag{C.25a}$$

$$\overset{\#}{C}{}^\alpha{}_{\mu\nu} := \frac{1}{2} g^{\alpha\beta}\Big(\overset{\#}{\nabla}_\mu g_{\beta\nu} + \overset{\#}{\nabla}_\nu g_{\mu\beta} - \overset{\#}{\nabla}_\beta g_{\mu\nu}\Big) \tag{C.25b}$$

$$= -\frac{1}{2} h^{\alpha\beta}\Big(\nabla_\mu h_{\beta\nu} + \nabla_\nu h_{\mu\beta} - \nabla_\beta h_{\mu\nu}\Big), \tag{C.25c}$$

$$\widetilde{C}^\alpha{}_{\mu\nu} := \frac{1}{2} f^{\alpha\beta}\Big(\overset{\#}{\nabla}_\mu f_{\beta\nu} + \overset{\#}{\nabla}_\nu f_{\mu\beta} - \overset{\#}{\nabla}_\beta f_{\mu\nu}\Big) \tag{C.25d}$$

$$= -\frac{1}{2} h^{\alpha\beta}\Big(\widetilde{\nabla}_\mu h_{\beta\nu} + \widetilde{\nabla}_\nu h_{\mu\beta} - \widetilde{\nabla}_\beta h_{\mu\nu}\Big), \tag{C.25e}$$

$$\overset{\#}{\Gamma}{}^\alpha = \Gamma^\alpha - g^{\mu\nu}\overset{\#}{\Gamma}{}^\alpha{}_{\mu\nu} + h^{\mu\nu}\Gamma^\alpha{}_{\mu\nu} - \Big(g^{\mu\nu} + h^{\mu\nu}\Big)\overset{\#}{C}{}^\alpha{}_{\mu\nu}, \tag{C.25f}$$

$$\overset{\#}{\Gamma}{}^\alpha = \widetilde{\Gamma}^\alpha - f^{\mu\nu}\overset{\#}{\Gamma}{}^\alpha{}_{\mu\nu} + h^{\mu\nu}\widetilde{\Gamma}^\alpha{}_{\mu\nu} - \Big(f^{\mu\nu} + h^{\mu\nu}\Big)\widetilde{C}^\alpha{}_{\mu\nu}. \tag{C.25g}$$

Therefore, we can gauge fix with respect to any of the metrics.

Appendix D

More on bimetric gauge pathologies

In Section 6.1, we discussed the bimetric gauge pathologies, defined as coordinate singularities arising when the foliation becomes null with respect to any of the metrics g, f, h. In this appendix, we list all possible cases in which a spacelike foliation can become null in BR.

Consider the metric g (the same holds for the other metrics). Since

$$\mathtt{t}^\alpha = \frac{1}{\alpha^2}\big(-1, \beta^i\big), \quad g(\mathtt{t}, \mathtt{t}) = -\frac{1}{\alpha^2}. \tag{D.1}$$

the foliation becomes g-null at $\hat{t}$ if

$$\lim_{t\to\hat{t}} \alpha = \infty, \quad \exists i \in \{1,2,3\} : \lim_{t\to\hat{t}} \frac{\beta^i}{\alpha^2} < \infty. \tag{D.2}$$

This implies that at least one component of the shift has to diverge as well. If only the lapse diverges, then $\mathtt{t}$ becomes the null vector and the foliation is not regular. We do not consider this case here.

To study all possible bimetric gauge pathologies, we use the relations (6.11) between $\mathtt{t}, \widetilde{\mathtt{t}}, \overset{\#}{\mathtt{t}}$ to deduce,

$$f\big(\overset{\#}{\mathtt{t}}, \overset{\#}{\mathtt{t}}\big) = g\big(\mathtt{t}, \mathtt{t}\big) = h\big(\overset{\#}{\mathtt{t}}, \mathtt{t}\big) = -\frac{1}{\alpha^2}, \tag{D.3a}$$

$$g\big(\overset{\#}{\mathtt{t}}, \overset{\#}{\mathtt{t}}\big) = f\big(\widetilde{\mathtt{t}}, \widetilde{\mathtt{t}}\big) = h\big(\overset{\#}{\mathtt{t}}, \widetilde{\mathtt{t}}\big) = -\frac{1}{\widetilde{\alpha}^2}, \tag{D.3b}$$

$$f\big(\widetilde{\mathtt{t}}, \overset{\#}{\mathtt{t}}\big) = g\big(\mathtt{t}, \overset{\#}{\mathtt{t}}\big) = h\big(\overset{\#}{\mathtt{t}}, \overset{\#}{\mathtt{t}}\big) = -\frac{1}{H^2}. \tag{D.3c}$$

We also remind the reader that the three lapse functions are related by,

$$\alpha^2 = H^2\lambda W, \quad \widetilde{\alpha}^2 = \frac{H^2\lambda}{W}, \quad \alpha = W\widetilde{\alpha}, \quad H^2 = \frac{\alpha\widetilde{\alpha}}{\lambda}, \tag{D.4}$$

the three shift vectors are related by,

$$\beta = q + \frac{\alpha}{\lambda}e^{-1}\mathbf{p}, \qquad \widetilde{\beta} = q - \frac{\widetilde{\alpha}}{\lambda}m^{-1}\mathbf{p}, \qquad \beta = \widetilde{\beta} + \left[\frac{\alpha}{\lambda}e^{-1} + \frac{\widetilde{\alpha}}{\lambda}m^{-1}\right]\mathbf{p} \tag{D.5}$$

and that $|\mathbf{p}| \equiv \mathbf{p}^\top \boldsymbol{\delta} \mathbf{p}$ diverges if and only if $\lambda = \left(1 + |\mathbf{p}|^2\right)^{1/2}$ diverges, and if and only if one of its components diverges. Also, two causal (i.e, timelike or null) vectors are orthogonal if and only if they are null and collinear [PR84, p. 4].

