national accelerator laboratory

FERMILAB-Conf-72/056-T

NAL-THY-56
June 7, 1972

EXOTIC APPLICATIONS OF LIGHT-CONE ALGEBRA

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Lectures delivered at the XI Internationale Universitätswochen fur Kernphysik, Schladming, Austria, February 1972.

Operated by Universities Research Association Inc. Under Contract with the United States Atomic Energy Commissio

I. INTRODUCTION

The parton model[1] and its formal companion, the light-cone algebra,[2,3,4] were invented to explain the scaling behavior of deep inelastic electroproduction. Both models relate scattering on physical, structured hadrons to scattering from some underlying point-like constituent hadrons ("partons"). It is by now common knowledge that in such schemes, scaling behavior for photon-induced reactions implies similar behavior for neutrino-induced reactions and that if a sufficiently simple internal-symmetry structure is assigned to the partons, sum rules exist relating the various possible photon-nucleon and neutrino-nucleon cross-sections to one another. Indeed, these predictions are regarded as the major test of the parton model/light-cone algebra and everybody is breathlessly awaiting the advent of the deep-inelastic neutrino data to see if they are satisfied.

In the meantime, attempts have been made to push the model beyond the simple applications to total inelastic cross-sections described above. On the whole, the motivation for this attempt is "because it's there," but in some measure it is stimulated by the hope that a crucial test not involving neutrinos can be found. Two sorts of result have been obtained: (a) inequalities on the total inelastic cross-sections and (b) extensions of the model to semi-inclusive cross-sections (in which some final state particles are observed). The inequalities are based on exactly the same physics as the classic tests, but were explicitly noted only recently.

They turn out in some cases to be quite stringent and, in one example, near to being violated by experiment. The extension to semi-inclusive reactions requires some new physics and turns out to have interesting implications concerning the physical reality of the underlying "partons". Both developments taken together form a largish body of new results, not yet generally familiar (hence the designation "exotic" in our title) but of considerable importance to both experimentalists and theoreticians interested in the problem of scaling. They will be the subject of our lectures.

II WHY THE LIGHT CONE?

As far as the "classic" applications to total inelastic reactions are concerned, the parton model and the light-cone algebra are fully equivalent. In my opinion, however, the light-cone algebra is pedagogically advantageous in that the fundamental assumption can be stated precisely, as can the circumstances in which it may be applied. Therefore, throughout these lectures, we shall take the light-cone algebra point of view, noting, where necessary, divergences from the parton model.

Let us begin by reviewing what we mean by the light-cone algebra and how it is applied to total inelastic cross-sections. We are interested either in $\ell + N \to \ell' + \underline{\overline{X}}$ or $\nu + N \to \ell + \underline{\overline{X}}$ where ℓ is a charged lepton, N a nucleon, and $\underline{\overline{X}}$ stands for unobserved hadrons. If q is the four-momentum transfer to the leptons and J_μ the current (either electromagnetic or weak) to which the leptons couple, then the unknown hadronic part of the cross-section is

where the latter equality holds if, as it always the case, $q_o > 0$. If we are interested in cross-sections on an unpolarized nucleon target, we may average over the nucleon spin (this spin average is implicitly assumed henceforth). Then $W_{\mu\nu}$ may be written in terms of scalar structure functions as

where the W_i are functions of q^2 and $q \cdot p$ (the mass and lab energy, respectively, carried by the current). The structure functions W_4 and W_5 are essentially irrevelant because the leptonic tensor, $L_{\mu\nu}$, by which we multiply $W_{\mu\nu}$ to get the cross-section, is conserved in the limit of zero lepton mass and so satisfies $q^\mu L_{\mu\nu} = 0(m_\ell)$. W_3 is a parity-violating object and so will be zero when J_μ is the electromagnetic current. Therefore, inelastic electron scattering is determined by two structure functions, W_1 and W_2, and one can easily work out the explicit formula

where θ is the lab scattering angle of the electron. It is also convenient to consider the cross-sections induced by transversely and longitudinally polarized photons. If we choose the target rest frame,

and define virtual photon polarization vectors by $q \cdot \epsilon = 0$, $|\epsilon^2| = 1$, a natural choice is

Then the longitudinal and transverse combinations of structure functions are

Reference to Eq. 2.1 shows that both W_T and W_L must be positive!

It was suggested some time ago by Bjorken[5] that the limit $q^2 \to -\infty$, $-q^2/2p\cdot q = \omega$ fixed, would be an especially useful one for the study of structure functions of the type we are interested in. The reason can be seen if we adopt the reference frome of Eq. 2.2 and rewrite Eq. 2.1 as

In the Bjorken limit, $q_+ \to (q\cdot p/m) \to \infty$ and $q_- \to -\omega m/2$, so that

Since $q \cdot p \to \infty$, the asymptotic behavior of $W_{\mu\nu}$ is determined entirely by the discontinuities of F. These in turn must come from singularities of the integrand, $\langle p | J_\mu^+(x) J_\nu(0) | p \rangle$. Barring pathological behavior, the product of two local operators, $A(x) B(y)$, is expected to be singular only at light-like separation, $(x - y)^2 = 0$. Therefore, in the Bjorken limit, the behavior of $W_{\mu\nu}$ is determined by the singularity of $J_\mu^+(x) J_\nu(0)$ on the light-cone, $x^2 = 0$.[6]

We have, of course, no a priori information about this singularity: The singularity at the tip of the light-cone, $x = 0$, is related to equal-time commutators of the currents, about which we might have canonical information. However, the singularity at finite distances from the tip is important in the Bjorken limit and unconstrained by canonical commutators.

