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## Article

# On Propagation in Loop Quantum Gravity

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**Abstract:** A rigorous implementation of the Wheeler–DeWitt equations was derived in the context of Loop Quantum Gravity (LQG) and was coined Quantum Spin Dynamics (QSD). The Hamiltonian constraint of QSD was criticised as being too local and to prevent “propagation” in canonical LQG. That criticism was based on an algorithm developed for QSD for generating solutions to the Wheeler–DeWitt equations. The fine details of that algorithm could not be worked out because the QSD Hamiltonian constraint makes crucial use of the volume operator, which cannot be diagonalised analytically. In this paper, we consider the  $U(1)^3$  model for Euclidean vacuum LQG which consists of replacing the structure group  $SU(2)$  by  $U(1)^3$  and otherwise keeps all properties of the  $SU(2)$  theory intact. This enables analytical calculations and the fine details of the algorithm *into* be worked out. By considering one of the simplest possible non-trivial classes of solutions based on very small graphs, we show that (1) an infinite number of solutions *ingexist* which are (2) generically *not normalisable* with respect to the inner product on the space of spatially diffeomorphism invariant distributions and (3) generically display *propagation*. Due to the closeness of the  $U(1)^3$  model to Euclidean LQG, it is extremely likely that all three properties hold also in the  $SU(2)$  case and even more so in physical Lorentzian LQG. These arguments can in principle be made water tight using modern numerical (e.g., ML or QC) methods combined with the techniques developed in this paper which we reserve for future work.



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**Keywords:** canonical quantum gravity; Wheeler DeWitt equation; propagation

## 1. Introduction

One of the most important unsettled research questions in Loop Quantum Gravity (LQG) [1–4] is the precise implementation of the quantum dynamics, i.e., the quantum Einstein equations or Wheeler–DeWitt equations. A concrete derivation of such equations has been given in [5,6]. It starts from the classical Hamiltonian constraint, which is then discretised in terms of non-Abelian magnetic holonomy and electric flux variables familiar from lattice gauge theory [7], which allow to define a regularised operator on the dense domain  $\mathcal{D}$  of the kinematical Hilbert space  $\mathcal{H}_{\text{kin}}$  consisting of the span of spin network functions [8–10]. The corresponding regulator is essentially the spatial extension of loops associated to vertices and pairs of adjacent edges of the graph on which the operator acts, and the corresponding Wilson loop functions is an approximant to the curvature that appears in the Hamiltonian constraint. It is possible to remove the regulator using an operator topology (coined URST in [5,6]) that exploits the fact that a solution to all quantum constraints must be spatially diffeomorphism invariant and that therefore the subspace of spatially diffeomorphism invariant vectors in the algebraic dual  $\mathcal{D}^*$  of  $\mathcal{D}$  is naturally available in order to define a kind of weak\*–topology. An important property of the classical Hamiltonian constraint is that it is a scalar density of weight, one which is kept in the quantum theory and grants the associated covariance conditions.

Thus, ref. [5,6] provides a concrete proposal (modulo the axiom of choice) for the Hamiltonian constraint of vacuum quantum gravity with a Euclidean or Lorentzian signature in the continuum, which is densely defined on  $\mathcal{D}$ . It is free of anomalies in the sense

that the action of its commutators on the algebraic dual annihilate spatially diffeomorphism invariant elements. However, the Hamiltonian constraint operator constructed in [5,6] is not yet entirely satisfactory for the following reason: While the dual of its commutators annihilate spatially diffeomorphism invariant distributions and thus must be a linear combination of spatial diffeomorphism constraints [8], that linear combination does not qualify as the linear combination that one would expect from a direct quantisation of the corresponding classical Poisson bracket [11] performed by the same methods. Using the analogy of Lie algebras (or algebroids), the quantum constraint algebra closes, but it closes with the wrong structure constants (or functions), the correct ones representing the hypersurface deformation algebra.

Correspondingly, several methods for improvement have been suggested in the literature. These efforts can be subdivided into two classes: In the first class, the issue of the structure functions is avoided altogether; in the second, the correctness of the structure constants is used as a guiding principle to adjust the fine details of the construction proposed in [5,6]. In historical order, the first class comprises the master constraint approach [12] and the reduced phase space quantisation approach [13], while the second class comprises the electric shift approach [14,15] (see also [16] for a preliminary attempt) and the Hamiltonian renormalisation approaching [17] to both the constraint of [5,6] and the physical Hamiltonian of [13] (see also the references in all four manuscripts). Common to the [5,6,14,15] versions of the Hamiltonian constraint is that the Hamiltonian constraint acts non-trivially only at the vertices of the graph, and its action on a given vertex deforms the graph in an open neighbourhood of that vertex while not modifying the graph in neighbourhoods of the other vertices (in [5,6,14,15], this deformation is encoded by the loop approximant to the curvature that one uses). By contrast, in [12,13,17], such a deformation is not considered, as the loops involved are part of the same lattice on which the graph in question is defined where the lattice plays the role of a resolution scale in the sense of the Wilsonian point of view of renormalisation. At that scale, one computes the matrix elements of the Hamiltonian (constraint) (the continuum operator is then some kind of inductive limit of these quadratic forms).

In that sense, the criticism conveyed in [18] should apply to the works [5,6,14,15]: the arguments in [18] are based on [19], where an algorithm is sketched for how to construct solutions in the algebraic dual of both the spatial diffeomorphism constraint of [8] and the Hamiltonian constraint of [5,6]. The construction [19] can be sketched as follows: A solution of the spatial diffeomorphism constraint is a linear combination of the diffeomorphism group averagings  $\eta(S)$  of spin network functions (SNWF)  $S$  (moulo graph symmetries—see [8] for details) which forms an orthonormal basis of the kinematical Hilbert space. Given such an element  $\Psi$ , one must impose that its evaluation on  $C(N)S$  vanishes for all  $N, S$  where  $C(N)$  is the Hamiltonian constraint with lapse function (smearing function of the Hamiltonian constraint)  $N$  and  $S$  is any spin network function. Since  $N$  can be chosen to be supported only in the vicinity of a vertex  $v$  of the graph  $\gamma(S)$  on which  $S$  is supported and since  $C(N)$  acts non-trivially only at those vertices (with action  $C_v$ ), we have equivalently that  $\Psi(C_v S) = 0$  for all  $S$  and all  $v \in V(\gamma(S))$  (vertex set of the graph  $\gamma(S)$  underlying  $S$ ). Since  $C_v$  modifies  $\gamma(S)$  and the colourings  $m(S)$  of  $\gamma(S)$  (by spin quantum numbers on edges and intertwiners at vertices) only in the vicinity of  $v$  leaving the rest of  $S$  unmodified (modulo diffeomorphisms), the following strategy suggests itself:  $C_v S$  can be decomposed into SNWF  $S'$  over new graphs  $\gamma_v$ . Due to the details of the graph deformations  $\gamma_v$  of [5,6] which adds “extraordinary edges”, there are graphs  $\gamma^{(0)}$  which can never be of the form  $\gamma_v$ , and thus, the diffeomorphism group averagings of SNWF over such  $\gamma^{(0)}$  are exact solutions to all constraint equations. We call these solutions “topological” because they are even normalisable with respect to the inner product on the space of solutions to the spatial diffeomorphism constraint which is not to be expected for generic solutions. To construct more interesting solutions, we are thus led to consider (diffeomorphism averages) of graphs which are of the form  $\gamma^{(1)} = \gamma_v^{(0)}$ ,  $v \in V(\gamma^{(0)})$ ; from those graphs, we generate graphs of the form  $\gamma^{(2)} = \gamma_v^{(1)}$ ,  $v \in V(\gamma^{(1)})$ , and proceeding inductively, we generate graphs of

the form  $\gamma^{(n)}$ . We may develop genealogical language and call graphs of the respective form  $\gamma^{(0)}$  primordial,  $\gamma^{(n)}$ ,  $n > 0$  descendants of generation  $n$ , and  $\gamma^{(n+1)}$  children of the parents  $\gamma^{(n)}$ . To classify solutions of the Hamiltonian constraint, it is therefore useful to keep track of the generation  $n$  of a (diffeomorphism average of a) graph descending from a primordial one. Similarly, we have a genealogy of SNWF which is different from that for graphs because also the colourings are involved. No confusion arises if we explicitly say child graph or child SNWF, etc.

A particular feature of the operator [5,6] is that the action of  $C_v$  on SNWF over  $\gamma_{v'}$ ,  $v' \in V(\gamma)$  is trivial for  $v \in V(\gamma_{v'}) - V(\gamma)$  and for  $v \in V(\gamma)$ , it equals the action of  $C_v$  on SNWF over  $\gamma$  (modulo diffeomorphisms). It is precisely this feature which makes the action of  $C(N)$  anomaly free. Thus, regarding modulo diffeomorphisms, all one needs to know in order to let the Hamiltonian constraint act on a SNWF, when evaluated on a diffeomorphism invariant distribution, is a neighbourhood of  $S$  (graph and colourings) of each vertex whose modulo diffeomorphisms can be chosen arbitrarily “small”. For the same reason, it is irrelevant that the axiom of choice is involved in the definition of  $C$  because different choices are washed away by the diffeomorphism averaging and the diffeomorphism invariant characteristics do not need the axiom of choice.

As shown in [19], the space of solutions to all constraints for the Euclidean version acquires the following structure<sup>1</sup>:

- i It is a linear combination of solutions of generation  $n$  for  $n = 0, 1, 2, \dots$ . Here, a solution of generation  $n$  is a linear combination of diffeomorphism averages of SNWF of generation  $n$ .
- ii Such a solution of generation  $n$  is itself a linear combination of elementary solutions of generation  $n$ . An elementary solution of generation  $n$  is a linear combination of diffeomorphism averages of SNWF which are the  $n$ -th generation children of SNWF with the same primordial parent graph.<sup>2</sup> In this paper, as we use the non-symmetric version of the Hamiltonian constraint [5,6], and we take this effect into account when we construct our solutions. We discuss these matters further in Section 3.4.

For the Lorentzian version property, (i) does not hold, as the solutions cannot be built from elementary solutions of constant generation  $n$ , necessarily more generations are involved and it cannot be excluded that a non-trivial typical solution is a linear combination of diffeomorphism averages of SNWF of unbounded generation. On the other hand, property (ii) still holds: elementary solutions are built from the same primordial graph.

Now, assume that in addition, the following property also holds [18] (irrespective of considering the Euclidean or Lorentzian version):

- iii For an elementary solution  $\Psi = \sum_{l \in L} \kappa_l l$  where  $l \in L$  is of the form  $\eta(S)$  and  $\gamma(S)$  is some descendant of the same primordial graph  $\gamma^{(0)}$ , the label set  $L$  can be partitioned into subsets  $L_v$  with  $v \in V(\gamma^{(0)})$  such that the constraint equations resulting from  $C_v$  have non-trivial influence only on  $\kappa_l$  for  $l \in L_v$ .

If this property, which due to the local action of  $C_v$  appears to be obvious, would hold, then a local perturbation of the solution, i.e., a variation of the  $\kappa_l$  for  $l \in L_v$  has no influence on the  $\kappa_{l'}$ ,  $l' \in L_{v'}$ ,  $v' \neq v$ . That is [18], the conclusion would be that there is *no propagation* or *absence of long-range correlations* in the space of solutions to the Hamiltonian constraint of [5,6] in the sense that the structure of the solution at vertex  $v$  is local to  $v$  and has no effect at any other vertex  $v'$ . Similar remarks would hold for the Hamiltonian constraint of [14,15].

In what follows, we focus on the property of propagation in the context of the constraint action of [5,6]. With regard to propagation in the context of the constraint action of [14,15], we restrict ourselves to a few remarks in Section 3.4, these remarks being indicative of a reasonable expectation that this constraint action does display propagation. A detailed analysis of propagation in the context of the constraint [14,15] is left for future work.

Accordingly, the main purpose of the present paper is twofold:

- I First, we show that assumption (iii) above is generically violated in LQG, independent of the signature. In fact, this was implicitly known since the appearance of [5,6].
- II Second, even if assumption (iii) is violated, the conclusion stated in [18] may still hold, as assumption (iii) is not obviously a necessary condition to hold for the absence of propagation to occur. However, we will show that also the conclusion is *extremely likely to be false* in LQG independent of the signature.

In that sense, propagation in QSD was present ever since it was proposed.

Let us explain this in more detail:

The mechanism behind the violation of condition (iii) is the diffeomorphism averaging that occurs in the solution to all constraints which prohibits a partition labelled by vertices of the underlying primordial graph as described above and leads to *non-unique parentage* at the level of diffeomorphism averages of SNWF. The fact that non-unique parentage is the *key* ingredient for propagation in the context of LQG constraint quantisation methods was first realised in the toy model context of Parameterised Field Theory [21]. In fact, the mechanism of non-unique parentage can also seen to be responsible for the absence of anomalies [5,6] in the sense described above and does not rely on the details of the action of the Hamiltonian constraint.

Still, it may accidentally happen that long-range correlations are absent in solutions to the Hamiltonian constraint, and it is here where details matter. The reason why in contrast to (i), we cannot make a statement with certainty in (ii) is due to the fact that the volume operator is a central ingredient of the operators [5,6] whose spectrum cannot be computed analytically. Accordingly, precise computations and estimates are not possible, thus prohibiting explicit solutions. To make progress, in this paper, we consider the  $U(1)^3$  model of Euclidean quantum gravity invented in [22] which shares almost all of its features except that the non-Abelian group  $SU(2)$  is replaced by  $U(1)^3$ . We do this for the sole purpose of being able to compute the corresponding volume spectrum analytically. We are then able, for sufficiently simple primordial graphs, to solve the Hamiltonian constraint equations analytically, thus allowing to compute the linear combinations of diffeomorphism invariant distributions that define the solutions explicitly together with the complete parametrisation of the freedom involved. This parametrisation freedom manifests the presence of long-range correlations or propagation. While the explicit calculations are confined to a particularly simple primordial graph, it follows from the solution-generating technique that we develop that propagation happens also for higher generations and for arbitrarily long iterations and gluings of those simple graphs filling all of a Cauchy surface arbitrarily densely. It will also become clear from the example how to compute the solutions explicitly in the general case. These strong statements can so far not be made within the  $SU(2)$  theory due to the complexity of the volume spectrum. However, and this is the sense in which we use the term “extremely likely” above, it is beyond reasonable doubt that the spectrum of the  $SU(2)$  volume operator does not conspire in such a way as to render the conclusions derived for the  $U(1)^3$  model invalid. Since the Lorentzian Hamiltonian constraint uses the Euclidean version and multiple commutators thereof with the volume operator [5,6], the same conclusion is extremely likely to hold for Lorentzian LQG.

As a by-product of our computations, we are also able to show that an infinite number of non-trivial solutions *exist* which are *not normalisable* with respect to the inner product on the space of spatially diffeomorphism invariant distributions [8]. The possibility of non-existence of non-trivial (i.e., non-topological) or absence of non-normalisable solutions can again not be ruled out for the  $SU(2)$  theory of [19] with certainty but is again beyond reasonable doubt.

The lesson learnt from the present work is that, modulo the reservation just spelled out, at least with respect to the existence, normalisability and propagation aspect, the constraint operator [5,6] does not suffer from pathologies, and we expect that modifications thereof that faithfully implement the hypersurface algebra will share this property.



The architecture of this article is as follows:

In Section 2, we recall the essential ingredients of LQG necessary for the understanding of the present work, sketch the structure of the solutions of the Hamiltonian constraint and define the notion of propagation desired of solutions of the Hamiltonian constraint in LQG.

In Section 3, we define the simple primordial graph and its descendants of generation, one for which we perform explicit calculations in the  $U(1)^3$  model of Euclidian LQG. We establish *existence, non-normalisability and the presence of long range correlations* in its solution space.

In Section 4, we conclude and give an outlook into further study. We state in more detail why these features extend to generic arbitrarily large graphs and are extremely likely to the  $SU(2)$  case in both Euclidean and Lorentzian signatures. In particular, we encourage the application of numerical methods in LQG [23–26] perhaps combined with modern machine learning [27,28] or quantum computing [29–31] techniques in order to perform an actual  $SU(2)$  calculation involving the  $SU(2)$  volume operator at least numerically using the explicitly known matrix elements of its fourth power [32] in order to complete the rigorous proof in the  $SU(2)$  theory.

## 2. LQG, Quantum Einstein Equations and Propagation

We begin by recalling the essential notions from LQG and the corresponding  $U(1)^3$  model. See [5,6] for details on the Hamiltonian constraint, reference [17] for a recent low-technical review of LQG and [33–35] for more details on the  $U(1)^3$  model invented in [22]. After that, we review the solution-generating algorithm of [19] and the notion of propagation defined in [18] slightly adapted to our language.

### 2.1. Essentials from LQG

A closed graph  $\gamma$  is the union of its semi-analytic [9,10] oriented edges  $e \in E(\gamma)$  that intersect nowhere except in its endpoints, which are called vertices  $v \in V(\gamma)$ . Each of its edges  $e$  carries a colouring by an irreducible representation of the underlying gauge group  $G$ , which is assumed to be compact, that is, a half-integral spin quantum number  $j_e$  in case of  $G = SU(2)$  and a vector  $m_e \in \mathbb{Z}^3$  in case of  $G = U(1)^3$ . We are interested in solutions to all constraints and thus for reasons of Yang–Mills type of gauge invariance, each vertex  $v$  is coloured by an intertwiner that intertwines the tensor product of the representations labelling the outgoing edges and the contragredients of the representations labelling the incoming edges with the trivial representation. In case of  $SU(2)$ , the intertwiners  $\iota_v$  are essentially invariant tensors built from Clebsch–Gordon coefficients, in case of  $U(1)^3$ , it is simply a Kronecker symbol  $\delta_v$  that imposes that the sum of incoming charge vectors equals the sum of outgoing charge vectors. A spin network function (SNWF)  $S$  is a function of the underlying connection  $A$ , which is constructed from the corresponding holonomies  $A(e)$  along the edges  $e$  by plugging them into the matrix element functions of the corresponding irreducible representation and contracting their matrix element indices with the intertwiner indices. They are normalised with respect to the product Haar measure where the number of factors equals the number of edges. The kinematical Hilbert space  $\mathcal{H}_{\text{kin}}$  can be defined by declaring the SNWF to be an orthonormal basis. For  $SU(2)$ , we would label  $S = T_{\gamma,j,\iota}$  where  $j, \iota$  stands for the collection of spins and intertwiners, respectively, for  $U(1)^3$ , we write  $S = T_{\gamma,m}$ , dropping the trivial intertwiner label and assuming that the constraints on the collection of charges  $m$  hold. To have a unified notation, we introduce the compound label  $m = (j, \iota)$  also in the  $SU(2)$  case. The span  $\mathcal{D}$  of SNWF defines a dense and invariant domain for the Hamiltonian constraint [5,6]. We define  $\gamma(S) = \gamma$ ,  $m(S) = m$ .

The group of spatial diffeomorphisms  $\varphi \in \text{Diff}(\sigma)$  of the model Cauchy surface  $\sigma$  underlying the canonical formulation of LQG is represented unitarily on  $\mathcal{H}_{\text{kin}}$ , and  $U(\varphi)$  acts on SNWF by dragging  $\gamma$  and its colourings along, i.e.,  $\gamma \mapsto \varphi(\gamma)$ ,  $j_{\varphi(e)} = j_e$ ,  $\iota_{\varphi(v)} = \iota_v$  for  $SU(2)$  and  $m_{\varphi(e)} = m_e$  for  $U(1)^3$ . With some care [8], one can “average” SNWF over  $\text{Diff}(\sigma)$ , resulting in distributions  $\eta(S)$  on  $\mathcal{D}$ : that is, elements of the algebraic dual  $\mathcal{D}^*$  of

$\mathcal{D}$  (linear functionals on  $\mathcal{D}$  without continuity conditions). In case that  $\gamma(S)$  has no graph symmetries (i.e., diffeomorphisms that preserve  $\gamma$  as a set but permute the edges), the linear functional  $\eta(S)$  is defined by  $\eta(S)[S']$  to equal unity when  $S, S'$  differs by a diffeomorphism and zero otherwise (we will only need  $S$  without graph symmetries in this paper). One parameter group of  $\text{Diff}(\sigma)$  does not act strongly continuously on the kinematical Hilbert space so that there is no spatial diffeomorphism constraint operator, but one may still impose the spatial diffeomorphism constraint on  $\Psi \in \mathcal{D}^*$  in the form  $\Psi[(U(\varphi) - \text{id}_{\mathcal{H}_{\text{kin}}})S] = 0$  for all  $S$  and  $\varphi$ . Thus, the general solution to those constraints are linear combinations of the  $\eta(S)$ , and their linear span can be equipped with the inner product  $\langle \eta(S), \eta(S') \rangle_{\text{Diff}} = \eta(S')[S]$  (again, modulo details associated with graph symmetries) with respect to which the  $\eta[S]$  forms an (almost) orthonormal basis.