We consider all cases one by one. We assume that when multiple lapses diverge simultaneously, they do it with the same order of growth, otherwise the one diverging faster will determine the configuration.

$\Sigma_{\hat{t}}$ **is g-null.** It holds,

$$\lim_{t\to\hat{t}} \alpha = \infty, \quad 0 < \lim_{t\to\hat{t}} \widetilde{\alpha} = b < \infty, \quad 0 < \lim_{t\to\hat{t}} H = c < \infty, \tag{D.6}$$

which implies

$$c^2 = \lim_{t\to\hat{t}} \frac{\alpha\widetilde{\alpha}}{\lambda} = b \lim_{t\to\hat{t}} \frac{\alpha}{\lambda}. \tag{D.7}$$

This implies that $\lambda = \mathcal{O}(\alpha)$, and $\lim_{t\to\hat{t}} \lambda = \infty \Longleftrightarrow \lim_{t\to\hat{t}} |\mathbf{p}| = \infty$.

This is consistent with (D.5), since at least one component of $\mathbf{p}$ has to diverge if the same component of β also diverges with finite $\widetilde{\beta}, q, \widetilde{\alpha}$ and α/λ.

$\Sigma_{\hat{t}}$ **is f-null.** It holds,

$$0 < \lim_{t\to\hat{t}} \alpha = a < \infty, \quad \lim_{t\to\hat{t}} \widetilde{\alpha} = \infty, \quad 0 < \lim_{t\to\hat{t}} H = c < \infty, \tag{D.8}$$

which implies

$$c^2 = \lim_{t\to\hat{t}} \frac{\alpha\widetilde{\alpha}}{\lambda} = a \lim_{t\to\hat{t}} \frac{\widetilde{\alpha}}{\lambda}. \tag{D.9}$$

This implies that $\lambda = \mathcal{O}(\widetilde{\alpha})$, and $\lim_{t\to\hat{t}} \lambda = \infty \Longleftrightarrow \lim_{t\to\hat{t}} |\mathbf{p}| = \infty$.

This is consistent with (D.5), since at least one component of $\mathbf{p}$ has to diverge if the same component of $\widetilde{\beta}$ also diverges with finite β, q, α and $\widetilde{\alpha}/\lambda$.

$\Sigma_{\hat{t}}$ **is h-null.** It holds,

$$0 < \lim_{t\to\hat{t}} \alpha = a < \infty, \quad 0 < \lim_{t\to\hat{t}} \widetilde{\alpha} = b < \infty, \quad \lim_{t\to\hat{t}} H = \infty, \tag{D.10}$$

which implies

$$\lim_{t\to\hat{t}} f\big(\overset{\#}{\mathtt{t}}, \overset{\#}{\mathtt{t}}\big) = \lim_{t\to\hat{t}} g\big(\mathtt{t}, \mathtt{t}\big) = -\lim_{t\to\hat{t}} \frac{1}{\alpha^2} = -\frac{1}{a^2} < 0, \tag{D.11a}$$

$$\lim_{t\to\hat{t}} g\big(\overset{\#}{\widetilde{\mathtt{t}}}, \overset{\#}{\widetilde{\mathtt{t}}}\big) = \lim_{t\to\hat{t}} f\big(\widetilde{\mathtt{t}}, \widetilde{\mathtt{t}}\big) = -\lim_{t\to\hat{t}} \frac{1}{\widetilde{\alpha}^2} = -\frac{1}{b^2} < 0, \tag{D.11b}$$

$$\lim_{t\to\hat{t}} f\big(\widetilde{\mathtt{t}}, \overset{\#}{\mathtt{t}}\big) = \lim_{t\to\hat{t}} g\big(\mathtt{t}, \overset{\#}{\widetilde{\mathtt{t}}}\big) = -\lim_{t\to\hat{t}} \frac{1}{H^2} = 0^-. \tag{D.11c}$$

The norms and scalar products in (D.11) mean that the limits of $\widetilde{\mathtt{t}}, \overset{\#}{\mathtt{t}}$ are f-timelike and f-orthogonal; also, the limits of $\mathtt{t}, \overset{\#}{\widetilde{\mathtt{t}}}$ are g-timelike and g-orthogonal. This is impossible, since two timelike vectors cannot be orthogonal. Therefore, the case (D.10) is impossible.

$\Sigma_{\hat{t}}$ **is** g, f**-null.** It holds,

$$\lim_{t\to\hat{t}} \alpha = \infty, \quad \lim_{t\to\hat{t}} \widetilde{\alpha} = \infty, \quad 0 < \lim_{t\to\hat{t}} H = c < \infty, \tag{D.12}$$

which implies

$$c^2 = \lim_{t\to\hat{t}} \frac{\alpha\widetilde{\alpha}}{\lambda}. \tag{D.13}$$

This implies that $\lambda = \mathcal{O}(\alpha\widetilde{\alpha})$, and $\lim_{t\to\hat{t}} \lambda = \infty \iff \lim_{t\to\hat{t}} |\mathbf{p}| = \infty$.