III THE LIGHT CONE ALGEBRA

To proceed, we evidently need a sensible hypothesis about the singularity on the light cone of current products or commutators. Fritzsch and Gell-Mann[2] have suggested that we assume the operator structure of this singularity to be the same as in free field theory. The matrix elements of the relevant operators would of course not be given by free field theory, only certain algebraic relations between operators.

More concretely, let us imagine that the underlying field theory of the world is one of standard quarks interacting via neutral vector gluons. Then the $SU_3 \otimes SU_3$ currents are

ψ being the quark field, λ^a the SU_3 generating matrix and : the usual normal-ordering instruction. Although J_μ itself is normal-ordered, the product $J_\mu(x)\ J_\nu(y)$ is not and therefore has singularities. These singularities can be explicitly brought out by using the Wick expansion to reduce the product to fully normal-ordered form.

With this end in view, we define

where M is a numerical matrix. Then our problem is to reduce the product $J_M(x)\ J_N(y)$. Wick's theorem states that

where the contraction is defined by

If we throw out the pure c-number piece in Eq. 3.2 along with the finite parts, we have

The operator in brackets has finite matrix elements, because it is fully normal-ordered, and the entire singularity is contained in the explicit c-number functions. If we substitute appropriate values for M and N, we finally obtain

with

In order to perform this expansion, we have worked within free field theory, and in that context, the bilocal operators $\mathscr{F}^{\sigma}$ and $\mathscr{F}^{\sigma 5}$ have finite matrix elements.[7] The light-cone algebra proposal is that the relations, Eqn. 3.3, between currents and bilocal operators be assumed to hold in the real world as well, regarding the bilocal operators as independent new entities about which one knows only that their matrix elements are finite.

This assumption has immediate consequences for total inelastic cross-sections. Since we are interested in the spin-averaged nucleon matrix element of the current product, we need the corresponding matrix elements of the bilocal operators. Using the various available invariance principles, we find that

where f and g are unknown functions. The finiteness of the matrix elements of $\mathscr{F}^{\sigma}$ guarantee that we may expand f and g about $x^2 = 0$. Since we are interested in the leading singularity at $x^2 = 0$, we may simply set $x^2 = 0$ in f and g. Furthermore, it is a matter of simple algebra to show that the g form factor does not contribute to leading order in the Bjorken limit. Consequently, in the Bjorken limit, and

where $\tilde{f}$ is the Fourier transform of f(p·x) and ω is, as usual, $-q^2/2p\cdot q$.

A number of consequences follow from this result. First of all, it is apparent that the quantities having a finite scaling limit are W_1, $(q\cdot p)\, W_2$ and $(q\cdot p)\, W_3$ (quantities conventionally called F_1, F_2 and F_3). Second of all, there is a relation between F_1 and F_2, $2\omega F_1 - F_2 = 0$, which is recognized from Eq. 2.4 to be the statement that the scaling limit of the longitudinal structure function, W_L (defined in Eq. 2.4), is zero.

Finally, the otherwise independent structure functions F_1 and F_3 have a dependence on the SU_3 quantum numbers of the currents involved which is restricted by

If we refer to the definition (Eqn. 3.5) of $f_{\pm}^{c}$, we see that hypercharge conservation together with the choice of the isospin 1/2 nucleon as target particle guarantee that $\tilde{f}_{\pm}^{c} = 0$ except for c = 0, 8, 1, 2, 3. Isospin conservation then guarantees that $\tilde{f}^{1,2,3}$ are all described by the same form factor. Therefore there are a total of $2 \times 3 = 6$ independent unknown functions describing the scaling limit of all possible lepton-induced reactions on nucleons. Since there are more reactions than 6, this implies sum rules, of which the Llewellyn-Smith relation,[8]

is one.

So far, we have used only the Lorentz and internal symmetry structure of the bilocal operators $\mathscr{F}^{\sigma}$. We may want to take the explicit structure of Eq. 3.3 more seriously to the extend of accepting that where V_μ and A_μ are the $SU_3 \otimes SU_3$ generating currents. Then the form factor ϕ_c, defined by

is given by

Insofar as ϕ_c is known (as it is for baryon number, hypercharge and isospin current) this yields integral sum rules on the scaling limit of the structure functions.