The Euclidean Hamiltonian constraint  $C(N)$  of [5,6] for smearing (lapse) function  $N$  is densely defined on  $\mathcal{D}$  and for  $\text{SU}(2)$  and  $\text{U}(1)^3$  has the following general form

$$C(N) T_{\gamma,m} = \sum_{v \in V(\gamma)} N(v) C_{\gamma,v} T_{\gamma,m}, \quad C_{\gamma,v} = \sum_{e,e' \in E(\gamma); e \cap e' = v} C_{\gamma,v,e,e'} \quad (1)$$

We will spell out more details about  $C_{\gamma,v,e,e'}$  in the next section. For the current section, it is sufficient to mention that  $C_{\gamma,v,e,e'}$  depends on the volume operator localised at  $v$ , holonomies along partial segments  $s_{\gamma,v,e}$  outgoing from  $v$  of edges  $e$  adjacent to  $v$  and arcs  $a_{\gamma,v,e,e'}$  between the endpoints of  $s_{\gamma,v,e}, s_{\gamma,v,e'}$  which do not intersect  $\gamma$  anywhere else. The arcs form the extraordinary structure of a graph, and a graph without extraordinary edges is called primordial. The endpoints of extraordinary edges are called extraordinary vertices which are co-planar but not co-linear and tri-valent. Repeated actions of the Hamiltonian constraint do not act at the extraordinary vertices but create extraordinary edges ever closer to the vertices of the primordial graph. In addition to adding extraordinary edges,  $C_{\gamma,v,e,e'}$  changes the colourings on  $s_{\gamma,v,e}, s_{\gamma,v,e'}, v$ . Graphs or SNWF with  $n$  extraordinary edges that arise in the  $n$ -th action of the Hamiltonian action arising from SNWF over a primordial graph are called the  $n$ -th generation.

## 2.2. Solutions of the Quantum Einstein Equations and Propagation

By definition, a physical state  $\Psi$  is an element of  $\mathcal{D}^*$  which solves besides the spatial diffeomorphism constraint also  $\Psi[C(N)S] = 0$  for all  $S$  and  $N$ . By choosing lapse functions of compact support, equivalently,  $\Psi[C_{\gamma,v} T_{\gamma,m}] = 0$  for all  $\gamma, m, v \in V(\gamma)$ . We consider a linear combination  $\Psi$  of the  $\eta(S)$  with complex coefficients  $\kappa_{[S]}$  where  $[S]$  denotes the diffeomorphism orbit of  $S$ . It is clear that every  $\eta(S)$  with  $\gamma(S)$  primordial is an exact (we call it *topological*) solution as  $C_{\gamma,v}$  increases the number of extraordinary edges by one. Next, we consider linear combinations of  $\eta(S)$  with  $\gamma(S)$  having precisely one extraordinary edge, i.e., graphs in the first generation. The conditions on the complex coefficients are now only non-trivial when  $\gamma$  in  $\Psi[C_{\gamma,v} T_{\gamma,m}]$  is primordial. We try to construct a simple subset of all those solutions by a picking a single primordial graph  $\gamma^{(0)}$  and all its first generation descendants  $\gamma^{(1)}$  and all possible colourings of those forming SNWF  $S'$ . Denote by  $\Sigma([\gamma^{(0)}])$  all  $[S']$  for these  $S'$  and consider  $\Psi = \sum_{[S'] \in \Sigma([\gamma^{(0)}])} \kappa_{[S']} \eta(S')$ . Then, the constraint equation reduces to  $\Psi[C_{\gamma^{(0)},v} T_{\gamma^{(0)},m}] = 0$  for all  $m, v \in V(\gamma^{(0)})$ , which is still an infinite number of conditions. Similar simplifications can be achieved for solutions of higher generations and also for the Lorentzian constraint; see [19] for details.

Typically, the number of coefficients  $\kappa_{[S']}$  exceeds the number of  $S = (\gamma^{(0)}, m)$  for which the constraint equations are not automatically satisfied, at least in a naive counting (both numbers are infinite), simply because the number of children graphs together with all its colourings exceeds the number of parent graphs (in this case only one) together with all colourings. Hence, one expects a rich number of non-trivial solutions. In the  $\text{SU}(2)$  theory, this cannot be granted with certainty because the numbers  $\eta(S')[C_{\gamma,v,e,e'} T_{\gamma,m}]$  are not analytically available, and the naive counting involving the subtraction of infinities is a dangerous enterprise. Thus, even the question of *existence* is not entirely trivial to answer in the  $\text{SU}(2)$  theory. Next, it may or may not be true that there exist solutions with only a

finite number of non-vanishing coefficients  $\kappa_{[S]}$  (besides the topological solutions) which would be normalisable with respect to the diffeomorphism invariant inner product. If such solutions would exist, we would call them *bounded*, as they have finite norm with respect to the diffeomorphism invariant inner product. On physical grounds, one would expect the interesting solutions to be unbounded as the Hamiltonian constraint at constant lapse is a diffeomorphism invariant unbounded operator on the diffeomorphism invariant Hilbert space so that zero should not be exclusively in its point spectrum. Finally, we come to the central question of the present work:

For simplicity, we consider the problem of constructing solutions involving only diffeomorphism averages of graphs with one extraordinary edge (first generation children). Similar remarks hold for solutions involving only diffeomorphism averages of constant generation  $n$  [19]. To further simplify the analysis, we may focus on children graphs  $\gamma^{(1)} \in \Gamma(\gamma^{(0)})$  which result from a single primordial parent graph  $\gamma^{(0)}$  via the action of the Hamiltonian constraint. As the action of the Hamiltonian constraint is local to the vertices,  $\Gamma(\gamma^{(0)})$  is the disjoint union of subsets  $\Gamma_v(\gamma^{(0)})$ ,  $v \in V(\gamma^{(0)})$  where  $\gamma_v^{(1)} \in \Gamma_v(\gamma^{(0)})$  is a graph label that appears in the SNWF decomposition of the vectors  $C_{\gamma^{(0)},v} T_{\gamma^{(0)},m}$  for some  $m$ . Let us denote for a graph  $\gamma$  by  $M(\gamma)$  the set of its possible colourings  $m$ . Then, the Ansatz for candidate solution reads in more detail

$$\Psi = \sum_{v \in V(\gamma^{(0)})} \sum_{\gamma_v^{(1)} \in \Gamma_v(\gamma^{(0)})} \sum_{m_v \in M(\gamma_v^{(1)})} \kappa_{v, \gamma_v^{(1)}, m_v} \eta(T_{\gamma_v^{(1)}, m_v}) \quad (2)$$

which involves a countably infinite sum<sup>3</sup> of diffeomorphism invariant distributions. Then, the quantum Einstein equations reduce to the equations

$$\Psi[C_{\gamma^{(0)},v} T_{\gamma^{(0)},m}] = 0 \quad \forall \quad v \in V(\gamma^{(0)}), \quad m \in M(\gamma^{(0)}) \quad (3)$$

because for any SNWF over a graph not in the diffeomorphism class  $[\gamma^{(0)}]$ , the equation is identically satisfied, and for any graph diffeomorphic to  $\gamma^{(0)}$ , the equations are strictly identical. Suppose now that

$$\eta(T_{\gamma_v^{(1)}, m_v})[C_{\gamma^{(0)},v'} T_{\gamma^{(0)},m}] \propto \delta_{v,v'} \quad (4)$$

then (3) splits into the  $|V(\gamma^{(0)})|$  independent sets of equations

$$\sum_{\gamma_v^{(1)} \in \Gamma_v(\gamma^{(0)})} \sum_{m_v \in M(\gamma_v^{(1)})} \kappa_{v, \gamma_v^{(1)}, m_v} \eta(T_{\gamma_v^{(1)}, m_v})[C_{\gamma^{(0)},v'} T_{\gamma^{(0)},m}] = 0 \quad (5)$$

which only involves the coefficients  $\kappa_{v, \gamma_v^{(1)}, m_v}$ . That is, the assumption (4) leads to a decoupling of the system (3) with respect to the vertex label, and the sets of equations (5) can be solved independently. In particular, setting  $\kappa_{v', \gamma_{v'}^{(1)}, m_{v'}} = 0$  for all  $v' \neq v_0$  and solving (5) for  $v = v_0 \in V(\gamma^{(0)})$  would yield a solution. Assuming that non-trivial solutions exist, the solution coefficients  $\kappa_{v, \gamma_v^{(1)}, m_v}$  will involve (typically infinitely many) free parameters which we collectively denote by  $\alpha_v$  and which parametrise the kernel of (5). These  $\alpha_v$  therefore correspond to *observables*<sup>4</sup>, i.e., gauge invariants which are unconstrained and invariant by the constraints and their gauge motions and which keep some degree of locality as they are associated with the vertex  $v$  in the diffeomorphism class of  $\gamma^{(0)}$ . Let us denote by  $\{\Psi(\{\alpha_v\}_{v \in V(\gamma^{(0)})})\}$  the complete space of solutions so obtained. Then, perturbing  $\Psi(\{\alpha_v\})$  with respect to  $\alpha_v$  has no influence on any other  $\alpha_{v'}$ ,  $v' \neq v$ . Since the  $\alpha_v$  parametrise solutions of the quantum Einstein equations, they parametrise, likely very indirectly, histories of spacetime metrics on the manifold  $\mathbb{R} \times \sigma$ . Then, the discussion suggests that performing a spatially local perturbation of a quantum solution has no global effect. If this picture is correct, then in the Lorentzian theory, we would conclude that the quantum solutions are



incompatible with the classical solutions of Einstein's equations which are known to be well posed in globally hyperbolic spacetimes which are the spacetimes considered in canonical quantum gravity (the perturbation of an entire history does not have compact support and thus has an unbounded domain of dependence). In particular, thinking of the  $\alpha_v$  as relational observables that depend on an intrinsic physical time parameter  $\tau$ , a variation of  $\tau$  should affect all  $\alpha_v$ . While these remarks are not entirely conclusive, the absence of propagation or long-range correlations that follows from (4) is at least worrisome.

This potential problem spelled out in [18] rests on the assumption (4), which seems to be quite reasonable, as the Hamiltonian constraint acts arbitrarily closely to  $v$  modulo diffeomorphisms. The catch is that the notion of closeness becomes void after diffeomorphism averaging. Thus, it may happen that while  $\gamma_v^{(1)} \in \Gamma_v(\gamma^{(0)})$  and  $\gamma_{v'}^{(1)} \in \Gamma_{v'}(\gamma^{(0)})$  are different graphs  $\gamma_v^{(1)} \neq \gamma_{v'}^{(1)}$  for  $v \neq v'$  and the corresponding SNWF are orthogonal, still  $[\gamma_v^{(1)}] = [\gamma_{v'}^{(1)}]$ ! The atomic prime example for this effect is a graph  $\gamma^{(0)}$  with two vertices  $v, v'$  and four non-co-planar edges between them (to have non-vanishing volume). Pick two of its edges  $e, e'$ . Then, there are contributions from  $C_{\gamma^{(0)}, v, e, e'}$  and  $C_{\gamma^{(0)}, v', e, e'}$  that attach arcs  $a_{\gamma^{(0)}, v, e, e'}$  and  $a_{\gamma^{(0)}, v', e, e'}$ , respectively, to  $\gamma^{(0)}$ , defining one of the possible  $\gamma_v^{(1)}, \gamma_{v'}^{(1)}$ , respectively. However,  $[\gamma_v^{(1)}] = [\gamma_{v'}^{(1)}]$ ! Thus, while the arcs labelled by  $v, v'$  are very "close" to  $v, v'$ , respectively, they are also very close to  $v', v$  modulo diffeomorphisms. Therefore, we may find  $T_{\gamma^{(0)}, m}, T_{\gamma^{(0)}, m'}$  with  $m \neq m'$  but such that the SNWF decompositions of  $C_{\gamma^{(0)}, v} T_{\gamma^{(0)}, m}$  and  $C_{\gamma^{(0)}, v'} T_{\gamma^{(0)}, m'}$  contains SNWF  $S, S'$  with  $[S] = [S']$  although  $\langle S, S' \rangle_{\mathcal{H}_{\text{kin}}} = 0$ . Abusing the notation, we call this effect the *non-unique parentage* of SNWF where the abuse refers to the fact that not  $S = S'$  but only  $[S] = [S']$ . The effect is triggered by the mechanism of diffeomorphism averaging. Indeed, the same mechanism is responsible for the fact that the algebra of Hamiltonian constraints closes modulo diffeomorphisms [5,6]. Therefore, assumption (4) is *false* whenever there is non-unique parentage. Whenever there is non-unique parentage, even the Ansatz (2) is strictly speaking incorrect, as there is an over-counting involved: Since some of the  $\eta(T_{\gamma_v^{(1)}, m_v}), \eta(T_{\gamma_{v'}^{(1)}, m_{v'}})$  for different  $v, v'$  and certain  $m_v \in M(\gamma_v^{(1)}), m_{v'} \in M(\gamma_{v'}^{(1)})$  are in fact identical, the separate coefficients  $\kappa_{v, \gamma_v^{(1)}, m_v}, \kappa_{v', \gamma_{v'}^{(1)}, m_{v'}}$  collapse to a single coefficient  $\kappa_{v, \gamma_v^{(1)}, m_v} + \kappa_{v', \gamma_{v'}^{(1)}, m_{v'}}$ . This fact is the technical reason for the *coupling* between the constraint equations resulting from different  $v, v'$ . Note that non-unique parentage implying the failure of (4) is a generic feature of LQG and does not depend on the fine details of the quantum dynamics.

We finish this section by providing a concrete technical definition for the presence of propagation in a single first-generation solution to the Hamiltonian constraint which is motivated by the above discussion [18], and the further analysis in this paper. Its virtue is that it is free of any details of how one actually finds solutions and thus can be stated rather non-technically. A second more technical definition of propagation in a whole class of solutions which is closer to the actual construction of solutions will be deferred to Section 3.9 after we have illustrated the construction algorithm. Both definitions are to be considered as working definitions that may have to be refined in the future as we gain more experience with propagation in ever more complicated solution classes.

### 2.3. Working Definition of Propagation

#### 2.3.1. Preliminaries

A candidate first-generation solution  $\Psi$  of the quantum Einstein equations based on a primordial parent graph  $\gamma^{(0)}$  can be written as a linear combination with non-redundant coefficients  $\kappa_{\bar{v}, l}$  where  $\bar{v}$  runs through a subset  $V$  of  $V(\gamma^{(0)})$  and at given  $\bar{v}$ ,  $l$  runs through a subset  $L_{\bar{v}}$  of pairs  $\gamma_{\bar{v}}^{(1)}, m_{\bar{v}}$  with  $\gamma_{\bar{v}}^{(1)} \in \Gamma_{\bar{v}}(\gamma^{(0)})$  and  $m_{\bar{v}} \in M(\gamma_{\bar{v}}^{(1)})$  taking into account non-unique parentage (i.e., there is no overcounting), that is,

$$\Psi = \sum_{\bar{v} \in V} \sum_{l \in L_{\bar{v}}} \kappa_{\bar{v}, l} \eta(T_l) \quad (6)$$

We define  $C_v, \Psi_v, V_v, C_{v_1 \wedge v_2}, \Psi_{v_1 \wedge v_2}$  as follows:

Fix a vertex  $v \in V(\gamma^{(0)})$  and consider the set  $C_v$  of elements  $\eta(T_l)$  where  $\eta(T_l) \in C_v$  if there exists a primordial parent based on  $\gamma^{(0)}$  on which the action of the constraint at vertex  $v$  produces, up to the action of a diffeomorphism, the state  $T_l$ .

We define the restriction of  $\Psi$  to a vertex  $v \in V(\gamma^{(0)})$  to be the state  $\Psi_v$  obtained by setting to zero in (6), the coefficients of all  $\eta(T_l) \notin C_v$ .

Define the set  $V_v \subset V(\gamma^{(0)})$  where  $v' \in V_v$  if  $\Psi_v$  fails to annihilated by the constraint action at  $v'$ .

Define the set  $C_{v_1 \wedge v_2}$  to be the set of all  $\eta(T_l)$  in (6) for which  $T_l$  (up to the action of a diffeomorphism) is produced by the action of the constraint at  $v_1$  as well as at  $v_2$ . Define the restriction  $\Psi_{v_1 \wedge v_2}$  of  $\Psi$  to be the state obtained by setting to zero in (6) the coefficients of all  $\eta(T_l) \notin C_{v_1 \wedge v_2}$ .

### 2.3.2. Definition of Propagation

If there exists  $v \in V(\gamma^{(0)})$  such that  $V_v \neq \emptyset$ , we say that  $\Psi$  encodes propagation. If  $V_v = \emptyset$  for some  $v \in V(\gamma^{(0)})$ , we say that there is no propagation from  $v$ . If  $V_v = \emptyset$  for every  $v \in V(\gamma^{(0)})$ , we say that the  $\Psi$  does not encode propagation.

### 2.3.3. Definition of Propagation Distance

Fix  $v \in V(\gamma^{(0)})$  and consider any other  $v' \in V(\gamma^{(0)})$ . If  $\Psi_{v \wedge v'}$  does not solve the equations at  $v'$ , we say that there is *immediate propagation from  $v$  to  $v'$*  else that there is no immediate propagation from  $v$  to  $v'$ . Note that there can be immediate propagation from  $v$  to  $v'$  as well as from  $v'$  to  $v$  and there could also be ‘one-way’ immediate propagation only from  $v$  to  $v'$  but not vice versa.

Note also that  $\Psi_{v \wedge v'}$  solves the constraint equations at  $v'$  for all  $v' \notin V_v$ . This follows immediately from the fact that (a)  $\Psi_v$  solves these equations at such  $v'$  and (b) the equations for  $\Psi_v$  at any  $v' \neq v$  only involve elements of  $C_{v \wedge v'}$ . It follows that for  $v' \notin V_v$ , there is no immediate propagation from  $v$  to  $v'$ . It also follows from (b) and the definition of  $V_v$  that there is immediate propagation from  $v$  to any element of  $V_v$ .

A chain  $C_{v_1, v_2}$  of propagation of length  $n$  from  $v_1$  to  $v_n$  is a set of vertices  $\{v_i\} \subset V(\gamma^{(0)})$  such that there is immediate propagation from  $v_i$  to  $v_{i+1}$ ,  $i = 1, \dots, n-1$ . If there exists a chain  $C_{v_1, v_2}$ , then we say that there is propagation from  $v_1$  to  $v_2$ . If there is propagation from  $v_1$  to  $v_2$ , the *propagation distance* from  $v_1$  to  $v_2$  is the length of the shortest chain from  $v_1$  to  $v_2$ .

### 2.3.4. Comments

The various choices of  $V$  (see the discussion before (6)) are a direct consequence of non-unique parentage, and thus, any choice of  $V$  as above is a valid choice capturing the idea that the corresponding term in (6) is produced by the action of the constraint at  $v$ . Note, however, that the role of the set  $V$  is only to provide an explicit labelling index for the coefficients which appear in (6). The set  $V$  plays no role in the above definition of propagation.

The definition of propagation is based directly on the coupling between the constraint equations at different vertices. The intuitive picture underlying the definition as formulated above is as follows. We think of a ‘perturbation’ or ‘disturbance’ (roughly speaking, the extraordinary edge between two vertices) as already being present in the solution  $\Psi$  with each child  $\eta(T_l)$  thought of as the quantum analog of canonical data on a slice. The way we determine how the perturbation ‘evolves’ depends on how we view these ‘quantum slices’. Consider a vertex  $v$  with the children obtained through the constraint action at  $v$  being viewed as encoding a perturbation/disturbance/signal originating at  $v$ . Suppose  $v' \in V_v$ . Then, this signal, encoded in the precise combination of children with parental vertex  $v$ , propagates through  $v'$  to a precise combination of children with parental vertex  $v'$ . If this signal was absent, the effect of its propagation (encoded in the precise coefficients of the children obtained through the insertion of extraordinary edges on edges which do not

connect  $v'$  to  $v$ ) would also be absent (i.e., the combination of children with parental vertex  $v$   $\Psi_{v \wedge v'}$  do not solve the equations at  $v'$  but need to be augmented by the combination of the remaining children with parental vertex at  $v'$ ). Note that since  $v'$  shares some parentage with  $v$ , it must be connected to and hence an ‘immediate’ neighbor of  $v$ : hence the nomenclature ‘immediate propagation’. On the other hand, if  $v' \notin V_v$ , then either  $v$  and  $v'$  are not nearest neighbors (so there is no question of immediate propagation), or they are nearest neighbors, but the presence of children with parental vertex  $v$  does not prevent the precise combination of the rest of the children with parental vertex  $v'$  from solving the equations at  $v'$ , i.e., not only does  $\Psi_{v'}$  automatically solve these equations but so does  $\Psi_{v'} - \Psi_{v \wedge v'}$ . In the latter case, the disturbance between  $v$  and  $v'$  (encoded in  $\Psi_{v \wedge v'}$ ) may be viewed as not propagating beyond  $v'$ .

Again, with more wording, it is obvious how to generalise this to solutions of higher generation or to solutions which involve more than one generation. In contrast to the presence of non-unique parentage, the presence of propagation does depend on the details of the dynamics, as we need access to explicit solutions to determine if they encode propagation. These are difficult to construct in a  $SU(2)$  calculation for the reason that the volume operator is not analytically diagonalisable. Thus, one has to resort to numerical techniques. To motivate an in-depth numerical analysis for  $SU(2)$ , we turn to the  $U(1)^3$  theory in the next section and study an almost atomic example for which a multiparameter class of solutions can be constructed.

