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Stable bundles on hypercomplex surfaces

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Abstract: A hypercomplex manifold is a manifold equipped with three complex structures I, J, K satisfying the quaternionic relations. Let M be a 4-dimensional compact smooth manifold equipped with a hypercomplex structure, and E be a vector bundle on M. We show that the moduli space of anti-self-dual connections on E is also hypercomplex, and admits a strong HKT metric. We also study manifolds with (4,4)-supersymmetry, that is, Riemannian manifolds equipped with a pair of strong HKT-structures that have opposite torsion. In the language of Hitchin's and Gualtieri's generalized complex geometry, (4,4)-manifolds are called "generalized hyperkähler manifolds". We show that the moduli space of anti-self-dual connections on M is a (4,4)-manifold if M is equipped with a (4,4)-structure.

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1. Introduction

1.1. Instanton moduli and stable holomorphic bundles

Ever since it was established by Donaldson and Uhlenbeck–Yau, the correspondence between instantons and stable holomorphic vector bundles on Kähler manifolds has been a constant source of new information about both instantons and holomorphic vector bundles.

One of the most immediate applications of this correspondence is the following. Let (X, g) be a compact Riemannian 4-dimensional manifold admitting complex structures I_1, I_2 such that the metric g is Kähler with respect to both I_1 and I_2. This happens, for instance, when X is a hyperkähler 4-manifold, that is, a K3 surface or a compact complex torus.

The moduli space $\mathcal{M}$ of instantons on X depends only on a metric. From the Donaldson–Uhlenbeck–Yau theorem, we obtain that $\mathcal{M}$, as a topological space, is identified with the moduli of stable holomorphic bundles E with $c_1(E) = 0$ on (X, I_1) and on (X, I_2). Therefore, $\mathcal{M}$ is equipped with a pair of complex structures, one induced from I_1, another from I_2.

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When X is hyperkähler, this can be used to show that the moduli of instantons is hyperkähler as well. This result was obtained by A. Tyurin ([30]) and generalized in [31] to hyperkähler manifolds of arbitrary dimension.

For non-Kähler complex manifolds, a version of the Donaldson–Uhlenbeck–Yau theorem was obtained by Buchdahl [6] for surfaces and by Li and Yau [25] for general Hermitian manifolds. In this context, the correspondence between instantons and stable holomorphic vector bundles is usually called *the Kobayashi-Hitchin correspondence*. This result is not new, but the full impact of the Buchdahl–Li–Yau theorem in the geometry of non-Kähler manifolds is still not completely realized, though a book by Lübke and Teleman ([26]) studies it in wonderful detail.

Let (X, I, g) be a compact complex Hermitian manifold, and $\omega \in \Lambda^{1,1}(X)$ its Hermitian form. If $\partial\bar{\partial}(\omega^{\dim X - 1}) = 0$, then the Hermitian metric on X is called *a Gauduchon metric*. P. Gauduchon ([12]) has proven that such a metric exists in each conformal class, and is unique up to a constant.

When (X, I, g) is equipped with a Gauduchon metric, the Li-Yau theorem identifies the instanton moduli space with the space of stable holomorphic bundles (see Section 3 for details).

In this context, an instanton is a Hermitian bundle E with a connection A whose curvature 2-form F_A is of type $(1,1)$ and is pointwise orthogonal to the Hermitian form:

$$F_A \in \Lambda^{1,1}(X), \quad F_A \perp \omega.$$

When X is a complex surface, these conditions are equivalent to the anti–self–duality of A (see Section 3).

Now, assume that (X, g) is a Riemannian manifold admitting two complex structures I_1, I_2, such that g is Hermitian and Gauduchon with respect to both I_1 and I_2. Then the Buchdahl-Li-Yau theorem implies that the moduli $\mathcal{M}$ of anti–seld–dual connections on X is equipped with two complex structures, induced by I_1 and I_2. It is not generally known how these complex structures relate to each other. However, if X is equipped with an additional geometric structure (HKT- or bi-Hermitian), then it is possible to recover a similar structure on the moduli space.

1.2. Bismut connection and HKT-structures

Definition 1.1.
Let (M, I, g) be a complex Hermitian manifold. A connection $\nabla : TM \longrightarrow TM \otimes \Lambda^1 M$ is called *Hermitian* if $\nabla I = \nabla g = 0$. Consider its torsion $T_1 \in \Lambda^2 \otimes TM$, and let $T \in \Lambda^2 \otimes \Lambda^1 M$ be the tensor obtained from T_1 via the isomorphism $TM \cong \Lambda^1 M$ provided by g. A Hermitian connection is called *a Bismut connection*, or *a connection with skew-symmetric torsion*, if T is *skew-symmetric*, that is, lies in $\Lambda^3 M \subset \Lambda^2 \otimes \Lambda^1 M$. The 3-form T is called *the torsion form* of Bismut connection.