The results summarized here are what is usually regarded as the content of the light cone algebra as far as total inelastic reactions are concerned. To a certain extent they depend on our choice, Eq. 3.1, of the structure of the currents in terms of underlying fields. As a rule one can say that: a) the result that W_1, $(q \cdot p)\, W_2$ and $(q \cdot p)\, W_3$ have finite scaling limits does not depend on the underlying fields,

b) the result $W_L = 0$ depends only on the underlying fields being spin $\frac{1}{2}$, and c) the relations, Eq. 3.7, between reactions initiated by different currents depend on the specific choice of standard quarks for the underlying fields. At this point, it should be noted that the standard quark parton model gives the same results as we have found via the light cone algebra. The reason for this and the relevant translation formulae will appear naturally in the next section.

IV POSITIVITY CONDITIONS

Any quantity of the type

($|p, s>$ *is a state of momentum* p *and internal* symmetry index s) is obviously a positive matrix in the sense that

for an arbitrary vector ξ. The total inelastic lepton cross-sections we have been dealing with are precisely of the above form (physically we are only interested in the case $r = s$) and the scaling limit of the structure functions, in particular, must satisfy constraints following from Eq. 4.1. On the other hand, the light-cone algebra assumption gives us a specific form for these structure functions in terms of a set of arbitrary form factors $\tilde{f}_{\pm}^{c}(\omega)$. In order for positivity and the light-cone algebra to be consistent, we therefore expect the possible range of variation of the $\tilde{f}_{\pm}^{c}$ to be restricted, with corresponding limitations on the independent measurable structure functions.

We have already seen a limited application of this fact in the observation that both W_T and W_L must be positive (Eq. 2.4). Much stronger relations may however be found and it is to this development, initiated by Nachtmann,[9] that we now turn.

We shall be interested in objects like Eq. 4.2 in the scaling limit and shall accordingly make use of the light-cone results of the previous section. First, we note that according to Eqs. 3.3, 3.4, current products of the form $(V + A)_\mu (V - A)_\nu$ vanish in the scaling limit. Therefore we may study the products $(V + A)_\mu (V + A)_\nu$ and $(V - A)_\mu (V - A)_\nu$ independently.

The relevant scaling limits of Eq. 4.1 are, according to Eqs. 3.6

where the plus/minus sign refers to choosing J equal to V + A or V - A, respectively, and $f_{\pm}^{c,rs}$ is defined by the obvious modification of Eq. 3.5:

If we work in the rest frame of p, it is easy to see that in the Bjorken limit

where we have written the dependence on μ and ν in explicit matrix form and

This four by four matrix has the eigenvectors $(1, 0, 0, \pm 1)$ and $(0, 1, \pm i, 0)$ with eigenvalues 2A, 0, A + B, A - B, independent of the form of the matrices A and B. Thus, the matrix $W_{\mu\nu}^{ar,bs}$ is positive if and only if the matrices $(A \pm B)_{ar,bs}$ are positive. If we make use of Eq. 4.3 this can be further reduced to the requirement that

be positive, where $\phi_{\pm} = \tilde{f}_{+} \pm \tilde{f}_{-}$.

We are only interested in diagonal matrix elements in the spece of nucleons, so that $\phi_{\pm}^{c,\ rs}$ will appear in physical cross-sections only for c = 0, 8, 3 and r = s = neutron or proton. In fact, if we make use of SU_2 invariance, it is apparent that $\phi_{\pm}^{\substack{8\\o}} = \phi_{\pm}^{\substack{8\\o},pp} = \phi_{\pm}^{\substack{8\\o},nn}$, $\phi_{\pm}^{3} = \phi_{\pm}^{3,pp} = -\phi_{\pm}^{3,nn}$. Thus, there are only $2 \times 3 = 6$ independent quantities out of which all structure functions must be constructed. Typical relations are

On the other hand, the matrix $F^{\pm}_{ar,bs}$, thought of as a matrix in $8 \times 2 = 16$ dimensions (8 for currents, 2 for nucleons) must be positive. To determine the constraints this places on $\phi_{\pm}^{0,3,8}$, it is necessary to diagonalize $F^{\pm}_{ar,bs}$. Hypercharge and isospin conservation allow one to cast F into block diagonal, as in Table I. Each entry is a linear combination of the parameters $\phi^{0,3,8}$. Therefore positiveity of F^+ yields six "linear" conditions of the form $\sum_{i=0,3,8} c_i \phi_+^i \geq 0$ plus one "quadratic" condition (arising from diagonalizing the I = 1/2 block), while positivity of F^- yields a corresponding set of conditions on the ϕ_- .