This is consistent with (D.5), since at least one component of $\mathbf{p}$ has to diverge if the same components of $\beta, \widetilde{\beta}$ also diverge with finite H, q.

$\Sigma_{\hat{t}}$ **is** g, h**-null.** It holds,

$$\lim_{t\to\hat{t}} \alpha = \infty, \quad 0 < \lim_{t\to\hat{t}} \widetilde{\alpha} = b < \infty, \quad \lim_{t\to\hat{t}} H = \infty, \tag{D.14}$$

which implies

$$\lim_{t\to\hat{t}} f\big(\overset{\#}{\mathtt{t}}, \overset{\#}{\mathtt{t}}\big) = \lim_{t\to\hat{t}} g\big(\mathtt{t}, \mathtt{t}\big) = -\lim_{t\to\hat{t}} \frac{1}{\alpha^2} = 0^-, \tag{D.15a}$$

$$\lim_{t\to\hat{t}} g\big(\overset{\#}{\mathtt{t}}, \overset{\#}{\mathtt{t}}\big) = \lim_{t\to\hat{t}} f\big(\widetilde{\mathtt{t}}, \widetilde{\mathtt{t}}\big) = -\lim_{t\to\hat{t}} \frac{1}{\widetilde{\alpha}^2} = -\frac{1}{b^2} < 0, \tag{D.15b}$$

$$\lim_{t\to\hat{t}} f\big(\widetilde{\mathtt{t}}, \overset{\#}{\mathtt{t}}\big) = \lim_{t\to\hat{t}} g\big(\mathtt{t}, \overset{\#}{\mathtt{t}}\big) = -\lim_{t\to\hat{t}} \frac{1}{H^2} = 0^-. \tag{D.15c}$$

The norms and scalar products in (D.15) mean that the limit of $\widetilde{\mathtt{t}}$ is f-timelike and the limit of $\overset{\#}{\mathtt{t}}$ is f-null. In addition, they are f-orthogonal. This is impossible, since a timelike and a null vector cannot be orthogonal. Therefore, the case (D.14) is impossible. An analogous reasoning applies to $\mathtt{t}, \overset{\#}{\mathtt{t}}$ with respect to g.

$\Sigma_{\hat{t}}$ **is** f, h**-null.** It holds,

$$0 < \lim_{t\to\hat{t}} \alpha = a < \infty, \quad \lim_{t\to\hat{t}} \widetilde{\alpha} = \infty, \quad \lim_{t\to\hat{t}} H = \infty, \tag{D.16}$$

which implies

$$\lim_{t\to\hat{t}} f\big(\overset{\#}{\mathtt{t}}, \overset{\#}{\mathtt{t}}\big) = \lim_{t\to\hat{t}} g\big(\mathtt{t}, \mathtt{t}\big) = -\lim_{t\to\hat{t}} \frac{1}{\alpha^2} = -\frac{1}{a^2} < 0, \tag{D.17a}$$

$$\lim_{t\to\hat{t}} g\big(\overset{\#}{\mathtt{t}}, \overset{\#}{\mathtt{t}}\big) = \lim_{t\to\hat{t}} f\big(\widetilde{\mathtt{t}}, \widetilde{\mathtt{t}}\big) = -\lim_{t\to\hat{t}} \frac{1}{\widetilde{\alpha}^2} = 0^-, \tag{D.17b}$$

$$\lim_{t\to\hat{t}} f\big(\widetilde{\mathtt{t}}, \overset{\#}{\mathtt{t}}\big) = \lim_{t\to\hat{t}} g\big(\mathtt{t}, \overset{\#}{\mathtt{t}}\big) = -\lim_{t\to\hat{t}} \frac{1}{H^2} = 0^-. \tag{D.17c}$$

The norms and scalar products in (D.17) mean that the limit of $\mathtt{t}$ is g-timelike and the limit of $\overset{\#}{\mathtt{t}}$ is g-null. In addition, they are g-orthogonal. This is impossible, since a timelike and a null vector cannot be orthogonal. Therefore, the case (D.16) is impossible. An analogous reasoning applies to $\widetilde{\mathtt{t}}, \overset{\#}{\mathtt{t}}$ with respect to f.

$\Sigma_{\hat{t}}$ **is** g, f, h**-null.** It holds,

$$\lim_{t\to\hat{t}} \alpha = \infty, \quad \lim_{t\to\hat{t}} \widetilde{\alpha} = \infty, \quad \lim_{t\to\hat{t}} H = \infty, \tag{D.18}$$

which implies

$$\lim_{t\to\hat{t}} f\big(\overset{\#}{\mathtt{t}}, \overset{\#}{\mathtt{t}}\big) = \lim_{t\to\hat{t}} g\big(\mathtt{t}, \mathtt{t}\big) = \lim_{t\to\hat{t}} h\big(\overset{\#}{\mathtt{t}}, \mathtt{t}\big) = -\lim_{t\to\hat{t}} \frac{1}{\alpha^2} = 0^-, \tag{D.19a}$$

