Discrete Random Spacetimes:
Covariance and Quantization in Growth Dynamics for Causal Sets

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Abstract

Within causal set theory, growth dynamics form a novel realisation of the path integral for quantum gravity. This thesis presents recent developments in growth dynamics, focusing on general covariance and quantization. In causal set theory, where spacetime takes the form of a discrete causal set, general covariance takes the form of label-invariance. Here we present the first manifestly covariant growth dynamics, namely models of random unlabeled graphs. These models, like their label-dependent predecessors, are classically stochastic. The decoherence functional offers a stochastic-like formulation of quantum theory particularly suited to quantum gravity and a proposal for a decoherence functional for causal sets based on growth dynamics has previously been put forward, but it was shown to fail in certain cases due to a technical pathology. We provide new criteria for when the procedure is well-defined and apply these to obtain the first known examples of quantum dynamics for causal sets. We generalise the construction of the decoherence functional to a wide class of growth dynamics and discuss its application to dynamics which give rise to bouncing cosmologies. Finally, we explore whether growth dynamics, labeled and unlabeled, can accommodate cosmologies in which time has no beginning.

List of publications

This thesis is based on the following publications:

F. Dowker, N. Imambaccus, A. Owens, R. Sorkin and S. Zalel, *A manifestly covariant framework for causal set dynamics*, *Class. Quantum Grav.* 37 (2020) 085003.

S. Surya and S. Zalel, *A Criterion for Covariance in Complex Sequential Growth Models*, *Class. Quantum Grav.* 37 (2020) 195030.

S. Zalel, *The structure of covtree: searching for manifestly covariant causal set dynamics*, *Class. Quantum Grav.* 38 (2021) 015001.

B. Bento, F. Dowker and S. Zalel, *If time had no beginning: growth dynamics for past-infinite causal sets*, in preparation (2021).

Originality declaration: I confirm that this work is my own, and that contributions from others have been appropriately referenced. A portion of the material in this thesis has appeared in the publications listed above, but this document as a whole has not been submitted for publication or for degree assessment elsewhere.

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Conventions

0 is contained in the set of natural numbers, $\mathbb{N} = \{0, 1, 2, ...\}$, in the set of negative integers, $\mathbb{Z}^- = \{0, -1, -2, ...\}$, and in the set of non-negative real numbers, $\mathbb{R}^+$.

Preface

The causal set approach to quantum gravity (also known as "causal set theory") postulates that spacetime is fundamentally discrete. This leap—motivated by theorems in Lorentzian geometry [Hawking, 2014; Kronheimer and Penrose, 1967; Malament, 1977], by the search for a resolution to the infinities of General Relativity and quantum field theory and by the Bekenstein-Hawking formula for the entropy of black holes—is bold and holds huge promise to unify General Relativity with quantum mechanics, a problem described by J.A. Wheeler as the "greatest crisis of physics of all time" [Rees et al., 1974].

In causal set theory, the smooth continuum spacetime of General Relativity is replaced with a "causal set"—a large collection of discrete "elements" organised by means of "relations" into a locally finite partial order [Bombelli et al., 1987]. Physically, each element is a discrete unit of spacetime volume, an event localised in space and time. The relation, denoted by $\prec$, is a causal relation, *i.e.* if elements x and y are related as $x \prec y$ then we say "x precedes y" or "x is in the past of y". In this way, the atomicity of quantum theory and the causality of General Relativity are married in an underlying reality which is purely combinatorial and requires no continuum concept to describe it.[1]

This novel framework is geared towards high (Planckian) energy scales and complements the continuum methods prevalent in modern theoretical physics, as evidenced by major successes which reach beyond causal set theory and into other areas of theoretical physics and cosmology. These include a formulation of quantum field theory [Sorkin, 2011a, 2017] which led to a proposal for a distinguished vacuum state for quantum fields in curved spacetime [Afshordi et al., 2012; Surya et al., 2019]; insights into quantum fields on topology-changing spacetimes [Buck et al., 2017] and into entanglement entropy

[1]It is worth noting that unlike other discretizations (*e.g.* regular lattice), in causal set theory Lorentz invariance is preserved in an appropriate sense [Bombelli et al., 2009b; Dowker, 2011].

[Sorkin, 2014; Sorkin and Yazdi, 2018]; a concrete study of the Hartle-Hawking wavefunction [Glaser and Surya, 2016]; a proposal which resolves the current tensions between the different measurements of the cosmological constant [Ahmed et al., 2004; Zwane et al., 2018]; and a successful prediction for the magnitude of the cosmological constant [Sorkin, 1990]—the only prediction by any quantum gravity theory to be verified so far.

Research in causal set theory explores a host of interrelated themes including[2]: how continuum attributes such as geometry, topology, *etc.* emerge from a causal set on macroscopic scales [Bombelli and Meyer, 1989; Bombelli et al., 2009a; Eichhorn et al., 2017, 2019a,b; Major et al., 2007, 2009]; the behaviour of matter fields living on a causal set [Dable-Heath et al., 2020; Johnston, 2010; X et al., 2017]; phenomenology [Dowker, 2013; Dowker et al., 2010a]; the causal set action [Benincasa and Dowker, 2010; Buck et al., 2015; Dowker and Glaser, 2013; Machet and Wang, 2021]; and growth dynamics, pioneered in [Rideout and Sorkin, 2000]. This thesis presents a contribution to the latter.

[2]The following citations are not exhaustive. For a comprehensive review of causal set theory see [Surya, 2019].

Chapter 1

Introduction

There are many forks on the road to a theory of quantum gravity. The current state of play in which quantum matter is coupled to a classical spacetime leaves much to the imagination and has led to a proliferation of ideas. But a common expectation shared by many is that spacetime itself should be governed by a quantum dynamics and exhibit quantum fluctuations on small length scales that would be most keenly felt in regimes of strong gravity, for example near the singularity of a black hole or in the early universe.

Even with this consensus, the quantization of gravity is plagued with obstructions. The non-linearity of the Einstein-Hilbert action is an impediment to the operator formalism [Hawking, 1978]; the general covariance of General Relativity hinders the canonical quantization via the so-called "problem of time" [Carlip, 1990; Isham, 1992]; spacetimes with exotic topologies do not admit a foliation into spatial hypersurfaces and thus resist the canonical formalism altogether [Anderson and DeWitt, 1986]; and the observer-based interpretation of quantum mechanics is incompatible with the regimes we wish to explore since they host no observers [Bell, 2004].

These struggles have led some to believe that the path integral is the most suitable framework for a quantum theory of gravity [Gibbons and Turok, 2008; Hawking, 1978; Sorkin, 1997]. Perhaps its most appealing feature in the context of gravity is its compatibility with general covariance: the integral sums over *complete* spacetime histories and therefore does not require a foliation or a distinguished time parameter while covariant "observables" can be defined independently of observers as attributes of histories. But establishing the path integral as the fundamental foundation of quantum mechanics remains a challenge and in its application to gravity it is not clear which histories should be contained in the domain of the integral nor whether the integration measure is well-defined [Gibbons et al., 1978, 1987]. Thus, one may be justified in regarding the path integral as a guiding principle rather than an exact prescription.

In the causal set approach, where the continuum spacetime of General Relativity is replaced by a discrete causal set, the path integral is replaced by a "sum-over-histories" whose precise definition and interpretation pose a momentous challenge. This is the challenge that the growth dynamics program aims to meet.

A growth dynamics is a probabilistic process in which a causal set comes into being *ex nihilo* by accretion of elements. This growth process plays a dual role: it embodies the sum-over-histories (*e.g.* by providing a mechanism from which the action is emergent) and it offers a novel route for accounting for the passage of time within physics [Dowker, 2014, 2020; Norton, 2018; Sorkin, 2007; Wuthrich and Callender, 2017]. Crucially, the growth does not happen *in* time—it constitutes the passage of time. The birth of an element is the *happening* of that event, while the existence of an element signifies that the event has *already happened*. Thus heuristically, the growth process is a physical process whose phenomenological manifestations is the passage of time.[1]

Motivated by Quantum Measure Theory—an approach to quantum foundations which views quantum mechanics as a generalised form of classical stochastic theory [Sorkin, 2012, 1994]—the growth dynamics program has furthered our understanding of the causal set sum-over-histories by exploiting tools from probability theory, measure theory and the theory of random walks. Still, many open questions remain and these can be roughly partitioned into three overarching themes:

> *What is the domain of the sum-over-histories?* The domain of the sum corresponds to the sample space of the growth process (*i.e.* to the causal sets which it can grow). Should it contain "labeled" or "unlabeled" causal sets[2]? Should it contain all infinite causal sets or only those which are past-finite?

[1]While the work presented here is motivated by this interpretation, our results are not contingent on it.

[2]We make these notions precise in chapter 2.

What is the amplitude by which each history should be weighted? In the Lagrangian formulation of dynamics, it is the role of the action to pick out the histories which best describe physical reality. In the growth dynamics framework, this role is played by the transition probabilities which govern the stochastic growth of the causal set. How should these transition probabilities be constrained to obtain physical dynamics? And how can these transition probabilities be generalised into transition *amplitudes* so that the resulting dynamics exhibits quantum interference?

What are the physical observables? In the growth framework, the observables are sets of histories (known in the probability literature as "events"). While these can be formally defined as elements of a σ-algebra, it is a challenge to seek out their physical interpretation in a setting where key concepts such as "local" and "global" are up for debate.

1.1 Synopsis

This thesis presents recent developments in growth dynamics for causal sets, focusing on general covariance and quantization. In chapter 2 we discuss how label-invariance plays the role of general covariance in causal set theory, and introduce some terminology which will underpin the discussion in the subsequent chapters. In chapter 3 we review the Classical Sequential Growth (CSG) models, with emphasis on three key features which will serve as foundation for the new work presented thereafter: label-dependence, observables and cosmic renormalisation. In chapter 4, building on the discussion of label-invariance as general covariance and of observables in CSG models, we present the first manifestly covariant (label-independent) growth dynamics and extend the cosmic renormalisation to these new models. The covariant models, like their label-dependent predecessors, are classically stochastic. The decoherence functional offers a stochastic-like formulation of quantum theory particularly suited to quantum gravity and a proposal for a decoherence functional for causal sets based on the CSG models has previously been put forward, but it was shown to fail in certain cases due to a technical pathology. In chapter 5 we provide new criteria for when the procedure is well-defined, apply these to obtain the first known examples of quantum dynamics for causal sets and generalise the construction of the decoherence functional to a wide class of growth dynamics. In chapter 6, motivated by cosmic renormalisation, we identify a class of observables within classical labeled dynamics which give rise to bouncing cosmologies and apply our results from the previous chapter in discussing the quantization of these models. In chapter 7, we explore whether growth dynamics, labeled and unlabeled, can accommodate cosmologies in which time has no beginning. We conclude in chapter 8. Glossaries of causal set and measure theory terminology are provided in appendices A.1 and A.2.

Chapter 2

Labels and label-invariance

In causal set theory—where the continuum spacetime of General Relativity is replaced by a discrete causal set—spacetime points and their coordinates are replaced by causal set elements and their "labels". Thus, Einstein's struggle between the formulation of generally covariant laws of nature and the intrinsic (in)distinguishability of spacetime points [Rovelli, 1991; Schilpp, 1949; Stachel, 2014] manifests in causal set theory as tension between label-dependence and label-independence. This tension can be expressed via a myriad of interrelated questions: What is the physical status which one should assign to the causal set elements and to their labels? Should we conflate the "label" of an element with the "intrinsic identity" of an element or should they be considered separately? What are the precise mathematical concepts which are best suited for formulating a physical theory of causal sets? In particular, could the theory be formulated without any reference to "labels"?

Mathematically, the elements of a causal set are distinguishable (in so far as a causal set is a set) and the notion of labeling these elements appears in pure mathematics and in its applications, including in causal set theory where labels naturally arise within the Classical Sequential Growth models of [Rideout and Sorkin, 2000]. Physically, it is a postulate of causal set theory that no information is contained in any individual identity or label of the elements, so that stripped from ordering the elements are physically indistinguishable. This combination of label-dependent mathematics and label-independent physics led to understanding general covariance within causal set theory as label-invariance [Brightwell et al., 2002, 2003; Rideout and Sorkin, 2000].

This chapter is dedicated to choosing the right concepts for the discussion of general covariance in causal set theory. We review the evolution of the relevant concepts through the literature (section 2.1) and present the definitions coined by the author of this thesis and her collaborators during their work (section 2.2). These definitions will form the basic vocabulary for the remainder of this thesis.

2.1 A brief history of labelings

A partial order is a pair, $(\Pi, \prec)$, where $\prec$ is a transitive, irreflexive relation on the ground-set Π. A linear order (also known as a total order) is a partial order in which any two elements are comparable. In the following, we use $\lll$ to denote a linear order on a generic ground-set and reserve the use of $<$ to denote the usual linear order on the integers.

A "labeling" of a set Π is a mapping λ from Π to an index set $\mathcal{I}$. When Π carries additional structure, it may be desirable that the labeling reflect this additional structure. In particular, when labeling a partially ordered set $(\Pi, \prec)$ one often endows the index set with a total order $\lll$ and requires that the labeling is order-preserving, *i.e.* $x \prec y \implies \lambda(x) \lll \lambda(y)\ \forall x, y \in \Pi$. In this way, the labeling extends the partial order into a linear order and thus can be thought of as a "linear extension". We now survey the usage of the terms "linear extension" and "labeling" in the relevant mathematics and physics literature.

Perhaps the simplest definition of linear extension is,

Definition 2.1.1 (Linear extension (definition 1))**.** *A linear extension of a partial order* $(\Pi, \prec)$ *is a linear order* $(\Pi, \lll)$ *such that* $x \prec y \implies x \lll y\ \forall x, y \in \Pi$ *[Alon et al., 1994; Brightwell, 1994].*

Definition 2.1.1 can be abstracted by allowing the ground-set of the linear order to be any set $\mathcal{I}$ satisfying $|\mathcal{I}| = |\Pi|$. In this case the term "linear extension" does not refer to the linear order $(\mathcal{I}, \lll)$ but to an order-preserving bijection from $(\Pi, \prec)$ to $(\mathcal{I}, \lll)$,

Definition 2.1.2 (Linear extension (definition 2))**.** *A linear extension of* $(\Pi, \prec)$ *is a* *bijection λ from Π to some linear order $(\mathcal{I}, \ll)$ such that $x \prec y \implies \lambda(x) \ll \lambda(y)\ \forall x, y \in \Pi$ [Brightwell, 1988].*

Therefore, we can think of a linear extension as a labeling by a totally ordered index set. A special case of the linear extension is the "natural extension", or "natural labeling" as it is known in the physics literature,

Definition 2.1.3 (Natural labeling)**.** *A natural labeling or natural extension is a bijection λ from Π to the linear order $(\mathbb{N}, <)$ such that $x \prec y \implies \lambda(x) < \lambda(y)\ \forall x, y \in \Pi$ [Brightwell and Luczak, 2011].*

In words, a natural labeling is an enumeration of the causal set elements which respects the partial order $\prec$. It is worth noting that a natural labeling is sometimes defined as a mapping *from* $\mathbb{N}$ *to* Π so that its *inverse* is order-preserving.

The notion of natural labeling is a useful one and can be adapted to partial orders of finite cardinality n by replacing $(\mathbb{N}, <)$ with one of the ordered intervals $(\{1, ..., n\}, <)$ or $(\{0, 1, ..., n-1\}, <)$. The former is common in the mathematics literature [Brightwell and Winkler, 1991] while the latter is common in the physics literature [Brightwell et al., 2003; Rideout and Sorkin, 2000]. We leave discussion of partial orders which are not of order-type $\mathbb{N}$ (*e.g.* $(\mathbb{Z}, <)$) to chapter 7.

Equipped with the notion of labeling, one may use the term "labeled partial order" to mean a partial order together with a natural labeling of it [Brightwell et al., 2003], which suggests that we should think of a labeled partial order as a triple $(\Pi, \prec, \lambda)$. In practice, one often discusses labeled partial orders without specifying the ground-set Π by repackaging the information contained in the order relation $\prec$ and the natural labeling λ into a partial order on a set of natural numbers. For example, if $|\Pi| = |\mathbb{N}|$ and $\lambda : \mathbb{N} \to \Pi$ then the corresponding labeled partial order is given simply as the partial order $(\mathbb{N}, \prec_\lambda)$, where $x \prec_\lambda y \iff \lambda(x) \prec \lambda(y)$ [Ash and McDonald, 2003; Brightwell et al., 2003; Sorkin, 2011b].

The term "unlabeled partial order" is borrowed from graph theory and means that the elements of the partial order are indistinguishable when stripped of the relation $\prec$. Therefore, an "unlabeled partial order" is not in fact a partial order but an order-isomorphism equivalence class of partial orders. Which partial orders are contained in a given equivalence class depends on one's universe of discourse (*e.g.* partial orders on a specified ground-set).

2.2 Labeled causets, orders and their stems

We now present the terminology distilled by the author and her collaborators for the purpose of discussing general covariance in causal set theory in a way which is both mathematically precise and physically clear. This is the vocabulary used in the remainder of this thesis, unless specified otherwise.

Recall that a causal set (or causet) is a locally finite partial order. For any natural number n, let $[0, n]$ denote the set $\{0, 1, ..., n\}$ (devoid of any ordering).

Definition 2.2.1 (Labeled causet)**.** *A labeled causet is any causet* $([0, n], \prec)$ *or* $(\mathbb{N}, \prec)$ *satisfying* $x \prec y \implies x < y$*.*

Definition 2.2.2 (n-causet)**.** *An* n*-causet is a labeled causet of cardinality* n*.*

Given some $n > 0$, we denote the set of n-causets by $\tilde{\Omega}(n)$. The set of finite labeled causets is denoted by $\tilde{\Omega}(\mathbb{N})$, *i.e.* $\tilde{\Omega}(\mathbb{N}) := \bigcup_{n>0} \tilde{\Omega}(n)$. The set of all infinite labeled causets is denoted by $\tilde{\Omega}$.

Our universe of discourse contains all labeled causets and their subcausets. (Note that a subcauset of a labeled causet is not necessarily a labeled causet because its ground-set may not be an interval of integers of the form $\{0, 1, ..., n\}$.) We denote labeled causets and their subcausets by capital Roman letters with a tilde, *e.g.* $\tilde{C}$. We often (but not always) use a subscript to denote the cardinality of an n-causet, *e.g.* $\tilde{C}_n$.

Definition 2.2.3 (Order)**.** *An order (or "unlabeled causet") is an order-isomorphism equivalence class of labeled causets.*

We denote orders by capital Roman letters without a tilde. Given an order, the Hasse diagrams of its representatives differ from each other only by the labeling of nodes (*i.e.* they are graph-isomorphic). Therefore, we represent an order by a Hasse diagram without node labels (Fig.2.1).

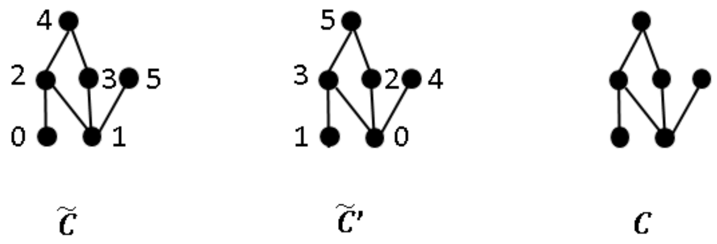


Figure 2.1: $\tilde{C}$ and $\tilde{C}'$ are order-isomorphic labeled causets. Each is a representative of the order C, shown on the right as a Hasse diagram without labels.

Definition 2.2.4 (Cardinality of an order)**.** *The cardinality of an order is defined to be the cardinality of a representative of it.*

We denote the cardinality of an order C by $|C|$.

Definition 2.2.5 (n-order)**.** *An n-order is an order of cardinality n.*

In other words, an n-order is an order whose representatives are n-causets. We often (but not always) use a subscript to specify the cardinality of an n-order, *e.g.* C_n.

We use $\Omega(n)$, $\Omega(\mathbb{N})$ and Ω to denote the set of n-orders, the set of finite orders and the set of infinite orders, respectively. Note that these are equivalent to the quotient spaces $\tilde{\Omega}(n)/\cong$, $\tilde{\Omega}(\mathbb{N})/\cong$ and $\tilde{\Omega}/\cong$, where $\cong$ denotes equivalence under order-isomorphism.

In similarity to the way an order "inherits" the cardinality of its representatives, the width and height of an order are defined to be those of its representatives. Likewise, an order is future-finite if its representatives are future-finite *etc.* We may also refer to an element of an order, meaning an element of a representative of it—the meaning should be clear from the context.

Finally, we discuss the "stems" of labeled causets and extend the concept of "stem" to orders. A stem in a labeled causet $\tilde{C}$ is a finite subcauset $\tilde{D} \subseteq \tilde{C}$ which contains its own past, *i.e.* if $x \in \tilde{D}$ and $y \prec x$ in $\tilde{C}$ then $y \in \tilde{D}$. In particular, for any finite integer n satisfying $0 \leq n \leq |\tilde{C}|$, the restriction $\tilde{C}|_{[0,n]}$ is a stem in $\tilde{C}$.[1]

Definition 2.2.6 (Stems)**.** *A finite order S is a stem in the order C if there exists a representative of S which is a stem in some representative of C. When S is a stem C, we may also say that S is a stem in any representative $\tilde{C}$ of C.*

[1] $\tilde{C}|_{[0,n]}$ denotes the restriction of $\tilde{C}$ to the interval $[0, n]$.

Hence, the meaning of "stem" depends on the context (Fig.2.2).

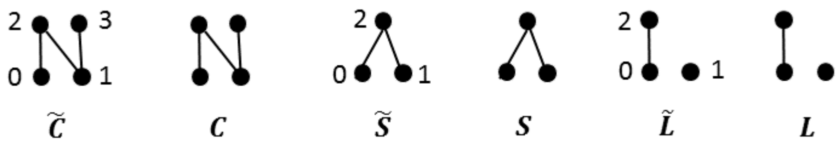


Figure 2.2: Labeled causets $\tilde{C}$, $\tilde{S}$ and $\tilde{L}$ are representatives of orders C, S and L, respectively. $\tilde{S}$ is a stem in $\tilde{C}$. $\tilde{L}$ is not a subcauset of $\tilde{C}$ so it is not a stem in $\tilde{C}$. S and L are stems in $\tilde{C}$ and in C.

Definition 2.2.7 (n-stem)**.** *An n-stem is a stem of cardinality n.*

In the literature, a "rogue" is a causet which cannot be fully specified by its stems [Brightwell et al., 2003]. We can rephrase this definition using our terminology: an infinite causet $\tilde{C} \in \tilde{\Omega}$ is a rogue if there exists some $\tilde{D} \in \tilde{\Omega}$ such that $\tilde{C} \ncong \tilde{D}$ and $S \in \Omega(n)$ is a stem in $\tilde{D}$ if and only if S is a stem in $\tilde{C}$. In that case we say that $\tilde{C}$ and $\tilde{D}$ are "equivalent rogues" or a "rogue pair". The term "rogue" can also be used to describe orders: we say that an order is a rogue if its representatives are rogues. Equivalently, C and D are a rogue pair when $S \in \Omega(n)$ is a stem in D if and only if S is a stem in C (Fig.2.3). Rogue equivalence, denoted by $C \sim_R D$, is an equivalence relation on Ω.

Figure 2.3: C is a countable union of 2-chains and D is the union of C with a single unrelated element. C and D have the same stems—any union of finitely many 2-chains and a finite, unrelated antichain—hence, C and D are equivalent rogues.

Chapter 3

CSG models: a modern review

Classical Sequential Growth (CSG) models are the archetype of growth dynamics for causal sets. They are inspired by the notion of *becoming*, namely that the causal set "comes into being" through a sequential accretion of elements, somewhat akin to a tree growing at its tips. Since their introduction in [Rideout and Sorkin, 2000], they have been the subject of much further study and have proved to be a fruitful arena in which to explore pertinent questions, including those relating to general covariance [Brightwell and Luczak, 2011, 2012] and generally covariant observables [Brightwell et al., 2002, 2003; Dowker and Surya, 2006]; causality [Dowker and Surya, 2006; Varadarajan and Rideout, 2006]; the continuum approximation [Brightwell and Georgiou, 2010; Rideout and Sorkin, 2001]; quantal dynamics [Dowker et al., 2010c]; the nature of time [Dowker, 2014, 2020; Sorkin, 2007; Wuthrich and Callender, 2017]; and cosmology [Ahmed and Rideout, 2010; Ash and McDonald, 2003, 2005; Bombelli et al., 2008; Dowker and Zalel, 2017; Martin et al., 2001; Sorkin, 2000].

This chapter is a modern review of the CSG models. We introduce the CSG models (section 3.1), collate their representations (section 3.2) and present examples and variations (section 3.3). We then review the construction of covariant observables (section 3.4) and of cosmic renormalisation (section 3.5). This chapter relies on the terminology established in chapter 2 (section 2.2) and serves as a foundation for the new work presented in chapters 4-7.

3.1 Key features

Classical Sequential Growth (CSG) models are models of causal set *growth* in which causal set elements are born *sequentially* (*i.e.* one after the other, in a sequence) and form relations with each other according to model-dependent *classical* probabilities (*i.e.* according to probabilities which obey the usual rules of probability theory so that the model is classically stochastic and does not display quantum interference effects). Each CSG model is encoded in a countable set of coupling constants from which the probabilities can be computed. The functional form of the probabilities ensures that the models satisfy mathematical constraints motivated by general covariance and local causality, earmarking the CSG models as interesting models for cosmology.

The growth process: A CSG process is made up of stages. Starting with a single element, at each stage a new element is born. The stages are labeled by the strictly positive integers, 1,2,3,... At the beginning of stage n, the growing causet contains n elements. In the limit $n \to \infty$, the process generates an infinite causal set.

The ground-set: For concreteness, we choose the initial element to be the integer 0, and we choose the element born at stage n to be the integer n. Namely, one starts with a causet containing the single element, 0. During stage 1, a new element—the integer 1—is born. At the subsequent stage, the element 2 is born, *etc.*, so that by the end of stage n the causet has the ground-set $\{0, 1, ..., n\}$. In the limit $n \to \infty$, the process generates an infinite causal set with ground set $\mathbb{N}$.

Internal temporality: CSG models satisfy the condition of "internal temporality" which states that an element cannot be born to the past of an already-existing element. Physically, the birth of an element is the *happening* of an event, while the existence of an element signifies that the event has *already happened*. Internal temporality reflects the intuitive notion that an event cannot happen before an event in its causal past happens.

Labeled Causets: Our choice of ground-set together with the condition of internal temporality mean that CSG models grow labeled causets: the process produces an n-

causet by the beginning of stage n and an infinite labeled causet in the $n \to \infty$ limit.

Parents, children and stems: We can think of stage n as a transition between labeled causets: $\tilde{C}_n \to \tilde{C}_{n+1}$. We call $\tilde{C}_n$ and $\tilde{C}_{n+1}$ the "parent" and the "child", respectively. Internal temporality is equivalent to requiring that the parent be a stem in the child. Thus, each child has a unique parent. A parent has as many children as it has stems since for each stem $\tilde{S} \subseteq \tilde{C}_n$ there exists a unique child in which $past(n) = \tilde{S}$.

The transition probabilities: Given some parent-child pair, $\tilde{C}_n$ and $\tilde{C}_{n+1}$, each CSG model furnishes an answer to the question "What is the probability $\mathbb{P}(\tilde{C}_n \to \tilde{C}_{n+1})$ to transition from $\tilde{C}_n$ to $\tilde{C}_{n+1}$?" via the following relations,

$$\mathbb{P}(\tilde{C}_n \to \tilde{C}_{n+1}) = \frac{\lambda(\varpi, m)}{\lambda(n, 0)}, \tag{3.1}$$

$$\lambda(k, p) := \sum_{i=0}^{k-p} \binom{k-p}{i} t_{p+i}, \tag{3.2}$$

where ϖ and m are the number of relations and links, respectively, formed by the new-born element and $(t_0 > 0, t_1, t_2, ...)$ is an infinite sequence of real non-negative couplings which defines the specific CSG model. An illustration is given in figure 3.1. The probability $\mathbb{P}(\tilde{C}_n)$ of growing the n-causet $\tilde{C}_n$ by the beginning of stage n is equal to the following product of transition probabilities,

$$\mathbb{P}(\tilde{C}_n) = \prod_{i=1}^{n-1} \mathbb{P}(\tilde{C}_i \to \tilde{C}_{i+1}), \tag{3.3}$$

where for each i, $\tilde{C}_i$ is the unique parent of $\tilde{C}_{i+1}$.

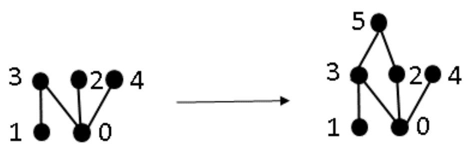


Figure 3.1: An illustration of the CSG transition probabilities. A transition between a parent and its child is shown. During the transition, the element $n = 5$ is born. It forms $m = 2$ links (with elements 2 and 3) and $\varpi = 4$ relations (with elements 0,1,2 and 3). Thus the probability for the transition is equal to $\frac{\lambda(4,2)}{\lambda(5,0)}$, as given by equation (3.1).

General covariance and causality: The CSG transition probabilities (3.1) are exactly the transition probabilities which satisfy both the "discrete general covariance" and the "Bell causality" conditions, discrete analogues of general covariance and local causality. "Discrete general covariance" is a label-invariance condition which states that order-isomorphic n-causets are equally likely to be grown by the process,

$$\tilde{C}_n \cong \tilde{C}'_n \implies \mathbb{P}(\tilde{C}_n) = \mathbb{P}(\tilde{C}'_n). \tag{3.4}$$

The "Bell causality" condition is a generalisation of Bell's local causality condition [Bell, 2004] to a discrete framework in which the causal structure is itself dynamical. It states that a transition probability can depend only on the past of the element which is born in the transition and takes the form of an equality between ratios of transition probabilities,

$$\frac{\mathbb{P}(\tilde{C}_n \to \tilde{C}_{n+1})}{\mathbb{P}(\tilde{C}_n \to \tilde{C}'_{n+1})} = \frac{\mathbb{P}(\tilde{B}_l \to \tilde{B}_{l+1})}{\mathbb{P}(\tilde{B}_l \to \tilde{B}'_{l+1})}, \tag{3.5}$$

where $\tilde{B}_{l+1}$, $\tilde{B}'_{l+1}$ and $\tilde{B}_l$ are obtained from $\tilde{C}_{n+1}$, $\tilde{C}'_{n+1}$ and $\tilde{C}_n$, respectively, by removing the "spectators" (elements which are unrelated to the new-born element in both transitions $\tilde{C}_n \to \tilde{C}_{n+1}$ and $\tilde{C}_n \to \tilde{C}'_{n+1}$) and relabeling consistently to form a new pair of transitions, $\tilde{B}_l \to \tilde{B}_{l+1}$ and $\tilde{B}_l \to \tilde{B}'_{l+1}$. An example is shown in figure 3.2. While $\tilde{B}_{l+1}$, $\tilde{B}'_{l+1}$ and $\tilde{B}_l$ will depend on our choice of relabeling, the right hand side of (3.5) is label-independent when discrete general covariance holds.

