

E. N. Glass

Acceleration and classical electromagnetic radiation

Received: 16 December 2006 / Accepted: 22 February 2007
© Springer Science+Business Media, LLC 2008

Abstract Classical radiation from an accelerated charge is reviewed along with the reciprocal topic of accelerated observers detecting radiation from a static charge.

Keywords Field theory, Radiation, Accelerating charges

1 Introduction

It is generally accepted by all physicists that when a charge is accelerated in an inertial frame, observers in that frame detect the emission of electromagnetic radiation. During the first half of the twentieth century, measurements and theory combined to form a complete framework for this part of classical electrodynamics. The radiated power of an accelerated charge is expressed by the Larmor formula and goes as the square of the charge times the square of the acceleration. A clear analysis leading to the Larmor formula is given in Jackson's text [1] and in Rohrlich's text [2], to cite only two of many. Classical radiation from an accelerated charge is reviewed in Part I.

Radiation can be observed when the relative acceleration between charge and observer is non-zero. Part II reviews the issue of accelerated observers detecting radiation from static charges. Mashhoon [3; 4] has emphasized the concept of locality and its role in measurements made by noninertial observers.

For a different view of radiation detected by accelerating observers, free fall in a Reissner-Nordström (RN) manifold is studied. Details of the RN metric are presented in Appendix A, and a radial free fall frame is developed in Appendix B.

This review commemerates Bahram Mashhoon's 60th birthday

E. N. Glass Department of Physics University of Michigan Ann Arbor, MI USA
englass@umich.edu · E. N. Glass Department of Physics University of Windsor Windsor, ON Canada

Fig. 1 Forward null cone

2 Part I

The conformal symmetry of Maxwell's equations in Minkowski spacetime allows a motionless Coulomb charge to be mapped to uniformly accelerated hyperbolic motion. (For finite times this is a special case of general bounded accelerations). This motion satisfies Rohrlich's local criterion [2] for electromagnetic radiation which requires projecting the electromagnetic stress–energy 4-momentum of a point charge into an inertial observer's rest frame. The Lorentz scalar obtained by projection is then integrated over a 2-sphere whose radius can be much smaller than the radiation wavelength. The integral provides the relativistic Larmor formula for radiated power (Gaussian units, metric signature -2)

$$\mathcal{R} = -\frac{2}{3} \frac{q^2}{c^3} a_\mu a^\mu \quad (1)$$

where a^μ is the 4-acceleration of charge q . An accelerating charge satisfies this criterion and has non-zero \mathcal{R} .

In the following, a brief derivation of the field and energy–momentum of a charge moving on a curved path in Minkowski spacetime is given.

2.1 Radiative field

Charge q has 4-current

$$j^\mu(x) = q \int_{-\infty}^{+\infty} v^\mu(s) \delta^{(4)}[x - z(s)] ds$$

where s is the proper time along the worldline of q with tangent $v^\mu(s) = dz^\mu/ds$, $v^\mu v_\mu = c^2$ ($c = 1$ hereafter).

Maxwell's equations (and the Lorenz gauge) provide

$$\square A^\mu(x) = 4\pi j^\mu(x).$$

Green's identity gives the retarded vector potential in terms of the 4-current

$$A_{\text{ret}}^\mu(x) = 4\pi \int_{\text{4-vol}} D_{\text{ret}}(x - x') j^\mu(x') d^4 x'$$

(note that μ is a Cartesian index which passes through the integral). D_{ret} is the retarded Jordan-Pauli function [5].

$$D_{\text{ret}}(x - z) = \frac{1}{2\pi} \theta(\tau) \delta[(x^\nu - z^\nu)(x_\nu - z_\nu)] \quad (2)$$

where $\tau = x^0 - z^0$, with step function $\theta(\tau) := \begin{cases} 1 & \tau > 0 \\ 0 & \tau < 0 \end{cases}$.

$$A_{\text{ret}}^\mu(x) = 2q \int_{-\infty}^{+\infty} v^\mu(s) \delta[(x^v - z^v)(x_v - z_v)] ds. \quad (3)$$

Since a forward null cone can emanate from each point on the worldline, we label the worldline points with u , such that $s > u$. Let $f^2(s)$ be the square of the spacelike distance between null cone field point x^v and worldline point z^v : $f^2(s) = -(x^\mu - z^\mu)(x^v - z^v)\eta_{\mu\nu}$ (Fig. 1). It follows that

$$\left[2f \frac{df}{ds} \right]_{s=u} = 2v_v(u)(x^v - z^v),$$

which motivates the definition $R := v_v(x^v - z^v)$, the distance along the null cone (the retarded distance between x^v and z^v). Note that, in the rest frame of the charge, $R = x^0 - z^0$. From Eq. (3) we obtain

$$A_{\text{ret}}^\mu(x) = 2q \int_{-\infty}^{+\infty} v^\mu \frac{\delta(s-u)}{2v_\alpha(u)(x^\alpha - z^\alpha)} ds = q \frac{v^\mu(u)}{R} \quad (4)$$

the retarded Lienard–Wiechart potential.