$$\lim_{t\to\hat{t}} g\big(\overset{\#}{\mathtt{t}}, \overset{\#}{\mathtt{t}}\big) = \lim_{t\to\hat{t}} f\big(\widetilde{\mathtt{t}}, \widetilde{\mathtt{t}}\big) = \lim_{t\to\hat{t}} h\big(\overset{\#}{\mathtt{t}}, \widetilde{\mathtt{t}}\big) = -\lim_{t\to\hat{t}} \frac{1}{\widetilde{\alpha}^2} = 0^-, \tag{D.19b}$$

$$\lim_{t\to\hat{t}} f\big(\widetilde{\mathtt{t}}, \overset{\#}{\mathtt{t}}\big) = \lim_{t\to\hat{t}} g\big(\mathtt{t}, \overset{\#}{\mathtt{t}}\big) = \lim_{t\to\hat{t}} h\big(\overset{\#}{\mathtt{t}}, \overset{\#}{\mathtt{t}}\big) = -\lim_{t\to\hat{t}} \frac{1}{H^2} = 0^-. \tag{D.19c}$$

The norms and scalar products in (D.19) mean that $\mathtt{t}, \widetilde{\mathtt{t}}, \overset{\#}{\mathtt{t}}$ are null and orthogonal with respect to the three metrics. This means that they are collinear, and define the same null direction $\mathcal{N}$. Hence, the three null cones share the direction $\mathcal{N}$.

Also,

$$0 < \lim_{t\to\hat{t}} \frac{\alpha}{\widetilde{\alpha}} = \lim_{t\to\hat{t}} W = w < \infty, \tag{D.20a}$$

$$0 < \lim_{t\to\hat{t}} \frac{\widetilde{\alpha}^2}{H^2} = \lim_{t\to\hat{t}} \frac{\widetilde{\alpha}^2\lambda}{\alpha\widetilde{\alpha}} = \lim_{t\to\hat{t}} \frac{\widetilde{\alpha}^2\lambda}{W\widetilde{\alpha}^2} = \frac{1}{w} \lim_{t\to\hat{t}} \lambda < \infty, \tag{D.20b}$$

which implies that $0 < \lim_{t\to\hat{t}} \lambda = \lambda_0 < \infty$. Hence W and $\mathbf{p}$ are finite at $t = \hat{t}$ and since the null cones of the three metrics share a null direction, the square root S can have algebraic types IIa or III (see Figure 6.1). From (D.5), we see that at least one component of each shift vector $\beta, \widetilde{\beta}, q$ has to diverge, with finite $\mathbf{p}$.

Appendix E

Numerical details

In this Appendix we discuss some numerical issues in obtaining the Newtonian and curvature potential at the end of Section 6.4.

First of all, all the computations are made with an internal working precision of 16 digits, which corresponds to the double floating point representation of a real number. In `bim-solver`, some steps have been made to implement `quadmath`, that is, the quadruple floating point precision representation of a real number. However, this is still matter of ongoing work and it was not used to obtain the solutions presented in Paper VIII and in this thesis.

For the models BR1, BR2, BR3, BR4 and GR (not for the reference GR solution, though) the radial grid has 250000 points covering the range $r/\mathsf{L} \in \left[0, 10^4\right]$ with a step size $\Delta r = 0.004$. The time step for the evolution was $\Delta t = 0.0025$, corresponding to a Courant–Lax factor of 1/16 to improve stability. We use the fourth-order Kreiss–Oliger dissipation coefficient equal to 0.03, and at each time step we smooth the data with a Savitzky–Golay filter, to filter out unwanted short-wavelength oscillations induced by the numerical approximation of radial derivatives in the sixth-order finite difference method. The maximal slicing boundary value problem is solved with a fourth-order finite difference method described, for example, in [Ric16, p. 702]. GR, BR1 and BR2 are evolved in time using the Method of Lines with a third order Runge–Kutta method, whereas BR3 and BR4 are evolved in time with a fourth order Runge–Kutta method, since this improves the accuracy of the solutions for these models. Time integration with the fifth order Runge–Kutta–Dormand–Prince does not improve sensibly the accuracy compared with the fourth order Runge–Kutta, for the solutions in the BR3 and BR4 models.

The equations are solved as an initial value problem, specifying at $r = 0$ the values of the radial derivatives to zero (Neumann condition) due to the parity conditions, and the values of the conformal factors listed in Table E.1. The latter are determined by the following procedure: For every bimetric model, first guesses for the values are chosen and the equations are integrated inside the radial interval $r/\mathsf{L} \in \left[0, 10^4\right]$, which corresponds to the extent of the radial grid in `bim-solver`. Then, a best fit is performed to the data in the interval $r/\mathsf{L} \in \left[9 \times 10^3, 10^4\right]$ with the function $a + b/r$, with $a, b \in \mathbb{R}$ fitting parameters. The choice of the fitting function is justified by the expectation that, very far from the source, the average trend of the conformal factors should resemble that of a Newtonian potential. As we saw previously, in BR this fitting function does not capture all the features of the conformal factors, since oscillations are present. Since we are only interested to the trend of the functions, before fitting the data we apply a lowpass filter to them, to filter out the oscillations and consider only the average trend. Then, we consider the limit of the fitted functions when $r \to \infty$, which is equal to a. If $a < 1$, the initial value for the conformal factors is increased, and if $a > 1$ it is decreased, until $|a - 1| < \epsilon$ with $\epsilon = c \times 10^{-5}$ our tolerance and $c \in [1, 2]$.