A CSG model as a random walk: A CSG model can be viewed as a random walk up "labeled poscau", the partial order of finite labeled causets where each child is directly above their parent,

Definition 3.1.1 (Labeled poscau). *Labeled poscau is the partial order* $(\tilde{\Omega}(\mathbb{N}), \prec)$*, where* $\tilde{S} \prec \tilde{R}$ *if and only if* $\tilde{S}$ *is a stem in* $\tilde{R}$.[1]

The first three levels of labeled poscau are shown in figure 3.3. A CSG model is a directed random walk on labeled poscau with transition probabilities of the form (3.1).

[1] We use the symbol $\prec$ to denote the relation for several different partial orders in this work. The meaning of $\prec$ in each case is to be inferred from the context.

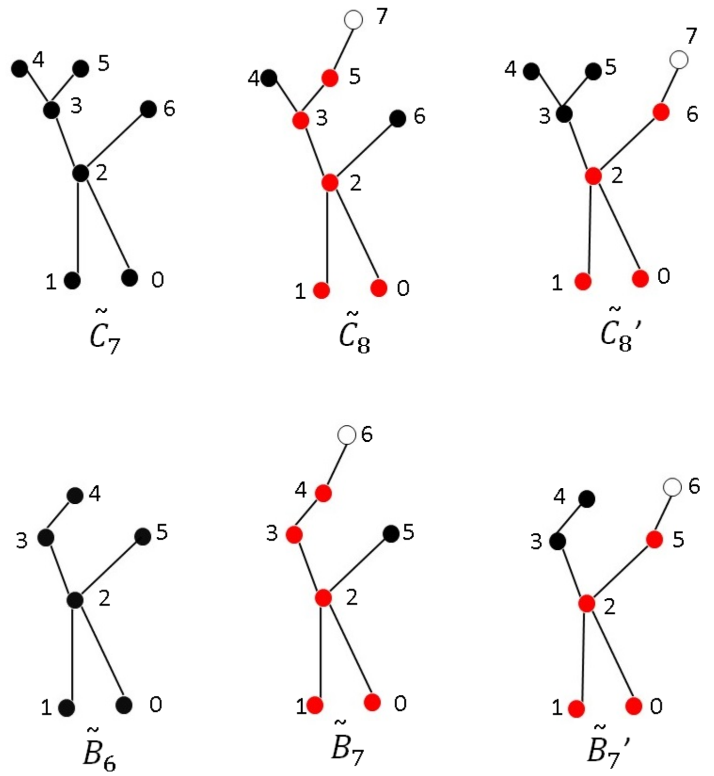


Figure 3.2: An illustration of the Bell causality condition. The parent $\tilde{C}_7$ and two of its children are shown on the top line. From these, the parent $\tilde{B}_6$ and two of its children, shown on the bottom line, can be constructed by removing the spectators (in this example, the element 4 is the only spectator) and relabeling. The new-born element in each child is shown in white. The past of the new-born element in each transition is shown in red.

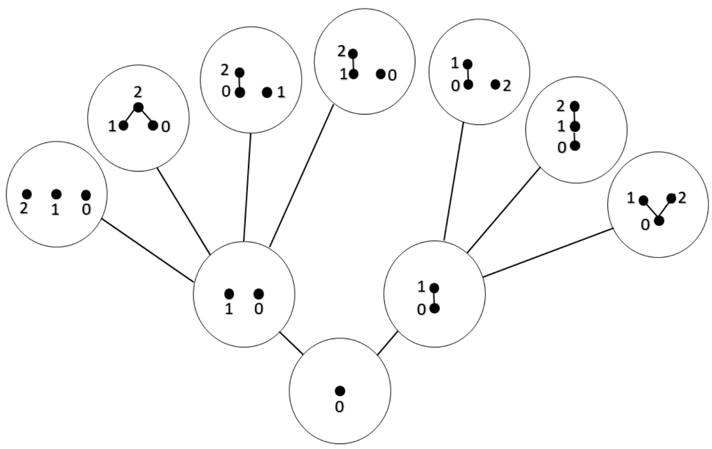


Figure 3.3: The first three levels of labeled poscau. The random walk starts at the root and proceeds upwards.

3.2 Parameterisations

The t-parameterisation: A CSG model can be parameterised by a countable sequence of real non-negative couplings $(t_0, t_1, t_2, ...)$ satisfying $t_0 > 0$ from which the transition probabilities can be computed via equations (3.1)-(3.2). This is a projective parameterisation, since any two sequences related by an overall positive factor, $(t_k) = c(t'_k)$, yield the same transition probabilities and thus parameterise the same model [Martin et al., 2001]. For simplicity, we often fix $t_0 = 1$.

This parameterisation has a meaningful interpretation in the context of growth: at stage n, a subset $\tilde{R} \subseteq \tilde{C}_n$ is selected with relative probability $t_{|\tilde{R}|}$ and the new element n is put above all elements which are below or equal to some element in $\tilde{R}$. $\tilde{R}$ acts as a "proto-past" from which the past of n is constructed and the transition probabilities (3.1) reflect this: the numerator is a sum over all proto-pasts which contribute to a particular transition $\tilde{C}_n \to \tilde{C}_{n+1}$ while the denominator sums over the proto-pasts which contribute to all possible transitions from the parent $\tilde{C}_n$. An illustration is shown in figure 3.4.

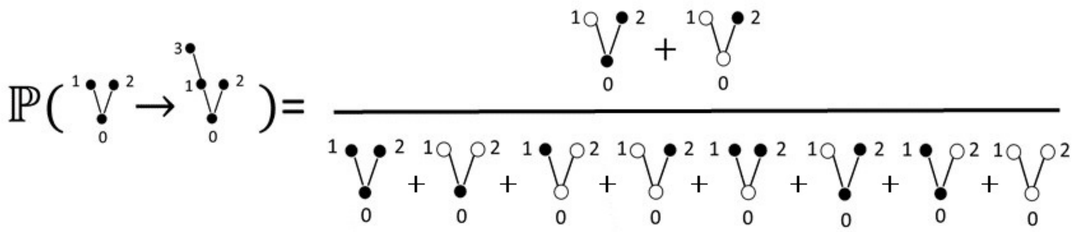


Figure 3.4: A pictorial representation of transition probability (3.1). Each diagram on the right hand side represents a proto-past $\tilde{R}$ (shown in white) and contributes a factor of $t_{|\tilde{R}|}$. The numerator is a sum over all proto-pasts which contribute to the transition on the left hand side. The denominator sums over all possible proto-pasts.

The q-parameterisation: A CSG model can be parameterised by a countable sequence $(q_0, q_1, q_2, ...)$ related to $(t_0, t_1, ...)$ via a binomial transform,

$$\begin{aligned} \frac{1}{q_n} &= \sum_{k=0}^{n} \binom{n}{k} t_k, \\ t_k &= \sum_{n=0}^{k} (-)^{k-n} \binom{k}{n} \frac{1}{q_n}. \end{aligned} \tag{3.6}$$

The q_n are strictly positive[2] and each q_n is bounded from above by a function of $q_0, q_1, ..., q_{n-1}$. This is a projective parameterisation. When fixing $t_0 = q_0 = 1$, q_n is the probability to transition from the n-antichain to the $(n+1)$-antichain [Rideout and Sorkin, 2000].

The p-parameterisation: A CSG model can be parameterised by a countable sequence $(p_1, p_2, ...)$ related to $(t_0, t_1, ...)$ via,

$$p_m = \frac{\sum_{k=1}^{m} \binom{m-1}{k-1} t_k}{\lambda(m,0)}. \tag{3.7}$$

The p_m satisfy $0 \leq p_m < 1$ and each p_m is bounded from below by $p_1, ..., p_{m-1}$ (*e.g.* $p_2 \geq \frac{p_1}{p_1+1}$). p_m is the probability of transition from the m-chain to the $(m+1)$-chain. Equivalently, p_m is the probability that a given element $n < m$ is included in the proto-past of element m,

$$p_m = \frac{\mathbb{E}|\tilde{R}|}{m} = \frac{\sum_{k=0}^{m} \binom{m}{k} k t_k}{m\lambda(m,0)}, \tag{3.8}$$

where $\tilde{R}$ is the proto-past of the element m and $\mathbb{E}$ denotes an expectation value [Brightwell and Luczak, 2016].

3.3 Examples and variations

We present key examples of CSG models and introduce the family of Originary CSG models. For simplicity, we set $t_0 = q_0 = 1$.

The Dust Universe: This CSG model is given by $t_0 = 1, t_k = 0 \;\forall\; k > 0$ or equivalently $q_n = 1 \;\forall\; n \geq 0$. It produces an infinite antichain with unit probability.

The Infinite Forest of Infinite Trees: $t_0 = t_1 = 1$, $t_k = 0 \;\forall\; k > 1$, this model generates a disjoint union of infinitely many trees with no maximal elements with unit probability.

[2] This is a consequence of Bell causality (3.5). Weaker variations of Bell causality allow $q_n = 0$ [Dowker and Surya, 2006; Varadarajan and Rideout, 2006].

Transitive Percolation: Transitive Percolation (TP) is a 1-parameter family of CSG models given in the various parameterisations by,

$$
\begin{aligned}
& t_k = t^k,\ t \in \mathbb{R}^+, \\
& q_n = q^n,\ 0 < q \leq 1, \\
& p_m = p,\ 0 \leq p < 1,
\end{aligned} \tag{3.9}
$$

where the relationship between the parameterisations is given by $p = \frac{t}{1+t}$ and $q = 1 - p$. In this special case, the transition probabilities (3.1) can be recast into the form,

$$
\mathbb{P}(\tilde{C}_n \to \tilde{C}_{n+1}) = p^m q^{n-\varpi}. \tag{3.10}
$$

The interpretation of equation (3.10) is that the new element born in the transition forms a relation with each already-existing element with probability p independently and then the transitive closure is taken to obtain $\tilde{C}_{n+1}$. This reflects the "local" nature of TP and all other CSG models can be seen as non-local generalisations of it in which the probability of forming a relation with a given element depends on whether or not a relation is formed with each of the other elements.

The Dust Universe is the special case where $p = t = 0$. When $p \neq 0$, the model is probabilistic but produces a causet with infinitely many posts with unit probability [Alon et al., 1994]. TP is also known as the model of "random graph orders".

Originary CSG Models: Originary CSG (OCSG) models are a family of growth models which differ from the CSG models only by the requirement that $t_0 = 0$ and $t_1 > 0$. These models are generally probabilistic but the causets that they produce are always originary.

3.4 Observables

In which sense is a CSG model a dynamics for causal sets? In the sense that it provides answers to the physical questions that one could ask about the causet spacetime.

Each question corresponds to a measurable event[3], or "observable", while each answer corresponds to the probability measure assigned to the observable by the CSG model.

The sample space, $\tilde{\Omega}$: The sample space is the set of all infinite causets which can be grown by a CSG process in the $n \to \infty$ limit, *i.e.* it is the set of infinite labeled causets, $\tilde{\Omega}$.

The labeled algebra, $\tilde{\mathcal{R}}$: For each $\tilde{C}_n$, its "cylinder set" is defined as,

$$cyl(\tilde{C}_n) := \{\tilde{C} \in \tilde{\Omega} | \tilde{C}_n \text{ is a stem in } \tilde{C}\}. \tag{3.11}$$

We denote the σ-algebra generated by the cylinder sets by $\tilde{\mathcal{R}}$. $(\tilde{\Omega}, \tilde{\mathcal{R}})$ is a measurable space on which each CSG model induces a unique probability measure $\tilde{\mu}$ satisfying,

$$\tilde{\mu}(cyl(\tilde{C}_n)) = \mathbb{P}(\tilde{C}_n) \; \forall \; \tilde{C}_n \in \tilde{\Omega}(\mathbb{N}), \tag{3.12}$$

where $\mathbb{P}(\tilde{C}_n)$ is given by equation (3.3). This construction is guaranteed by standard results in measure theory [Kolmogorov and Fomin, 1975].

The covariant algebra, $\mathcal{R}$: We say that an event $\tilde{\mathcal{E}} \in \tilde{\mathcal{R}}$ is covariant if it does not distinguish between order-isomorphic causets, namely if whenever $\tilde{C} \in \tilde{\mathcal{E}}$ and $\tilde{C} \cong \tilde{D}$ then $\tilde{D} \in \tilde{\mathcal{E}}$. The collection of covariant events is a σ-algebra and we denote it by $\mathcal{R}$ [Brightwell et al., 2003]. $(\tilde{\Omega}, \mathcal{R})$ is a measurable space on which each CSG model induces a probability measure μ via the restriction of $\tilde{\mu}$ to $\mathcal{R}$.

The stem algebra, $\mathcal{R}(\mathcal{S})$: For each n-order C_n, its "stem set" is defined as,

$$stem(C_n) := \{\tilde{D} \in \tilde{\Omega} \mid C_n \text{ is a stem in } \tilde{D}\}, \tag{3.13}$$

and is equal to the union of cylinder sets of the labeled causets in which C_n is a stem. We denote the σ-algebra generated by the stem sets by $\mathcal{R}(\mathcal{S})$ and note that $\mathcal{R}(\mathcal{S}) \subset \mathcal{R}$

[3]See appendix A.2 for a glossary of measure theory terminology.

[Brightwell et al., 2003]. $(\tilde{\Omega}, \mathcal{R}(\mathcal{S}))$ is a measurable space on which each CSG model induces a probability measure $\mu_{\mathcal{S}}$ via the restriction of $\tilde{\mu}$ (or μ) to $\mathcal{R}(\mathcal{S})$.

Observables: Which of the σ-algebras $\tilde{\mathcal{R}} \supset \mathcal{R} \supset \mathcal{R}(\mathcal{S})$ should be crowned the "algebra of observables"? The labeled algebra $\tilde{\mathcal{R}}$ is too large, since it contains events which are not covariant and hence unphysical. The covariant algebra $\mathcal{R}$ is a good candidate since (by definition) it contains exactly the covariant events, but it provides no useful information about the physical interpretation of these events. The stem algebra $\mathcal{R}(\mathcal{S})$ may be the best candidate of the three since it contains covariant events with a clear physical interpretation: each event corresponds to a logical combination of (at most countably many) statements about which finite orders are stems in the causet $\tilde{C}$ grown by the process in the $n \to \infty$ limit, *e.g.* the event $\big(stem(A) \cap stem(B)\big) \in \mathcal{R}(\mathcal{S})$ is interpreted as "both A and B are stems in $\tilde{C}$". This kinematical heuristic is strengthened by the dynamical result that in every CSG model it suffices to know the measure on $\mathcal{R}(\mathcal{S})$ in order to reconstruct the measure of every event in $\mathcal{R}$ [Brightwell et al., 2003]. Thus, in a precise sense the events in $\mathcal{R}(\mathcal{S})$ exhaust the set of observables in any CSG model.

Quotient Spaces: One can conceive of $\mathcal{R}$ as a σ-algebra on Ω, the set of infinite orders, via the projection $p : \tilde{\Omega} \to \Omega$ which assigns to each causet $\tilde{C}$ the order C of which it is a representative. Whether a covariant event is a set of causets or a set of orders is of no consequence for our purposes and we will use the two interchangeably. Similarly, since $\mathcal{R}(\mathcal{S})$ contains exactly all the covariant events which do not distinguish between equivalent rogues (*i.e.* $\mathcal{E} \in \mathcal{R}(\mathcal{S}) \iff$ whenever $C \in \mathcal{E}$ and $C \sim_R D$ then $D \in \mathcal{E}$) it can be thought of as a σ-algebra on $\Omega / \sim_R$, the space of rogue equivalence classes.

3.5 Cosmic renormalisation

The "cosmic renormalisation" transformations are a family of transformations which act in the space of $(t_0, t_1, ...)$. They play a central role in causal set cosmology.

The transformations: Define the linear transformation M,

$$M : t_k \mapsto t_k + t_{k+1}. \tag{3.14}$$

For any integer $a \geq 0$, the "post renormalisation transformation" is given by [Martin et al., 2001],

$$S_a : (t_k) \mapsto (t_k^{(a)}), \quad t_k^{(a)} = \begin{cases} 0 & : k = 0 \\ M^a(t_k) = \sum_{l=0}^{a} \binom{a}{l} t_{k+l} & : k > 0 \end{cases} \tag{3.15}$$

For any pair of integers $a \geq r > 0$, the "break renormalisation transformation" is given by [Dowker and Zalel, 2017],

$$Q_{a,r} : (t_k) \mapsto (t_k^{(a,r)}), \quad t_k^{(a,r)} = \begin{cases} M^{a-r}(t_r) = \sum_{l=0}^{a-r} \binom{a-r}{l} t_{r+l} & : k = 0 \\ M^a(t_k) = \sum_{l=0}^{a} \binom{a}{l} t_{k+l} & : k > 0 \end{cases} \tag{3.16}$$

The transformations satisfy the following composition rules,

$$\begin{aligned} S_b S_a &= S_b Q_{a,r} = S_{a+b}, \\ Q_{b,s} Q_{a,r} &= Q_{b,s} S_a = Q_{a+b,s}. \end{aligned} \tag{3.17}$$

A growth model (t_k) is a stationary point of S_a if it is mapped onto itself (*i.e.* if $(t_k^{(a)}) = c(t_k)$ for some positive constant c). The stationary points of $Q_{a,r}$ are similarly defined. For any $a > 0$, the stationary points of S_a are the Originary Transitive Percolation models ($t_0 = 0, t_k = t^k \; \forall \; k > 0$) and the stationary points of $Q_{a,r}$ are the CSG models defined by $t_0 = 1$, $t_k = \left(\frac{1+t}{t}\right)^r t^k \; \forall \; k > 0$. Neither transformation has cycles of length greater than 1.

Physical interpretation: The renormalisation transformations express a relationship between different growth models which arises via a process of conditioning. Consider some CSG dynamics and condition that the element a is a post with some fixed past $\tilde{A}$ in the growing causet.[4] Then, the growth of the future of the post "decouples" from the past and can be treated independently by deleting the past of the post and relabeling the remaining elements $n \to n - a$. This new growth process is governed by conditional transition probabilities which can be expressed in the form (3.1)-(3.2), with (t_k) replaced by the "renormalised" or "effective" couplings $(t_k^{(a)})$ given in (3.15). The coupling $t_k^{(a)}$ is the relative probability that a proto-past of cardinality k is chosen from the undeleted elements. It is a remarkable feature of CSG models that the effective dynamics depends only on a, *i.e.* only on the cardinality of the past, not on its causal structure.

Similarly, when conditioning on a break with a fixed past $\tilde{A}$, the growth of the future decouples from the past and can be treated independently by deleting the past and relabeling $n \to n - a$, where $a := |\tilde{A}|$. The new transition probabilities are given by (3.1)-(3.2), with (t_k) replaced by the $(t_k^{(a,r)})$ of (3.16), where r is the number of maximal elements of $\tilde{A}$.

Note that the event "a is a post" is the same as the event "there is a break with a past of cardinality $a + 1$ and 1 maximal element". Thus, S_a and $Q_{a+1,1}$ give two equivalent descriptions of the same system. The former yields an originary effective dynamics (since $t_0^{(a)} = 0$), ensuring that every element is born above the post. The latter yields a non-originary dynamics, where the post condition is enforced at the level of the kinematics. (Another way to say this is, in the former formulation the post is not "deleted" while in the latter it is.)

The composition rules (3.17) express the behaviour of the dynamics under successive conditionings, *e.g* $S_b S_a = S_{a+b}$ is the statement that first conditioning on a being a post and then conditioning on the b^{th} element to be born to the future of a being a post is equivalent to conditioning that the element $a + b$ is a post, reflecting that the effective couplings depend only on the cardinality of the past of the post, not on its ordering. Illustrations are shown in figures 3.5-3.6.

Application to cosmology: Posts and breaks are discrete analogues of Big Crunch-Big Bang singularities. When composed in a sequence, they are used to model "bounc-

[4]When $a = 0$, $\tilde{A} = \emptyset$.

ing" or "cyclic" cosmologies in which the universe goes through subsequent epochs of expansion and collapse. Conditioning on such a bouncing cosmology, the corresponding renormalisation transformations create a flow in parameter space echoing proposals by J. A. Wheeler, L. Smolin and others that the parameters of nature are modified each time the universe is "squeezed through" a singularity [Bičák, 2009; Rees et al., 1974; Smolin, 1992, 2008]. This evolutionary mechanism is at the heart of the causal set cosmological paradigm, a heuristic which aims to explain the emergence of a flat, homogeneous and isotropic cosmos directly from the quantum gravity era[5] [Sorkin, 2000]. The narrative is that, given that the renormalisation flow generated by the successive epochs has certain features[6], then the dynamics evolves into growing larger, flatter epochs as the universe cycles repeatedly. It is then only a matter of time until the universe displays the desired behaviour.

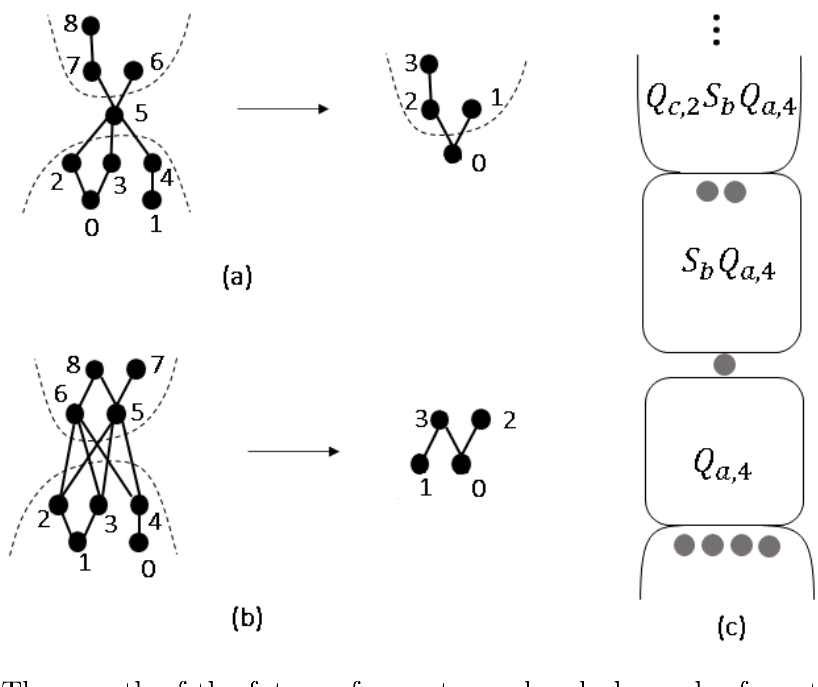


Figure 3.5: The growth of the future of a post or a break decouples from the past and can be treated independently. Figures (a) and (b) illustrate the process of deleting the past and relabeling the remaining elements $n \to n - a$, where a is the cardinality of the past. Figure (c) shows a sequence of epochs separated by posts and breaks and the effective dynamics which governs each epoch.

[5]This paradigm pertains only to the causal set spacetime, not to any matter living on it. Whether a causal set is enough to give rise to matter degrees of freedom [Rideout and Sorkin, 2000] or whether one requires additional structure such as a field living on the causal set is still unknown. Whichever the case may be, this simplified cosmological paradigm will act as a guide to building a causal set cosmology.

[6]For instance, that its stationary points grow causal sets with the desired cosmological features, that the basin of attraction of these stationary points is large and that it contains an abundance of dynamics which are likely to give rise to posts/breaks.

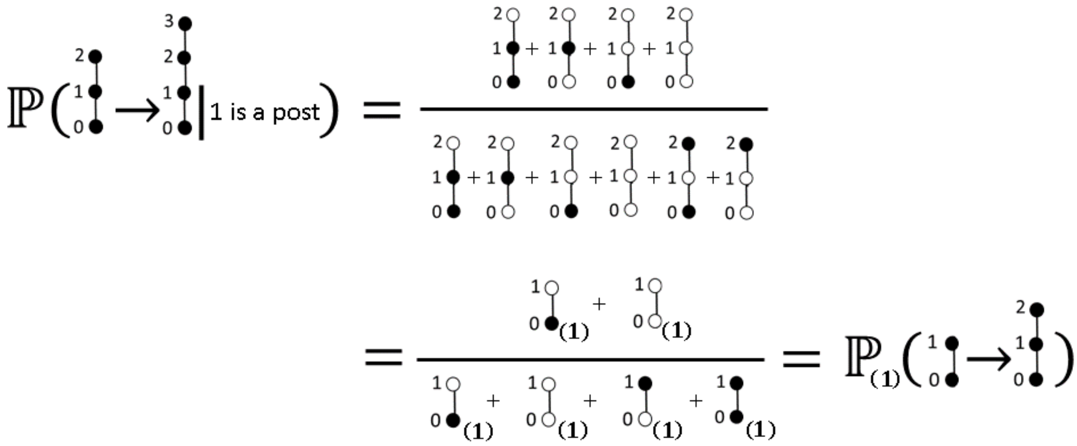


Figure 3.6: A pictorial representation of the relationship between (t_k) and $(t_k^{(a)})$ for $a = 1$. Each diagram represents a proto-past $\tilde{R}$ shown in white (cf. Fig.3.4). First line: each diagram on the right hand side contributes a factor of $t_{|\tilde{R}|}$. The numerator is a sum over all proto-pasts which contribute to the transition on the left hand side. The denominator is a sum over all possible proto-pasts consistent with the condition that 1 is a post. Second line: each diagram on the left hand side contributes a factor of $t^{(1)}_{|\tilde{R}|}$, as indicated by the subscripts. The numerator is a sum over all proto-pasts which contribute to the transition on the right hand side and the denominator sums over all possible proto-pasts.

Chapter 4

Manifestly covariant dynamics

Einstein's efforts to distinguish between the physical and the unphysical were central to the development of the generally covariant field equations of General Relativity. Indeed, this struggle between gauge and physical degrees of freedom is inherent in the nature of General Relativity as a gauge theory and one might expect that grappling with the corresponding issues within quantum gravity will be important to its development too. One facet of this struggle can be summarised by the question: can the laws of physics be formulated using only physical degrees of freedom? [Berkovits, 2000; Carlip, 1990; Witten, 1989] Physics has thus far favoured gauge theories, rendering this problem extremely difficult. But there is an expectation shared by some workers that the difficulty stems from the incompleteness of our current theories and that a resolution will come hand in hand with a theory of quantum gravity.

In causal set theory—where general covariance takes the form of label invariance, a discrete precursor to the continuous diffeomorphism invariance of General Relativity[1]—the CSG models are a gauge formulation (since they refer to labeled causets) and our question can be framed as: is it possible to define physically interesting causal set growth dynamics which refer only to orders, not to labeled causets? We call such dynamics "manifestly covariant".

From the start, one can identify three challenges. First, orders are mathematically more difficult to handle (*e.g.* enumerating unlabeled graphs is generally more difficult

[1]Whether label invariance alone is enough to give rise to diffeomorphism invariance is subsumed in the question of whether causal sets can give rise to a continuum approximation at all.

than enumerating labeled graphs [Harary and Palmer, 1973]), echoing the difficulties often encountered in physics when working with global degrees of freedom. Second, physical dynamics are often arrived at by imposing invariance under gauge transformations. This is the case in the CSG models, where the discrete general covariance condition (3.4) is akin to the requirement that the Einstein-Hilbert action is invariant under diffeomorphisms. Therefore even if a label-independent framework did exist at the level of the kinematics, how are the physically meaningful dynamics to be picked out from the plethora of available models? Finally, our intuitive notion of growth is inherently sequential: elements are born one after the other in a kind of global time which renders the elements distinguishable (e.g. in a CSG model, the element n is born at stage n). How does one reconcile a physical process of becoming with manifest covariance? The prevailing view in causal set theory is that one should seek a form of "asynchronous becoming", namely a growth process in which elements are born in a partial (not a total) order [Dowker, 2014, 2020; Sorkin, 2007]. What could it mean for elements to be born in a partial order? It is the role of mathematics to make sense of notions which lie beyond our everyday experience, and it may be that new mathematics is what is needed to better understand asynchronous becoming and its consequences for the nature of time. It has been suggested in [Wuthrich and Callender, 2017] that this could be achieved via a "novel and exotic" framework in which questions such as "which element is born at stage n?" are left unanswered (not because of ignorance but because they are unphysical), but there has been no concrete proposal for such a framework until now.

This chapter presents original work on manifestly covariant growth dynamics. We begin with a technical formulation of the problem before defining a new structure which we call "covtree" and proving that it provides a manifestly covariant framework for growth dynamics (section 4.1). We obtain key results regarding cosmic renormalisation and the growth of posts and breaks within this new framework (section 4.2) and illustrate with a toy example how a better understanding of the structure of covtree can be used to solve for physically interesting dynamics (section 4.3).

4.1 Introducing covtree

We seek a manifestly covariant analogue of the CSG models. Let us make this statement precise. Recall that a CSG model is formally a measure space $(\tilde{\Omega}, \tilde{\mathcal{R}}, \tilde{\mu})$ where $\tilde{\mu}$ is related to the CSG transition probabilities via (3.12). However, $\tilde{\mathcal{R}}$ contains events which are not covariant and hence unphysical. At the physical level, a CSG model can be thought of as the measure space $(\Omega, \mathcal{R}, \mu)$ where μ is the restriction of $\tilde{\mu}$ to the covariant algebra $\mathcal{R}$. Finally, we can further economise by denoting a CSG model by $(\Omega, \mathcal{R}(\mathcal{S}), \mu_{\mathcal{S}})$ where $\mu_{\mathcal{S}}$ is the restriction of μ to $\mathcal{R}(\mathcal{S})$, since one can show that in this case μ is the unique extension of $\mu_{\mathcal{S}}$ to $\mathcal{R}$ [2] [Brightwell et al., 2003]. This has the advantage that each event in $\mathcal{R}(\mathcal{S})$ has a physically meaningful interpretation: each event corresponds to a logical combination of statements about which finite orders are stems in the growing order. Although this latter formulation is manifestly covariant at the level of the kinematics (*i.e.* the sample space is the space of orders, Ω), the measure $\mu_{\mathcal{S}}$ is defined via a label-dependent mechanism, *i.e.* via a random walk up labeled poscau which gives rise to the measure $\tilde{\mu}$ which is then restricted to $\mu_{\mathcal{S}}$. We seek to define a measure on $\mathcal{R}(\mathcal{S})$ via a random walk on a new tree structure which makes no reference to labeled causets, only to orders. To be interpreted as a growth dynamics, the new tree must be equipped with a map from Ω to the set of infinite upward-going paths. This map should be an isomorphism between the tree's topological σ-algebra[3] and $\mathcal{R}(\mathcal{S})$ so that a measure induced by a random walk on the former can be carried over to the latter. In particular, the map should be surjective. This guarantees that some order is grown in every realisation of the process or equivalently that $\mu_{\mathcal{S}}(\emptyset) = 0$. Finally to allow us to interpret the random walk as a physical process, each node should carry a clear physical meaning. To illustrate this final point, let us consider "poscau" [4],

Definition 4.1.1 (Poscau). *Poscau is a partial order on finite orders,* $(\Omega(\mathbb{N}), \prec)$*, where*

[2] The proof can be sketched as follows: one shows that (i) in any CSG model $\mu(\Theta) = 0$, where $\Theta \in \mathcal{R}$ is the set of rogues, and (ii) that for every $\mathcal{A} \in \mathcal{R}$ the event $\mathcal{A} \setminus \Theta \in \mathcal{R}(\mathcal{S})$. Thus, $\mu(\mathcal{A}) = \mu(\mathcal{A} \setminus \Theta) = \mu_{\mathcal{S}}(\mathcal{A} \setminus \Theta)$, where the first equality follows from (i) and the second from (ii). Therefore the measure of every event in $\mathcal{R}$ is fixed by $\mu_{\mathcal{S}}$ [Brightwell et al., 2003].

