Cluster structure of light nuclei

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Abstract. The cluster structure of $k\alpha$ nuclei with k=2, 3, 4 (^{8}Be, ^{12}C, ^{16}O) is briefly reviewed. Recent work on ^{20}Ne (k=5) is presented. Current work on systems $k\alpha+x$ neutrons or protons, x=1 (^{9}Be, ^{9}B; ^{13}C, ^{13}N; ^{21}Ne, ^{21}Na) is summarized.

1. Introduction

The cluster structure of light nuclei has been the subject of many investigations since the seminal work of Wheeler [1]. In 1965, Brink [2],[3] suggested specific geometric configurations for nuclei composed of k α-particles, hereafter referred as $k\alpha$ nuclei. In particular, the suggested configurations of the ground state were for k=2 (^{8}Be) a dumbbell configuration with Z_2 symmetry, for k=3 (^{12}C) an equilateral triangle with D_{3h} symmetry, for k=4 (^{16}O) a tetrahedron with T_d symmetry, for k=5 (^{20}Ne) a trigonal bipyramid with D_3 (or D_{3h}) symmetry and for k=6 (^{24}Mg) a triaxial bipyramid with D_{2h} symmetry. Since 2001, we have started a systematic investigation of these cluster structures. A summary of work for k=2, 3, 4 (^{8}Be, ^{12}C, ^{16}O) and recent work on k=5 (^{20}Ne) is given in the following sections [4-7]. In 1996, von Oertzen [8] suggested to extend Brink's idea to $k\alpha+x$ neutrons (or protons) structures. Since 2017, we have started a systematic investigation of these structures in terms of a cluster shell model [9]. A summary of work performed so far for x=1 systems (^{9}Be, ^{9}B; ^{13}C, ^{13}N; ^{21}Ne, ^{21}Na) [10-12] will be given in the following sections.

2. Structures of type $k\alpha$

In order to understand what are the signatures of specific geometric configurations for nuclei composed of k α-particles and to provide explicit analytic formulas which can be easily compared with experiment, we have developed methods to determine the allowed states of a quantum system with discrete symmetry group, G.

2.1. ^8Be (Z_2), ^{12}C (D_{3h}), ^{16}O (T_d)

For ^{8}Be with Z_2 symmetry, the allowed representations are denoted by A, and the allowed values of the angular momentum, L, and parity, P, are A: $L^P=0^+, 2^+, 4^+, \ldots$

For ^{12}C with D_{3h} symmetry, the allowed representations are denoted by A (singly degenerate) and E (doubly degenerate), and the allowed values of angular momentum, L, and parity, P, are A: $L^P=0^+, 2^+, 3^-, 4^\pm, \ldots$; E: $L^P=1^-, 2^\pm, 3^\pm, \ldots$. Note that both positive and negative parity states sit in the same representation (rotational band). This property is called parity doubling.

For ^{16}O with T_d symmetry, the allowed representations are denoted by A (singly degenerate), E (doubly degenerate) and F (triply degenerate), and the allowed values of the angular momentum, L, and parity, P, are A: $L^P=0^+, 3^-, 4^+, 6^\pm, \ldots$; E: $L^P=2^\pm, 4^\pm, 5^\pm, 6^\pm, \ldots$; F: $L^P=1^-, 2^+, 3^\pm, 4^\pm, 5^{-\pm}, \ldots$. Again both positive and negative parity states sit in the same representation. (The group T_d has two singly degenerate representations A_1, A_2, one doubly degenerate representation E, and two triply degenerate representations F_1, F_2, but only $A_1 \equiv A$, E and $F_2 \equiv F$ appear in the vibrational, bosonic, states).

One can then provide analytic formulas for energy levels:

$$Z_2: \quad E(v,L) = E_0 + \omega(v+\frac{1}{2}) + BL(L+1) \tag{1a}$$

$$D_{3h}: \quad E(v_1,v_2,L,K) = E_0 + \omega_1(v_1+\frac{1}{2}) + \omega_2(v_2+1) + BL(L+1) + (A-B)(K\mp 2\ell_2)^2 \tag{1b}$$

$$T_d: \quad E(v_1,v_2,v_3,L) = E_0 + \omega_1(v_1+\frac{1}{2}) + \omega_2(v_2+1) + \omega_3(v_3+\frac{3}{2}) + BL(L+1) \tag{1c}$$

and for B(EL) values along the rotational ground state band:

$$Z_2: \quad B(EL;0\to L) = \left(\frac{Ze\beta^L}{2}\right)^2 \frac{2L+1}{4\pi}\left[2+2P_L(-1)\right] \tag{2a}$$

$$D_{3h}: \quad B(EL;0\to L) = \left(\frac{Ze\beta^L}{3}\right)^2 \frac{2L+1}{4\pi}\left[3+6P_L(-\frac{1}{2})\right] \tag{2b}$$

$$T_d: \quad B(EL;0\to L) = \left(\frac{Ze\beta^L}{4}\right)^2 \frac{2L+1}{4\pi}\left[4+12P_L(-\frac{1}{3})\right] \tag{2c}$$

One can also obtain analytic expressions for form factors in electron scattering but they will not be given here for lack of space. They are reviewed in [4].

The expressions (1) and (2) can be compared with experiment. The occurrence of Z_2 symmetry in the energy levels of ^{8}Be has been emphasized by many authors. The evidence here is meager since only few energy levels are known. Fig. 1 shows a comparison between theory and experiment.

The occurrence of D_{3h} symmetry in ^{12}C has been the subject of many recent investigations [13], [14]. Fig. 2 shows clear evidence for the occurrence of this symmetry both in the ground state, A representation, and in the vibrations, A and E representations.

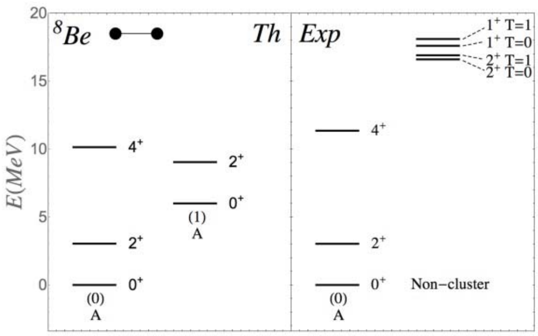

Figure 1. Comparison between theoretical and experimental energy levels in ^{8}Be. Reproduced from [4] with permission.

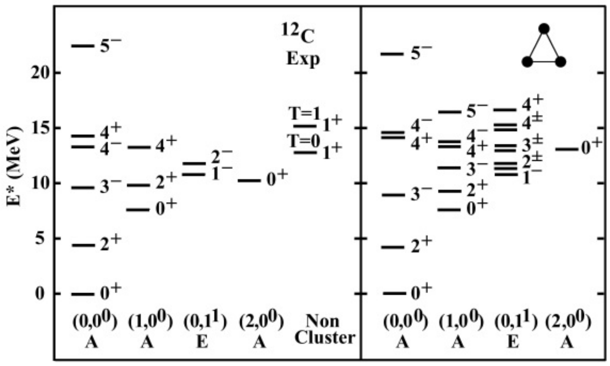

Figure 2. Comparison between theoretical and experimental energy levels in ^{12}C. Reproduced from [14] with permission.

The occurrence of T_d symmetry in ^{16}O was emphasized by Robson [15] and it has been more recently revisited in [6]. Fig. 3 shows clear evidence for the occurrence of T_d symmetry both in the ground state band, A representation, and in the vibrations, A, E and F representations.

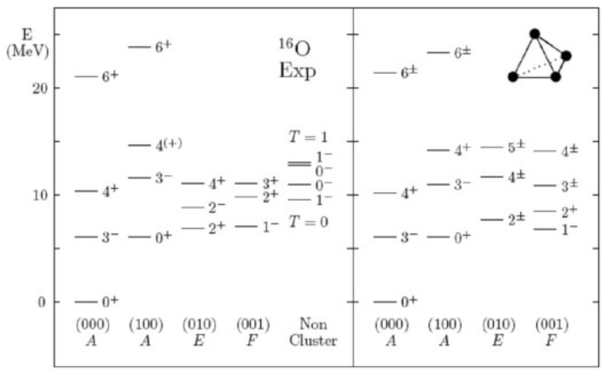

Figure 3. Comparison between theoretical and experimental energy levels in ^{12}C. Reproduced from [4] with permission.

B(EL) values also provide strong evidence for cluster structure. Table 1 gives a comparison between theory and experiment for B(EL) values in the ground state band of ^{8}Be. In this and in the following tables, B(EL) values are in e^2fm^{2L} and E is in keV. In table 1, E(Th*) is calculated by E(keV)=510L(L+1) and B(EL;$L^P\to0^+$) by Eq.(2a) with β=2.0 fm.