Similarly, to determine the outer boundary condition in solving the maximal slicing elliptic equation for the lapse, a first guess was chosen, and then the lapse was fitted with the function $a - b/r$. The value of a is the limit of the lapse when $r \to \infty$, which we want to be 1 up to some tolerance, the latter being roughly the same as for the conformal factors. The boundary condition on the lapse is imposed at $r/\mathsf{L} = 2000$, in order to postpone the time when the constraint violations due to the open boundary conditions at $r/\mathsf{L} = 10^4$ affect the solution to the maximal slicing boundary value problem.

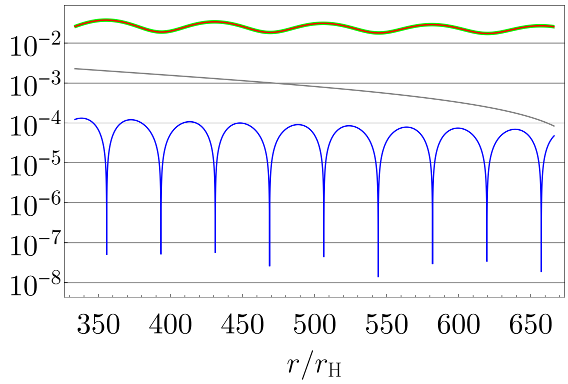
In order to apply the perturbative analysis, the metric fields must be equal to the Minkowski metric in spherical coordinates plus a small perturbation. That this is the case for the models we consider in Section 6.4 is shown in Figure E.1. The perturbations are at most of order 10^{-2}.

In addition, Figure E.2 and Figure E.3 show the constraint violations corresponding to the configurations illustrated in Figure 6.11 and Figure 6.12. The constraint violations are very small for all the bimetric models, showing that the oscillatory behavior we see in Figure 6.11 and Figure 6.12 is not a numerical artifact, but carries physical information. Note that the violations of the Hamiltonian constraints are very similar. The same is true for the momentum constraints, hence we only show the violations of one momentum constraint in the plots.

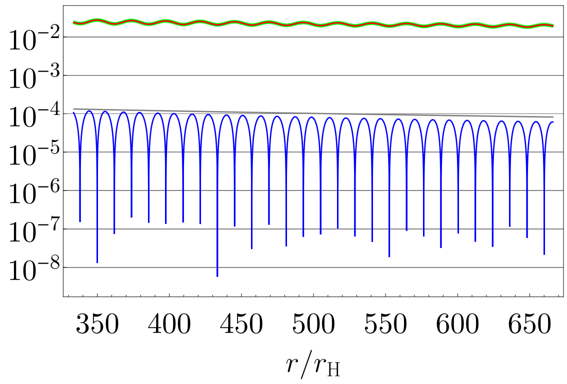
As a confirmation of the accuracy of the solution, we also show the residuals of the time integration for the configuration in Figure 6.12, in Figure E.4. The residuals can be thought of as the “evolution equations violations,” and they should always be much smaller then the time derivative of the considered dynamical variable (see Appendix D of Paper III and references therein for more details). The residuals are computed by approximating the time derivatives of the grid values of the fields with a sixth-order finite difference method. The BR3 solutions have the smallest residual, which are always at least 2 order of magnitude smaller than the time derivative of the considered field, as it is shown in Figure E.4 for a. That is why we showed the evolution of the curvature potential for this model. The residuals of b and the other fields are similar to those of a, hence they are not plotted here. This confirms that the oscillations in time are not numerical artifacts.

Model	$\beta_{(n)}$ parameters	Initial conditions
BR1	$\beta_{(1)} = -\beta_{(2)} = 5 \cdot 10^{-5}$ $\beta_{(3)} = 4 \cdot 10^{-5}$ $\upbeta = \mathrm{i} \cdot 3.162 \cdot 10^{-3}$	$\psi_\gamma(r=0) = \frac{120413330}{111057291}$ $\psi_\varphi(r=0) = \frac{117387983}{108266795}$
BR2	$\beta_{(1)} = 5 \cdot 10^{-4}$ $\beta_{(2)} = -3 \cdot 10^{-4}$ $\beta_{(3)} = 0$ $\upbeta = \mathrm{i} \cdot 10^{-2}$	$\psi_\gamma(r=0) = \frac{752828032}{695134343}$ $\psi_\varphi(r=0) = \frac{92742323}{85634756}$
BR3	$\beta_{(1)} = 5 \cdot 10^{-4}$ $\beta_{(2)} = -2 \cdot 10^{-2}$ $\beta_{(3)} = 3 \cdot 10^{-2}$ $\upbeta = \mathrm{i} \cdot 9.747 \cdot 10^{-2}$	$\psi_\gamma(r=0) = \frac{95805272}{88709821}$ $\psi_\varphi(r=0) = \frac{83512493}{77327305}$
BR4	$\beta_{(1)} = 5 \cdot 10^{-4}$ $\beta_{(2)} = -2 \cdot 10^{-2}$ $\beta_{(3)} = 0$ $\upbeta = \mathrm{i} \cdot 1.987 \cdot 10^{-1}$	$\psi_\gamma(r=0) = \frac{57261815}{53512611}$ $\psi_\varphi(r=0) = \frac{96660265}{90331276}$
GR	$\beta_{(1)} = \beta_{(2)} = \beta_{(3)} = \upbeta = 0$	$\psi_\gamma(r=0) = \frac{95436538}{84233961}$ $\psi_\varphi(r=0) = \frac{282793406}{249597917}$

