

Generalization of the Proca Action

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Abstract. We consider the Lagrangian of a vector field with derivative self-interactions with a priori arbitrary coefficients. Starting with a flat space-time we show that for a special choice of the coefficients of the self-interactions the ghost-like pathologies disappear. This constitutes the Galileon-type generalization of the Proca action with only three propagating physical degrees of freedom. The longitudinal mode of the vector field is associated to the usual Galileon interactions for a specific choice of the overall functions. In difference to a scalar Galileon theory, the generalized Proca field has more free parameters. We then extend this analysis to a curved background. The resulting theory is the Horndeski Proca action with second order equations of motion on curved space-times.

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1 Introduction

Motivated by the work from de Rham and Gabadadze for the generalization of the Fierz-Pauli action for a massive graviton [1], we investigate here the generalization of the Proca action for a massive vector field with derivative self-interactions. We will be addressing the natural question of what is the Lagrangian for a self-interacting vector field with second order equations of motion yielding three propagating physical degrees of freedom. We will call them the "vector Galileons", since they contain derivative self-interactions for the vector field and the longitudinal mode corresponds to a Galileon [2].

In the standard relativistic quantum field theory we describe particles with local covariant field operators like scalars, vectors, tensors... etc. The finite-dimensional representation of the Lorentz group dictates to us the number of propagating degrees of freedom. For a massless spin-1 field the theory needs to have the gauge symmetry in order to have the Lorentz invariance manifestly built in. The theory describes then a massless spin-1 field with two propagating degrees of freedom $h = \pm 1$. On the other hand, for a massive spin-1 field we have $(2s + 1)$, i.e three, propagating physical degrees of freedom. The Proca action is the theory describing a massive vector field, which propagates the corresponding three polarizations (two transverse plus one longitudinal). The mass term breaks explicitly the $U(1)$ gauge invariance such that the longitudinal mode propagates as well. However, the zero component of the vector field does not propagate. So therefore it is a natural question to investigate also the existence of derivative interactions for the vector field with still only three propagating degrees of freedom. This is exactly what we aim in this paper: we want to find the generalization of the Proca action for a massive vector field with derivative self-interactions. The standard Proca action is given by

$$\mathcal{S}_{\text{Proca}} = \int d^4x \left[-\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}m^2A^2 \right] \quad (1.1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In this theory, the temporal component of the vector field does not propagate and generates a primary constraint. The consistency condition of this primary constraint generates a secondary constraint, whose Poisson bracket with the primary constraint is proportional to the mass so that only in the massless case it corresponds to a first class constraint generating a gauge symmetry. Gauge invariance is just a redundancy

in the description of massless particles describing the same physical state. Therefore, we can use the Stueckelberg trick to restore the gauge invariance. In the case of the standard Proca action we can restore the gauge invariance by adding an additional scalar field via $A_\mu \rightarrow A_\mu + \partial_\mu \pi$. This trick does not change the number of propagating physical degrees of freedom. We add one additional scalar degree of freedom but we restore the gauge invariance which guarantees the existence of only two physical degrees of freedom for the vector field, making it in total three physical degrees of freedom. When we now take the mass going zero limit $m \rightarrow 0$ the Lagrangian results in a theory of a massless scalar field completely decoupled from a massless vector field for a conserved source. This is the reason why there is no vDVZ discontinuity in the case of the Proca field for conserved sources. This is different for the massive graviton. There, the helicity-0 degree of freedom does not decouple and gives rise to an additional fifth force which has to be screened via Vainshtein mechanism. In the case of the vector field, we do not need any Vainshtein mechanism since there is no observational difference between a massless and massive vector fields for conserved sources. In the generalized Proca action that we construct here, the longitudinal mode of the vector field has exactly the same interactions as a Galileon scalar field for a specific choice of the overall functions. The Galileon theory is an important class of infra-red modifications of general relativity. The Galileon interactions were introduced as a natural extension of the decoupling limit of the DGP model [2]. It is constructed as an effective field theory for a scalar field by the restriction of the invariance under internal Galilean and shift transformations and second order equations of motion. This effective action is local and contains higher order derivatives. Nevertheless, these interactions come in a very specific way such that they only give rise to second order equations of motion. The allowed interactions for the Galileon were originally determined order by order by writing down all the possible contractions for the derivative scalar field interactions and finding the proper coefficients giving rise to second order equations of motion. However, de Rham and Tolley could construct an unified class of four dimensional effective theories starting from a higher dimensional setup and show that these effective theories reproduce successfully all the interaction terms of the Galileon in the non-relativistic limit [3]. In a similar way we wonder whether or not one could construct the generalized Proca action that we are proposing here from a higher dimensional set-up which we will investigate in a future work. Naively, we would think, that, starting from a higher dimensional set-up with manifestly covariant Lovelock invariants, one would only construct terms which are gauge invariant after dimensional reduction. There is only one possible non-minimal interaction which fulfills this requirement, namely the contraction of two field strength tensors with the dual Riemann tensor. Therefore, this specific interaction with gauge invariance could be easily constructed from a higher dimensional set-up. However, it would be worth to study, if the other not gauge invariant non-minimal couplings could be constructed by dimensional reduction.

