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Rotating particles in AdS: Holography at weak gauge coupling and without conformal symmetry

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We consider the gauge/gravity correspondence between maximally supersymmetric Yang–Mills theory in $(p+1)$ dimensions and superstring theory on the near-horizon limit of the Dp-brane solution. The string-frame metric is $\mathrm{AdS}_{p+2} \times S^{8-p}$ times a Weyl factor, and there is no conformal symmetry except for $p = 3$. In a previous paper by one of the present authors, the free-field result of gauge theory has been reproduced from string theory for a particular operator that has angular momentum along S^{8-p}. In this paper, we extend this result to operators that have angular momenta along AdS_{p+2}. Our approach is based on a Euclidean formulation proposed by Dobashi et al. [Nucl. Phys. B **665**, 94 (2003)] and on the "string bit" picture. We first show that the spinning string solution in Lorentzian AdS, found by Gubser et al. [Nucl. Phys. B **636**, 99 (2002)], can be recast in a form that connects two points on the boundary of Euclidean AdS. The transition amplitudes of such strings can be interpreted as gauge theory correlators. We study the case of zero gauge coupling by ignoring interactions among string bits (massless particles in 10D spacetime that constitute a string), and show that the free-field results of gauge theory are reproduced.

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1. Introduction

Gauge/gravity correspondence provides concrete realizations of the holographic principle [1,2], a highly non-trivial proposal that states that quantum gravity should be described by degrees of freedom localized on the spatial boundary. Since Maldacena's original proposal of the anti-de Sitter/conformal field theory (AdS/CFT) correspondence [3], numerous examples of gauge/gravity correspondence have been proposed. This has led to exciting developments in quantum gravity and in strongly coupled quantum systems. An example of the former would be the discovery of the chaos bound [4]; this was found through clever use of the Ryu–Takayanagi formula for entanglement entropy [5,6] in a two-sided black hole spacetime that is holographically dual to the thermofield double state [7]. Examples of the latter include the Sakai–Sugimoto models for quantum chromodynamics (QCD) [8,9], which have provided new viewpoints in terms of strings and branes for problems in quantum field theory.

In spite of such developments, gauge/gravity correspondence has not been proven. In our opinion, there are unclear issues associated with the cases without conformal symmetry and/or at weak gauge coupling; clarification of these issues could show us a way towards proving gauge/gravity correspondence.

Conformal symmetry has been extremely helpful in formulating and testing gauge/gravity correspondence. Although the holographic principle should be a concept independent of the conformal symmetry, the cases without conformal symmetry are much less understood than those with it. By correctly formulating the correspondence without conformal symmetry, we would be able to gather important data suggesting how gauge/gravity correspondence works.

Gauge/gravity correspondence at weak gauge (or 't Hooft) coupling is poorly understood at present. Some people even doubt whether a correspondence exists in this region, but it is inconceivable that a proof of gauge/gravity correspondence is possible without a clear understanding of this region. Weak gauge coupling is supposed to correspond to weak string tension. In this region, the energy levels of string excited states are smaller than the mass scales for the supergravity modes (the lowest modes of strings). Thus, the supergravity approximation is not sufficient, and we will need the string worldsheet theory.

The purpose of this paper is twofold. One purpose is to establish a formalism for computing gauge-theory correlators using string worldsheet theory in the cases without conformal symmetry. The other purpose is to apply this formalism to the case of zero gauge coupling, with an additional assumption of the "string bit". We will reproduce the free-field result of gauge theory correlators from string theory, extending the result obtained in a previous paper by one of the present authors [10] for a particular operator to the general class of operators.

In Sect. 1.1 below, we will remind the reader of basic facts about gauge/gravity correspondence. This part can be skipped if it is not necessary for the reader. In Sect. 1.2, we will introduce the concrete example studied in this paper, namely the gauge/gravity correspondence associated with Dp-branes. This part is a review of old work, but may contain subjects that are not widely known. Then, in Sect. 1.3, we will describe the aim of the present work.

1.1 *Conformal and non-conformal*

Conformal symmetry has played important roles in formulating and testing gauge/gravity correspondence, such as the following: (1) The first hint of the equivalence of two completely different theories was that the symmetries on both sides match [3]: The isometry group of $(d+1)$-dimensional anti-de Sitter (AdS) space and the conformal group in d dimensions are isomorphic. (2) The dictionary between the bulk fields and the boundary operators, namely, the coupling between them in the Gubser–Klebanov–Polyakov–Witten (GKPW) prescription [11,12], is determined from the requirement that they should belong to the same representations of the (super)conformal group (see, e.g., Ref. [13]). (3) In theories with superconformal symmetry, there are powerful non-renormalization theorems. For instance, the scaling dimensions of the so-called BPS operators, which correspond to supergravity (SUGRA) modes, are not renormalized from their free-field values. The fact that the gauge-theory correlators calculated by the GKPW prescription using the tree-level SUGRA have the free-field scaling dimensions provides an important consistency check of AdS/CFT correspondence. (4) There have been highly nontrivial tests of AdS/CFT correspondence from calculations of quantities interpolating between weak and strong couplings. In such analyses, conformal symmetry played essential technical roles. Examples include the calculation of the expectation values of Wilson loops by using conformal transformations of the shape of the loops (see, e.g., Ref. [14]) and the calculation of cusp anomalous dimensions using integrability (see, e.g., Ref. [15]).

In theories without conformal symmetry, we do not have these (at least not obviously), and it is not straightforward to study gauge/gravity correspondence.

In fact, there have been many examples of gauge/gravity correspondence without conformal symmetry. However, most of them are associated with theories with conformal symmetry (though they involve highly non-trivial and interesting ideas), in the sense that they are continuous deformations of theories with conformal symmetry, and the symmetry is restored in some limit. Examples of such interesting theories include: (a) attempts at holographic duals of QCD-like theories, starting from the work by Witten [16], by compactifying spatial directions in the conformally invariant 6D theory; (b) studies of RG flows by constructing geometries that interpolate between two AdS regions [17–19]; and (c) applications of gauge/gravity correspondence to nuclear or condensed matter physics (see, e.g., Refs. [20,21] for reviews), in which attention is focused on the vicinity of the quantum phase transition points at which conformal symmetry is realized.

There are very few examples of gauge/gravity correspondence that do not have conformal symmetry from the outset. One such example is described below. Through the study of such a theory, we hope to develop the formalism and techniques applicable to non-conformal cases in general.

1.2 *Gauge/gravity correspondence for Dp-branes*

In this paper, we will study the gauge/gravity correspondence between maximally supersymmetric $SU(N)$ Yang–Mills theories in $(p+1)$ dimensions and superstring theories on the near-horizon limit of the Dp-brane solutions, proposed by Itzhaki et al. [22]. The former theory is an open-string description on the worldvolume of Dp-branes, and the latter is a closed-string description treating the Dp-branes as classical solutions in supergravity. The $p = 3$ case is conformally invariant, and is the most typical example of AdS/CFT correspondence [3]. For $p \neq 3$, the gauge coupling has non-zero dimension, and the theory is not conformally invariant. Without conformal symmetry, the analysis for $p \neq 3$ is not easy, but the correspondence is as well-motivated as in the $p = 3$ case. We expect the $p \neq 3$ cases to yield useful data about the mechanism of gauge/gravity correspondence. In this paper, we will study this example of gauge/gravity correspondence.

Supergravity analysis based on the GKPW prescription has been performed on the near-horizon Dp-brane background [23–25], and it has been found that the operators in the maximally supersymmetric Yang–Mills theories in $(p + 1)$ dimensions that correspond to SUGRA modes have power-law correlators, even though there is no conformal symmetry for $p \neq 3$. This is the result supposed to be valid at strong 't Hooft coupling. The power is in general a fractional number, different from the free-field value. This power law has not been understood analytically in gauge theory, but for some operators in $p = 0$, it has been confirmed to a high precision by the Monte Carlo simulation in gauge theory [26,27].

A problem with the cases without conformal symmetry has been that it is not clear how to perform bulk analysis based on the string worldsheet. For theories with conformal symmetry, one can identify the energy with respect to the global time in AdS with the scaling dimension of the corresponding operator in gauge theory. This identification is based on the isomorphism of the symmetry groups, and cannot be applied to the case without conformal symmetry. Although the superstring action on AdS has complicated non-linear interactions and is difficult to solve, there has been significant progress based on semi-classical approximations, especially[1]

[1] Also, in other cases such as AdS_3/CFT_2 [28–30] and the D1–D5 system (see, e.g., Ref. [31]), detailed worldsheet analyses have been performed.

in the case of $AdS_5 \times S^5$: (a) In the limit of large angular momenta along S^5, Berenstein, Maldacena, and Nastase (BMN) found the spectrum of quadratic fluctuations around a pointlike classical configuration of the string moving along S^5, and obtained the scaling dimensions of the corresponding operators (the so-called BMN operators). This includes higher string excitations, not only the supergravity modes [32]. (b) In the limit of large angular momenta along AdS_5, Gubser, Klebanov, and Polyakov (GKP) identified classical solutions of strings with such momenta, which are folded and spinning in AdS, and obtained the scaling dimensions of the corresponding operators in gauge theories. This result is expected to capture the properties of non-supersymmetric theory as well [33].

To the best of our knowledge, the only formalism for the worldsheet analysis applicable to the non-conformal cases are the one proposed by Dobashi, Shimada, and Yoneya (DSY) [34]. In the original paper of DSY, the BMN operators in the conformally invariant $p = 3$ case were studied. They consider Euclidean AdS in order to make contact with the GKPW prescription, which is formulated in the Euclidean background. There is a geodesic in Euclidean AdS that connects two points on the boundary. By performing a semi-classical approximation (similar to the one performed by BMN) of a superstring along this geodesic, one can compute the transition amplitudes of the Euclidean string. The amplitude is interpreted as the correlation function of the corresponding BMN operator. The results thus obtained are consistent with the ones obtained by BMN [32]; furthermore, some puzzles have been solved (see Refs. [34–36]). This formalism does not use conformal symmetry and is based on an intuitively clear relation between the bulk and boundary; thus it is applicable to the non-conformal cases. Asano, Yoneya, and one of the present authors have applied this to the study of the BMN operators for general p ($0 \leq p \leq 4$) [37,38]. When applied to the SUGRA modes, this method gives a result consistent with the one obtained by the GKPW prescription. The bulk analyses described above are supposed to give results for gauge theory in the large-N limit with strong 't Hooft coupling.

1.3 *Aim of the present work*

As reviewed above, it is essentially understood how to compute correlation functions at strong 't Hooft coupling from the bulk (using supergravity or string theory), including the cases of $p \neq 3$ that are not conformally invariant. On the other hand, it was not known how to obtain the free-field results of gauge theory from string theory except for the BPS operators for $p = 3$, whose scaling dimensions are protected.

In a previous paper by one of the present authors [10], the BMN operators in the $(p + 1)$-dimensional super-Yang–Mills theory have been studied at zero gauge coupling in string theory. This is based on the "string bit" picture, summarized as follows: Consider single trace operators, $\mathrm{Tr}(Z^J)$, where Z is a complex combination of two of the $(9 - p)$ scalar fields in gauge theory, say, $Z = \phi_8 + i\phi_9$, defined in a manner similar to the $p = 3$ case, following BMN. We will call this a BMN operator. (It might be common to use this terminology only when J is large, but here we will continue using it when J is not necessarily large.) This operator corresponds to a string state with J units of angular momentum along S^{8-p}. We assume that the spatial direction of the worldsheet is discretized into J bits, each of which has a single unit of angular momentum.