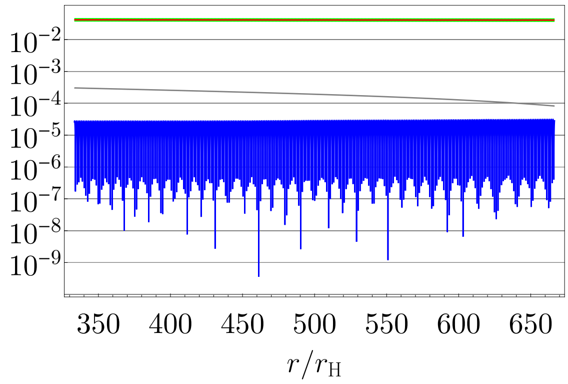
Table E.1: Initial conditions and $\beta_{(i)}$ parameters for the asymptotically flat solutions to the Hamiltonian constraints (6.108). The initial conditions are completed by the Neumann conditions $\partial_r \psi_\gamma(r=0) = \partial_r \psi_\varphi(r=0) = 0$ for all the models. Defining $\upbeta \equiv \sqrt{\beta_{(1)} + 2\beta_{(2)} + \beta(3)}$, the bimetric models are ordered in increasing order of $\upbeta$. They all have $\kappa_g = \kappa_f = 8\pi$ and $\ell = 1\,\mathrm{m}$.



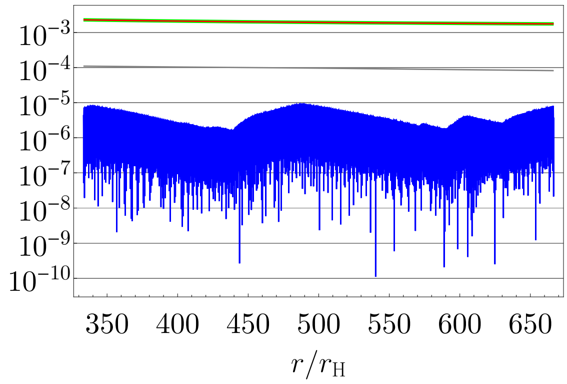
(a) BR1



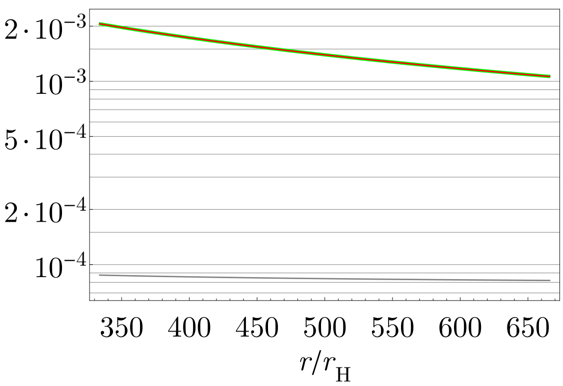
(b) BR2



(c) BR3



(d) BR4



(e) GR

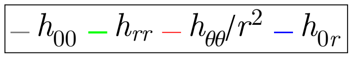
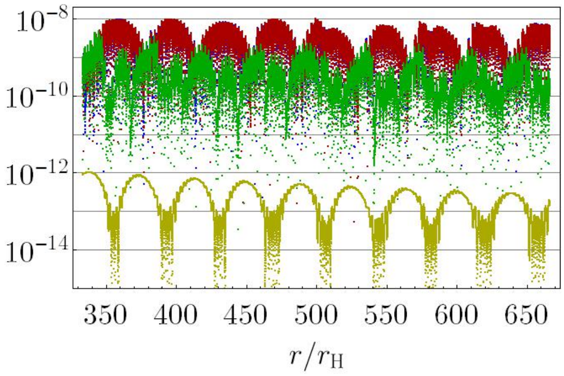
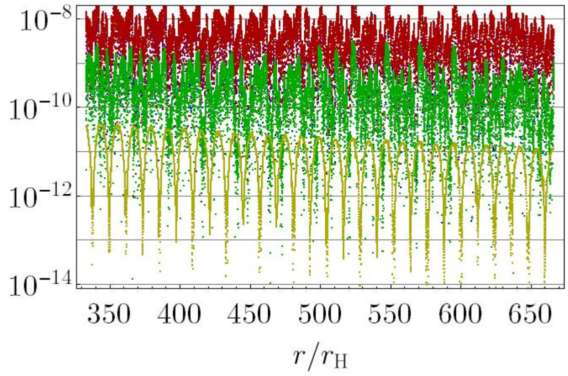


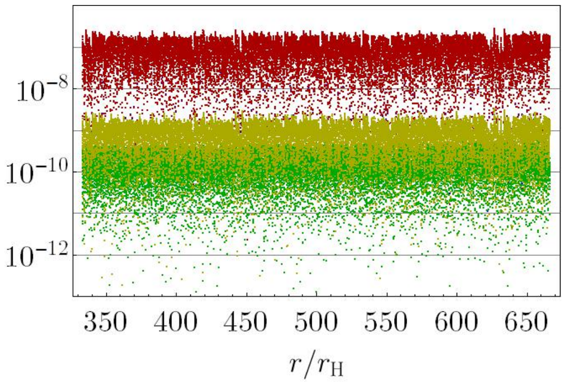
Figure E.1: Perturbations of the metric functions around the Minkowski metric in spherical coordinates. They are $\lesssim 10^{-2}$, justifying the use of the perturbative analysis.