The Galileon interactions present a subclass of Horndeski interactions which describe scalar-tensor interactions with at most second order equations of motion on curved backgrounds [4]. Interestingly, a subclass of Horndeski scalar-tensor interactions [5] can also be constructed by covariantizing the decoupling limit of massive gravity [6]. In the literature there has been some attempts to find a theory for vector fields which is equivalent to scalar Galileons, i.e. to find the "vector Galileons" besides the Maxwell kinetic term with second order equations of motion on flat space-times [7]. There, the authors were interested in derivative self-interactions for the vector field with gauge symmetry yielding only two propagating degrees of freedom. They concluded that the Maxwell kinetic term is the only allowed

interaction and wrote a no-go theorem for generalized vector Galileons.¹ However, on curved background there only exists one term respecting the gauge symmetry, which is given by the non-minimal coupling between the field strength tensor and the dual Riemann tensor [9, 10]. However, if one gives up on the gauge invariance meaning that we allow for terms which are not invariant under $A_\mu \rightarrow A_\mu + \partial_\mu \theta$ then one can indeed construct "vector Galileons" on flat-spacetimes or Horndeski vector interactions on curved backgrounds giving rise to three propagating physical degrees of freedom with second order equations of motion. We will illustrate this in this work.

We will use the $(+,-,-,-)$ notation for the signature of the metric and furthermore the conventions $\partial \cdot A = \partial_\mu A^\mu$, $(\partial \cdot A)^2 = \partial_\mu A^\mu \partial_\nu A^\nu \dots$ etc. throughout the paper.

2 The theory of generalized Proca field

Now we want to generalize the Proca action (1.1) to include derivative self-interactions of the vector field, but without changing the number of propagating degrees of freedom. When we construct the derivative self-interactions, we will be applying one derivative acting on the vector field at a time. If we would apply more derivatives on the vector field, this would give rise to ghost degrees of freedom. In other words, when we write the vector field in terms of the Stueckelberg field $A_\mu = \partial_\mu \pi$, second derivatives applied on the vector field would give rise to higher order derivatives applied on the scalar Stueckelberg and hence rendering it to a ghost. In order to obtain such interactions, we will analyze all the possible Lorentz invariant terms that can be built at each order and constrain the interactions to remove the ghost-instabilities. The Lagrangian for the generalized Proca vector field with derivative self-interactions is given by

$$\mathcal{L}_{\text{gen.Proca}} = -\frac{1}{4}F_{\mu\nu}^2 + \sum_{n=2}^5 \alpha_n \mathcal{L}_n \quad (2.1)$$

where the self-interactions of the vector field are

$$\begin{aligned} \mathcal{L}_2 &= f_2 \\ \mathcal{L}_3 &= f_3 \partial \cdot A \\ \mathcal{L}_4 &= f_4 [(\partial \cdot A)^2 + c_2 \partial_\rho A_\sigma \partial^\rho A^\sigma - (1 + c_2) \partial_\rho A_\sigma \partial^\sigma A^\rho] \\ \mathcal{L}_5 &= f_5 [(\partial \cdot A)^3 - 3d_2 (\partial \cdot A) \partial_\rho A_\sigma \partial^\rho A^\sigma - 3(1 - d_2) (\partial \cdot A) \partial_\rho A_\sigma \partial^\sigma A^\rho \\ &\quad + 2 \left(1 - \frac{3d_2}{2}\right) \partial_\rho A_\sigma \partial^\gamma A^\rho \partial^\sigma A_\gamma + 2 \left(\frac{3d_2}{2}\right) \partial_\rho A_\sigma \partial^\gamma A^\rho \partial_\gamma A^\sigma] \end{aligned} \quad (2.2)$$

with $\partial \cdot A = \partial_\mu A^\mu$ and where the functions $f_{2,3,4,5}$ are arbitrary functions. Let us first emphasize the dependences of these functions. First of all, they all can depend on $A^2 = A_\mu A^\mu$. Nevertheless, the function f_2 is special in the sense that it is the only function which is not multiplied by any term with derivatives acting on the vector field. Therefore, this function f_2 can also have dependence on all the possible terms which have U(1) symmetry like $F^2 = F^{\mu\nu} F_{\mu\nu}$, $FF^* = F^{\mu\nu} F_{\mu\nu}^*$... etc (where F^* is the dual of F). Furthermore, the function f_2 can depend on terms which does not contain any time derivative applying on the

¹A similar no-go theorem has been also studied in [8] in the context of massive graviton

temporal component A_0 of the vector field like for instance $A_\mu A_\nu F^{\mu\rho} F_\rho^\nu$. This is not true for the remaining functions $f_{3,4,5}$.

$$f_2 = f_2(A^2, A \cdot F, F^2, FF^*) \quad f_{3,4,5} = f_{3,4,5}(A^2) \quad (2.3)$$

where $A \cdot F$ encodes all the possible contractions between A 's and F 's. For instance the function f_2 can naturally depend on terms like $A^2 F^2$, $A^2 F^4$, $A^4 F^4$, $A_\mu A_\nu F^{\mu\rho} F_\rho^\nu \dots$ etc while the remaining functions $f_{3,4,5}$ can only depend on A^2 since these functions are multiplied by terms which contain derivatives acting on the vector field. These functions do not change the number of propagating physical degrees of freedom since they do not contain any dynamics for the temporal component of the vector field. We will comment more on that in section 4. The second Lagrangian \mathcal{L}_2 naturally contains the mass $\frac{1}{2}m^2 A^2$ and potential terms $V(A^2)$ for the vector field in the function f_2 . In the next section we will illustrate order by order why these interactions give rise to only three propagating degrees of freedom and illustrate the absence of ghost instabilities. Note also the appearance of the two free parameters c_2 and d_2 . It means that the "vector Galileons" contain more free parameters then the usual scalar Galileon theory. The interactions can be also expressed in terms of the Levi-Civita tensors