This example opens up the possibility of an alternate definition of propagation based on a class of solutions rather than a single fixed solution as above. As already mentioned, we shall discuss this alternate definition after we construct the class of solutions alluded to above. Since the definition in this section is based on a single solution, we refer to it as a definition of *intrinsic* propagation in contrast to the alternative, ‘space of solutions’-dependent definition, which we refer to as a definition of *extrinsic* propagation. We emphasise again that propagation is a subtle notion, and its capture in a complete definition is expected to rely on experience with a variety of concrete examples. Consequently, both the intrinsic definition and the extrinsic one are to be taken as working definitions to be modified in response to inputs in the future.

### 3. Demonstration of Existence, Non-Normalisability and Propagation for $U(1)^3$ Model

In this section, we will study the question of existence, boundedness and propagation for a generation 1 class of solutions descending from a concrete and simple primordial  $\gamma^{(0)}$ . We work within the  $U(1)^3$  theory for which all constraint equations can be solved explicitly. Thus, the colourings  $m, n, \dots$  are now valued in  $\mathbb{Z}^3$ . Accordingly, we speak of charge network functions (CNWF) rather than SNWF.

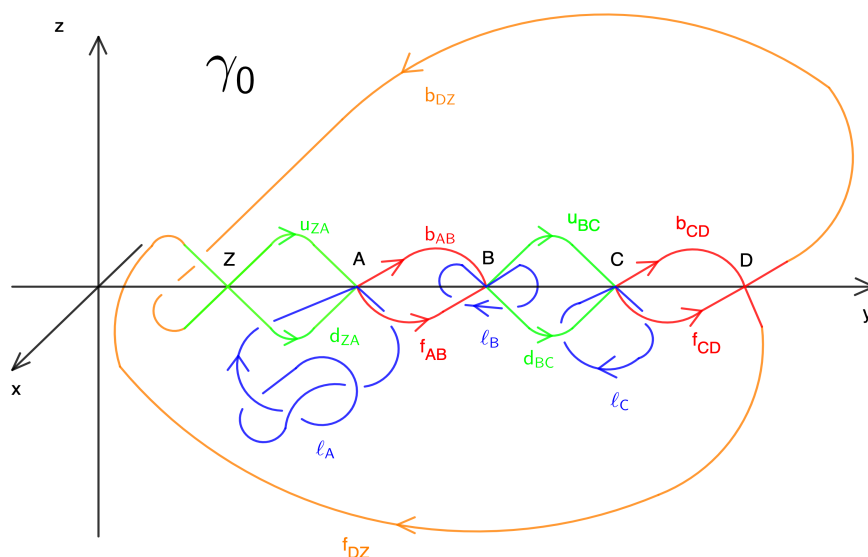
#### 3.1. The Primordial Parent Graph

In what follows, we provide the necessary and sufficient information on the parent graph, such that the action of the Hamiltonian constraint is unambiguously defined:

We consider a closed graph  $\gamma_0$  with five vertices, three of which are six-valent and two of which are four-valent, see Figure 1. The graph is chosen so small that it fits into a single chart and, to be very explicit, we choose a right-oriented coordinate frame such that on a piece of paper, the y-axis points to the right, the z-axis points upwards, and the x-axis points towards the observer out of the sheet of the paper plane. The five vertices all lie on the y-axis, and we label them  $Z, A, B, C, D$  from left to right. There are two edges each between the pairs of vertices  $(Z, A)$ ,  $(A, B)$ ,  $(B, C)$ ,  $(C, D)$  and  $(D, Z)$ , respectively, and no others that connect different vertices, i.e., there are altogether ten such edges. We choose the pair of edges between  $Z, A$  and  $B, C$ , respectively, to lie in the y,z plane, while we choose the pair of edges between  $A, B$  and  $C, D$ , respectively, to lie in the x,y plane. The pair of edges between  $D, Z$  is such that the four edges at  $Z$  and  $D$ , respectively, have co-planar tangents there. All edges are semi-analytic and are oriented such that the respective pairs point from  $Z$  to  $A$ , from  $A$  to  $B$ , from  $B$  to  $C$ , from  $C$  to  $D$  and from  $D$  to  $Z$ . In addition, at

vertices  $A, B, C$  there is a semi-analytic loop attached, intersecting the graph nowhere else as specified below.

More specifically, the beginning segments of the pair of edges between  $D, Z$  lie in the  $x, y$  plane, and their end segment lies in the  $y, z$  plane, while in between, they wind around accordingly without intersecting or knotting. The other edges also do not knot. The two edges between  $Z, A$  and  $B, C$ , respectively, are such that one of them lies above (“up”) the  $x, y$  plane and the other lies below (“down”). Accordingly, we label them  $u_{ZA}, d_{ZA}$  and  $u_{BC}, d_{BC}$ , respectively. Likewise, the two edges between  $A, B$  and  $C, D$ , respectively, are such that one of them lies above the  $y, z$  plane (“front” of the sheet) and the other lies below (“back” of the sheet). Accordingly, we label them  $f_{AB}, b_{AB}$  and  $f_{CD}, b_{CD}$ , respectively. Finally, the pairs of edges  $u_*, d_*, * \in \{ZA, BC\}$  intersect in both their beginning and final point at a non-vanishing angle (say  $\pi/2$ ), and likewise, the pairs of edges  $f_*, b_*, * \in \{AB, CD\}$ . The edges between  $D, Z$  will be denoted  $f_{DZ}, b_{DZ}$ , respectively, where  $f_{DZ}$  is of f-type when leaving  $D$  and of u-type when entering  $Z$ , while  $b_{DZ}$  is of b-type when leaving  $D$  and of d-type when entering  $Z$ . The loops are labelled  $l_A, l_B, l_C$ , respectively, and have the following properties:  $l_A$  has a beginning analytic segment which is tangent to  $u_{ZA}$  at  $A$  and an end segment tangent to  $b_{AB}$  at  $A$  and is semi-analytic in between. Likewise,  $l_B$  is tangent to  $f_{AB}$  in  $B$  in its beginning and to  $d_{BC}$  at  $B$  in its end, while  $l_C$  is tangent to  $u_{BC}$  at  $C$  in the beginning and to  $b_{CD}$  at  $C$  in its end. The loop  $l_A$  is knotted, while  $l_B, l_C$  are unknotted.



**Figure 1.** The parent graph. Green lines are in the  $y, z$  plane and red lines are in the  $x, y$  plane. Orange lines interchange between these planes. The u (d) lines have positive (negative)  $z$  coordinates, while the f (b) lines have positive (negative)  $x$  coordinates. The blue loops are tangent to edges at their endpoints as indicated.

### 3.2. The Parent Charges

In the  $U(1)^3$  theory, the edges of charge network functions (CNWF) carry a charge vector, i.e., a triple of integers. Note that the charge vector by definition is non-trivial; i.e., at least one of those integers is supposed to be non-vanishing; otherwise, the CNWF is defined over the smaller graph with the uncharged edge dropped. We denote the charge vectors of the edges  $u_{ZA}, f_{AB}, u_{BC}, f_{CD}, b_{DZ}$ , respectively, by  $m_{ZA}, m_{AB}, m_{BC}, m_{CD}, m_{DZ}$  and of the edges  $d_{ZA}, b_{AB}, d_{BC}, b_{CD}, f_{DZ}$ , respectively, by  $n_{ZA}, n_{AB}, n_{BC}, n_{CD}, n_{DZ}$ . The charge vectors of the loops  $l_A, l_B, l_C$  are denoted, respectively, by  $c_A, c_B, c_C$ .

We only consider CNWF, which are solutions to the  $U(1)^3$  Gauss constraint, which imposes the constraint that for all  $*$   $\in \{ZA, AB, BC, CD, DZ\}$ , the vector

$$N_* := m_* + n_* \equiv N \quad (7)$$

is the same vector  $N$ , i.e., independent of the label  $*$ . There is no constraint on the loop charges. Accordingly, the CNWF over  $\gamma_0$  is unambiguously labelled by nine vectors in  $\mathbb{Z}^3$ , say the five  $m_*$ , the three  $c_*$  and  $N$ .

### 3.3. Definition of the $U(1)^3$ Hamiltonian Constraint

The  $U(1)^3$  analog of the (smeared) Euclidean Hamiltonian constraint is defined on a general CNWF  $T_{\gamma,m}$  up to a state vector independent factor by

$$\begin{aligned} C(f) T_{\gamma,m} &= \sum_{v \in V(\gamma)} \frac{f(v)}{[2i] T(v)} C_v T_{\gamma,m} \\ C_v &= \sum_{e_1 \cap e_2 \cap e_3 = v} C_{v,e_1,e_2,e_3} \\ C_{v,e_1,e_2,e_3} &= \sum_{I,J,K,L \in \{1,2,3\}} \epsilon^{IJK} [h_{\alpha_{\gamma,v,e_1,e_3}}^I - (h_{\alpha_{\gamma,v,e_1,e_3}}^I)^{-1}] h_{s_{\gamma,v,e_3}}^L [V_v, (h_{s_{\gamma,v,e_3}}^L)^{-1}] \end{aligned} \quad (8)$$

where the notation is as follows:  $V(\gamma)$  is the set of vertices of  $\gamma$  and  $E(\gamma)$  is the set of its oriented edges. The natural number  $T(v)$  is the number of unordered triples of edges adjacent to  $v$  which have linearly independent tangents at  $v$ , and the second sum in (8) is the sum over those triples of edges. If an edge  $e$  is adjacent to  $v$ , we define  $s_{\gamma,v,e}$  to be a segment of  $e$  connected to  $v$  but not to the other endpoint of  $e$  with an orientation outgoing from  $v$ . Given two edges  $e, e'$  adjacent to  $v$ , we consider a loop  $\alpha_{\gamma,v,e,e'}$  starting in  $v$  along  $s_{\gamma,v,e}$  and ending in  $v$  along  $s_{\gamma,v,e'}^{-1}$ . To complete the loop, there is a connecting arc  $a_{\gamma,v,e,e'}$  from the endpoint of  $s_{\gamma,v,e}$  to the endpoint of  $s_{\gamma,v,e'}$  which is unknotted and intersects  $\gamma$  nowhere else. Thus,

$$\alpha_{\gamma,v,e,e'} = s_{\gamma,v,e} \circ a_{\gamma,v,e,e'} \circ s_{\gamma,v,e'}^{-1} \quad (9)$$

We require  $a_{\gamma,v,e,e'} = a_{\gamma,v,e',e}^{-1}$  so that also  $\alpha_{\gamma,v,e,e'} = \alpha_{\gamma,v,e',e}^{-1}$ . The ordering within the triple is such that the tangents of  $s_{\gamma,v,e_1}, s_{\gamma,v,e_2}, s_{\gamma,v,e_3}$  at  $v$  form a right oriented basis if they are linearly independent. The routing of the arc through the edges incident at  $v$  is described in detail in [5,6] and reduces for the case that  $e, e'$  lie in a coordinate plane to let the arc also lie in that plane. Finally, by  $h_p^j$ , we denote the holonomy of the connection  $A^j$ ,  $j = 1, 2, 3$  along the path  $p$ .

The operator  $V_v$  appearing in (8) is the  $U(1)^3$  analog of the Ashtekar–Lewandowski volume operator, which reads explicitly up to a state vector independent factor

$$V_v = \left| \frac{1}{3!} \sum_{e_1 \cap e_2 \cap e_3 = v} \sigma(s_1, s_2, s_3) \epsilon_{jkl} X_{s_1}^j X_{s_2}^k X_{s_3}^l \right|^{1/2} \quad (10)$$

where  $s_I$  is the shorthand for  $s_{\gamma,v,e_I}$ , the integer  $\sigma(s_1, s_2, s_3)$  is the sign of the determinant of the matrix of column vectors  $(\dot{s}_1(0), \dot{s}_2(0), \dot{s}_3(0))$  and  $X_s^j = -h_s^j \frac{\partial}{\partial h_s^j}$ . Note that in (10), we sum over all triples of edges, not only those whose tangents are linearly independent and not only those whose ordered tangents are right oriented. The whole purpose of considering the  $U(1)^3$  truncation of the actual  $SU(2)$  theory is that (10) is diagonal on CNWF, while in the  $SU(2)$  theory,  $X_s^j$  is replaced by a right invariant vector field on  $SU(2)$  so that (10) needs the spectral theorem for an explicit evaluation which except for specific spin configuration is not possible analytically.

Specifically, suppose that an edge  $e$  is adjacent to  $v$ , carries charge  $m_e$  and is outgoing from (incoming to)  $v$  so that  $e = s^{\mathbb{Z}_{\gamma,v,e}} \circ e'$  with  $\mathbb{Z}_{\gamma,v,e} = +1$  ( $\mathbb{Z}_{\gamma,v,e} = -1$ ) where  $e'$  is disjoint



from  $v$ . Then,  $X_s^j$  is diagonal on the corresponding CNWF with eigenvalue  $z_{\gamma,v,e} m_e^j$ , and the eigenvalue of the volume operator is given explicitly by

$$\nu_v = |Q_v|^{1/2}, \quad Q_v = \frac{1}{3!} \sum_{e_1 \cap e_2 \cap e_3 = v} \left[ \prod_{l=1}^3 z_l \right] \sigma(s_1, s_2, s_3) \det(m_{e_1}, m_{e_2}, m_{e_3}) \quad (11)$$

### 3.4. Notes on Closability

For completeness, although not necessary for the understanding of the rest of this paper, we note that the operator (8) is not symmetric in an obvious way. As mentioned, the arguments of [20] even suggest that the Hamiltonian constraint operator *must not be* symmetric. Nevertheless, the way it stands, it is not even closable (i.e., its adjoint is densely defined), which would be a prerequisite for having a symmetric operator. This is because it may happen that in the decomposition into CNWF  $T_{\gamma'}$  of its action on a CNWF  $T_{\gamma}$ , some  $\gamma'$  appear that contain the extraordinary edges that the Hamiltonian constraint adds but not all of the beginning segments  $s_{\gamma,v,e}$ . Clearly, this ‘edge disappearance’ occurs only if these segments in  $T_{\gamma}$  happen to be labelled by charges which are cancelled by the charges carried by the corresponding segments of the loops added by the constraint action. Since such  $T_{\gamma'}$  can be produced from an uncountably infinite number of mutually orthogonal  $T_{\gamma}$  (e.g., all  $\gamma$  that differ from a given  $\gamma_0$  by deforming just  $s_{\gamma_0,v,e}$  into  $s_{\gamma,v,e}$ ), the adjoint of (8) is not densely defined on CNWF.

In the context of gauge group  $SU(2)$  (i.e., full blown gravity rather than the  $U(1)^3$  model), this disappearance of edges with consequent non-closability can occur with the constraint action constructed in the arXiv version of Reference [19] whenever the segments of  $T_{\gamma}$  carry spin  $\frac{1}{2}$ . This phenomenon implies that the solutions tracing to a single unique primordial graph do not exhaust the space of solutions. However, one may, as in the work in this paper on the  $U(1)^3$  model, restrict attention to the class of solutions from a unique primordial graph. The results of this paper then strongly indicate that even this restricted class contains propagating solutions.

Various proposals have been made in the literature to make the operator closable in the  $SU(2)$  case, which applies immediately to the simpler context of  $U(1)^3$ . Perhaps the minimal correction to (9) that makes this possible is as follows (see [36] for details): Simply substitute  $H_{v,e_1,e_2,e_3}$  in (8) by

$$H'_{v,e_1,e_2,e_3} = P_{v,e_1} P_{v,e_2} P_{v,e_3} H_{v,e_1,e_2,e_3}, \quad P_{v,e_1} := 1 - \theta(\Delta_{s_{\gamma,v,e_1}}) \quad (12)$$

Here,  $\theta(x) = 1$  for  $x \geq 1$  and  $\theta(x) = 0$  otherwise is the step function and  $\Delta_s = \sum_j (X_s^j)^2$  the Laplacian for  $s$ . These spectral projections avoid the effect just described. The semiclassical properties of these projections are discussed in [36]; suffice it to say here that we may interpret  $\Delta_s$  as the quantisation of  $-\sum_j E_j(S_s)^2$  where  $S_s$  is an arbitrarily small but finite surface intersection  $s$  transversally. If the classical 3-metric is non-degenerate, then  $\theta(-\sum_j E_j(S)^2) = 0$ , and in that sense, the modification is justified.

With this modification, the operator becomes closable and could be symmetrically ordered; see the third reference of [1–4] for the technical statement. The conclusions of the rest of this paper apply to both the non-closable version (8) and the closable modification just discussed because they both exhibit the phenomenon of non-unique parentage. By contrast, the closable Euclidean operators in the published version of [19] based on so-called double kinks instead of the loops considered here or in [37,38] based on loops which intersect the original graph in only one vertex, both by design have unique parentage and therefore do not exhibit propagation.

Recently, the constraint action construction techniques of [5,6] have been combined with key geometrical insights into the classical constraint action developed in [39]. The resulting constraint action [14,15] <sup>5</sup> differs from the ones mentioned hitherto in several ways. In particular, the loops added by the constraint are labelled by spin representations which are tailored to the labels of the parent state being acted upon. As a result, there are

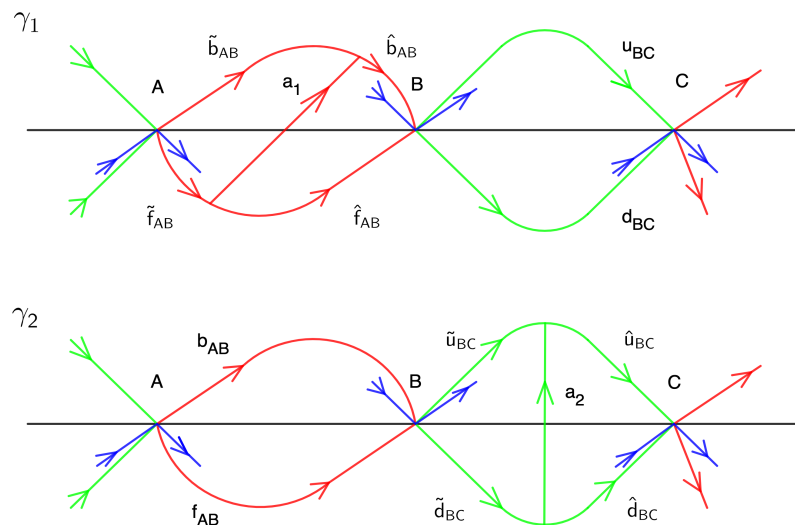
*always* children in which edge segments disappear. Since a detailed analysis of propagation for the constraint action of [14,15] constitutes an interesting but as yet open problem, we restrict ourselves to the following remarks:

- (i) The constraint action creates children with displaced vertices (called ‘electric diffeomorphism type children’ in [14,15]) and two classes of children obtained by the addition of extraordinary edges. One class of these children has disappearing edges, the second does not. A preliminary analysis suggests that non-unique parentage is associated with both these classes of children but not with electric diffeomorphism type children.
- (ii) As indicated in our introductory remarks in Section 1, while non-unique parentage invalidates the arguments of [18], the existence of explicit solutions to the constraints involving children of non-unique parentage must be established, which is a task that is still incomplete in the context of [14,15].
- (iii) In the context of ‘disappearing edges’, non-unique parents are expected to be labelled, generically, by diffeomorphically distinct graphs. Since our definition of propagation in this work is based on a fixed primordial graph structure, it is necessary to generalise this definition appropriately. We note here that a qualitative and intuitive description of propagation in the context of the variable parental graph structure is provided in [40] in the context of a novel  $U(1)^3$  constraint action distinct from the QSD-type action of this paper.

### 3.5. Evaluation of the Hamiltonian Constraint on the Chosen Parent CNWFs

By construction, the tangents of all edges adjacent to the vertices  $Z, D$  are co-planar so that the operator  $[V_v, (h_{s_{\gamma, v, e}})^{-1}]$  with  $e$  adjacent to  $v$  vanishes there. Accordingly, the Hamiltonian constraint has non-trivial action only at vertices  $A, B, C$ . At each of these vertices, the Hamiltonian constraint produces children graphs which correspond to gluing in an arc of the above type between pairs of adjacent edges. The non-uniqueness of parentage and the non-trivial correlation between the actions of the Hamiltonian constraint at different vertices when computing solutions to the quantum Einstein equations here can be made explicit and transparent. Specifically, a child CNWF with an arc in between  $A, B$  can come from either the action at  $A$  or  $B$  but originating from two different CNWF over  $\gamma_0$ , that is, with different parental charges so that these two parental CNWF cannot be related by a diffeomorphism. The same applies to a child with an arc between  $B, C$ . Accordingly, when constructing a solution to the Hamiltonian and spatial diffeomorphism constraint as a superposition of diffeomorphism averages of child CNWF of the above type, the coefficients in that superposition are constrained not by mutually disjoint sets of constraints, one set for each vertex, but rather by a coupled system of equations. In what follows, we will make this explicit and show that the resulting system of equations admits non-trivial solutions.

We will thus consider child graphs  $\gamma_1$  and  $\gamma_2$ , respectively, where  $\gamma_1$  differs from the parental graph  $\gamma_0$  by an arc  $a_1$  between interior points of  $f_{AB}$  and  $b_{AB}$  in the  $x, y$  plane, respectively, starting from the point on  $f_{AB}$ , see Figure 2, while  $\gamma_2$  differs from the parental graph  $\gamma_0$  by an arc  $a_2$  between interior points on  $d_{BC}$  and  $u_{BC}$  in the  $y, z$  plane starting from a point on  $d_{BC}$ . The action at  $B$  on  $\gamma_0$  produces both  $\gamma_1$  and  $\gamma_2$  types of graphs plus additional ones  $\gamma'_B$ , while the action at  $A$  only produces the  $\gamma_1$  type and additional ones  $\gamma'_A$ , and the action at  $C$  only produces the  $\gamma_2$  type and additional ones  $\gamma'_C$ . In what follows, the  $\gamma'_A, \gamma'_B, \gamma'_C$  contributions to the Hamiltonian constraint will be ignored, because we construct distributional solutions out of diffeomorphism averages of CNWF over  $\gamma_1, \gamma_2$  which, as distributions, have trivial action on CNWF over  $\gamma'_*$ ,  $* \in \{A, B, C\}$ .