Theorem 1.1.
Let (M, I, g) be a complex Hermitian manifold. Then M admits a Bismut connection ∇, which is unique. Moreover, its torsion form is equal to $Id\omega$.

Proof. See [3], [9]. □

Remark 1.1.
Clearly, if $d\omega = 0$, then the Bismut connection is torsion-free, and thus coincides with the Levi-Civita connection. Theorem 1.1 can therefore be used to show that the Levi-Civita connection on a Kähler manifold satisfies $\nabla I = 0$.

Connections with skew-symmetric torsion play an important role in string physics (see for example [21]). In the physics literature, a complex Hermitian manifold (M, I, g) with a Bismut connection is called *a KT-manifold* (Kähler torsion manifold). If, in addition, the torsion 3–form is closed, then (M, I, g) is called *a strong KT-manifold*. By Theorem 1.1, a manifold is therefore strong KT if and only if $\partial\bar{\partial}\omega = 0$. For complex surfaces, this is equivalent to g being a Gauduchon metric.

There are several other structures based on Bismut connections which are even more important.

Definition 1.2.
Let (M, g) be a Riemannian manifold, and I_-, I_+ be Hermitian complex structures. Consider the corresponding Bismut

connections, and suppose that their torsion 3-forms satisfy $T_+ = -T_-$, $dT_\pm = 0$. Then (M, g, I_+, I_-) is called *bi-Hermitian*.

Bi-Hermitian structures appear naturally in several different (and seemingly unrelated) contexts. In differential geometry, these were studied by Apostolov, Gauduchon and Grantcharov ([1]), who obtained classification results in the case when $\dim_\mathbb{R} M = 4$; they showed in particular that if M is of Kähler type, then it is a rational surface, a torus or a K3 surface. In physics, such structures were studied as early as 1984 by Gates, Hull and Roček ([10]), in connection to $D = 2$, $N = 4$ supersymmetric σ-models. More recently, bi-Hermitian manifolds have appeared both in mathematics and in string physics, due to the work of N. Hitchin and M. Gualtieri on generalized complex geometry. In his Ph. D. thesis, [16], Gualtieri explored the notion of generalized complex manifold, which was first developed by Hitchin ([17]). He defined generalized Kähler manifolds, and described them in terms of more classical differential-geometric structures. More precisely, Gualtieri found that a generalized Kähler structure on a manifold M is uniquely determined by a bi-Hermitian structure on M whose torsion form T_+ (called *flux* by physicists) is exact. There is also a slight generalization of generalized Kähler structures, called *twisted generalized Kähler structures*, and these are equivalent to bi-Hermitian structures with arbitrary (not necessarily exact) torsion form.

In this sense, the notions of generalized Kähler structure and bi-Hermitian structure are synonymous.

Another notion, also due to physicists, is the notion of HKT-manifold, which was suggested by Howe and Papadopoulos in [19] and has been much studied since then.

Definition 1.3.

Let (M, g) be a Riemannian manifold, and I, J, K be complex structures on (M, g) which are Hermitian and satisfy the quaternionic relations $IJ = -JI = K$. Then (M, g, I, J, K) is called *a quaternionic Hermitian hypercomplex manifold*. If, in addition, the Bismut connections associated to I, J and K coincide, then (M, g, I, J, K) is called *an HKT-manifold*; and if the Bismut torsion is closed, then (M, g, I, J, K) is called *strong HKT*.

For more details and examples of hypercomplex manifolds and HKT-geometry, please see Section 2.

Remark 1.2.

An orthogonal connection is uniquely determined by its torsion (see for example [9]). Therefore, the Bismut connections associated to I, J, K are equal if and only if the corresponding torsion forms are equal:

$$I d\omega_I = J d\omega_J = K d\omega_K.$$

Consequently, a hypercomplex Hermitian structure I, J, K on a Riemannian manifold (M, g) is HKT with respect to g if and only if the torsion 3-forms corresponding to I, J, K are equal.

We finally consider (4,4)-supersymmetry structures on Riemannian manifolds. These structures were also introduced by Gates, Hull and Roček in [10], and can be formulated in Hitchin's and Gualtieri's language as *generalized hyperkähler structures*. These structures were explored in more detail in [7]; also see [20] and [14].

Definition 1.4.