$$\begin{aligned} \mathcal{L}_2 &= -\frac{f_2}{24} \mathcal{E}^{\mu\nu\alpha\beta} \mathcal{E}_{\mu\nu\alpha\beta} = f_2 \\ \mathcal{L}_3 &= -\frac{f_3}{6} \mathcal{E}^{\mu\nu\alpha\beta} \mathcal{E}^\rho_{\nu\alpha\beta} \partial_\mu A_\rho = f_3 \partial \cdot A \\ \mathcal{L}_4 &= -\frac{f_4}{2} \left(\mathcal{E}^{\mu\nu\alpha\beta} \mathcal{E}^{\rho\sigma}_{\alpha\beta} \partial_\mu A_\rho \partial_\nu A_\sigma + c_2 \mathcal{E}^{\mu\nu\alpha\beta} \mathcal{E}^{\rho\sigma}_{\alpha\beta} \partial_\mu A_\nu \partial_\rho A_\sigma \right) \\ &= f_4 \left[(\partial \cdot A)^2 + c_2 \partial_\rho A_\sigma \partial^\rho A^\sigma - (1 + c_2) \partial_\rho A_\sigma \partial^\sigma A^\rho \right] \\ \mathcal{L}_5 &= -f_5 \left[\left(1 - \frac{3}{2}d_2\right) \mathcal{E}^{\mu\nu\alpha\beta} \mathcal{E}^{\rho\sigma\gamma}_{\beta} \partial_\mu A_\rho \partial_\nu A_\sigma \partial_\alpha A_\gamma + \frac{3}{2}d_2 \mathcal{E}^{\mu\nu\alpha\beta} \mathcal{E}^{\rho\sigma\gamma}_{\beta} \partial_\mu A_\rho \partial_\nu A_\sigma \partial_\gamma A_\alpha \right] \\ &= f_5 \left[(\partial \cdot A)^3 - 3d_2 (\partial \cdot A) \partial_\rho A_\sigma \partial^\rho A^\sigma - 3(1 - d_2) (\partial \cdot A) \partial_\rho A_\sigma \partial^\sigma A^\rho \right. \\ &\quad \left. + 2 \left(1 - \frac{3d_2}{2}\right) \partial_\rho A_\sigma \partial^\gamma A^\rho \partial^\sigma A_\gamma + 2 \left(\frac{3d_2}{2}\right) \partial_\rho A_\sigma \partial^\gamma A^\rho \partial_\gamma A^\sigma \right] \end{aligned} \quad (2.4)$$

The Lagrangians $\mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4, \mathcal{L}_5$ in (2.2) propagate only three degrees of freedom. Higher order interactions beyond the quintic order are trivial in four dimensions, being just total derivatives, hence the series stops here. Expressed in terms of the Levi-Civita tensors this means that we run out of the indices.

When we wrote the derivative self-interactions in terms of the Levi-Civita tensors, the indices of the potential interactions were always contracted with each other. Without loss of generality consider for example the special choice for the functions $f_{2,3,4,5} = (A^2)$ and the special choice for the parameters $c_2 = 1$ and $d_2 = 1$, then in this case, we could either consider contractions in the functions as $A_\mu A^\mu$ or contract the indices of these two vector fields with the Levi-Civita tensor as well. One might wonder, if it yields different interactions once the indices of the term (A^2) for example are contracted with the Levi-Civita tensors as

well.

$$\begin{aligned}
\mathcal{L}_2^{al} &= -\frac{1}{6}\mathcal{E}^{\mu\nu\alpha\beta}\mathcal{E}^\rho_{\nu\alpha\beta}A_\mu A_\rho = (A^2) \\
\mathcal{L}_3^{al} &= -\frac{1}{2}\mathcal{E}^{\mu\nu\alpha\beta}\mathcal{E}^{\rho\sigma}_{\alpha\beta}A_\mu A_\rho \partial_\nu A_\sigma = (A^2)(\partial \cdot A) - A^\mu A^\nu \partial_\nu A_\mu \\
\mathcal{L}_4^{al} &= -\mathcal{E}^{\mu\nu\alpha\beta}\mathcal{E}^{\rho\sigma\delta}_{\beta}A_\mu A_\rho \partial_\nu A_\sigma \partial_\alpha A_\delta \\
&= (A^2)[(\partial \cdot A)^2 - \partial_\rho A_\sigma \partial^\sigma A^\rho] - 2A^\mu A^\nu \partial_\nu A_\mu (\partial \cdot A) + 2A^\mu A^\nu \partial_\nu A_\rho \partial^\rho A_\mu \\
\mathcal{L}_5^{al} &= \mathcal{E}^{\mu\nu\alpha\beta}\mathcal{E}^{\rho\sigma\delta\gamma}A_\mu A_\rho \partial_\nu A_\sigma \partial_\alpha A_\delta \partial_\beta A_\gamma \\
&= (A^2)[-(\partial \cdot A)^3 + 3(\partial \cdot A)\partial_\rho A_\sigma \partial^\sigma A^\rho - 2\partial_\rho A_\sigma \partial^\gamma A^\rho \partial^\sigma A_\gamma] \\
&\quad + 3A^\mu A^\nu \partial_\nu A_\mu (\partial \cdot A)^2 - 6A^\mu A^\nu \partial_\nu A_\rho \partial^\rho A_\mu (\partial \cdot A) + 6A^\mu A^\nu \partial_\nu A_\rho \partial^\rho A_\gamma \partial^\gamma A_\mu \\
&\quad - 3A^\mu A^\nu \partial_\nu A_\mu \partial_\rho A_\sigma \partial^\sigma A^\rho
\end{aligned} \tag{2.5}$$

But on closer inspection one can see that they give rise to exactly the same interactions once integrations by part are performed. In the following we will start with the general interactions order by order with arbitrary coefficients and demonstrate that imposing the absence of the unphysical degree of freedom gives rise to the above Lagrangian with only three propagating physical degrees of freedom.