[3] This is the σ-algebra generated by the "cylinder sets", where the cylinder set of a given node is the set of all paths which contain it.

[4] Rideout and Sorkin originally used poscau to introduce the CSG models.

$A \prec B$ if and only if A is a stem in B.

Albeit not a tree, there is a natural correspondence between poscau paths and infinite orders. Hence if one identifies each node A in poscau with the set $stem(A)$ (cf. (3.13)), one might be tempted to try to define a dynamics as a random walk up poscau where arriving at the node A corresponds to the occurrence of the covariant event $stem(A)$. Intuitively, this does not work because our physical interpretation of the nodes implies that each order contains only one n-stem for each $n > 0$ (which is clearly untrue). Thinking in this way, however, suggests the solution: the walk should be on a tree formed of countably many levels in which the nodes in level n are not single n-orders but *sets* of n-orders. Each set of n-orders in level n will correspond to the covariant event "the n-stems of the growing order are the elements of this set." We call this tree "covtree", short for "covariant tree".

This section is dedicated to defining covtree (4.1.1-4.1.2), to proving the existence of a surjection from Ω to the set of covtree paths and to proving that covtree's topological σ-algebra is equivalent to $\mathcal{R}(\mathcal{S})$ (4.1.3).

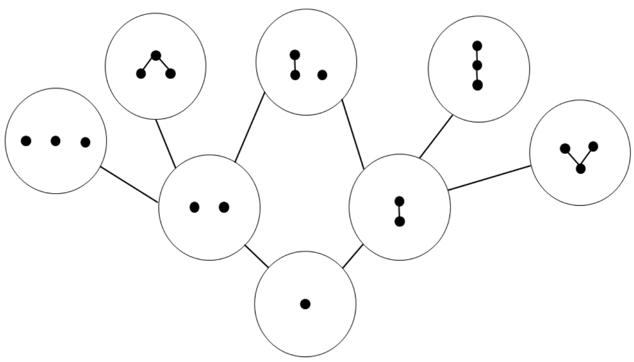


Figure 4.1: The first three levels of poscau.

4.1.1 Certificates

This section introduces the notion of "certificate" which will play a key role in the definition of covtree and in its interpretation as a framework for growth dynamics.

Let $\Gamma_n \subseteq \Omega(n)$ be a non-empty set of n-orders.

Definition 4.1.2 (Certificate)**.** *A finite or infinite order C is a certificate of Γ_n if Γ_n is the set of all n-stems in C.*

Given some Γ_n, it may or may not have a certificate. We will be interested in those Γ_n which do have a certificate.

Definition 4.1.3 (Λ, the collection of certified sets)**.** *Λ is the collection of sets of n-orders, for all n, for which there exists a certificate:*

$$\Lambda := \bigcup_{n>0} \{\Gamma_n \subseteq \Omega(n) | \exists \textit{ a certificate for } \Gamma_n\}. \tag{4.1}$$

One can show that each $\Gamma_n \in \Lambda$ has infinitely many certificates, including infinitely many finite certificates and infinitely many infinite certificates. We will often work with the "minimal" certificates:

Definition 4.1.4 (Minimal certificate)**.** *Given some $\Gamma_n \in \Lambda$, we order its finite certificates as follows: let C, C' be finite certificates of Γ_n, then $C \preceq C'$ if C is a stem in C'. A minimal certificate of Γ_n is minimal in this partial order of certificates.*

At times it may be easier to work with labeled causets rather than with orders. To this end we define the labeled analogue of the certificate.

Definition 4.1.5 (Labeled certificate)**.** *A labeled certificate of Γ_n is a representative of a certificate of Γ_n. A labeled minimal certificate of Γ_n is a representative of a minimal certificate of Γ_n.*

Illustrations are shown in figure 4.2.

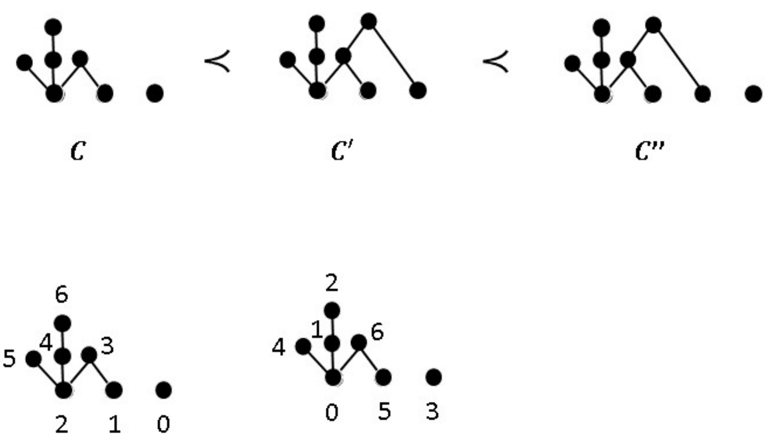


Figure 4.2: Certificates. C, C' and C'' are finite certificates of $\Omega(3)$={ , . . ., $\vee$, $\wedge$, . }. The $\prec$ relation indicates inclusion by stem. C is a minimal certificate of $\Omega(3)$. The labeled causets shown are representatives of C and hence are labeled minimal certificates of $\Omega(3)$.

4.1.2 Definition of covtree

We begin by introducing the map $\mathcal{O}$.

Definition 4.1.6 (The map $\mathcal{O}$)**.** *For any $n > 1$ and any Γ_n, the map $\mathcal{O}$ takes Γ_n to the set of $(n-1)$-stems of elements of Γ_n:*

$$\mathcal{O}(\Gamma_n) := \{B_{n-1} \in \Omega(n-1) \mid \exists\ A_n \in \Gamma_n \text{ s.t. } B_{n-1} \textit{ is a stem in } A_n\}\,. \tag{4.2}$$

An illustration is shown in figure 4.3. The exponentiation $\mathcal{O}^k$ takes Γ_n to the set of $(n-k)$-stems of elements of Γ_n. If C is a certificate of Γ_n, then C is also a certificate of $\mathcal{O}^k(\Gamma_n)$ for any $k < n$.[5] The converse is not true: if C is a certificate of $\mathcal{O}(\Gamma_n)$, then C may or may not be a certificate of Γ_n (in fact, Γ_n may have no certificates at all).

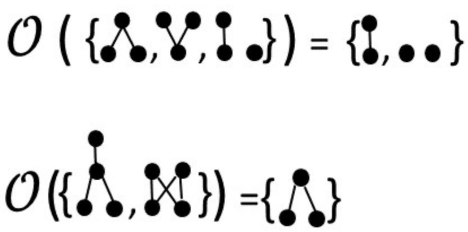


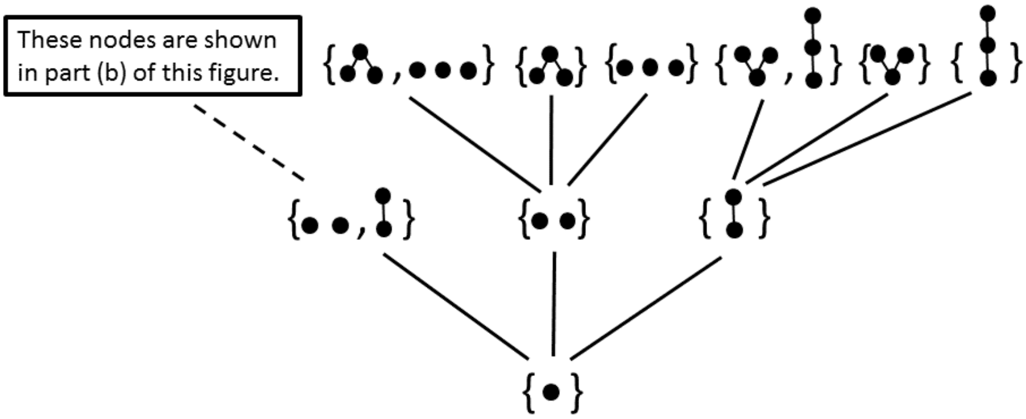
Figure 4.3: Illustration of the map $\mathcal{O}$.

[5]The proof may be summarised by the mnemonic: a stem in a stem is a stem, not a stem in any stem is not a stem

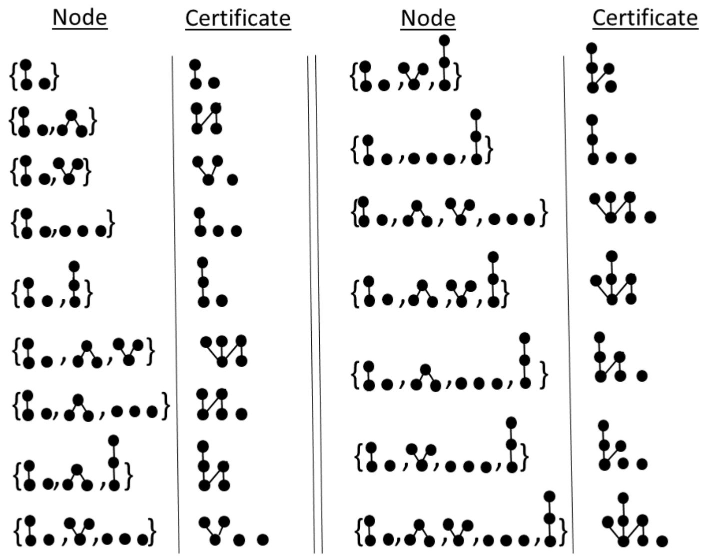
Definition 4.1.7 (Covtree)**.** *Covtree is the partial order* $(\Lambda, \prec)$*, where* $\Gamma_n \prec \Gamma_m$ *if and only if* $n < m$ *and* $\mathcal{O}^{m-n}(\Gamma_m) = \Gamma_n$*.*

We note some key points about covtree:

- Covtree is the partial order on Λ defined by putting each Γ_n directly above $\mathcal{O}(\Gamma_n)$ and taking the transitive closure. Thus, covtree is a tree.
- Covtree has no maximal nodes. Every covtree node is contained in uncountably many inextendible upward-going paths.
- We label the levels of covtree by 1,2,... where level 1 contains the root. The nodes at level n are the sets of n-orders which have certificates (this is the motivation for the term certificate: a certificate of Γ_n certifies that Γ_n is a node in covtree.)
- A certificate of a node Γ_n is also a certificate of every node below Γ_n.
- Given a node Γ_n, repeated applications of $\mathcal{O}$ generate the unique path downwards from Γ_n to the root.
- In order to construct level n of covtree, one considers all the non-empty subsets of $\Omega(n)$. These are the "candidate nodes" for level n. To determine whether a candidate node is a node in covtree one needs to determine whether it has a certificate. In general, this is a difficult problem.
- Given any n-order C_n, the set $\{C_n\}$ is a node at level n since C_n is a certificate of $\{C_n\}$.
- The first three levels of covtree are shown in figure 4.4. Levels 1 and 2 contain all candidate nodes, while level 3 contains 22 nodes out of 31 candidates. The 9 "non-nodes" are shown in figure 4.5.



(a) The structure of the first three levels of covtree.



(b) The level 3 nodes which are directly above the node { I, .. } are shown together with their respective certificates.

Figure 4.4: The first three levels of covtree.

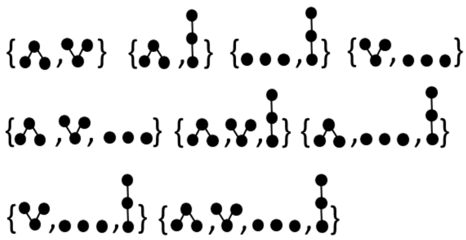


Figure 4.5: The sets shown in the figure have no certificates and therefore are not nodes. For every set shown, if an order contains all the elements of that set as stems then it also contains as a stem.

4.1.3 The sample space

We now prove that there is a surjection from Ω to the space of inextendible upward-going covtree paths.

Let $\mathcal{P}$ denote an inextendible covtree path from the origin upwards, $\Gamma_1 \prec \Gamma_2 \prec \dots$

Definition 4.1.8 (Certificate of path)**.** *An infinite order C is a certificate of $\mathcal{P}$ if it is a certificate of every node in $\mathcal{P}$. A labeled certificate of $\mathcal{P}$ is a representative of a certificate of $\mathcal{P}$.*

We propose that the surjection we seek is given by the correspondence between paths and their certificates. In the following we show that this correspondence is indeed a surjection, since it is a function on Ω (lemma 4.1.9) and it is surjective (theorem 4.1.10). Thus, each infinite order $C \in \Omega$ is mapped onto the path $\mathcal{P}$ of which it is a certificate. This establishes Ω as the sample space of the covtree growth process. The narrative is that in any realisation of the process, the growing order is the certificate of the path traced by the random walk. However, we must point out two subtleties with this interpretation. First, a path may have more than one certificate: C and C' are two distinct certificates of the same path if and only if they are equivalent rogues. In other words, since equivalent rogues cannot be distinguished by their stems, the covtree process cannot distinguish between equivalent rogues. How one should resolve this depends on the physical interpretation that one assigns to rogues. If one believes that rogues are unphysical and should never be grown by the process[6] then the resolution can be to only consider random walks in which the measure of the set of the "rogue paths" is null. An alternative is to allow rogues to arise but propose that they are physically indistinguishable (since they can only be distinguished globally, not by any local observer living on them). This is equivalent to replacing Ω with the space of rogue equivalence classes $\Omega/{\sim_R}$, where each path corresponds to exactly one class. The second subtlety relates to the notion of growth. To what extent can we say that an order is *growing* as the covtree walk advances? At stage n, we do not know which finite order has grown thus far nor its cardinality, only

[6] Reasons to think this include that in the CSG models rogues never happen, *i.e.* $\mu(\Theta) = 0$ where Θ is the set of rogues, and that every rogue contains an infinite antichain corresponding to infinite space [Brightwell et al., 2003].

which n-stems it contains. While in the CSG models the growth is explicit, on covtree it is implicit or "vague" [Wuthrich and Callender, 2017]. But if there is a process of growth which can be associated with a covtree walk, then it may be that it is this quality of vagueness which embodies asynchronous becoming.

Lemma 4.1.9. *Let C be an infinite order. Then C is a certificate of exactly one path $\mathcal{P}$.*

Proof. By definition of certificate 4.1.3, C is a certificate of exactly one node Γ_n for each $n > 0$. Hence, C is a certificate of at most one path. Additionally, if C is a certificate of Γ_n then C is a certificate of $\mathcal{O}(\Gamma_n)$. Hence the nodes of which C is a certificate form a path $\mathcal{P}$ of which C is a certificate. □

Theorem 4.1.10. *Every path $\mathcal{P}$ has at least one certificate.*

We begin with some lemmas.

Lemma 4.1.11. *Let $\Gamma_n = \{A_n^1, ..., A_n^k\}$ be a set containing $k \geq 1$ n-orders and let C be a minimal certificate of it. Then $n \leq |C| \leq kn$. Additionally, $|C| = n$ if and only if $k = 1$.*

Proof. Clearly, $|C| \geq n$. Now, consider a labeled representative $\tilde{C}$ of C. For each A_n^i, $i = 1, 2, \ldots k$, let $\tilde{A}_n^i$ denote a stem in $\tilde{C}$ which is order-isomorphic to a representative of A_n^i and take their union $\tilde{U} := \bigcup_i \tilde{A}_n^i$ (*i.e.* $\tilde{U}$ is the subcauset of $\tilde{C}$ defined by $x \in \tilde{U} \iff x \in \tilde{A}_n^i$ for some i). Let U denote the order of which $\tilde{U}$ is (order-isomorphic to) a representative. Then U is a stem in C, U is a certificate of Γ_n and $|U| \leq kn$. Since C is a minimal certificate, $C = U$ and hence $|C| \leq kn$. Finally, if $|C| = n$ then C is the only n-stem in C and therefore $\Gamma_n = \{A_n^1\}$, where $A_n^1 = C$. □

Lemma 4.1.12. *Let $\Gamma_n \in \mathcal{P}$. Then there exists a node $\Gamma_m \in \mathcal{P}$ which contains a certificate of Γ_n as an element.*

Proof. When $\Gamma_n = \{C_n\}$ for some order C_n, a node with the required property is $\Gamma_{n+1} \in \mathcal{P}$ since it follows from definition 4.1.6 that C_n is the unique n-order in every $C_{n+1} \in \Gamma_{n+1}$ and therefore each C_{n+1} is a certificate of Γ_n.

Suppose Γ_n contains $k > 1$ orders and recall that if C is a minimal certificate of Γ_n then $n < |C| \leq nk$ (lemma 4.1.11). Define $N := nk$ and consider $\Gamma_N \in \mathcal{P}$. Let D be a finite certificate of Γ_N and hence of Γ_n (since a certificate of a node is also a certificate of all the nodes below it). By definition of minimal certificate 4.1.4, at least one minimal certificate of Γ_n occurs as a stem in D. Choose one, call it C, let $m := |C|$ and consider $\Gamma_m \in \mathcal{P}$. C is an m-stem in D. Γ_m is the set of all m-stems of D and so C is an element of Γ_m. $\square$

Lemma 4.1.13. *Given a path $\mathcal{P}$, there exists a sequence of labeled causets $\tilde{C}_{m_1}, \tilde{C}_{m_2}, \ldots$, such that, for all $k > 0$,*

(i) $|\tilde{C}_{m_k}| = m_k$;

(ii) $m_k < m_{k+1}$;

(iii) $\tilde{C}_{m_k}$ *is a stem in* $\tilde{C}_{m_{k+1}}$;

(iv) $\tilde{C}_{m_{k+1}}$ *is a labeled certificate of* $\Gamma_{m_k} \in \mathcal{P}$;

(v) $C_{m_{k+1}}$, *the order of which* $\tilde{C}_{m_{k+1}}$ *is a representative, is an element of* $\Gamma_{m_{k+1}} \in \mathcal{P}$.

Proof. The required sequence $\tilde{C}_{m_1}, \tilde{C}_{m_2}, \ldots$ is constructed by the following inductive algorithm.

Step 1:

1.0) Pick some non-zero natural number m_0 to start and consider $\Gamma_{m_0} \in \mathcal{P}$.

1.1) By lemma 4.1.12, there exists an $m_1 > m_0$ such that $\Gamma_{m_1} \in \mathcal{P}$ contains a certificate of Γ_{m_0}. Call that certificate C_{m_1}.

1.2) Pick a representative of C_{m_1} and call it $\tilde{C}_{m_1}$.

1.3) Go to step 2.

Step $k > 1$:

k.1) By lemma 4.1.12, there exists an $m_k > m_{k-1}$ and such that $\Gamma_{m_k} \in \mathcal{P}$ contains a certificate of $\Gamma_{m_{k-1}} \in \mathcal{P}$ as an element. Call that certificate C_{m_k}.

k.2) Pick a representative $\tilde{C}_{m_k}$ of C_{m_k} such that $\tilde{C}_{m_{k-1}}$ is a stem in $\tilde{C}_{m_k}$ (this is always possible because $C_{m_{k-1}}$ is a stem in C_{m_k} because C_{m_k} is a certificate of $\Gamma_{m_{k-1}} \in \mathcal{P}$ and $C_{m_{k-1}} \in \Gamma_{m_{k-1}}$).

k.3) Go to step $k + 1$. $\square$

Proof of Theorem 4.1.10. Consider a sequence $\tilde{C}_{m_1}, \tilde{C}_{m_2}, \dots$ generated for $\mathcal{P}$ by the algorithm in lemma 4.1.3. The union $\tilde{C} := \bigcup_{k=1}^{\infty} \tilde{C}_{m_k}$ is a labeled certificate of $\mathcal{P}$. The order C of which $\tilde{C}$ is a representative is a certificate of $\mathcal{P}$. □

An alternative proof is given in appendix 4.A.

4.1.4 The algebra and measure

Having established Ω as the sample space, we now discuss the σ-algebra. To each covtree node assign its "cylinder set" [7], the set of all paths $\mathcal{P}$ which contain it. The collection of all cylinder sets generates covtree's topological[8] σ-algebra. Under the surjection which maps an infinite order to the path of which it is a certificate, the cylinder set of the node Γ_n is the image of the set $cert(\Gamma_n)$ defined by,

Definition 4.1.14 (Certificate set)**.** *For each covtree node Γ_n, its certificate set, $cert(\Gamma_n)$, is the set containing all its infinite certificates,*

$$cert(\Gamma_n) := \{C \in \Omega \mid C \text{ is a certificate of } \Gamma_n\}. \tag{4.3}$$

Thus the σ-algebra generated by the certificate sets is the algebra of measurable events, or observables, in a covtree growth process. This σ-algebra is in fact $\mathcal{R}(\mathcal{S})$ (as we prove in lemma 4.1.15). Standard results in measure theory ensure that each covtree random walk (defined by a complete set of covtree transition probabilities) gives rise to a unique measure on $\mathcal{R}(\mathcal{S})$, where the measure of $cert(\Gamma_n) \in \mathcal{R}(\mathcal{S})$ is equal to the probability of reaching Γ_n (*i.e.* to the product of transition probabilities on the path from the root to Γ_n).[9] Thus, we did what we set out to do: we found a label-independent method to define measures on $\mathcal{R}(\mathcal{S})$. We obtained a manifestly covariant framework for causal set growth dynamics.

[7] "Cylinder set" is a generic term in stochastic processes and should not be confused with its specific usage in (3.11). The meaning should be clear from the context.

[8] It is called "topological" because the cylinder sets are the open balls under the metric topology given by the metric $d(\mathcal{P}, \mathcal{P}') = 1/2^n$, where n is the number of nodes shared by $\mathcal{P}$ and $\mathcal{P}'$.

[9] Due to covtree's tree structure, the certificate sets (together with the empty set) form a semi-ring on which each covtree walk defines a countably-additive "pre-measure" [Lindstrøm, 2017] which has a unique extension to $\mathcal{R}(\mathcal{S})$ by the Kolmogorov extension theorem [Kolmogorov and Fomin, 1975].

We now have two ways of defining measures on $\mathcal{R}(\mathcal{S})$: via a restriction of a measure $\tilde{\mu}$ on the labeled algebra $\tilde{\mathcal{R}}$ (where $\tilde{\mu}$ arises from a random walk on labeled poscau[10]) or directly via a covtree random walk. Do these methods give rise to different classes of measures? By reverse engineering our construction, one can show that every measure on $\mathcal{R}(\mathcal{S})$ can be derived from a covtree walk. Additionally, every measure on $\mathcal{R}(\mathcal{S})$ possesses some extension[11] to $\tilde{\mathcal{R}}$ (as we prove in lemma 4.1.16), meaning that every measure on $\mathcal{R}(\mathcal{S})$ can be obtained via a restriction of some $\tilde{\mu}$. The set of manifestly covariant dynamics and the set of dynamics obtained via the labeled procedure are equal! For every walk on labeled poscau—whether it satisfies discrete general covariance or not—there exists a covtree walk which produces the same measure on $\mathcal{R}(\mathcal{S})$. There is no easy relationship between the discrete general covariance condition on a labeled poscau walk and the manifest covariance of a covtree walk.

Lemma 4.1.15. *Let Σ denote the collection of all certificate sets and let $\mathcal{R}(\Sigma)$ denote the σ-algebra generated by Σ. Then $\mathcal{R}(\Sigma) = \mathcal{R}(\mathcal{S})$.*

Proof. We will show that any stem set (cf. equation (3.13)) can be constructed by a finite number of set operations on the certificate sets and vice versa, and the result follows.

Consider an n-order B_n. Let Γ^i_n denote the covtree nodes which contain B_n, where i labels the individual nodes. Suppose $C \in \mathit{cert}(\Gamma^i_n)$ for some i. Then B_n is a stem in C and hence $C \in \mathit{stem}(B_n)$. Suppose $C \notin \mathit{cert}(\Gamma^i_n)$ for all i. Then B_n is not a stem in C and hence $C \notin \mathit{stem}(B_n)$. It follows that $\mathit{stem}(B_n) = \bigcup_i \mathit{cert}(\Gamma^i_n) \implies \mathcal{R}(\mathcal{S}) \subseteq \mathcal{R}(\Sigma)$.

Consider some node $\Gamma_n = \{A^1_n, ..., A^k_n\}$ in covtree. Let $\Omega(n) \setminus \Gamma_n = \{B^1_n, ..., B^l_n\}$. Suppose $C \in \mathit{cert}(\Gamma_n)$. Then $A^1_n, ..., A^k_n$ are stems in C, and $B^1_n, ..., B^l_n$ are not stems in C. Hence $C \in \bigcap_{i=1}^{k} \mathit{stem}(A^i_n) \setminus \bigcup_{j=1}^{l} \mathit{stem}(B^j_n)$. Suppose $C \notin \mathit{cert}(\Gamma_n)$. Then either (i) there exists some $A^i_n \in \Gamma_n$ which is not a stem in $C \implies C \notin \bigcap_{i=1}^{k} \mathit{stem}(A^i_n)$, or (ii) there exists some $B^j_n \in \Omega(n) \setminus \Gamma_n$ which is a stem in $C \implies C \in \bigcup_{j=1}^{l} \mathit{stem}(B^j_n)$. It follows that, $\mathit{cert}(\Gamma_n) = \bigcap_{i=1}^{k} \mathit{stem}(A^i_n) \setminus \bigcup_{j=1}^{l} \mathit{stem}(B^j_n) \implies \mathcal{R}(\Sigma) \subseteq \mathcal{R}(\mathcal{S})$. □

Lemma 4.1.16. *For every measure $\mu_{\mathcal{S}}$ on $\mathcal{R}(\mathcal{S})$ there exists an extension $\tilde{\mu}$ to $\tilde{\mathcal{R}}$.*

[10] The random walk need not be a CSG model nor must it satisfy any physical conditions.
[11] The extension may or may not be unique.

Proof. First note that there is a metric on $\tilde{\Omega}$ with respect to which $(\tilde{\Omega}, \tilde{\mathcal{R}})$ is a Polish space [Brightwell et al., 2003]. Since every Polish space is a Lusin space [Schwartz, 1973], $(\tilde{\Omega}, \tilde{\mathcal{R}})$ is a Lusin space. Note also that $\mathcal{R}(\mathcal{S})$ is a separable sub-σ-algebra of $\tilde{\mathcal{R}}$ since there exists a countable collection of subsets of $\tilde{\Omega}$ which generates $\mathcal{R}(\mathcal{S})$, namely $\mathcal{S}$ (or Σ). The result follows from the theorem that if $(Y, \mathcal{B})$ is a Lusin space, then every measure defined on a separable sub-σ-algebra of $\mathcal{B}$ can be extended to $\mathcal{B}$ [Landers and Rogge, 1974]. □

4.2 Covtree and causal set cosmology

There is no reason to expect that a generic covtree walk will be physically interesting: the class of such walks is too vast to be interesting. We need physically motivated conditions to restrict the models to a sub-class worth studying.

One challenge lies in the translation of physically desirable conditions (*e.g.* that manifold-like[12] orders are preferred by the dynamics) into conditions on covtree transition probabilities. Doing so requires an understanding of the relationship between paths and their certificates (*e.g.* which paths have manifold-like certificates). Closely related challenges include formulating a causality condition on covtree and understanding what additional constraints general covariance may impose on the transition probabilities (cf. the discrete general covariance condition in the CSG models).

In addition to the relationship between paths and their certificates, an understanding of the structure of covtree is also important for constraining the dynamics. For example, any dynamics should satisfy the Markov-sum-rule: the sum of the transition probabilities from any node Γ_n must equal 1. But with no knowledge of the number of nodes directly above Γ_n or of the relation they bear to it, this constraint is intractable. (In contrast, in the case of the CSG models knowing that the children of $\tilde{C}_n$ are in 1-to-1 correspondence with the stems in $\tilde{C}_n$ allows to solve the Markov-sum-rule.)

In addressing these challenges, one might be tempted to construct covtree explicitly. Indeed, we presented the first three levels (Fig.4.4), but brute force methods come up short in going to higher levels as the number of candidate nodes at level n increases

[12] We say an order C is manifold-like if a representative of C can be faithfully embedded into a four-dimensional Lorentzian manifold.

rapidly as $2^{|\Omega(n)|} - 1$, where $|\Omega(3)| = 5, |\Omega(5)| = 63$ and $|\Omega(16)| = 4483130665195087$ [The On-Line Encyclopedia of Integer Sequences]. In the remainder of this chapter, we make progress by focusing on structural properties which are independent of level.

In this section, guided by the narrative of causal set cosmology (section 3.5) which relies on the existence of posts and breaks and on the associated cosmic renormalisation, we identify the covtree paths whose certificates contain posts and breaks (section 4.2.1), show that covtree has a self-similar structure (section 4.2.2), use this self-similarity to obtain a covariant analogue of cosmic renormalisation and conclude with a discussion of open questions, including a proposal for a causality condition for manifestly covariant dynamics (section 4.2.3).

4.2.1 Certificates with posts and breaks

We begin with two definitions.

Definition 4.2.1 (A-break/A-post)**.** *Let A denote a finite order and let $\tilde{A}$ be some representative of it. We say that an order C contains an A-break (A-post) if it has a representative which contains an $\tilde{A}$-break ($\tilde{A}$-post).*[13]

Definition 4.2.2 (Covering causet/order)**.** *Given an n-causet $\tilde{C}_n$, its covering causet $\widehat{\tilde{C}_n}$ is the $(n+1)$-causet formed by putting the element n above every element of $\tilde{C}_n$. Similarly, $\widehat{C_n}$ is the covering order of C_n, where $\widehat{\tilde{C}_n}$ and $\tilde{C}_n$ are representatives of the respective orders.*

An illustration is shown in figure 4.6.

We can now state our main theorem which identifies those covtree paths whose certificates contain posts and breaks,

Theorem 4.2.3. *Let the order C be a certificate of the path $\mathcal{P}$.*

1. *C contains an A-break if and only if $\{\widehat{A}\}$ is a node in $\mathcal{P}$.*

2. *C contains an A-post if and only if $\{\hat{\hat{A}}\}$ is a node in $\mathcal{P}$ (where $\hat{\hat{A}}$ is the covering order of the covering order of A).*

[13]Recall that an $\tilde{A}$-break is a break with past $\tilde{A}$. $\tilde{A}$-post is similarly defined.