This looks (and is) very complicated. Fortunately, there is a trick to reduce the complexity of these conditions and, as a side benefit, establish a connection with the parton model.[10] To see how this goes, we rewrite Eq. 4.4 as

with

Then the condition that $F^{\pm}_{ar,bs}$ be positive is equivalent to for an arbitrary octet of 3×3 matrices M^a. It is of course trivial to make a special choice of the matrices M^a so that these conditions become

with

So, if the 16 dimensional matrix $F^{\pm}_{ar,bs}$ is positive, then so is the 2×3 = 6 dimensional matrix $\Phi^{\alpha r,\beta s}_{\pm}$. Since α and β carry the same isospin as quarks (I = 0, I = 1/2) we see that, if r, s refer to nucleon, we can completely diagonalize $\Phi^{\alpha r,\beta s}_{\pm}$ and that it has three eigenvalues, each of which is a linear combination of the quantities $\phi^{0,3,8}_{\pm}$ introduced before. Thus we actually have precisely as many linear positivity conditions (6) as independent quantities determining the structure functions. That these conditions are the entire content of Eq. 4.7 is guaranteed by the fact that Eq. 4.8 implies Eq. 4.7.

The extraction of the content of these equations is easy. We have three conditions of the form $\underset{i\,=\,0,8,3}{} \xi_i\, \phi_+^i \geq 0$ and three similar conditions for ϕ_-^i. If we define $\vec{\phi}^{\pm} = (\phi_0^{\pm}, \phi_8^{\pm}, \phi_3^{\pm})$ and $\vec{e} = (\xi_0, \xi_8, \xi_3)$, then we have the geometrical problem of characterizing the three-vectors $\vec{\phi}^{\pm}$ which satisfy $\vec{e}_i^{\,\pm} \cdot \vec{\phi}^{\pm} \geq 0$ for given vectors $\vec{e}_i^{\,\pm}$, $i = 1, 2, 3$. It is easy to show that the solution is

$\vec{f}_i^{\,\pm} = \epsilon_{ijk}\, \vec{e}_j^{\,\pm} \times \vec{e}_k^{\,\pm}$ and $\alpha_i^{\pm} \geq 0$. If we go back to the expression of individual structure functions as linear combinations of the components of $\vec{\phi}^{\pm}$, we can discover that they may be written in terms of the arbitrary positive parameters $\alpha_i^{\pm}$ as in Table 2. Had we assumed SU_3 symmetry and taken the entire baryon octet as target, we would have gotten slightly different results.

There are various ways of exploiting this result. Consider first of all just $F_1^{\gamma p}$ and $F_1^{\gamma n}$. If we form $\rho F_1^{\gamma n} - F_1^{\gamma p}$, we easily see that it is positive for $\rho > \frac{1}{4}$. By similar arguments one can show that $F_1^{\gamma p}/F_1^{\gamma n} > \frac{1}{4}$ so that in general

Had we used SU_3, we would have been able to replace the lower bound by $1/3$! This relation is particularly interesting because experiment indicates that $F_1^{\gamma n}/F_1^{\gamma p}$ decreases rapidly as ω approaches 1 (threshold) and may well go below 1/4. This is obviously a crucial test for the quark-based parton model.

In this context we note that, according to Table II,

Therefore,

This means that whenever $F_1^{\gamma n}/F_1^{\gamma p}$ is near its lower bound of 1/4, one must find that the νp structure functions approach zero.

For the moment the experimental data one needs to test these, and other, relations following from Table 1 is not available, but we can expect to have it before too long. It is to be remarked that the inequalities we have so far discussed are local relations - they are true for each value of ω. Table I, of course, still holds for integrals of the F_i over positive weight functions. Since total cross sections are such integrals, and are easier to measure than the $F_i(\omega)$, more easily testable inequalities can be found in this fashion.

Next, we propose to discuss how this problem appears in the context of the parton model. In the parton model, one assumes the existence of free point-like constituents within the target hadron and assumes that the interaction with the current is entirely via these constituents. We have been discussion the process $a+r \rightarrow b+s$ where a and b refer to the current and r and s to the target hadron. Evidently the parton model result is

where $\phi_{ar}(\bar{\phi}_{ar})$ is the amplitude for finding a quark (antiquark), a, in hadron, r, and the parentheses refer to the free quark-current scattering cross section. The elementary quark-current scattering cross-sections are easily seen to be

Therefore, in the parton model, the structure functions for scattering off hadron targets satisfy

with

This, of course, has exactly the form of Eq. 3.7 which was derived from the light-cone algebra. From this point of view, the positivity of the matrices $G_{\pm}^{ar,\beta s} = (G_{\pm}^{rs})_{a\beta}$ in the 6-dimensional space spanned by the indices a, r is trivial - it follows from the positivity of the elementary quark-current cross-sections. It obviously implies, via the arguments given earlier, the positivity of the physical cross sections. Because of the formal identity of Eqs. 3.7 and 4.9 the positivity conditions following from the quark model and the light-cone algebra must be the same.