The worldline is now parametrized by u , i.e. $z^\alpha(u)$. The distance between points on the null cone is zero. $(x^\alpha - z^\alpha)(x_\alpha - z_\alpha) = 0$. The gradient $\partial/\partial x^\mu$ of the zero product yields

$$\begin{aligned} \left(\delta_\mu^\alpha - \frac{dz^\alpha}{du} \partial_\mu u \right) (x_\alpha - z_\alpha) &= 0 \\ (\delta_\mu^\alpha - v^\alpha \partial_\mu u) (x_\alpha - z_\alpha) &= 0 \\ (x_\mu - z_\mu) - R \partial_\mu u &= 0. \end{aligned}$$

The null vector k_μ along which the radiation propagates on the forward light cone is defined as

$$k_\mu := \partial_\mu u = (x_\mu - z_\mu)/R. \quad (5)$$

From the definition of R it follows that

$$\partial_\mu R = R k_\mu (a_\alpha k^\alpha) + v_\mu - k_\mu$$

where $a^\alpha := dv^\alpha/du$ is the acceleration along the worldline. A unit spacelike vector is defined as $\hat{n}_\alpha := k_\alpha - v_\alpha$ such that

$$\begin{aligned} \hat{n}_\alpha \hat{n}^\alpha &= -1, & \hat{n}_\alpha v^\alpha &= 0, & \hat{n}_\alpha k^\alpha &= -1, & k^\alpha v_\alpha &= 1, \\ v^\alpha v_\alpha &= 1, & k_\alpha k^\alpha &= 0. \end{aligned}$$

It is convenient to define a curvature scalar $\kappa := a_\alpha k^\alpha$. The gradient of R can be rewritten as

$$\partial_\mu R = \kappa R k_\mu - \hat{n}_\mu.$$

Differentiation of the Lienard-Wiechart potential (4) provides

$$A_{\text{ret}}^{\nu,\mu}(x) = \frac{q}{R} k^\mu (a^\nu - \kappa v^\nu) + \frac{q}{R^2} \hat{n}^\mu v^\nu.$$

Define spacelike $y^\nu := a^\nu - \kappa v^\nu$, orthogonal to k^ν . The electromagnetic field is

$$\begin{aligned} F_{\text{ret}}^{\mu\nu} &= 2A_{\text{ret}}^{[\nu,\mu]} \\ &= 2\frac{q}{R} k^{[\mu} y^{\nu]} + 2\frac{q}{R^2} \hat{n}^{[\mu} v^{\nu]} \end{aligned} \quad (6)$$

showing the $1/R$ radiation field and the $1/R^2$ velocity field [6].

2.2 Energy-momentum

The symmetric energy-momentum tensor, for signature $[+,-,-,-]$, is

$$4\pi T^{\mu\nu} = -F^{\mu\alpha} F_\alpha^\nu + \frac{1}{4} \eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}.$$

The radiation part of $F_{\text{ret}}^{\mu\nu}$ has energy-momentum

$$4\pi T^{\mu\nu} = \frac{q^2}{R^2} (y_\alpha y^\alpha) k^\mu k^\nu. \quad (7)$$

Consider a 4-volume bounded by two null cones \mathcal{N}_1 and \mathcal{N}_2 and two timelike $R = \text{const}$ surfaces Σ_1 and Σ_2 with normal n^α (Fig. 2). The limit $R \rightarrow \infty$ slides Σ_1 and Σ_2 to null infinity while maintaining a finite radial separation. Since $\partial_\nu T^{\mu\nu} =$