3 The propagation of three degrees of freedom

The simplest modification of the Proca action (1.1) is of course promoting the mass term to an arbitrary function f_2 which contains amongst others the mass term and the potential interactions for the vector field $f_2 \supset V(A^2)$, since this trivially does not modify the number of degrees of freedom. As we already emphasized, this function can also contain gauge invariant interactions which are invariant under the $U(1)$ transformations and terms which do not contain any dynamics for the temporal component of the vector field, i.e. terms of the form $f_2 \supset F^2 + FF^* + A^2F^2 + A^2FF^* + A_\mu A_\nu F^{\rho\mu}F_\rho^\nu + \dots$.

The first term that we can have to the next order in the vector field is simply

$$\mathcal{L}_3 = f_3 \partial \cdot A \tag{3.1}$$

with f_3 an arbitrary function of the vector field norm $f_3(A^2)$. It is a trivial observation that in (3.1) the temporal component of the vector field A_0 does not propagate, even if we include the Maxwell kinetic term, and it acts as a lagrange multiplier. The easiest way to see it is by computing the corresponding Hessian, which vanishes trivially. Also notice that the presence of the function f_3 is crucial since if it was simply a constant, that term would be a total divergence and, thus, with no contribution to the field equations.

To next order, the independent interaction terms that we can have are given by

$$\mathcal{L}_4 = f_4 [c_1(\partial \cdot A)^2 + c_2\partial_\rho A_\sigma \partial^\rho A^\sigma + c_3\partial_\rho A_\sigma \partial^\sigma A^\rho] \tag{3.2}$$

with a priori free parameters c_1 , c_2 and c_3 and f_4 an arbitrary function depending on $f_4(A^2)$. Now, we need to fix the parameters such that only three physical degrees of freedom propagate, i.e., such that we still have a second class constraint. In order to eliminate one propagating degree of freedom, we need a constraint equation, which is guaranteed if the

determinant of the Hessian matrix vanishes. The Hessian matrix for (3.2) is given by

$$H_{\mathcal{L}_4}^{\mu\nu} = \frac{\partial^2 \mathcal{L}_4}{\partial \dot{A}_\mu \partial \dot{A}_\nu} = f_4 \begin{pmatrix} 2(c_1 + c_2 + c_3) & 0 & 0 & 0 \\ 0 & -2c_2 & 0 & 0 \\ 0 & 0 & -2c_2 & 0 \\ 0 & 0 & 0 & -2c_2 \end{pmatrix} \quad (3.3)$$

For a vanishing determinant of the Hessian matrix we have two possibilities. First possibility corresponds to choosing $c_2 = 0$. In this case the Hessian matrix contains three vanishing eigenvalues corresponding to three constraints. Therefore, if we choose $c_2 = 0$, only the zero component of the vector field propagate while the other three degrees of freedom do not propagate. This is not what we are looking for, therefore we disregard this choice. The other possibility for a vanishing determinant of the Hessian matrix corresponds to $c_1 + c_2 + c_3 = 0$. Without loss of generality we can set $c_1 = 1$ and therefore $c_3 = -(1 + c_2)$. In this case the Hessian matrix only contains one vanishing eigenvalue and hence only one propagating constraint. This case corresponds to three propagating degrees of freedom with the Lagrangian at this order given by:

$$\mathcal{L}_4 = f_4 \left[(\partial \cdot A)^2 + c_2 \partial_\rho A_\sigma \partial^\rho A^\sigma - (1 + c_2) \partial_\rho A_\sigma \partial^\sigma A^\rho \right] \quad (3.4)$$

Note that we can write these interactions also as

$$\mathcal{L}_4 = f_4 \left[(\partial \cdot A)^2 - \partial_\rho A_\sigma \partial^\sigma A^\rho + c_2 F_{\rho\sigma}^2 \right] \quad (3.5)$$

which then in principal could be absorbed into $f_2 \supset F^2$ since the function f_2 depends in general on the gauge invariant quantities which do not contain any dynamics for the A_0 degree of freedom and therefore will not change the number of propagating degrees of freedom. One can either include the interaction $c_2 F_{\rho\sigma}^2$ into the function f_2 or leave it at the order of \mathcal{L}_4 , but not both at the same to avoid redundancy. The vanishing of the determinant of the Hessian matrix guarantees the existence of a constraint. To find the expression for the constraint, we have to compute the conjugate momentum $\Pi_{\mathcal{L}_4}^\mu = \frac{\partial \mathcal{L}_4}{\partial \dot{A}_\mu}$. The zero component of the conjugate momentum is given by

$$\Pi_{\mathcal{L}_4}^0 = -2f_4 \nabla \mathbf{A} \quad (3.6)$$

As one can see, the zero component of the conjugate momentum does not contain any time derivative yielding the constraint equation

$$\mathcal{C}_1 = \Pi_{\mathcal{L}_4}^0 + 2f_4 \nabla \mathbf{A}. \quad (3.7)$$

This constraint equation will generate a secondary constraint given by

$$\{H, \mathcal{C}_1\} = \frac{\partial H}{\partial A_\mu} \frac{\partial \mathcal{C}_1}{\partial \Pi^\mu} - \frac{\partial H}{\partial \Pi^\mu} \frac{\partial \mathcal{C}_1}{\partial A_\mu} \quad (3.8)$$

or equivalently one can obtain the secondary constraint by calculating the time derivative of the conjugate momentum $\dot{\Pi}^\mu$ and use the Hamiltonian equations $\frac{\partial H}{\partial A_\mu} = -\dot{\Pi}^\mu$ and $\frac{\partial H}{\partial \Pi^\mu} = \dot{A}_\mu$. We have checked explicitly the existence of the secondary constraint and therefore the Lagrangian \mathcal{L}_4 possesses only three propagating degrees of freedom.