**Figure 2.** The figure shows the two contributing children graphs. Only the subdivided edges and arcs are highlighted; the rest of the graphs are identical to the parent graph.

### Action at A:

We notice that with the orientation choices made, the following triples of edges adjacent at and outgoing from  $A$  are such that their tangents form a right-oriented triple of linearly independent vectors at  $A$

$$(f_{AB}, b_{AB}, u_{ZA}^{-1}), (b_{AB}, f_{AB}, d_{ZA}^{-1}), (d_{ZA}^{-1}, u_{ZA}^{-1}, b_{AB}), (u_{ZA}^{-1}, d_{ZA}^{-1}, f_{AB}) \quad (13)$$

The orientations of those edges coincides with those of the outgoing segments from  $A$  whence  $z_{\gamma_0, A, u_{ZA}} = z_{\gamma_0, A, d_{ZA}} = -1$  and  $z_{\gamma_0, A, f_{AB}} = z_{\gamma, A, b_{AB}} = +1$ . Thus, the volume eigenvalue at  $A$  derives as  $\nu_A = |Q_A|^{1/2}$  with

$$\begin{aligned} Q_A &= Q(m_{ZA}, n_{ZA}, m_{AB}, n_{AB}, c_A, c'_A)_{c'_A=c_A}; \quad Q(m_{ZA}, n_{ZA}, m_{AB}, n_{AB}, c_A, c'_A) \\ &= -[\det(m_{AB}, n_{AB} + c'_A, m_{ZA} + c_A) + \det(n_{AB} + c'_A, m_{AB}, n_{ZA})] \\ &\quad + [\det(n_{ZA}, m_{ZA} + c_A, n_{AB} + c'_A) + \det(m_{ZA} + c_A, n_{ZA}, m_{AB})] \end{aligned} \quad (14)$$

where the factor  $3!$  was cancelled because the permutation of edges within the above triples all gives the same contribution. We have exploited that while the tangents of  $u_{ZA}, l_A$  point into the same direction at  $A$  for the beginning segment of  $l_A$ , the loop is here outgoing, while  $u_{ZA}$  is ingoing. Likewise, while the tangents of  $b_{AB}, l_A$  point into the same directions at  $A$  for the end segment of  $l_A$ , the loop is here ingoing while  $b_{AB}$  is outgoing. The number of arguments of the function  $Q$  is redundant due to (7), but we will keep it for reasons of more transparent bookkeeping. We will use  $\nu = \sqrt{|Q|}$  in what follows.

Thus, the  $\gamma_1$  type contribution of the Hamiltonian constraint  $C_A$  at vertex  $A$  is given by (we drop the common factor  $T(v) = 12$ ,  $v = A, B, C$  in what follows as it can be absorbed into the lapse function and  $m$  stands collectively for all charges on all edges of  $\gamma_0$ )

$$\begin{aligned} C_A T_{\gamma_0, m} &= \sum_l \{ [v(m_{ZA} + \delta_l, n_{ZA}, m_{AB}, n_{AB}, c_A, c_A) - v(m_{ZA}, n_{ZA}, m_{AB}, n_{AB}, c_A, c_A)] [h_{\alpha_A}^l - (h_{\alpha_A}^l)^{-1}] \\ &\quad + [v(m_{ZA}, n_{ZA} + \delta_l, m_{AB}, n_{AB}, c_A, c_A) + v(m_{ZA}, n_{ZA}, m_{AB}, n_{AB}, c_A - \delta_l, c_A) \\ &\quad - 2v(m_{ZA}, n_{ZA}, m_{AB}, n_{AB}, c_A, c_A)] [(h_{\alpha_A}^l)^{-1} - h_{\alpha_A}^l] \} T_{\gamma_0, m} \\ &= \sum_l [v(m_{ZA} + \delta_l, n_{ZA}, m_{AB}, n_{AB}, c_A, c_A) - v(m_{ZA}, n_{ZA}, m_{AB}, n_{AB}, c_A - \delta_l, c_A) \\ &\quad - v(m_{ZA}, n_{ZA} + \delta_l, m_{AB}, n_{AB}, c_A, c_A) + v(m_{ZA}, n_{ZA}, m_{AB}, n_{AB}, c_A, c_A)] [(h_{\alpha_A}^l)^{-1} - h_{\alpha_A}^l] T_{\gamma_0, m} \\ &=: \sum_l d_A^l(m_{ZA}, n_{ZA}, m_{AB}, n_{AB}, c_A) [h_{\alpha_A}^l - (h_{\alpha_A}^l)^{-1}] T_{\gamma_0, m} \end{aligned} \quad (15)$$

where  $\alpha_A = s_{\gamma_0,A,f_{AB}} \circ a_1 \circ s_{\gamma_0,A,b_{AB}}^{-1}$  and  $\delta_l \in \mathbb{Z}^3$  are the vector with component  $[\delta_l]^j = \delta_l^j$ . In (15), we have considered the beginning segments  $s$  of the edges  $u_{ZA}, d_{ZA}, l_A$  that have linearly independent tangents at  $A$  together with the tangents of  $f_{AB}, b_{AB}$  (that is why there is no contribution from the end segment of  $l_A$ ). Their holonomies along  $s$  with outgoing orientation and charge  $\pm\delta_l$  enters the commutator with the volume operator explaining the argument shifts by  $\pm\delta_l$  in the functions  $\nu$  displayed. Then, the loop  $\alpha_A$  or  $\alpha_A^{-1}$ , respectively, becomes attached when the tangents of  $s, f_{AB}, b_{AB}$  in this order are right or left oriented, respectively.

#### Action at C:

The situation here with respect to orientations of edges and charges is exactly as at  $A$  with the substitutions of labellings  $ZA \rightarrow BC$  and  $AB \rightarrow CD$  and  $A \rightarrow C$ . Therefore,

$$Q_C = Q(m_{BC}, n_{BC}, m_{CD}, n_{CD}, c_C, c'_C)_{c'_C=c_C} \quad (16)$$

Thus, the  $\gamma_2$  type contribution of the Hamiltonian constraint  $C_C$  at vertex  $C$  is given by

$$\begin{aligned} C_C T_{\gamma_0,m} &= \sum_l \{ [\nu(m_{BC}, n_{BC}, m_{CD}, n_{CD} - \delta_l, c_C, c_C) - \nu(m_{BC}, n_{BC}, m_{CD}, n_{CD}, c_C, c_C)] [h_{\alpha_C}^l - (h_{\alpha_C}^l)^{-1}] \\ &\quad + [\nu(m_{BC}, n_{BC}, m_{CD} - \delta_l, n_{CD}, c_C, c_C) + \nu(m_{BC}, n_{BC}, m_{CD}, n_{CD}, c_C, c_C + \delta_l) \\ &\quad - 2\nu(m_{BC}, n_{BC}, m_{CD}, n_{CD}, c_C, c_C)] [(h_{\alpha_C}^l)^{-1} - (h_{\alpha_C}^l)] \} T_{\gamma_0,m} \\ &= \sum_l [\nu(m_{BC}, n_{BC}, m_{CD}, n_{CD} - \delta_l, c_C, c_C) - \nu(m_{BC}, n_{BC}, m_{CD}, n_{CD}, c_C, c_C + \delta_l) \\ &\quad - \nu(m_{BC}, n_{BC}, m_{CD} - \delta_l, n_{CD}, c_C, c_C) + \nu(m_{BC}, n_{BC}, m_{CD}, n_{CD}, c_C, c_C)] [h_{\alpha_C}^l - (h_{\alpha_C}^l)^{-1}] T_{\gamma_0,m} \\ &=: \sum_l d_C^l(m_{BC}, n_{BC}, m_{CD}, n_{CD}, c_C) [h_{\alpha_C}^l - (h_{\alpha_C}^l)^{-1}] T_{\gamma_0,m} \end{aligned} \quad (17)$$

where  $\alpha_C = s_{\gamma_0,C,d_{BC}} \circ a_2 \circ s_{\gamma_0,C,u_{BC}}^{-1}$ . We have exploited that it is now the end segment of  $l_C$  which contributes.

#### Action at B:

The tangents of the following triples of edges are right oriented and outgoing from  $B$

$$(d_{BC}, u_{BC}, f_{AB}^{-1}), (u_{BC}, d_{BC}, b_{AB}^{-1}), (f_{AB}^{-1}, b_{AB}^{-1}, d_{BC}), (b_{AB}^{-1}, f_{AB}^{-1}, u_{BC}), \quad (18)$$

The orientations of those edges coincide with those of the outgoing segments from  $B$  whence  $z_{\gamma_0,B,f_{AB}} = z_{\gamma_0,B,b_{AB}} = -1$  and  $z_{\gamma_0,B,u_{BC}} = z_{\gamma_0,B,d_{BC}} = +1$ . Thus, the volume eigenvalue at  $B$  derives as  $\nu = |Q|^{1/2}$  where

$$Q_B = Q(m_{AB}, n_{AB}, m_{BC}, n_{BC}, c_B, c'_B)_{c'_B=c_B} \quad (19)$$

where it was exploited that while the tangents of  $f_{AB}$  and the beginning segment of  $l_B$  as well as the tangents of  $d_{BC}$  and the end segment of  $l_B$  point into the same direction at  $B$ ,  $f_{AB}$  is ingoing while the beginning segment of  $l_B$  is outgoing from  $B$  and  $d_{BC}$  is outgoing while the end segment of  $l_B$  is ingoing at  $B$ .

Thus, following the same arguments as at vertices  $A, C$ , the  $\gamma_1$ -type contribution of the Hamiltonian constraint at  $B$  is given by

$$\begin{aligned} C_B^1 T_{\gamma_0,m} &= \sum_l \{ [\nu(m_{AB}, n_{AB}, m_{BC}, n_{BC} - \delta_l, c_B, c_B) - \nu(m_{AB}, n_{AB}, m_{BC}, n_{BC})] [h_{\alpha_B}^l - (h_{\alpha_B}^l)^{-1}] \\ &\quad + [\nu(m_{AB}, n_{AB}, m_{BC} - \delta_l, n_{BC}, c_B, c_B) + \nu(m_{AB}, n_{AB}, m_{BC}, n_{BC}, c_B, c_B + \delta_l) \\ &\quad - 2\nu(m_{AB}, n_{AB}, m_{BC}, n_{BC}, c_B, c_B)] [(h_{\alpha_B}^l)^{-1} - (h_{\alpha_B}^l)] \} T_{\gamma_0,m} \\ &= \sum_l [\nu(m_{AB}, n_{AB}, m_{BC}, n_{BC} - \delta_l, c_B, c_B) - \nu(m_{AB}, n_{AB}, m_{BC}, n_{BC}, c_B, c_B + \delta_l) \\ &\quad - \nu(m_{AB}, n_{AB}, m_{BC} - \delta_l, n_{BC}, c_B, c_B) + \nu(m_{AB}, n_{AB}, m_{BC}, n_{BC}, c_B, c_B)] [h_{\alpha_B}^l - (h_{\alpha_B}^l)^{-1}] T_{\gamma_0,m} \\ &=: \sum_l d_{B,1}^l(m_{AB}, n_{AB}, m_{BC}, n_{BC}, c_B) [h_{\alpha_B}^l - (h_{\alpha_B}^l)^{-1}] T_{\gamma_0,m} \end{aligned} \quad (20)$$

where  $\alpha_B^1 = s_{\gamma_0, B, f_{AB}}^{-1} \circ a_1 \circ s_{\gamma_0, v, b_{AB}}$ . We are abusing the notation, since at the level of CNWF, the arc  $a_1$  coming from  $B$  is generally different from the one coming from  $A$ , but after diffeomorphism averaging, they are identified, and this is what matters in what follows.

Likewise, the  $\gamma_2$  type contribution of the Hamiltonian constraint at  $B$  is given by

$$\begin{aligned} C_B^2 T_{\gamma_0, m} &= \sum_I \{ [v(m_{AB} + \delta_I, n_{AB}, m_{BC}, n_{BC}, c_B, c_B) - v(m_{AB}, n_{AB}, m_{BC}, n_{BC}, c_B, c_B)] [h_{\alpha_B^2}^I - (h_{\alpha_B^2}^I)^{-1}] \\ &\quad + [v(m_{AB}, n_{AB} + \delta_I, m_{BC}, n_{BC}) + v(m_{AB}, n_{AB}, m_{BC}, n_{BC}, c_B - \delta_I, c_B)] \\ &\quad - 2v(m_{AB}, n_{AB}, m_{BC}, n_{BC}, c_B, c_B)] [(h_{\alpha_B^2}^I)^{-1} - h_{\alpha_B^2}^I] \} T_{\gamma_0, m} \\ &= \sum_I [v(m_{AB} + \delta_I, n_{AB}, m_{BC}, n_{BC}, c_B, c_B) - v(m_{AB}, n_{AB} + \delta_I, m_{BC}, n_{BC}, c_B, c_B) \\ &\quad - v(m_{AB}, n_{AB}, m_{BC}, n_{BC}, c_B - \delta_I, c_B) + v(m_{AB}, n_{AB}, m_{BC}, n_{BC}, c_B, c_B)] [h_{\alpha_B^2}^I - (h_{\alpha_B^2}^I)^{-1}] T_{\gamma_0, m} \\ &=: \sum_I d_{B,2}^I(m_{AB}, n_{AB}, m_{BC}, n_{BC}) [h_{\alpha_B^2}^I - (h_{\alpha_B^2}^I)^{-1}] T_{\gamma_0, m} \end{aligned} \quad (21)$$

where  $\alpha_B^2 = s_{\gamma_0, B, d_{BC}} \circ a_2 \circ s_{\gamma_0, v, u_{BC}}^{-1}$ . Again, at the level of CNWF, the arc  $a_2$  coming from  $B$  is generally different from the one coming from  $C$ , but after diffeomorphism averaging, they are identified.

### 3.6. Properties of the Coefficients $d_{B,I}^I$

The twelve coefficients  $d_{A,I}^I, d_{C,I}^I, d_{B,I}^I; I = 1, 2$  computed in the previous subsection are quite complicated. They have the following general structure: they are linear combinations with coefficients  $\pm 1$  of the differences

$$v_*(m + \sigma \delta_I, n, c) - v_*(m, n, c), v_*(m, n + \sigma \delta_I, c) - v_*(\sigma \delta_I, n, c), v_*(m, n, c + \sigma \delta_I) - v_*(m, n, c) \quad (22)$$

where  $*$   $\in \{A, C, (B, 1), (B, 2)\}$  and  $\sigma = \pm 1$ . For each value of  $*$ , the function  $v_*$  is a function of five charges as follows: Respectively, the variables  $m, n, c$  coincide with the charges  $m = m_{ZA}, n = n_{ZA}, c = c_A$  for  $*$   $= A$  with  $m_{AB}, n_{AB}$  fixed,  $m = m_{CD}, n = n_{CD}, c = c_C$  for  $*$   $= C$  with  $m_{BC}, n_{BC}$  fixed,  $m = m_{BC}, n = n_{BC}, c = c_B$  for  $*$   $= (B, 1)$  with  $m_{AB}, n_{AB}$  fixed,  $m = m_{AB}, n = n_{AB}, c = c_B$  for  $*$   $= (B, 2)$  with  $m_{BC}, n_{BC}$  fixed. The functions  $v_*$  themselves are square roots of the modulus of a linear combinations with coefficients  $\pm 1$  of four determinants built from the five charge vectors involved, or equivalently, it is the fourth root of a positive bilinear expression in those four determinants with coefficients  $1, \pm 2$ . The more explicit form of those coefficients can be found in Appendix A.

In what follows, the charges  $\Delta := \{m_{ZA}, m_{CD}, m_{ZD}, c_A, c_B, c_C, N\}$  will be fixed and  $n_* = N - m_*$  with  $*$   $= ZA, AB, BC, CD, DZ$  due to gauge invariance so that the only variables are  $m_{AB}, m_{BC}$ . We consider the vector “fields”  $\mathbb{Z}^3 \rightarrow \mathbb{R}^3$  given by  $m_{AB} \mapsto d_A(m_{AB}), m_{BC} \mapsto d_C(m_{BC})$  and  $\mathbb{Z}^6 \rightarrow \mathbb{R}^3$  given by  $(m_{AB}, m_{BC}) \mapsto d_{B,I}(m_{AB}, m_{BC}); I = 1, 2$ . What we want to show is that all vector fields  $d_A, d_C, d_{B,I}$  are non-vanishing at “generic points” and moreover that  $d_{B,1}, d_{B,2}$  are linearly independent at “generic points”. By generic points, we mean all of  $\mathbb{Z}^3$  or  $\mathbb{Z}^6$ , respectively, except for “lower dimensional” sublattices of  $\mathbb{Z}^3$  or  $\mathbb{Z}^6$ , respectively, or even better on finite subsets of those, respectively. Here, a lattice of dimension  $k \leq l$  in  $\mathbb{Z}^l$  is a subset of the form  $a_1 e_1 + \dots + a_k e_k$  with variable  $a_1, \dots, a_k \in \mathbb{Z}$  and fixed linearly independent (over  $\mathbb{Z}$ )  $e_1, \dots, e_k \in \mathbb{Z}^l$ . This would imply that (1)  $d_A, d_C$  are non-vanishing “almost everywhere” and thus are not compactly supported and (2) that  $d_{B,1}, d_{B,2}$  are linearly independent “almost everywhere”.

We will not be able to give a strict proof of these statements, but we will give arguments for why this is very plausible. The subsequent discussion will also show why it is difficult to turn the plausibility arguments into a strict proof: We enter difficult questions of algebraic geometry which can be addressed in principle but require much more work. The arising questions and conjectured answers could benefit from machine learning techniques, which we advertise at this point.

To begin with, the vector condition  $d_A = 0$  is a set of three equations for three unknowns  $m_{AB}, l = 1, 2, 3$  and thus will typically have only a finite number of solutions even if we allow  $m_{AB} \in \mathbb{C}^3$ . The catch is in the word “typically”. In principle, it could



happen that  $d_A$  depends on  $m_{AB}$  in a sufficiently degenerate way so that the statements that we would like to prove do not hold, although this is intuitively hard to imagine. The reason why this appears unlikely is as follows: To investigate these questions analytically, one would for instance (extending  $m_{AB}$  to  $\mathbb{R}^3$ ) compute the Hessian  $H_A = \det(\frac{\partial d_A}{\partial m_{AB}})$  from whose zeroes one would infer where the map  $d_A$  fails to be injective. However, computing that Hessian and its zeroes is even more difficult than determining the zeroes of  $d_A$  directly. In order to say more than “by inspection  $d_A$  depends on  $m_{AB}$  sufficiently non-degenerately”, we consider solving the equations  $d_A^l = 0$ . We may write (see Appendix A)  $d_A^l = v_1^l + v_2^l - v_3^l - v_4^l$  where  $v_\mu^l = |Q_\mu^l|^{1/2}$ ,  $\mu = 1, 2, 3, 4$  and  $Q_\mu^l$  has the form  $a_j^{l,\mu} m_{AB}^j + b^{l,\mu}$  where  $a_j^{l,\mu}, b^{l,\mu}$  are integers depending on  $\Delta$ . By inspection (see Appendix A), the co-normals  $\vec{a}^\mu \in \mathbb{Z}^3$  are different for different  $\mu$  because they involve charge shifts in different entries of the determinants involved; hence, the affine polynomials  $Q_\mu^l$  have different level hypersurfaces. Accordingly, for each  $l$ , these four polynomials  $Q_\mu^l$  and thus the  $v_\mu^l$  are algebraically independent and the fact that the level hypersurfaces are different suggests that the four  $v_\mu^l$  cancel for each  $l$  at most at finitely many points.

To investigate this further, dropping the index  $l$  for notational simplicity, we obtain the condition  $v_1 + v_2 = v_3 + v_4$ . As all  $v$  are not negative, this is equivalent with

$$v_3^2 + v_4^2 - v_1^2 - v_2^2 = 2(v_1 v_2 - v_3 v_4) \quad (23)$$

which implies upon squaring

$$-(v_3^2 + v_4^2 - v_1^2 - v_2^2)^2 + 4(v_1^2 v_2^2 + v_3^2 v_4^2) = 8v_1 v_2 v_3 v_4 \quad (24)$$

and squaring again once more thereby introducing  $q_\mu := |Q_\mu| = v_\mu^2$

$$[4(q_1 q_2 + q_3 q_4) - (q_1 + q_2 - q_3 - q_4)^2]^2 = 64 q_1 q_2 q_3 q_4 \quad (25)$$

This equation is still not a polynomial in  $m_{AB}$  because of the modulus involved and one cannot get rid of it by squaring once more, the equation is simply *not of algebraic type*. In order to turn it into a polynomial, we write  $q_\mu = \epsilon_\mu Q_\mu$  with  $\epsilon_\mu \in \{0, \pm 1\}$ . Then, at fixed  $\epsilon_\mu$ , (25) becomes a quartic polynomial in  $m_{AB}$ . Suppose one finds a solution of (25) at fixed  $\epsilon_\mu$ . Then, to make it a valid solution, one has to check that  $\epsilon_\mu Q_\mu(m_{AB}) > 0$  if  $\epsilon_\mu \neq 0$  and  $Q_\mu(m_{AB}) = 0$  for  $\epsilon_\mu = 0$ .