In addition to being a kinematical statement, theorem 4.2.3 has implications for the search for physical dynamics, for our understanding of observables and for the development of a covariant analogue of the cosmic renormalisation transformation. Starting with the former, our theorem implies that a covtree dynamics in which an infinite sequence of breaks happens with unit probability is one which, with unit probability, passes through infinitely many nodes of the form $\{\widehat{A}\}$. Such dynamics are of interest to causal set cosmology, since they give rise to bouncing universes. In the future, our theorem could be used in combination with theorems in probability theory and stochastic process (*e.g.* various zero-one laws) to solve for covtree random walks which give rise to bouncing cosmologies. Secondly, our theorem implies that the event "the growing order contains an A-break" can be affirmed or denied at a finite stage of the covtree process. On the contrary, in a CSG model the labeled event "the growing causet contains an $\tilde{A}$-break" cannot be affirmed at a finite stage (although it can be denied). While what we mean by "local" in this framework is open to interpretation, one could argue that the occurrence of a post or a break is a non-local event (cf. causal horizon in the continuum) and that it is the covariant nature of covtree which allows us to access such non-local information which is inaccessible in the labeled growth process. Finally, we discuss in detail the theorem's application to cosmic renormalisation in section 4.2.3.

We now prove theorem 4.2.3, beginning with a lemma.

Lemma 4.2.4. *Let $\tilde{C}$ and $\tilde{A}$ be labeled causets. The following statements are equivalent:*

(i) *$\tilde{C}$ contains an $\tilde{A}$-break,*

(ii) *every $(|\tilde{A}|+1)$-stem in $\tilde{C}$ is isomorphic to $\widehat{\tilde{A}}$,*

(iii) *$\tilde{A}$ is the unique $|\tilde{A}|$-stem in $\tilde{C}$.*

Proof. Let $\tilde{A}$ be a stem in $\tilde{C}$. Let x denote a minimal element in $\tilde{C}\setminus\tilde{A}$. Let a denote an element in $\tilde{A}$.

$(i) \implies (ii)$ It follows from the definition of an $\tilde{A}$-break that every $(|\tilde{A}|+1)$-stem in $\tilde{C}$ is of the form $\tilde{A}\cup\{x\}$ and that each such stem is isomorphic to $\widehat{\tilde{A}}$.

$(ii) \implies (iii)$ Suppose for contradiction that $\tilde{D} \neq \tilde{A}$ is an $|\tilde{A}|$-stem in $\tilde{C}$. Let y be minimal in $\tilde{D} \setminus \tilde{A}$. Then $\tilde{A} \cup \{y\}$ is an $(|\tilde{A}|+1)$-stem in $\tilde{C}$. By assumption (ii), $y \succ a$ for all a in $\tilde{A}$, and therefore (by definition of stem) $\tilde{A} \cup \{y\} \subseteq \tilde{D}$ which in turn implies that $|\tilde{D}| > |\tilde{A}|$. Contradiction.

$(iii) \implies (i)$ By assumption (iii), $x \succ a$ for all x and a, and hence (by definition of break) $\tilde{C}$ contains an $\tilde{A}$-break. □

The following covariant statement is a corollary:

Corollary 4.2.5. *An order C contains an A-break if and only if $\widehat{A}$ is its unique* $(|A|+1)$-*stem.*

Proof of theorem 4.2.3. Let the order C be a certificate of the path $\mathcal{P}$. Recall that by definition 4.1.8, the set of n-stems of C is the node Γ_n in $\mathcal{P}$.

To prove part 1, suppose C contains an A-break. Then by corollary 4.2.5, the set of $(|A|+1)$-stems of C is $\{\widehat{A}\}$. By definition of certificate, $\{\widehat{A}\}$ is in $\mathcal{P}$. Now suppose $\{\widehat{A}\}$ is in $\mathcal{P}$. Then by definition of certificate, $\{\widehat{A}\}$ is the set of $(|A|+1)$-stems of C. By corollary 4.2.5, C contains an A-break.

Part 2 follows from the fact that C contains an A-post if and only if C contains an $\widehat{A}$-break. □

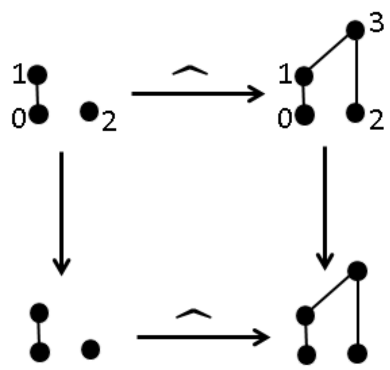


Figure 4.6: The relationship between an order, its covering order and their representatives.

4.2.2 Covtree self-similarity

Covtree is itself a causal set whose ground-set is Λ (definitions 4.1.3 and 4.1.7). This section is dedicated to proving that covtree has a "self-similar" structure. We begin with a definition.

Definition 4.2.6 (Self-similar causal set). *A causal set is self-similar if it contains infinitely many copies of itself.*

For any finite order A, let $\Lambda_A \subset \Lambda$ be the convex subcauset of covtree which contains the node $\{\widehat{A}\}$ and everything above it. We will show that:

Theorem 4.2.7. *For any finite order A, Λ_A is a copy of covtree. Thus, covtree is self-similar.*

An illustration of covtree's self-similar structure is shown in figure 4.7.

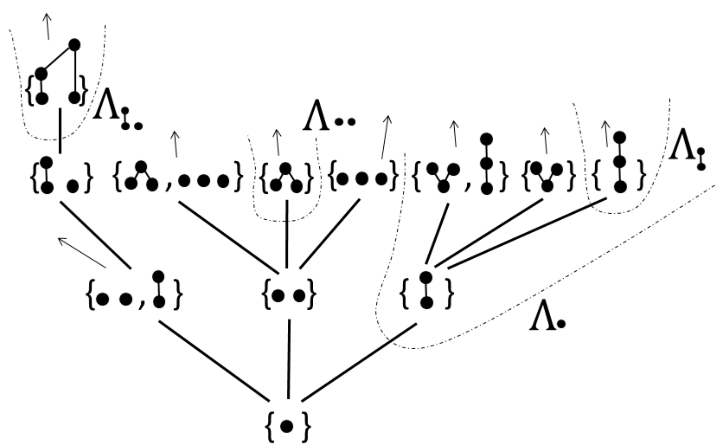


Figure 4.7: The self-similar structure of covtree. The figure displays the first two levels of covtree in full and selected nodes from levels 3 and 4. The arrows indicate additional nodes not shown in the figure. The dashed lines indicate where a new copy of covtree begins. The ground-set Λ_A of each copy is indicated next to each dashed line. Figuratively, we can write $\Lambda = \Lambda_\emptyset$.

To prove theorem 4.2.7, we will need the following definition:

Definition 4.2.8. *Given a finite order A and a set $\Gamma_n \in \Lambda$, the map $\mathcal{G}_A$ takes Γ_n to $\mathcal{G}_A(\Gamma_n)$, the set of orders which contain a break with past A and future $B_n \in \Gamma_n$, i.e.*

$$\mathcal{G}_A(\Gamma_n) := \{C \mid C \text{ is an order which contains a break with past } A \text{ and future } B_n \in \Gamma_n\}.$$

Examples are shown in figure 4.8.

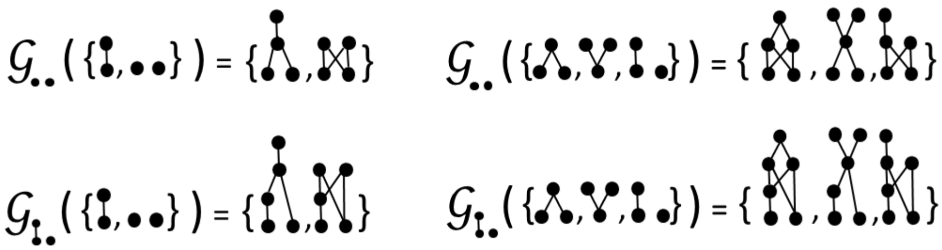


Figure 4.8: Illustration of the operation $\mathcal{G}_A$.

Proof of theorem 4.2.7: We will show that, for any finite order A,

(*i*) $\mathcal{G}_A(\Lambda) = \Lambda_A$, and

(*ii*) the map $\mathcal{G}_A : \Lambda \to \Lambda_A$ is an order-isomorphism,

and the result follows.

To prove part (*i*), we first show that $\mathcal{G}_A(\Lambda) \subseteq \Lambda_A$. Let $\Gamma_n \in \Lambda$ and let the m-order C_m be a certificate of Γ_n. Let D_k denote the order of cardinality $k = m + |A|$ which contains a break with past A and future C_m. Then D_k is a certificate of $\mathcal{G}_A(\Gamma_n) \implies \mathcal{G}_A(\Gamma_n) \in \Lambda$. Next, note that $\widehat{A}$ is the unique $|\widehat{A}|$-stem in every order in $\mathcal{G}_A(\Gamma_n) \implies \mathcal{G}_A(\Gamma_n) \in \Lambda_A$.

Second, we show that $\Lambda_A \subseteq \mathcal{G}_A(\Lambda)$. Let $\Gamma_n \in \Lambda_A$ and let the p-order E_p be a certificate of Γ_n. Necessarily, E_p contains a break with past A and some future B. Let Γ_l denote the set of l-stems of B, where $l = n - |A|$. Then $\Gamma_l \in \Lambda$ and $\Gamma_n = \mathcal{G}_A(\Gamma_l)$.

To prove part (*ii*), we use the commutativity of the operations $\mathcal{O}$ and $\mathcal{G}_A$ to show that $\mathcal{G}_A : \Lambda \to \Lambda_A$ is order-preserving. Suppose $\Gamma_n \prec \Gamma_{n+1}$, then by definition of covtree we have that $\Gamma_n = \mathcal{O}(\Gamma_{n+1})$, and therefore

$$\mathcal{G}_A(\Gamma_n) = \mathcal{G}_A(\mathcal{O}(\Gamma_{n+1})) = \mathcal{O}(\mathcal{G}_A(\Gamma_{n+1})) \implies \mathcal{G}_A(\Gamma_n) \prec \mathcal{G}_A(\Gamma_{n+1}).$$

Now suppose $\mathcal{G}_A(\Gamma_n) \prec \mathcal{G}_A(\Gamma_{n+1})$. Then

$$\mathcal{G}_A(\Gamma_n) = \mathcal{O}(\mathcal{G}_A(\Gamma_{n+1})) = \mathcal{G}_A(\mathcal{O}(\Gamma_{n+1})) \implies \Gamma_n \prec \Gamma_{n+1}.$$

□

4.2.3 Covariant cosmic renormalisation

In this section we apply theorems 4.2.3 and 4.2.7 to obtain the covariant counterpart of cosmic renormalisation and conclude with a discussion of open questions. In the following, we denote a covtree transition probability by $\mathbb{P}(\Gamma_n \to \Gamma_{n+1})$. We use $\{\mathbb{P}\}$ to denote a complete set of covtree transition probabilities (or a "covariant dynamics").

Renormalisation after a break: In the CSG models, cosmic renormalisation arises by first conditioning that the growing causet contains a break with a particular past $\tilde{A}$ and then "deleting and relabeling" the elements in the past of the break to obtain a new growth process governed by an effective dynamics. The covariant analogue of the first part is conditioning that the growing order contains an A-break, for some finite order A. Theorem 4.2.3 tells us that this condition is equivalent to requiring that the random walk pass through the node $\{\widehat{A}\}$. The covariant analogue of "deleting and relabeling" is acting on each node in Λ_A with $\mathcal{G}_A^{-1}$ (the inverse of definition 4.2.8) since this effectively "deletes" the past A of the break. Theorem 4.2.7 tells us that $\mathcal{G}_A^{-1}$ maps Λ_A to Λ, so that the growth of the future of the break (previously described by a walk on Λ_A) can now be treated independently of the past as a walk on the whole of Λ governed by a new set of "effective" transition probabilities. Given a dynamics $\{\mathbb{P}\}$, the effective dynamics $\{\mathbb{P}_A\}$ obtained by conditioning on an A-break is given simply by,

$$R_A : \{\mathbb{P}\} \mapsto \{\mathbb{P}_A\},\ \mathbb{P}_A(\Gamma_n \to \Gamma_{n+1}) = \mathbb{P}(\mathcal{G}_A(\Gamma_n) \to \mathcal{G}_A(\Gamma_{n+1})). \tag{4.4}$$

For a generic covtree dynamics the functional relationship between $\{\mathbb{P}\}$ and $\{\mathbb{P}_A\}$ can depend on any feature of A, and this is signified by the label A on the transformation R_A and on the effective transition probabilities $\{\mathbb{P}_A\}$.

The transformation R_A acts on a set of transition probabilities, not on a set of couplings. To emphasise this, we call R_A a "similarity" transformation rather than a "renormalisation" transformation. If (as one hopes) in the future we are able to characterise a covtree dynamics by a set of couplings, it may be possible to write the similarity transformation as a renormalisation transformation which acts on the couplings directly.

Similarly, we reserve the term "stationary point" for a set of couplings which is mapped onto itself by the renormalisation transformation. If a covtree dynamics is mapped onto itself by a similarity transformation R_A, we say that it is "self-similar" with respect to R_A. A covtree dynamics $\{\mathbb{P}\}$ is self-similar with respect to R_A if and only if it satisfies the condition

$$\mathbb{P}(\Gamma_n \to \Gamma_{n+1}) = \mathbb{P}(\mathcal{G}_A(\Gamma_n) \to \mathcal{G}_A(\Gamma_{n+1})) \tag{4.5}$$

for every n and every transition $\Gamma_n \to \Gamma_{n+1}$. Constructing a self-similar dynamics is simple: assign any set of transition probabilities to the transitions which lie outside Λ_A, and then use equality (4.5) to set the transition probabilities in Λ_A.

It is possible to use this procedure to fix the transition probabilities in Λ_A for every A simultaneously, thus constructing a dynamics which is self-similar with respect to R_A for all A. We call such dynamics "maximally self-similar".

A dynamics cannot be self-similar with respect to a unique transformation R_A. If a dynamics is self-similar with respect to some transformation R_A then it is also self-similar with respect to $(R_A)^n$, for any positive integer n. But $(R_A)^n$ is itself a similarity transformation: $(R_A)^n = R_{A^n}$, where we define A^n to be the order of cardinality $n|A|$ which is a stack of n copies of A separated by breaks. Therefore there exists no dynamics which is self-similar with respect to a unique transformation. We say that a dynamics $\{\mathbb{P}\}$ is "minimally self-similar" if there exists a unique order A such that $\{\mathbb{P}\}$ is only self-similar with respect to $(R_A)^n$ for all n.

Some self-similar dynamics are neither minimally nor maximally self-similar. Consider a (finite or infinite) collection $\{\widehat{A}\}, \{\widehat{B}\}, ...$ of covtree nodes. Does there exist a dynamics which is only self-similar with respect to $R_A, R_B, ...$ and their respective powers? If every pair of nodes are unrelated in covtree then the answer is yes. On the other hand, suppose that $\{\widehat{A}\} \prec \{\widehat{B}\}$. Then B contains an A-break, and let us denote the future of the break by D. Then $\mathcal{G}_B = \mathcal{G}_D\mathcal{G}_A$, and a dynamics is self-similar with respect to R_A and R_B if and only if it is self-similar also with respect to R_D.

Renormalisation after a post: Since the occurrence of an A-post is equivalent to the occurrence of an $\widehat{A}$-break, the effective dynamics obtained by conditioning on an A-post is given by transformation $R_{\widehat{A}}$, obtained from transformation (4.4) via $A \to \widehat{A}$, and given explicitly by,

$$R_{\widehat{A}} : \{\mathbb{P}\} \to \{\mathbb{P}_{\widehat{A}}\},\ \mathbb{P}_{\widehat{A}}(\Gamma_n \to \Gamma_{n+1}) = \mathbb{P}(\mathcal{G}_{\widehat{A}}(\Gamma_n) \to \mathcal{G}_{\widehat{A}}(\Gamma_{n+1})). \tag{4.6}$$

$R_{\widehat{A}}$ is the covariant counterpart of the labeled *non-originary* transformation $Q_{a+1,1}$ (cf. (3.16)). Additionally, an *originary* formulation also exists in the covariant case and is analogous to the labeled transformation S_a (cf. (3.15)). We say that a covtree dynamics $\{\mathbb{P}\}$ is "originary" if $\mathbb{P}(\Gamma_1 \to \{⫯\}) = 1$. In the originary viewpoint of the A-post condition, a covtree dynamics $\{\mathbb{P}\}$ is mapped onto an *originary* covtree dynamics $\{\mathbb{P}'_A\}$ via the transformation[14]:

$$\begin{aligned}
&T_A : \{\mathbb{P}\} \to \{\mathbb{P}'_A\},\\
&\mathbb{P}'_A(\Gamma_1 \to \ \{⫯\}\ ) = 1,\\
&\mathbb{P}'_A(\Gamma_n \to \Gamma_{n+1}) = \mathbb{P}(\mathcal{G}_A(\Gamma_n) \to \mathcal{G}_A(\Gamma_{n+1}))\ \forall\ \Gamma_n \succeq \ \{⫯\}\ ,\\
&\mathbb{P}'_A(\Gamma_n \to \Gamma_{n+1}) = 0 \text{ otherwise.}
\end{aligned} \tag{4.7}$$

A summary of the covariant similarity transformations derived in this section and of their labeled counterparts is shown in table 4.1.

	break	**post**	
		non-originary	*originary*
labeled	$Q_{a,r}$ (3.16)	$Q_{a+1,1}$ (3.16)	S_a (3.15)
covariant	R_A (4.4)	$R_{\widehat{A}}$ (4.6)	T_A (4.7)

Table 4.1: Summary of transformations. The first row lists the renormalisation transformations which act on CSG couplings. The second row lists the similarity transformations which act on covtree transition probabilities.

Discussion: While our success in adapting the cosmic renormalisation to the covtree framework bodes well for a covariant causal set cosmology, our results will remain purely formal until we are able to identify a class of physical covtree dynamics with which to work. Having said that, our similarity transformations can themselves be exploited to

[14] The apostrophe on the transition probabilities $\{\mathbb{P}'_A\}$ is used to distinguish between the images of $\{\mathbb{P}\}$ under R_A and T_A.

better understand the form that CSG models take on covtree and to obtain new classes of physical dynamics by considering how these dynamics transform:

1. We have seen that the labeled break transformation $Q_{a,r}$ depends on the past of the break only via its cardinality and number of maximal elements. Is the condition on a covtree dynamics $\{\mathbb{P}\}$ that $\{\mathbb{P}_A\} = \{\mathbb{P}_B\}$ if and only if A and B have the same cardinality and number of maximal elements necessary for $\{\mathbb{P}\}$ to be a CSG dynamics? Is it sufficient?

2. Recall that the action of $Q_{a,r}$ can be factorised in terms of the transformation M of (3.14). Does this property bear any relation to the constraint on a covtree dynamics $\{\mathbb{P}\}$ that, for any finite order A, the renormalisation transformation can be factorised as $R_A = R^{|A|}$ for some transformation R?

3. If a given CSG model is a stationary point of a labeled transformation, is its corresponding covtree dynamics self-similar?

4. Recall that a CSG dynamics is a stationary point of both $Q_{a,r}$ and $Q_{b,s}$ if and only if $r = s$. Is the condition on a covtree dynamics $\{\mathbb{P}\}$ that $\{\mathbb{P}\} = \{\mathbb{P}_A\} = \{\mathbb{P}_B\}$ only if A and B have the same number r of maximal elements necessary for $\{\mathbb{P}\}$ to be a CSG dynamics? Is it sufficient? Such a dynamics is neither maximally nor minimally self-similar, and it follows from our previous analysis that for any $r > 0$ there exists a family of self-similar dynamics which satisfy this condition. Are these CSG dynamics, or is the relationship between the labeled and covariant formulations more complex?

5. Let us consider the post condition in the originary formulation. A covtree dynamics $\{\mathbb{P}\}$ flows to a self-similar dynamics under T_A if

$$(T_A)^k[\mathbb{P}(\Gamma_n \to \Gamma_{n+1})] \to (T_A)^{k+1}[\mathbb{P}(\Gamma_n \to \Gamma_{n+1})] \quad \text{as} \quad k \to \infty. \tag{4.8}$$

If $\{\mathbb{P}\}$ arrives at a self-similar dynamics after N applications of T_A, expression (4.8) simplifies to:

$$\begin{aligned} &(T_A)^k[\mathbb{P}(\Gamma_n \to \Gamma_{n+1})] = (T_A)^{k+1}[\mathbb{P}(\Gamma_n \to \Gamma_{n+1})] \;\forall\; k \geq N \\ \implies &\mathbb{P}(\mathcal{G}_A^k(\Gamma_n) \to \mathcal{G}_A^k(\Gamma_{n+1})) = \mathbb{P}(\mathcal{G}_A^{k+1}(\Gamma_n) \to \mathcal{G}_A^{k+1}(\Gamma_{n+1})) \;\forall\; k \geq N. \end{aligned} \tag{4.9}$$

The $N = 1$ case is of special interest to us. It is easy to show that, under an application of S_a, a Transitive Percolation (TP) dynamics (given by $t_k = t^k$ for some constant t) is mapped onto the corresponding Originary Transitive Percolation (OTP) dynamics ($t_0 = 0, t_k = t^k$ for $k > 0$), where the latter is a stationary point of S_a. Does this mean that for a covtree dynamics $\{\mathbb{P}\}$ to be a TP dynamics it must satisfy condition (4.9) with $N = 1$?

6. It is known that OTP gives rise to infinitely many posts with unit probability [Alon et al., 1994]. Is this property related to the fact that OTP is a stationary point? Do self-similar covtree dynamics give rise to an infinite sequence of posts or breaks? Could this be a feature of the maximally self-similar dynamics?

7. When the transformation R_A factorises as $R_A = R^{|A|}$, the effective dynamics is independent of the causal structure of the past. Therefore, could the condition that R_A factorises be interpreted as a causality condition on covtree dynamics?

4.3 Further structure of covtree

In this section we present additional results about the structure of covtree and about certificates. We conclude with a toy example illustrating how an understanding of the structure of covtree can be a useful tool for constraining covtree dynamics.

4.3.1 Nodes

Here we list properties which pertain to nodes, including criteria for a set of n-orders to be a node, properties of minimal certificates and a study of direct descendants and valency.

Definition 4.3.1 (Singleton and doublet). *A node Γ_n in covtree is a singleton if it contains a single n-order. A node Γ_n in covtree is a doublet if it contains exactly two n-orders.*

We start with the simple property:

Property 1. *For any finite order C_n, there is a singleton node $\{C_n\}$ in covtree. C_n is the unique minimal certificate of $\{C_n\}$.*

Recall that $\Gamma_n \succ \Gamma_m$ in covtree if and only if every certificate of Γ_n is a certificate of Γ_m. Therefore $\{C_n\} \succ \Gamma_m$ in covtree if and only if C_n is a certificate of Γ_m. Since every node in covtree has countably many finite certificates, we find that:

Property 2. *Every node in covtree has countably many singleton descendants.*

Next we note that, since C_n is the only n-stem in its covering order $\widehat{C_n}$, the node $\{\widehat{C_n}\}$ is directly above $\{C_n\}$ in covtree and therefore:

Property 3. *Every singleton has at least one direct descendant which is a singleton.*

Moreover,

Property 4. *If $\{\widehat{C_n}\}$ is the only singleton directly above $\{C_n\}$ then $\{\widehat{C_n}\}$ is the only node directly above $\{C_n\}$.*

To see this, assume for contradiction that there exists some node $\{A^1_{n+1}, ..., A^k_{n+1}\}$ which is directly above $\{C_n\}$, where $k \geq 2$. It follows from the definition of covtree that C_n is the unique n-stem in every $A^i_{n+1} \in \{A^1_{n+1}, ..., A^k_{n+1}\}$ and therefore each of the nodes $\{A^1_{n+1}\}, ..., \{A^k_{n+1}\}$ are singletons directly above $\{C_n\}$, which is a contradiction.

Every singleton with valency greater than one has at least one direct descendant which is a doublet since:

Property 5. *If $\{D_{n+1}\} \succ \{C_n\}$ and $D_{n+1} \neq \widehat{C_n}$ then $\{\widehat{C_n}, D_{n+1}\} \succ \{C_n\}$.*

A corollary of properties 4 and 5 is:

Property 6. *No singleton has a valency of 2.*

Singletons which possess property 4 are the only nodes in covtree which have exactly one direct descendant since:

Property 7. *Only singletons can have exactly one direct descendant in covtree.*

To prove this, consider the node $\Gamma_n = \{A_n^1, ..., A_n^k\}$ with $k \geq 2$, and assume for contradiction that Γ_{n+1} is the only direct descendant of Γ_n.

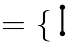
First, we show that Γ_{n+1} contains the covering order $\widehat{A_n^i}$ of every $A_n^i \in \Gamma_n$. Suppose for contradiction that Γ_{n+1} does not contain the covering order $\widehat{A_n^i}$ of some $A_n^i \in \Gamma_n$. Let C_m be an m-order that is a certificate of Γ_{n+1}. Then there exists an $(m+1)$-order D_{m+1} which contains C_m and $\widehat{A_n^i}$ as stems (a representative of D_{m+1} can be constructed by taking a representative of C_m and adding an element above the stem which is isomorphic to a representative of A_n^i). D_{m+1} is a certificate of $\Gamma'_{n+1} = \Gamma_{n+1} \cup \{\widehat{A_n^i}\}$, and $\Gamma'_{n+1} \succ \Gamma_n$. This contradicts the assumption that Γ_{n+1} is the only direct descendant of Γ_n . Hence Γ_{n+1} contains the covering order $\widehat{A_n^i}$ for all $A_n^i \in \Gamma_n$.

Now we show that, since Γ_{n+1} contains the covering order $\widehat{A_n^i}$ of some $A_n^i \in \Gamma_n$, it cannot be the only direct descendant of Γ_n. Let the p-order E_p be a minimal certificate of Γ_n. Therefore $\Gamma_n \prec \Gamma_{n+1} \prec \{E_p\}$ and hence E_p is a certificate of Γ_{n+1}. Then there exists a $(p-1)$-order F_{p-1} such that F_{p-1} is a stem in E_p *and* $\widehat{A_n^i}$ is not a stem in F_{p-1} (a causet isomorphic to a representative of F_{p-1} can be constructed by taking a representative of E_p and removing the element which is maximal in the stem isomorphic to a representative of A_n^i). Then F_{p-1} is a certificate of Γ_n, which contradicts the assumption that E_p is a minimal certificate of Γ_n. Therefore, only singletons can have exactly one direct descendant in covtree.

Additionally,

Property 8. *For any $k \geq 1$ there is a singleton $\{C_n\}$ in covtree with k singletons directly above it.*

An immediate corollary is that the valency of singletons is unbounded. (Note that k is not the valency of $\{C_n\}$, for if $k \geq 2$ then $\{C_n\}$ has additional direct descendants which are not singletons, cf. property 5.)

An example of a singleton node with 1 singleton directly above it is $\Gamma_4 = \{$$\}$. To show that the statement is true for $k > 1$, we construct a countable sequence of singletons $\{C_{n_2}\}, \{C_{n_3}\}, ..., \{C_{n_k}\}, ...$ such that $\{C_{n_k}\}$ has k singletons directly above it. Figure 4.9 shows the first three singletons in the sequence and their respective singleton descendants.

Let us explain the pattern shown in figure 4.9. Each order C_{n_k} has cardinality $n_k = 4k - 1$. A representative $\tilde{C}_{n_k}$ of C_{n_k}, contains $\frac{n_k - 1}{2}$ elements in level 1, $\frac{n_k - 1}{2}$ elements in level 2, and a single element in level 3. Each element in level 1 is below two elements in level 2. Each element in level 2 is above two elements in level 1. The element in level 3 is above all but one of the elements in level 2.

We construct the singleton descendants of $\{C_{n_k}\}$ as follows. Construct a new causet from $\tilde{C}_{n_k}$ by adding a new element directly above all but one of the level 2 elements subject to the constraint that no level 2 element is maximal in the resulting causet. There are $2k - 2$ ways to do this, leading to a collection of $2k - 2$ causets. One can show that each of these causets is order-isomorphic to exactly one other in the collection, and taking the corresponding orders gives $k - 1$ distinct $(n_k + 1)$-orders whose only n_k-stem is C_{n_k}. A singleton containing each of these orders is directly above $\{C_{n_k}\}$. The additional singleton descendant is $\{\widehat{C_{n_k}}\}$.

Similarly,

Property 9. *For any integer $k \geq 1$ there exists a doublet in covtree with k singletons directly above it.*

One can construct an infinite sequence of doublets, $\{C_{m_1}, D_{m_1}\}, \{C_{m_2}, D_{m_2}\}, ..., \{C_{m_k}, D_{m_k}\}, ...$, such that the k^{th} doublet in the sequence has k singletons directly above it. Figure 4.10 shows the first three doublets in the sequence and their direct singleton descendants.

A representative $\tilde{C}_{m_k}$ of C_{m_k} has cardinality $m_k = 4k + 5$ and partial ordering as described below property 8 (with n_k replaced by m_k). A representative of D_{m_k} has these same properties, except that the element in level 3 is above all but *two* of the elements in level 2. The two elements missed out must have a common element in their pasts.

We construct the singleton descendants of $\{C_{m_k}, D_{m_k}\}$ as follows. Construct a new causet from $\tilde{C}_{m_k}$ by adding a new element directly above all but two of the level 2 elements subject to the constraints that no level 2 element is maximal in the resulting causet *and*

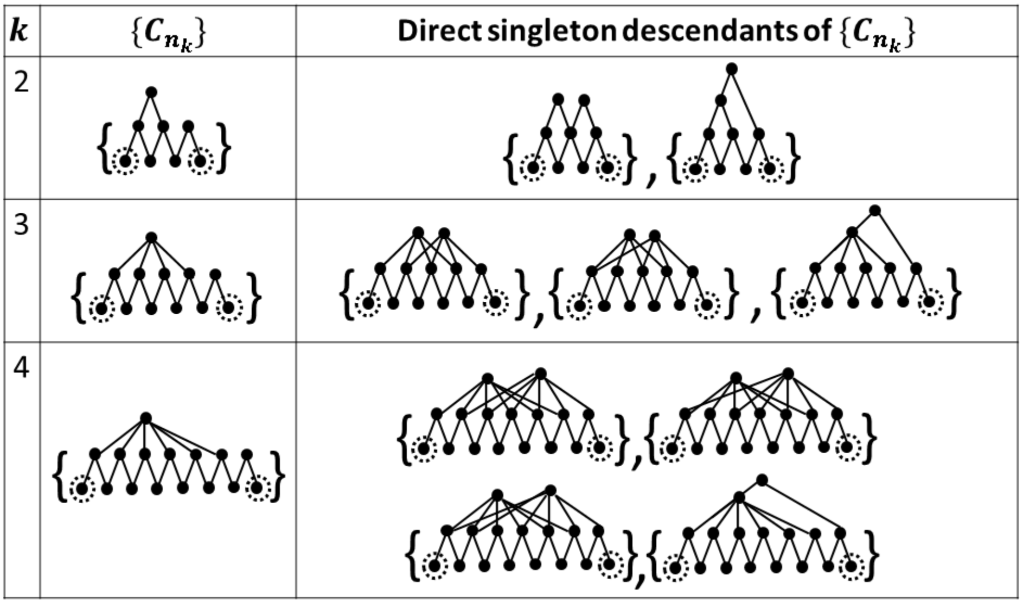


Figure 4.9: Illustration of property 8. The elements circled by a dotted line are identified with each other.