The reasons for considering the string bits are twofold: First, on the gauge-theory side, if one wants to represent the string worldsheet by the cyclic sequence of the fields inside the trace, one can only represent J sites. The string bit picture is mentioned in the original paper by BMN [32]. It also appears in recent papers such as Refs. [39] and [40] in contexts related to but somewhat different from ours[2].

In our opinion, there is another reason for considering string bits, on the string theory side: To represent a state with non-zero angular momentum, one usually inserts one or more creation operators that have appropriate angular momentum. Those operators are the Fourier modes with respect to the spatial direction on the worldsheet. In other words, they are obtained by smearing local operators on the worldsheet. Before smearing, each operator inserted on the worldsheet can be regarded as a particle (bit) with a unit angular momentum, interacting with other bits via strings connecting them. At the weak gauge coupling limit, the unit of angular momentum is large[3], and the interactions among the bits are weak. Thus, it would be appropriate to treat these bits as discrete objects.

In the previous paper [10], the case of zero coupling was studied by ignoring the interactions among the bits. Then, following the DSY formalism [34,37] mentioned in the last subsection, the amplitude for the collection of bits was computed, and interpreted as a gauge-theory correlator. By properly taking into account the zero-point energies due to the fluctuations, the free-field result of the $(p+1)$-dimensional field theory was reproduced.

The result of Ref. [10] was obtained for one particular operator. In this paper we would like to show that the free-field result can be obtained for general operators. We consider operators that have angular momentum along AdS, typically of the form $\sum_i \mathrm{Tr}(\phi_i D^S \phi_i)$, where ϕ_i are scalar fields in the $(p+1)$-dimensional gauge theory, and D denotes a complex combination of gauge covariant derivatives in two directions, say, $D = D_1 + iD_2$. This type of operator is called a Gubser, Klebanov, and Polyakov (GKP) operator [33]. (It might be common to use this terminology only when S is large and angular momentum along the S^{8-p} directions is zero, but we will use it loosely whenever an operator has non-zero angular momentum along AdS_{p+2}.) At strong 't Hooft coupling for $p = 3$, the GKP operators are described by strings that are folded and spinning in AdS, as found in a seminal paper by GKP [33]. In the present paper we first clarify how to describe such operators in the bulk for $p \neq 3$ (in which we cannot rely on the identification between the energy in terms of global time and the scaling dimension). Our approach is an extension of the Euclidean formulation of DSY, in which an angle along which the string is moving is taken to be imaginary, in order to keep the angular momentum real in the Euclidean setting. In DSY, only one angle was imaginary, but in our work, two of the angles are taken to be imaginary. In this formulation, we will find a solution in which a string worldsheet connects two points on the boundary; the center of the string follows a trajectory in AdS similar to the one for the BMN operators.

Then, we consider zero gauge coupling in this framework. As in the previous paper [10], we assume that a bit has a unit angular momentum along S^{8-p}. Calculating the amplitudes by ignoring the interactions among the bits, we will find the free-field results both for $p = 3$ and for $p \neq 3$. (For $p = 3$, we will also present an analysis for zero gauge coupling in the Lorentzian formulation.) We regard these results to be indications of the validity of the approach initiated in Ref. [10].

[2]We thank Yasuaki Hikida for pointing out the former reference to us.

[3]In this limit in which the string length is much larger than the AdS radius, stretched strings within a patch with the AdS size cannot be ignored. The string bit picture may provide one approach for understanding "sub-AdS locality", known to be a difficult problem in gauge/gravity correspondence [41].

We should mention that our result is based on two assumptions that have not been proven at present. One is that we assume that the near-horizon Dp-brane background does not receive $\alpha^{'}$ corrections for general p. The $p = 3$ case is believed to be $\alpha^{'}$ exact [42,43], but for $p \neq 3$ it is not known, to the best of our knowledge. The other assumption is that when we compute the zero-point energy of a single bit, we take into account the fluctuations up to the quadratic order around the classical trajectory. For $p = 3$, contributions from the bosonic and fermionic fluctuations cancel each other because there is effectively a worldsheet supersymmetry [38,44–46]. But for $p \neq 3$, we do not have a clear explanation why it is enough to stop at the quadratic order. Since we are not considering large angular momenta (because we are considering a single bit), there is no small parameter, so higher-order terms could in principle contribute. One way to estimate the order would be as follows: a bit can be regarded as a massive particle in AdS_{p+2}; its mass is given by the angular momentum on S^{8-p} and is of the order of $m \sim 1/L$, where L is the radius of AdS_{p+2} and S^{8-p}. Thus, there is an expansion parameter $1/m \sim L$, but each order is associated with the corresponding factor of $1/L$, since the expansion along the geodesic in AdS_{p+2} is essentially an expansion in curvature. In this counting, each term is of the order of unity. (This is also clear from the fact that the action, $m \int d\tau \sqrt{g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}$, of a massive particle on AdS_{p+2} has no dependence on L, since $m \sim 1/L$, and $\sqrt{g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}} \sim L$.)

1.4 *Organization of this paper*

This paper is organized as follows. In Sect. 2, we will briefly review the gauge/gravity correspondence for Dp-branes. The basic features of the near-horizon limit of the Dp-brane solution will be described. In Sect. 3, as preparation, we will consider the case of strong gauge coupling. We will first review the Lorentzian solution found by Gubser, Klebanov, and Polyakov, then rotate some coordinates to the imaginary directions, and obtain a string configuration that connects two points on the boundary. We will show that the string amplitude can be interpreted as the gauge-theory correlator. Our analysis here is based on the $p = 3$ case, but this formalism is applicable to the cases without conformal symmetry. In Sect. 4, we consider weak gauge coupling. We study the zero coupling case by considering the string bits without interactions among them. We first describe the analysis for $p = 3$ using global time in Lorentzian AdS. From the energy of the rotating particle, we obtain the free-field result in gauge theory. We then study the case of general p. We compute the amplitude for particles moving along a trajectory connecting two points on the boundary, and show that this reproduces the free-field result in gauge theory. In Sect. 5, we conclude, and mention directions for future research.

2. Gauge/gravity correspondence

A gauge/gravity correspondence associated with the Dp-branes has been proposed by Itzhaki et al. [22]. We will assume $0 \leq p \leq 4$. Here we will review only the basic facts, and refer the readers to the original papers for details[4]: for the supergravity analysis based on the GKPW prescription, see Refs. [23–25]; for the worldsheet analysis, see Refs. [37,38,50]; for the tests by Monte Carlo simulations in gauge theory, see Refs. [26,27]; for the correspondence at weak gauge coupling, see Ref. [10].

[4]See also Refs. [47–49] for "generalized conformal symmetry", which motivated the following analyses.

The metric and the dilaton for the zero-temperature Dp-brane solution in the string frame is given by[5]

$$ds^2 = H^{-1/2}\left(-dt^2 + dx_a^2\right) + H^{1/2}\left(dr^2 + r^2 d\Omega_{8-p}^2\right),$$
$$e^{\phi} = g_s H^{\frac{3-p}{4}}, \quad H = 1 + \frac{q}{r^{7-p}} \tag{1}$$

where $a = 1, \ldots, p$ and $q = \tilde{c}_p g_s N \ell_s^{7-p}$ with $\tilde{c}_p = 2^{6-p}\pi^{(5-p)/2}\Gamma(7-p)/2$. The integer N denotes the number of Dp-branes, ℓ_s is the string length, and g_s is related to the Yang–Mills coupling by $g_{\rm YM}^2 = (2\pi)^{p-2} g_s \ell_s^{p-3}$. We consider the near-horizon limit $r \ll q^{1/(7-p)}$, and take $H \to q/r^{7-p}$.

For $p = 3$, the near-horizon geometry is $\mathrm{AdS}_5 \times S^5$. For $p \neq 3$, it is related to $\mathrm{AdS}_{p+2} \times S^{8-p}$ by a Weyl transformation[6]:

$$ds^2 = H^{1/2} r^2 \left[\left(\frac{2}{5-p}\right)^2 \left(\frac{dt^2 + dx_a^2 + dz^2}{z^2}\right) + d\Omega_{8-p}^2\right]. \tag{2}$$

The radial variable z in the Poincaré coordinates for AdS_{p+2} is defined by

$$z = \frac{2}{5-p}(g_s N)^{1/2} l_s^{(7-p)/2} r^{-(5-p)/2} = \frac{2}{5-p} H^{1/2} r. \tag{3}$$

For $p = 3$, the Weyl factor $H^{1/2}r^2$ is constant. For $p \neq 3$, there is no AdS isometry, since the Weyl factor, dilaton, and the gauge fields do not have such symmetry. The radius of AdS_{p+2} is $\frac{2}{5-p}$ times the radius of S^{8-p}, as we see from the relative factor between the first and the second terms in Eq. (2).

To the best of our knowledge, it is not known whether there are α' corrections to the background (2) for general p. We assume that there are no corrections, and continue using this background when the curvature is strong.

The string coupling e^{ϕ} and the curvature of the background depend on the position for $p \neq 3$. String coupling is weak, $e^{\phi} \ll 1$, away from the center $r/\ell_s \gg N^{\frac{-4}{(3-p)(7-p)}}(g_s N)^{1/(3-p)}$ for $p < 3$, and away from the boundary $r/\ell_s \ll N^{\frac{4}{(p-3)(7-p)}}(g_s N)^{-1/(p-3)}$ for $p > 3$. To study the curvature, it would be helpful to consider an effective curvature radius $\tilde{L}(r)$ given by the Weyl factor in Eq. (2), which can be written as

$$\tilde{L}^2(r) \equiv H^{1/2} r^2 \propto \left(g_s N \ell_s^4\right)^{1/2} \left(\frac{\ell_s}{r}\right)^{(3-p)/2}. \tag{4}$$

The curvature is weak relative to the string scale, $\tilde{L}(r) \gg \ell_s$, away from the boundary $r/\ell_s \ll (g_s N)^{1/(3-p)}$ for $p < 3$, and away from the center $r/\ell_s \gg (g_s N)^{-1/(p-3)}$ for $p > 3$. If one wants to use the tree-level supergravity approximation, we will need these conditions to be satisfied in a large portion of the near-horizon region. This is achieved if we take $N \to \infty$ with $g_s N$ (i.e., the 't Hooft coupling $g_{\rm YM}^2 N$) fixed but large [23][7]. In this paper, we will always ignore string loops. This corresponds to taking $N \to \infty$ in gauge theory. In Sect. 4, we will consider weak gauge coupling $g_s \to 0$ (or weak 't Hooft coupling $g_s N \to 0$), which corresponds to strong curvature $\tilde{L}(r)/\ell_s \to 0$. We will study this region using string theory based on the string bit picture, not relying on the supergravity approximation. When gauge coupling is zero, the effective string tension, which is of the order of $\tilde{L}^2(r)/\ell_s^2$, is zero.

[5]The time coordinate t in this equation corresponds to time in the Poincaré coordinates, which will be denoted as $t_{\rm P}$ in Sect. 3. The time for global coordinates will be denoted as t there.

[6]For $p = 5$, the background is Weyl equivalent to a linear dilaton background.

[7]The singularity at the center $r = 0$ causes no problem in the GKPW prescription, since we take the wave functions that decay exponentially at the center [12].