For the next order interactions we write down all the possible contractions between the derivative self-interactions which gives:

$$\begin{aligned} \mathcal{L}_5 = f_5 & \left[d_1 (\partial \cdot A)^3 - 3d_2 (\partial \cdot A) \partial_\rho A_\sigma \partial^\rho A^\sigma - 3d_3 (\partial \cdot A) \partial_\rho A_\sigma \partial^\sigma A^\rho \right. \\ & \left. + 2d_4 \partial_\rho A_\sigma \partial^\gamma A^\rho \partial^\sigma A_\gamma + 2d_5 \partial_\rho A_\sigma \partial^\gamma A^\rho \partial_\gamma A^\sigma \right] \end{aligned} \quad (3.9)$$

with a priori the arbitrary parameters d_1, d_2, d_3, d_4 and d_5 and function f_5 depending only on A^2 . In this quintic Lagrangian (3.9) the additional possible term $\partial_\sigma A_\rho \partial^\gamma A^\rho \partial_\gamma A^\sigma$ is actually equal to $\partial_\rho A_\sigma \partial^\gamma A^\rho \partial_\gamma A^\sigma$ since $\partial^\gamma A^\rho \partial_\gamma A^\sigma$ is symmetric under the exchange of ρ and σ . The Hessian matrix for this quintic Lagrangian is giving by

$$\begin{aligned}
H_{\mathcal{L}_5}^{00} &= -6(d_1 - d_2 - d_3)(\nabla \mathbf{A}) + 6(d_1 - 3d_2 - 3d_3 + 2(d_4 + d_5))\dot{A}_t \\
H_{\mathcal{L}_5}^{0i} &= H_{\mathcal{L}_5}^{i0} = (6d_3 - 2(3d_4 + d_5))A_{t,i} + 2(3d_2 - 2d_4)A_{i,t} \\
H_{\mathcal{L}_5}^{11} &= -6d_2 A_\alpha^2 (A_{z,z} + A_{y,y}) - 2(3d_2 - 2d_4)(A_{x,x} - A_{t,t}) \\
H_{\mathcal{L}_5}^{12} &= H_{\mathcal{L}_5}^{21} = 2d_5(A_{x,y} + A_{y,x}) \\
H_{\mathcal{L}_5}^{13} &= H_{\mathcal{L}_5}^{31} = 2d_5(A_{x,z} + A_{z,x}) \\
H_{\mathcal{L}_5}^{22} &= 2(-3d_2 A_{z,z} + (-3d_2 + 2d_5)A_{y,y} - 3d_2 A_{x,x} + (3d_2 - 2d_5)A_{t,t}) \\
H_{\mathcal{L}_5}^{23} &= H_{\mathcal{L}_5}^{32} = 2d_5(A_{y,z} + A_{z,y}) \\
H_{\mathcal{L}_5}^{33} &= (-6d_2 + 4d_5)A_{z,z} - 6d_2(A_{y,y} + A_{x,x}) + 2(3d_2 - 2d_5)A_{t,t}
\end{aligned} \tag{3.10}$$

In order to have only three propagating degrees of freedom the parameters need to fulfill the following conditions

$$\begin{aligned}
d_1 - d_2 - d_3 &= 0, & d_1 - 3d_2 - 3d_3 + 2(d_4 + d_5) &= 0, \\
3d_3 - 3d_4 - d_5 &= 0, & 3d_2 - 2d_5 &= 0
\end{aligned} \tag{3.11}$$

which are fulfilled by choosing (again without loss of generality we can choose $d_1 = 1$)

$$d_3 = 1 - d_2, \quad d_4 = 1 - \frac{3d_2}{2}, \quad d_5 = \frac{3d_2}{2} \tag{3.12}$$

Hence, the quintic Lagrangian with only three propagating physical degrees of freedom is given by

$$\begin{aligned}
\mathcal{L}_5 = f_5 & \left[(\partial \cdot A)^3 - 3d_2(\partial \cdot A)\partial_\rho A_\sigma \partial^\rho A^\sigma - 3(1 - d_2)(\partial \cdot A)\partial_\rho A_\sigma \partial^\sigma A^\rho \right. \\
& \left. + 2\left(1 - \frac{3d_2}{2}\right)\partial_\rho A_\sigma \partial^\gamma A^\rho \partial^\sigma A_\gamma + 2\left(\frac{3d_2}{2}\right)\partial_\rho A_\sigma \partial^\gamma A^\rho \partial_\gamma A^\sigma \right]
\end{aligned} \tag{3.13}$$

Analogously, we can write these interactions also as

$$\begin{aligned}
\mathcal{L}_5 = f_5 & \left[(\partial \cdot A)^3 - 3(\partial \cdot A)\partial_\rho A_\sigma \partial^\sigma A^\rho + 2\partial_\rho A_\sigma \partial^\gamma A^\rho \partial^\sigma A_\gamma \right. \\
& \left. - \frac{3d_2}{2}(\partial \cdot A)F_{\rho\sigma}^2 + 3d_2\partial_\sigma A_\gamma F_\rho^\sigma F^{\rho\gamma} \right].
\end{aligned} \tag{3.14}$$