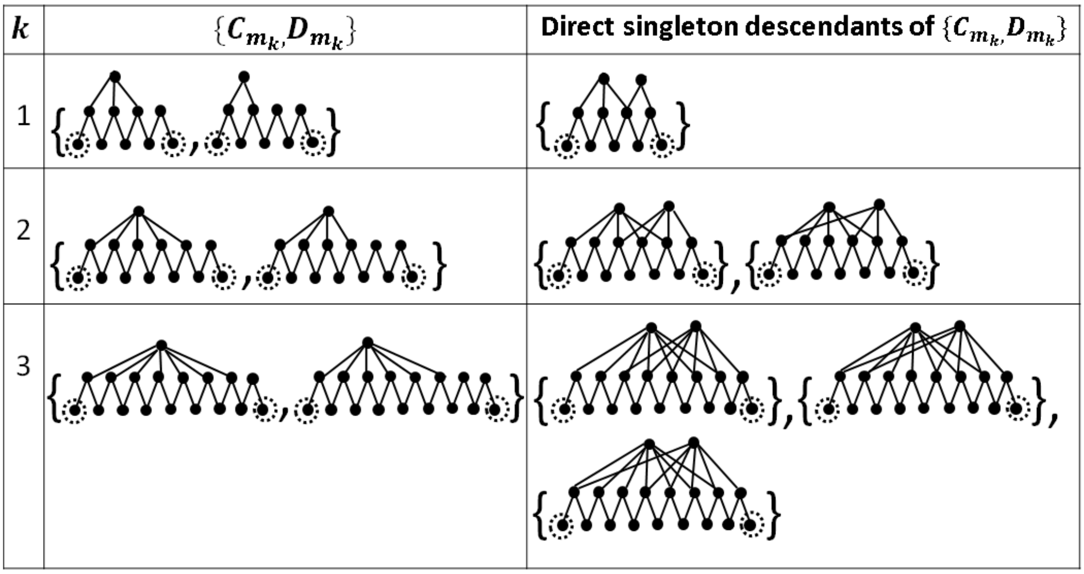


Figure 4.10: Illustration of property 9. The elements circled by a dotted line are identified with each other.

that the two elements which are missed out have a common element in their pasts. In this way, one generates a collection of $2k$ causets. One can show that each of these causets is isomorphic to exactly one other in the collection, and taking the corresponding orders gives k distinct (m_k+1)-orders whose set of m_k-stems is $\{C_{m_k}, D_{m_k}\}$. This proves property 9.

A key hurdle in the construction of covtree is understanding which sets of n-orders are covtree nodes. The following property gives a necessary condition in the case of doublets:

Property 10. *$\{A_n, B_n\}$ is a doublet in covtree only if there exists an $(n-1)$-order S which is a stem in both A_n and B_n.*

To prove property 10, let $\tilde{E}$ be a labeled minimal certificate of $\{A_n, B_n\}$. Let $\tilde{A}_n$ and $\tilde{B}_n$ be stems in $\tilde{E}$ which are isomorphic to representatives of A_n and B_n, respectively. Define $\tilde{S} := \tilde{A}_n \cap \tilde{B}_n$. We will show that $|\tilde{S}| = n-1$ and property 10 follows.

Note that $\tilde{E} = \tilde{A}_n \cup \tilde{B}_n$, for otherwise $\tilde{E}$ would not be minimal. Define $k := n - |\tilde{S}|$ and suppose for contradiction that $1 < k \leq n$. Let $\tilde{A}_n \setminus \tilde{B}_n = \{v_1, v_2, ..., v_k\}$ and $\tilde{B}_n \setminus \tilde{A}_n = \{w_1, w_2, ..., w_k\}$. Without loss of generality, let $\tilde{F} = \tilde{S} \cup \{v_1, v_2, ..., v_{k-1}, w_1\}$ be an n-stem in $\tilde{E}$. Then either $\tilde{F} \cong \tilde{A}_n$ or $\tilde{F} \cong \tilde{B}_n$. Suppose $\tilde{F} \cong \tilde{A}_n$. Then $\tilde{E} \setminus \{v_k\}$ is isomorphic to a labeled certificate of $\{A_n, B_n\}$ and is a stem in $\tilde{E}$, which contradicts the assumption that $\tilde{E}$ is minimal. Suppose $\tilde{F} \cong \tilde{B}_n$. Then $\tilde{E} \setminus \{w_2, w_3, ..., w_k\}$ is isomorphic to a labeled certificate of $\{A_n, B_n\}$ and is a stem in $\tilde{E}$, which is again a contradiction. Hence $|\tilde{S}| = n-1$, which completes the proof.

Property 11 is a corollary:

Property 11. *If Γ_n is a doublet in covtree then all minimal certificates of Γ_n are $(n+1)$-orders.*

Therefore, if Γ_n is a doublet in covtree and $\Gamma_n \prec \Gamma_{n+1}$ then Γ_{n+1} contains some minimal certificate of Γ_n. It is a corollary of properties 9 and 11 that for any integer $k \geq 1$ there exists a doublet in covtree with k minimal certificates.

4.3.2 Paths

Here we present properties of certain covtree paths and their certificates.

Property 12. *In covtree, there are infinite upward-going paths from the origin in which every node is a singleton.*

We call the subset of covtree which contains exactly all these paths "singtree", since it is a tree of singletons. Figure 4.11 shows the first three levels of singtree.

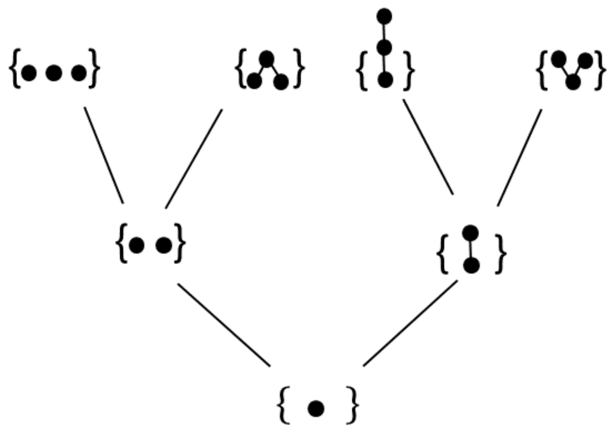


Figure 4.11: The first three levels of singtree.

To discuss singtree we will need the concept of the Newtonian order.

Definition 4.3.2 (Newtonian causet/order)**.** *A Newtonian causet is a causet in which every element in level k is above every element in level $k-1$. A Newtonian order is an order whose representatives are Newtonian.*

In a Newtonian causet, every pair of elements which are unrelated have the same past and the same future, alluding to a notion of a Newtonian global time, hence its name[15]. A Newtonian causet is a "stack of antichains", and for any natural number N, the union of the first N levels is a past of a break. The local finiteness condition implies that every level whose elements are not maximal must be finite.

To characterise the nodes of singtree and the certificates of singtree paths we will need the following lemma about Newtonian orders:

Lemma 4.3.3. *The following properties of a finite or infinite order, C, are equivalent:*

(i) C has a unique representative,

[15] A Newtonian order is *not* a good approximation of continuum Euclidean space.

(ii) for every natural number $n \leq |C|$ there is a unique n-order which is a stem in C,

(iii) C is Newtonian.

Proof. Let $\tilde{C}$ denote a representative of C. Let $L(x)$ denote the level of element x (*e.g.* $L(x) = 1$ if x is minimal).

$(i) \implies (ii)$ Suppose for contradiction that C has two n-stems, $C_n \neq C'_n$, for some $n \leq |C|$. Then there is a representative of C whose restriction to the interval $\{0, 1, ..., n-1\}$ is a representative of C_n, and similarly for C'_n. Hence C has at least two representatives. Contradiction.

$(ii) \implies (iii)$ Suppose for contradiction that C is not Newtonian, *i.e.* there exist some $x, y \in \tilde{C}$ such that $L(x) > L(y)$ and $x \not\succ y$. Let $\tilde{S}$ be the union of the inclusive past of x with levels $1, 2, ..., L(y)$. Note that $\tilde{S} \subset \tilde{C}$ is a stem in $\tilde{C}$. Then $\tilde{S} \setminus \{x\}$ and $\tilde{S} \setminus \{y\}$ are non-isomorphic stems in $\tilde{C}$. (To see that they are non-isomorphic, note that $\tilde{S} \setminus \{x\}$ contains $L(x) - 1$ levels and $\tilde{S} \setminus \{y\}$ contains $L(x)$ levels.) Hence C has at least two n-orders as stems, where $n = |\tilde{S}| - 1$. Contradiction.

$(iii) \implies (i)$ Let $f : \tilde{C} \to \tilde{C}'$ be an order-isomorphism between two representatives of C. The Newtonian condition restricts the action of f on each level in $\tilde{C}$ to be a permutation, and therefore f must be an automorphism. Hence C has a unique representative.

□

The equivalence of statements (ii) and (iii) in lemma 4.3.3 implies that:

Property 13. *A singleton $\{C_n\}$ is in singtree if and only if C_n is Newtonian.*

If $\{C_n\}$ is a node in singtree then it has exactly two direct descendants in singtree: $\{\widehat{C_n}\}$ and $\{D_{n+1}\}$, where D_{n+1} is the Newtonian order whose representative is constructed from a representative of C_n by adding a new element to its maximal level. If $\{C_n\}$ is a node in singtree then it has exactly three direct descendants in covtree: its singtree descendants, $\{\widehat{C_n}\}$ and $\{D_{n+1}\}$, and the doublet $\{\widehat{C_n}, D_{n+1}\}$.

A second corollary of lemma 4.3.3 is:

Property 14. *An infinite order C is Newtonian if and only if it is a certificate of a singtree path.*

Given property 14, it is now a simple matter to solve for the family of covtree dynamics in which the set of non-Newtonian orders is null: it is the set of covtree walks in which the walker stays in singtree with probability 1, *i.e.*

$$\mathbb{P}(\Gamma_n) = 0 \ \forall \ \Gamma_n \text{ not in singtree.} \tag{4.10}$$

This family of Newtonian dynamics acts as a proof of principle, illustrating how an understanding of covtree could allow one to solve for a dynamics with particular features. But, since these dynamics are unphysical, this is very much a case of "looking under the lamp-post". Where are we to look if not under the lamp-post? One avenue for exploration is to ask: what role, if any, do rogues play in the physics of covtree walks?

Since in CSG models the set of rogues is null [Brightwell et al., 2003], identifying covtree dynamics which possess this property is a step towards understanding what form CSG dynamics take on covtree. Moreover, if following [Brightwell et al., 2003] we are to choose $\mathcal{R}$ to be our σ-algebra of observables then—unless the covtree measure on $\mathcal{R}(\mathcal{S})$ has a unique extension to $\mathcal{R}$—one is faced with ambiguities both in interpretation and calculation. It is sufficient that the set of rogues be null for there to exist a unique extension, and therefore rogue-free dynamics are compatible with this approach. Finally, rogue-free dynamics are cosmologically interesting. A rogue causet contains an infinite level and as a result cannot contain an infinite sequence of posts or breaks. Therefore, that the dynamics is rogue-free is a necessary condition for the dynamics to be relevant for our cosmological paradigm.

One can draw an analogy between the condition that the set of rogues is null and the condition that the set of non-Newtonian orders is null: the former is the condition that the set of paths with more than one certificate is null, the latter the condition that the set of paths with more than one *labeled* certificate is null. However, while we were able to solve for the latter, solving for the former poses a new challenge because it is a limiting condition: at no finite stage of the covtree walk can the claim that the growing order is a rogue be verfied or falsified. This is because for every node in covtree there exist both an infinite certificate which is a rogue and an infinite certificate which is not a rogue.

This means that there is no rogue analogue to singtree. Instead, we must look for other ways to obtain rogue-free dynamics. Pursuing the strictly stronger condition that the dynamics gives rise to infinitely many posts or breaks with unit probability is a promising route. We already know the defining feature of these covtree walks: that, with unit probability, they pass through infinitely many nodes of the form $\{\widehat{A}\}$. The challenge ahead is to formulate this feature in terms of covtree transition probabilities.

4.A Supplementary material

Alternative proof to theorem 4.1.10. Recall that $\sim_R$ denotes the rogue equivalence relation, and let $p : \Omega \to \Omega/\sim_R$ be the associated canonical quotient map. Let $[A]_R$ denote an element of $\Omega/\sim_R$, where $A \in \Omega$ is a representative of $[A]_R$.

Define the following metric on $\Omega/\sim_R$: $d([A]_R, [B]_R) = \frac{1}{2^n}$, where n is the highest integer such that the set of n-stems of A is the set of n-stems of B. One can show that $(\Omega/\sim_R, d)$ is a complete metric space.

Let $[cert(\Gamma_n)]_R \subset \Omega/\sim_R$ denote the image of the certificate set $cert(\Gamma_n) \subset \Omega$ under the quotient map p. Then one can show that $[cert(\Gamma_n)]_R$ is both open[16] and closed in $(\Omega/\sim_R, d)$.

The diameter of $[cert(\Gamma_n)]_R$, $d([cert(\Gamma_n)]_R)$, is defined to be the maximum distance between any two elements in $[cert(\Gamma_n)]_R$ and is equal to $\frac{1}{2^n}$. Hence $d([cert(\Gamma_n)]_R) \to 0$ as $n \to \infty$.

Given a path in covtree, $\mathcal{P} = \Gamma_1 \prec \Gamma_2 \prec ...$, then the following is a nested sequence: $[cert(\Gamma_1)]_R \supset [cert(\Gamma_2)]_R \supset \dots$.

Now, Cantor's Lemma states that a metric space (X, d) is complete if and only if, for every nested sequence $\{F_n\}_{n\geq 1}$ of nonempty closed subsets of X, that is, (a) $F_1 \supseteq F_2 \supseteq \dots$ and (b) $d(F_n) \to 0$ as $n \to \infty$, the intersection $\bigcap_{n=1}^{\infty} F_n$ contains one and only one point [Shirali and Vasudeva, 2006]. □

[16] $[cert(\Gamma_n)]_R$ are exactly the open balls under the metric topology.

Chapter 5

Quantum dynamics

If spacetime is itself quantal (rather than a classical background for quantum matter) then it is to be governed by a quantum dynamics. In the causal set approach, discreteness and non-locality resist the canonical Hamiltonian time-evolution. Instead, we seek to "quantize" the growth dynamics via a formulation of quantum mechanics which does not rely on *states* but on *histories*[1] by means of a decoherence functional [Dowker et al., 2010b; Halliwell, 1995; Hartle, 1992; Sorkin, 1994],

Definition 5.0.1 (Decoherence functional)**.** *Let X be a space of histories, and let $\mathfrak{A}$ be an event algebra of subsets of X. A decoherence functional is a complex function $D : \mathfrak{A} \times \mathfrak{A} \to \mathbb{C}$ which is*

1. *finitely-biadditive: for any $\alpha, \beta, \gamma \in \mathfrak{A}$ where $\alpha \cap \beta = 0$*

$$D(\alpha + \beta, \gamma) = D(\alpha, \gamma) + D(\beta, \gamma) \text{ and } D(\gamma, \alpha + \beta) = D(\gamma, \alpha) + D(\gamma, \beta);$$

2. *Hermitian: for any $\alpha, \beta \in \mathfrak{A}$ $D(\alpha, \beta) = D(\beta, \alpha)^*$;*

3. *normalised: $D(X, X) = 1$;*

4. *strongly positive: for any finite subset $\{\alpha_i\}$ of $\mathfrak{A}$ the matrix $M_{ij} \equiv D(\alpha_i, \alpha_j)$ is positive semi-definite (i.e. M_{ij} has only non-negative eigenvalues).*

[1] A state in a Hilbert space gives a description of reality at a particular moment in time (*e.g.* at time t the particle is at spatial position x), while a history is a complete description of reality (*e.g.* a complete trajectory of a particle).

The decoherence functional encodes the quantum mechanical interference between pairs of events. That it has two arguments is a reflection of the nature of quantum mechanics, namely that interference is displayed between pairs of events but not between members of a larger collection of events[2].

Obtaining a decoherence functional for causal sets has been a longstanding obstruction in the path to quantum gravity. A proposal for deriving a decoherence functional from the CSG models was first put forward in [Dowker et al., 2010c]. In this proposal, the t_k parameters take complex values and give rise to a complex measure from which a decoherence functional can be obtained. But in the same paper it was shown by example that there are cases in which this procedure fails because the complex measure does not extend to the full σ-algebra and it remained unclear whether there were any cases in which the procedure was successful.

In the remainder of this chapter we review the proposal and the problem of extension (section 5.1), provide criteria for when the proposal gives rise to a decoherence functional on the labeled σ-algebra $\tilde{\mathcal{R}}$ (section 5.2) and apply our criteria to provide the first known examples of decoherence functionals, and hence of quantum dynamics for causal sets (section 5.3). We conclude by generalising our results to a broad class of σ-algebras (section 5.4).

5.1 A decoherence functional from CSG models

In this section we review the proposal of [Dowker et al., 2010c] for deriving a decoherence functional from the CSG models. We conclude with a statement about the range of validity of the proposal.

We begin by defining the "complex CSG models". Complex CSG models are a generalisation of the CSG models in which the t_k are allowed to take complex values. Thus the real transition probabilities $\mathbb{P}(\tilde{C}_n \to \tilde{C}_{n+1})$ are replaced by complex transition amplitudes

[2]Classical stochastic phenomena do not display any interference so they can be described by a function whose argument is a single event, *i.e.* by a probability measure.

$A(\tilde{C}_n \to \tilde{C}_{n+1})$ of the form,

$$A(\tilde{C}_n \to \tilde{C}_{n+1}) = \frac{\lambda(\varpi, m)}{\lambda(n, 0)}, \tag{5.1}$$

where $\lambda(k,p)$ is as defined in equation (3.2). The amplitude to reach a given $\tilde{C}_n$ by stage n is denoted by $A(\tilde{C}_n)$ and is equal to the product of transition amplitudes along the path from the empty set to $\tilde{C}_n$. As with the (real) CSG models, $(t_0, t_1, t_2, ...)$ is a projective parametrisation and we will assume $t_0 = 1$, unless explicitly stated otherwise. Note that equation (5.1) is well-defined only when $\lambda(n,0) \neq 0$, *i.e* when,

$$\sum_{k=0}^{n} \binom{n}{k} t_k \neq 0. \tag{5.2}$$

We will only consider complex CSG models in which inequality (5.2) holds for all n. In these models, the amplitudes satisfy the sum rule,

$$\sum_i A(\tilde{C}_n \to \tilde{C}^i_{n+1}) = 1, \tag{5.3}$$

where i labels the children of $\tilde{C}_n$. Equation (5.3) is reminiscent of the classical Markov-sum-rule and we define,

Definition 5.1.1 (Complex Markovian dynamics)**.** *A complex Markovian dynamics is a complete set of transition amplitudes $A(\tilde{C}_n \to \tilde{C}_{n+1})$ satisfying the sum rule* (5.3).

The complex CSG models are the subset of complex Markovian dynamics satisfying (5.1).

Let $\mathfrak{a}$ denote the collection of finite unions of cylinder sets, and note that it is an event algebra (but not a σ-algebra). Given a complex Markovian dynamics one can define the complex measure $\tilde{\mu}_v$ on $\mathfrak{a}$ via,[3]

$$\tilde{\mu}_v(cyl(\tilde{C}_n)) := A(\tilde{C}_n) \; \forall \tilde{C}_n \in \tilde{\Omega}(\mathbb{N}). \tag{5.4}$$

If $\tilde{\mu}_v$ extends to a complex measure $\tilde{\mu}_{v,ex}$ on $\tilde{\mathcal{R}}$, then one can define a decoherence

[3]The subscript v is used to distinguish $\tilde{\mu}_v$ from the real probability measure given in (3.12).

functional on $\tilde{\mathcal{R}} \times \tilde{\mathcal{R}}$,

$$D(\tilde{\mathcal{E}}, \tilde{\mathcal{H}}) := \tilde{\mu}^*_{v,ex}(\tilde{\mathcal{E}})\tilde{\mu}_{v,ex}(\tilde{\mathcal{H}}) \ \forall \ \tilde{\mathcal{E}}, \tilde{\mathcal{H}} \in \tilde{\mathcal{R}}, \tag{5.5}$$

where * denotes complex conjugation. The key phrase here is *if $\tilde{\mu}_v$ extends*, and indeed it was shown in [Dowker et al., 2010c] that there are complex CSG models in which it does not. When does $\tilde{\mu}_v$ possess an extension? We begin with a definition.

Definition 5.1.2 (Variation and bounded variation)**.** *Let X be a space of histories, and let $\mathfrak{A}$ be an event algebra of subsets of X. Let F be a complex measure on $\mathfrak{A}$. The variation of F is the extended non-negative function $|F|$ whose value on a set $\alpha \in \mathfrak{A}$ is given by*

$$|F|(\alpha) = \sup_{\pi} \sum_{\beta \in \pi} |F(\beta)|, \tag{5.6}$$

where the supremum is taken over all partitions π of α into a finite number of pairwise disjoint members of $\mathfrak{A}$. If $|F|(X) < \infty$ then F is said to be of bounded variation.

We can now state our answer:

Lemma 5.1.3. *$\tilde{\mu}_v$ extends to a complex measure $\tilde{\mu}_{v,ex}$ on $\tilde{\mathcal{R}}$ if and only if $\tilde{\mu}_v$ is of bounded variation. When the extension exists, it is unique.*

We leave the proof to appendix 5.A.

5.2 Two theorems

In this section, we develop criteria for $\tilde{\mu}_v$ to be of bounded variation in a complex Markovian dynamics (theorem 5.2.1) and in CSG models (theorem 5.2.7).

Given an n-causet $\tilde{C}_n$, we denote its children by $\tilde{C}^i_{n+1}$, where i labels the individual children. For each complex Markovian dynamics, define the mapping ζ from the set of finite labeled causets to the non-negative real numbers,

$$
\begin{aligned}
&\zeta : \tilde{\Omega}(\mathbb{N}) \to \mathbb{R}^+, \\
&\zeta(\tilde{C}_n) := \sum_i |A(\tilde{C}_n \to \tilde{C}^i_{n+1})| - |\sum_i A(\tilde{C}_n \to \tilde{C}^i_{n+1})| \\
&\qquad\quad = \sum_i |A(\tilde{C}_n \to \tilde{C}^i_{n+1})| - 1,
\end{aligned} \tag{5.7}
$$

where the equality follows from (5.3). For each $n > 0$ we define the maximum and minimum of ζ over $\tilde{\Omega}(n)$,

$$
\begin{aligned}
\zeta_n^{max} &:= \max_{\tilde{C}_n \in \tilde{\Omega}(n)} \zeta(\tilde{C}_n), \\
\zeta_n^{min} &:= \min_{\tilde{C}_n \in \tilde{\Omega}(n)} \zeta(\tilde{C}_n).
\end{aligned} \tag{5.8}
$$

We can now state our first theorem.

Theorem 5.2.1. *In a complex Markovian dynamics, if* $\sum_{n=1}^{\infty} \zeta_n^{max}$ *converges then* $\tilde{\mu}_v$ *is of bounded variation. If* $\sum_{n=1}^{\infty} \zeta_n^{min}$ *diverges then* $\tilde{\mu}_v$ *is not of bounded variation.*

In order to develop some intuition, let us restrict our attention to complex CSG models. Then one can think of the function ζ as a measure of how "different" a complex CSG model is to a (non-complex) CSG model. Roughly speaking, the larger the relative phases between the couplings—the further the departure of a complex CSG model from the (non-complex) CSG models—the larger ζ becomes. When all the t_k have vanishing phases then $\zeta(\tilde{C}_n) = 0$ for all $\tilde{C}_n$. In this case $\sum_{n=1}^{\infty} \zeta_n^{max} = 0$, $\tilde{\mu}_v$ is a real probability measure of bounded variation and a decoherence functional on $\tilde{\mathcal{R}} \times \tilde{\mathcal{R}}$ can be defined. These models define a quantum dynamics in the sense that there is intereference between pairs of alternatives (*i.e.* the decoherence functional is not diagonal). But since all the amplitudes have vanishing phases these models only exhibit *constructive* interference. Therefore they cannot display the typical quantum behaviour which arises from *destructive* interference (cf. the double-slit experiment)[4]. For this reason, the interesting cases are those in which some of the t_k have a non-vanishing phase. When some of the t_k have

[4]Another way to state this is that in these dynamics there is no set of zero measure which has a susbset of non-zero measure.

non-vanishing phases then $\zeta(\tilde{C}_n) > 0$ for some or all of the $\tilde{C}_n$. Therefore $\sum_{n=1}^{\infty} \zeta_n^{min}$ may diverge in which case one cannot define a decoherence functional via equation (5.5). Theorem 5.2.1 makes this notion precise by quantifying the behaviour that ζ must show in order for $\tilde{\mu}_v$ to be of bounded variation.

We now prove theorem 5.2.1 through a series of lemmas. Recall that $\tilde{\Omega}$ is the set of infinite labeled causets and $\mathfrak{a}$ is the event algebra which contains all finite unions of cylinder sets. Let $\tilde{\mathcal{E}}$ denote an element of $\mathfrak{a}$. We refer to a cylinder set $cyl(\tilde{C}_n)$ as a "cylinder set at level n". Let π denote a partition of $\tilde{\Omega}$ into a finite number of pairwise disjoint members of $\mathfrak{a}$. For each $n > 0$, define,

$$S_n := \sum_{\tilde{C}_n \in \tilde{\Omega}(n)} |\tilde{\mu}_v(cyl(\tilde{C}_n))| = \sum_{\tilde{C}_n \in \tilde{\Omega}(n)} |A(\tilde{C}_n)|. \tag{5.9}$$

Lemma 5.2.2. *For any finite partition π of $\tilde{\Omega}$ there exists an $n \in \mathbb{N}$ such that $S_n \geq \sum_{\tilde{\mathcal{E}} \in \pi} |\tilde{\mu}_v(\tilde{\mathcal{E}})|$.*

Proof. We note that for any $\tilde{\mathcal{E}} \in \mathfrak{a}$ there exists a $p \in \mathbb{N}$ such that $\tilde{\mathcal{E}}$ can be written as a union of cylinder sets at level $m \, \forall \, m > p$. Therefore, there exists an $n \in \mathbb{N}$ such that every $\tilde{\mathcal{E}} \in \pi$ can be written as a union of cylinder sets at level n. Therefore, $\sum_{\tilde{\mathcal{E}} \in \pi} |\tilde{\mu}_v(\tilde{\mathcal{E}})| \leq S_n$, where the inequality is saturated in the special cases when $\tilde{\mu}_v(\tilde{\mathcal{E}}) \in \mathbb{R}^+ \, \forall \, \tilde{\mathcal{E}} \in \pi$ or when $\pi = \{cyl(\tilde{C}_n) | \tilde{C}_n \in \tilde{\Omega}(n)\}$. □

Since $|\tilde{\mu}_v|(\tilde{\Omega}) = \sup_\pi \sum_{\tilde{\mathcal{E}} \in \pi} |\tilde{\mu}_v(\tilde{\mathcal{E}})|$ (definition 5.1.2), a direct corollary of lemma 5.2.2 is that $|\tilde{\mu}_v|(\tilde{\Omega}) = \sup_n S_n$ and therefore,

Corollary 5.2.3. $|\tilde{\mu}_v|(\tilde{\Omega}) < \infty$ *if and only if* $\sup_n S_n < \infty$.

Corollary 5.2.3 relates $|\tilde{\mu}_v|$ to the S_n. In the following, we translate this into a relation between $|\tilde{\mu}_v|$ and ζ.

Lemma 5.2.4. $(1 + \zeta_{n-1}^{min}) S_{n-1} \leq S_n \leq (1 + \zeta_{n-1}^{max}) S_{n-1}$.

Proof.

$$\begin{aligned} S_n &= \sum_{\tilde{C}_n \in \tilde{\Omega}(n)} |A(\tilde{C}_n)| \\ &= \sum_{\tilde{C}_{n-1} \in \tilde{\Omega}(n-1)} |A(\tilde{C}_{n-1})| \sum_i |A(\tilde{C}_{n-1} \to \tilde{C}_n^i)| \\ &= \sum_{\tilde{C}_{n-1} \in \tilde{\Omega}(n-1)} |A(\tilde{C}_{n-1})|(1 + \zeta(\tilde{C}_{n-1})), \end{aligned} \tag{5.10}$$

where it is understood that for each $\tilde{C}_{n-1} \in \tilde{\Omega}(n-1)$ the label i labels the children of it (so that i depends on $\tilde{C}_{n-1}$ and the sums in the second line of equation (5.10) cannot be commuted).

Therefore,

$$\begin{aligned} S_n &\leq \sum_{\tilde{C}_{n-1} \in \tilde{\Omega}(n-1)} |A(\tilde{C}_{n-1})|(1 + \zeta_{n-1}^{max}) \\ \implies S_n &\leq (1 + \zeta_{n-1}^{max}) S_{n-1}. \end{aligned} \tag{5.11}$$

Similarly,

$$\begin{aligned} S_n &\geq \sum_{\tilde{C}_{n-1} \in \tilde{\Omega}(n-1)} |A(\tilde{C}_{n-1})|(1 + \zeta_{n-1}^{min}) \\ \implies S_n &\geq (1 + \zeta_{n-1}^{min}) S_{n-1}. \end{aligned} \tag{5.12}$$

□

Lemma 5.2.4 implies that the sequence $\{S_n\}_{n>0}$ is monotonically increasing over n and therefore,

Corollary 5.2.5. $|\tilde{\mu}_v|(\tilde{\Omega}) < \infty$ *if and only if* $lim_{n\to\infty} S_n < \infty$.

All that remains in order to prove theorem 5.2.1 is to show that if $\sum_{n=1}^{\infty} \zeta_n^{max}$ converges then $lim_{n\to\infty} S_n < \infty$, and if $\sum_{n=1}^{\infty} \zeta_n^{min}$ diverges then $lim_{n\to\infty} S_n = \infty$.

Lemma 5.2.6. $\prod_{r=1}^{n-1}(1 + \zeta_r^{min}) \leq S_n \leq \prod_{r=1}^{n-1}(1 + \zeta_r^{max})$.

Proof. By lemma 5.2.4 we have,

$$\begin{aligned} S_n &\leq (1+\zeta_{n-1}^{max})S_{n-1} \\ &\leq (1+\zeta_{n-1}^{max})(1+\zeta_{n-2}^{max})S_{n-2} \\ &\ \vdots \\ &\leq \prod_{r=1}^{n-1}(1+\zeta_r^{max})S_1. \end{aligned} \tag{5.13}$$

Since $S_1 = 1$ we arrive at the result that $S_n \leq \prod_{r=1}^{n-1}(1+\zeta_r^{max})$. That $\prod_{r=1}^{n-1}(1+\zeta_r^{min}) \leq S_n$ can be similarly derived. □

Given an infinite product $\prod_1^\infty(1+a_n)$, if all a_n are positive, or all are negative, a necessary and sufficient condition for the convergence of $\prod_1^\infty(1+a_n)$ is that $\sum a_n$ shall converge [Jeffreys and Swirles, 1966]. Therefore we have,

$$\begin{aligned} &\sum_{n=1}^{\infty}\zeta_n^{max} < \infty \implies lim_{n\to\infty}S_n < \infty \implies |\tilde{\mu}_v|(\tilde{\Omega}) < \infty, \\ &\sum_{n=1}^{\infty}\zeta_n^{min} = \infty \implies lim_{n\to\infty}S_n = \infty \implies |\tilde{\mu}_v|(\tilde{\Omega}) = \infty, \end{aligned} \tag{5.14}$$

where in each line, the first implication (“ $\implies$ ”) is by lemma 5.2.6 and the second by corollary 5.2.5. This proves theorem 5.2.1.