The gauge theory that corresponds to superstring theory on the above background is the maximally supersymmetric $SU(N)$ Yang–Mills theory in $(p + 1)$ dimensions. This theory is obtained by dimensional reduction from the super-Yang–Mills theory in (9+1) dimensions, and has $(9 - p)$ scalar fields, $\phi_{p+1}, \ldots, \phi_9$, in addition to the gauge field in $(p + 1)$ dimensions. There is $SO(9 - p)$ global symmetry that rotates the scalar fields among themselves. We will consider a complex combination of two scalar fields, say, $Z \equiv \phi_8 + i\phi_9$, and also a complex combination of gauge covariant derivatives in two directions, say, $D \equiv D_1 + iD_2$.

3. Spinning strings (strong gauge coupling)

The purpose of this section is to establish a method for calculating the correlators of operators that have angular momentum along AdS_{p+2} from string theory that is applicable to the $p \neq 3$ cases that are not conformally invariant. Our method is an extension of the prescription proposed by Dobashi et al. which has so far been formulated only for the BMN operators (those with angular momenta along S^{8-p}). In this formulation, we will consider Euclidean AdS. In the limit of strong 't Hooft coupling, correlation functions of operators of the form $\sum_i \mathrm{Tr}(\phi_i D^S \phi_i)$ can be calculated by evaluating the action of classical string solutions. For $p = 3$, the string solution is given from the one obtained by GKP [33] by rotating some coordinates to imaginary values. This connects two points on the boundary. We will defer the study of classical solutions for $p \neq 3$ to future work, but this formalism is applicable to the cases without conformal symmetry.

In Sect. 3.1, we review GKP's analysis performed in Lorentzian AdS for $p = 3$. Then, in Sect. 3.2, we obtain a solution that connects two points on the boundary in the Euclidean formulation. We will see that the Euclidean amplitude for this string can be written in the form of gauge-theory correlator.

3.1 *Lorentzian signature*

We write the $\mathrm{AdS}_5 \times S^5$ using the global coordinates for AdS[8],

$$ds^2 = L^2 \Big\{ -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho \left(\cos^2\theta\, d\psi^2 + d\theta^2 + \sin^2\theta d\phi^2\right) + \cos^2\tilde{\theta}\, d\tilde{\psi}^2 + d\tilde{\theta}^2 + \sin^2\tilde{\theta} d\Omega_3^2 \Big\} \ , \tag{5}$$

since the symmetry is manifest in this coordinate system.

The bosonic part of the string action is given by

$$I = \frac{1}{4\pi\alpha'} \int d\tau \int_0^{2\pi\alpha'} d\sigma \sqrt{-h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}. \tag{6}$$

In addition to the equation of motion for X^μ, there is a constraint obtained by varying the action with respect to the worldsheet metric $h^{\alpha\beta}$:

$$-\frac{1}{2} h_{\alpha\beta} \partial^\gamma X^\mu \partial_\gamma X^\nu g_{\mu\nu} + \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu} = 0. \tag{7}$$

In this section we will take the conformal gauge, $\sqrt{-h} h^{\alpha\beta} = \eta^{\alpha\beta}$, in which the above constraint becomes

$$\dot{X}^\mu \dot{X}^\nu g_{\mu\nu} = -X^{\mu\prime} X^{\nu\prime} g_{\mu\nu}, \tag{8}$$

[8]The time coordinate t without a subscript denotes time in global coordinates in this section and Sect. 4.1.

$$\dot{X}^{\mu} X^{\nu\prime} g_{\mu\nu} = 0. \tag{9}$$

On the background (5), the momenta corresponding to the translations in t, ψ, and $\tilde{\psi}$ are conserved. We will call them E, S, and J, respectively:

$$E \equiv P_t \equiv \frac{\delta S}{\delta \dot{t}} = \frac{L^2}{2\pi\alpha'} \int_0^{2\pi\alpha'} d\sigma \cosh^2\rho\, \dot{t}, \tag{10}$$

$$S \equiv P_\psi \equiv -\frac{\delta S}{\delta \dot{\psi}} = \frac{L^2}{2\pi\alpha'} \int_0^{2\pi\alpha'} d\sigma \sinh^2\rho\, \cos^2\theta \dot{\psi}, \tag{11}$$

$$J \equiv P_{\tilde{\psi}} \equiv -\frac{\delta S}{\delta \dot{\tilde{\psi}}} = \frac{L^2}{2\pi\alpha'} \int_0^{2\pi\alpha'} d\sigma \cos^2\tilde{\theta}\, \dot{\tilde{\psi}}. \tag{12}$$

Following GKP [33] (also allowing the motion along S^5 [51–53]), we take the following ansatz for classical solutions (with $\theta = \tilde{\theta} = 0$):

$$t = \tau, \quad \psi = \omega\tau, \quad \rho = \rho(\sigma), \quad \tilde{\psi} = \tilde{\omega}\tau. \tag{13}$$

Since t and ψ are functions of τ only, and ρ is a function of σ only, the constraint (9) is satisfied. The other constraint (8) gives the relation

$$d\sigma = \frac{d\rho}{\sqrt{\cosh^2\rho - \tilde{\omega}^2 - \omega^2 \sinh^2\rho}}, \tag{14}$$

which implicitly determines ρ as a function of σ. The string is folded and stretched: the points $\sigma = 0$ and $\sigma = \pi\alpha'$ on the worldsheet are at the center of the string, $\rho = 0$; the points $\sigma = \pi\alpha'/2$ and $\sigma = 3\pi\alpha'/2$ are at the end, $\rho = \rho_0$.

From Eqs. (10), (11), and (12), one can find the relations among E, S, J. For example, as described in Ref. [33] (for $J = 0$), for small S one has $E^2 \sim S$; for large S one has $E - S \sim \ln S$. By identifying the energy with the scaling dimension, $\Delta \sim E$, strong-coupling results of gauge theory have been obtained [33].

3.2 *Euclidean signature*

We would like to relate GKP's string solution [33] with the GKPW prescription [11,12] based on supergravity. Since the latter is defined on Euclidean AdS, we replace $t \to it_{\rm E}$ in Eq. (5). To study strings (or particles) on this background, we will take the worldsheet (or worldline) time imaginary also, $\tau \to i\tau_{\rm E}$. In this Euclidean calculation, the angular momenta J and S should be kept real, since these are quantum numbers that specify the representation of the symmetry group and have direct meanings in gauge theory. Accordingly, as we see from Eqs. (11) and (12), the angular variables ψ and $\tilde{\psi}$ should be taken imaginary[9], since τ is now imaginary. This is a straightforward extension of the prescription of Dobashi et al. [34], in which one angle was taken to be imaginary, to the case of two angles. The Euclidean solution takes the form

$$t_{\rm E} = \tau_{\rm E}, \quad \psi = i\omega\tau_{\rm E}, \quad \rho = \rho(\sigma), \quad \tilde{\psi} = i\tilde{\omega}\tau_{\rm E}, \tag{15}$$

with ω and $\tilde{\omega}$ being real; these are the same as the angular velocities for the Lorentzian solution.

By solving the constraint (8), which now becomes

$$\partial_{\tau_{\rm E}} X^{\mu} \partial_{\tau_{\rm E}} X^{\nu} g_{\mu\nu} = X^{\mu\prime} X^{\nu\prime} g_{\mu\nu}, \tag{16}$$

we obtain the relation between ρ and σ. It is of the same form as Eq. (14).

[9] We regard the momentum representation as fundamental for some variables such as ψ and $\tilde{\psi}$ here. Thus, we do not particularly pursue physical meanings for the imaginary values of ψ and $\tilde{\psi}$.

The Euclidean version of E, which we will call H, is

$$H = \frac{L^2}{2\pi\alpha'}\int_0^{2\pi\alpha'} d\sigma \cosh^2\rho \frac{dt_{\rm E}}{d\tau_{\rm E}} = \frac{2L^2}{\pi\alpha'}\int_0^{\rho_0} d\rho \frac{\cosh^2\rho}{\sqrt{\cosh^2\rho - \tilde{\omega}^2 - \omega^2\sinh^2\rho}}, \tag{17}$$

and the angular momentum along ψ is

$$S = -i\frac{L^2}{2\pi\alpha'}\int_0^{2\pi\alpha'} d\sigma \sinh^2\rho \frac{d\psi}{d\tau_{\rm E}} = \frac{2L^2}{\pi\alpha'}\omega\int_0^{\rho_0} d\rho \frac{\sinh^2\rho}{\sqrt{\cosh^2\rho - \tilde{\omega}^2 - \omega^2\sinh^2\rho}}. \tag{18}$$

3.2.1 *Poincaré coordinates.* We now rewrite the above solution in Poincaré coordinates, so that the correspondence with gauge theory becomes clear. For clarity, we present the coordinate transformations using the embedding coordinates explicitly. We will write the formulas for general p in this subsection.

The Euclidean AdS_{p+2} is represented as a hyperboloid,

$$-X_{p+2}^2 + X_0^2 + \sum_{a=1}^{p+1} X_a^2 = -L^2, \tag{19}$$

in the $p+3$-dimensional flat space, $ds^2 = -dX_{p+2}^2 + dX_0^2 + \sum_{a=1}^{p+1} dX_a^2$. The hyperboloid is parametrized by the global coordinates as

$$\begin{aligned} X_{p+2} &= L\cosh\rho\cosh t_{\rm E}, \\ X_0 &= L\cosh\rho\sinh t_{\rm E}, \\ X_a &= L\sinh\rho\Omega_a, \end{aligned} \tag{20}$$

where $\sum_{a=1}^{p+1}\Omega_a^2 = 1$.

The solution (15) effectively has angular variable ψ imaginary. If we choose ψ to parametrize the rotation in the X_1-X_2 plane as follows:

$$X_1 = L\sinh\rho\cos\theta\cos\psi, \quad X_2 = L\sinh\rho\cos\theta\sin\psi, \tag{21}$$

the coordinate X_2 effectively becomes imaginary if ψ is imaginary. In the following, X_2, and also x_2 defined below, are understood to be imaginary[10].

The Poincaré coordinates are defined as

$$\begin{aligned} X_{p+2} &= \frac{z}{2}\left(1 + \frac{L^2 + \sum_i x_i^2 + t_{\rm P}^2}{z^2}\right), \\ X_0 &= L\frac{t_{\rm P}}{z}, \qquad X_i = L\frac{x_i}{z}, \\ X_{p+1} &= \frac{z}{2}\left(1 + \frac{-L^2 + \sum_i x_i^2 + t_{\rm P}^2}{z^2}\right). \end{aligned} \tag{22}$$

Equating Eqs. (20) and (22), we obtain

$$t_{\rm P} = \tilde{\ell}\tanh\tau_{\rm E}, \tag{23}$$

$$z = \frac{\tilde{\ell}}{\cosh\rho\cosh\tau_{\rm E}}, \tag{24}$$

$$\sum_i x_i^2 = z^2\sinh^2\rho = \frac{\tilde{\ell}^2\tanh^2\rho}{\cosh^2\tau_{\rm E}}, \tag{25}$$

[10] If we define a real variable $\hat{x}_2$ by $x_2 = i\hat{x}_2$, the expression $\sum_i x_i^2$ below means $x_1^2 - \hat{x}_2^2 + x_3^2 + \cdots$.

where $\tilde{\ell}$ is an arbitrary constant, and we have used $t_{\rm E} = \tau_{\rm E}$ from Eq. (16).