Let (M, g) be a Riemannian manifold, and let I_+, J_+, K_+ and I_-, J_-, K_- be two triples of complex structures on (M, g) which are hypercomplex Hermitian and HKT with respect to g. Denote by T_+, T_- the corresponding torsion forms. Then $(M, g, I_+, J_+, K_+, I_-, J_-, K_-)$ is called *a (4,4)-manifold*, or *a generalized hyperkähler manifold* if the torsion forms $T_\pm$ are closed and satisfy $T_+ = -T_-$.

Remark 1.3.

Note that (4,4)-manifolds are equipped with a plethora of bi-Hermitian structures. Indeed, take a complex structure V_+ induced by the first quaternion action, and a complex structure U_- induced by the second one. Then the Bismut torsion of V_+ is equal to T_+, and the Bismut torsion of U_+ is equal to T_-, implying that (M, g, V_+, U_-) is a bi-Hermitian structure.

A trivial (and not very interesting) example of a (4,4)-manifold can be obtained starting from a hyperkähler manifold (M, I, J, K, g). Let I', J', K' be another hyperkähler structure on (M, g); such a triple I', J', K' can be obtained, for instance, by twisting I, J, K by a non-zero quaternion. Since M is Kähler, the Bismut connection coincides with the Levi-Civita connection, and its torsion vanishes. Then $(M, g, I, J, K, I', J', K')$ is a (4,4)-manifold. Clearly, all (4,4)-manifolds with trivial torsion are obtained this way.

There are not many examples of (4,4)-manifolds with non-trivial torsion. In fact, all known examples (except those provided by Theorem 1.3) are homogeneous or the product of a homogeneous (4,4)-manifold and a hyperkähler manifold. Examples of homogeneous (4,4)-manifolds are not very difficult to obtain. Let G be a semisimple Lie group admitting a left-invariant hypercomplex structure I_+, J_+, K_+ (such hypercomplex structures were constructed and completely classified by D. Joyce in [22]). Replacing the left multiplication by the right one, we may also choose a right-invariant hypercomplex structure I_-, J_-, K_- on G. The Killing metric g on G is HKT (see for example [15]), and its torsion T_+ is equal to the fundamental 3-form of G. In particular, T_+ is closed. If we replace the left group multiplication by the right one, then the fundamental 3-form of G becomes the opposite of the previous one. The torsion form T_- of I_-, J_-, K_- therefore satisfies $T_- = -T_+$, and $(G, g, I_+, J_+, K_+, I_-, J_-, K_-)$ is a (4,4)-manifold.

The main results of the paper are the following. Consider a bi-Hermitian manifold X of real dimension 4. It was then shown in [18] that the moduli space of anti-self-dual connections on X is also a bi-Hermitian space. We prove similar results for hypercomplex, HKT, and (4,4)-structures:

Theorem 1.2.
Let (X, I, J, K, g) be a compact strong HKT-manifold of real dimension 4, and E be a smooth complex vector bundle on X. Denote by $\mathcal{M}$ the moduli space of gauge-equivalence classes of anti-self-dual connections (instantons) on E. Then $\mathcal{M}$ is equipped with a natural strong HKT-structure.

Proof. See section 3.2. □

Remark 1.4.
Compact hypercomplex 4-manifolds were classified in [4], where it was shown that a compact hypercomplex 4-manifold is either a torus, a K3-surface, or a special type of Hopf surface (see section 2.2). Each of these manifolds admits a strong HKT-structure (see section 2.2). Therefore, the moduli of stable holomorphic $SL(n, \mathbb{C})$-bundles on a given hypercomplex surface is again hypercomplex.

From Theorem 1.2, we deduce the following theorem.

Theorem 1.3.
Let $(X, I_\pm, J_\pm, K_\pm, g)$ be a compact (4,4)-manifold of real dimension 4, and E be a smooth complex vector bundle on X. Denote by $\mathcal{M}$ the moduli space of gauge-equivalence classes of anti-self-dual connections (instantons) on E. Then $\mathcal{M}$ is equipped with a natural (4,4)-structure.

Proof. See section 2.2. □

2. Hypercomplex structures and HKT-metrics

2.1. Hypercomplex manifolds

A smooth manifold M equipped with three complex structure operators $I, J, K : TM \to TM$ that satisfy the quaternionic identities

$$IJ = -JI = K \tag{1}$$

is said to be *hypercomplex* or to admit a *hypercomplex structure*. The complex structures I, J, and K induce other almost complex structures on M of the form $L := aI + bJ + cK$ for all real numbers a, b, c such that $a^2 + b^2 + c^2 = 1$; that these

almost complex structures are in fact integrable follows from Obata [23, 28]. Given such a complex structure L on M, we will denote by (M, L) the manifold M considered as a complex manifold with respect to L.

In this paper, we study the moduli spaces of instantons (solutions to the anti–self–dual Yang–Mills equations) on compact hypercomplex 4–manifolds; we show, in particular, that these moduli spaces admit a natural hypercomplex structure which is induced from the hypercomplex structure on the 4–manifolds.