Table 1. Comparison between theory and experiment for B(EL) values in the ground state band ^{8}Be.

B(EL;$L^P\to0^+$)	Th	Exp	E(L^P)	Th*	Exp
B(E2;$2^+\to0^+$)	20.4	20.0±0.8 (21.9(9)W.u.)	E(2^+)	3060	3030
B(E2;$4^+\to0^+$)	326.1		E(4^+)	10200	11350

Table 2 gives a comparison between theory and experiment in the ground sate band of ^{12}C. Here E(Th*)=730L(L+1) and β=1.9fm. Note the large value of B(E3) which is a signature of D_{3h} symmetry.

Table 2. Comparison between theory and experiment for B(EL) values in the ground state band of ^{12}C.

B(EL;L^P→0$^+$)	Th	Exp	E(L^P)	Th*	Exp
B(E2;2$^+$→0$^+$)	9.3	7.6±0.4 [4.65(26)W.u.]	E(2$^+$)	4400	4439
B(E3;3$^-$→0$^+$)	84	103±17 [12(2)W.u.]	E(3$^-$)	9640	9641
B(E4;4$^+$→0$^+$)	68		E(4$^+$)	14670	14080

Table 3 gives a comparison between theory and experiment in the ground state band of ^{16}O. Here E(Th*)=511L(L+1) and β=1.9fm extracted from the elastic form factor in electron scattering. Note the large value of B(E4) which is a signature of T_d symmetry.

Table 3. Comparison between theory and experiment for B(EL)values in the ground state band of ^{16}O.

B(EL;L^P→0$^+$)	Th	Exp	E(L^P)	Th*	Exp
B(E3;3$^-$→0$^+$)	181	205±10 [13.5(7)W.u.]	E(3$^-$)	6132	6130
B(E4;4$^+$→0$^+$)	338	378±133 [3.7(13)W.u.]	E(4$^+$)	10220	10356
B(E6;6$^+$→0$^+$)	8245		E(6$^+$)	21462	21052

2.2. ^{20}*Ne* (D_{3h})

The determination of expected states L^P for the *5α* configuration with D_{3h} symmetry (bi-pyramid, fig.4) requires a study of all possible vibrations of this structure. These vibrations can be classified by

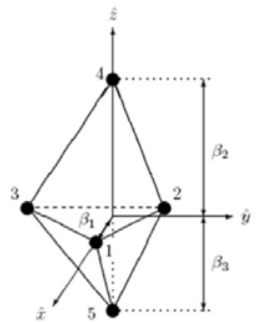

Figure 4. A bi-pyramid with D_3 symmetry. If β_2=β_3 the bi-pyramid has D_{3h} symmetry (considered here). Reproduced from [7] with permission.

the representations Γ of D_{3h}. Each representation Γ of D_{3h} (called here irrep in short) contains a discrete (but infinite) number of values of K^P, where K is the projection of the angular momentum on the symmetry axis *z* and P is the parity. The results are given in table 4, where the representations are labelled either by their crystal physics notation, first column, or by the molecular physics notation, second column. The representations with a prime are related to those with a double prime by a change in parity P→-P. On top of each value of K there is a rotational band with L=K, K+1, K+2, …, except for K=0 for which the values of L are even or odd. Note that only the representations Γ_1≡A'$_1$, Γ_4≡A'$_2$, Γ_5≡E', Γ_6≡E" appear for vibrational (bosonic) states. Also, the representations Γ_1 and Γ_5 which appear in the vibrational states of the triangle were denoted by Γ_1≡A, Γ_5≡E in Sect.2.1.

Table 4. K^P bands in a representation Γ of D_{3h}

Γ	Γ	K^P
Γ_1	A'$_1$	0^+(L=even), 3^-, 6^+, …
Γ_2	A"$_1$	0^-(L=even), 3^+, 6^-, …
Γ_3	A'$_2$	0^+(L=odd), 3^-, 6^+, …
Γ_4	A"$_2$	0^-(L=odd), 3^+, 6^-, …
Γ_5	E'	1^-, 2^+, 4^+, 5^-, …
Γ_6	E"	1^+, 2^-, 4^-, 5^+, …

The 5α bi-pyramidal configuration has 6 vibrational states (3 singly degenerate and 3 double degenerate). The energies are given by

$$E([v],K,L)=E_0+\sum_{i=1}^{6}\omega_i v_i+B_{x[v]}L(L+1)+\left[B_z-B_x\right]_{[v]}K^2 \qquad (3)$$

where [v]≡[v_1,v_2,v_3,v_4,v_5,v_6]. A remarkable feature of ^{20}Ne is that all 6 vibrations appear to have been observed with vibrational frequency ω and rotational parameter B as given in Table 5. Note that all vibrational states appear to have about the same frequency ω~6-8 MeV and rotational parameter B~120-140 keV, with exception of the v_6 (K^P = 2^-) vibration which has a lower frequency and larger rotational parameter. The observed rotation-vibration bands are also shown in fig.5.