From Eq. (25), we see that the spatial extent of the string reduces to a point in terms of the coordinate x_i, as the string approaches the boundary $z \to 0$ (or $\tau \to \pm\infty$), in the following sense: the string occupies $0 \leq \rho \leq \rho_0$ (with ρ_0 being a finite constant), and the center of the string, $\rho = 0$, is at $x_i = 0$; Eq. (25) shows that the value of $\sum_i x_i^2$ corresponding to the endpoint of the string, $\rho = \rho_0$, goes to zero[11] as $z \to 0$ (or $|\tau| \to \infty$). This fact is in comfort with the fact that we represent this state as a local operator in gauge theory.

3.2.2 *Gauge-theory correlator.* By substituting $\tau_{\rm E} = \pm\infty$ in Eq. (23), we see that the coordinate distance between the two points, $t_{\rm i}$ and $t_{\rm f}$ (both with $x_i = 0$), on the boundary is given by

$$|t_{\rm f} - t_{\rm i}| = 2\tilde{\ell}. \tag{26}$$

In fact, Eq. (25) shows that the center of mass of the string follows the same trajectory as the trajectory of a particle that has an angular momentum along S^{8-p} (and not along AdS_{p+2}). The latter was first studied for $p = 3$ by Dobashi et al. [34], and then generalized to $p \neq 3$ by Asano et al. [37]. This will be also described in Sect. 4 of this paper.

The Euclidean amplitude for a string that propagates from a point on the boundary to another point on the boundary is interpreted as the two-point function of gauge theory. In the classical approximation, the bulk amplitude is

$$e^{-S_{\rm cl}} = e^{-\int_{-T}^{T} d\tau_{\rm E} H(\tau_{\rm E})}, \tag{27}$$

where $H(\tau_{\rm E})$ is the Hamiltonian at Euclidean worldsheet time $\tau_{\rm E}$. This is just the Euclidean version of the global energy given in Eq. (17), since the worldsheet time is equal to the global time, in our classical solution (15).

We introduce a cutoff T for the worldsheet time, $-T \leq \tau \leq T$. We also introduce a cutoff for the radial coordinate, $z \geq 1/\Lambda$. This IR cutoff in the bulk is interpreted as a UV cutoff in gauge theory [41]. The relation between the two cutoffs can be read off from Eqs. (23) and (24) in the $|T| \to \infty$ limit as

$$2\tilde{\ell}\Lambda = e^{T}. \tag{28}$$

For the solution (15), the Hamiltonian is independent of the worldsheet time. Therefore, it can be taken out of the integral, and the amplitude can be written as

$$e^{-\int_{-T}^{T} d\tau_{\rm E} H(\tau_{\rm E})} = e^{-2HT} = \frac{1}{(\Lambda |t_{\rm f} - t_{\rm i}|)^{2H}}. \tag{29}$$

The worldsheet Hamiltonian gives the scaling dimension, and we have recovered the result of GKP.

This formalism should be applicable to the $p \neq 3$ case without conformal symmetry. In that case, the worldsheet Hamiltonian will not be time independent in general; thus the above integral has to be evaluated explicitly. In addition, finding the string solution is more challenging than $p = 3$, since the AdS isometry is not the symmetry of the string due to the position-

[11] If x_i were all real, this would really mean that the string reduces to a single point at the boundary. In our case where $x_2 = i\hat{x}_2$ is imaginary, the string is really stretched in the "lightlike" direction in the x_1-$\hat{x}_2$ plane. However, as mentioned in a previous footnote, we do not attach a particular physical meaning to imaginary values of the coordinates. Thus, if the invariant distance $\sum_i x_i^2$ is zero, we interpret it as a single point.

dependent Weyl factor. One approach would be to study the limit of a short string by using an approximate form of the geometry near the center of the string. This subject is under study [54] and will be reported elsewhere.

4. Rotating particles (weak gauge coupling)

Let us now consider weak gauge coupling. In this paper, we will concentrate on the case where the gauge coupling is strictly zero. On the string theory side, this corresponds to the case where the string tension is zero. If the spatial direction of the worldsheet is discretized into bits, the interactions among the bits can be ignored in this limit. The previous paper [10] considered a state that has angular momentum only along S^{8-p}, but here we will extend the analysis to those that also have angular momentum along AdS_{p+2}. As in Ref. [10], we assume that one bit carries[12] a single unit of angular momentum along S^{8-p}.

We first study the $p = 3$ case using the Lorentzian formulation, by identifying the AdS energy in terms of the global time with the scaling dimension. We then study the case of general p using the Euclidean formulation that does not rely on conformal symmetry.

4.1 $p = 3$

A bit is a massless particle in 10D spacetime [10], namely, $\mathrm{AdS}_5 \times S^5$ in this case. We will fix the angular momentum in the S^5 direction. Then, we can perform a Kaluza–Klein reduction and treat a bit as a massive particle on AdS_5. More precisely, by considering the Routh function (in which a Legendre transformation is performed for an angle in the S^5 direction) in 10 dimensions, we obtain the action of a massive particle on AdS_5 (see Sect. 2.2 of Ref. [37]). The mass is given by $m = J/L$, where J is an integer and L is the radius of S^5, which is equal to the radius of AdS_5. One bit has $J = 1$ [10], but we will keep m in the formulas below for the time being.

The action for a single bit is given by

$$I_{\rm bit}[X^\mu] = \frac{1}{2}\int d\tau \left[\frac{1}{\eta(\tau)} g_{\rho\sigma}(X^\mu(\tau))\dot{X}^\rho(\tau)\dot{X}^\sigma(\tau) - \eta(\tau)m^2\right], \tag{30}$$

and the corresponding equations of motion are

$$g_{\rho\sigma}\dot{X}^\rho\dot{X}^\sigma = -\eta^2 m^2, \tag{31}$$

$$\dot{X}^\rho\partial_\rho\dot{X}^\mu = -\Gamma^\mu{}_{\rho\sigma}\dot{X}^\rho\dot{X}^\sigma. \tag{32}$$

For AdS_5 in the global coordinates (5), the single-bit action becomes

$$I_{\rm bit}[X^\mu] = \frac{1}{2}\int d\tau \left[\frac{L^2}{\eta}\Big(-\cosh^2\rho\, \dot{t}^2 + \dot{\rho}^2 + \sinh^2\rho\, \dot{\psi}^2 + (\text{remaining angular part})\Big) - \eta m^2\right].$$

The following canonical momenta are conserved, corresponding to the isometry of the background:

$$E \equiv -p_t = \frac{L^2}{\eta}\cosh^2\rho\, \dot{t}, \tag{33}$$

$$S \equiv p_\psi = \frac{L^2}{\eta}\sinh^2\rho\, \dot{\psi}. \tag{34}$$

[12] Here, we are considering the orbital angular momentum, since a bit (particle) is pointlike. This is in contrast to the stretched string spinning around its center of mass, studied in the previous section.

Let us take $\rho = \text{const.}$ as an ansatz for a classical solution. Setting $\dot\rho = 0$ in the constraint (31), we find

$$-\frac{E^2}{\cosh^2\rho} + \frac{S^2}{\sinh^2\rho} = -m^2L^2. \tag{35}$$

In addition, in order to have $\ddot\rho = 0$, the equation of motion for ρ,

$$\begin{aligned}\ddot\rho &= \cosh\rho\sinh\rho\left[-\dot t^2 + \dot\psi^2\right]\\ &= \cosh\rho\sinh\rho\,\frac{\eta^2}{L^4}\left[-\frac{E^2}{\cosh^4\rho} + \frac{S^2}{\sinh^4\rho}\right],\end{aligned} \tag{36}$$

indicates

$$-\frac{E^2}{\cosh^4\rho} + \frac{S^2}{\sinh^4\rho} = 0\,. \tag{37}$$

Equations (35) and (37) lead to[13]

$$\begin{aligned}E^2 \;&=\; m^2L^2\cosh^4\rho,\quad S^2 = m^2L^2\sinh^4\rho\\ &\Longleftrightarrow E = mL\cosh^2\rho,\quad |S| = mL\sinh^2\rho.\end{aligned} \tag{38}$$

Thus,

$$E = |S| + mL. \tag{39}$$

One bit has a single unit of angular momentum on S^5, and $m = 1/L$, so its energy is

$$E_{\text{bit}} = |S| + 1. \tag{40}$$

Now consider a collection of n non-interacting bits. Its total energy $E = \sum_{i=1}^{n} E_i$ can be written as

$$E = \sum_{i=1}^{n} |S_i| + n, \tag{41}$$

where $|S_i|$ is the magnitude of the angular momentum along AdS_5 (which is integer in our convention) carried by the ith bit. The n bits can have different directions of angular momenta. One bit has a single unit of angular momentum along S^5, and contributes 1 to the energy; summing this over the n bits, we obtain n in the last term of Eq. (41). In the special case where all the bits have angular momenta in the same direction both for AdS_5 and S^5, which corresponds to an operator[14] such as $\mathrm{Tr}(Z^{J-k}D^SZ^k)$, the total energy is

$$E = |S| + J, \tag{42}$$

where $|S|$ and J are the magnitudes of the total angular momenta along AdS_5 and S^5, respectively.

The above results for the non-interacting bits correctly reproduce the free-field results in gauge theory, if we identify the global energy E with the scaling dimension Δ of the corresponding operator. A single bit is assumed to correspond to a single scalar field inside the trace, which contributes 1 to the scaling dimension in the free theory in (3+1) dimensions. The ith bit with angular momentum S_i along AdS_5 can be realized by applying $|S_i|$ gauge covariant derivatives

[13]The energy takes only positive values $E \geq 0$, but the angular momentum S could be positive or negative.

[14]Here we only mean that the number of the Z field in the trace is J, and the number of the covariant derivative D is S. At the strictly zero gauge coupling considered in this paper, it would be difficult to distinguish different operators within this class.

on the ith scalar field. In the free theory, the covariant derivative is just a partial derivative, which contributes 1 to the scaling dimension. Thus, Eq. (41) (or Eq. (42) for a special case) gives the correct free-field result under the identification $\Delta = E$.

4.2 *General p*

Let us now consider the case of general p. We assume $p \geq 2$, so that we have two spatial directions in gauge theory in which the string (or particle) can rotate.

The background geometry is conformal to the $\mathrm{AdS}_{p+2} \times S^{8-p}$ spacetime[15],

$$ds^2 = L^2 r^{-\frac{3-p}{2}} \left[\left(\frac{2}{5-p} \right)^2 \frac{1}{z^2} \left\{ -dt^2 + dz^2 + dx_a^2 \right\} + \left(d\theta^2 + \cos^2\theta d\psi^2 + \sin^2\theta d\Omega_{6-p}^2 \right) \right], \tag{43}$$

where $a = 1, \ldots, p$. For general p, there is no conformal symmetry (or AdS isometry), and we cannot use the identification $\Delta = E$, so we follow the DSY prescription [34,37,38]. As explained in Sect. 3, gauge-theory correlators are obtained by calculating the transition amplitude from the path integral in the multiply Wick-rotated background:

$$\langle t_f^E, J, \Theta_f | t_i^E, J, \Theta_i \rangle = \langle it_f, J, \Theta_f | it_i, J, \Theta_i \rangle = \int \mathcal{D}X e^{-(I + J\psi_f - J\psi_i + S\Theta_f - S\Theta_i)}. \tag{44}$$

The role of the term $J\psi_f - J\psi_i$ (or $S\Theta_f - S\Theta_i$) on the exponent on the right-hand side is to fix the angular momentum in the ψ (or Θ) direction to J. This term can be equivalently represented by using the Routh function defined by $I + \int d\tau (J\dot{\psi} + S\dot{\Theta})$ instead of the action I. See Sect. 2.2 of Ref. [37].