As a final remark, we want to show how simply the bound on $F_1^{\gamma n}/F_1^{\gamma p}$ can be understood in the parton model. Let the densities in the parton of the p, n and λ quarks (charges 2/3, -1/3, -1/3 respectively) be a, b and c. They $F_1^{\gamma p} = (\frac{2}{3})^2 a + (\frac{1}{3})^2 b + (\frac{1}{3})^2 c$. Since the neutron is a 180° isospin rotation of the proton, the roles of n and p quarks are interchanged and $F_1^{\gamma N} = (\frac{1}{3})^2 a + (\frac{2}{3})^2 b + (\frac{1}{3})^2 c$. Thus

and since a, b and c are all positive,

V SEMI INCLUSIVE REACTIONS

The total inelastic cross-sections we have been considering so far are interesting enough, but rather limited in scope: after all, there are only two possible targets (neutron and proton) and two possible projectiles (electron and neutrino)! It was first suggested by Ellis that, with modest extensions of the hypothesis stated in Section 2, one could discuss processes of the type (current) + (target) → (specific observed hadrons) + (unobserved hadrons). Experimentally, such reactions are the next easiest thing to do after the total cross-section experiments and will be done in the near future. It is therefore most appropriate at this time to discuss what results might be expected.

For definiteness, we shall consider the reaction $J(q) + N(p) \rightarrow \pi(r) + \underline{\overline{X}}$ where: J, N and π are the incoming current, target nucleon and outgoing observed pion, respectively; q, p and r are the respective momenta of these objects; and $\underline{\overline{X}}$ stands for unobserved hadrons. Nothing stops us from observing more than one outgoing hadron, and the extension of the arguments which follow to such a case will be easy.

The cross-section for this reaction will be given by

the obvious analog of the hadron tensor measured in the total inelastic reactions. We shall take J_μ to be the electromagentic current in order to simplify the kinematics somewhat: $W_{\mu\nu}$ then satisfies $q^\mu W_{\mu\nu} = q^\mu W_{\nu\mu} = 0$ and parity conservation holds. In order to measure $W_{\mu\nu}$ one performs the reaction $e(\ell) + N(p) \rightarrow e(\ell') + \pi(r) + \underline{\overline{X}}$. If the one-photon-exchange approximation is valid, this cross-section is proportional to

The vector ϵ^{μ}_{lep} can be expressed as a known linear combination of the complete set q_{μ}, ϵ^{+}_{μ}, ϵ^{-}_{μ}, ϵ^{0}_{μ}, where the ϵ's are the right-handed, left-handed and longitudinal polarization vectors associated with q. Since $q^{\mu}W_{\mu\nu} = 0$, I may be expressed as a linear combination of the helicity cross-sections

Parity invariance ($W_{ab} = (-)^{a-b} W_{ba}$) reduces the number of independent quantities from 9 to 4: W_{++}, W_{00}, W_{+0}, W_{+-}.

The experimentally measurable quantity, I, is then a linear combination of these helicity cross-sections with known coefficients. If one deals with unpolarized electron beams, the relation turns out to be[12, 13]

where: $\nu = q \cdot p/m$, θ is the lab scattering angle of the lepton and ϕ is basically the rotation angle of the plane containing the leptons about their momentum transfer to the hadrons. The important point is that if you bodily rotate the lepton plane about q, so that q stays fixed, then all the arguments on which I depends, except ϕ, remain unchanged. Thus, one can experimentally separate the helicity flip 0 (ϕ indep.) helicity flip 1 (cos ϕ dependent) and helicity flip 2 (cos 2ϕ dependent) cross sections.

Before proceeding to our light-cone analysis, we say a word about what we expect to happen in the scaling limit: Since the total longitudinal cross-section vanishes (Sec. 3), so must any partial longitudinal cross-section. Hence, both W_{oo} and W_{+o} should vanish relative to W_{++} and W_{+-}. Neither positivity nor our work on total cross-sections leads us to expect that W_{++} and W_{+-} should behave differently from one another. Hence, we expect some cos 2ϕ dependence in I.

To see what the light-cone has to say about this, we first note that the matrix element $<N(p)|J_{\mu}(o)|\pi(r)\underline{\overline{X}}>$ may be written in the form

where J_{π} is the pion source in the usual sense of field theory. Consequently, $W_{\mu\nu}(pqr)$ may be written

To follow the argument of Section 2 as closely as possible, let us consider a limit in which r and p are held fixed and q gets large in such a way that both $q^2/p\cdot q$ and $q^2/r\cdot q$ are held fixed (we shall again call this the Bjorken limit). This is achieved by the choice

$$p = (m, o, o, o)$$

$$r = (\text{ fixed })$$

$$q = (q_o, o, o, q_3).$$

Then the analysis of the asymptotic behavior of $W_{\mu\nu}(qp)$ which follows Eq. 2.5 is directly applicable here. Consequently, the Bjorken limit of $W_{\mu\nu}(p, q, r)$ is determined by the singularities in x of

This in the end boils down to looking at the singularities of multiple operator products such as $J_{\mu}(x)\, J_{\pi}(z)\, J_{\nu}(o)\, J_{\pi}(z')$ and relevant permutations thereof.

Let us imagine that the underlying fields of the theory are spin 1/2 and take the attitude advocated by Fritzsch and Gell-Mann for the study of operator products: calculate the free-field operator structure of the leading singularity and assume it to be true of the physical currents. The calculation is a bit more complicated in this case, but the underlying assumption seems no more radical.