With this understanding, we are left with solving the following problem: Let  $x = m_{AB}^1$ ,  $y = m_{AB}^2$ ,  $z = m_{AB}^3$ . Then, for each choice of  $\epsilon_\mu$  ( $3^4 = 81$  possibilities), we are looking for the zeroes  $(x, y, z) \in \mathbb{Z}^3$  of a non-homogeneous quartic polynomial  $P(x, y, z)$  with integer coefficients. In the theory of algebraic geometry, this known as a *diophantine equation*. The condition  $P(x, y, z) = 0$  defines an algebraic variety (surface) in  $\mathbb{R}^3$ , and we are looking for its integral points. We simplify the problem and look at the quartic polynomials  $p_0(x, y) := P(x, y, 0)$  with coefficients in  $\mathbb{Z}$  and for  $z \neq 0$  at  $p_z(X, Y) := P(x, y, z)/z^4$ ,  $X = x/z$ ,  $Y = y/z$  with coefficients in  $\mathbb{Q}$ , which defines algebraic curves  $C$ . For this situation, we have the following deep statement in number theory.

**Theorem 1 (Falting).** *Let  $C$  be a non-singular algebraic curve over the rationals (i.e., a curve in the plane defined by the zeroes of a polynomial equation in  $x, y$  with rational coefficients and such that it is not self intersecting). If the curve has algebraic genus  $g > 1$ , then  $\mathbb{Q}^2 \cap C$  is finite. Here, the algebraic genus for non-singular  $C$  is  $g = (n - 1)(n - 2)/2$  where  $n$  is the degree of the polynomial.*

The theorem takes a step towards proving that Fermat’s famous equation  $x^n + y^n = z^n$  has at most finitely many positive integer solutions  $x, y, z > 0$  when  $n \geq 3$ : After dividing by  $z$ , this becomes  $X^n + Y^n = 1$  which has at most finitely many rational solutions  $X = p/q$ ,  $Y = r/s$  with  $p, q$  and  $r, s$  relative prime. Then, the solution to the original

problem is  $x = a(ps), y = a(rq), z = a(qs)$ ,  $a \in \mathbb{N}$ . Note that any homogeneous equation such as Fermat's has infinitely many if it has one, which is not true for non-homogeneous equations, and the hard part of the proof is thus to show that there is not a single one. The algebraic genus can be extended to algebraic curves with singularities and is generically reduced by them [41].

Note that instead of introducing the  $\epsilon_\mu$ , one could momentarily drop the requirement that the integers  $q_\mu$  are not negative and consider the solutions of the quartic (25). If one could argue as above and conclude that the quartic has only finitely many solutions, one would then conclude that it also has only a finite number of solutions with  $q_\mu \geq 0$ . Then, for each of these leftover configurations, one would be left with solving the twelve equations  $Q_\mu^l = \pm q_\mu^l$  where we reintroduced the index  $l$ . As the  $Q_\mu^l$  are algebraically independent, this system is again very unlikely to have more than a finite number of solutions. In order to run this argument, one would need to have an extension of Falting's theorem at one's disposal for algebraic surfaces rather than curves because (25) involves four  $q_\mu$  rather than three  $m_{AB}^l$  (and for what follows more generally higher dimensional algebraic varieties defined by integer coefficient polynomials). This would also ease the proof in terms of  $m_{AB}$  so that one does not have to argue via the curves. Unfortunately, we were not able to find such an extension in the literature.

Applied to our problem, assuming that the algebraic curves in question are singularity free, we would conclude that we obtain only a finite number of solutions of  $p_0(x, y) = 0$  for  $z = 0$  and of  $p_z(X, Y) = 0$  for  $z \neq 0$  and thus most only finitely many 1-dimensional sublattices of  $\mathbb{Z}^3$  as solutions (note that only those of these solutions are admissible for which the above condition involving the  $\epsilon_\mu$  is met). However, note that this is true for each direction  $l$  in charge space separately. As these finitely many discrete lines for different  $l$  are unlikely to coincide, we expect that there are at most finitely many solutions of  $d_A = 0$ . The same reasoning applies to  $d_C$ .

Another qualitative argument would be as follows: Compute the polynomials  $P_l(x, y, z) = 0$  as above for each  $l$  and each choice of the  $\epsilon_\mu$ . These define three distinct algebraic surfaces. The intersection of two surfaces is one dimensional unless the two polynomials coincide, which they do not (if they would intersect in a common face, then by analyticity, they would coincide everywhere). The intersection with the third is then also at most one dimensional, more likely a discrete set of points. From that curve or set, one would need to pick the integer points if they exist at all.

To apply this number theoretic argument to  $d_{B,1}, d_{B,2}$  is significantly harder. As for the vanishing of  $d_{B,I}$ , this again leads to a quartic equation but now for a five-dimensional (generalised—as it is no longer defined by an algebraic equation) variety. Using the above heuristic, one would argue that each  $d_{B,I}$  vanishes at most on the integer points of a four-dimensional, more likely three-dimensional, variety  $S_I$ , which is different for the two choices of  $I$ . For the points outside of the union  $S = S_1 \cup S_2$  of these sets, the condition that  $d_{B,1}, d_{B,2}$  be co-linear can be stated as  $d_{B,1} \times d_{B,2} = 0$ . On  $\mathbb{Z}^6 - S$ , only two of these three equations are algebraically independent. However, this time, it is not possible to cast these into two algebraic equations in six variables by squaring the equations sufficiently often. The equation to be solved is simply not algebraic any more, and almost nothing can be said about its solution structure.

Yet, one expects that these two conditions single out an at most four-dimensional generalised variety of which one would again need to determine the integer points which then very likely reduces this to a finite set.

In summary, the analysis of the singularity structure of the vector fields  $d_A, d_C, d_{B,I}$  leads to hard questions in number theory and algebraic geometry which are beyond the scope of this paper. However, the heuristic arguments given, backed up by some results from algebraic geometry, make it appear extremely likely that  $d_A, d_B$  are nowhere vanishing and that  $d_{B,1}, d_{B,2}$  are nowhere linearly dependent except on lower dimensional sublattices and more likely on finite subsets. In other words, the violation of these conditions, if possible at all, appears to be a tremendous number theoretic accident, and

moreover, one can exploit the freedom in the choice of  $\Delta$  to downsize this set of violating points even further.

We will use the assumption of non-singularity and linear independence of the vector fields at generic points in the sense described as a plausible conjecture in what follows. To attempt a strict proof will require deeper methods from algebraic geometry and number theory and could benefit from numerical machine learning techniques.

### 3.7. Decomposition into CNWF

We must write the functions  $[h_\beta^I - (h_\beta^I)^{-1}] T_{\gamma_0, m}$  in terms of CNWF for the loops  $\beta = \alpha_A, \alpha_B^I, \alpha_C, I = 1, 2$  that appear in (15), (17), (20) and (21). For  $\gamma_1$ , we see that the edges  $f_{AB}, b_{AB}$  are split in halves  $f_{AB} = \tilde{f}_{AB} \circ \hat{f}_{AB}, b_{AB} = \tilde{b}_{AB} \circ \hat{b}_{AB}$  and that the arc  $a_1$  runs from  $\tilde{f}_{AB} \cap \tilde{b}_{AB}$  to  $\hat{f}_{AB} \cap \hat{b}_{AB}$ . We thus need to introduce charge labels  $\tilde{m}_{AB}, \hat{m}_{AB}, \tilde{n}_{AB}, \hat{n}_{AB}, m_1$  for  $\tilde{f}_{AB}, \hat{f}_{AB}, \tilde{b}_{AB}, \hat{b}_{AB}, a_1$ . Likewise, for  $\gamma_2$ , we see that the edges  $u_{BC}, d_{BC}$  are split in halves  $u_{BC} = \tilde{u}_{BC} \circ \hat{u}_{BC}, d_{BC} = \tilde{d}_{BC} \circ \hat{d}_{BC}$  and that the arc  $a_2$  runs from  $\tilde{d}_{BC} \cap \tilde{u}_{BC}$  to  $\hat{d}_{BC} \cap \hat{u}_{BC}$ . We thus need to introduce charge labels  $\tilde{m}_{BC}, \hat{m}_{BC}, \tilde{n}_{BC}, \hat{n}_{BC}, m_2$  for  $\tilde{u}_{BC}, \hat{u}_{BC}, \tilde{d}_{BC}, \hat{d}_{BC}$ , and  $a_2$ , respectively. Due to gauge invariance, these charges are subject to the constraints

$$\tilde{m}_{AB} = m_1 + \hat{m}_{AB}, \tilde{n}_{AB} = -m_1 + \hat{n}_{AB}, \tilde{m}_{BC} = -m_2 + \hat{m}_{BC}, \tilde{n}_{BC} = m_2 + \hat{n}_{BC}, \quad (26)$$

The factor  $(h_{\alpha_A}^I)^\sigma, \sigma = \pm 1$  changes the charges  $\tilde{m}_{AB} = \hat{m}_{AB} = m_{AB}, \tilde{n}_{AB} = \hat{n}_{AB} = n_{AB}, m_1 = 0$  of  $T_{\gamma_0, m}$  to

$$\tilde{m}_{AB} = m_{AB} + \sigma \delta_I, \hat{m}_{AB} = m_{AB}, \tilde{n}_{AB} = n_{AB} - \sigma \delta_I, \hat{n}_{AB} = n_{AB}, m_1 = \sigma \delta_I \quad (27)$$

in agreement with (26). The other charges are unchanged. We denote the CNWF with the changed labels (27) by  $T_{\gamma_1, m}^{A, I, \sigma}$ .

The consideration for the other loops is similar: The factor  $(h_{\alpha_B}^I)^\sigma$  leads to the change of charges

$$\tilde{m}_{AB} = m_{AB}, \hat{m}_{AB} = m_{AB} - \sigma \delta_I, \tilde{n}_{AB} = n_{AB}, \hat{n}_{AB} = n_{AB} + \sigma \delta_I, m_1 = \sigma \delta_I \quad (28)$$

We denote the CNWF with the changed labels (28) by  $T_{\gamma_1, m}^{B, I, \sigma}$ .

The factor  $(h_{\alpha_B}^I)^\sigma$  leads to the change of charges

$$\tilde{m}_{BC} = m_{BC} - \sigma \delta_I, \hat{m}_{BC} = m_{BC}, \tilde{n}_{BC} = n_{BC} + \sigma \delta_I, \hat{n}_{BC} = n_{BC}, m_2 = \sigma \delta_I \quad (29)$$

We denote the CNWF with the changed labels (29) by  $T_{\gamma_2, m}^{B, I, \sigma}$ .

Finally, the factor  $(h_{\alpha_C}^I)^\sigma$  leads to the change of charges

$$\tilde{m}_{BC} = m_{BC}, \hat{m}_{BC} = m_{BC} + \sigma \delta_I, \tilde{n}_{BC} = n_{BC}, \hat{n}_{BC} = n_{BC} - \sigma \delta_I, m_2 = \sigma \delta_I \quad (30)$$

We denote the CNWF with the changed labels (30) by  $T_{\gamma_2, m}^{C, I, \sigma}$ .

Note that still for  $* \in \{AB, BC\}$

$$\tilde{m}_* + \tilde{n}_* = \hat{m}_* + \hat{n}_* = N \quad (31)$$

We may summarise the relevant contributions to the action of the Hamiltonian constraint at vertices  $A, B, C$  as

$$\begin{aligned} C_A T_{\gamma_0, m} &= \sum_{l, \sigma} \sigma d_A^l(m_{ZA}, n_{ZA}, m_{AB}, n_{AB}, c_A) T_{\gamma_1, m}^{A, l, \sigma}, \\ C_C T_{\gamma_0, m} &= \sum_{l, \sigma} \sigma d_C^l(m_{BC}, n_{BC}, m_{CD}, n_{CD}, c_C) T_{\gamma_2, m}^{C, l, \sigma}, \\ C_B^I T_{\gamma_0, m} &= \sum_{l, \sigma} \sigma d_{B, I}^l(m_{AB}, n_{AB}, m_{BC}, n_{BC}, c_B) T_{\gamma_1, m}^{B, l, \sigma}, \end{aligned} \quad (32)$$

for  $I = 1, 2$ .

Note that the labelling of the vector states  $T_{\gamma_1, m}^{A, l, \sigma}, T_{\gamma_1, m'}^{B, l', \sigma'}$  using labels A,B presents an overcounting; i.e., there are linear relations between them. The same applies to  $T_{\gamma_2, m}^{C, l, \sigma}, T_{\gamma_2, m'}^{B, l', \sigma'}$  with respect to labels C,B. This occurs precisely due to the phenomenon of non-unique parentage and will play a crucial role when solving the Hamiltonian constraint in the next subsections. On the other hand, the labels  $(l, \sigma)$  are taking six distinct possible values labelling the six distinct possible charges  $\sigma\delta_l$  on the arcs and are not redundant.

Before closing this subsection, note that the children CNWF arising from the parent CNWF  $T_{\gamma_0, m}$  can be such that one of the edges in the pairs  $(\tilde{f}_{AB}, \tilde{b}_{AB}), (\hat{f}_{AB}, \hat{b}_{AB})$  or  $(\tilde{u}_{BC}, \tilde{d}_{BC}), (\hat{u}_{BC}, \hat{d}_{BC})$ , but not both if  $N \neq 0$  can become uncharged in the child by the action of the Hamiltonian constraint if the corresponding charge vector is of the form  $\pm\delta_l$  in the parent. Since our solutions are by definition such that all segments of  $\gamma_0$  in the graphs  $\gamma_1, \gamma_2$  are charged, we do not need to worry about such solutions with ‘edge evaporation’.

### 3.8. Statement of Quantum Einstein Equations for an Example Class of Solutions

The Ansatz for an example solution to the Quantum Einstein Equations exhibiting propagation is the linear functional

$$\Psi_{\Delta} = \sum_{M \in \mathbb{Z}^6; I=1,2; l=1,2,3; \sigma=\pm 1} \kappa_M^{I, l, \sigma} \eta_{M; \Delta}^{I, l, \sigma} \quad (33)$$

The notation is as follows: Consider CNWF  $T_{\gamma_I, M; \Delta}^{l, \sigma}$  over  $\gamma_I$  where  $\gamma_I$  are the child graphs described above and with fixed values of the data  $\Delta := (N, \{m_*\}_{* \in \{ZA, CD, DZ\}}, \{c_*\}_{* \in \{A, B, C\}})$  but with variable charges  $M = (M_{AB}, M_{BC}) \in \mathbb{Z}^6$  where for  $I = 1$

$$\tilde{m}_{AB} = M_{AB}, \hat{m}_{AB} = M_{AB} - \sigma\delta_l, m_1 = \sigma\delta_l, m_{BC} = M_{BC} \quad (34)$$

while for  $I = 2$

$$m_{AB} = M_{AB}, \tilde{m}_{BC} = M_{BC} - \sigma\delta_l, \hat{m}_{BC} = M_{BC}, m_2 = \sigma\delta_l \quad (35)$$

and of course  $m_* + n_* = \hat{m}_* + \hat{n}_* = \tilde{m}_* + \tilde{n}_* = N$  for all  $* \in \{ZA, AB, BC, CD, DZ\}$ . Then,  $\eta_{M; \Delta}^{I, l, \sigma}$  is the diffeomorphism average of  $T_{\gamma_I, M; \Delta}^{l, \sigma}$  normalised such that that for any CNWF  $T_{\gamma, m}$ , we have

$$\eta_{M; \Delta}^{I, l, \sigma}[T_{\gamma, m}] = \delta_{[\gamma_I], [\gamma]} \langle T_{\gamma_I, M; \Delta}^{l, \sigma}, T_{\gamma_I, m} \rangle \quad (36)$$

where  $\langle \cdot, \cdot \rangle$  is the inner product on the kinematical LQG Hilbert space and  $[\gamma]$  is the set  $\{\varphi(\gamma), \varphi \in \text{Diff}(\sigma)\}$  with  $\text{Diff}(\sigma)$  denoting the group of semi-analytic diffeomorphisms of the spatial manifold  $\sigma$  underlying the canonical formulation. Note that we dropped the vertex label  $* \in \{A, B, C\}$  in  $T_{\gamma_I, M; \Delta}^{l, \sigma}$  introduced in the previous subsection, thus removing the overcounting. In (36), we exploited that the result vanishes unless the diffeomorphism classes of  $\gamma, \gamma_I$  coincide.

In (36), we have also exploited that  $\gamma_0, \gamma_1, \gamma_2$  cannot have any graph symmetries [8] which would otherwise unnecessarily complicate our discussion so that there is precisely one term in the distribution  $\eta_{M; \Delta}^{I, l, \sigma}$  that contributes. Here, a graph symmetry of a graph  $\gamma$  is semi-analytic diffeomorphism  $\varphi$  that preserves  $\gamma$  as a set but permutes its vertices and edges possibly including orientation reversal. An orientation reversal for an edge  $e$  of a CNWF over  $\gamma$  is equivalent to a charge reflection  $m_e \mapsto -m_e$ , and a permutation of edges leads to a corresponding permutation of charges. The existence of graph symmetries implies that CNWF over the same graph but different charges are in the same diffeomorphism orbit, and thus, their diffeomorphism averages are the same. Thus, (36) would have to be corrected by a sum over the corresponding possible charge configurations that are identified under diffeomorphism averaging.

To achieve a trivial graph symmetry group was the reason to add the extra structure  $u_{ZA}, d_{ZA}, f_{CD}, b_{CD}, l_A, l_B, l_C$  to  $\gamma_0$ : If we would drop these and connect A,C by lines

$u_{CA}, d_{CA}$  similar to  $f_{ZA}, b_{ZA}$ , then reflection in the  $y, z$  plane would exchange  $f_{AB}, b_{AB}$  and map  $a_1$  to  $a_1^{-1}$ , reflection in the  $x, y$  plane would exchange  $u_{BC}, d_{BC}$  and map  $a_2$  to  $a_2^{-1}$  and, e.g., rotation by  $\pi$  about the  $z$ -axis through  $B$  was followed by rotation by  $\pi/2$  about the  $y$ -axis, which would map  $(f_{AB}, b_{AB}, u_{BC}, d_{BC})$  to  $(u_{BC}^{-1}, d_{BC}^{-1}, f_{AB}^{-1}, b_{AB}^{-1})$  and similar maps for  $u_{CA}, d_{CA}$  leading to corresponding complicated identifications of charges in  $\eta_M^{I, \sigma}$ . With the extra structure provided, the latter type of identifications and other similar ones are suppressed since a semi-analytic diffeomorphism cannot un-knot a loop. Furthermore, while a semi-analytic diffeomorphism can be local, it still has to be at least  $C^1$ . This prevents, e.g., the existence of a local semi-analytic diffeomorphism which preserves all parts of say  $\gamma_1$  while interchanging  $f_{AB}, b_{AB}$  for it would have to preserve in particular  $u_{BC}, d_{BC}, l_B$  at  $B$ . To see this in detail, consider some parametrisation  $[0, 1] \ni t \mapsto s_e(t)$  where  $s_e$  is an analytic segment incident at and outgoing from  $B$ , respectively, of the edge  $e$  where  $e$  is one of  $f_{AB}^{-1}, b_{AB}^{-1}, u_{BC}, d_{BC}$  or the beginning segment of  $l_B$ . Suppose there exists a semi-analytic diffeomorphism  $\varphi$  with

$$\begin{aligned} \varphi(s_{f_{AB}^{-1}}(t)) &= b_{AB}^{-1}(f_1(t)), \quad \varphi(s_{b_{AB}^{-1}}(t)) = f_{AB}^{-1}(f_2(t)), \quad \varphi(s_{u_{BC}}(t)) = u_{BC}(f_3(t)), \\ \varphi(s_{d_{BC}}(t)) &= d_{BC}(f_4(t)), \quad \varphi(s_{l_B}(t)) = l_B(f_5(t)), \end{aligned} \quad (37)$$

where  $f_i$  are reparametrisations of  $[0, 1]$ , in particular  $f_i(0) = 0$ ,  $df_i/dt > 0$ . Let  $M$  be the matrix  $(\partial\varphi/\partial x)(B)$  and  $\lambda_I = (df_i/dt)(0) > 0$ . Taking the derivatives of (37) at  $t = 0$  gives

$$\begin{aligned} M(b_1 - b_2) &= \lambda_1(-b_1 - b_2), \quad M(-b_1 - b_2) = \lambda_2(b_1 - b_2), \quad M(b_3 + b_2) = \lambda_3(b_3 + b_2), \\ M(-b_3 + b_2) &= \lambda_4(-b_3 + b_2), \quad M(-b_1 + b_2) = \lambda_5(-b_1 + b_2) \end{aligned} \quad (38)$$

where  $b_j$  denotes the Cartesian basis of  $\mathbb{R}^3$ . The first and fifth equation are in contradiction leading to  $\lambda_1 = \lambda_5 = 0$ . Without the fifth equation, the first four equations would be satisfied for  $\lambda_I = 1$  and  $M$  the reflection at the  $y, z$  plane. Likewise, a simultaneous interchange of  $f_{AB}, b_{AB}$  and  $u_{BC}, d_{BC}$  while preserving  $l_B$  would lead to a contradiction between the first and fifth equation. An interchange of  $u_{BC}, d_{BC}$  preserving all other parts of the graph, in particular also the end segment of  $l_B$ , would lead to

$$\begin{aligned} M(b_1 - b_2) &= \lambda_1(b_1 - b_2), \quad M(-b_1 - b_2) = \lambda_2(-b_1 - b_2), \quad M(b_3 + b_2) = \lambda_3(-b_3 + b_2), \\ M(-b_3 + b_2) &= \lambda_4(b_3 + b_2), \quad M(-b_3 + b_2) = \lambda_5(-b_3 + b_2) \end{aligned} \quad (39)$$

and now, the fourth and fifth equation are in contradiction, leading to  $\lambda_4 = \lambda_5 = 0$ . Without the fifth equation, the first four equations can be satisfied by  $\lambda_I = 1$  and  $M$  the reflection at the  $x, y$  plane. Finally, a mapping between the pairs  $(f_{AB}, u_{AB})$  and  $(u_{BC}, d_{BC})$  possibly with interchanges within a pair and possibly with orientation reversal would need to map the whole structure to the one with the sequence of vertices reversed preserving  $B$ , in particular  $l_A \mapsto l_C^{\pm 1}$ ,  $l_B \mapsto l_B^{\pm 1}$ ,  $l_C \mapsto l_A^{\pm 1}$ , but this cannot be a graph symmetry because  $l_A, l_C$  have different topology.