Compact hypercomplex 4–manifolds were classified by Boyer who showed that if (X, I, J, K) is a compact hypercomplex 4–manifold, then X is either a torus, a $K3$ surface, or a quaternionic Hopf surface (see [4], Theorem 1). Recall that a *quaternionic Hopf surface X* can be defined as the quotient of the non–zero quaternions $\mathbb{H} - \{0\}$ by a cyclic group generated by some $q \in \mathbb{H}$ with $|q| > 1$, where $\langle q \rangle$ acts on $\mathbb{H} - \{0\}$ by right multiplication:

$$X := (\mathbb{H} - \{0\})/\langle q \rangle. \tag{2}$$

Note that the action of left multiplication by i, j, and k commutes with the action of right multiplication by q; hence, the hypercomplex structure $\{I, J, K\}$ on $\mathbb{H}$ induced by left multiplication by i, j, k, respectively, descends to a hypercomplex structure on the Hopf surface. Furthermore, any quaternionic Hopf surface in Ma. Kato's classification ([24], Proposition 8) is isomorphic to a finite cover of a Hopf surface of the form (2), thus acquiring the same hypercomplex structure from $\mathbb{H}$.

It has been known for some time that instanton moduli spaces on tori and K3 surfaces admit hypercomplex structures. In this article, we show that this is true for hypercomplex Hopf surfaces; this is done by identifying the instanton moduli spaces with moduli spaces of stable bundles, implying, in particular, that we will consider metrics on these 4–manifolds which are Hermitian[1] with respect to every complex structure (see section 3.1).

In [22], page 747, D. Joyce suggested that the space of instantons on quaternionic Hopf surfaces can be obtained through quaternionic reduction. Similar results were obtained independently by Oliver Nash and Gil Cavalcanti in unpublished papers [27] and [8], using the methods of hypercomplex reduction (Nash) and reduction of Courant algebroids applied to generalized Kähler geometry (Cavalcanti).

A Riemannian metric g on a hypercomplex manifold (M, I, J, K) is called *hyperhermitian* or *quaternionic Hermitian* if it is Hermitian with respect to every complex structure L on M induced by I, J, K. In addition, if a hyperhermitian metric g is Kähler[2] for all complex structures on M, then it is called *hyperkähler*; the Euclidean metric on $\mathbb{H}$ is an example of a hyperkähler metric. Note that hyperhermitian metrics exist on all hypercomplex manifolds (M, I, J, K); indeed, one can construct a hyperhermitian metric on M by taking any Riemannian metric on M and averaging it over the natural $SU(2)$–action on M (induced by multiplication by the quaternions). However, hyperkähler metrics only exist if the underlying manifold admits Kähler metrics; for instance, quaternionic Hopf surfaces do not admit Kähler metrics since they have odd first Betti number, implying that they do not admit Kähler structures.

One can endow tori and $K3$ surfaces with hyperkähler metrics (for details, see [2]); quaternionic Hopf surfaces are therefore the only compact hypercomplex 4–manifolds on which hyperhermitian metrics are never hyperkähler. One can nonetheless construct hyperhermitian metrics on quaternionic Hopf surfaces which are Gauduchon[3] with respect to every complex structure. Consider, for instance, a quaternionic Hopf surface of type (2). Let r be the Euclidean length on $\mathbb{H}$ and let $\varphi := r^2$. The 2–forms

$$\omega_L := \frac{dd^c_L \varphi}{\varphi},$$

where d^c_L denotes the twisted differential, are then $\langle q \rangle$–invariant, and descend to 2–forms on X which induce the same metric g on X, that is, $g(\cdot, \cdot) = \omega_L(\cdot, L \cdot)$ for all complex structures L. The metric g is thus hyperhermitian. Moreover, a

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[1] *A Riemannian metric g on a smooth manifold M with complex structure L is called* Hermitian *if $g(LX, LY) = g(X, Y)$ for all vector fields X and Y on M.*

[2] *One can associate to any Hermitian metric g on (M, L) the 2–form $\omega_L(\cdot, \cdot) := g(L\cdot, \cdot)$, called the* Hermitian form *of g. A Hermitian metric g is then said to be* Kähler *if its Hermitian form ω_L is d–closed.*

[3] *A Hermitian metric g on an n–dimensional complex manifold (M, L) is called* Gauduchon *if the $(n-1)$–th power its Hermitian form ω_L is dd^c_L–closed, where d^c_L is the twisted differential which acts as $(-1)^m L \circ d \circ L$ on m–forms. Note that although Kähler metrics are Gauduchon, the converse is in general not true.*

direct computation shows that

$$d_I^c \omega_I = d_J^c \omega_J = d_K^c \omega_K = H,$$

where H is a d-closed 3-form, implying that g is Gauduchon with respect to every complex structure on X induced by I, J, K.