The Hessian matrix with this chosen parameters then becomes

$$H_{\mathcal{L}_5}^{\mu\nu} = f_5(A^2) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -6d_2(A_{z,z} + A_{y,y}) & 3d_2(A_{x,y} + A_{y,x}) & 3d_2(A_{x,z} + A_{z,x}) \\ 0 & 3d_2(A_{x,y} + A_{y,x}) & -6d_2(A_{z,z} + A_{x,x}) & 3d_2(A_{y,z} + A_{z,y}) \\ 0 & 3d_2(A_{x,z} + A_{z,x}) & 3d_2(A_{y,z} + A_{z,y}) & -6d_2(A_{y,y} + A_{x,x}) \end{pmatrix} \tag{3.15}$$

with a vanishing determinant $\det(H_{\mathcal{L}_5}^{\mu\nu}) = 0$. As required, the Hessian matrix only contains one vanishing eigenvalue and hence only one propagating constraint which is again given by the corresponding zero component of the conjugate momentum $\Pi_{\mathcal{L}_5}^\mu = \frac{\partial \mathcal{L}_5}{\partial \dot{A}_\mu}$

$$\begin{aligned}
\Pi_{\mathcal{L}_5}^0 = -3f_5(A^2)(d_2(A_{x,z}^2 + A_{y,z}^2 + A_{x,y}^2) - 2A_{z,z}A_{y,z} - 2(-1 + d_2)A_{y,z}A_{z,y} + d_2A_{z,y}^2 + d_2A_{z,x}^2 \\
- 2(A_{z,z} + A_{y,y})A_{x,x} + 2A_{x,y}A_{y,x} - 2d_2A_{x,y}A_{y,x} + d_2A_{y,x}^2 - 2(-1 + d_2)A_{x,z}A_{z,x})
\end{aligned} \tag{3.16}$$

As you can see, there is no time derivatives appearing in the expression of the zero component of the conjugate momentum, representing the constraint equation. Associated to this constraint, there will be a secondary constraint guaranteeing the propagation of the constraint equation and removing the unphysical degree of freedom.

Our construction guarantees the existence of the constraint system which removes one of the four degrees of freedom and the remaining physical degrees of freedom possess only second order equations of motion which ensures that they are not ghostlike degrees of freedom. One distinguishes two types of ghost degrees of freedom. The one we removed here in our construction is like the Boulware-Deser ghost in massive gravity representing the unphysical degree of freedom of the vector field. However, the remaining three physical degrees of freedom might not always behave healthy around some given specific backgrounds. Once we construct our Lagrangian for the three physical propagating degrees of freedom, it would be interesting to study them around specific backgrounds like cosmological backgrounds or spherically symmetric backgrounds and constraint further the parameters to avoid ghost and/or Laplacian instabilities for these three physical degrees of freedom.

4 Special case of the functions $f_{2,3,4,5} = A^2$

In this section we will pay attention to the special case where the arbitrary functions are chosen to be $f_{2,3,4,5} = A^2$. In this case the four Lagrangians $\mathcal{L}_{2,3,4,5}$ in (2.2) simply become

$$\begin{aligned}\mathcal{L}_2 &= A^2 \\ \mathcal{L}_3 &= A^2(\partial \cdot A) \\ \mathcal{L}_4 &= A^2 [(\partial \cdot A)^2 + c_2 \partial_\rho A_\sigma \partial^\rho A^\sigma - (1 + c_2) \partial_\rho A_\sigma \partial^\sigma A^\rho] \\ \mathcal{L}_5 &= A^2 [(\partial \cdot A)^3 - 3d_2(\partial \cdot A) \partial_\rho A_\sigma \partial^\rho A^\sigma - 3(1 - d_2)(\partial \cdot A) \partial_\rho A_\sigma \partial^\sigma A^\rho \\ &\quad + 2 \left(1 - \frac{3d_2}{2}\right) \partial_\rho A_\sigma \partial^\gamma A^\rho \partial^\sigma A_\gamma + 2 \left(\frac{3d_2}{2}\right) \partial_\rho A_\sigma \partial^\gamma A^\rho \partial_\gamma A^\sigma]\end{aligned}\quad (4.1)$$

We can restore the $U(1)$ gauge symmetry using the Stueckelberg trick by adding an additional scalar field via $A_\mu \rightarrow A_\mu + \partial_\mu \pi$. To zeroth order in A_μ we extract out only the longitudinal mode of the vector field and recover exactly the Galileon interactions

$$\begin{aligned}\mathcal{L}_2 &= (\partial\pi)^2 \\ \mathcal{L}_3 &= (\partial\pi)^2 \square\pi \\ \mathcal{L}_4 &= (\partial\pi)^2 [(\square\pi)^2 - (\partial_\mu \partial_\nu \pi)^2] \\ \mathcal{L}_5 &= (\partial\pi)^2 [(\square\pi)^3 - 3\square\pi(\partial_\mu \partial_\nu \pi)^2 + 2(\partial_\mu \partial_\nu \pi)^3]\end{aligned}\quad (4.2)$$

Note that after introducing the gauge symmetry the dependence of the free parameters c_2 and d_2 disappears. Similarly, to first order in A_μ we obtain the following scalar-vector interactions