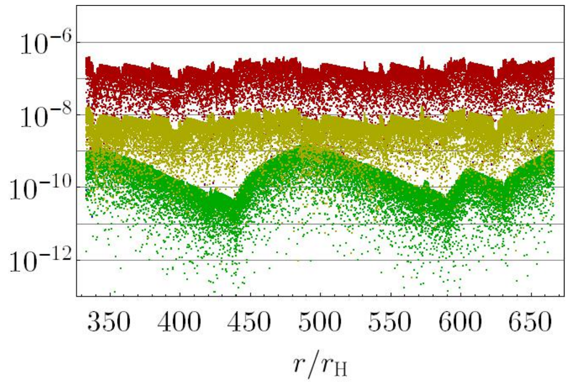
(a) BR1



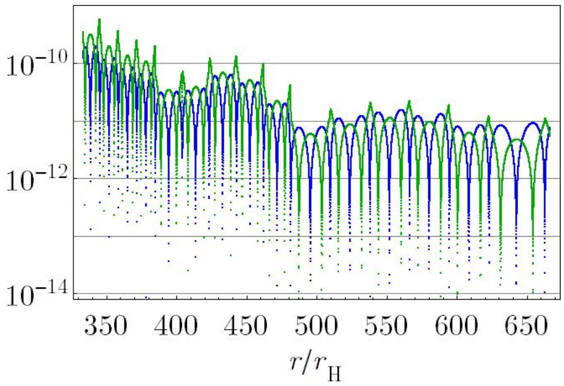
(b) BR2



(c) BR3



(d) BR4



(e) GR

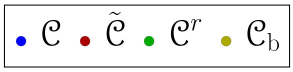


Figure E.2: Constraint violations for the configurations shown in Figure 6.11. They are much smaller than the perturbations in Figure E.1, confirming that our analysis is reliable.