2.2. HKT-metrics and (4,4)-symmetry

Consider a hypercomplex manifold (M, I, J, K). A hyperhermitian metric g on M is then called an *HKT-metric* if

$$d_I^c \omega_I = d_J^c \omega_J = d_K^c \omega_K = H,$$

for some 3-form H, where ω_L is the Hermitian form of g and d_L^c is the twisted differential, corresponding to the complex structures $L = I, J, K$. Moreover, if H is d-closed, then g is said to be a *strong HKT-metric*. Note that for any complex structure L on M, the skew-symmetric torsion of the Bismut connection on (M, L) is equal to the 3-form $-2H$ (see [19], or [15] Proposition 1). Furthermore, an HKT-metric g is hyperkähler if and only if $H = 0$ (hyperkähler metrics are in fact strong HKT). However, on a manifold that does not admit Kähler metrics, one has $H \neq 0$, hence the terminology HKT which stands for "hyperkähler metric with torsion".

There exists another characterisation of HKT-metrics. Let g be a hyperhermitian metric on (M, I, J, K) and let ω_I, ω_J, and ω_K be its Hermitian forms for I, J, and K, respectively. Set

$$\Omega := \omega_J + \sqrt{-1}\omega_K.$$

Then Ω is a $(2, 0)$-form on (M, I), which can be used to determine whether the metric is HKT. Indeed, one can show that the metric g is HKT if and only if it satisfies the condition $\partial\Omega = 0$, where $\partial = \frac{1}{2}(d + \sqrt{-1}d_I^c)$ (see [19] and [15], Proposition 2). Consequently, since $\partial\Omega$ is a $(3, 0)$-form on (M, I), if M is a 4-manifold, any hyperhermitian structure is an HKT-structure. This implies that on hypercomplex 4-manifolds, strong HKT-metrics are equivalent to hyperhermitian metrics that are Gauduchon with respect to all complex structures. Every hypercomplex compact 4-manifold therefore admits a strong HKT-metric: referring to section 2.1, tori and K3 surfaces admit hyperkähler metrics, and quaternionic Hopf surfaces admit metrics which are Gauduchon with respect to every complex structure.

Let us now consider the quaternionic Hopf surface

$$X := (\mathbb{H} - \{0\})/\langle q \rangle$$

with $q \in \mathbb{R}$. One can then endow X with two natural hypercomplex structures. We have seen that left multiplication by i, j, and k on $\mathbb{H}$ induces a hypercomplex on X, which we now denote I_+, J_+, K_+. The other hypercomplex structure on X corresponds to right multiplication by i, j, and k on $\mathbb{H}$ (since q is real, its action on $\mathbb{H}$ commutes with the action of right multiplication by i, k, and k); we will denote this second hypercomplex structure I_-, J_-, K_-. Note that any hypercomplex structure on X induced by one of these two hypercomplex structures is orientation preserving. Moreover, one can verify that the 2-forms $(dd_L^c\varphi)/\varphi$, where $\varphi = r^2$ and r is the Euclidean length on $\mathbb{H}$, induce the same metric on X which is a strong HKT-metric for both hypercomplex structures I_+, J_+, K_+ and I_-, J_-, K_-. In fact,

$$d_{I_+}^c \omega_{I_+} = d_{J_+}^c \omega_{J_+} = d_{K_+}^c \omega_{K_+} = H,$$

and

$$d_{I_-}^c \omega_{I_-} = d_{J_-}^c \omega_{J_-} = d_{K_-}^c \omega_{K_-} = -H,$$

for some d-closed 3-form H. Finally, the two families of hypercomplex structures are independent, in the sense that no complex structure induced by I_+, J_+, K_+ can be written as a linear combination of I^-, J^-, K^-, and vice-versa. Hence, every pair (L_+, L_-) with $L = I, J,$ or K defines a bi-Hermitian structure X. A pair of strong HKT structures that satisfy the above properties is known as a $(4, 4)$-structure (see Definition 1.4). The Hopf surface endowed with its two hypercomplex structures I_+, J_+, K_+ and I_-, J_-, K_- is then an example of $(4, 4)$-symmetry.

It then follows from Theorems 1.2 and 3.4 that the natural L^2-metric g_{L^2} on the instanton moduli space $\mathcal{M}$ is strong HKT with respect to the hypercomplex structures induced by both I_+, J_+, K_+ and I_-, J_-, K_-. Moreover, a result of Hitchin's [18] shows that each pair (L_+, L_-) induces a bi-Hermitian structure on $\mathcal{M}$ with respect to the L^2 metric g_{L^2}. This implies that the instanton moduli space also admits a $(4, 4)$-structure.