Let us set

and use the Wick theorem to serch for singularities in the product $J_{\mu}(x)\, J_{\pi}(z)\, J_{\nu}(y)\, J_{\pi}(z')$. There are two relevant kinds of contraction in the complete expansion of the product: those which connect $J_{\mu}(x)$ and $J_{\nu}(y)$ only, and those in which the connection between fields at x and y is indirect,

Terms in the first category have exactly the same structure as the singularities of the product $J_\mu(x)\, J_\nu(y)$ - the J_π's are just spectators as far as the singularity structure is concerned. The second type of term is more complicated and will not in general have the same Lorentz or internal symmetry structure as the first type. On the other hand, it will be less singular: For arbitrary z, Eq. (5. 3) has no singularity in x-y. Only if z lies along the line joining x and y will there be a singularity. In the matrix element, Eq. (5. 2), one integrates over all z, thereby including such points, but since they have volume zero the degree of singularity of their contribution to the matrix element is reduced below that of Eq. (5. 5). Therefore, the leading singularity is given by contractions between the currents themselves, and its Lorentz and SU_3 structure will be precisely as calculated in Section 3 in the study of the simple product $J_\mu(x) J_\nu(y)$. The spectator particle serves only to provide new variables on which the matrix elements of the "bilocal operators" may depend. One gets the correct answer by going back to Eq. (5. 2) and pretending that one can reduce the J_π back into the states,

and then using the expressions of Eq. (3. 3) for the leading light-cone singularity of the current product.

Following this prescription, we find that, in the limit $q^2 \to -\infty$, with $q^2/q \cdot p$ and $q^2/q \cdot r$ fixed,

We have neglected to write the SU_3 content of $\mathcal{F}^{\beta}$ and have used the fact that since no spin variables are present, there is no possible matrix element of $\mathcal{F}^{5\beta}$. This can then be written in terms of scalar structure functions:

The scalar form factors depend on x^2 as well, but since we want the leading light-cone singularity, we are entitled to set $x^2 = 0$. There is another covariant, x^{β}, but it does not contribute at the same level as the two we have explicitly written. Finally, if we make the obvious Fourier decomposition of f and g, we may write.

where we have dropped terms in the delta-function which do not survive in the Bjorken limit.

This result now should be converted to statements about the helicity cross-sections defined earlier. We work in the target rest frame so that

and find that

The vanishing cross-sections are expected to vanish like $1/q^2$. The cross-section W_{0+} satisfies the positivity bound $| W_{+0} | \leq \sqrt{W_{++}} \sqrt{W_{00}}$ and so presumably vanishes like $1\sqrt{q^2}$.

These results are all as expected, except for the prediction that W_{+-} vanishes. If we refer back to the expression (Eq. 5.1) for the electron scattering cross-section, we see that the vanishing of W_{+0} and W_{+-} means that this cross-section is independent of the angle ϕ describing the orientation of the electron plane about q. This is a reasonably clear experimental signal and a non-trivial test of the model. The vanishing of W_{+-} is rather clearly related to the assumption that the underlying field is spin 1/2: If the absoprtion and re-emission in the forward direction of the virtual photon takes place on a spin-1/2 particle, helicity change 2 is impossible. Presumably, any light-cone or parton model with underlying fields of spin less than 1 would yield the result $W_{+-} \to 0$, although if spin 0 fields are present, it will no longer be true that W_{00} and W_{0+} vanish.

One may feel uneasy about this derivation because, on top of the assumption that the singularity structure of operator products is as in free field theory, we have had to make a specific assumption about the structure of the pion source (J_π) in terms of underlying fields. Since particle sources, as opposed to $SU_3 \times SU_3$ currents, are only defined on mass shell one is unhappy about an argument which relies on a choice of their off-mass-shell behavior.

Fortunately, an alternate approach[14] to the problem, which avoids this difficulty, can be found. It is most simply discussed in the framework of the reaction $e + \bar{e} \to \pi + \underline{\overline{X}}$ (semi-inclusive annihilation). On the assumption that this proceeds via one-photon exchange, the cross-section is determined by the hadron tensor

where q is the total lepton momentum and $\nu = q \cdot p$. This is kinematically very similar to the total cross-section discussed in Section 3 except that in this case $q^2 > 0$ and the allowed range of $\omega = q^2/2p \cdot q$ is

$1 < \omega \leq \sqrt{q^2}/2\mu\pi$.. The cross-section is

where θ is the c.m. angle of the observed pion with respect to the incoming electron and $\overline{F}_{T,L}$ are transverse and longitudinal cross-sections defined by

Now $\overline{W}_{\mu\nu}$ is the discontinuity of the forward virtual Compton scattering amplitude, $T_{\mu\nu}$, across the cut in $(q-p^2)$. Since $q^2 > 0$, the full discontinuity of $T_{\mu\nu}$ contains other pieces which are not experimentally interesting and are represented, together with the interesting one, in Fig. 1

Since $\overline{T}_{\mu\nu}$ is just a matrix element of a product of currents, its scaling limit ($q^2 \to +\infty$, $q^2/2q\cdot p$ fixed) may be studied with the help of the techniques of Section 3. Naturally one finds that $\overline{T}_1$ and $\nu\overline{T}_2$ scale and that their longitudinal combination vanishes. One's natural inclination is to assume that each piece of the discontinuity must have the same behavior as the amplitude itself and to carry these results over th $W_{\mu\nu}$ without further ado.