As in the last subsection, we will study the case of zero gauge coupling by ignoring interactions among the bits. The action for a single bit on the background (43) is

$$I = \int_{-T}^{T} d\tau \, \frac{\tilde{\alpha}}{2} L^2 \left[\left(\frac{2}{5-p} \right)^2 \frac{1}{z^2} \left\{ \dot{t}^2 + \dot{z}^2 + \dot{R}^2 - R^2 \dot{\Theta}^2 + \cdots \right\} + \left(\dot{\theta}^2 - \cos^2\theta \dot{\psi}^2 + \cdots \right) \right]. \tag{45}$$

The factor $\tilde{\alpha}$ is a constant to be determined later. The coordinates R and Θ are the radius and angle defined from two of the coordinates x_a.

Here we have absorbed the Weyl factor in the metric (43) by a choice of the einbein (which is an overall factor in the action, since we are considering massless particles in 10D) [37,38]. As a result, the action (45) is formally the same as the one in $\mathrm{AdS}_{p+2} \times S^{8-p}$. In string theory this amounts to taking the worldsheet metric so that $\sqrt{h}h^{\tau\tau} = \bar{r}^{\frac{3-p}{2}}$ (with $h^{\tau\sigma} = 0$), to absorb the overall factor $\bar{r}^{-\frac{3-p}{2}}$ in the metric. Then, $\sqrt{h}h^{\sigma\sigma} = 1/(\sqrt{h}h^{\tau\tau}) = \bar{r}^{-\frac{3-p}{2}}$, and the coefficient of $(\partial_\sigma x^i)^2$ in the worldsheet action gets a time-dependent factor $\bar{r}^{-(3-p)}$ [37,38][16]. Thus, the effect of the Weyl factor appears for excited states of the string (for which $\partial_\sigma x^i \neq 0$). In the string bit picture, coefficients of the interaction terms among the bits will have time dependence for $p \neq 3$.

[15] Up to now we have put tildes on the angles in S^{8-p}, but in this subsection we will omit the tildes and denote them as θ, ψ to simplify the notations.

[16] If one takes the conformal gauge $\sqrt{h}h^{\alpha\beta} = \delta^{\alpha\beta}$, both $(\partial_\tau x^i)^2$ and $(\partial_\sigma x^i)^2$ have time-dependent coefficients, but the final result of the calculation is of course independent of the gauge choice [37,38].

The equations of motion are

$$\frac{d}{d\tau}\left(\frac{\dot{t}}{z^2}\right)=0,\ \ \frac{d}{d\tau}\left(\frac{\dot{z}}{z^2}\right)=-\frac{1}{z^3}\{\dot{t}^2+\dot{z}^2+\dot{R}^2-R^2\dot{\Theta}^2+\cdots\},\ \ \frac{d}{d\tau}\left(\frac{\dot{R}}{z^2}\right)=-\frac{R}{z^2}\dot{\Theta}^2,$$
$$\frac{d}{d\tau}\left(\frac{R^2}{z^2}\dot{\Theta}\right)=0,\ \ \frac{d}{d\tau}\dot{\theta}=-\sin\theta\cos\theta\dot{\psi}^2,\ \ \frac{d}{d\tau}\dot{\psi}=0, \tag{46}$$

and the constraint is

$$\left(\frac{2}{5-p}\right)^2\frac{1}{z^2}\left\{\dot{t}^2+\dot{z}^2+\dot{R}^2-R^2\dot{\Theta}^2+\cdots\right\}+(\dot{\theta}^2-\cos^2\theta\dot{\psi}^2+\cdots)=0. \tag{47}$$

4.2.1 *The $S = 0$ case.* The solution for $S = 0$ has been obtained by Asano et al. [37,38];

$$t=\tilde{\ell}\tanh\tau,\ \ z=\frac{\tilde{\ell}}{\cosh\tau},\ \ R=\Theta=0,$$
$$\theta=0,\ \ \psi=\frac{2}{5-p}\tau, \tag{48}$$

where $\tilde{\ell}$ parametrizes the separation between t_f and t_i, i.e.,

$$|t_f-t_i|=2\tilde{\ell}. \tag{49}$$

The solution (48) is represented as a half circle $t^2+z^2=\tilde{\ell}^2$ in the coordinate space (t, z).

As in the last section, we introduce the IR cutoff at $z=\frac{1}{\Lambda}$, which is related to the worldsheet cutoff T as

$$2\tilde{\ell}\Lambda=e^T. \tag{50}$$

The transition amplitude at the zero-loop level on the worldsheet becomes

$$\langle t_f, J, 0|t_i, J, 0\rangle\simeq e^{-(I+J(\psi_f-\psi_i))}=e^{-\frac{4}{5-p}JT}$$
$$=\frac{1}{\Lambda^{\frac{4}{5-p}J}}\frac{1}{|t_f-t_i|^{\frac{4}{5-p}J}}\quad\text{(at zero loop)} \tag{51}$$

where we have used Eqs. (49) and (50) in the last line to rewrite the amplitude in terms of the variables in gauge theory. We have the factor J on the exponent, because there are J non-interacting bits, each giving the contribution described above. The expression (51) could be regarded as the leading part in the large-J limit of the correlator of $\mathrm{Tr}(Z^J)$ at strong gauge coupling [37,38].

Let us consider the one-loop contribution on the worldsheet, following Ref. [10]. The action of a single bit at the quadratic level of the bosonic fluctuations around the classical trajectory (48) is [37,38]

$$I^{(2)}=\frac{\tilde{\alpha}}{2}\int_{-T}^{T}d\tau\left\{\dot{x}_a^2+m_x^2x_a^2+\dot{y}_i^2+m_y^2y_i^2\right\}. \tag{52}$$

We have $(p+1)$ fields x_a with mass $m_x=1$ that comes from fluctuations along AdS_{p+2}, and $(7-p)$ fields y_i with mass $m_y=\frac{2}{5-p}$ that come from fluctuations along S^{8-p}. There are also eight fermionic fluctuations with mass $m_f=\frac{(7-p)}{2(5-p)}$. For the quadratic action of the fermionic fluctuations, see Ref. [38].

In Ref. [10], the modification to the correlator (51) for $\mathrm{Tr}(Z^J)$ due to the one-loop contribution on the worldsheet has been obtained by an operator method, by including the zero-point energies of the bosonic and fermionic fluctuations (which are harmonic oscillators) for each bit.

For $p = 3$, the contributions from the bosonic and fermionic fluctuations cancel each other [38], since there is effectively worldsheet supersymmetry.

Here we will derive the same result as Ref. [10] from the Euclidean path integral of harmonic oscillators with the boundary condition $x(T) = x(-T) = 0$. Since the complete set of bases may be given by

$$\left\{\cos\frac{\pi}{2T}(2\bar{k}+1)\tau,\quad \sin\frac{\pi}{2T}(2\tilde{k})\tau\right\}_{\bar{k},\tilde{k}} \tag{53}$$

the eigenvalues of the operator $\left(-\frac{d^2}{d\tau^2}+m^2\right)$ are given by

$$\lambda_k = \frac{\pi^2}{4T^2}k^2 + m^2 \ \ (k \in \mathbb{Z}_{\geq 0}). \tag{54}$$

Therefore, the path integral is

$$\begin{aligned} Z^{(2)} &= \int \mathcal{D}x\, e^{-\frac{1}{2}\int_{-T}^{T} d\tau\left\{\dot{x}^2+m^2x^2\right\}} \\ &= \prod_{\bar{k},\tilde{k}} \int dA \int dB e^{-A^2\left(\frac{\pi^2}{4T^2}(2\bar{k}+1)^2+m^2\right)T} e^{-B^2\left(\frac{\pi^2}{4T^2}(2\tilde{k})^2+m^2\right)T} \\ &= \prod_k \sqrt{\frac{\pi}{\lambda_k T}}, \end{aligned} \tag{55}$$

i.e.,

$$\log Z^{(2)} = -\frac{1}{2}\sum_k \log\lambda_k + (\text{divergent part}). \tag{56}$$

Here, we apply the zeta function regularization method. Redefine $Z^{(2)}$ by

$$\log Z^{(2)} \equiv \frac{1}{2}\left.\frac{d\zeta(s)}{ds}\right|_{s=0} \tag{57}$$

where the generalized zeta function is given by

$$\zeta(s) \equiv \sum_k \lambda_k^{-s}. \tag{58}$$

Now make k continuous by considering large T, i.e.,

$$\begin{aligned} \zeta(s) &= \sum_k \left(\frac{\pi^2}{4T^2}k^2+m^2\right)^{-s} = \frac{2T}{\pi}\sum_K \left(K^2+m^2\right)^{-s}\Delta K \\ &\to \frac{2T}{\pi}\int_0^\infty dK\left(K^2+m^2\right)^{-s} \end{aligned} \tag{59}$$

where we have changed the variable $K \equiv \frac{\pi}{2T}k$ and $\Delta K = \frac{\pi}{2T}$. Then,

$$\zeta(s) = \frac{2T}{\pi}\int_0^\infty dK\left(K^2+m^2\right)^{-s} = \frac{T}{\sqrt{\pi}}m^{1-2s}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} \tag{60}$$

$$\simeq -2mTs + 4mT(-1+\log 2+\log m)s^2 + \cdots . \tag{61}$$

Eventually,

$$Z^{(2)} = e^{\frac{1}{2}\left.\frac{d\zeta(s)}{ds}\right|_{s=0}} = e^{-mT}. \tag{62}$$

Note that this is independent of the overall factor of the harmonic oscillator action. Therefore, the one-loop contribution of the bosonic part for J bits is

$$Z^{(2)}_{J\,\text{bit(boson)}} = e^{-J((p+1)m_x+(7-p)m_y)T} = e^{-\frac{-p^2+2p+19}{5-p}JT}. \tag{63}$$

Similarly, the one-loop contribution of the fermionic part is

$$Z^{(2)}_{J\,\text{bit(fermion)}} = e^{J8m_f T} = e^{\frac{28-4p}{(5-p)}JT}. \tag{64}$$

Then

$$Z^{(2)}_{J\,\text{bit}} = Z^{(2)}_{J\,\text{bit(boson)}} Z^{(2)}_{J\,\text{bit(fermion)}} = e^{\frac{(p-3)^2}{5-p}JT}. \tag{65}$$

Combining the zero- and one-loop contributions on the worldsheet, the amplitude becomes

$$\begin{aligned}\langle t_f, J, 0|t_i, J, 0\rangle &\simeq e^{-(I+J(\psi_f-\psi_i))} Z^{(2)}_{J\,\text{bit}} \\ &= e^{-(p-1)JT} \\ &= \frac{1}{\Lambda^{(p-1)J}} \frac{1}{|t_f - t_i|^{(p-1)J}}.\end{aligned} \tag{66}$$

This reproduces the free-field result for the two-point function of $\text{Tr}(Z^J)$ in the $(p+1)$-dimensional gauge theory [10]: from dimensional analysis, a scalar field has dimension $(p-1)/2$ in the free theory, and the operator consists of J scalar fields.