3. Instanton moduli spaces

3.1. Hermitian-Einstein connections and stable bundles

Let X be a compact complex surface with fixed Gauduchon metric g and Hermitian form ω; in particular, we have $\partial\bar{\partial}\omega = 0$. Consider a smooth vector bundle E on X and let h be a Hermitian metric in E. The space of h-unitary connections in E is denoted $\mathcal{A}(E, h)$.

Recall that a connection A in $\mathcal{A}(E, h)$ is called g-*Hermitian-Einstein* if its curvature 2-form F_A is of type $(1,1)$ and satisfies

$$\sqrt{-1}\Lambda_g F_A = \gamma_A \cdot \mathrm{id}_E$$

for some $\gamma_A \in \mathbb{R}$, where Λ_g is the contraction of 2-forms by ω. Note that all Hermitian-Einstein connections are integrable and therefore induce holomorphic structures in E. Moreover, irreducible g-Hermitian-Einstein connections give rise to g-stable holomorphic structures in E, where g-stability is defined as follows.

Stability with respect to Gauduchon metrics is an extension of Mumford-Takemoto stability and thus requires the notion of degree. Given the Gauduchon metric g on X, the *degree* of a holomorphic line bundle L on X is defined, up to a multiplicative constant, by

$$\deg L := \frac{1}{2\pi} \int_M F \wedge \omega,$$

where F is the curvature of a Hermitian connection on L, compatible with $\bar{\partial}_L$. Since any two such forms F differ by a $\partial\bar{\partial}$-exact form and $\partial\bar{\partial}\omega = 0$, the degree does not depend on the choice of connection.

Note that flat line bundle have degree zero since the curvature of any connection on such bundles is zero; in particular, the trivial line bundle has degree zero. Furthermore, if the metric g is Kähler, then we get the usual topological degree; otherwise, the degree is not a topological invariant, as it can take continua of values in $\mathbb{R}$.

The *degree* of a torsion-free coherent sheaf $\mathcal{E}$ on X is given by

$$\deg(\mathcal{E}) := \deg(\det\mathcal{E}),$$

where $\det\mathcal{E}$ is the determinant line bundle of $\mathcal{E}$, and the *slope of* $\mathcal{E}$ is defined as

$$\mu(\mathcal{E}) := \deg(\mathcal{E})/\mathrm{rk}(\mathcal{E}).$$

A torsion-free coherent sheaf $\mathcal{E}$ on X is then said to be g-*(semi)stable* if and only if for every proper coherent subsheaf $\mathcal{S} \subset \mathcal{E}$ we have

$$\mu(\mathcal{S}) \leq \mu(\mathcal{E}),$$

with strict inequality for g-stable bundles. One can then show that a holomorphic vector bundle $\mathcal{E}$ is g-stable if and only if it admits an irreducible g-Hermitian-Einstein connection; this was done by Buchdahl [6] for surfaces and Li and Yau [25] for all Hermitian manifolds. The one-to-one correspondence between irreducible Hermitian-Einstein connections and stable holomorphic structures in a smooth vector bundle is known both as the Donaldson-Uhlenbeck-Yau correspondence and the Kobayashi-Hitchin correspondence; a comprehensive reference on the subject is the book [26].

Recall that a connection A on E is called *anti-self-dual* (ASD) if its curvature F_A satisfies the equation:

$$*F_A = -F_A,$$

or equivalently, if F_A is a matrix of anti-self-dual 2-forms. Since the anti-self-dual forms on a Hermitian 4-manifold (M, J, g) consist of $(1,1)$-forms which are orthogonal to the Hermitian form ω of g, a connection A is ASD if and only if it is g-Hermitian-Einstein and $\Lambda_g F_A = 0$. Irreducible anti-self-dual connections in E therefore induce g-stable holomorphic structures of degree zero in E.

Let us now consider a compact hypercomplex 4-manifold (X, I, J, K) equipped with a strong HKT-metric g. Let E be a smooth complex vector bundle on X and h be a Hermitian metric in E. An h-unitary connection A in E is said to be *hyperholomorphic* if it is integrable with respect to every complex structure L on X. Note that anti-self-dual forms on X are of type $(1,1)$ with respect every complex structure L on X induced by I, J, K. Anti-self-dual connections in E are therefore hyperholomorphic. The converse is also true:

Theorem 3.1.
Let E be a vector bundle with Hermitian metric h on a compact strong HKT 4-manifold (X, I, J, K, g). Then, an h-unitary connection A in E is hyperholomorphic if and only if A is anti-self-dual.