This would be wrong, however, because the discontinuity of $\overline{T}_{\mu\nu}$ in the scaling limit and for $\omega > 1$ is actually zero![4] If we consider, for example, the full discontinuity, $\bar{t}_1$, or $\bar{T}_1$, we find that it has support properties

while being symmetric under crossing: $\bar{t}_1(\nu, \omega) = \bar{t}_1(-\nu, -\omega)$. In the scaling limit, $\nu \to \infty$, $\bar{t}_1(\nu, \omega) \to \bar{t}_1(\omega)$. On the other hand, if we let $\nu \to -\infty$, $\bar{t}_1(\nu, \omega)$ should approach the same function $\bar{t}_1(\omega)$. Hence, crossing implies that $\bar{t}_1(\omega)$ is even in ω and the support properties of $\bar{t}_1(\nu, \omega)$ imply that $\bar{t}_1(\omega) = 0$ for $|\omega| > 1$.

To get around this difficulty, it is necessary to identify the individual pieces of the discontinuity of the scaling limit of $\bar{T}_{\mu\nu}$. This is trivially possible if we are willing to assert that the bilocal $\mathscr{F}_\mu(x, y)$ appearing in Eq. (3.5) is actually the product of local operators at the points x and y. Once this is the case, we can insert complete sets of intermediate states inside the bilocal and pick off the relevant discontinuity without ever having to say anything specific about the source function of the observed pion. The actual predictions for the scaling limit of $\bar{W}_1$ and $\bar{W}_2$ are as expected. The virtue of this approach is that instead of having to make a new assumption for each process one wants to consider one needs only one universal hypothesis about the nature of the bilocal operator.

Positivity constraints exist for these semi-inclusive cross-sections also.[15] Indeed, according to Eq. (5.6) the situation for reactions of the of the type $\gamma + A \rightarrow B$ (unobserved) is the same as for total cross-sections of the type $\gamma + (A\overline{B}) \rightarrow$ (unobserved). The simple quark-model argument given in the last paragraph of Sec. 4 indicates that

for any target a, where a_r is the 180° isospin rotation of a. Consequently, for semi-inclusive cross-sections we will find positivity constraints such as

Many other testable relations can be found, and it will be interesting to see how they compare with experiment.

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TABLE I

	$J^{\eta}\otimes N$	$(J^{\pi}\otimes N)_{1/2}$	$(J^{\pi}\otimes N)_{3/2}$	$(J^{K}\otimes N)_{1}$	$(J^{K}\otimes N)'_{0}$	$(J^{\overline{K}}\otimes N)_{1}$	$(J^{\overline{K}}\otimes N)_{0}$
$J^{\eta}\otimes N$	---	---	0	0	0	0	0
$(J^{\pi}\otimes N)_{1/2}$	---	---	0	0	0	0	0
$(J^{\pi}\otimes N)_{3/2}$	0	0	---	0	0	0	0
$(J^{K}\otimes N)_{1}$	0	0	0	---	0	0	0
$(J^{K}\otimes N)_{0}$	0	0	0	0	---	0	0
$(J^{\overline{K}}\otimes N)_{1}$	0	0	0	0	0	---	0
$(J^{\overline{K}}\otimes N)_{0}$	0	0	0	0	0	0	---

TABLE II

$F_1^{\gamma p}$ =	$\frac{1}{6}\alpha_1^+$	$+\ \frac{1}{9}\alpha_2^+$	$+\ \frac{1}{18}\alpha_3^+$	$+\ \frac{1}{4}\alpha_1^-$	$+\ \frac{1}{36}\alpha_2^-$	$+\ \frac{1}{18}\alpha_3^-$
$F_1^{\gamma \eta}$ =	$\frac{1}{4}\alpha_1^+$	$+\ \frac{1}{36}\alpha_2^+$	$+\ \frac{1}{18}\alpha_3^+$	$+\ \frac{1}{6}\alpha_1^-$	$+\ \frac{1}{9}\alpha_2^-$	$+\ \frac{1}{18}\alpha_3^-$
$F_1^{\nu p}$ =	$1\ \alpha_1^+$	$+\ 0\ \alpha_2^+$	$+\ 0\ \alpha_3^+$	$+\ 1\ \alpha_1^-$	$+\ 0\ \alpha_2^-$	$+\ 0\ \alpha_3^-$
$F_1^{\nu \eta}$ =	$\frac{1}{2}\alpha_1^+$	$+\ \frac{1}{2}\alpha_2^+$	$+\ 0\ \alpha_3^+$	$+\ \frac{1}{2}\alpha_1^-$	$+\ \frac{1}{2}\alpha_2^-$	$+\ 0\ \alpha_3^-$
$F_3^{\nu p}$ =	$-1\ \alpha_1^+$	$+\ 0\ \alpha_2^+$	$+\ 0\ \alpha_3^+$	$+\ 1\ \alpha_1^-$	$+\ 0\ \alpha_2^-$	$+\ 0\ \alpha_3^-$
$F_3^{\nu \eta}$ =	$-\frac{1}{2}\alpha_1^+$	$+\ -\frac{1}{2}\alpha_2^+$	$+\ 0\ \alpha_3^+$	$+\ \frac{1}{2}\alpha_1^-$	$+\ \frac{1}{2}\alpha_2^-$	$+\ 0\ \alpha_3^-$