4.2.2 *The $S \neq 0$ case.* In this case, the solution is

$$\begin{aligned}&t = \tilde{\ell}\tanh\tau, \quad z = \frac{\sqrt{\tilde{\ell}^2 - B^2}}{\cosh\tau}, \quad R = \frac{B}{\cosh\tau}, \quad \Theta = \tau, \\ &\theta = 0, \quad \psi = \frac{2}{5-p}\tau.\end{aligned} \tag{67}$$

Like the solution (48) for $S = 0$, this is represented as a half circle, now given by $t^2 + z^2 + R^2 = \tilde{\ell}^2$ in the coordinate space (t, z, R). Since J and S are given by

$$J = \tilde{\alpha}L^2\dot{\psi} = \tilde{\alpha}L^2\frac{2}{5-p}, \tag{68}$$

$$S = \tilde{\alpha}L^2\left(\frac{2}{5-p}\right)^2\frac{R^2}{z^2}\dot{\Theta} = \tilde{\alpha}L^2\left(\frac{2}{5-p}\right)^2\frac{B^2}{\tilde{\ell}^2 - B^2}, \tag{69}$$

the constants $\tilde{\alpha}$ and B are related to J, S, and $\tilde{\ell}$ via

$$\tilde{\alpha} = \frac{5-p}{2}\frac{J}{L^2}, \tag{70}$$

$$B = \sqrt{\frac{S(5-p)}{2J + S(5-p)}}\,\tilde{\ell}\,. \tag{71}$$

We introduce the IR cutoff at $z = \frac{1}{\Lambda}$. The relation between Λ and T can be read off from Eq. (67) in the $|T| \to \infty$ limit, and becomes

$$2\tilde{\ell}\Lambda\sqrt{\frac{2J}{2J + S(5-p)}} = e^T. \tag{72}$$

At the zero-loop level on the worldsheet, the transition amplitude is given by

$$\begin{aligned}\langle t_f, J, S|t_i, J, S\rangle &\simeq e^{-(I+J(\psi_f-\psi_i)+S(\Theta_f-\Theta_i))} \\ &= e^{-(\frac{4}{5-p}J+2S)T} \\ &= \left(\frac{2J+S(5-p)}{2J}\right)^{\frac{2}{5-p}J+S}\frac{1}{\Lambda^{\frac{4}{5-p}J+2S}}\frac{1}{|t_f - t_i|^{\frac{4}{5-p}J+2S}} \quad \text{(at zero loop)}.\end{aligned} \tag{73}$$

The amplitude including the one-loop contribution can be obtained by expanding the coordinates of the particle to the quadratic order in fluctuations around the classical solution (67) by doing an analysis similar to the one for $S = 0$ [38]. As described in Appendix A, we find the following spectrum of the bosonic fluctuations: there are $(p-1)$ fields with mass 1 (from the fluctuations along S^p within AdS_{p+2}), one field with mass 2 (from the one along the radial direction ρ in AdS_{p+2}), and $(7-p)$ fields with mass $\frac{2}{5-p}$ (from the ones along S^{8-p}). We have eight fermionic fluctuations with mass $\frac{7-p}{2(5-p)}$. This spectrum is in contrast to the $S=0$ case[17], as described below Eq. (52). Here, we have two fewer fields with mass 1 than in the $S=0$ case, but have one extra field with mass 2. These compensate each other, making the zero-point energy (the one-loop contribution) for $S\neq 0$ equal to the one for $S=0$.

By including the one-loop contribution, the amplitude becomes

$$\begin{aligned} \langle t_f, J, S | t_i, J, S\rangle &\simeq e^{-(I+J(\psi_f-\psi_i)+S(\Theta_f-\Theta_i))} Z^{(2)}_{J\,\mathrm{bit}} \\ &= e^{-((p-1)J+2S)T} \\ &= \left(\frac{2J+S(5-p)}{2J}\right)^{\frac{p-1}{2}J+S} \frac{1}{\Lambda^{(p-1)J+2S}} \frac{1}{|t_f-t_i|^{(p-1)J+2S}}. \end{aligned} \tag{74}$$

This agrees with the free-field result of the correlator for the operator of the form $\mathrm{Tr}(Z^{J-k}D^S Z^k)$: At zero coupling, the covariant derivative D is just a partial derivative. Inserting one derivative in the operator increases two powers of the coordinate distance in the two-point function.

5. Conclusions

In this paper, we have considered the gauge/gravity correspondence between maximally supersymmetry Yang–Mills theories in $(p+1)$ dimensions and superstrings on the near-horizon limit of the Dp-brane solutions. We computed two-point functions of operators with angular momentum along the AdS directions from the string worldsheet theory.

First, we considered the conventional continuum string theory, which should correspond to the strongly coupled gauge theory. We considered the conformally invariant case of $p=3$, and first considered the folded and spinning string solution found by Gubser et al. [33]. We rotated some coordinates to imaginary, and obtained a string configuration that connects two points on the boundary. We explained that gauge-theory correlators can be obtained as transition amplitudes for the Euclidean string. This formalism is not based on the identification of the energy with the global time, and can be applied to theories without conformal symmetry. We will defer the analysis of $p\neq 3$ for future study. This is an interesting technical challenge, since the isometry of $\mathrm{AdS}_{p+2}\times S^{8-p}$ is not a symmetry of string action due to the position-dependent Weyl factor.

Then, we considered the limit of zero gauge coupling. As in the previous paper [10], we assumed that the string is made of bits, each of which has a single unit of angular momentum along S^{8-p}. In the limit of weak gauge coupling, the string tension is small compared with the scale of angular momenta. At zero coupling, we computed the amplitude by ignoring the interactions among bits. We obtained the free-field correlator of gauge theory, extending the result of Ref. [10] to general operators. We regard this result as an indication of the validity of the approach for weak gauge coupling initiated in Ref. [10].

[17]The reason that we have one less (seven) bosonic fluctuation than in the $S=0$ case (eight) is that we are fixing one more angular momentum here, introducing one more constraint among the fluctuations.

As mentioned at the end of the introduction, our result is based on two unproven assumptions. It is important to clarify whether these assumptions are true. One assumption is that the near-horizon Dp-brane background does not receive $\alpha^{'}$ corrections for general p. This does not seem very unrealistic, since the geometry for $p \neq 3$ has a tensor structure similar to $p = 3$, which is believed to be $\alpha^{'}$ exact [42,43]. Another assumption is that taking into account the quantum effect of fluctuations up to the quadratic order gives the correct answer (when computing the amplitude for a string bit). At the moment, we do not have a clear justification for this. By performing exact quantization of a superparticle in AdS_{p+2} (without the Weyl factor since this does not contribute to the single-bit action, as explained in Sect. 4), we should be able to clarify this point. Alternatively we could compute the next order in the expansion. Since the contributions from the bosonic and fermionic fluctuations cancel each other at all orders for $p = 3$, it might be possible that there is only contribution at the quadratic order for $p \neq 3$.

There are two main directions for future research. One direction is towards understanding the weak-coupling limit of gauge/gravity correspondence [54]. We will be able to incorporate the interactions among the string bits perturbatively. It is highly important whether the perturbative expansion in string tension agrees with the perturbative expansion in gauge theory. If they agree, this can be regarded as a proof of the gauge/gravity correspondence.

Another direction is towards the understanding of gauge theory without conformal symmetry at strong gauge coupling. The string solution along the lines of Sect. 3 for $p \neq 3$ is under study [54]. It is important to study the large-S behavior of that solution. For $p = 3$ there is a characteristic large-S behavior in the scaling dimensions of the form $\log S$. In gauge theory, this comes from gauge fields propagating in the internal lines. Large-S behavior for $p \neq 3$ will give new information for the structure of these gauge theories.

Apart from the above two problems, it would be interesting to extend the analysis in this paper to more general backgrounds. Our formalism of computing the gauge-theory correlator from the string amplitude, described in Sect. 3, would be applicable to general backgrounds without conformal symmetry. Also, the approach to weakly coupled gauge theories based on the string bit picture, described in Sect. 4 and in Ref. [10], will be applicable for backgrounds in which the two assumptions mentioned above are satisfied.

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Appendix A. Fluctuations around classical solution of a particle

In this appendix, we will obtain the massless particle action at the quadratic order in the fluctuations around a classical solution on the near-horizon limit of the Dp-brane solution. We are really interested in the background with Euclidean time and imaginary angular coordinates, but here we will do the analysis in the usual Lorentzian background, since the fluctuations around the former can be obtained simply by rotating the time and two angles to imaginary values.

The bosonic part of the massless particle action in the 10D spacetime is

$$I_{\rm b}=\frac{L^2}{2}\int d\tau\,\eta^{-1}f^2(r)\Big[c^2\Big\{-\cosh^2\rho\,\dot{t}^2+\dot{\rho}^2+\sinh^2\rho\,\left(\cos^2\theta\,\dot{\psi}^2+\dot{\theta}^2+\sin^2\theta\,\dot{\Omega}^2_{p-2}\right)\Big\}$$
$$+\cos^2\tilde{\theta}\,\dot{\tilde{\psi}}^2+\dot{\tilde{\theta}}^2+\sin^2\tilde{\theta}\dot{\tilde{\Omega}}^2_{6-p}\Big], \tag{A1}$$

where

$$c=\frac{2}{5-p}.$$

As in the main text, the coordinates without tildes are for AdS_{p+2} and those with tildes are for S^{8-p}. In this appendix, we take the coordinates to be dimensionless, assuming that they are measured in units of $L=(g_sN)^{1/(7-p)}\ell_s$. We will use global coordinates for AdS_{p+2}, since the classical solution is simple in this coordinate system. The symbol $\dot{\Omega}^2_{p-2}$ is a shorthand for the kinetic term along the $(p-2)$-dimensional sphere. Though we will not need its explicit form in the present analysis, it is given by, e.g.,

$$\dot{\Omega}^2_{p-2}=\dot{\phi}_1^2+(\sin^2\phi_1)\dot{\phi}_2^2+\cdots+(\sin^2\phi_1\cdots\sin^2\phi_{p-3})\dot{\phi}^2_{p-2},$$

if we use a standard metric on S^{p-2} using angular coordinates $\phi_1,\ldots,\phi_{p-2}$. The quantity $\dot{\tilde{\Omega}}^2_{6-p}$ is defined similarly.