Fig. 2 4-Volume bounded by $\mathcal{N}_1, \mathcal{N}_2, \Sigma_1, \Sigma_2$

0, integration of $T^{\mu\nu}$ is cast onto the boundaries Σ_1 and Σ_2 at null infinity. The volume between Σ_1 and Σ_2 is $R^2 d\Omega du$ (with $d\Omega = \sin\vartheta d\vartheta d\varphi$). The energy and momentum radiated to null infinity in the interval between \mathcal{N}_1 and \mathcal{N}_2 is

$$\begin{aligned} \frac{d\mathcal{E}^\mu}{du} &= \lim_{R \rightarrow \infty} \frac{1}{4\pi} \oint T^{\mu\nu} n_\nu R^2 d\Omega \\ &= -q^2 (y_\alpha y^\alpha) k^\mu = -q^2 (a_\alpha a^\alpha + \kappa^2) k^\mu \end{aligned} \quad (8)$$

Since $k^0 > 0$, it follows that $d\mathcal{E}^0/du \geq 0$ in all inertial frames. Therefore $d\mathcal{E}^\mu/du$ is a timelike vector, which implies there is no inertial observer for whom the total momentum decreases without a corresponding energy loss.

3 Part II

Since an electric charge accelerating in an inertial frame emits electromagnetic radiation, the reciprocal question is whether an accelerated observer in an inertial frame, moving past a static charge (i.e. moving through an electrostatic field) detects electromagnetic radiation? We review the arguments for this question.

It is known that observers who accelerate through empty Minkowski space observe a heat bath [7; 8]. This supports the notion that such observers can see radiation in an electrostatic field, but this is a quantum effect, outside the scope of this review.

Rohrlich [9], in a study of the equivalence principle, asks and answers the question “does radiation from a uniformly accelerated charge contradict the equivalence principle?” The equivalence principle can be defined as the existence of a local spacetime region where gravitational tidal forces vanish, i.e. $R_{\mu\nu\alpha\beta} \delta x^\mu \delta x^\nu \delta x^\alpha \delta x^\beta \sim 0$. He then shows that in such a region, a charge in hyperbolic motion satisfies a local criterion [2] for radiation. Rohrlich further concludes that a uniformly accelerating detector **will** absorb electromagnetic energy from a static charge. Fugmann and Kretzschmar [10] have generalized Rohrlich’s results to arbitrary acceleration.

Mould [11] has provided invariant criteria to identify the absorption characteristics of such detectors. Mould constructs a simple antenna and computes the absorbed energy from a radiating charge on a hyperbolic trajectory. He shows that a freely falling detector **will** absorb energy from a static electric field.

Taken together, Rohrlich and Mould have fully answered the question of whether accelerated observers detect radiation from static charges. In the following, we affirm their answer from a different perspective.

3.1 Free fall

A freely falling observer is attached to a family of local inertial reference frames, isomorphic to the frames in which Maxwell’s equations are expressed and verified. Although Newtonian gravity is completely adequate for this topic, we introduce

the complexity of Einstein's gravitation in order to have analytic expressions for observers freely falling in a static electric field with non-uniform acceleration.

The RN metric g^{RN} describes an electrostatic, spherically symmetric, spacetime. Inside the RN trapped surface the source has parameters (m_0, q) , with q the source of a static Coulomb field. The inertial frame which exists at asymptotic spatial infinity can be used to span the RN spacetime. We ask whether an accelerated observer in this spacetime can see electromagnetic radiation?

Details of the RN spacetime are given in Appendix A.

3.2 Maxwell field seen from free fall

The Maxwell tensor for the RN Coulomb field, from Eq. (A3), is

$$F_{\alpha\beta}^{\text{RN}} = -\left(\frac{q}{r^2}\right) 2\hat{r}_{[\alpha}\hat{r}_{\beta]}$$

with basis vectors which span the RN metric in curvature coordinates. Since the RN spacetime is constructed with its source (m_0, q) a priori trapped, an infalling observer cannot see the charge itself. The observer can only take local samples of the surrounding static Coulomb field which can be projected on a forward or backward light cone emanating from the observer's position. Here we will use a forward cone. An expansion of $R_q(r)$ from Eq. (B7) provides

$$R_q = \left(\frac{m_0^2}{2q^3} \right) r^2 + O(r^3).$$

Thus

$$r^2 = 2 \left(\frac{q^3}{m_0^2} \right) R_q + O(R_q^2). \quad (9)$$

From Appendix B, we find the relationship between the static RN frame and the radial free fall null frame

$$\hat{t}_\alpha = [(1 + A_q^{1/2})/2] L_\alpha + (1 + A_q^{1/2})^{-1} N_\alpha, \quad (10)$$

$$\hat{r}_\alpha = (1 + A_q^{1/2})^{-1} N_\alpha - [(1 + A_q^{1/2})/2] L_\alpha. \quad (11)$$

It follows that

$$2\hat{t}_{[\alpha} \hat{r}_{\beta]} = L_{[\alpha} N_{\beta]}.$$

The free fall Maxwell tensor is

$$F_{\alpha\beta}^{\text{RN-ff}} = \left[-\frac{1}{2} \frac{(m_0^2/q^2)}{R_q} + O(1/R_q^2) \right] L_{[\alpha} N_{\beta]}, \quad (12)$$

and so, to first order, a free fall observer views the Coulomb field as a $1/R_q$ wave zone radiation field.