Next, we restrict ourselves to the complex CSG models and use the functional dependence of the amplitude A on the t_k (5.1) to identify ζ_n^{max} and ζ_n^{min} for all n. Let $\tilde{C}_n^a$ and $\tilde{C}_n^c$ denote the n-antichain and n-chain, respectively. Define the abbreviations,

$$\begin{aligned} \zeta_n^a &:= \zeta(\tilde{C}_n^a) \\ \zeta_n^c &:= \zeta(\tilde{C}_n^c) \end{aligned} \tag{5.15}$$

Theorem 5.2.7. *In any complex CSG model, $\zeta_n^{max} = \zeta_n^a$ and $\zeta_n^{min} = \zeta_n^c$.*

Proof. First, we show that $\zeta_n^{max} = \zeta_n^a$. Given some parent $\tilde{C}_n$, every choice of proto-past in $\tilde{C}_n$ leads to a unique transition if and only if $\tilde{C}_n = \tilde{C}_n^a$. Hence we have,

$$\zeta_n^a = -1 + \frac{\sum_{k=0}^n \binom{n}{k}|t_k|}{|\lambda(n,0)|} \geq \zeta(\tilde{C}_n) \ \forall \tilde{C}_n \in \tilde{\Omega}(n), \tag{5.16}$$

where the inequality follows from the triangle inequality.

Next, we proceed by induction to show that $\zeta_n^{min} = \zeta_n^c$. One can show that $\zeta_n^{min} = \zeta_n^c$ when $n = 1, 2$. Suppose true for all $n = 1, 2, 3, ..., r-1$, and note that

$$\begin{aligned} \zeta_r^c &= -1 + \frac{|\lambda(r-1,0)|}{|\lambda(r,0)|}(\zeta_{r-1}^{min} + 1) + \frac{|U|}{|\lambda(r,0)|}, \\ U &= \sum_{k=0}^{r-1} \binom{r-1}{k} t_{k+1}, \end{aligned} \tag{5.17}$$

where U is the relative probability of the transition $\tilde{C}_r^c \to \tilde{C}_{r+1}^c$ in which the new element r is born above the element $r-1$.

Consider some $\tilde{C}_r \in \tilde{\Omega}(r)$. Let $\tilde{C}_{r-1}$ denote the restriction of $\tilde{C}_r$ to $\{0, 1, ..., r-2\}$ and note that $\tilde{C}_{r-1}$ is a stem in $\tilde{C}_r$. Then we have,

$$\begin{aligned} \zeta(\tilde{C}_r) &= -1 + \frac{|\lambda(r-1,0)|}{|\lambda(r,0)|}(\zeta(\tilde{C}_{r-1}) + 1) + \frac{|V|}{|\lambda(r,0)|} \\ &\geq -1 + \frac{|\lambda(r-1,0)|}{|\lambda(r,0)|}(\zeta_{r-1}^{min} + 1) + \frac{|V|}{|\lambda(r,0)|}, \end{aligned} \tag{5.18}$$

where on the first line, the last term on the RHS contains the contributions from all transitions in which the new element r is born above the element $r-1$. If $\tilde{C}_r$ has a single maximal element then $V = U \implies \zeta(\tilde{C}_r) \geq \zeta_r^c$. If $\tilde{C}_r$ has at least two maximal elements it follows from the triangle inequality that $|V| \geq |U| \implies \zeta(\tilde{C}_r) \geq \zeta_r^c$. This completes the proof. □

5.3 Examples

We now apply theorems 5.2.1 and 5.2.7 to determine whether various families of complex CSG models give rise to a decoherence functional.

Dynamics with vanishing phases: As we have seen in the previous section, when all the t_k have vanishing phases then $\zeta(\tilde{C}_n) = 0$ for all $\tilde{C}_n$. Therefore, in these cases a decoherence functional on $\tilde{\mathcal{R}} \times \tilde{\mathcal{R}}$ can be defined.

Dynamics with two non-zero couplings: The simplest case with non-vanishing phases occurs when only two of the couplings are non-vanishing: $t_0 = 1$, $t_k = se^{i\phi}$ ($s > 0$ and $0 \leq \phi < 2\pi$) for some $k > 0$ and $t_l = 0$ otherwise. In this case,

$$\zeta_n^a = \frac{1 + \binom{n}{k}s}{\sqrt{1 + 2s\binom{n}{k}\cos\phi + s^2\binom{n}{k}^2}} - 1, \tag{5.19}$$

and it can be shown by the comparison test against $\sum \frac{1}{n^x}$ (with $x > 1$) that $\sum \zeta_n^a$ converges for all values of s and ϕ when $k \geq 2$. Let us sketch the proof. Since $\zeta_n^a \geq 0$, $\zeta_n^a \leq \frac{1}{n^x}$ if and only if,

$$\left(-\frac{2}{n^x} - \frac{1}{n^{2x}}\right)\left(1 + s^2\binom{n}{k}^2\right) + 2s\binom{n}{k}\left((1 - \cos\phi) - \left(-\frac{2}{n^x} - \frac{1}{n^{2x}}\right)\cos\phi\right) \leq 0. \tag{5.20}$$

When $n \gg k$, $\binom{n}{k} \sim \frac{n^k}{k!}$ and the dominant contribution to the LHS is

$$\approx \frac{2s}{k!}n^k\left(-\frac{s}{k!}n^{k-x} + (1 - \cos\phi)\right), \tag{5.21}$$

which is negative for large n if $k > x > 1$ for all values of s and ϕ. Indeed one can show that given some s, ϕ and $k > 1$ there exists some $N(k, s, \phi) \in \mathbb{N}$ such that $\zeta_n^a \leq \frac{1}{n^x}$ for all $n > N(k, s, \phi)$ and some $x > 1$. Thus, the decoherence functional is well-defined for any choice of t_k as long as $k \geq 2$.

When $k = 1$, it can be shown by the comparison test against $\sum \frac{1}{n^x}$ (with $x < 1$) that $\sum \zeta_n^c$ diverges when $\phi \neq 0$. In this case,

$$\zeta_n^c = \frac{ns + 1}{\sqrt{1 + n^2s^2 + 2ns\cos(\phi)}} - 1, \tag{5.22}$$

and the dominant contribution at large n scales as $\frac{1}{n}$. A decoherence functional cannot be defined in this case.

The qualitative difference between the $k > 1$ case which gives rise to a decoherence functional and the $k = 1$ case which does not stems from the fact that for large n the number of 2-element subsets of an n-causet is significantly greater than the number of 1-element subsets [Georgiou, 2005]. When $k > 1$, the amplitudes at large n are dominated by contributions from t_k and the effect of the phase difference between t_0 and t_k becomes negligible, leading to small ζ values. When $k = 1$, the number of 1-element subsets does not grow fast enough with n to drown out the contribution from t_0 and the phase difference between t_0 and t_1 registers as large ζ values.

Dynamics with N non-zero couplings: Our example of two non-zero couplings can be generalised to any finite number of non-zero couplings. Consider a complex CSG model with N non-zero couplings $t_0, t_{k_1}, t_{k_2}, \dots t_{k_N}$, where $k_N > k_{N-1} \dots > k_1 > 0$. For each i, let $t_{k_i} = s_i e^{i\phi_i}$. In this case,

$$\zeta_n^a = \frac{1 + \sum_{i=1}^N \binom{n}{k_i} s_i}{|1 + \sum_{i=1}^N \binom{n}{k_i} s_i e^{i\phi_i}|} - 1, \tag{5.23}$$

and $\zeta_n^a \leq \frac{1}{n^x}$ (with $x > 1$) if and only if,

$$\left(-\frac{2}{n^x} - \frac{1}{n^{2x}}\right)\left(1 + \sum_i \binom{n}{k_i}^2 s_i^2\right) + 2\sum_i \binom{n}{k_i} s_i \left(1 - \cos\phi_i + \left(-\frac{2}{n^x} - \frac{1}{n^{2x}}\right)\cos\phi_i\right)$$
$$+ 2\sum_{i,j,i\neq j} \binom{n}{k_i}\binom{n}{k_j} s_i s_j \left(1 - \cos(\phi_i - \phi_j) + \left(-\frac{2}{n^x} - \frac{1}{n^{2x}}\right)\cos(\phi_i - \phi_j)\right) \leq 0. \tag{5.24}$$

For $n \gg N > 1$, the dominant contributions to the LHS arise from t_{k_N} and $t_{k_{N-1}}$ and are given by,

$$-\frac{2s_N^2}{(k_N!)^2} n^{2k_N - x} + \frac{2s_N s_{N-1}}{k_N! k_{N-1}!} n^{k_N + k_{N-1}} (1 - \cos(\phi_N - \phi_{N-1})), \tag{5.25}$$

which is negative when $k_N - k_{N-1} > x$ for all values of s_i, ϕ_i. Choosing $1 < x < 2$, we see that a decoherence functional can be defined whenever $k_N - k_{N-1} > 1$. Intuitively, the condition that $k_N - k_{N-1} > 1$ ensures that the difference between the number of N-element and $(N-1)$-element subsets of an n-causet $(n > N)$ is large enough for t_N to dominate so that the difference in the phases of t_{k_N} and $t_{k_{N-1}}$ is ironed out leading to ζ

values small enough for convergence.

Dynamics with an infinite number of non-zero couplings: We can use the intuition we developed in the previous cases to infer the behaviour of ζ in complex CSG models which have infinitely many non-zero couplings. If there exists an N such that all the t_k with $k > N$ have the same phase (or are vanishing) then at $n > N$ the contributions to the transition amplitudes will be dominated by collinear couplings leading to small ζ values and a well-defined decoherence functional. More generally, to define a decoherence functional the phase difference between t_k and t_{k+1} has to go to zero fast enough with k.

Complex Transitive Percolation: Complex Transitive Percolation (CTP) is defined by $t_k = t^k$ for some constant $t \in \mathbb{C}$. Therefore, when t is not real and positive, the phase of t_k oscillates with k, the phase difference between t_k and t_{k+1} does not approach zero and a decoherence functional cannot be defined. Indeed, it was shown in [Dowker et al., 2010c] that a decoherence functional cannot be defined in this case, and our results shed light on why this is the case. To confirm that our intuition is correct and to give a new proof of this result, we sketch how in this case $\lim_{n\to\infty} \zeta_n^{min} > 0$ and hence $\sum_n \zeta_n^{min}$ diverges. We can write $t = \frac{1-q}{q}$, $q \in \mathbb{C}$, where t is real and positive unless (i) $q \in \mathbb{C}$ and $|q| > 1$, (ii) $q \notin \mathbb{R}^+$ and $|q| = 1$, or (iii) $q \notin \mathbb{R}^+$ and $|q| < 1$. The minimum of ζ is given by,

$$\zeta_n^{min} = -1 + |q|^n + |q|^{n-1}|1-q| \sum_{k=0}^{n-1} |q|^{-k}, \tag{5.26}$$

and therefore in case (i),

$$\zeta_n^{min} = (1-|q|^n)\left(\frac{|1-q|}{1-|q|} - 1\right) \implies \lim_{n\to\infty} \zeta_n^{min} = -sign\left(\frac{|1-q|}{1-|q|} - 1\right)\infty = \infty, \tag{5.27}$$

in case (ii),

$$\zeta_n^{min} = n|1-q| \implies \lim_{n\to\infty} \zeta_n^{min} = \infty, \tag{5.28}$$

and in case (iii),

$$\zeta_n^{min} = (1 - |q|^n)(\frac{|1-q|}{1-|q|} - 1) \implies \lim_{n\to\infty} \zeta_n^{min} = \frac{|1-q|}{1-|q|} - 1 > 0. \tag{5.29}$$

5.4 Beyond labeled poscau

As defined in 5.1.1, complex Markovian dynamics are complex random walks on *labeled poscau*. We now generalise the usage of the term "complex Markovian dynamics" by allowing to replace labeled poscau with any tree $\mathscr{T}$ which (i) contains no maximal elements, and (ii) in which every node has a finite valency. A complex Markovian dynamics on $\mathscr{T}$ is a complete set of complex transition amplitudes $A(x \to y)$ satisfying $\sum_y A(x \to y) = 1$ for all $x, y \in \mathscr{T}$ and y directly above x.

Given any $\mathscr{T}$ and a complex Markovian dynamics, define,

$$\begin{aligned} &\zeta : \mathscr{T} \to \mathbb{R}^+, \\ &\zeta(x) := \sum_y |A(x \to y)| - 1, \end{aligned} \tag{5.30}$$

$$\begin{aligned} \zeta_n^{max} &:= \max_{x\in\mathscr{T}_n} \zeta(x), \\ \zeta_n^{min} &:= \min_{x\in\mathscr{T}_n} \zeta(x), \end{aligned} \tag{5.31}$$

where $\mathscr{T}_n \subset \mathscr{T}$ is the set of nodes at level n. With these definitions, we can now interpret theorem 5.2.1 in the context of a complex Markovian dynamics on any $\mathscr{T}$. The generalised proof can be obtained from the proof in the text by replacing labeled poscau's sample space, cylinder sets and nodes with those of $\mathscr{T}$.

This generalisation paves the way for novel quantum dynamics. In particular, one could consider complex Markovian dynamics on covtree, although currently this remains a purely formal development. We discuss applications of the generalised theorem to other tree structures in section 6.2.

5.A Supplementary material

Proof of lemma 5.1.3: Let F be a complex measure on an algebra $\mathfrak{A}$. If F has an extension to the σ-algebra generated by $\mathfrak{A}$ then F has bounded variation [Dowker et al., 2010c; Rudin, 1987]. We now show that bounded variation is also sufficient for extension. The Caratheodory-Hahn-Kluvanek (CHK) extension theorem (theorem 2 in [Diestel and Uhl, 1977]) states that a bounded weakly countably additive vector measure $F : \mathfrak{A} \to X$ has a unique countably additive extension $\bar{F} : \Sigma \to X$ to the σ-algebra Σ generated by $\mathfrak{A}$ if and only if F is strongly additive. Note that bounded variation $\implies$ strong additivity $\implies$ boundedness [Diestel and Uhl, 1977]. Note also that countable additivity $\implies$ weak countable additivity. Finally in any complex Markovian dynamics, $\tilde{\mu}_v$ is countably additive since no cylinder set can be written as a countable disjoint union of cylinder sets [Lindstrøm, 2017]. Therefore, when $\tilde{\mu}_v$ has bounded variation then $\tilde{\mu}_v$ is a bounded weakly countably additive vector measure and it is strongly additive, and therefore it possesses a unique extension by the CHK extension theorem. □

Chapter 6

Labeled dynamics with posts and breaks

In causal set theory, cycles of cosmic expansion and collapse are modeled by sequences of posts and breaks and a special role is played by dynamics which favour cyclic causal set spacetimes [Ahmed and Rideout, 2010; Ash and McDonald, 2003, 2005; Bombelli et al., 2008; Dowker and Zalel, 2017; Martin et al., 2001; Sorkin, 2000].

This chapter is dedicated to the occurrence of posts and breaks in labeled dynamics (where by a "labeled dynamics" we mean a random walk on labeled poscau). We compute the probability that the element n is a post in any CSG dynamics (section 6.1). We then consider two classes of "cyclic dynamics" and identify their respective σ-algebras of observables (section 6.2.1). We discuss the random walks which induce a probability measure on these algebras (section 6.2.2) and apply the results of chapter 5 to study their complex counterparts (section 6.2.3).

6.1 Probability of a post

In this section we consider CSG dynamics. For each $n \geq 0$, define:

$$\lambda^{(n)}(i,0) := \sum_{k=0}^{i} \binom{i}{k} t_k^{(n)} = \sum_{k=1}^{i} \binom{i}{k} t_k^{(n)}, \tag{6.1}$$

where $t_k^{(n)}$ are the effective couplings given in (3.15).

We will show that,

Lemma 6.1.1. *In any CSG dynamics and any fixed $n \geq 0$, the probability that the element n is a post in the completed causal set is,*

$$\mathbb{P}(n \text{ is a post}) = F(n) \prod_{i=1}^{\infty} \frac{\lambda^{(n)}(i,0)}{\lambda(n+i,0)}, \tag{6.2}$$

where $F(n)$ is the probability that n is related to all other existing elements at the end of stage n and is given by,

$$F(n) = \begin{cases} 1 & : n = 0, \\ \sum_{\tilde{C}_n \in \tilde{\Omega}(n)} \mathbb{P}(\tilde{C}_n) \frac{\lambda(n, M(\tilde{C}_n))}{\lambda(n,0)} & : n > 0, \end{cases} \tag{6.3}$$

and $M(\tilde{C}_n)$ denotes the number of maximal elements in $\tilde{C}_n$.

We note that (6.2) agrees with two previously known special cases: the probability that the element 0 is a post in any CSG dynamics and the probability that $n \geq 0$ is a post in Transitive Percolation. In any CSG dynamics,

$$\mathbb{P}(0 \text{ is a post}) = \prod_{i=1}^{\infty} 1 - q_i, \tag{6.4}$$

where $q_i = \frac{t_0}{\lambda(i,0)}$ is the probability that the element i is born unrelated to all the existing elements. In the special case of Transitive Percolation, order-reversal invariance[1] implies that,

$$\sum_{\tilde{C}_n} \mathbb{P}(\tilde{C}_n) \frac{\lambda(n, M(\tilde{C}_n))}{\lambda(n,0)} = \prod_{i=1}^{n} 1 - q^i \tag{6.5}$$

and therefore for any $n > 0$ [Bombelli et al., 2008],

$$\mathbb{P}(n \text{ is a post}) = \prod_{i=1}^{n} 1 - q^i \prod_{i=1}^{\infty} 1 - q^i = (q,q)_n (q,q)_\infty, \tag{6.6}$$

where in the last line we used the definition of the q-Pochhammer symbol, $(q,q)_n \equiv \prod_{i=1}^{n} 1 - q^i$. $(q,q)_n$ decreases with n, and so the probability of n being a post decreases

[1] Two causets $\tilde{C}_n$ and $\tilde{C}'_n$ are related by order-reversal when $x \prec y$ in $\tilde{C}_n \iff y \prec x$ in $\tilde{C}'_n$. A dynamics is order-reversal invariant if $\mathbb{P}(\tilde{C}_n) = \mathbb{P}(\tilde{C}'_n)$ whenever $\tilde{C}_n$ and $\tilde{C}'_n$ are related by order-reversal.

with n, but is bounded below by $(q,q)^2_\infty > 0$ for all $0 < q < 1$. Hence whilst the probability that n is a post decreases with n, it is finite for all n.

Proof of lemma 6.1.1. Given $s > r > n$, we write $[r,s] \succ n$ to mean that every element in the interval $[r,s]$ is above n. We write $\{r,...\} \succ n$ to mean that every element greater or equal to r is above n.

First note that for any $s > n+1$,

$$\mathbb{P}(s \succ n \,\big|[n+1,s-1] \succ n) = \frac{\sum_{k=1}^{s-n}\binom{s-n}{k}\sum_{l=0}^{n}\binom{n}{l}t_{k+l}}{\lambda(s,0)} = \frac{\lambda^{(n)}(s-n,0)}{\lambda(s,0)}, \tag{6.7}$$

where the sum over k counts the non-empty subsets of $\{n, n+1, ..., s-1\}$ and the sum over l counts all subsets of $\{0,...,n-1\}$ so that together they count all possible proto-pasts consistent with the condition $s \succ n$, and we have performed the sum over l to obtain $t_k^{(n)}$ via equation (3.15) and then performed the sum over k to obtain $\lambda^{(n)}(s-n,0)$ via (6.1). Similarly,

$$\mathbb{P}(n+1 \succ n) = \frac{\sum_{l=0}^{n}\binom{n}{l}t_{l+1}}{\lambda(n+1,0)} = \frac{\lambda^{(n)}(1,0)}{\lambda(n+1,0)}. \tag{6.8}$$

Define $F(n)$ to be the probability that n is related to all $x < n$ and note that it is given by expression (6.3). Thus,

$$\begin{aligned}\mathbb{P}(n \text{ is a post}) =& F(n)\mathbb{P}(\{n+1,...\} \succ n)\\ =& F(n)\mathbb{P}(n+1 \succ n)\prod_{i=2}^{\infty}\mathbb{P}(n+i \succ n \,\big|[n+1,n+i-1] \succ n)\\ =& F(n)\prod_{i=1}^{\infty}\frac{\lambda^{(n)}(i,0)}{\lambda(n+i,0)}.\end{aligned} \tag{6.9}$$

□

6.2 Cyclic dynamics

This section is about labeled dynamics which generate infinitely many breaks or posts with unit probability. We begin by considering classical random walks on labeled poscau (*i.e.* a complete set of transition probabilities $\mathbb{P}(\tilde{C}_n \to \tilde{C}_{n+1})$), or equivalently a probability measure $(\tilde{\Omega}, \tilde{\mathcal{R}}, \tilde{\mu})$ (where $\tilde{\mu}\big(cyl(\tilde{C}_n)\big) = \mathbb{P}(\tilde{C}_n)$). Since we will be concerned with the covariant observables of these models, we may refer to $\mathcal{R}$, the covariant algebra, and to μ, the restriction of $\tilde{\mu}$ to $\mathcal{R}$. We will restrict our attention to two families of labeled dynamics which are of particular interest to cosmology. Let $\mathcal{B}_\infty \in \mathcal{R}$ denote the event that there are infinitely many breaks,

$$\mathcal{B}_\infty = \{\tilde{C} \in \tilde{\Omega} | \tilde{C} \text{ contains infinitely many breaks}\}, \tag{6.10}$$

and similarly, $\mathcal{P}_\infty \in \mathcal{R}$ denotes the event that there are infinitely many posts. We will consider the family of dynamics which satisfy $\mu(\mathcal{B}_\infty) = 1$ and the family of dynamics which satisfy the strictly stronger condition $\mu(\mathcal{P}_\infty) = 1$. We call models in either of these families "cyclic". A cyclic model may or may not be a CSG model and a CSG model may or may not be cyclic, though the best-understood growth dynamics—Transitive Percolation—is both, since it is a CSG model which almost surely gives rise to infinitely many posts.

6.2.1 Observables

Consider the family of cyclic dynamics satisfying $\mu(\mathcal{B}_\infty) = 1$. What are the physical observables in these models? $\mathcal{R}$ is the algebra of covariant events[2] but the condition $\mu(\mathcal{B}_\infty) = 1$ enables us to identify a strictly smaller algebra as the algebra of observables[3]. We propose that the set of observables in cyclic models satisfying $\mu(\mathcal{B}_\infty) = 1$ are those events which do not distinguish between causal sets which contain finitely many breaks, since such causal sets cannot be generated by the growth process. More precisely, we propose that the physical observables are the events $A \in \mathcal{R}$ which satisfy $\mathcal{B}^c_\infty \subseteq A$ or $\mathcal{B}^c_\infty \subseteq A^c$, where the superscript ^{c} denotes the set complement. Indeed, using the standard

[2]This is a consequence of the "kinematics", namely the structure of labeled poscau, which gives rise to the measurable space $(\tilde{\Omega}, \tilde{\mathcal{R}})$. It is independent of the measure μ.

[3]This is analogous to the CSG models, where $\mu(\Theta) = 0$ was used to identify $\mathcal{R}(\mathcal{S}) \subset \mathcal{R}$ as the algebra of observables (see sections 3.4 and 4.1).

set theory results,

$$
\begin{aligned}
&X \subseteq Y \iff Y^c \subseteq X^c, \\
&(\bigcup_i X_i)^c = \bigcap_i X_i^c,
\end{aligned}
\tag{6.11}
$$

one can show that the set of all such observables is a σ-algebra and we denote it by $\mathcal{R}_b$,

$$
\mathcal{R}_b := \{A \in \mathcal{R} | \mathcal{B}_\infty^c \subseteq A \text{ or } \mathcal{B}_\infty^c \subseteq A^c\}. \tag{6.12}
$$

For each $E \in \mathcal{R}$, $E \cap \mathcal{B}_\infty \in \mathcal{R}_b$. Therefore in any dynamics satisfying $\mu(\mathcal{B}_\infty) = 1$, the measure of each $E \in \mathcal{R}$ is uniquely determined by the measure of an event in $\mathcal{R}_b$ via $\mu(E) = \mu(E \cap \mathcal{B}_\infty)$. It is in this sense that $\mathcal{R}_b$ exhausts the σ-algebra of observables in dynamics which almost surely give rise to infinitely many breaks. The discussion for the family of dynamics satisfying $\mu(\mathcal{P}_\infty) = 1$ is completely analogous, and in this case the algebra of observables is given by,

$$
\mathcal{R}_p := \{A \in \mathcal{R} | \mathcal{P}_\infty^c \subseteq A \text{ or } \mathcal{P}_\infty^c \subseteq A^c\}. \tag{6.13}
$$

Now, we prove theorems 6.2.1-6.2.2 which provide further insight into the nature of the algebras $\mathcal{R}_b$ and $\mathcal{R}_p$. The following terminology will be useful. Recall that for any $n > 1$ the labeled causet $\tilde{C}_n \in \tilde{\Omega}(\mathbb{N})$ is a covering causet (of some causet $\tilde{C}_{n-1}$) if it has a unique maximal element (definition 4.2.2). We say that $\tilde{C}_n$ is a "super-covering causet" if it is a covering causet of a covering causet. In our convention, the 1-causet $\tilde{C}_1$ is both a covering and a super-covering causet. We say that $cyl(\tilde{C}_n)$ is a covering (super-covering) cylinder set if $\tilde{C}_n$ is a covering (super-covering) causet. Let $\tilde{\mathcal{R}}_b$ and $\tilde{\mathcal{R}}_p$ denote the σ-algebras generated by the covering and the super-covering cylinder sets, respectively. The corresponding covariant algebras are $\tilde{\mathcal{R}}_b \cap \mathcal{R}$ and $\tilde{\mathcal{R}}_p \cap \mathcal{R}$.

Theorem 6.2.1. $\mathcal{R}_b = \tilde{\mathcal{R}}_b \cap \mathcal{R}$.

Theorem 6.2.2. $\mathcal{R}_p = \tilde{\mathcal{R}}_p \cap \mathcal{R}$.

Thus, $\mathcal{R}_b$ ($\mathcal{R}_p$) is the collection of covariant events which can be built from covering (super-covering) cylinder sets by a countable number of set operations. Note that the connection between covering causets and breaks (super-covering causets and posts) is reminiscent of the covariant theorem 4.2.3, with the important difference that in the covariant case we can make statements about a finite number of breaks while in the labeled case we cannot.

We now prove theorem 6.2.1. The proof of theorem 6.2.2 is analogous. Let $\tilde{\mathcal{F}} \subset \tilde{\Omega}$ denote the set of all infinite causets which contain finitely many covering causets as stems.[4]

Lemma 6.2.3. *$\tilde{C} \in \mathcal{B}^c_\infty$ if and only if there exists some $\tilde{D} \in \tilde{\mathcal{F}}$ such that $\tilde{D} \cong \tilde{C}$.*

An illustration is shown in figure 6.1.

Proof. Note that $\tilde{\mathcal{F}} \subset \mathcal{B}^c_\infty$, since a causet with infinitely many breaks necessarily contains infinitely many covering causets as stems. Therefore, if there exists some $\tilde{D} \in \tilde{\mathcal{F}}$ such that $\tilde{D} \cong \tilde{C}$ then $\tilde{C} \in \mathcal{B}^c_\infty$.

To prove the converse, let $\tilde{C} \in \mathcal{B}^c_\infty$ and let N be the cardinality of the past of the last break in $\tilde{C}$. Suppose $\tilde{C}$ contains infinitely many covering causets as stems and consider the infinite sequence,

$$n_1 < n_2 < ... \equiv (n_i)$$

of integers $n_i > N+1$ for which the restriction $\tilde{C}|_{[0,n_i]}$ is a covering causet (*i.e.* $n_i \succ x \; \forall \; x < n_i$).

We now construct a bijection $g : \tilde{C} \to \mathbb{N}$ and define $\tilde{D} \in \tilde{\Omega}$ to be the infinite causet in which $x \prec y \iff g^{-1}(x) \prec g^{-1}(y)$ in $\tilde{C}$. The definition of $\tilde{D}$ ensures that $\tilde{D} \cong \tilde{C}$ and our construction of g ensures that $\tilde{D} \in \tilde{\mathcal{F}}$. This proves the claim.

Set $g(x) = x \; \forall \; x < n_1$. Set $g(n_1) = n_1 + 1$ (this ensures that $\tilde{D}|_{[0,n_1]}$ will not be a covering causet). Let n_1^* denote the smallest integer greater than n_1 which satisfies both $n_1^* \natural n_1$ and $n_1^* \not\succ y$ for at least one $y < n_1 - 1$ in $\tilde{C}$.[5] (Such an integer surely exists because otherwise $\tilde{C}$ would contain a break with past $[0, n_1 - 1]$, which is a contradiction.) Set $g(x) = x + 1 \; \forall \; n_1 < x < n_1^*$ and $g(n_1^*) = n_1$.

[4] Another way to say this is, $\tilde{C} \in \tilde{\mathcal{F}} \iff \exists \; m \in \mathbb{N}$ such that, for all $n > m$, $\tilde{C}|_{[0,n]}$ contains at least two maximal elements.

[5] Where for any two elements a, b the notation $a \natural b$ denotes that a and b are unrelated, *i.e.* $a \not\succ b$ and $a \not\prec b$.

Now, define m_2 to be equal to the smallest entry in (n_i) which is larger than n_1^*. Set $g(x) = x \; \forall \; n_1^* < x < m_2$. Set $g(m_2) = m_2 + 1$ (this ensures that $\tilde{D}|_{[0,m_2]}$ will not be a covering causet). Let n_2^* denote the smallest integer greater than m_2 which satisfies both $n_2^* \natural m_2$ and $m_2^* \not\succ y$ for at least one $y < m_2 - 1$ in $\tilde{C}$. Set $g(x) = x + 1 \; \forall \; m_2 < x < n_2^*$ and $g(n_2^*) = m_2$. Repeat this process with $m_2 \to m_3, n_2^* \to n_3^*$ *etc.* $\square$

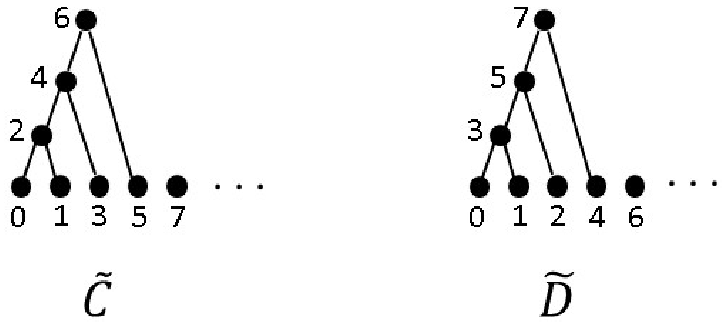


Figure 6.1: Illustration of lemma 6.2.3. $\tilde{C} \in \mathcal{B}_\infty^c$, since it contains no breaks. $\tilde{C} \notin \tilde{\mathcal{F}}$, since $\tilde{C}|_{[0,2n]}$ is a covering causet $\forall \; n \in \mathbb{N}$. $\tilde{D} \in \tilde{\mathcal{F}}$, since $\tilde{D}|_{[0,n]}$ is *not* a covering causet $\forall \; n > 0$.