$$\begin{aligned}\mathcal{L}_2 &= 2A^\mu \partial_\mu \pi \\ \mathcal{L}_3 &= (\partial\pi)^2 (\partial \cdot A) + 2\square\pi \partial_\mu \pi A^\mu \\ \mathcal{L}_4 &= 2(\partial\pi)^2 \square\pi (\partial \cdot A) + 2(\square\pi)^2 \partial_\mu \pi A^\mu - 2(\partial\pi)^2 \partial_\mu \partial_\nu \pi \partial^\nu A^\mu - 2(\partial_\mu \partial_\nu \pi)^2 \partial_\rho \pi A^\rho \\ \mathcal{L}_5 &= 6(\partial\pi)^2 \partial_\sigma \partial_\beta \pi \partial_\alpha \partial^\sigma \pi \partial^\beta A^\alpha + 2A^\alpha \partial_\alpha \pi (\square\pi)^3 + 3(\partial \cdot A) (\partial\pi)^2 (\square\pi)^2 - 6\partial_\alpha \partial_\beta \pi \partial^\beta A^\alpha (\partial\pi)^2 \square\pi \\ &\quad + 4A^\alpha \partial_\alpha \pi \partial^\rho \partial^\beta \pi \partial_\sigma \partial_\beta \pi \partial^\sigma \partial_\rho \pi - 3(2A^\alpha \partial_\alpha \pi \square\pi + (\partial \cdot A) (\partial\pi)^2) (\partial_\rho \partial_\sigma \pi)^2\end{aligned}\quad (4.3)$$

This is another way of observing that the interactions we found for the vector field indeed only propagate three degrees of freedom, since when we plug in the longitudinal mode, we obtain the Galileon interaction with at most second order equations of motion. The terms for the vector field $\partial_\rho A_\sigma \partial^\rho A^\sigma$ and $\partial_\rho A_\sigma \partial^\sigma A^\rho$ are not the same, but when we replace $A_\mu = \partial_\mu \pi$, they are since the derivatives acting on the scalar field commute $\partial_\mu \partial_\nu \pi = \partial_\nu \partial_\mu \pi$ on flat space-time. This has a huge consequence: *the interactions for the vector field have more free parameters than the Galileon interactions*. It means that if we had started with the Galileon interactions and performed the replacement $\partial_\mu \pi \rightarrow A_\mu$ we would have been missing some of the interactions which also yield three propagating degrees of freedom. The vector interaction have two more free parameters (namely what we called c_2 and d_2 in (4.1)). In fact, an alternative way of finding our generalized Proca action is by restoring the U(1) gauge invariance and imposing that the Stueckelberg field propagates only one degree of freedom, i.e., it satisfies second order field equations. One must be careful though, since in addition to the pure Stueckelberg sector, it is also necessarily to analyse the terms mixing the Stueckelberg field and the vector field. For \mathcal{L}_4 , no additional constraints arise from the mixing terms, since we obtain terms of the general form $K^{\mu\nu}(A_\mu) \partial_\mu \pi \partial_\nu \pi$, which automatically leads to second order contributions for π . However, for \mathcal{L}_5 we obtain terms like $K^{\alpha\beta\gamma\delta}(A_\mu) \partial_\alpha \partial_\beta \pi \partial_\gamma \partial_\delta \pi$ so we need to impose the tensor $K^{\alpha\beta\gamma\delta}(A_\mu)$ to have the correct structure. It is also worth to emphasize one more time that the arbitrary functions $f_{2,3,4,5}$ appearing in our generalized Proca action have been chosen to be A^2 in this section to be able to relate them to the Galileon interactions. In the Stueckelberg language, this is so in order to guarantee the second order nature of the field equations with respect to π . There are however additional contributions upon which the functions might depend without altering the number of degrees of freedom. Such terms are those for which the Stueckelberg field give a trivial contribution, i.e., those which are U(1) gauge invariant. Therefore, the function f_2 could actually depend also on the combinations F^2 or FF^* . It can naturally also depend on any possible contraction between A_μ and $F_{\mu\nu}$ as well, in a way like for example $A_\mu A_\nu F^{\mu\alpha} F_\alpha^\nu \dots$ etc. From the vector field perspective, these terms do not contain time derivatives of A_0 , so that it will not spoil the existence of the constraints. Indeed, if you look at the interactions in \mathcal{L}_4 which are proportional to the parameter c_2 then you trivially recognize that these terms are just $c_2 F^2$. Since function f_2 also depends on F^2 then the term for instance in \mathcal{L}_4 could be absorbed into $f_2(A^2, F^2)$. One must be cautious however, since arbitrary functions of such invariants typically give rise to violations of the hyperbolicity of the field equations and hence to superluminal propagation, which we do not discuss in this work.

The equations of motion for the Lagrangian of the derivative self-interacting vector field (4.1) on top of the Maxwell kinetic term are given by