Table 5. Observed vibrational states in ^{20}Ne

	Γ	K^P	ω(MeV)	B(keV)
g.s.	A'$_1$	0^+	0.00	212
v_1	A"$_2$	0^-	5.52	137
v_2	A'$_1$	0^+	6.72	127
v_3	A'$_1$	0^+	7.19	130
v_4	E'	1^-	8.59	134
		2^+	8.53	112
v_5	E'	1^-	8.42	147
		2^+	8.71	130
v_6	E"	1^+	9.68	124
		2^-	4.10	145

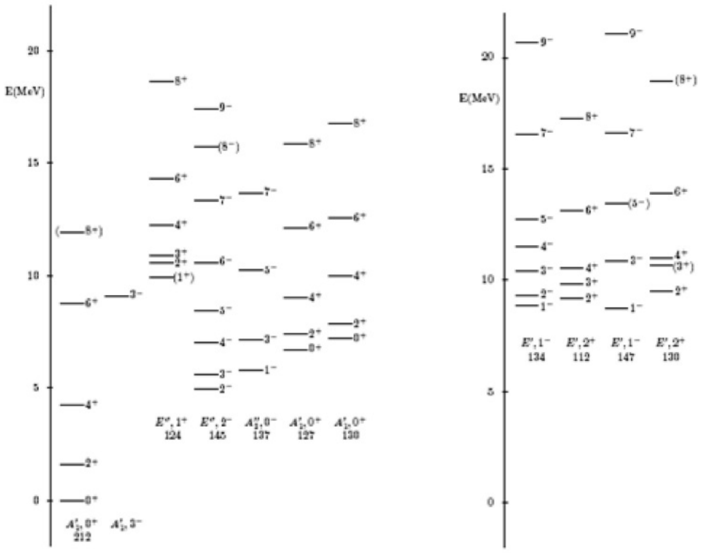

Figure 5. Observed rotation-vibration bands in ^{20}Ne. Reproduced from [7] with permission.

3. Alpha-clustering in *kα+x* structures

These structures were suggested by von Oertzen [8].

3.1. x=1

Adding one neutron (proton) to the *kα* nuclei discussed in Sect.2 gives ^{9}Be, ^{9}B (2α+1 neutron or proton); ^{13}C, ^{13}N (3α+1 neutron or proton); ^{17}O, ^{17}F (4α+1 neutron or proton); ^{21}Ne, ^{21}Na (5α+1 neutron or proton). Work in ^{9}Be, ^{9}B has been completed [10]. Fig.6 shows a comparison between experimental and theoretical levels in ^{9}Be.

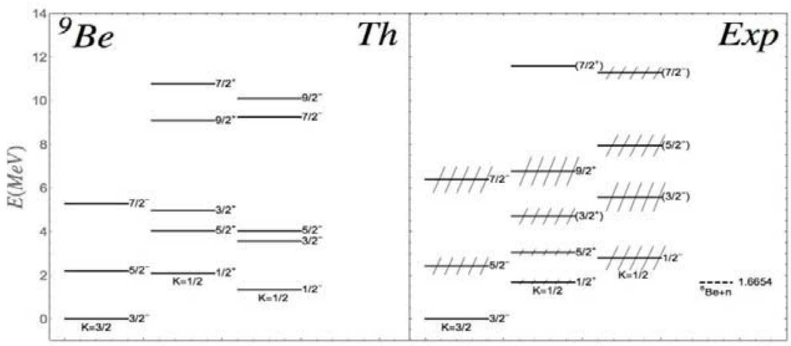

Figure 6. Comparison between theoretical and experimental energy levels of ^{9}Be. The dashed region is given by the width of the states. Reproduced from [10] with permission.