FIGURE CAPTIONS

Figure 1: The four independent contributions to the discontinuity of the virtual Compton scattering amplitude.

Table I: Block diagonalization, via isospin and hypercharge, of $F^{\pm}_{ar,bs}$

Table II: Measurable structure functions in terms of the positive parameners $\alpha^{\pm}_{i}$.

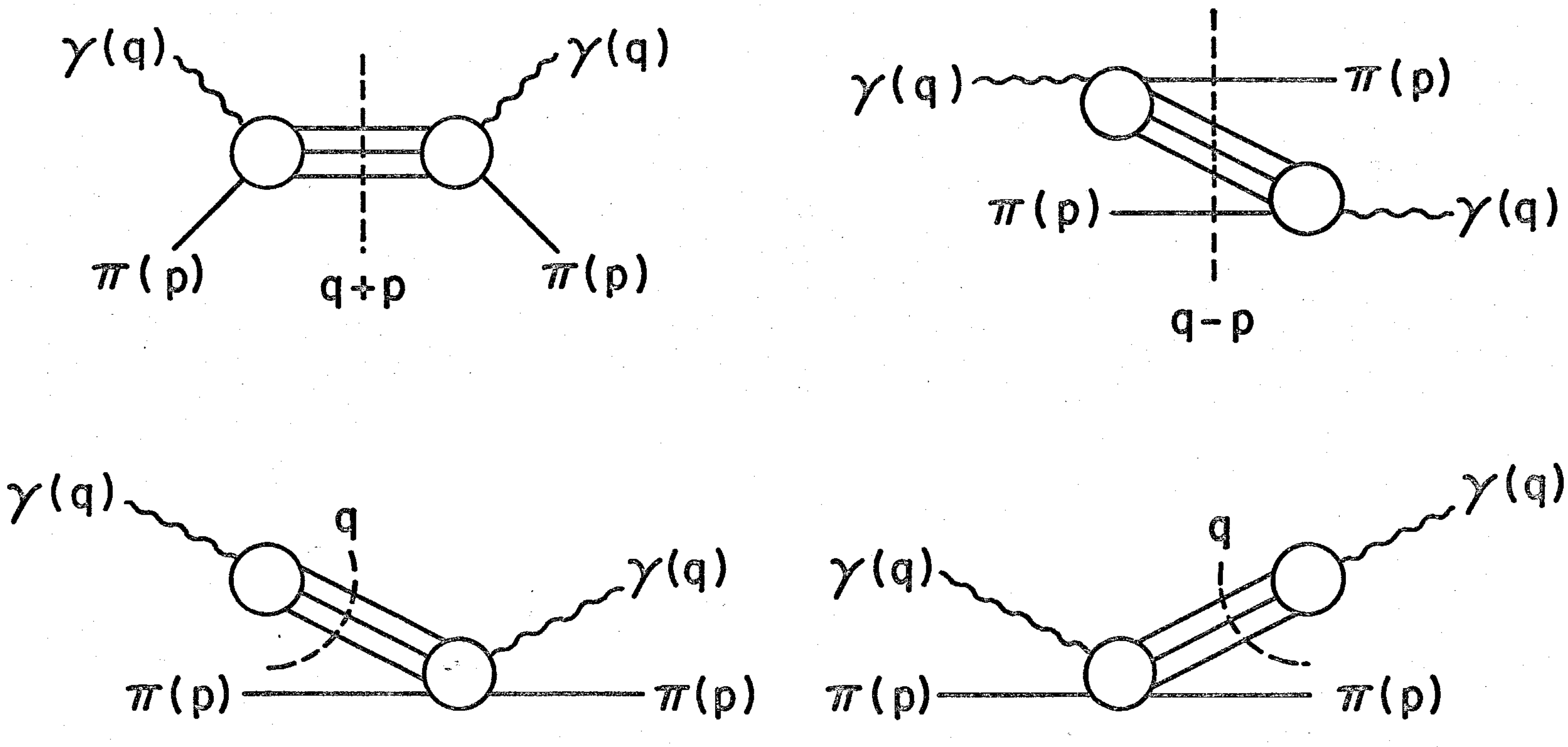

Figure 1