Corollary 6.2.4. *If $A \in \mathcal{R}$ and $\tilde{\mathcal{F}} \subset A$ then $\mathcal{B}_\infty^c \subseteq A$.*

Lemma 6.2.5. *Let $A \in \tilde{\mathcal{R}}_b \cap \mathcal{R}$. $A \cap \mathcal{B}_\infty^c \neq \emptyset \implies \mathcal{B}_\infty^c \subseteq A$.*

Proof. For each covering causet $\tilde{C}_n \in \tilde{\Omega}(\mathbb{N})$ let $S(\tilde{C}_n) \subset \tilde{\Omega}(\mathbb{N})$ denote the set of covering causets with cardinality greater than n whose restriction to $[0, n-1]$ is $\tilde{C}_n$, and define

$$\Gamma_{\tilde{C}_n} := cyl(\tilde{C}_n) \setminus \bigcup_{\tilde{D}_m \in S(\tilde{C}_n)} cyl(\tilde{D}_m). \tag{6.14}$$

$\Gamma_{\tilde{C}_n}$ is the set of infinite causets which (i) contain the covering causet $\tilde{C}_n$ as a stem and (ii) contain no covering causet of cardinality greater than n as a stem. We will use the following properties:

1. Each $\Gamma_{\tilde{C}_n}$ is an atom of $\tilde{\mathcal{R}}_b$ (*i.e.* the elements of $\Gamma_{\tilde{C}_n}$ cannot be separated by the covering cylinder sets).

2. The collection of all the $\Gamma_{\tilde{C}_n}$ is a partition of $\tilde{\mathcal{F}}$ (since every $\tilde{C} \in \tilde{\mathcal{F}}$ is contained in some $\Gamma_{\tilde{C}_n}$ and for any two covering causets $\tilde{C}_n \neq \tilde{D}_m$ we have $\Gamma_{\tilde{C}_n} \cap \Gamma_{\tilde{D}_m} = \emptyset$).

Given a covering causet $\tilde{C}_n$, let $\tilde{X}_{\tilde{C}_n} \in \tilde{\Omega}$ denote the infinite causet whose restriction to $[0, n-1]$ is $\tilde{C}_n$ and in which all elements $m \geq n$ are unrelated to all others. Let $\tilde{X}'_{\tilde{C}_n}$ denote a causet isomorphic to $\tilde{X}_{\tilde{C}_n}$ in which the element 0 is unrelated to all others. Then $\tilde{X}_{\tilde{C}_n} \in \Gamma_{\tilde{C}_n}$ and $\tilde{X}'_{\tilde{C}_n} \in \Gamma_{\tilde{C}_1}$.

Suppose $\Gamma_{\tilde{C}_1} \subset A \in \tilde{\mathcal{R}}_b \cap \mathcal{R}$. Since $\Gamma_{\tilde{C}_1} \subset A$ we have $\tilde{X}'_{\tilde{C}_n} \in A$ for all covering causets $\tilde{C}_n$. Since A is covariant, $\tilde{X}_{\tilde{C}_n} \in A$ for every covering $\tilde{C}_n$. Hence, by property 1, $\Gamma_{\tilde{C}_n} \subset A$ for every covering $\tilde{C}_n$. By property 2, $\tilde{\mathcal{F}} \subset A$ and by corollary 6.2.4, $\mathcal{B}^c_\infty \subseteq A$.

Now, consider any $A \in \tilde{\mathcal{R}}_b \cap \mathcal{R}$ for which $A \cap \mathcal{B}^c_\infty \neq \emptyset$ and let $\tilde{C} \in A \cap \mathcal{B}^c_\infty$. Then there exists some $\tilde{D} \cong \tilde{C}$ and some $\tilde{C}_n$ such that $\tilde{D} \in \Gamma_{\tilde{C}_n}$ and therefore $\Gamma_{\tilde{C}_n} \subset A$. Hence $\tilde{X}_{\tilde{C}_n} \in \Gamma_{\tilde{C}_n}$. Since A is covariant, $\tilde{X}'_{\tilde{C}_n} \in A$. Therefore $\Gamma_{\tilde{C}_1} \subset A$, which completes the proof. □

Lemma 6.2.6. $\mathcal{B}_\infty \in \tilde{\mathcal{R}}_b$.

Proof. Let $\tilde{C}$ be a finite or infinite causet. A "segment" in $\tilde{C}$ is a subcauset which lies between two consecutive breaks. More precisely, if $\{A_1, B_1\}, \{A_2, B_2\}, ...$ are breaks in $\tilde{C}$ then the segments of $\tilde{C}$ are $A_1, A_2 \setminus A_1, A_2 \setminus A_3, ...$ If $\tilde{C}$ contains $0 < k < \infty$ breaks then B_k is also a segment. If $\tilde{C}$ contains no breaks then $\tilde{C}$ is the only segment in $\tilde{C}$.

Consider the collection of finite labeled causets which contain no breaks. We enumerate these causets using the label $i \in \mathbb{N}$, so that $\tilde{C}_{n_i}$ is the i^{th} finite causet which contains no breaks and its cardinality is n_i. We will use the string $\tilde{C}_{n_{i_1}} ... \tilde{C}_{n_{i_{k-2}}} \widehat{\tilde{C}_{n i_{k-1}}}$ to represent the finite covering causet whose k^{th} segment is (canonically isomorphic to) $\tilde{C}_{n_{i_k}}$.

For each finite order $\tilde{C}_{n_{i_1}}$ which contains no breaks, define the set,

$$\mathcal{B}_1(\tilde{C}_{n_{i_1}}) := \bigcap_{k=2}^{\infty} \bigcup cyl(\tilde{C}_{n_{i_1}} \tilde{C}_{n_{i_2}} ... \tilde{C}_{n_{i_{k-1}}} \widehat{\tilde{C}_{n_{i_k}}}), \tag{6.15}$$

where the union is over all sequences $(i_2, ..., i_k)$ of natural numbers. Note that if $\tilde{C} \in \mathcal{B}_1(\tilde{C}_{n_{i_1}})$ then $\tilde{C}_{n_{i_1}}$ is the first segment in $\tilde{C}$, and therefore $\tilde{C}$ contains at least one break. Additionally, if $\tilde{C} \in \mathcal{B}_\infty$ and $\tilde{C}_{n_{i_1}}$ is the first segment in $\tilde{C}$ then $\tilde{C} \in \mathcal{B}_1(\tilde{C}_{n_{i_1}})$.

We now generalise (6.15) to any $l \geq 1$. Given a string $\tilde{C}_{n_{i_1}} ... \tilde{C}_{n_{i_l}}$ of finite causets which contain no breaks define the set

$$\mathcal{B}_l(\tilde{C}_{n_{i_1}}...\tilde{C}_{n_{i_l}}) := \bigcap_{k=l+1}^{\infty} \bigcup cyl(\tilde{C}_{n_{i_1}}...\tilde{C}_{n_{i_l}}\tilde{C}_{n_{i_{l+1}}}...\widehat{\tilde{C}_{n_{i_k}}}) \tag{6.16}$$

where the union is over all sequences $(i_{l+1}, ..., i_k)$ of natural numbers. If $\tilde{C} \in \mathcal{B}_l(\tilde{C}_{n_{i_1}}...\tilde{C}_{n_{i_l}})$ then $\tilde{C}_{n_{i_1}}, ..., \tilde{C}_{n_{i_l}}$ are the first l segments in $\tilde{C}$ (and therefore $\tilde{C}$ contains at least l breaks). Additionally, if $\tilde{C} \in \mathcal{B}_\infty$ and $\tilde{C}_{n_{i_1}}, ..., \tilde{C}_{n_{i_l}}$ are the first l segments in $\tilde{C}$ then $\tilde{C} \in \mathcal{B}_l(\tilde{C}_{n_{i_1}}...\tilde{C}_{n_{i_l}})$. Define,

$$\mathcal{B}_l := \bigcup \mathcal{B}_l(\tilde{C}_{n_{i_1}}\tilde{C}_{n_{i_2}}...\tilde{C}_{n_{i_l}}), \tag{6.17}$$

where the union is over all sequences $(i_1, ..., i_l)$. For each l, $\mathcal{B}_l \supset \mathcal{B}_\infty$ and if $\tilde{C} \in \mathcal{B}_l$ then $\tilde{C}$ contains at least l breaks. Therefore,

$$\mathcal{B}_\infty = \bigcap_{l=1}^{\infty} \mathcal{B}_l. \tag{6.18}$$

□

Proof of theorem 6.2.1. By lemma 6.2.5, $\tilde{\mathcal{R}}_b \cap \mathcal{R} \subseteq \mathcal{R}_b$. To prove the converse, it is sufficient to prove that $A \in \tilde{\mathcal{R}}_b \cap \mathcal{R}$ whenever $A \in \mathcal{R}$ and $A \subseteq \mathcal{B}_\infty$. Let $A \in \mathcal{R}$ and $A \subseteq \mathcal{B}_\infty$. Then $A \in \mathcal{R}(\mathcal{S})$ and moreover A is in the restriction of $\mathcal{R}(\mathcal{S})$ to $\mathcal{B}_\infty$, *i.e.* $A \in \{E \cap \mathcal{B}_\infty | E \in \mathcal{R}(\mathcal{S})\}$, and therefore A can be obtained from countably many set operations on sets of the form $stem(C_n) \cap \mathcal{B}_\infty = (\bigcup cyl(\tilde{D}_m)) \cap \mathcal{B}_\infty$, where the union is over all covering cylinder sets contained in $stem(C_n)$. It follows from lemma 6.2.6 that every set of the form $(\bigcup cyl(\tilde{D}_m)) \cap \mathcal{B}_\infty$ is contained in $\tilde{\mathcal{R}}_b \cap \mathcal{R}$. Thus $A \in \tilde{\mathcal{R}}_b \cap \mathcal{R}$. □

6.2.2 Classical dynamics

Our new understanding of the covariant algebra $\mathcal{R}_b$ as a meaningful algebra of observables strongly suggests that we should seek to understand the measures which can be defined on it. We will consider measures which can be obtained via a restriction of a measure on the non-covariant algebras $\tilde{\mathcal{R}}$ or $\tilde{\mathcal{R}}_b$. The advantage in doing so is that we know that these algebras are countably generated by the cylinder sets and covering cylinder sets, respectively, and that therefore each measure can be conceived of as a random walk up a tree. A measure on $\tilde{\mathcal{R}}$ corresponds to a walk up labeled poscau, while a measure on $\tilde{\mathcal{R}}_b$ corresponds to a walk up a new tree which we dub "reduced poscau".

We can think of reduced poscau as obtained from labeled poscau by merging groups of nodes into one, so that in reduced poscau each node at level n is a collection of n-causets. Covering causets are never merged with others, so each covering causet is contained in a node on its own. Given a covering causet, all its children (except for the one which is itself a covering causet) are merged into one node. Thus, in reduced poscau each covering causet has two nodes directly above it. A node at level n which contains r causets has $r+1$ nodes above it: r nodes each of which contains a single covering causet (one for each causet in the parent node) and an additional node which contains all other $(n+1)$-causets in which the elements of the parent node are stems. The meaning of each node is "one of these causets is a stem in the growing causet". The first three levels of reduced poscau are shown in figure 6.2.

Each classical walk on reduced poscau induces a probability measure on $\tilde{\mathcal{R}}_b$, where the measure of a covering cylinder set is equal to the probability of reaching the corresponding node. Each walk on labeled poscau induces a walk on reduced poscau by fixing the probability of reaching each covering causet. The converse is also true, each (classical) walk on reduced poscau induces a (classical) walk on labeled poscau.[6] Thus walks on labeled poscau and on reduced poscau yield the same class of probability measures on the covariant $\mathcal{R}_b$. At the classical level, the only advantage of working directly with reduced poscau is that its structure could potentially lead to new physical constraints and hence to identifying new classes of physically interesting dynamics.

Note that the discussion relating to measures on $\mathcal{R}_p$ is completely analogous. In that case, measures on $\tilde{\mathcal{R}}_p$ correspond to walks up "super-reduced poscau" whose description is obtained from the above by replacing "covering" with "super-covering" (so only super-covering causets are contained in nodes of their own).

[6] The proof is analogous to the proof that every probability measure on $\mathcal{R}(\mathcal{S})$ has an extension to $\tilde{\mathcal{R}}$ (cf. lemma 4.1.16).

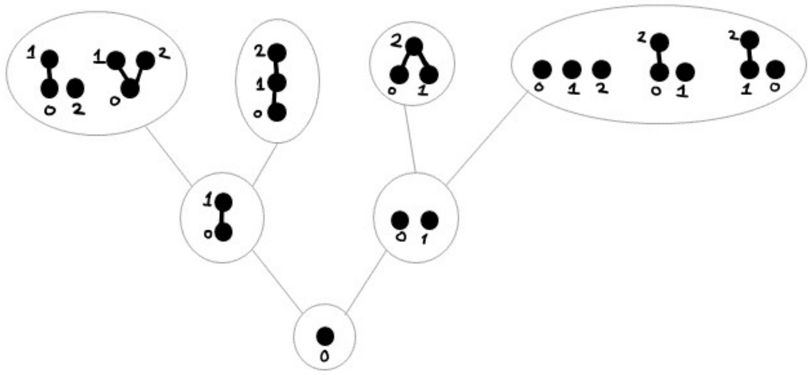


Figure 6.2: The first three levels of reduced poscau.

6.2.3 Quantization

As discussed in section 6.2.2, each probability measure on $\tilde{\mathcal{R}}_b$ extends to $\tilde{\mathcal{R}}$. However, it is unknown to the author of this thesis whether every *complex* measure on $\tilde{\mathcal{R}}_b$ extends to $\tilde{\mathcal{R}}$. If there exist complex measures on $\tilde{\mathcal{R}}_b$ which do not extend to $\tilde{\mathcal{R}}$, then our new understanding of the covariant algebra $\mathcal{R}_b$ as a meaningful algebra of observables combined with the understanding of complex measures as quantum dynamics (cf. chapter 5) strongly suggest that we should search for them.

For example, Transitive Percolation is a cyclic dynamics whose complex counterpart, Complex Transitive Percolation, does not induce a measure on $\tilde{\mathcal{R}}$ (cf. section 5.3). But can it induce a measure on the smaller algebra $\tilde{\mathcal{R}}_b$? Consider the complex Markovian dynamics induced on reduced poscau by Complex Transitive Percolation. In this dynamics, the amplitude to transition from a covering causet to its covering causet descendant is equal to $p \in \mathbb{C}$, hence the function ζ (cf. equation (5.30)) takes the value $|p|+|1-p|-1$ on each covering causet (since each covering causet has valency equal to 2). We solved numerically for ζ_n^{min} (cf. equation (5.31)) as a function of p for $n = 2, 3, 4$ and found that $\zeta_2^{min} = |p|+|1-p|-1 \implies \zeta_3^{min} = |p|+|1-p|-1 \implies \zeta_4^{min} = |p|+|1-p|-1$, as shown by the nested regions in figure 6.3a. Our result suggests that for any $p \in \mathbb{C}$ satisfying $p \notin [0,1]$, there exists an $m \in \mathbb{N}$ such that $\zeta_n^{min} = |p| + |1-p| - 1$ for all $n > m$. If this is borne out then, by theorem 5.2.1, Complex Transitive Percolation cannot induce a measure on $\tilde{\mathcal{R}}_b$ when p is not a real number satisfying $0 \leq p \leq 1$.

Can Complex Transitive Percolation induce a measure on the smaller algebra $\tilde{\mathcal{R}}_p$?

Theorem 5.2.1 cannot answer this question because every level $n > 1$ in super-reduced poscau contains nodes with valency equal to 1. On these nodes, ζ vanishes and therefore $\sum_n \zeta_n^{min} = 0$, so the theorem cannot be used to prove for the lack of bounded variation.[7] To conclude this discussion, we sketch how the derivation of theorem 5.2.1 can be amended to provide further scrutiny in the special case where there are nodes with valency equal to 1.

We build on the definitions and notation of section 5.4. Let $\mathscr{T}$ denote a tree satisfying the criteria outlined in section 5.4 and let $\mathscr{T}_n \subset \mathscr{T}$ denote the set of nodes at level n. Let x_n denote a node at level n. Let $\overline{\mathscr{T}_n} \subset \mathscr{T}_n$ be the set of level n nodes which have valency greater than 1, and define,

$$\overline{\zeta_n^{min}} = \min_{x_n \in \overline{\mathscr{T}_n}} \zeta(x_n). \tag{6.19}$$

Recall that $S_n = \sum_{x_n \in \mathscr{T}_n} |A(x_n)|$ and define,

$$\begin{aligned} \overline{S_n} &:= \sum_{x_n \in \overline{\mathscr{T}_n}} |A(x_n)|, \\ S_n^{v=1} &:= S_n - \overline{S_n}. \end{aligned} \tag{6.20}$$

Then in analogy to lemma 5.2.4, one can show that,

$$S_n \geq (1 + \overline{\zeta_{n-1}^{min}}) S_{n-1} - \overline{\zeta_{n-1}^{min}} S_{n-1}^{v=1}. \tag{6.21}$$

In analogy to lemma 5.2.6 one can show that,

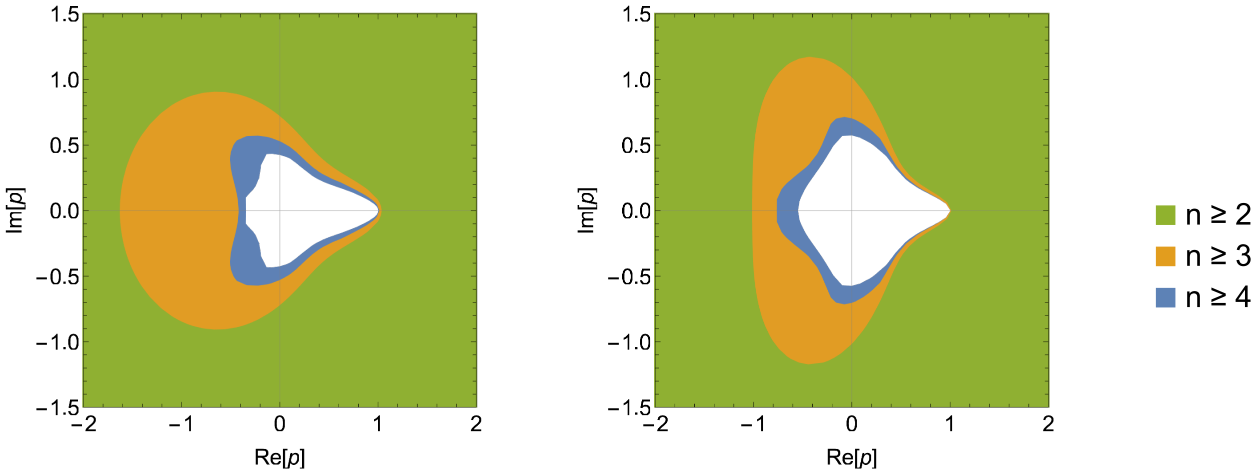
$$S_n \geq \prod_{r=1}^{n-1} (1 + \overline{\zeta_r^{min}}) - \sum_{r=1}^{n-1} \left[S_{n-r}^{v=1} \, \overline{\zeta_{n-r}^{min}} \, \prod_{i=1}^{r-1} (1 + \overline{\zeta_{n-r+i}^{min}}) \right]. \tag{6.22}$$

To prove the lack of bounded variation, one needs to show that $\sup_n S_n = \infty$.

We now return to Complex Transitive Percolation on super-reduced poscau. Numerical solutions for $\overline{\zeta_n^{min}}$ as a function of p for $n = 2, 3, 4$ (shown in figure 6.3b) suggest that for any $p \in \mathbb{C}$, there exists an $m \in \mathbb{N}$ such that $\overline{\zeta_n^{min}} = |p| + |1-p| - 1$ for all $n > m$. If this

[7] That the theorem cannot be used to prove for bounded variation when p is not a real number satisfying $0 \leq p \leq 1$ follows from $\zeta_n^{max} \geq |p| + |1-p| - 1 > 0$ for all n, since ζ is equal to $|p| + |1-p| - 1$ on every super-covering causet under Complex Transitive Percolation.

is borne out then, when $p \notin [0, 1]$, the first term in (6.22) diverges as $n \to \infty$. Whether $\sup_n S_n = \infty$ will depend on the behaviour of $S^{v=1}_{n-r}$ in the second term. We note that, since $S^{v=1}_{n-r}$ is the sum over absolute values of amplitudes of reaching a super-covering causet by stage $n - r - 1$, it could be computed using techniques similar to those used to obtain the probability of a post in section 6.1. We leave this calculation for future work.



(a) Reduced poscau. **(b)** Super-reduced poscau.

Figure 6.3: Complex Transitive Percolation. Figure (a) shows the values of $p \in \mathbb{C}$ for which $\zeta^{min}_n = |p| + |1 - p| - 1$ when $n = 2, 3, 4$ on reduced poscau. For example, if p is in the orange region then $\zeta^{min}_2 < |p| + |1 - p| - 1$ and $\zeta^{min}_3 = \zeta^{min}_4 = |p| + |1 - p| - 1$. Figure (b) shows the values of $p \in \mathbb{C}$ for which $\underline{\zeta^{min}_n} = |p| + |1 - p| - 1$ when $n = 2, 3, 4$ on super-reduced poscau.

Chapter 7

Two-way infinite dynamics

In this chapter we consider whether the growth dynamics paradigm is compatible with two-way infinite cosmologies (*i.e.* cosmologies in which time has neither a beginning nor an end).[1] We adapt our terminology to this new set-up (section 7.1) and modify the CSG models to accommodate growth of two-way infinite causets (section 7.2). Already from the outset, conceptual problems arise. Perhaps the most pressing of these is that our new framework requires that elements be born to the past of existing ones, making it (nearly if not entirely) impossible to conceive of the growth process as a physical phenomenon. However, we are able to identify a set of physically meaningful observables—namely the "convex-events" which describe the convex suborders contained in the growing causal set—which sets the stage for our pursuit of a manifestly covariant framework for two-way infinite growth. We construct a variation of covtree, show that the resulting framework is compatible with two-way infinite growth and prove that the observables in this case are exactly the formerly identified convex-events (section 7.3).

7.1 Preliminaries

In section 2.2 we introduced the terminology of "labeled causet" and "order" which laid the foundation for the constructions in chapters 3-6. Now we generalise these concepts to facilitate discussion of two-way infinite causal sets.

[1]The possibility of generalising growth dynamics to past-infinite cosmologies was also suggested in [Wuthrich and Callender, 2017].

We emphasise that the definition of labeled causets which we give here is different to definition 2.2.1, and correspondingly the definitions deriving from labeled causets (e.g. the definition of an n-order) and the symbols we use to denote spaces of labeled causets and orders (e.g. $\tilde{\Omega}(n)$ and Ω) take a different meaning in this chapter than they did in previous chapters.

For any pair of integers $k \leq l$, let $[k, l] := \{k, k+1, ..., l-1, l\}$ (devoid of any ordering).

Definition 7.1.1 (Labeled causet)**.** *A labeled causet is any causet $([k, l], \prec)$ or $(\mathbb{Z}, \prec)$ or $(\mathbb{Z}^-, \prec)$ or $(\mathbb{N}, \prec)$ satisfying $x \prec y \implies x < y$. An n-causet is a labeled causet of cardinality n.*

We denote labeled causets and their subcausets by capital Roman letters with a tilde, *e.g.* $\tilde{C}$. We often (but not always) use a subscript to denote the cardinality of an n-causet.

We will use the following notation to denote certain collections of labeled causets:

for any $n > 0$, $\tilde{\Omega}(n)$ is the set of n-causets,

$\tilde{\Omega}_{\mathbb{N}}$ is the set of labeled causets with ground-set $\mathbb{N}$,

$\tilde{\Omega}_{\mathbb{Z}}$ is the set of labeled causets with ground-set $\mathbb{Z}$,

$\tilde{\Omega}_{\mathbb{Z}^-}$ is the set of labeled causets with ground-set $\mathbb{Z}^-$,

$\tilde{\Omega}$ is the set of all infinite causets.

Thus, $\tilde{\Omega} = \tilde{\Omega}_{\mathbb{N}} \sqcup \tilde{\Omega}_{\mathbb{Z}} \sqcup \tilde{\Omega}_{\mathbb{Z}^-}$ (note that the union is disjoint).

Definition 7.1.2 (Order)**.** *An order is an order-isomorphism class of labeled causets.*

We denote orders by capital Roman letters without a tilde.

We will use the following notation to denote certain collections of orders:

for any $n > 0$, $\Omega(n)$ is the set of n-orders,

$\Omega := \tilde{\Omega}/\cong$ is the set of all infinite orders,

$\Omega_{\mathbb{N}}$ is the set of orders which have a representative in $\tilde{\Omega}_{\mathbb{N}}$,

$\Omega_{\mathbb{Z}^-}$ is the set of orders which have a representative in $\tilde{\Omega}_{\mathbb{Z}^-}$,

$\Omega_{\mathbb{Z}}$ is the set of orders which have a representative in $\tilde{\Omega}_{\mathbb{Z}}$.

Thus, $\Omega = \Omega_{\mathbb{Z}} \cup \Omega_{\mathbb{Z}^-} \cup \Omega_{\mathbb{N}}$ (where the union is *not* disjoint).

The following lemma provides a characterisation of infinite causets:

Lemma 7.1.3. *Let* Π *be any countably infinite causet.*

- Π *is past-finite if and only if* $\Pi \cong \tilde{C}$ *for some* $\tilde{C} \in \tilde{\Omega}_{\mathbb{N}}$.[2]
- Π *is future-finite if and only if* $\Pi \cong \tilde{C}$ *for some* $\tilde{C} \in \tilde{\Omega}_{\mathbb{Z}^-}$.
- *If* $\Pi \cong \tilde{C}$ *for some* $\tilde{C} \in \tilde{\Omega}_{\mathbb{Z}}$ *then* (*at least*)[3] *one of the following holds:*[4]

  (i) Π *is two-way infinite,*

  (ii) Π *is past-finite and has infinitely many minimal elements,*

  (iii) Π *is future-finite and has infinitely many maximal elements.*

Lemma 7.1.3 ensures that all order-types are included in $\tilde{\Omega}$ and hence in Ω (*i.e.* we are not missing any partial order structures by working only with labeled causets).

The following is a covariant corollary:

Corollary 7.1.4. *Let* C *be an infinite order.*

- C *is past-finite if and only if* $C \in \Omega_{\mathbb{N}}$.
- C *is future-finite if and only if* $C \in \Omega_{\mathbb{Z}^-}$.
- *If* $C \in \Omega_{\mathbb{Z}}$ *then* (*at least*) *one of the following holds:*

  (i) C *is two-way infinite,*

  (ii) C *is past-finite and has infinitely many minimal elements,*

  (iii) C *is future-finite and has infinitely many maximal elements.*

Finally, we will also need the notions of an n-suborder and convex-rogue. Let C and D be orders with representative $\tilde{C}$ and $\tilde{D}$, respectively.

[2] From [Brightwell and Luczak, 2011].

[3] Families (ii) and (iii) are disjoint from (i) but not from each other, *e.g.* the infinite antichain is contained in both.

[4] From [Gupta, 2018; Honan, 2018].

Definition 7.1.5 (Convex suborder/n-suborder)**.** *C is a convex suborder in D if $\tilde{D}$ contains a copy of $\tilde{C}$. In that case we may also say that C is a convex suborder in $\tilde{D}$. If additionally C is an n-order, we say that C is an n-suborder in D or in $\tilde{D}$.*

Definition 7.1.6 (Convex-rogue)**.** *C is a convex-rogue if there exists another order $D \ncong C$ which has the same n-suborders as C for all n. In that case we say that C and D are a convex-rogue pair. We may also refer to $\tilde{C}$ and $\tilde{D}$ as convex-rogues or as a convex-rogue pair.*

An example of a convex-rogue pair is shown in figure 7.1.

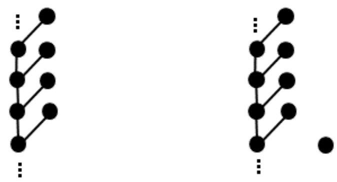


Figure 7.1: The "infinite comb" (left) and the infinite comb disjoint union a single element (right) are convex-rogues since they contain the same convex suborders as each other.

7.2 Labeled growth

In this section we generalise the sequential growth paradigm to accommodate two-way infinite cosmologies (section 7.2.1). We consider what form various physical conditions take in this new framework and whether they are satisfied by a naive generalisation of the CSG models, and we identify a class of observables for the new models (section 7.2.2).

7.2.1 Alternating growth

The labeled growth models considered thus far, cannot generate past-infinite causal sets since they satisfy the Internal Temporality condition which states that a new element cannot be born to the past of an existing one.[5] Indeed, the first challenge in generalising the labeled models to the two-way infinite case is generalising the condition of Internal Temporality. If we are to both generate two-way infinite causal sets and keep the essence of sequential growth (*i.e.* that starting from a single element, new elements are born one

[5]See section 3.1.

by one), we must loosen the condition of Internal Temporality to allow elements to be born to the past of already-existing ones. Although at first this may seem like a complete departure from Internal Temporality and from the physical motivation behind it, it is not necessarily so. In the past-finite case, Internal Temporality reduces to the requirement that the growing causal set is an element of $\tilde{\Omega}_{\mathbb{N}}$. Reformulating Internal Temporality as a statement about labeled causets reveals a candidate generalisation of it to the two-way infinite case: the requirement that the growing causal set is contained in $\tilde{\Omega}_{\mathbb{Z}}$. Some freedom remains in how to translate this condition back into a statement about the birth of elements. To fix this freedom we choose that the positive and negative integers are born in an alternating sequence, $0, -1, 1, -2, 2...$, so that at stage n, if n is even the element $\frac{n}{2}$ is born and if n is odd the element $-\frac{n+1}{2}$ is born. Internal Temporality then becomes: positive elements cannot be born to the past of existing elements, negative elements cannot be born to the future of existing elements. In particular, no element can be born in-between two existing elements so that at each stage the finite causet is a convex subcauset of the growing infinite causet. We call a transition in which a positive element is born a "forward transition". Similarly, a "backward transition" is one in which a negative element is born. A transition $\tilde{C}_n \to \tilde{C}_{n+1}$ is forward when n is even and backward when n is odd.