Proof.   See for example [31], sections 1 and 2, for a proof in the case where (X, I, J, K) admits a hyperkähler metric. The arguments used in [31] extend, however, to all hypercomplex 4-manifolds. □

Consequently, since anti-self-dual forms on a hyperhermitian 4-manifold (X, I, J, K, g) are orthogonal to the Hermitian form ω_L of g for all complex structures L on X, we see that a connection in E induces a g-stable holomorphic structure in E with respect to all complex structures on X if and only if it is anti-self-dual.

In the next section, we study moduli spaces of connections on compact strong HKT 4-manifolds (X, I, J, K, g). We show in particular that these moduli spaces admit hypercomplex structures; this is done by identifying these moduli spaces with moduli spaces of g-stable holomorphic structures, for all complex structures on the 4-manifolds. We therefore only consider connections that induce g-stable holomorphic structures for all complex structures on X, i.e., anti-self-dual connections.

3.2. Hypercomplex structures and HKT-metrics

Let (X, I, J, K, g) be a compact strong HKT 4-manifold and let E be a smooth complex vector bundle on X. The space $\mathcal{A}^{ASD}$ of all irreducible anti-self-dual connections (instantons) in E is then a $\mathcal{G}$-principal bundle, where $\mathcal{G}$ is the group of gauge transformations of E. The quotient space

$$\mathcal{M} := \mathcal{A}^{ASD}/\mathcal{G}$$

is the moduli space of gauge equivalence classes of anti-self-dual connections in E. Moreover, the L^2 metric

$$(a_1, a_2) = -\int_M \mathrm{tr}(a_1 \wedge *a_2) \tag{3}$$

on $\mathcal{A}^{ASD}$ descends to a metric g_{L^2} on the moduli space $\mathcal{M}$.

Referring to the previous section, instantons correspond to connections in E that are integrable with respect to every complex structure on X induced by the quaternions. The moduli space $\mathcal{M}$ can therefore be identified via the Kobayashi-Hitchin correspondence with the moduli space $\mathcal{M}_L^{st}$ of isomorphism classes of g-stable holomorphic structures in E, for any complex structure L on X. The moduli space $\mathcal{M}$ thus inherits a natural complex structure from $\mathcal{M}_L^{st}$, for every complex structure L on X, which can be described as follows.

Recall that the tangent space to the moduli space $\mathcal{M}$ at any point $[A]$ can be identified with the horizontal subspace at A of any connection on the principal $\mathcal{G}$-bundle

$$\mathcal{P} := \mathcal{A}^{ASD} \to \mathcal{M}.$$

Moreover, since the difference between any two connections in E is a 1-form with values in $sl(E)$, where sl denotes trace-free endomorphisms, then every element of $\mathcal{A}^{ASD}$ is of the form $A + a$ for some $a \in A^1(sl(E))$. Suppose that one fixes a complex structure L on X. The horizontal subspace at A is then chosen to be the set of 1-forms a such that $\Lambda_g d_L^c a = 0$ (where the subscript A in d_L^c is suppressed for clarity), whereas the vertical subspace is the tangent space of the $\mathcal{G}$-orbit through A, giving us the following local model:

$$T_{[A]}\mathcal{M} = \{a \in A^1(sl(E)) \mid d_A^+ a = 0 \text{ and } \Lambda_g d_L^c a = 0\} \tag{4}$$

(see for example [26] for more details).

The advantage of using this particular connection on the $\mathcal{G}$-bundle $\mathcal{P}$ is twofold. The complex structure $\tilde{L}$ on $\mathcal{M}$ induced from the natural complex structure on $\mathcal{M}_L^{st}$ has a very simple expression at any given point $[A]$. Indeed, note that the

complex structure L on X decomposes $A^d(sl(E))$ into components $A^{p,q}(sl(E))$ with $p + q = d$; given this decomposition, the complex structure $\tilde{L}$ on $\mathcal{M}$ is the operator

$$\tilde{L}(a) := \sqrt{-1}\left(a^{0,1} - a^{1,0}\right),\tag{5}$$

for any $a \in T_{[A]}\mathcal{M}$. Furthermore, the metric g_{L^2} on $\mathcal{M}$ is Hermitian with respect to $\tilde{L}$, and has the following properties:

Theorem 3.2 (Lübke-Teleman).
The natural L^2 metric g_{L^2} on $(\mathcal{M}, \tilde{L})$ is Hermitian and its Hermitian form $\tilde{\omega}_{\tilde{L}}$ is such that:

(i) *Let θ be the curvature of the connection on the principal $\mathcal{G}$-bundle $\mathcal{P}$ which has horizontal subspaces (4), and denote by $\bar{a}_i$ any horizontal lift of $a_i \in T_{[A]}\mathcal{M}$ to $\mathcal{A}^{ASD}$. Then,*

$$\tilde{\omega}_{\tilde{L}}(a_1, a_2) = \int_M \omega_L \wedge \operatorname{tr}(\bar{a}_1 \wedge \bar{a}_2),$$

and

$$d^c_{\tilde{L}}\tilde{\omega}(a_1, a_2, a_3) = \frac{1}{3} \sum_{\sigma \in S_3} (-1)^{\operatorname{sign}\sigma} \int_M d^c_L \omega \wedge \operatorname{tr}\left(\theta(\bar{a}_{\sigma(1)}, \bar{a}_{\sigma(2)})\bar{a}_{\sigma(3)}\right).\tag{6}$$

(ii) $dd^c_{\tilde{L}}\tilde{\omega}_{\tilde{L}} = 0$.

Proof. For details see [26], Theorem 5.3.6 and Lemma 5.3.7. □

Remarks 3.1.
Given any element $a \in A^1(sl(E))$ we have:

$$d^*_A a = \Lambda_g d^c_L a + *(d^c_L \omega_L \wedge a).$$

The horizontal slices (4) can then be described as

$$\{a \in A^1(sl(E)) \mid d^+_A a = 0 \text{ and } d^*_A a = *(d^c_L \omega_L \wedge a)\},\tag{7}$$

which implies the following.
(i) If the metric g is Kähler with respect to L, then we have

$$d^*_A a = \Lambda_g d^c_L a$$

since $d^c_L \omega = 0$; in this case (7) reduces to

$$\{a \in A^1(sl(E)) \mid d^+_A a = 0 \text{ and } d^*_A a = 0\},$$

so that (4) is the usual local model for the tangent space to $\mathcal{M}$ (namely the orthogonal complement in $\mathcal{A}^{ASD}$ to the tangent space of the gauge orbit at A, with respect to the L^2 metric (3)).
(ii) If the metric g is strong HKT, then

$$d^c_I \omega_I = d^c_J \omega_J = d^c_K \omega_K = H\tag{8}$$

for some d-closed 3-form H on X. Consequently, given the description (7) of the horizontal spaces, we see that the tangent space to $\mathcal{M}$ at $[A]$ is the same for all complex structures L; one can therefore compose the complex structures $\tilde{L}$ on $\mathcal{M}$. Moreover, our choice of connection on $\mathcal{P}$ is independent of the complex structure, so that its connection matrix θ is the same for all complex structures L. This, combined with (8), (6), and Theorem 3.2 (ii), gives us that

$$d^c_{\tilde{I}}\omega_{\tilde{I}} = d^c_{\tilde{J}}\omega_{\tilde{J}} = d^c_{\tilde{K}}\omega_{\tilde{K}} = \tilde{H},$$

where $\tilde{H}$ is a d-closed 3-form.

Referring to Remark 3.1 (i), the complex structures $\tilde{I}$, $\tilde{J}$, and $\tilde{K}$ on $\mathcal{M}$ can be composed; moreover, these complex structures satisfy the quaternionic identities (1), since we have the following:

Lemma 3.1.

The complex structures I and J on X induce complex structures $\tilde{I}$ and $\tilde{J}$ on $\mathcal{M}$ that anti-commute.

Proof. A section a of $A^1(sl(E))$ can be written locally as

$$a = \Sigma a_i \otimes s_i,$$

with $a_i \in A^1(X)$ and $s_i \in sl(E)$. For any complex structure L on X, one therefore has

$$\tilde{L}(a) = -\Sigma L(a_i) \otimes s_i,$$

which is independent of the local trivialisation. Hence, since $IJ = -JI$, we have that $\tilde{I}\tilde{J}(a) = -\tilde{J}\tilde{I}(a)$ for all $a \in A^1(sl(E))$. □

We therefore have the following results:

Theorem 3.3.

Let (X, I, J, K) be a compact hypercomplex 4-manifold and let g be a strong HKT-metric on X. Let E be a fixed smooth complex vector bundle on X. The moduli space $\mathcal{M}$ of gauge-equivalence classes of anti-self-dual connections on E then admits a hypercomplex structure. □

Theorem 3.4.

Let (X, I, J, K) be a compact hypercomplex 4-manifold and let g be a strong HKT-metric on X. Fix a smooth complex vector bundle E on X and consider the moduli space $\mathcal{M}$ of gauge-equivalence classes of anti-self-dual connections on E. The L^2 metric on $\mathcal{M}$ is then strong HKT.

Proof. This follows from Theorem 3.2 and Remark 3.1 (ii). □

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