Preliminary work in ^{13}C, ^{13}N has also been completed [11]. It has been found that these nuclei display strong evidence for the occurrence of D'_{3h} symmetry, where D'_{3h} is the double group of D_{3h} with both spinor and tensor representations. This group describes fermionic states. Its representations are

denoted by $E^{(-)}_{1/2}$, $E^{(+)}_{1/2}$, $E_{3/2}$. The values of K^P contained in these representations are $E^{(-)}_{1/2}$: $1/2^-$, $5/2^+$, $7/2^+$, … , $E^{(+)}_{1/2}$: $1/2^+$, $5/2^-$, $7/2^-$, … , $E_{3/2}$: $3/2^-$, $3/2^+$, $9/2^+$, … . Note the alternating parity of these values of K^P which is a signature of triangular symmetry. In fig. 7 a comparison between the observed rotational bands and those expected on the basis of D'_{3h} symmetry is shown. The observation of the side bands is evidence for the occurrence of D'_{3h} symmetry.
Work is in progress for ^{17}O, ^{17}F. Work in ^{21}Ne, ^{21}Na has been completed [12], but it will not be discussed here for lack of space.

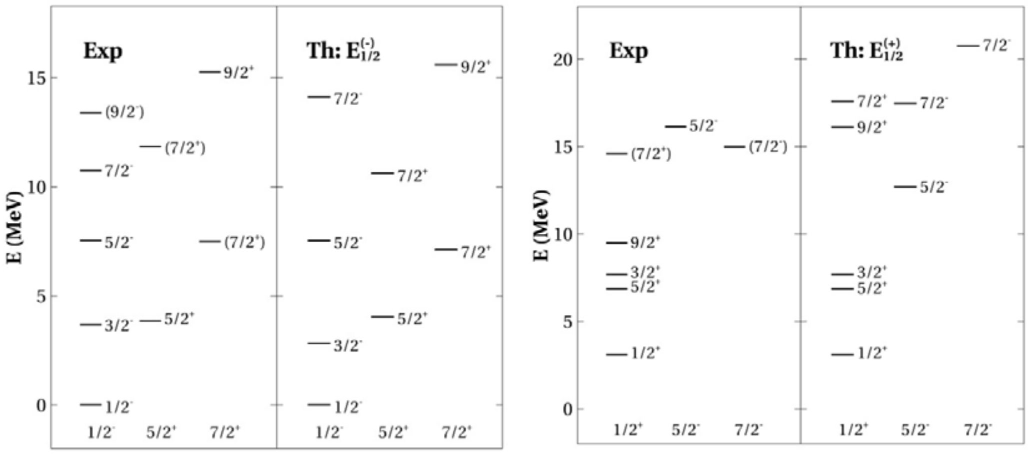

Figure 7. Left: Comparison between experimental and theoretical energy levels for the ground state band of ^{13}C assigned to the representation $E^{(-)}_{1/2}$ of D'_{3h}. Right: Same as left but for the first excited band assigned to the representation $E^{(+)}_{1/2}$ of D'_{3h}. Reproduced from [11] with permission.

3.2. x=2

Adding two particles to the $k\alpha$ structures of Sect. 2 gives ^{10}Be, ^{10}B, ^{10}C (2α+2 particles); ^{14}C, ^{14}N, ^{14}O (3α+2 particles); ^{18}O, ^{18}F, ^{18}Ne (4α+2 particles); ^{22}Ne, ^{22}Na, ^{22}Mg (5α+2 particles). Work on ^{10}Be within the framework of the cluster shell model [9] has been initiated, as well as a preliminary investigation of ^{14}C and ^{22}Ne. Clustering appears to be present in these nuclei.

4. Summary and conclusions

Signatures of clustering are properties of spectra and electromagnetic transition rates, especially parity doubling, unusual structure of rotational bands, and enhanced E2, E3, E4, E5, … transitions. The signature of clustering that distinguishes it from other collective models is the occurrence of vibrations with specific values of K^P and L^P given by the discrete symmetry of clustering. Clustering appears to extend at least to angular momentum J=6 in k=2, 3, 4, 5 systems, and up to excitation energies of the order of 25 MeV. (Recent unpublished work on k=6 systems, appears to indicate that also A=24 nuclei are clusterized.) Cluster appears to survive the addition of fermions, at least up to x=1. (Recent unpublished work appears to indicate that also x=2 display clustering phenomena.) Discrete symmetries appear to play a crucial role in the structure of light nuclei. These nuclei are further examples of symmetry in physics.

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