The equations of motion for t, ψ, $\tilde{\psi}$ tell us that the global energy E and angular momenta S and J, defined as follows, are conserved:

$$E\equiv P_t=-\frac{\partial\mathcal{L}}{\partial\dot{t}}=L^2c^2\eta^{-1}f^2(r)\cosh^2\rho\,\dot{t}, \tag{A2}$$

$$S\equiv P_\psi=\frac{\partial\mathcal{L}}{\partial\dot{\psi}}=L^2c^2\eta^{-1}f^2(r)\sinh^2\rho\cos^2\theta\,\dot{\psi}, \tag{A3}$$

$$J\equiv P_{\tilde{\psi}}=\frac{\partial\mathcal{L}}{\partial\dot{\tilde{\psi}}}=L^2\eta^{-1}f^2(r)\cos^2\tilde{\theta}\,\dot{\tilde{\psi}}. \tag{A4}$$

There is a massless constraint obtained by varying the action with respect to η:

$$c^2\Big\{-\cosh^2\rho\,\dot{t}^2+\dot{\rho}^2+\sinh^2\rho\,\left(\cos^2\theta\,\dot{\psi}^2+\dot{\theta}^2+\sin^2\theta\,\dot{\Omega}^2_{p-2}\right)\Big\}$$
$$+\left(\cos^2\tilde{\theta}\,\dot{\tilde{\psi}}^2+\dot{\tilde{\theta}}^2+\sin^2\tilde{\theta}\dot{\tilde{\Omega}}^2_{6-p}\right)=0. \tag{A5}$$

The equation of motion for ρ is

$$-\ddot{\rho}+\cosh\rho\sinh\rho\left\{-\dot{t}^2+\cos^2\theta\,\dot{\psi}^2+\dot{\theta}^2+\sin^2\theta\,\dot{\Omega}^2_{p-2}\right\}=0. \tag{A6}$$

We will consider the following classical solution, which represents a particle rotating in AdS with a fixed radius ρ_0, and also rotating along S^{8-p}:

$$t=\tau,\quad \rho=\rho_0,\quad \psi=\tau,\quad \theta=0,$$
$$\tilde{\psi}=c\tau,\quad \tilde{\theta}=0. \tag{A7}$$

The worldline metric is chosen as follows:

$$\eta=f^2(\bar{r})=\bar{r}^{-\frac{3-p}{4}}, \tag{A8}$$

where $\bar{r}$ is a function of τ, which is given by inserting the classical solution (A7) into the coordinate transformations (given in the main text) from the global coordinates on AdS to the radial coordinate r. With this choice of η, the Weyl factor $f(\bar{r})$ disappears from the action. One can see that this solution (A7) satisfies the constraint (A5) and the equation of motion (A6), and that the momenta E, S, J are conserved.

When we Wick-rotate the spacetime coordinates t, ψ, $\tilde{\psi}$, and also the worldline time τ, the classical solution (A7) remains a solution. This solution can be expressed in Poincaré coordinates by performing the coordinate transformations (20), (22) and is given by Eqs. (23), (24), (25) with $\rho = \rho_0$ (or equivalently, as Eq. (67) with $B = \tilde{\ell} \tanh \rho_0$). This trajectory starts from a point on the boundary and returns to another point on the boundary, unlike the Lorentzian solution that rotates near the center of AdS. Therefore, the singularity at the center for the $p \neq 3$ case causes no problem.

A1 Bosonic fluctuations

We now expand the fields (coordinates) of the particle in powers of $1/L$:

$$x^{\mu}_{(\text{tot})} = x^{\mu}_{(0)} + \frac{1}{L} x^{\mu} + \frac{1}{L^2} x^{\mu}_{(2)} + \cdots . \tag{A9}$$

The fields without any subscripts above are really the "total" field. They will be denoted with the subscript (tot) hereafter. For notational simplicity, we will omit the subscript (1) on the first-order fluctuations, which will be used most often in the following. $x^{\mu}_{(0)}$ represents the classical solution (A7).

Before expanding the action, let us first study the constraints that should be satisfied by the fluctuations. The massless constraint (A5) at the first order in $1/L$ becomes

$$c \left\{ -(\cosh^2 \rho_0)\dot{t} + (\sinh^2 \rho_0)\dot{\psi} \right\} + \dot{\tilde{\psi}} = 0. \tag{A10}$$

Now, we assume that the values of the angular momenta S and J are fixed at the zeroth order, and will not change at higher orders, since these are the quantities that we are interested in. (On the other hand, we do not fix the value of E.) By expanding S in Eq. (A3) to the first order and setting it to zero, we get

$$(\sinh \rho_0) \left\{ (2 \cosh \rho_0)\rho + (\sinh \rho_0)\dot{\psi} \right\} = 0. \tag{A11}$$

Also, by expanding J in Eq. (A4) to the first order and setting it to zero, we get

$$\dot{\tilde{\psi}} = 0. \tag{A12}$$

From Eqs. (A10), (A11), and (A12), assuming $\rho_0 \neq 0$, we obtain

$$\dot{t} = -2(\tanh \rho_0)\rho, \tag{A13}$$

$$\dot{\psi} = -2 \left(\frac{1}{\tanh \rho_0} \right) \rho. \tag{A14}$$

When $S = 0$ (i.e., $\rho_0 = 0$), we get $\dot{t} = \dot{\tilde{\psi}} = 0$ from Eqs. (A10) and (A12). (In that case, Eq. (A11) is trivially satisfied since $\sinh \rho_0 = 0$.)

We will now expand the action as

$$I_{\text{b}} = L^2 I_{\text{b}(0)} + L I_{\text{b}(1)} + I_{\text{b}(2)} + \cdots .$$

The action at the first order $I_{(1)}$ vanishes due to the massless constraint described above. The terms relevant for the calculation of the second-order action are as follows[18]:

$$I_{\rm b} = \frac{L^2c^2}{2}\int d\tau\left[-\left\{\cosh^2\rho_0 + (2\cosh\rho_0\sinh\rho_0)\frac{\rho}{L} + (2\cosh^2\rho_0 - 1)\frac{\rho^2}{L^2} + \cdots\right\}\left(1+\frac{\dot{t}}{L}\right)^2\right.$$
$$+\frac{\dot{\rho}^2}{L^2} + \left\{\sinh^2\rho_0 + (2\cosh\rho_0\sinh\rho_0)\frac{\rho}{L} + (2\cosh^2\rho_0 - 1)\frac{\rho^2}{L^2} + \cdots\right\}$$
$$\times\left\{\left(1-\frac{\theta^2}{L^2}+\cdots\right)\left(1+\frac{\dot{\psi}}{L}\right)^2 + \frac{\dot{\theta}^2}{L^2} + \left(\frac{\theta^2}{L^2}+\cdots\right)\dot{\Omega}^2_{p-2}\right\}$$
$$\left.+\frac{1}{c^2}\left\{\left(1-\frac{\tilde{\theta}^2}{L^2}+\cdots\right)\left(c+\frac{\dot{\tilde{\psi}}}{L}\right)^2 + \frac{\dot{\tilde{\theta}}^2}{L^2} + \left(\frac{\tilde{\theta}^2}{L^2}+\cdots\right)\dot{\tilde{\Omega}}^2_{6-p}\right\}\right]. \tag{A15}$$

Taking the quadratic part (the order-L^0 terms), we get

$$I_{\rm b(2)} = \frac{c^2}{2}\int d\tau\left[-(\cosh^2\rho_0)\dot{t}^2 + \dot{\rho}^2 + 4(\cosh\rho_0\sinh\rho_0)\rho(-\dot{t}+\dot{\psi})\right.$$
$$+(\sinh^2\rho_0)\left\{\dot{\psi}^2 + \dot{\theta}^2 + \theta^2\dot{\Omega}^2_{p-2} - \theta^2\right\}$$
$$\left.+\frac{1}{c^2}\left\{\dot{\tilde{\psi}}^2 + \dot{\tilde{\theta}}^2 + \tilde{\theta}^2\dot{\tilde{\Omega}}^2_{6-p} - c^2\tilde{\theta}^2\right\}\right]. \tag{A16}$$

Using the relations (A13)–(A12) to eliminate $\dot{t}$, $\dot{\psi}$, $\dot{\tilde{\psi}}$, it becomes

$$I_{\rm b(2)} = \frac{c^2}{2}\int d\tau\left[\dot{\rho}^2 - 4\rho^2 + (\sinh^2\rho_0)\left(\dot{\theta}^2 + \theta^2\dot{\Omega}^2_{p-2} - \theta^2\right)\right.$$
$$\left.+\frac{1}{c^2}\left\{\dot{\tilde{\theta}}^2 + \tilde{\theta}^2\dot{\tilde{\Omega}}^2_{6-p} - c^2\tilde{\theta}^2\right\}\right]. \tag{A17}$$

By defining the fields (Cartesian coordinates) x_a and y_l as

$$x_a = c(\sinh\rho_0)\theta\Omega_a \qquad (a = 1,\ldots,p-1), \tag{A18}$$

$$y_l = \tilde{\theta}\tilde{\Omega}_l \qquad (l = 1,\ldots,7-p), \tag{A19}$$

where Ω_a ($\tilde{\Omega}_l$) denotes a vector with unit length $\sum_{a=1}^{p-1}\Omega_a^2 = 1$ ($\sum_{l=1}^{6-p}\tilde{\Omega}_l^2 = 1$), representing a point on S^{p-2} (S^{6-p}), and by making a suitable redefinition of fields by constant factors, the action becomes

$$I_{\rm b(2)} = \frac{1}{2}\int d\tau\left[\dot{\rho}^2 - 4\rho^2 + \sum_{a=1}^{p-1}\left(\dot{x}_a^2 - x_a^2\right) + \sum_{l=1}^{7-p}\left(\dot{y}_l^2 - c^2y_l^2\right)\right]. \tag{A20}$$

When $S = 0$, there are $(p+1)$ fields with mass 1, coming from the fluctuations in the AdS_{p+2} directions [38][19]. By contrast, we have $(p-1)$ fields (x_a) with mass 1 from the angular directions

[18]We do not assign any factors of L on $\dot{\Omega}^2_{p-2}$ and $\dot{\tilde{\Omega}}^2_{6-p}$, which are (squares of the τ-derivative of) angles whose corresponding radius is classically zero ($\theta_{(0)} = \tilde{\theta}_{(0)} = 0$). We will assign a factor of $1/L$ to the Cartesian coordinates x_a and y_l (whose origin corresponds to $\theta = 0$ and $\tilde{\theta} = 0$) defined in Eqs. (A18) and (A19). Since θ and $\tilde{\theta}$ already have a factor of $1/L$, we do not put any factor of L on Ω_a and $\tilde{\Omega}_l$.

[19]The result for $S = 0$ is obtained as follows. We have $\rho_0 = 0$ and $\dot{t} = \dot{\tilde{\psi}} = 0$, as explained above. We assign no factors of L on the fluctuations of the angles on S^p whose corresponding radius is zero classically $\rho_{(0)} = 0$, for the same reason as mentioned in the previous footnote. The factor in the curly bracket in the third line of Eq. (A15) is now replaced with $\dot{\Omega}^2_p$ (without any factor of L). Then, from the

along S^p in AdS_{p+2} and one field (from the radial direction ρ) with mass 2 (mass-squared =4). The mass of the fluctuations in the S^{8-p} directions is the same as in the $S = 0$ case; we have (7 − p) fields with mass $c = \frac{2}{5-p}$.

The fact that there is one less physical degree of freedom (seven in total) of fluctuations than in the $S = 0$ case (eight) is that we are fixing one more angular momentum S now, introducing an extra constraint (A11). The one-loop quantum contribution to the amplitude (the contribution from the zero-point energy of fluctuations) is the same as in the $S = 0$ case, since the contribution from the field ρ compensate for the one less degree of freedom.

The quadratic action for the fluctuations around the tunneling null geodesic with Euclidean time and imaginary angular directions is given simply by changing the signs of the mass-squared terms in Eq. (A20).

A2 Fermionic fluctuations

We will find that the mass of the fermionic fluctuations for $S \neq 0$ is the same as in the $S = 0$ case obtained in Ref. [38]. Here we will summarize only the main points, and refer the reader to Ref. [38] for more details.