$$\begin{aligned}
\mathcal{E}_2 &= 2A_\mu \\
\mathcal{E}_3 &= 2A_\mu(\partial \cdot A) - 2A^\nu \partial_\mu A_\nu \\
\mathcal{E}_4 &= 2\left(A_\mu \left[(\partial \cdot A)^2 - (1+c_2)\partial_\rho A_\sigma \partial^\sigma A^\rho + c_2 \partial_\rho A_\sigma \partial^\sigma A^\rho\right] + c_2 A^2(-\square A_\mu + \partial_\nu \partial_\mu A^\nu)\right. \\
&\quad \left.- 2c_2 A^\rho \partial_\nu A_\rho \partial^\nu A_\mu - 2(\partial \cdot A) A^\rho \partial_\mu A_\rho + 2(1+c_2) A^\rho \partial_\nu A_\rho \partial_\mu A^\nu\right) \\
\mathcal{E}_5 &= 2A_\mu \left[(\partial \cdot A)^3 + 3(-1+d_2)(\partial \cdot A) \partial_\rho A_\sigma \partial^\sigma A^\rho - 3d_2(\partial \cdot A) \partial_\rho A_\sigma \partial^\rho A^\sigma + (2-3d_2) \partial_\rho A_\sigma \partial^\gamma A^\rho \partial^\sigma A_\gamma\right. \\
&\quad \left.+ 3d_2 \partial_\rho A^\sigma \partial^\rho A^\gamma \partial_\sigma A_\gamma\right] - 3A^\rho \left(-d_2(4\partial_\nu A_\rho \partial^\nu A_\mu (\partial \cdot A) - 2(\partial_\nu A_\mu \partial^\nu A_\sigma + \partial^\nu A_\mu \partial_\sigma A_\nu) \partial^\sigma A_\rho)\right. \\
&\quad \left.+ A_\rho(\partial^\nu A_\mu (\partial_\sigma \partial_\nu A^\sigma - \square A_\nu) + 2(\partial \cdot A)(\square A_\mu - \partial_\sigma \partial_\mu A^\sigma) + (\partial_\nu \partial_\mu A_\sigma - 2\partial_\sigma \partial_\nu A_\mu + \partial_\sigma \partial_\mu A_\nu) \partial^\sigma A^\nu)\right) \\
&\quad + 2((\partial \cdot A)^2 + ((-1+d_2)\partial_\nu A_\sigma - d_2 \partial_\sigma A_\nu) \partial^\sigma A^\nu) \partial_\mu A_\rho + (4(-1+d_2)\partial_\nu A_\rho (\partial \cdot A) \\
&\quad \left.+ d_2 A_\rho(-\partial_\sigma \partial_\nu A^\sigma + \square A_\nu) + 2((2-3d_2)\partial_\nu A_\sigma + d_2 \partial_\sigma A_\nu) \partial^\sigma A_\rho\right) \partial_\mu A^\nu \quad (4.4)
\end{aligned}$$

Note also that the equations of motion for the vector field does reproduce the equations of motion of the Galileon field if we take the divergence of it and replace $A_\mu = \partial_\mu \pi$.

5 Curved space-times

In the flat space-time the derivatives applied on the vector field were simply partial derivatives which commute. When we consider a general non-flat background the derivatives become covariant derivatives and therefore we have to add non-minimal couplings to the graviton in order to maintain second order equations of motion and healthy propagating degrees of freedom. When we generalize the derivative self-interactions in (5.1) on a curved space-time, the Lagrangian for the generalized Proca field becomes

$$\mathcal{L}_{\text{gen.Proca}}^{\text{curved}} = -\frac{1}{4}F_{\mu\nu}^2 + \sum_{n=2}^5 \beta_n \mathcal{L}_n \quad (5.1)$$

where now the self-interactions are encoded in the following Lagrangians

$$\begin{aligned} \mathcal{L}_2 &= G_2(X) \\ \mathcal{L}_3 &= G_3(X)(D_\mu A^\mu) \\ \mathcal{L}_4 &= G_4(X)R + G_{4,X} [(D_\mu A^\mu)^2 + c_2 D_\rho A_\sigma D^\sigma A^\rho - (1+c_2) D_\rho A_\sigma D^\sigma A^\rho] \\ \mathcal{L}_5 &= G_5(X)G_{\mu\nu} D^\mu A^\nu - \frac{1}{6}G_{5,X} \left[(D_\mu A^\mu)^3 - 3d_2(D_\mu A^\mu)D_\rho A_\sigma D^\rho A^\sigma - 3(1-d_2)(D_\mu A^\mu)D_\rho A_\sigma D^\sigma A^\rho \right. \\ &\quad \left. + 2\left(1 - \frac{3d_2}{2}\right)D_\rho A_\sigma D^\gamma A^\rho D^\sigma A_\gamma + 2\left(\frac{3d_2}{2}\right)D_\rho A_\sigma D^\gamma A^\rho D_\gamma A^\sigma \right] \end{aligned} \quad (5.2)$$

with the short cut $X = -\frac{1}{2}A_\mu^2$. These interactions give rise to the standard scalar Horndeski interactions for the longitudinal mode of the vector field. The non-minimal coupling between the field strength tensors and the dual Riemann tensor considered in [10] is already incorporated in the above interactions. Similarly the function G_2 does not need to depend only on X but can also depend on terms like $G^{\mu\nu} A_\mu A_\nu$ which does not contain any dynamics for the temporal component of the vector field. Note again the appearance of the two additional free parameters c_2 and d_2 as in flat space-time case. All these interactions give only rise to three propagating degrees of freedom in curved background.

6 Summary and discussion

In this paper we have constructed the generalized Proca action for a vector field with derivative self-interactions with only three propagating degrees of freedom. We started our analysis with the case of a flat Minkowski spacetime. We successfully showed that for appropriate choices of the coefficients of the derivative self-interactions that generalize the Proca action, one can construct a consistent and local theory of massive vector field without the presence of ghost-like instabilities. The resulting theory is simple and constitutes four Lagrangians for the self-interactions of the vector field. We were able to show that the constrained coefficients yield the necessary propagating constraint in order to remove the unphysical degree of freedom. These are the "vector Galileons" with three propagating degrees of freedom. At each order the Lagrangian has an overall function which depends on A^2 and the function

of the quadratic Lagrangian can also depend on all the possible terms invariant under $U(1)$ symmetry like for instance F^2 and FF^* ... etc. Similarly this function can also depend on any contractions between the vector field and the field strength tensor $A_\mu A_\nu F^{\mu\rho} F_\rho^\nu$ which does not contain any time derivative applied on the temporal component of the vector field. The dependence of the function f_2 on the gauge invariant terms or terms in which the zero component of the vector field does not have any dynamics, do not alter the number of propagating degrees of freedom. We have also shown, that these interactions have more free parameters than the corresponding scalar Galileon interactions. We then generalized our results to the case of curved space-time and obtained the corresponding "Horndeski vector" interactions. In the very latest stage of this work we became aware that a similar idea has been explored in [11] even though the interactions we constructed here are more general.

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