4 Summary

Classical radiation from an accelerating charge has been reviewed in Part I, and falls within the framework of standard Maxwell theory. Part II treats the reciprocal question of accelerated observers detecting radiation from a static charge. Rohrlich and Mould have shown that such radiation is detectable. Their answer has been affirmed here by studying a freely falling observer in the static Reissner–Nordström spacetime. The topic of radiation reaction has been omitted, but there exists voluminous literature about the Lorentz–Dirac equation. Within the citations at the end of this work, the interested reader can find additional references to all related topics.

Acknowledgments We thank Fritz Rohrlich for enlightening correspondence.

Appendix A: Reissner–Nordström spacetime

The RN metric in curvature coordinates has the standard form

$$ds_{\text{RN}}^2 = (1 - A_q) dt^2 - (1 - A_q)^{-1} dr^2 - r^2 d\Omega^2 \quad (\text{A1})$$

where $A_q = 2m_0/r - q^2/r^2$ and $d\Omega^2 = d\vartheta^2 + \sin^2 \vartheta d\varphi^2$.

Basis vectors, which span the RN metric and coincide with a Minkowski frame in the $r \rightarrow \infty$ limit, are

$$\hat{t}_\alpha dx^\alpha = (1 - A_q)^{1/2} dt, \quad (\text{A2a})$$

$$\hat{r}_\alpha dx^\alpha = (1 - A_q)^{-1/2} dr, \quad (\text{A2b})$$

$$\hat{\theta}_\alpha dx^\alpha = r d\vartheta, \quad (\text{A2c})$$

$$\hat{\phi}_\alpha dx^\alpha = r \sin \vartheta d\varphi, \quad (\text{A2d})$$

with

$$g_{\alpha\beta}^{\text{RN}} = \hat{t}_\alpha \hat{t}_\beta - \hat{r}_\alpha \hat{r}_\beta - \hat{\theta}_\alpha \hat{\theta}_\beta - \hat{\phi}_\alpha \hat{\phi}_\beta.$$

This frame spans the RN metric outside the trapped surface located at $1 - A_q = 0$.

The static Coulomb field has Maxwell tensor

$$F_{\alpha\beta}^{\text{RN}} = -(q/r^2)(\hat{t}_\alpha \hat{r}_\beta - \hat{r}_\alpha \hat{t}_\beta) \quad (\text{A3})$$

and trace-free Ricci tensor

$$R_{\alpha\beta}^{\text{RN}} = -(q^2/r^4)(\hat{t}_\alpha \hat{t}_\beta - \hat{r}_\alpha \hat{r}_\beta + \hat{\theta}_\alpha \hat{\theta}_\beta + \hat{\phi}_\alpha \hat{\phi}_\beta). \quad (\text{A4})$$

Killing observers $k^\beta \partial_\beta = (1 - A_q)^{-1/2} \partial_t$ find a radial electric field $E_\alpha = F_{\alpha\beta} k^\beta = (q/r^2) \delta_\alpha^{(r)}$. The RN charge is given by the integral of $F_{\alpha\beta}$ over any closed $t = \text{const}$, $r = \text{const}$, two-surface S^2 beyond the trapped surface:

$$\oint_{S^2} F_{\alpha\beta}^{\text{RN}} dS_{\alpha\beta} = 4\pi q.$$

Appendix B: Free fall frame

For observers falling from rest at infinity along radial geodesics, the RN metric can be written in terms of the free fall proper time τ [12] where $dt = d\tau - A_q^{1/2} (1 - A_q)^{-1} dr$ with $A_q = 2m_0/r - q^2/r^2$.

$$ds_{\text{RN-ff}}^2 = (1 - A_q) d\tau^2 - 2A_q^{1/2} d\tau dr - dr^2 - r^2 d\Omega^2. \quad (\text{B1})$$