Even if  $l_A, l_C$  would have the same topology so that a graph symmetry inducing such a pair exchange would be conceivable, a corresponding identification of charges on these edges possibly including reflections would be accompanied by an associated identification of charges between the pairs of edges  $(u_{ZA}, d_{ZA})$  and  $(f_{CD}, b_{CD})$ , respectively. Since we have fixed the data  $\Delta$ , if we pick the pair of charges  $(m_{ZA}, n_{ZA})$  and  $(m_{CD}, n_{CD})$ , respectively, such that there is no combination of permutation maps or single entry reflection maps between them, then at most, one of these two CNWFs is not annihilated by  $\eta_{M, \Delta}^{I, \sigma}$ . The same can be achieved by picking the charges on the loops such that there are no reflection or permutation maps between them. Thus, even in this case, (36) remains valid.

The  $\kappa_M^{I, \sigma}$  are complex coefficients that are to be determined by asking that  $\Psi$  obeys the quantum Einstein equations, that is

$$\Psi_\Delta[(U(\varphi) - \text{id}) T_{\gamma, m}] = 0 = \Psi_\Delta[C(f) T_{\gamma, m}] \quad \forall \quad \varphi \in \text{Diff}(\sigma), f, \gamma, m, \quad (40)$$



Here,  $U$  is the unitary representation of  $\text{Diff}(\sigma)$  densely defined by

$$U(\varphi) T_{\gamma, \{m_e\}} = T_{\varphi(\gamma), \{m_{\varphi(e)} = m_e\}} \quad (41)$$

from which we see that the first condition in (40) is already satisfied. The  $T_{\gamma, m}$  are without loss of generality solutions to the Gauss constraint (otherwise, (40) is again trivially solved).

Moreover,  $\kappa_M^{l, l, \sigma}$  is set to zero for those  $M$  for which not all edges of  $\gamma_I$  are charged in  $T_{\gamma_I, M; \Delta}^{l, \sigma}$  according to (34) and (35). Thus, we impose on the data  $\Delta$  that  $N \neq 0, m_* \neq 0, N; * \in \{ZA, CD, DZ\}$  and that for  $I = 1$  and each  $l, \sigma$  the coefficient  $\kappa_M^{l, l, \sigma}$  vanishes when

$$M_{AB} \in \{0, N, -\sigma\delta_l, N - \sigma\delta_l\}, \text{ or } M_{BC} \in \{0, N\} \quad (42)$$

and for  $I = 2$  and each  $l, \sigma$  the coefficient  $\kappa_M^{l, l, \sigma}$  vanishes when

$$M_{AB} \in \{0, N\}, \text{ or } M_{BC} \in \{0, N, \sigma\delta_l, N + \sigma\delta_l\} \quad (43)$$

This avoids that there can be child graphs with “erased edges” in addition to the ones considered. Note that (42) and (43) would not be necessary for the closable operator of Section 3.4 because by construction, the closable operator does not have SNWF in its image for which not all edges of the parent graph are charged. In what follows, we consider the non-closable operator for simplicity. In addition, note that child graphs with “erased edges” need to be carefully taken into account among the first generation graphs [19] when constructing higher-generation solutions.

To solve the Hamiltonian constraint, we note that due to our specific Ansatz,  $\Psi_\Delta[C(f) T_{\gamma, m}] \equiv 0$  is automatically satisfied unless the child graphs of  $\gamma$  that appear in the CNWF decomposition of  $C(f) T_{\gamma, m}$  lie in the diffeomorphism class of either  $\gamma_1$  or  $\gamma_2$ . This means that  $\gamma \in [\gamma_0]$  because the Hamiltonian action consists of adding arcs between pairs of edges meeting in at least tri-valent vertices. For  $\gamma \in [\gamma_0]$ , we may choose without loss of generality the representative  $\gamma = \gamma_0$ , and we can choose the support of  $f$  solely in mutually disjoint neighbourhoods of the vertices  $A, B, C$ , which means that we obtain the following system of linear equations

$$\Psi_\Delta[C_A T_{\gamma_0, m}] = 0, \quad \Psi_\Delta[(C_B^1 + C_B^2) T_{\gamma_0, m}] = 0, \quad \Psi_\Delta[C_C T_{\gamma_0, m}] = 0 \quad (44)$$

for all  $m_* \in \mathbb{Z}^3$ ,  $* \in \{ZA, AB, BC, CD, DZ\}$  and  $c_* \in \mathbb{Z}^3$ ,  $* \in \{A, B, C\}$ . Again, (44) is trivially satisfied unless  $(N := m_{ZA} + n_{ZA}, \{m_*\}_{* \in \{ZA, CD, DZ\}}, c_{* \in \{A, B, C\}}) = \Delta$ . To evaluate (44) for such CNWF, we notice

$$T_{\gamma_I, M; \Delta}^{l, \sigma} = T_{\gamma_I, m}^{B, l, \sigma} \quad (45)$$

where it is understood that the decomposition of the data contained in  $m$  into  $(m_{AB}, m_{BC})$  and the rest matches the data  $M$  and  $\Delta$  respectively. Thus

$$\begin{aligned} \langle T_{\gamma_1, M; \Delta}^{l, \sigma}, T_{\gamma_1, m}^{B, l', \sigma'} \rangle &= \delta_{l, l'} \delta_{\sigma, \sigma'} \delta_{M_{AB}, m_{AB}} \delta_{M_{BC}, m_{BC}} \\ \langle T_{\gamma_1, M; \Delta}^{l, \sigma}, T_{\gamma_1, m}^{A, l', \sigma'} \rangle &= \delta_{l, l'} \delta_{\sigma, \sigma'} \delta_{M_{AB}, m_{AB} + \sigma\delta_l} \delta_{M_{BC}, m_{BC}} \\ \langle T_{\gamma_2, M; \Delta}^{l, \sigma}, T_{\gamma_2, m}^{C, l', \sigma'} \rangle &= \delta_{l, l'} \delta_{\sigma, \sigma'} \delta_{M_{AB}, m_{AB}} \delta_{M_{BC}, m_{BC} + \sigma\delta_l} \\ \langle T_{\gamma_1, M; \Delta}^{l, \sigma}, T_{\gamma_2, m}^{C, l', \sigma'} \rangle &= \langle T_{\gamma_2, M; \Delta}^{l, \sigma}, T_{\gamma_1, m, N}^{A, l', \sigma'} \rangle = 0 \end{aligned} \quad (46)$$

We thus obtain the following system of non-trivial equations

$$\begin{aligned}
 0 &= \Psi_{\Delta}[C_A T_{\gamma_0, m}] = \sum_{l, \sigma} \sigma \kappa_{M_{AB}=m_{AB}+\sigma\delta_l, M_{BC}=m_{BC}}^{1, l, \sigma} d_A^l(m_{ZA}, n_{ZA}, m_{AB}, n_{AB}, c_A) \\
 0 &= \Psi_{\Delta}[C_C T_{\gamma_0, m}] = \sum_{l, \sigma} \sigma \kappa_{M_{AB}=m_{AB}, M_{BC}=m_{BC}+\sigma\delta_l}^{2, l, \sigma} d_C^l(m_{BC}, n_{BC}, m_{CD}, n_{CD}, c_C) \\
 0 &= \Psi_{\Delta}[(C_B^1 + C_B^2) T_{\gamma_0, m}] = \sum_{l, \sigma} \sigma \kappa_{M_{AB}=m_{AB}, M_{BC}=m_{BC}}^{l, l, \sigma} d_{B, I}^l(m_{AB}, n_{AB}, m_{BC}, n_{BC}, c_B)
 \end{aligned} \tag{47}$$

to be solved for all  $(m_{AB}, m_{BC}) \in \mathbb{Z}^6$ .

### 3.9. Solving the Example Class and Discussion

#### 3.9.1. Preparation

The structure of the solutions of the system (47) of course depends sensitively on the coefficients  $d_A^l, d_C^l, d_{B, I}^l$  with  $I = 1, 2$ . We note that  $d_A^l$  depends on  $m_{AB}; m_{ZA}, c_A, N$ , that  $d_B^l$  depends on  $m_{BC}; m_{CD}, c_B, N$  and that  $d_{B, I}^l$  depends on  $m_{AB}; m_{BC}, c_B, N$ . The dependence on  $N$  is via  $m_* + n_* = N$ .

To simplify the subsequent discussion, we will drop the dependence of these functions on the fixed structure  $\Delta$  and introduce the notation  $p := m_{AB}, q := m_{BC}$  in order not to clutter the formulae. Furthermore, following the discussion in Section 3.6, we assume that  $\Delta$  has been chosen such that the number of configurations  $p, q$  for which one or several of the  $\vec{d}_A, \vec{d}_C, \vec{d}_{B, 1}, \vec{d}_{B, 2}$  or that  $\vec{d}_{B, 1}, \vec{d}_{B, 2}$  are linearly dependent is a finite subset of  $\mathbb{Z}^3$  or  $\mathbb{Z}^6$ , respectively.

#### 3.9.2. Existence and Explicit Construction of Solutions

We note that we have coefficients  $\kappa_M^{l, l, \sigma}$  that depend on indices  $I \in \{1, 2\}, l \in \{1, 2, 3\}, \sigma \in \{+1, -1\}$  and  $M = (p, q) \in \mathbb{Z}^6$ , that is, 12 coefficients per point  $M \in \mathbb{Z}^6$ , except for configurations  $M$  depending on  $(I, l, \sigma)$ , which are excluded by (42) and (43). These coefficients are subject to the three constraints (47) for every point  $M \in \mathbb{Z}^6$  except for those such that  $C(f) T_{\gamma_0, m}$  has an expansion into CNWF over  $\gamma_1, \gamma_2$  such that for all appearing CNWF not all edges of  $\gamma_1, \gamma_2$  are charged. Thus, except for those exceptional points, we have roughly counting 12 times a  $\mathbb{Z}^6$  worth of complex variables and three times a  $\mathbb{Z}^6$  worth of equations and thus expect that the system (47) admits a rich (in fact infinite) number of solutions. However, not all of them may display propagation. To see this, let us refer to the third equation in (47) as the “B-equations” and the first and second equations in (47) as the “A-equations” and “C-equations”, respectively. Consider the subspace of solutions derived by setting  $\kappa^{2, l, \sigma} = 0$  and call the remaining equations the 1-equations. In a similar manner, we define the 2-equations by setting the coefficients  $\kappa^{1, l, \sigma} = 0$ . Note that the 1-equations, respectively, 2-equations only involve the ‘ $\kappa$ ’ coefficients for the child of type 1 and 2, respectively. Next, consider the 1-equations. The C equations are trivially satisfied, and we are left with the A-equations and the B-equations that now impose conditions only on  $\kappa^{1, l, \sigma}$ . Likewise, for the 2-equations, the A-equations are trivially satisfied, and we are left with the C-equations and the B-equations that now impose conditions only on  $\kappa^{2, l, \sigma}$ . In both cases, we obtain  $2\mathbb{Z}^6$  equations for the corresponding  $6\mathbb{Z}^6$  left over of  $\kappa$  coefficients, which should yield  $4\mathbb{Z}^6$  solutions to these equations and hence  $4\mathbb{Z}^6$  solutions to all A, B, and C equations. Altogether, we should thus obtain  $8\mathbb{Z}^6$  solutions which are not propagating, because each of them involves only a single child graph. This leaves us with a  $9\mathbb{Z}^6 - 8\mathbb{Z}^6$ , i.e., a  $\mathbb{Z}^6$  worth of solutions.

This expectation is indeed borne out by the detailed solution construction method below so that the propagation degree of the solution (to be defined below) class is only unity, i.e., only one complex variable can be accounted for propagation rather than nine.

We proceed as follows. Turning to the B-equations, we note that they are ultra-local with respect to both  $p, q$  and can be written more explicitly as

$$\vec{\Delta}_{p, q}^1 \cdot \vec{d}_{B, 1}(p, q) + \vec{\Delta}_{p, q}^2 \cdot \vec{d}_{B, 2}(p, q) = 0 \tag{48}$$

where we have introduced for  $I = 1, 2$

$$(\vec{\Delta}_{p,q}^I)^I := \kappa_{p,q}^{I,l,+} - \kappa_{p,q}^{I,l,-} \quad (49)$$

It can therefore be solved algebraically for each  $p, q$  separately, and we drop the dependence on  $p, q$  as far as the B-equations are concerned.

Case 0: Both  $\vec{d}_{B,I}$ ,  $I = 1, 2$  vanish.

Let  $\vec{d}_a$ ,  $a = 1, 2, 3$  be the Cartesian basis of  $\mathbb{R}^3$ . Then

$$\vec{\Delta}^I = \gamma^I \vec{d}_3 + \alpha^I \vec{d}_1 + \beta^I \vec{d}_2 \quad (50)$$

is the general solution with arbitrary 6 parameters  $\alpha^I, \beta^I, \gamma^I$ ;  $I = 1, 2$ . We arbitrarily set  $\beta^2 = \alpha^1 := \nu$  thus deliberately reducing it to five parameters, although this is not necessary.

Case 1a: Only  $\vec{d}_{B,2}$  vanishes.

Let  $\vec{d}_1 := \vec{d}_B^1 / \|\vec{d}_B^1\|$  and  $\vec{d}^2, \vec{d}^3$  be any two vectors such that  $\vec{d}_a$ ,  $a = 1, 2, 3$  is an orthonormal basis of  $\mathbb{R}^3$ . Then

$$\vec{\Delta}^1 = \gamma^1 \vec{d}_3 + \beta^1 \vec{d}_2, \quad \vec{\Delta}^2 = \gamma^2 \vec{d}_3 + \alpha^2 \vec{d}_1 + \beta^2 \vec{d}_2 \quad (51)$$

is the general solution with five arbitrary parameters  $\gamma^1, \gamma^2, \alpha^2, \beta^1, \nu := \beta^2$ .

Case 1b: Only  $\vec{d}_{B,1}$  vanishes.

Let  $\vec{d}_2 := \vec{d}_B^2 / \|\vec{d}_B^2\|$  and  $\vec{d}^1, \vec{d}^3$  be any two vectors such that  $\vec{d}_a$ ,  $a = 1, 2, 3$  is an orthonormal basis of  $\mathbb{R}^3$ . Then

$$\vec{\Delta}^1 = \gamma^1 \vec{d}_3 + \alpha^1 \vec{d}_1 + \beta^1 \vec{d}_2, \quad \vec{\Delta}^2 = \gamma^2 \vec{d}_3 + \alpha^2 \vec{d}_1 \quad (52)$$

is the general solution with five arbitrary parameters  $\gamma^1, \gamma^2, \alpha^2, \beta^1, \nu := \alpha^1$ .

Case 1c:  $\vec{d}_{B,I}$ ,  $I = 1, 2$  are co-linear but non-vanishing.

Thus, there is  $\mu \neq 0$  such that  $\vec{d}_{B,2} = \mu \vec{d}_{B,1}$ . Let  $\vec{d}_1 := \vec{d}_B^1 / \|\vec{d}_B^1\|$  and  $\vec{d}^2, \vec{d}^3$  be any two vectors such that  $\vec{d}_a$ ,  $a = 1, 2, 3$  is an orthonormal basis of  $\mathbb{R}^3$ . Then

$$\vec{\Delta}^1 = \gamma^1 \vec{d}_3 + (-\mu\nu) \vec{d}_1 + \beta^1 \vec{d}_2, \quad \vec{\Delta}^2 = \gamma^2 \vec{d}_3 + \nu \vec{d}_1 + \alpha^2 \vec{d}_2 \quad (53)$$

is the general solution with five arbitrary parameters  $\gamma^1, \gamma^2, \alpha^2, \beta^1, \nu$ .

Case 2:  $\vec{d}_{B,I}$ ,  $I = 1, 2$  are linearly independent. We write with  $\vec{d}_{B,3} := \vec{d}_{B,1} \times \vec{d}_{B,2}$  and expand

$$\vec{\Delta}^I = \gamma^I \vec{d}_{B,3} + \alpha^I \vec{d}_{B,1} + \beta^I \vec{d}_{B,2} \quad (54)$$

It follows

$$\|\vec{d}_{B,1}\|^2 \alpha^1 + \|\vec{d}_{B,2}\|^2 \beta^2 + (\vec{d}_{B,1} \cdot \vec{d}_{B,2}) (\alpha^2 + \beta^1) = 0 \quad (55)$$

Using  $\|\vec{d}_{B,1}\|^4 + \|\vec{d}_{B,2}\|^4 > 0$ , we find the general solution

$$\alpha^1 = \|\vec{d}_{B,1}\|^2 \mu + \|\vec{d}_{B,2}\|^2 \nu, \quad \beta^2 = \|\vec{d}_{B,2}\|^2 \mu - \|\vec{d}_{B,1}\|^2 \nu, \quad \mu = -\frac{(\vec{d}_{B,1} \cdot \vec{d}_{B,2}) (\alpha^2 + \beta^1)}{\|\vec{d}_{B,1}\|^4 + \|\vec{d}_{B,2}\|^4} \quad (56)$$

depending on five parameters  $\gamma^1, \gamma^2, \alpha^2, \beta^1, \nu$ .

To summarise: whatever the coefficients  $\vec{d}_{B,I}$ , we find an at least five parameter sets of solutions to all B-equations. Note that case 2, where  $\vec{d}_{B,1}(p, q), \vec{d}_{B,2}(p, q)$  are non-vanishing and linearly independent, is satisfied at generic points  $p, q$ , as argued in Section 3.6. There is *necessarily* a non-trivial coupling of  $\vec{\Delta}^1, \vec{\Delta}^2$  in cases 1c and 2.

Turning to the A-equations and C-equations, we consider the equivalent set of coefficient functions  $\Delta^{I,l}$  and  $\kappa^{I,l,-}$  where  $\kappa^{I,l,+} = \Delta^{I,l} + \kappa^{I,l,-}$  and  $\Delta^{I,l}$  is now parametrised

by a linear combination of the five free functions  $\gamma^l, \nu, \alpha^2, \beta^1$ . Accordingly, we write the A-equations and C-equations in (47) as

$$\begin{aligned} 0 &= \sum_l [\Delta_{p-\delta_l, q}^{1,l} + \kappa_{p-\delta_l, q}^{1,l,-} - \kappa_{p+\delta_l, q}^{1,l,-}] d_A^l(p, q) \\ 0 &= \sum_l [\Delta_{p, q+\delta_l}^{2,l} + \kappa_{p, q+\delta_l}^{2,l,-} - \kappa_{p, q-\delta_l}^{2,l,-}] d_C^l(p, q) \end{aligned} \quad (57)$$

We note that the first of these two equations is ultra-local in  $q$  and the second is in  $p$ . Moreover,  $\kappa^{1,l,-}, \gamma^1$  are only involved in the first equation,  $\kappa^{2,l,-}$ , and  $\gamma^2$  is only involved in the second, while generically, both depend on  $\alpha^2, \beta^1, \nu$  through which they are coupled in Case 2 (which is expected to be the generic case, see Section 3.6). It is therefore natural to solve the first and second equation, respectively, in terms of  $\kappa^{1,l,-}, \gamma^1$  and  $\kappa^{2,l,-}, \gamma^2$ , respectively. We will show that it is possible to solve both equations just in terms of  $\kappa^{1,l,-}, \kappa^{2,l,-}$ , respectively, thus leaving  $\gamma^1, \gamma^2$  untouched.