We will consider the Green–Schwarz type action for the fermionic particle, obtained from the Green–Schwarz superstring action at the quadratic order in the fermions [46] by ignoring the worldsheet spatial derivative. For type IIA theories, the fermionic part of the particle action is

$$I_{\mathrm{f}} = -\frac{i}{2\pi}\int d\tau\, \Theta^T \Gamma_0 \eta^{-1} \partial_\tau x^\mu \Gamma_\mu \mathcal{D}_\tau \Theta. \tag{A21}$$

We will take the vielbein $\eta = f^2(r)$ as in the bosonic case. Θ is a 32-component spinor in 10D spacetime. We take a real representation, $\Theta^* = \Theta$. The part $\mathcal{D}_\tau \Theta$ is defined as

$$\mathcal{D}_\tau \Theta = \nabla_\tau \Theta + \Omega \partial_\tau x^\mu \Gamma_\mu \Theta. \tag{A22}$$

The first term is the covariant derivative defined in the usual manner,

$$\nabla_\tau \Theta = \partial_\tau \Theta + \frac{1}{4}\partial_\tau x^\mu \omega_\mu{}^{\hat{\nu}\hat{\sigma}} \Gamma_{\hat{\nu}\hat{\sigma}}, \tag{A23}$$

where the indices with hats ($\hat{\mu}$, $\hat{\nu}$, ...) are those for the local Lorentz frame. The (p + 2)-form field strength (sourced by the Dp-brane with even p for type IIA; for $p = 4$, we consider the dual of the 6-form, which is a 4-form) enters the action through

$$\Omega = -\frac{1}{16}e^\phi \left(\Gamma_{11}\Gamma^{\mu_1\mu_2}F_{\mu_1\mu_2} - \frac{1}{12}\Gamma^{\mu_1\mu_2\mu_3\mu_4}F_{\mu_1\mu_2\mu_3\mu_4}\right). \tag{A24}$$

We write the vielbein as

$$e_\mu{}^{\hat{\nu}} = Lf(\bar{r})\hat{e}_\mu{}^{\hat{\nu}}, \tag{A25}$$

where the vielbein with the tilde $\hat{e}_\mu{}^{\hat{\nu}}$ is the one for $\mathrm{AdS}_{p+2} \times S^{8-p}$, represented as

$$\begin{aligned}&\hat{e}^{\hat{t}} = c\cosh\rho dt, \quad \hat{e}^{\hat{\rho}} = cd\rho, \quad \hat{e}^{\hat{\theta}} = c\sinh\rho d\theta, \quad \hat{e}^{\hat{\psi}} = c\sinh\rho\cos\theta d\psi,\\ &\hat{e}^{\hat{a}} = c\sinh\rho\sin\theta d\Omega_a, \quad \hat{e}^{\hat{\tilde{\psi}}} = \cos\tilde{\theta}d\tilde{\psi}, \quad \hat{e}^{\hat{\tilde{\theta}}} = d\tilde{\theta}, \quad \hat{e}^{\hat{l}} = \sin\tilde{\theta}d\Omega_l.\end{aligned} \tag{A26}$$

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first three lines in Eq. (A15), we obtain the part of $I_{\mathrm{b}(2)}$ containing the fluctuations from the AdS_{p+2} directions as follows:

$$\frac{c^2}{2}\int d\tau \left\{\dot{\rho}^2 - \rho^2 + \rho^2\dot{\Omega}_p^2\right\} = \frac{1}{2}\int d\tau \sum_{a=1}^{p+1}\left\{\dot{x}_a^2 - x_a^2\right\},$$

where $x_a = c\rho\Omega_a$, with Ω_a (a = 1, ..., p + 1) being a unit vector that parametrizes a point on S^p. The part of $I^{(2)}$ containing the fluctuations from the S^{8-p} directions is the same as in the $S \neq 0$ case.

For the vielbein of the form (A25), the spin connection is written as

$$\omega_\mu{}^{\hat{a}\hat{b}} = (\partial_\nu \log f)\left(\hat{e}^{\nu\hat{b}}\hat{e}^{\mu\hat{a}} - \hat{e}^{\nu\hat{a}}\hat{e}^{\mu\hat{b}}\right) + \hat{\omega}_\mu{}^{\hat{a}\hat{b}}, \tag{A27}$$

where the spin connection $\hat{\omega}_\mu{}^{\hat{a}\hat{b}}$ with the hat denotes the one for the vielbein (A26) for $\mathrm{AdS}_{p+2} \times S^{8-p}$.

Since the action (A21) is already quadratic in the fermions, we just have to substitute the classical solution (A7), which will be called $x^\mu_{(0)}$, into the bosonic fields x^μ. The following combination often appears in the action:

$$\partial_\tau x^\mu_{(0)}\Gamma_\mu = cf(\bar{r})\left(\cosh\rho_0\Gamma_{\hat{t}} + \sinh\rho_0\Gamma_{\hat{\psi}} + \Gamma_{\hat{\tilde{\psi}}}\right) = cf(\bar{r})\left(\hat{\Gamma}_{\hat{t}} + \Gamma_{\hat{\tilde{\psi}}}\right). \tag{A28}$$

In the last expression, we have defined the gamma matrix $\hat{\Gamma}_{\hat{t}}$ by the following transformation from $\Gamma_{\hat{t}}$ and $\Gamma_{\hat{\psi}}$:

$$\begin{pmatrix} \hat{\Gamma}_{\hat{t}} \\ \hat{\Gamma}_{\hat{\psi}} \end{pmatrix} = \begin{pmatrix} \cosh\rho_0 & \sinh\rho_0 \\ \sinh\rho_0 & \cosh\rho_0 \end{pmatrix}\begin{pmatrix} \Gamma_{\hat{t}} \\ \Gamma_{\hat{\psi}} \end{pmatrix}. \tag{A29}$$

This is a Lorentz boost in the $\hat{t}$–$\hat{\psi}$ plane, and the gamma matrices with the hat satisfy the same anti-commutation relations as the original ones. The difference to the $S = 0$ case studied in Ref. [38] is that we have $\hat{\Gamma}_{\hat{t}}$ in place of $\Gamma_{\hat{t}}$.

We take the following representation of the 10D gamma matrices. For the $\hat{t}$ and $\hat{\tilde{\psi}}$ directions, we take

$$\hat{\Gamma}_{\hat{t}} = \Gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Gamma_{\hat{\tilde{\psi}}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{A30}$$

where each block is a 16 × 16 matrix. For gamma matrices in the remaining eight directions collectively denoted by $\Gamma_{\hat{i}}$ (with the one for the $\hat{\psi}$ direction taken to be $\hat{\Gamma}_{\hat{\psi}}$ defined in Eq. (A29)), we take

$$\Gamma_{\hat{i}} = \begin{pmatrix} \gamma_i & 0 \\ 0 & -\gamma_i \end{pmatrix}. \tag{A31}$$

In this representation,

$$\Gamma_{11} = \begin{pmatrix} \gamma_9 & 0 \\ 0 & -\gamma_9 \end{pmatrix} \tag{A32}$$

with $\gamma_9 = \prod_{i=1}^{8}\gamma_i$. We decompose Θ as

$$\Theta = \begin{pmatrix} \hat{\theta} \\ \theta \end{pmatrix}. \tag{A33}$$

The action contains the following factor, which is proportional to a projection operator $\hat{\Gamma}_+ \equiv \Gamma_0(\hat{\Gamma}_{\hat{t}} + \Gamma_{\hat{\tilde{\psi}}})/2$:

$$\Gamma_0\partial_\tau x^\mu_{(0)}\Gamma_\mu = -2cLf(\bar{r})\hat{\Gamma}_+ = -2cLf(\bar{r})\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{A34}$$

This projects Θ on to the lower component, so only half the degrees of freedom θ appear in the action. The other half $\hat{\theta}$ is unphysical, and can be gauged away by the κ-symmetry of the Green–Schwarz action [38,46].

To rewrite the kinetic term (which depends on the covariant derivative ∇_τ), we use the following field redefinition:

$$\Theta^{(\mathrm{old})} = \left(\frac{Lf(\bar{r})}{2c}\right)^{\frac{1}{2}} e^{-\frac{\tau}{4}\partial_\tau x^\mu_{(0)}\hat{\omega}_\mu{}^{\hat{a}\hat{b}}\Gamma_{\hat{a}\hat{b}}}\Theta^{(\mathrm{new})}. \tag{A35}$$

We chose the first factor on the right-hand side so that the $f(\bar{r})$-dependent factor in the action disappears. At the same time, the τ-derivative of this factor cancels the first term in the spin connection (A27). (This corresponds to the fact that the massless spinor is Weyl invariant.) We chose the second factor so that the contribution to the covariant derivative that depends on the spin connection $\hat{\omega}_\mu{}^{\hat{a}\hat{b}}$ (which is constant when evaluated with the classical solution (A7)) is canceled. In terms of the redefined (new) field, the kinetic term becomes

$$I_{\mathrm{f,kin}} = +\frac{i}{2\pi}\int d\tau\, \Theta^T \hat{\Gamma}_+ \partial_\tau \Theta. \tag{A36}$$

The mass term of the fermion comes from the $(p+2)$-form field strength. We first note [38]

$$\Omega = \frac{7-p}{8} L^{-1} f^{-1}(\bar{r}) \Gamma_{(p)}, \tag{A37}$$

where

$$\Gamma_{(p=0)} = \Gamma_{11}\Gamma^{\hat{s}\hat{x}_1}, \quad \Gamma_{(p=2)} = -\Gamma^{\hat{s}\hat{x}_1\hat{x}_2\hat{x}_3}, \quad \Gamma_{(p=4)} = -\Gamma_{11}\Gamma^{\hat{s}\hat{x}_1\cdots\hat{x}_5}. \tag{A38}$$

Then, the Ω-dependent term of the action becomes

$$\begin{aligned} I_{\mathrm{f},\Omega} &= +\frac{i}{2\pi}\int d\tau\, \Theta^{(\mathrm{old})T}\eta^{-1}\left(\Gamma_0\partial_\tau x^\mu_{(0)}\Gamma_\mu\right)\Omega\Gamma_0\left(\Gamma_0\partial_\tau x^\nu_{(0)}\Gamma_\nu\right)\Theta^{(\mathrm{old})} \\ &= +\frac{i}{2\pi}\int d\tau\, \Theta^T\hat{\Gamma}_+\left(\frac{7-p}{2(5-p)}\right)\Gamma_{(p)}\Gamma_0\hat{\Gamma}_+\Theta, \end{aligned} \tag{A39}$$

where the field Θ in the last line is the new field defined in Eq. (A35). The $(p+2)$-form field strength enters in the action through $f(\bar{r})\Omega$. Since this is a constant, it is not affected by the classical solution of the bosonic field, and $I_{\mathrm{f},\,\Omega}$ takes the same value as in the $S=0$ case obtained in Ref. [38].

Finally, by combining the kinetic and mass terms, and writing the action in terms of the 16-component field θ, we get

$$I_{\mathrm{f}} = -\frac{i}{2\pi}\int d\tau\, \theta^T\left(\partial_\tau\theta + m_{\mathrm{f}(p)}\gamma_{(p)}\theta\right) \tag{A40}$$

where

$$m_{\mathrm{f}(p)} = \frac{7-p}{2(5-p)} \tag{A41}$$

and

$$\gamma_{(p=0)} = \gamma_9\gamma_1, \quad \gamma_{(p=2)} = \gamma_{123}, \quad \gamma_{(p=4)} = -\gamma_9\gamma_{12345}. \tag{A42}$$

The Euclidean action is obtained from Eq. (A41) by the standard procedure $\tau \to i\tau_{\mathrm{E}}$, and multiplying the action by $-i$. The quantization of fermion described by this action is straightforward (see Ref. [38]). The action for the type IIB theory can be obtained in exactly the same manner as above, and we find that the mass is also given by Eq. (A41) (with odd p) [38].

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