$g^{\text{RN-ff}}$ is spanned by

$$\hat{\tau}_\alpha dx^\alpha = (1 - A_q)^{1/2} d\tau - \left(\frac{A_q}{1 - A_q} \right)^{1/2} dr, \quad \hat{\tau}^\alpha \partial_\alpha = (1 - A_q)^{-1/2} \partial_\tau,$$

$$\hat{r}_\alpha dx^\alpha = (1 - A_q)^{-1/2} dr, \quad \hat{r}^\alpha \partial_\alpha = - \left[\frac{1 - A_q}{(1 - A_q)^{1/2}} \right] \partial_r - \left(\frac{A_q}{1 - A_q} \right)^{1/2} \partial_\tau,$$

$$\hat{\vartheta}_\alpha dx^\alpha = r d\vartheta, \quad \hat{\phi}_\alpha dx^\alpha = \sin \vartheta d\varphi$$

such that

$$g_{\alpha\beta}^{\text{RN-ff}} = \hat{\tau}_\alpha \hat{\tau}_\beta - \hat{r}_\alpha \hat{r}_\beta - \hat{\vartheta}_\alpha \hat{\vartheta}_\beta - \hat{\phi}_\alpha \hat{\phi}_\beta. \quad (\text{B2})$$

The radial free fall geodesics along radial lines \mathfrak{R} have unit tangent

$$\hat{v}^\alpha \partial_\alpha = \partial_\tau - A_q^{1/2} \partial_r, \quad \hat{v}_\alpha dx^\alpha = d\tau \quad (\text{B3})$$

which satisfy $\hat{v}^\alpha_{;\beta} \hat{v}^\beta = 0$. This can be written as

$$\frac{d\hat{v}^\alpha}{d\tau} + \Gamma_{\beta\nu}^\alpha \hat{v}^\beta \hat{v}^\nu = 0, \quad (\text{B4})$$

with radial acceleration

$$\ddot{r} = -\frac{1}{r^2} (m_0 - q^2/r). \quad (\text{B5})$$

Electromagnetic energy diminishes the gravitational effect of m_0 .

A null frame $[L, N, M, \bar{M}]$ along \mathfrak{R} is constructed from the orthonormal frame. L is given by

$$L_\alpha dx^\alpha = \left[\frac{(1-A_q)^{1/2}}{1+A_q^{1/2}} \right] d\tau - (1-A_q)^{-1/2} dr, \quad (\text{B6})$$

$$dR_q = (1-A_q)^{-1/2} dr,$$

where R_q is distance from the observer along the outgoing null direction.

$$R_q = \int_0^r [1 - 2m_0/r' + q^2/(r')^2]^{-1/2} dr' \quad (\text{B7})$$

$$= (r^2 - 2m_0 r + q^2)^{1/2} + m_0 \ln[r - m_0 + (r^2 - 2m_0 r + q^2)^{1/2}]$$

$$- q - m_0 \ln(q - m_0)$$

with constraint $m_0^2 > q^2$. The other null vectors are

$$2N_\alpha dx^\alpha = (1-A_q)^{1/2} [(1+A_q^{1/2}) d\tau - dr], \quad (\text{B8})$$

$$M_\alpha dx^\alpha = -(r/\sqrt{2})(d\vartheta + i \sin \vartheta d\varphi). \quad (\text{B9})$$

References

1. Jackson, J.D.: Classical Electrodynamics, 3rd edn, Chap. 14. Wiley, New York (1999)
2. Rohrlich, F.: Classical Charged Particles. Addison-Wesley, Reading (1965)
3. B. Mashhoon (1990) Limitations of spacetime measurements *Phys. Lett. A* **143** 176
4. B. Mashhoon (1990) The hypothesis of locality in relativistic physics *Phys. Lett. A* **145** 147
5. Heitler, W.: The Quantum Theory of Radiation, 3rd edn. Oxford University Press, Oxford (1960)
6. C. Teitelboim (1970) Splitting of the Maxwell tensor *Phys. Rev. D* **1** 1572
7. W.G. Unruh (1976) Notes on black-hole evaporation *Phys. Rev. D* **14** 870

8. P.C.W. Davies (1975) Scalar production in Schwarzschild and Rindler metrics *J. Phys. A* **8** 609
9. F. Rohrlich (1963) The principle of equivalence *Ann. Phys.* **22** 169
10. W. Fugmann M. Kretzschmar (1991) Classical electromagnetic radiation in noninertial reference frames *Il Nouv. Cim. B* **106** 351
11. R.A. Mould (1964) Internal absorption properties of accelerating detectors *Ann. Phys.* **27** 1
12. C. Doran (2000) New form of the Kerr solution *Phys. Rev. D* **61** 067503