Let us abbreviate

$$\rho_A^l = \kappa^{1,l,-}, \rho_C^l = \kappa^{2,l,-}, u_A^l(p, q) := \Delta_{p+\delta_l, q}^{1,l}, u_C^l(p, q) := -\Delta_{p, q+\delta_l}^{2,l} \quad (58)$$

and define the discrete derivatives

$$(\partial_l^A g)(p, q) := g(p + \delta_l, q) - g(p - \delta_l, q), (\partial_l^C g)(p, q) := g(p, q + \delta_l) - g(p, q - \delta_l) \quad (59)$$

to cast (57) into the form (at each  $(p, q)$ )

$$\sum_l d_A^l [\partial_l^A \rho_A^l - u_A^l] = 0, \sum_l d_C^l [\partial_l^C \rho_C^l - u_C^l] = 0 \quad (60)$$

As we wish to keep  $u_A^l, u_C^l$  unconstrained, we consider these as inhomogeneities in the linear discrete PDE problems (60) for the three functions  $\rho_A^l$  and  $\rho_C^l$ , respectively. Due to linearity, the general solution  $\rho_A^l$  is a linear combination of a particular solution to the inhomogeneous problem  $\rho_{A, \text{inhom}}^l$  and of the general solution to the homogeneous problem  $\rho_{A, \text{hom}}^l$ . If we had only one function of each  $\rho_A^l := \rho_A, \rho_C^l := \rho_C$  available, then the construction of the solution to the inhomogeneous problem would read

$$\sum_l d_A^l \partial_l^A \rho_A = f_A, \sum_l d_C^l \partial_l^C \rho_C = f_C \quad (61)$$

where

$$f_A(p, q) := \sum_l \Delta_{p+\delta_l, q}^{1,l} d_A^l(p, q), f_C(p, q) := -\sum_l \Delta_{p, q+\delta_l}^{2,l} d_C^l(p, q), \quad (62)$$

This is the discrete version of two independent PDE's of the form  $\vec{d} \cdot \nabla \rho = f$  with a vector field  $\vec{d}$ . One would then try to solve the discrete PDE's by a discrete version of the characteristic method [42] by inverting the logic of solving a PDE numerically using finite difference techniques. As is well known from the continuous version, this can lead to global existence questions depending on the details of  $\vec{d}$ .

Fortunately, as we have *three* functions rather than just one available, an inhomogeneous solution is readily provided explicitly: We simply solve for each  $l$  separately

$$\partial_l^A \rho_A^l = u_A^l, \partial_l^C \rho_C^l = u_C^l \quad (63)$$

These can be solved explicitly by integration if we prescribe "initial values". We will display this for the  $l = 1$  direction, while the  $l = 2, 3$  directions can be treated by cyclic permutation of the roles of  $l = 1, 2, 3$ . The integration of (63) for  $l = 1$  requires initial data  $\tau_{A,0}^1(p_2, p_3; q), \tau_{A,1}^1(p_2, p_3; q)$  for  $\rho_A^1(p, q)$  at  $p_1 = 0, 1$  and  $\tau_{C,0}^1(p; q_2, q_3), \tau_{C,q}^1(p; q_2, q_3)$

for  $\rho_C^1(p, q)$  at  $q_1 = 0, 1$ . One finds (we drop dependence on ultra local variables and on two or three components of the remaining charge vectors involved)

$$\begin{aligned}\rho_A^1(2p_1) &= \sum_{k=0}^{p_1-1} u_A^1((p)_1 = 2k+1) + \tau_{A,0}^1, \quad p_1 > 0 \\ \rho_A^1(2p_1+1) &= \sum_{k=1}^{p_1} u_A^1((p)_1 = 2k) + \tau_{A,1}^1, \quad p_1 > 0 \\ \rho_A^1(-2p_1) &= -\sum_{k=0}^{p_1-1} u_A^1((p)_1 = -(2k+1)) + \tau_{A,0}^1, \quad p_1 > 0 \\ \rho_A^1(-(2p_1+1)) &= -\sum_{k=0}^{p_1} u_A^1((p)_1 = -2k) + \tau_{A,1}^1, \quad p_1 \geq 0\end{aligned}\quad (64)$$

Similar for  $\rho_C^1, \tau_{C,0}^1, \tau_{C,1}^1$  with  $p_1$  substituted by  $q_1$ .

This proves the existence of an inhomogeneous solution with no restriction on  $\vec{\Delta}^I$  other than that coming from the B-equations, which were already solved. We are free to add a solution of the homogeneous equation

$$\sum_l d_A^l \partial_I^A \rho_A^l = 0, \quad \sum_l d_C^l \partial_I^C \rho_C^l = 0 \quad (65)$$

In contrast to the inhomogeneous solution just provided, the general solution of (65) will depend on the discrete “characteristics” (in the sense of PDE theory) determined by the discrete “vector fields”  $\vec{d}_A, \vec{d}_C$ . To see how one would proceed, consider the A-equation, start at  $p = 0$  and suppose that the component  $d_A^1(0)$  is not vanishing. Then, given an initial datum at  $p = 0, q$  for  $\rho_A^1$ , we can solve  $\rho_A^1$  along an “ $l = 1$  flow line”, in terms of  $\rho_A^2, \rho_A^3$  by (64) if we simply redefine  $u_A^1 := \frac{1}{d_A^1} \sum_{l=2,3} d_A^l \partial_I^A \rho_A^l$ . This works until  $d_A^1 = 0$ . When that happens, but  $d_A^2 \neq 0$ , we may similarly solve  $\rho_A^2$  in terms  $\rho_A^3$  along an “ $l = 2$  flow line” as long as  $d_A^2 \neq 0$  until  $d_A^1 \neq 0$  again or both  $d_A^1 = d_A^2 = 0$ . In the first case, we solve again along the  $l = 1$  line, in the second, if  $d_A^3 \neq 0$  we keep  $\rho_A^3$  constant along the  $l = 3$  line until either  $d_A^1 \neq 0$  again, in which case, we proceed propagating along  $l = 1$ , or  $d_A^1 = 0, d_A^2 \neq 0$  again in which case, we proceed propagating along  $l = 2$  or  $\vec{d}_A = 0$ . At the zeroes of the vector field, there are obviously no conditions on the  $\rho_A^l$ , and we must supply additional initial data there in order to extend the solution beyond. This way, one finds the solution along the “lexicographic integral curves” of  $\vec{d}_A$  (i.e., along directions in the charge space of its non-vanishing components ordered lexicographically), which is possible until they cross, which leads to “shocks”. We will not go into further details here but rather remark that as argued in Section 3.6, (i) the vector fields  $\vec{d}_A, \vec{d}_C$  and even their separate components have zeroes at most at isolated points, and (ii) we may always solve the homogeneous equation by the trivial solution. Then, still, the set of solutions to all constraints is infinite consisting of the five free parameters at each  $(p, q) \in \mathbb{Z}^6$  that are left over after solving the B-equations and the six parameters parametrising the initial data on two  $\mathbb{Z}^5$  surfaces, that is  $\tau_{A,0}^l, \tau_{A,1}^l, \tau_{C,0}^l, \tau_{C,1}^l$ ;  $l = 1, 2, 3$ .

There is one remaining subtlety: the equations (47) were derived under the assumption that (42) and (43) hold, i.e., that  $\kappa^{1,l,\sigma}(p, q)$  vanishes whenever  $p \in \{0, N, -\sigma\delta_l, N - \sigma\delta_l\}$  or  $q \in \{0, N\}$  and that  $\kappa^{2,l,\sigma}(p, q)$  vanishes whenever  $q \in \{0, N, \sigma\delta_l, N + \sigma\delta_l\}$  or  $p \in \{0, N\}$ . Since  $\Delta^{1,l} = \kappa^{1,l,+} - \kappa^{1,l,-}$ , we have that  $\kappa^{1,l,+} = \Delta^{1,l} + \kappa^{1,l,-}$ . This in fact couples the B-equations that relate  $\Delta^{1,l}, \Delta^{2,l}$  with the A-equations on  $\kappa^{1,l,-}$  and with the C-equations on  $\kappa^{2,l,-}$ . While this provides an additional propagation effect, it jeopardises the derivation above which rested on the assumption that we can consider the  $u_A^l \propto \Delta^{1,l}, u_C^l \propto \Delta^{2,l}$  as independent of  $\rho_A^l, \rho_C^l$ . In order to avoid this, we impose the stronger, more uniform condition that  $\kappa^{1,l,\sigma}$  vanishes whenever  $p$  or  $q$  lie in the set of six distinct (assuming  $|\vec{N}_k| > 2$  for  $k = 1, 2, 3$ ) points  $s_l := \{0, \delta_l, -\delta_l, N, N + \delta_l, N - \delta_l\}$ . That way, the vanishing condition on  $\kappa^{1,l,\sigma}$  no longer depends on the labels  $l, \sigma$  and now, the vanishing of  $\kappa^{1,l,+}, \kappa^{1,l,-}$  on  $S_l := s_l \times \mathbb{Z}^3 \cup \mathbb{Z}^3 \times s_l$  is equivalent to the vanishing of  $\Delta^{1,l}, \kappa^{1,l,-}$  on  $S_l$ . To satisfy



this stronger condition, we proceed as follows: The B-equations are ultra-local; thus, we simply set all five free coefficients that parametrise  $\Delta^{I,l}$  to zero on  $S_l$  because  $\Delta^{I,l}$  is homogeneously linear in those. This in fact implies that all  $\Delta^{I,l'}(p, q)$ ,  $I = 1, 2$ ,  $l' = 1, 2, 3$  vanish at any  $(p, q) \in S_l$ , not only the  $l' = l$  components. Considering the A-equations and if we set a possible homogeneous solution to zero, we must ensure that the solutions  $\rho_A^l, \rho_C^l$  vanish on  $S_l$ . Thus, explicitly, the solution  $\rho_A^1$  given in (64) must vanish on  $S_1$ . Recall that  $\rho_A^1$  depends on initial data  $\tau_{A,0}^1(p_2, p_3; q)$ ,  $\tau_{A,1}^1(p_2, p_3; q)$  on the hypersurfaces  $\Sigma_0^1 = \{(p_1 = 0, p_2, p_3; q); p_2, p_3 \in \mathbb{Z}, q \in \mathbb{Z}^3\}$  and  $\Sigma_1^1 = \{(p_1 = 1, p_2, p_3; q); p_2, p_3 \in \mathbb{Z}, q \in \mathbb{Z}^3\}$ . Now,  $s_1 = \{(\{0, \pm 1\}, 0, 0), (N_1 + \{0, \pm 1\}, N_2, N_3)\}$ ; thus, the points in  $s_1 \times \mathbb{Z}^3$  with  $(p_2, p_3) = (0, 0)$  and  $(p_2, p_3) = (N_2, N_3)$  lie on different  $l = 1$  lines. It is then straightforward to see that we can guarantee the vanishing condition on  $\rho_A^1$  at those points by simply restricting one of the initial data in (64) on each of the two distinct  $l = 1$  lines. On the other hand, the points in  $\mathbb{Z}^3 \times s_1$  fill out all of  $p$  space at six discrete values of  $q \in s_1$ . However, by (58) and by the stronger assumption on  $\Delta^{I,l}$  also, the function  $u_A^1$  vanishes there identically. We may therefore consistently set  $\rho_A^1 \equiv 0$  on  $\mathbb{Z}^3 \times s_1$ .

Summarising, for  $q \in s_1$ , we set  $\rho_A^1 \equiv 0$  for all  $p \in \mathbb{Z}^3$ . For  $p \in s_1$ ,  $q \in \mathbb{Z}^3 - s_1$ , we can grant  $\rho_A^1 = 0$  by choosing one of the initial data. Similar considerations hold for  $\rho_C^1$  and the other directions  $l = 2, 3$ . Note that if we would add a homogeneous solution, we could satisfy the vanishing conditions by similar restrictions. Finally, note that the sets  $S_l$  are of “measure zero” in  $\mathbb{Z}^6$  having “codimension” 3.

### 3.9.3. Non-Normalisability and Further Properties of the Solutions

The solution (64) shows the following features:

1. Even if the free data  $\gamma^1, \nu, \alpha^2, \beta^1$  on which  $u_A^1$  generically depends linearly and homogeneously and the initial data  $\tau_{A,0}^1, \tau_{A,1}^1$  have compact support with respect to both  $p, q$  (subject to the above vanishing conditions), the inhomogeneous solution  $\rho_A^1$  has non-compact support with respect to  $p_1$ . The solution just becomes constant, it “freezes”, with respect to  $p_1$  for sufficiently large  $|p_1|$  and has compact support with respect to  $p_2, p_3, q$ . Similar remarks hold for the inhomogeneous solution  $\rho_C^1$  with respect to  $q_1$  and the free data  $\gamma^2, \nu, \alpha^2, \beta^1$  on which  $u_C^1$  depends linearly and homogeneously and the initial data  $\tau_{C,0}^1, \tau_{C,1}^1$ . Identical remarks hold for  $l \neq 1$ . This relies on the non-vanishing of  $\vec{d}_A$  and/or  $\vec{d}_C$  at generic points as argued in Section 3.6, which excludes the possibility that these vector fields have compact support.
2. The constraint equations at vertex A and C, respectively, only involve  $\kappa^{1,l,\sigma}$  and  $\kappa^{2,l,\sigma}$ , respectively, and if there was no constraint equation at vertex B, the solution of those equations could be found independently of each other, and there would be no “propagation”. Assembling the various results, the generic solution found above has

the structure ( $\lambda := -\frac{\vec{d}_{B,1} \cdot \vec{d}_{B,2}}{||\vec{d}_{B,1}||^4 + ||\vec{d}_{B,2}||^4}$ )

$$\begin{aligned} \Delta^{1,l} &= \gamma^1 d_{B,3}^l + [\lambda ||d_{B,1}||^2 (\alpha^2 + \beta^1) + ||d_{B,2}||^2 \nu] d_{B,1}^l + \beta^1 d_{B,2}^l \\ \Delta^{2,l} &= \gamma^2 d_{B,3}^l + [\lambda ||d_{B,2}||^2 (\alpha^2 + \beta^1) - ||d_{B,1}||^2 \nu] d_{B,2}^l + \alpha^2 d_{B,1}^l \\ \kappa^{1,l,-} &= \rho_A^l = G_A^l [\Delta^{1,l}, \tau_{A,0}^l, \tau_{A,1}^l] \\ \kappa^{2,l,-} &= \rho_C^l = G_C^l [\Delta^{2,l}, \tau_{C,0}^l, \tau_{C,1}^l] \\ \kappa^{I,l,+} &= \Delta^{I,l} + \kappa^{I,l,-} \end{aligned} \quad (66)$$

where  $\alpha^{I=2}, \beta^{I=1}, \nu, \gamma^I$ ;  $I = 1, 2$  are five free functions on  $\mathbb{Z}^6$  and  $\tau_{*,I}^l$ ,  $l = 1, 2, 3$ ;  $* \in \{A, C\}$ ,  $I = 0, 1$  are 12 free functions on  $\mathbb{Z}^5$  (both subject to vanishing constraints on  $\mathbb{Z}^3$  hypersurfaces). The functions  $G_A^l, G_B^l$  are homogeneous linear aggregates of their arguments displayed. Modulo characteristic crossing issues, this is by far not the most general solution; we expect to be able to generically add homogeneous solutions to  $\kappa^{I,l,-}$  which depend on four free functions on  $\mathbb{Z}^6$ , but this possibility will not play any role for what follows, because the homogeneous parts can be solved independently at vertices A and C and thus do not contribute to propagation.

The index  $I = 1$  tells that the degree of freedom is associated to vertex  $A$  while the index  $I = 2$  tells that the degree of freedom is associated to vertex  $C$ . The degree of freedom  $\nu$  is part of both  $\alpha^1$  and  $\beta^2$  and cannot be associated to only one of those vertices. Therefore,  $\Delta^{1,I}$  depends only on data associated to  $A$  if we set  $\nu = \alpha^2 = 0$  and  $\Delta^{2,I}$  depends only on data associated to  $C$  if we set  $\nu = \beta^1 = 0$ . In this case,  $\Delta^{1,I} = \gamma^I b_{B,3}^I$  and therefore  $\kappa^{I,I,-}$  depends only on  $\gamma^I$  and thus also  $\kappa^{I,I,+}$  does. That is to say, if we set  $\alpha^2 = \beta^1 = \nu = 0$  and thus also  $\alpha^1 = \beta^2 = 0$ , then the constraint equations at vertex  $B$  are trivially satisfied and the constraint equations at vertices  $A$  and  $C$  can be solved independently of each other. As soon as one of  $\alpha^2, \beta^1, \nu$  is non-vanishing, this is no longer the case. In particular, if we perturb <sup>6</sup> the solution by varying the degree of freedom  $\beta^1$  associated to  $A$ , this perturbs not only  $\alpha^1$  but also  $\beta^2$  associated to  $C$  unless  $\vec{b}_{B,1} \cdot \vec{b}_{B,2} = 0$ , which will not be the case at generic values of  $p, q$ . Likewise, if we perturb the solution by varying the degree of freedom  $\alpha^2$  associated to  $C$ , this perturbs not only  $\beta^2$  but also  $\alpha^1$  associated to  $A$ . A perturbation of  $\nu$  affects both  $\alpha^1, \beta^2$  and can also not be localised to either of the two vertices  $A$  or  $C$ .

It is straightforward to check that the solution with one of  $\nu, \alpha^2, \beta^1$  non-vanishing encodes propagation in the ‘intrinsic’ sense of Section 2.3 and that in such a solution, there is immediate propagation from  $A$  to  $B$  and  $C$  to  $B$  but none from  $B$ . The maximum propagation distance is then just unity.

Since we not only have access to specific solutions but an entire class of solutions with free parameters, it is also possible to ask for a measure of the *strength* of ‘extrinsic’ propagation. A more precise definition can be found below, but the idea motivated by our example is as follows. We define the *propagation degree* of a set of solutions to be the minimal number of its free functions that must be fixed so that the parameters associated to each vertex depends on mutually disjoint sets of the remaining free parameters. This definition makes sense, because then, the parameters for each vertex completely decouple. We define the *propagation co-degree* as the set of all free parameters of a set of solutions minus the propagation degree.

In our case, the parameters that are associated to vertex  $A$  are the  $\kappa^{1,I,\sigma}$  and the parameters that are associated to vertex  $C$  are the  $\kappa^{2,I,\sigma}$ . By inspection of (66), we see that in the case  $\lambda = 0$  (i.e.,  $\vec{d}_{B,1} \cdot \vec{d}_{B,2} = 0$ ), the decoupling is achieved by setting  $\nu = 0$ . In case  $\lambda \neq 0$ , we may write

$$\begin{aligned}\beta^2 &:= \lambda \|d_{B,2}\|^2 (\alpha^2 + \beta^1) - \|d_{B,1}\|^2 \nu \\ \alpha^1 &:= \lambda \|d_{B,1}\|^2 (\alpha^2 + \beta^1) + \|d_{B,2}\|^2 \nu = \left[ \frac{\|d_{B,2}\|^4}{\|d_{B,1}\|^2} + \|d_{B,1}\|^2 \right] \lambda (\alpha^2 + \beta^1) - \frac{\|d_{B,2}\|^2}{\|d_{B,1}\|^2} \beta^2\end{aligned}\quad (67)$$

and consider  $\beta^2$  instead of  $\nu$  as free parameter while  $\alpha^1$  remains a dependent parameter. Then, we can make  $\Delta^{1,I}$  only depend on  $\beta^1$  and  $\Delta^{2,I}$  only depend on  $\alpha^2$  by fixing  $\beta^2 = \lambda \alpha^2 \left[ \frac{\|d_{B,1}\|^4}{\|d_{B,2}\|^2} + \|d_{B,2}\|^2 \right]$ .

We conclude that the propagation degree (to be defined below) of our set of solutions is unity. The co-degree lies between four and eight depending on how many free functions parametrise the set of homogeneous solutions (neglecting the ‘measure zero’ freedom/restrictions provided by initial data/vanishing conditions) to the constraint equations. Thus, at least two functions each can be localised to the vertices  $A$  and  $C$ , respectively, in the sense that these automatically satisfy the constraint equation at  $B$ . However, there is precisely one function that can be considered responsible for non-locality ( $\beta^2$  in the above parametrisation).

### 3.9.4. Propagation

The definition of propagation in Section 2.3 is based on the interpretation of the solution at fixed parameter values as already encoding a disturbance or perturbation and may be applied to any fixed solution with fixed parameter values. Since this definition of propagation is dependent on the intrinsic structure of a single solution, we referred to such propagation (see comments at the end of Section 2.3) as *intrinsic*. In contrast, we now

discuss a definition of ‘extrinsic’ propagation based on an alternate ‘extrinsic’ interpretation of perturbations as changes from one solution to another through small changes in free parameters. That ‘extrinsic’ notion of propagation is less abstract and more adapted to the actual construction of solutions about which the ‘intrinsic’ definition gives no information.