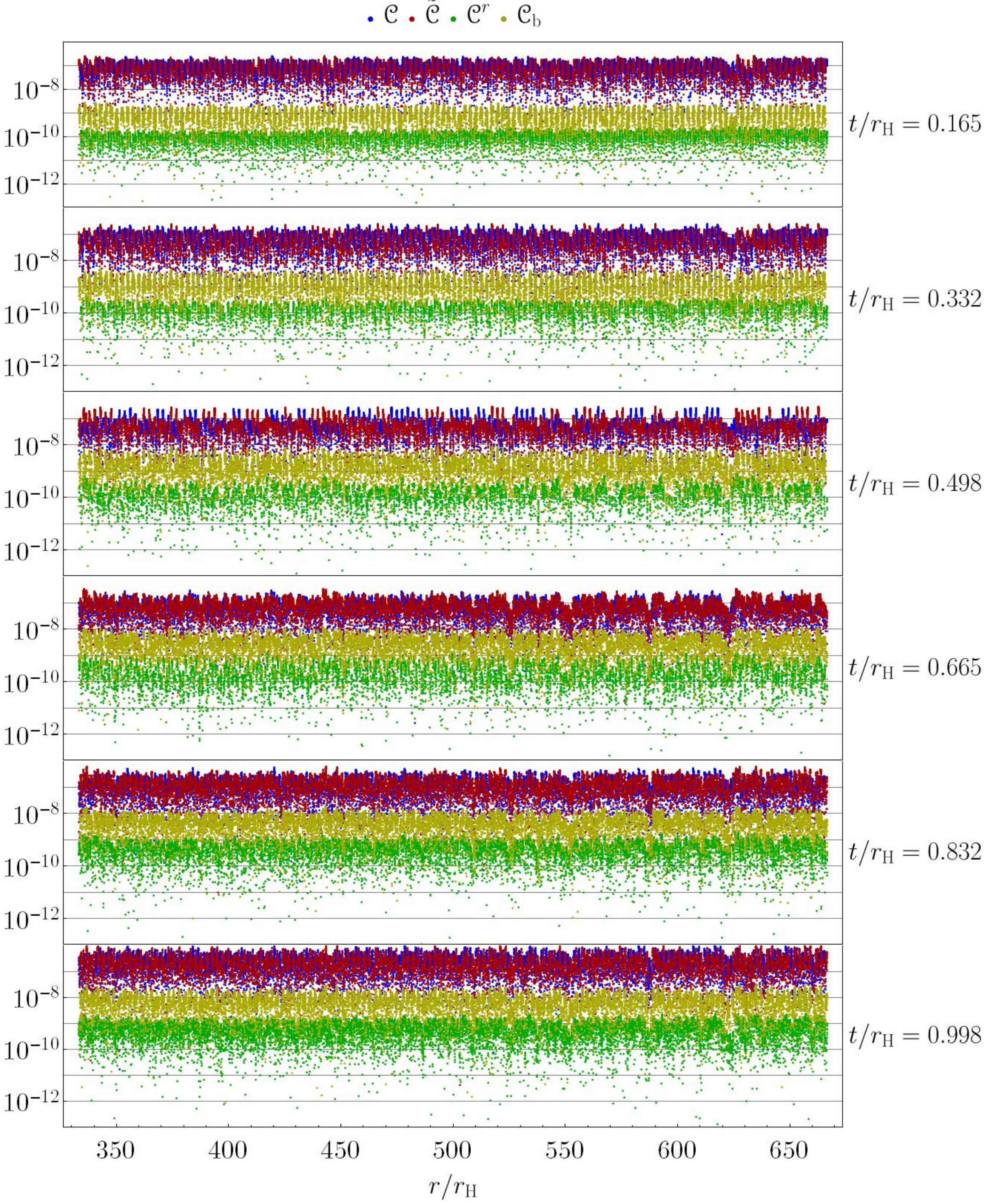


Figure E.3: Constraint violations for the BR3 model at the times at which the evolution of the curvature potential is shown in Figure 6.12. They are much smaller than the perturbations in Figure E.1, confirming that our analysis is reliable.

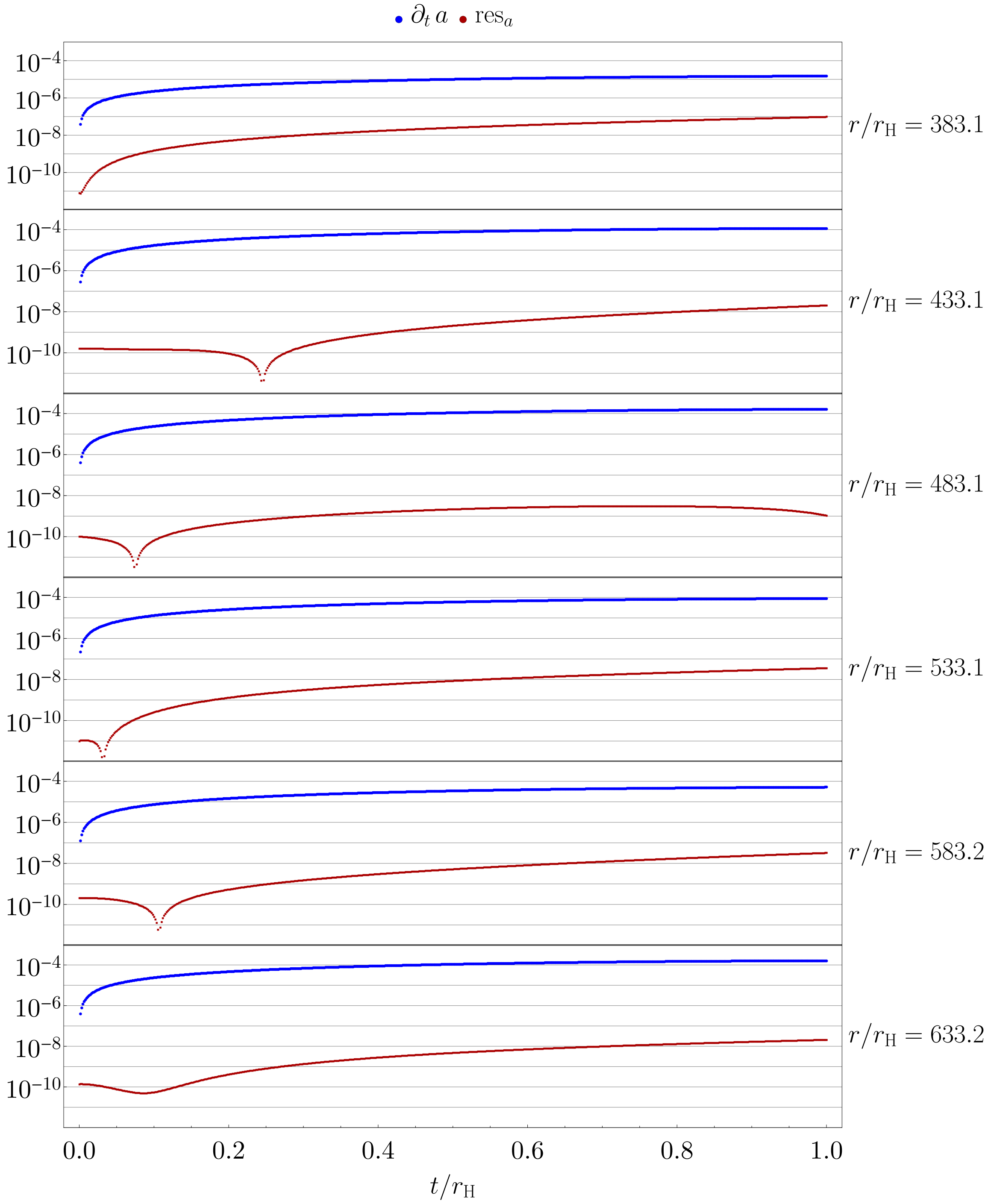


Figure E.4: Residuals of the numerical integration in time using the method of lines with a fourth order Runge–Kutta method, for different values of the radial grid point. The residual are always at least 2 orders of magnitude smaller than the derivative of a, which implies that the numerical integration is accurate enough for our purposes.

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Part III

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