### Definition 1.

1. Suppose that a candidate first generation solution  $\Psi$  of the quantum Einstein equations based on a primordial parent graph  $\gamma^{(0)}$  is a linear combination with non-redundant coefficients  $\kappa_{v,i,k}$  where  $v$  runs through a subset  $V$  of  $V(\gamma^{(0)})$  and at given  $v \in V$ ,  $i$  runs through an index set  $I_v$  labelling  $\gamma_v^{(1)} \in \Gamma_v(\gamma^{(0)})$  and the labels on the edges and vertices, respectively, of  $\gamma_v^{(1)}$  which are not edges and vertices of  $\gamma^{(0)}$ , respectively. Finally,  $k$  runs through a fixed subset  $L$  of the labels of SNWF over  $\gamma^{(0)}$ . It is being understood that gauge invariance is obeyed, and some of the  $\kappa_{v,i,k}$  are automatically vanishing. It is also understood that non-unique parentage is taken into account (i.e., there is no overcounting). That is,

$$\Psi = \sum_{v \in V} \sum_{i \in I_v} \sum_{k \in L} \kappa_{v,i,k} \eta(T_{i,k}) \quad (68)$$

Suppose that the general solution to the quantum Einstein equations imposes conditions on the  $\kappa_{v,i,k}$  to the effect that they become constrained, at generic points, as follows

$$\kappa_{v,i,k} = \sum_{a \in A} \sum_{k' \in L} z_{(v,i),a}(k, k') f_a(k') + \sum_{b \in B} \sum_{k' \in L_b} z_{(v,i),b}(k, k') g_b(k') \quad (69)$$

Here,  $f_a$  are free functions on all of  $L$ , while  $g_b$  are free functions on proper “lower-dimensional” subsets  $L_b \subset L$ . As  $L$  is discrete, lower dimensional means that  $L_b$  is parametrised by less discrete variables than  $L$  is, where those missing variables take infinitely many values in  $L$ . In addition, for each  $a \in A$ ,  $f_a$  is a linear combination of certain  $\kappa_{v,i,l}$  for some fixed  $v$  and some  $i \in I_v$  which label the same  $\gamma_a = \gamma_v^{(1)} \in \Gamma_v(\gamma^{(0)})$ . Let  $A_v \subset A$  be such that for each  $a \in A_v$  there exists at least one  $i \in I_v$  such that the function  $L \times L \rightarrow \mathbb{C}$  with  $(k, k') \mapsto z_{(v,i),a}(k, k')$  is not identically zero. Then,  $\Psi$  is said to display propagation with respect to the choice  $V$  if the  $A_v$  are not mutually disjoint.

2. The propagation degree  $p_V$  subordinate to a choice  $V$ , of a solution parametrised by the label set  $A$ , is the number of elements in the smallest set  $C \subset A$  such that the sets  $A_v - C$  are mutually disjoint. The maximal (minimal) propagation degree  $P(p)$  is the maximum (minimum) of  $p_V$  over all possible choices of  $V$ . The propagation co-degree is, respectively, the number of parameters in  $A$  minus the respective propagation degree.
3. A solution is said to be weakly propagating if  $P > 0$  (i.e., there exists  $V$  with  $p_V > 0$ ) and strongly propagating if  $p > 0$  (i.e., for all  $V$  we have  $p_V > 0$ ).
4. A vertex  $v' \in V$  is said to be correlated with  $v \in V$  if the sets  $A_v, A_{v'}$  are not mutually disjoint. Let  $V_v \subset V$  be the subset of vertices correlated with  $v$ . The correlation length with respect to  $V$  is the maximum of the set  $\{d(v, v'), v \in V, v' \in V_v\}$  where  $d(v, v')$  is the minimal number of edges in  $\gamma^{(0)}$  that one needs to traverse in order to reach  $v'$  from  $v$ .

This definition is also quite technical; thus, let us explain the intuitive meaning of the functions  $f_a, g_b$ : As the number of the functions  $l \mapsto \kappa_{v,i,l}$  is larger than the number of equations (one for each in  $v \in V(\gamma^{(0)})$  and  $l \in L$ ), one will try to solve the constraint equations by separating the  $\kappa_{v,i}$  into independent and dependent functions and solve the equations for the dependent functions. The  $\alpha_a$  captures those independent functions. However, as we have seen in the context of our example, the Hamiltonian constraint equations imposes conditions on the dependent  $\kappa_{v,i,l}$ , which can be considered as discrete partial differential equations (“partial difference equations”). The freedom captured by  $\beta_b$  corresponds to the initial data on measure subsets  $L_b$  that one needs to provide in a solution to those PDE’s (we may equivalently define  $g_b$  on all of  $L$  by extending it trivially to  $L - L_b$  in which case  $g_b$  is supported only on measure zero subsets). The fact that we

phrase propagation in terms of the set  $A$  only rather than also  $B$  is that the functions  $g_b$  have support on “measure zero subsets” of  $L$ . If we would include also  $B$  into the definition of propagation, this would increase the propagation degree; hence, using only  $A$  could be considered a minimal requirement. The physical significance of this choice is not clear at the moment, and it may turn out in the future that this has to be revisited.

Another potential issue is the dependence of the definition on the choice of parameterisation. We have reduced this choice, based on the structure of our example, so that each  $f_a$  is constructed out of  $\kappa_{v,i,l}$  in the manner described above. Whether this is too stringent a requirement to be realised in generic solution constructions or not stringent enough to prevent the notion of propagation being dependent on the available choices of parameterisation is also not clear at the moment. To see what the role of the  $f_a$  is in our example and how they are expressed in terms of the  $\kappa_{v,i,k}$ , we note that in our example,  $V = \{A, C\}$  corresponding to  $I = 1, 2$ ,  $L = \mathbb{Z}^6$  with coordinates  $k = (m_{AB} = p, m_{BC} = q)$  and  $I_A = I_C = \{+1, -1\} \times \{1, 2, 3\}$  with indices  $i = (\sigma, l)$ . We confine ourselves to the generic case that the vectors  $\vec{d}_{B,I}$  are linearly independent and set a possible homogeneous solution to zero. We pick as  $f_a$ ,  $a = \{1, 2, 3, 4, 5\}$  the functions  $f_1 = \beta^1$ ,  $f_2 = \gamma^1$ ,  $f_3 = \alpha^2$ ,  $f_4 = \beta^2$ ,  $f_5 = \gamma^2$  which allow us to write  $\vec{\Delta}^1 \equiv \vec{\Delta}^A$ ,  $\vec{\Delta}^2 \equiv \vec{\Delta}^B$  as linear combinations of the vectors  $\vec{d}_{B,1}$ ,  $\vec{d}_{B,2}$ ,  $\vec{d}_{B,3} := \vec{d}_{B,1} \times \vec{d}_{B,2}$  (which are prescribed by the dynamics) as above in terms of  $f_1, \dots, f_5$  while  $\alpha_1 := [\frac{d_4^2}{d_1^2 + d_1^2}] \lambda(f_3 + f_1) - \frac{d_2^2}{d_1^2} f_4$  is a dependent function, see (67). Here, we have abbreviated  $d_I^2 := \|\vec{d}_{B,I}\|^2$ ,  $I = 1, 2, 3$ . Let  $D$  be the  $2 \times 2$  matrix with entries  $D_{IJ} := \vec{d}_{B,I} \cdot \vec{d}_{B,J}$ ,  $I, J = 1, 2$  which is non-degenerate as  $\det(D) = \|\vec{d}_{B,3}\|^2$ . Let also  $\vec{\Delta}_A \cdot \vec{d}_{B,I} := \Delta_{A,I}$ ,  $\vec{\Delta}_C \cdot \vec{d}_{B,I} := \Delta_{C,I}$ ,  $I = 1, 2, 3$ . Then, we can write  $f_1, \dots, f_5$  in terms of  $\vec{\Delta}_A$ ,  $\vec{\Delta}_C$  as follows using the inverse  $D^{-1}$

$$f_2 = \frac{\Delta_{A,3}}{d_3^2}, f_5 = \frac{\Delta_{C,3}}{d_3^2}, f_1 = \frac{D_{11} \Delta_{A,2} - D_{12} \Delta_{A,1}}{d_3^2}, f_3 = \frac{D_{22} \Delta_{C,1} - D_{12} \Delta_{C,2}}{d_3^2}, f_4 = \frac{D_{11} \Delta_{C,2} - D_{12} \Delta_{C,1}}{d_3^2} \quad (70)$$

Since with  $\vec{\kappa}_A^\sigma := \vec{\kappa}^{1,\sigma}$ ,  $\vec{\kappa}_C^\sigma := \vec{\kappa}^{2,\sigma}$ , we have

$$\vec{\Delta}_A = \vec{\kappa}_A^+ - \vec{\kappa}_A^-, \vec{\Delta}_C = \vec{\kappa}_C^+ - \vec{\kappa}_C^- \quad (71)$$

we have demonstrated that  $f_1, f_2$  can be expressed as linear combinations of the  $\kappa_A^i$  which refer to the graph  $\gamma_A := \gamma_1$ , while  $f_3, f_4, f_5$  can be expressed as linear combinations of the  $\kappa_C^i$  which refer to the graph  $\gamma_C := \gamma_2$ . Therefore, for the index  $a = 1, 2$ , we have the fixed graph  $\gamma_a = \gamma_A$  associated to  $v = A$  and for the index  $a = 3, 4, 5$ , we have the fixed graph  $\gamma_b = \gamma_C$  associated to  $v = C$ .

As for the definition of intrinsic propagation, let us apply the above definition to our example. Accordingly, interpreting a perturbation of a solution of its free coefficients associated to  $A$  that affects its free coefficients associated to  $C$  as extrinsic propagation of the perturbation from  $A$  to  $C$  via  $B$ ; then, we see that extrinsic propagation is a generic feature in this example class of solutions. The crucial feature of the system (47) responsible for this is the non-trivial coupling between the  $\kappa_M^{1,l,\sigma}$  and  $\kappa_M^{2,l,\sigma}$  coefficients that is displayed in its third equation. The first and second condition, respectively, can be argued to provide a local condition on  $\kappa_M^{1,l,\sigma}$  and  $\kappa_M^{2,l,\sigma}$ , respectively, which solely arise due to the action of the Hamiltonian constraint at vertices  $A$  and  $C$ , respectively. However, the third equation which arises from its action at vertex  $B$  cannot be considered as local to  $B$  as it involves both  $\kappa_M^{1,l,\sigma}$ ,  $\kappa_M^{2,l,\sigma}$  whose “locality was already assigned” to vertex  $A, C$ , respectively. The reason for why this happens is due to the mechanism of non-unique parentage. A child vector state of the form  $T_{\gamma_1, M; \Delta}^{l,\sigma}$  is in the image of both  $C_A$  and  $C_B$  but from different parents. The parent  $T_{\gamma, m, N}$  for  $C_A$  must adapt the charges  $m_{AB}, m_{BC}$  according to  $m_{AB} = M_{AB} + \sigma \delta_l$ ,  $m_{BC} = M_{BC}$ , while the parent  $T_{\gamma, m, N}$  for  $C_B$  must adapt the charges according to  $m_{AB} = M_{AB}$ ,  $m_{BC} = M_{BC} - \sigma \delta_l$ . Likewise, a vector child state of the form  $T_{\gamma_2, M; \Delta}^{l,\sigma}$  is in the image of both  $C_C$  and  $C_B$  but from different parents. The parent  $T_{\gamma, m, N}$  for  $C_C$  must

adapt the charges  $m_{AB}, m_{BC}$  according to  $m_{AB} = M_{AB}$ ,  $m_{BC} = M_{BC} - \sigma\delta_l$ , while the parent  $T_{\gamma, m, N}$  for  $C_B$  must adapt the charges according to  $m_{AB} = M_{AB} + \sigma\delta_l$ ,  $m_{BC} = M_{BC}$ . This means that the coefficients *cannot be unambiguously associated to vertices* in the sense that one has disjoint sets of coefficients that are to be solved for vertex wise with no further conditions.

We conclude that the example studied displays intrinsic propagation in the sense of Section 2.3 (i.e., in the context of a fixed solution) as well as in the extrinsic sense above in terms of perturbations of free parameters, which define the space of solutions.

#### 4. Conclusions and Outlook

It is obvious that the propagation effect sketched above extends to more complicated graphs (e.g., those that are networks made out of arbitrarily long and knotted chains of four valent graphs intersecting in eight valent vertices and possibly with some knotted loops attached to make the graph symmetry group trivial) and will even be enhanced the higher the connectivity of the graphs and the valence of their vertices are, at least in the  $U(1)^3$  theory. That is, we expect that the propagation degree of a solution class, as well as the propagation distance in a (propagating) solution, that is labelled by more and more complicated graphs will drastically increase with the number and complexity of the graphs considered.

As far as the extension to  $SU(2)$  is concerned, note that the qualitative features of our example remain the same. The curvature factor in the Hamiltonian constraint now leads to shifts by  $\pm\frac{1}{2}$  in the spin quantum numbers on the edges of SNWF, and the arcs become charged with spin  $1/2$  rather than  $\pm\delta_l$ . The only difference is that in the  $SU(2)$  case, we are not able to compute the eigenvalues of the volume operator so easily, and this was used above to give an easy proof of the absence of the disjointness of sets of coefficients required for the presence of propagation within a solution. However, it is hard to imagine that there should be an accidental symmetry that would render magically all those matrix elements of the volume operator to zero which would lead to the absence of long-range correlations of the solution of the Hamiltonian constraint. At least, the present paper establishes that the burden of proof rests on those that argue that the locality of the action of the Hamiltonian constraint at vertices implies locality of its solution with no propagation. In fact, our paper suggests a numerical proof of propagation both for Lorentzian and Euclidean GR with gauge group  $SU(2)$  by diagonalising the volume operator on four valent vertices numerically, which is feasible since the matrix elements of its fourth power are available analytically [32]. We leave this to future work which may benefit from modern numerical [23–26] methods and machine learning [27,28] and/or quantum computing [29–31] techniques.

Finally, in our example, we considered only a tiny subset of solutions as we took only children of “first generation” into account. More generally, one can consider solutions that involve the images of an arbitrary number of applications of the Hamiltonian constraint on CNWF. In the Euclidean theory, the set of conditions to be met by a solution to the constraints does not mix those generations, and in that sense, the analysis is complete when one controls the one-generation case. However, this is no longer true in the Lorentzian theory, which does mix generations. Hence, we expect an even stronger notion of propagation for Lorentzian signature. See [19] for details.

We close by stressing again that propagation hinges on the presence of non-unique parentage. This is the case, modulo the above reservations, in QSD [5,6,19] and in the electric shift approach of [14,15] even for the Euclidean part of the constraint but generically not, e.g., for the symmetric Euclidean Hamiltonian constraint operator proposal of [37,38]: There, the loops created by the Hamiltonian constraint are attached only to the vertex of the original graph, i.e., they intersect the original graph in that single vertex and no other point of the graph. Moreover, they lie in the coordinate plane defined by the tangents of two edges adjacent to that vertex. Accordingly, unless there are graph symmetries, such a child graph has a unique parent graph even after diffeomorphism averaging. Therefore,

constraint equations at each vertex are local to that vertex. As the loop attachment does not change the spins of the edges of the parent SNWF, the Euclidean Hamiltonian constraint at a given vertex thus only imposes constraints on the intertwiners associated to that vertex, and that intertwiner space is finite dimensional. Thus, the solutions to the Euclidean Hamiltonian of [37,38] are normalisable with respect to the norm of the diffeomorphism invariant Hilbert space in contrast to the situation in [5,6] and the present paper.

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## Appendix A. Explicit form of $d_A^l, d_C^l, d_{B,1}^l, d_{B,2}^l$

From (14) and (15), we have that:

$$Q_A = (m_{ZA} + c_A, n_{ZA}, m_{AB} - n_{AB} - c'_A) - (m_{AB}, n_{AB} + c'_A, m_{ZA} + c_A - n_{ZA}) \quad (A1)$$

$$d_A^l := -d_{A,1}^l + d_{A,2}^l + d_{A,3}^l - d_{A,4}^l \quad (A2)$$

with

$$d_{A,1}^l = |(Q_A + (n_{AB} + c_A, n_{ZA} + m_{AB}, \delta_l) - (m_{AB}, n_{ZA}, \delta_l))|^{\frac{1}{2}} \quad (A3)$$

$$d_{A,2}^l = |(Q_A - (n_{AB} + c_A, n_{ZA} + m_{AB}, \delta_l) + (m_{AB}, n_{ZA}, \delta_l))|^{\frac{1}{2}} \quad (A4)$$

$$d_{A,3}^l = |(Q_A - (n_{AB} + c_A, m_{ZA} + c_A + m_{AB}, \delta_l) + (m_{AB}, m_{ZA} + c_A, \delta_l))|^{\frac{1}{2}} \quad (A5)$$

$$d_{A,4}^l = |Q_A|^{\frac{1}{2}} \quad (A6)$$

where we have found it convenient to denote the  $\nu$  functions in (15) by  $d_{A,i}^l$ . We shall continue to use such notation in what follows.

Next, from (16) and (17), we have that:

$$Q_C = (m_{BC} + c_C, n_{BC}, m_{CD} - n_{CD} - c'_C) - (m_{CD}, n_{CD} + c'_C, m_{BC} + c_C - n_{BC}) \quad (A7)$$

$$d_C^l := d_{C,1}^l - d_{C,2}^l - d_{C,3}^l + d_{C,4}^l \quad (A8)$$

with

$$d_{C,1}^l = |(Q_C + (m_{BC} + c_C, n_{BC} + m_{CD}, \delta_l) - (n_{BC}, m_{CD}, \delta_l))|^{\frac{1}{2}} \quad (A9)$$

$$d_{C,2}^l = |(Q_C - (m_{BC} + c_C, n_{BC} + m_{CD}, \delta_l) + (n_{BC}, m_{CD}, \delta_l))|^{\frac{1}{2}} \quad (A10)$$

$$d_{C,3}^l = |(Q_C - (m_{BC} + c_C, n_{BC} + c_C + n_{CD}, \delta_l) + (n_{BC}, n_{CD} + c_C, \delta_l))|^{\frac{1}{2}} \quad (A11)$$

$$d_{C,4}^l = |Q_C|^{\frac{1}{2}} \quad (A12)$$

Finally, from (19), (20) and (21), we have:

$$Q_B = (m_{AB} + c_B, n_{AB}, m_{BC} - n_{BC} - c'_B) - (m_{BC}, n_{BC} + c'_B, m_{AB} + c_B - n_{AB}) \quad (A13)$$

$$d_{B,1}^l := d_{B1,1}^l - d_{B1,2}^l - d_{B1,3}^l + d_{B1,4}^l \quad (A14)$$



with

$$d_{B1,1}^l = |(Q_B + (m_{AB} + c_B, n_{AB} + m_{BC}, \delta_l) - (n_{AB}, m_{BC}, \delta_l))|^{\frac{1}{2}} \quad (\text{A15})$$

$$d_{B1,2}^l = |(Q_B - (m_{AB} + c_B, n_{AB} + m_{BC}, \delta_l) + (n_{AB}, m_{BC}, \delta_l))|^{\frac{1}{2}} \quad (\text{A16})$$

$$d_{B1,3}^l = |(Q_B - (m_{AB} + c_B, n_{AB} + n_{BC} + c_B, \delta_l) + (n_{AB}, n_{BC} + c_B, \delta_l))|^{\frac{1}{2}} \quad (\text{A17})$$

$$d_{B1,4}^l = |Q_B|^{\frac{1}{2}} \quad (\text{A18})$$

$$d_{B2}^l := d_{B2,1}^l - d_{B2,2}^l - d_{B2,3}^l + d_{B2,4}^l \quad (\text{A19})$$

with

$$d_{B2,1}^l = |(Q_B - (m_{BC} + n_{AB}, n_{BC} + c_B, \delta_l) + (n_{AB}, m_{BC}, \delta_l))|^{\frac{1}{2}} \quad (\text{A20})$$

$$d_{B2,2}^l = |(Q_B - (m_{AB} + c_B, m_{BC}, \delta_l) + (m_{AB} + c_B + m_{BC}, n_{BC} + c_B, \delta_l))|^{\frac{1}{2}} \quad (\text{A21})$$

$$d_{B2,3}^l = |(Q_B + (m_{BC} + n_{AB}, n_{BC} + c_B, \delta_l) - (n_{AB}, m_{BC}, \delta_l))|^{\frac{1}{2}} \quad (\text{A22})$$

$$d_{B2,4}^l = |Q_B|^{\frac{1}{2}} \quad (\text{A23})$$

## Notes

- <sup>1</sup> Note that the arXiv version of [19] differs from its published version. Both versions discuss non-symmetric and symmetric operators, but these operators are different in the two versions. What follows applies to the non-symmetric operator in the arXiv version, which coincides with the operator of [5,6], which the discussion in [18] refers to. While surprisingly non-symmetric operators are the ones of physical interest due to the arguments given in [20], nevertheless, we will also comment on a symmetric operator which, however, is different from the one discussed in the arXiv version of [19]; see Section 3.4. Both  $U(1)^3$  analogs of the non-symmetric and the symmetric operator discussed in this paper display propagation.
- <sup>2</sup> This property holds only for the published version of [19] and not for the arXiv version. In the arXiv version, the statement in Definition 1.2 (iii) (namely that any spin net has a uniquely identified ‘source’) is in error. As we argue in this paper, the existence of spin nets with non-unique sources is an essential ingredient for propagation.
- <sup>3</sup> Strictly speaking, only if the graphs in question have no moduli [8] or if we work in a single superselected sector. The graphs that we use in concrete calculations in the next section do not have moduli because its vertices are either 4-valent or 6-valent but with only four distinct tangent directions of the adjacent edges.
- <sup>4</sup> The exchange operators defined in [18] are (linear) maps of these  $\alpha_v$  and since these map solutions to solutions, they provide Dirac observables.
- <sup>5</sup> The first paper in Reference [14,15] coBiomech’ Modeling Mechanobiolstrate anomaly-free constraint commutators for one of them, which is referred to as the ‘Mixed Action’ in the first. Our comments here pertain to this Mixed Action.
- <sup>6</sup> This notion of ‘extrinsic’ perturbation is distinct from the intrinsic disturbance/perturbation discussed in Section 2.3.

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