We dub the resulting dynamical framework "alternating growth". The alternating growth process can be represented as a random walk on "alternating poscau", a directed tree whose nodes are finite labeled causets. More precisely,

Definition 7.2.1. *Alternating poscau is the partial order on the collection of finite labeled causets whose ground set is* $[-n, n]$ *or* $[-n, n+1]$ *for any* $n > 0$*, where* $\tilde{S} \prec \tilde{R}$ *if and only if* $\tilde{S}$ *is a convex subcauset in* $\tilde{R}$*.*

The first three levels of alternating poscau are shown in figure 7.2. There is a bijection between the space of infinite paths and $\tilde{\Omega}_{\mathbb{Z}}$, where an infinite path $\tilde{C}_1 \prec \tilde{C}_2 \prec ...$ corresponds to $\tilde{C} = \bigcup_{n>0} \tilde{C}_n$. The measurable events are generated by the "cylinder sets" which are associated with each node, where $cyl(\tilde{C}_n) \subset \tilde{\Omega}_{\mathbb{Z}}$ is the set of causets which contain $\tilde{C}_n$ as a convex subcauset. Any random walk on alternating poscau is a well-defined measure on this measurable space (and vice versa, every measure corresponds to a unique random walk).

By generalising the Internal Temporality condition we are thus able to extend the sequential growth paradigm to include growth of two-way infinite causets. One could extend Internal Temporality in other ways, leading to new trees and sample spaces.[6] Finally, recall that by lemma 7.1.3 the collection of two-way infinite causets is only one of three families contained in the sample space $\tilde{\Omega}_{\mathbb{Z}}$, and therefore it will be up to the dynamics (*i.e.* the random walk) to pick out this family of causets (family (*i*) in lemma 7.1.3) over the alternatives.

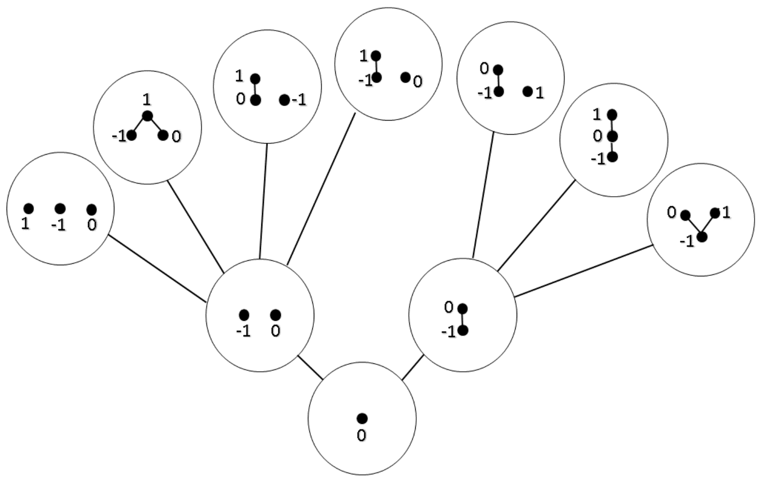


Figure 7.2: The first three levels of alternating poscau.

7.2.2 Alternating growth dynamics

While any random walk on alternating poscau gives rise to a well-defined measure on $\tilde{\Omega}_{\mathbb{Z}}$, not every such walk is of interest to physics and it remains for us to identify classes of interest. This is completely analogous to the past-finite case, where the CSG models were identified as a physically-meaningful subclass of random walks on labeled poscau. Could the CSG models be generalised for alternating growth? Indeed, equation (3.1) can be adapted for alternating growth simply by letting $\tilde{C}_n$ and $\tilde{C}_{n+1}$ denote nodes in alternating poscau—we dub this new family of models the "Alternating CSG" dynamics.

[6]See [Bento, 2018] for further variations, including sequential growth models in which the identity of the element born at each stage is determined at random.

Do the Alternating CSG dynamics retain the features that make CSG models interesting for physics? For example, can the Alternating CSG models be considered causal? To answer this question we must first understand what form causality takes within the alternating growth framework. In the remainder of this section we identify the form which four key attributes—covariance, causality, causal immortality and meaningful observables—take in the alternating growth framework and discuss whether the Alternating CSG models possess these attributes.

We will pay special attention to Transitive Percolation and its alternating growth counterpart, Alternating Transitive Percolation, since the two models share many similarities. For example, let $\tilde{C}_n$ and $\tilde{D}_n$ be nodes in labeled poscau and alternating poscau respectively, and let $\tilde{C}_n \cong \tilde{D}_n$. Then the probablity of reaching $\tilde{C}_n$ under a given CSG dynamics is equal to the probability of reaching $\tilde{D}_n$ under its alternating growth counterpart if and only if the dynamics in question is Transitive Percolation [Gupta, 2018; Honan, 2018].

Covariance: The discrete general covariance condition (3.4) can be generalised to the alternating sequential growth framework simply by letting $\tilde{C}_n$ and $\tilde{C}'_n$ denote nodes in alternating poscau.

Every CSG model satisfies the discrete general covariance condition. In contrast, the only Alternating CSG dynamics which satisfies discrete general covariance is Alternating Transitive Percolation:

Lemma 7.2.2. *An Alternating CSG model satisfies the discrete general covariance condition if and only if it is an Alternating Transitive Percolation model.*

Proof. First we show that an Alternating CSG model is not covariant if it is not Alternating Transitive Percolation. Consider the $(2n+1)$-order C which contains a $(2n)$-antichain of which n elements have a common ancestor, as shown in figure 7.3.
Let $\tilde{C}$ denote the representative of C which is grown in the alternating framework in the following way: the element 0 and the elements born in the first $2n-2$ stages form an antichain, the element born at stage $2n-1$ is born to the past of n of the existing elements, and the element born at stage $2n$ is unrelated to all existing elements. The

Figure 7.3: The $(2n+1)$-order C, which contains a $(2n)$-antichain of which n elements have a common ancestor.

probability of growing $\tilde{C}$ in an Alternating CSG dynamics is $\mathbb{P}(\tilde{C}) = t_0^{2n-2} t_n t_0 = t_0^{2n-1} t_n$. Let $\tilde{C}'$ denote another representative of C which is grown in the alternating framework in the following way: the elements born in forward transitions are all born to the future of the element 0, and the elements born in backward transitions are all born unrelated to all existing elements. The probability of growing $\tilde{C}'$ in an Alternating CSG dynamics is $\mathbb{P}(\tilde{C}') = t_1^n t_0^n$. An Alternating CSG model is covariant only if $\mathbb{P}(\tilde{C}) = \mathbb{P}(\tilde{C}')$, *i.e.* only if $\frac{t_1^n}{t_n} = t_0^{n-1}$. This is Alternating Transitive Percolation and can be cast into the form $t_n = t^n$ by setting $t_0 = 1$.

That Alternating Transitive Percolation is covariant follows from equation (3.10) since it implies that the probability of reaching some $\tilde{C}_n$ in alternating poscau is $\mathbb{P}(\tilde{C}_n) = p^L q^{\binom{n}{2} - R}$, where L and R are the number of links and relations in $\tilde{C}_n$, respectively. □

Causality: What form can the Bell causality condition take within the alternating growth framework? At first glance it may seem that we can simply interpret the labeled causets in equation (3.5) as nodes in alternating poscau, but this is not so. To see the problem, let $\tilde{C}_n$ be a node in alternating poscau, and let $\tilde{C}_{n+1}$ and $\tilde{C}'_{n+1}$ denote two of its children. Now, construct $\tilde{B}_l$ from $\tilde{C}_n$ by removing the spectators and relabeling. Next, remove the spectators from $\tilde{C}_{n+1}$ and relabel—this is where the problem arises since there may be no relabeling which produces a child of $\tilde{B}_l$. In particular, this failure occurs whenever the number of spectators is odd because in that case if $\tilde{C}_n \to \tilde{C}_{n+1}$ is a forward transition[7] then $\tilde{B}_l \to \tilde{B}_{l+1}$ must be a backward transition, which leads to a contradiction. An example is shown in figure 7.4. It is in these cases that the generalisation of equation (3.5) to the alternating dynamics becomes ill-defined. Instead, we will use a weakened causality condition (in similarity to the weakened causality conditions of [Dowker and Surya, 2006; Varadarajan and Rideout, 2006]) which states that an alternating dynamics

[7] Recall that a transition $\tilde{C}_n \to \tilde{C}_{n+1}$ is forward when n is even and backward when n is odd.

is causal if equation (3.5) is satisfied whenever it is well-defined.

Having arrived at a causality condition for the alternating framework, we can ask whether the Alternating CSG dynamics are causal. We begin by examining Alternating Transitive Percolation. Since the Alternating Transitive Percolation models are covariant (cf. lemma 7.2.2), the labeling ambiguity in the Bell causality condition is alleviated and we can use equation (3.10) to verify that equality (3.5) is satisfied,

$$\frac{\mathbb{P}(\tilde{C}_n \to \tilde{C}_{n+1})}{\mathbb{P}(\tilde{C}_n \to \tilde{C}'_{n+1})} = \frac{p^m q^{n-\varpi}}{p^{m'} q^{n-\varpi'}} = \frac{p^m q^{l-\varpi}}{p^{m'} q^{l-\varpi'}} = \frac{\mathbb{P}(\tilde{B}_l \to \tilde{B}_{l+1})}{\mathbb{P}(\tilde{B}_l \to \tilde{B}'_{l+1})}, \tag{7.1}$$

where ϖ' and m' denote the number of relations and links, respectively, formed by the element which is born at stage n in the transition $\tilde{C}_n \to \tilde{C}'_{n+1}$. In this sense, the Alternating Transitive Percolation models are causal. Since the remaining Alternating CSG dynamics are not covariant, to ascertain whether they are causal requires specifying a canonical choice of labeling by which $\tilde{B}_{l+1}$ should be obtained from $\tilde{C}_{n+1}$ *etc.*, which renders the Bell causality condition itself label-dependent and hence not covariant. Since causal structure is fundamentally frame-independent, this suggests that the Bell causality condition is only physically-meaningful within covariant growth dynamics.

While we have had some success in formally adapting equality (3.5) to the alternating growth framework, the physical interpretation of this new causality condition is far from clear. In the framework of past-finite growth, the Bell causality is interpreted as the statement that the probability for each transition depends only on the past of the new-born element. But this interpretation is obliterated in the alternating growth framework. In a forward transition, the transition probability depends only on the past of the new-born element—but not on its entire past, since some of it has not yet been determined. The situation is even worse in the backward transitions where the transition probabilities depend on the future of the new-born element. One resolution is to require that equality (3.5) holds only for the forward transitions (*i.e.* when n is even), leaving the backward transitions unconstrained by causality. Or it may be that we need an altogether new way of thinking about causality in the alternating framework.

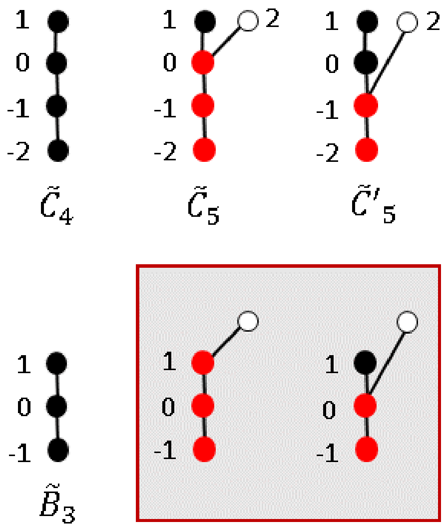


Figure 7.4: The Bell Causality condition is ill-defined in the alternating growth framework. The parent $\tilde{C}_4$ and two of its children are shown on the top line. The new-born element in each child is shown in white. The past of the new-born element in each transition is shown in red. $\tilde{B}_3$ is constructed from $\tilde{C}_4$ by removing the element 1 (*i.e.* the spectator) and relabeling. Removing the spectator from $\tilde{C}_5$ and $\tilde{C}_5'$ results in the causets shown in the box, but there is no relabeling of these which corresponds to children of $\tilde{B}_3$.

Causal immortality: In past-finite sequential growth, the sample space is the space of all infinite past-finite causets, $\tilde{\Omega}_{\mathbb{N}}$. This space contains a variety of cosmologies: some are future-infinite and some are future-finite, some contain infinite antichains and some do not. But in the CSG models, only a subset of all these potential configurations can be realised because the CSG models generate, with unit probability, causets which have no maximal elements [Brightwell et al., 2003]. We say that the CSG models have the property of "causal immortality" because the effect of each element/event reaches infinitely far into the future.

Similarly, in alternating sequential growth the sample space $\tilde{\Omega}_{\mathbb{Z}}$ contains several causal set families (given in lemma 7.1.3) but only a subset of these is realised by the Alternating CSG dynamics because these dynamics generate causets which have no maximal nor minimal elements, as we show in lemma 7.2.3. This "two-way causal immortality" means that not only does the effect of each event reaches infinitely far into the future, but that the cause of each event can be traced infinitely far into the past.

Lemma 7.2.3. *Every element in a causal set grown in Alternating CSG dynamics with* $t_k > 0$ *for some* $k > 0$ *almost surely has an element to its future and an element to its* *past.*

Proof. Consider a growth process with an Alternating CSG dynamics with $t_k > 0$ for some $k > 0$. Suppose that the labeled causet $\tilde{C}_n$ has been grown by the beginning of stage $n > k$, and let $x \in \tilde{C}_n$ be a maximal element.

First, we show that the probability that x is maximal in the complete causal set is zero. Let $r \geq n$ be an even integer. Then the probability that x is maximal at the end of stage r (given that x is maximal at the beginning of stage r) is,

$$1 - p_r = \frac{\lambda(r-1,0)}{\lambda(r,0)}. \tag{7.2}$$

where p_r is the effective parameter of [Brightwell and Luczak, 2016]. Therefore the probability that x is maximal in the complete causet is,

$$\lim_{s\to\infty} \mathbb{P}(x \text{ is maximal at end of stage } s) = \lim_{s\to\infty} \prod_{\text{even } n\leq r\leq s} 1 - p_r \tag{7.3}$$

which converges to a non-zero value if and only if the following series converges [Jeffreys and Swirles, 1966],

$$\lim_{s\to\infty} \sum_{\text{even } n\leq r\leq s}^{\infty} p_r. \tag{7.4}$$

Rearranging equation (7.2) we have,

$$p_r = \frac{\sum_{l=1}^{r}\binom{r-1}{l-1}t_l}{\lambda(r,0)} = \frac{1}{r}\left(\frac{\sum_{l=1}^{r}\frac{r!}{(r-l)!(l-1)!}t_l}{\lambda(r,0)}\right) \geq \frac{1}{r}\left(\frac{\sum_{l=1}^{r}\binom{r}{l}t_l}{t_0+\sum_{l=1}^{r}\binom{r}{l}t_l}\right) \geq \frac{1}{r}\left(\frac{t_k}{t_0+t_k}\right) \tag{7.5}$$

and therefore the series (7.4) is divergent and probability (7.3) vanishes.

The argument can be adapted to show that every element has an element to its past by letting x be a minimal element and letting r take odd values. □

Observables: In the past-finite CSG models, the role of observables is played by "stem-events", namely elements of the algebra $\mathcal{R}(\mathcal{S})$.[8] What are the observables within the alternating growth framework? The freedom to grow past-infinite causal sets means that

[8]See section 3.4.

the stem-events have a weak distinguishing power—they tell us nothing about the past-infinite part of a casual set and they cannot distinguish between causets which contain no minimal elements. We can make progress by noticing that stems are to past-finite growth what convex suborders are to alternating growth. The ordering of labeled poscau is determined by the stem relation (*i.e.* the order of labeled poscau is order-by-inclusion-as-stem, cf. definition 3.1.1), while the ordering of alternating poscau is determined by the convex relation (cf. definition 7.2.1). Each node in labeled poscau is a stem in the growing causet, while each node in alternating poscau is a convex subcauset in the growing causet. Therefore, it is reasonable to expect that "convex-events"—the convex analogue of stem-events—play the role of observables for alternating growth.

To make this precise, for each finite order C_n let $convex(C_n) \subset \tilde{\Omega}_{\mathbb{Z}}$ be the collection of causets which contain C_n as a convex suborder. A "convex-event" is any set which can be generated from the $convex(C_n)$'s via countable set operations (*i.e.* a convex-event is an element of the σ-algebra generated by the $convex(C_n)$'s). Each convex-event is a covariant measurable event with a clear physical meaning—it corresponds to a logical combination of statements about which finite orders are convex suborders in the growing causet.

Convex-events form an extensive class of observables which provide us with information about the structure of the causal set at arbitrarily early times/infinitely far into the past. But they are not fully-distinguishing as they cannot distinguish between pairs of convex-rogues (cf. definition 7.1.6). In the past-finite framework, the stem-events are also not fully-distinguishing since they fail to distinguish between pairs of rogues. But it was shown in [Brightwell et al., 2003] that in any CSG dynamics the set of rogues has measure zero and therefore, in a precise mathematical sense, the stem-events exhaust the set of physical observables in any CSG dynamics. Crucially, the result of [Brightwell et al., 2003] depends on the specifics of the CSG dynamics and does not hold for a generic random walk on labeled poscau but only for those models in which the set of rogues has measure zero.

Can convex-events similarly be considered to exhaust the set of physical observables in the alternating CSG dynamics? For Alternating Transitive Percolation, the answer is no. Like in Transitive Percolation, in Alternating Transitive Percolation, every finite order is almost surely a convex suborder in the growing causet (*i.e.* the measure of every $convex(C_n)$ is equal to 1) [Brightwell and Luczak, 2016]. This means that the convex-events cannot distinguish between the various possible outcomes of the growth process (which only generates convex-rogues) and therefore the dynamics is deterministic with respect to the convex-events. But not every Alternating CSG model is deterministic with respect to the convex-events:

Lemma 7.2.4. *A CSG dynamics is not deterministic with respect to convex-events when its couplings are given by,*

$$t_0 = 1 \text{ and } t_n = f(n)\lambda(n-1,0) \; \forall n \geq 1, \tag{7.6}$$

where $f(n)$ is a function satisfying $\sum \frac{1}{f(n)} < \infty$ (e.g. $f(n) = x^n$ with $x > 1$ or $f(n) = n^s$ with $s \geq 2$).

Proof. Let A_2 denote the 2-antichain order, and let C_∞ denote the two-way infinite chain order. Note that $\mathbb{P}(C_\infty) = 1 - \mu(convex(A_2))$, where $\mathbb{P}(C_\infty)$ is the probability of growing C_∞ and $\mu(convex(A_2))$ is the measure of $convex(A_2)$. Note also that $\mu(convex(A_2)) > t_0/\lambda(1,0) > 0$ in any CSG dynamics. We will show that in the dynamics above, $\mathbb{P}(C_\infty) > 0$ and therefore $0 < \mu(convex(A_2)) < 1$ and the result follows.

Note that $\mathbb{P}(C_\infty) = \prod_{n>0} p_n$, where (as in claim 7.2.3) p_n is the effective parameter given by,

$$p_n = \frac{\sum_{k=0}^{n-1} \binom{n-1}{k} t_{k+1}}{\lambda(n,0)} = \frac{\lambda(n,0) - \lambda(n-1,0)}{\lambda(n,0)}, \tag{7.7}$$

and the product converges to a non-zero value if and only if the series $\sum 1 - p_n$ converges [Jeffreys and Swirles, 1966]. We can write the m^{th} term of this series as,

$$1 - p_m = \frac{\lambda(m-1,0)}{\lambda(m,0)} = \left(\frac{\sum_{r=0}^{m-1} \binom{m}{r} t_r}{\lambda(m-1,0)} + \frac{t_m}{\lambda(m-1,0)} \right)^{-1}, \tag{7.8}$$

and then substitute the couplings given in (7.6) to find,

$$1 - p_m = \left(\frac{\sum_{r=0}^{m-1} \binom{m}{r} t_r}{\lambda(m-1,0)} + f(m) \right)^{-1} \leq \frac{1}{f(m)}. \tag{7.9}$$

It follows that in the models given in (7.6) the sum $\sum 1 - p_n$ converges by the comparison test against $\sum \frac{1}{f(n)}$ and hence $\mathbb{P}(C_\infty) > 0$. □

That the convex-events are in some sense interesting or sufficient to describe an alternating dynamics (*e.g.* that the dynamics almost surely does not generate convex-rogues, or that the dynamics is not deterministic with respect to the convex-events) can be used as a constraint or guiding principle in searching for new/physical alternating dynamics.

7.3 Manifestly covariant growth

How can the covariant framework of chapter 4 be adapted for two-way infinite growth? Pursuing the analogy between stems and convex suborders, here we propose a new covariant framework, dubbed $\mathbb{Z}$-covtree, whose sample space is $\Omega_\mathbb{Z}$ and whose set of observables is exactly the set of convex-events. While in the past-finite case, label-independence came at the cost of an explicit process of growth, in the two-way infinite case the masking of the birth of elements by the covariance is a gain, since it alleviates the difficulty of interpreting a growth in which a new element is born to the past of existing elements.

We begin by defining "convex-covtree", a variation of covtree in which stems are replaced by suborders (section 7.3.1). We prove that convex-covtree's sample space is strictly greater than $\Omega_\mathbb{Z}$ (section 7.3.2) and that it can be consistently truncated into a new tree, $\mathbb{Z}$-covtree, whose sample space is the desired $\Omega_\mathbb{Z}$ (sections 7.3.3-7.3.4).

7.3.1 Defining convex-covtree

Convex-covtree is a directed tree whose nodes at level n are certain subsets of $\Omega(n)$. Some $\Gamma_n \subset \Omega(n)$ is a node in convex-covtree if and only if it is the set of n-suborders of some order C. In the following, we formalise this notion.

Definition 7.3.1 (Convex-certificate). *An order C is a convex-certificate of Γ_n if Γ_n is the set of n-suborders of C. A labeled convex-certificate of Γ_n is a representative of a convex-certificate of Γ_n.*

A convex-certificate may be finite or infinite, and if it is infinite it may be past-finite, future-finite or neither. Note that some $\Gamma_n \subset \Omega(n)$ have no convex-certificates at all. If Γ_n has an infinite convex-certificate then it has a finite convex-certificate, but the converse is not true. Examples are shown in figure 7.5.

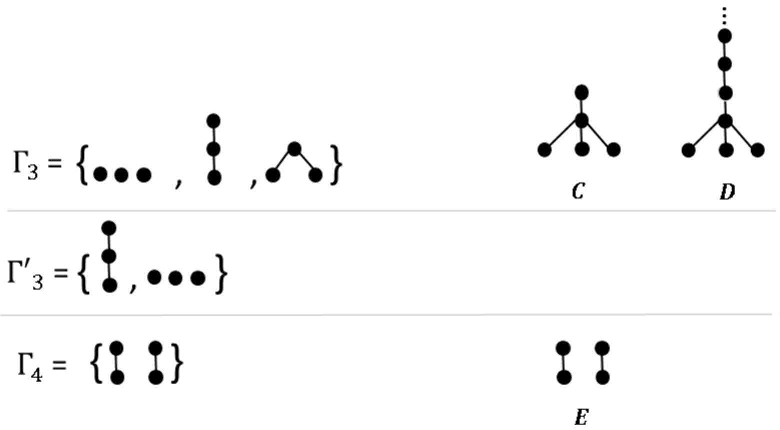


Figure 7.5: Illustration of convex-certificates. C and D are convex-certificates of Γ_3. Γ_3' has no convex-certificates since any order which contains the 3-chain and the 3-antichain as 3-suborders also contains a 2-chain plus a point as a 3-suborder. E is a convex-certificate of Γ_4. Γ_4 has no infinite convex-certificates.

Definition 7.3.2 (χ, the collection of convex-certified sets). *χ is the collection of sets of n-orders, for all n, for which there exists a convex-certificate:*

$$\chi := \bigcup_{n>0} \{\Gamma_n \subseteq \Omega(n) \mid \exists \text{ a convex-certificate of } \Gamma_n\}. \tag{7.10}$$

Definition 7.3.3 (The map $\mathcal{O}_c$). *For any $n > 1$ and any Γ_n, the map $\mathcal{O}_c$ takes Γ_n to the set of $(n-1)$-suborders of elements of Γ_n:*

$$\mathcal{O}_c(\Gamma_n) := \{B \in \Omega(n-1) \mid \exists\ A \in \Gamma_n \text{ s.t. } B \text{ is an } (n-1)\text{-suborder in } A\} \tag{7.11}$$

One way to think about the operation $\mathcal{O}_c$ on Γ_n is to pick an n-order in Γ_n and delete a maximal or minimal element of it to form an $(n-1)$-order. The set $\mathcal{O}_c(\Gamma_n)$ is the set of all $(n-1)$-orders which can be formed in this way. An illustration is shown in figure 7.6.

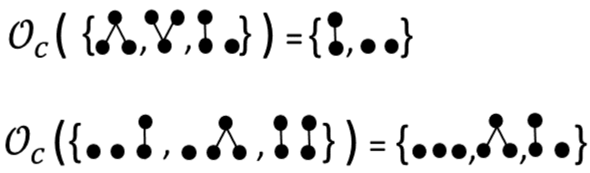


Figure 7.6: Illustration of the operation $\mathcal{O}_c$.

Definition 7.3.4 (Convex-covtree)**.** *Convex-covtree is the partial order* $(\chi, \prec)$*, where* $\Gamma_n \prec \Gamma_m$ *if and only if* $n < m$ *and* ${\mathcal{O}_c}^{m-n}(\Gamma_m) = \Gamma_n$*.*

The nodes in the first three levels of convex-covtree are shown in figure 7.7. We will also need the following definition,

Definition 7.3.5 (Convex-certificate of a path)**.** *An order* C *is a convex-certificate of a path* $\mathcal{P}$ *if it is a convex-certificate of every node in* $\mathcal{P}$*.*

The definition of convex-covtree is obtained from that of covtree (cf. chapter 4) simply by replacing n-stems with n-suborders. Indeed the two resulting structures share some similarities, including:

(1) if C is a (convex-)certificate of a node Γ_n in (convex-)covtree then C is a (convex-)certificate of all nodes below Γ_n,

(2) every inextendible path in (convex-)covtree has a (convex-)certificate.[9]

Properties (1) and (2) allow us to interpret a random walk on convex-covtree as a covariant process of growth: the growing order is a convex-certificate of the path which is traced by the random walk. Each node in the path corresponds to a covariant property of the growing order, *i.e.* Γ_n is the set of n-suborders of the growing order.

But *which* orders are produced by the process? In contrast to all existing growth models, a dynamics on convex-covtree can produce *finite* orders. This is because convex-covtree contains maximal nodes, meaning that some of its inextendible paths are finite.

[9] We will prove this in lemmas 7.3.9 and 7.3.11.

A finite inextendible path has only a finite convex-certificate, and so if a random walk traces a finite path then a finite order is generated. We leave further discussion of maximal nodes and finite inextendible paths to section 7.3.2. The convex-certificates of infinite paths are necessarily infinite (since they contain n-suborders for every $n > 0$) and every infinite order (past-finite, future-finite or neither) is a convex-certificate of some infinite path in convex-covtree.

In summary, the sample space is vast and contains all infinite orders and many (but not all) finite orders. In keeping with the motivation of this chapter—to develop growth dynamics for two-way infinite orders—it is natural to ask whether there is a way to consistently restrict the sample space to $\Omega_{\mathbb{Z}}$. We will show in section 7.3.3 that it is possible and that in this case the observables are the convex-events. We will also show that an inconsistency arises (property (2) no longer holds) when restricting the sample space to $\Omega_{\mathbb{N}}$, suggesting that convex suborders are unsuitable for describing past-finite growth.

7.3.2 The sample space of convex-covtree

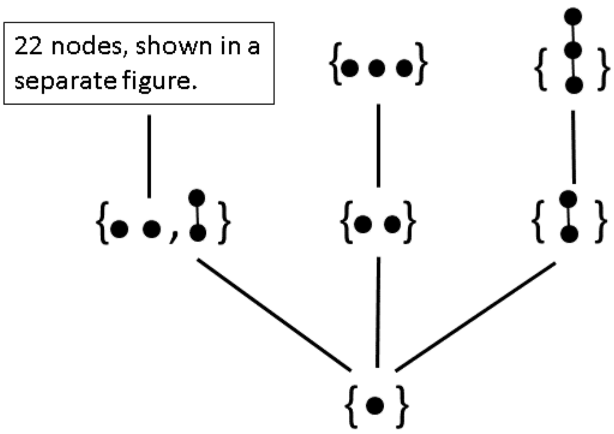
We begin by considering *finite* inextendible paths. Each finite inextendible path ends at a maximal node characterised by the following lemma.

Lemma 7.3.6. *If Γ_n is maximal in convex-covtree then it is a singleton $\Gamma_n = \{C_n\}$ whose only convex-certificate is C_n.*

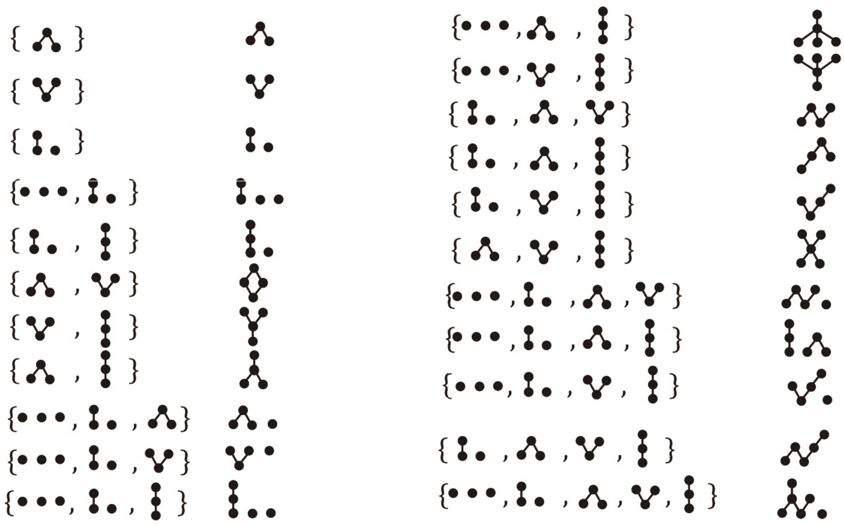
Proof. Let $\Gamma_n = \{C_n\}$ and let its only convex-certificate be C_n. Suppose for contradiction that $\Gamma_{n+1} \succ \Gamma_n$. Then there exists some D with cardinality $> n$ which is a convex-certificate of Γ_{n+1} and hence of Γ_n. Contradition. Therefore Γ_n is maximal.

Suppose that $\Gamma_n = \{C_n\}$ has a convex-certificate $D \neq C_n$. Then D has cardinality $> n$ and therefore $\{D\} \succ \Gamma_n \implies \Gamma_n$ is not maximal. Similarly, if Γ_n is not a singleton then it has a convex-certificate D with cardinality $> n \implies \{D\} \succ \Gamma_n$. □

The singleton $\Gamma_4 = \{$  $\}$ is an example of a maximal node. To see that Γ_4 has no convex-certificate of cardinality > 4 it is sufficient to attempt to construct such a convex-certificate by adding a single element to (a representative of) . For example, we can

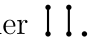


(a) The first three levels of convex-covtree.



(b) 22 nodes of convex-covtree and their convex-certificates. These are the level 3 nodes which appear directly above the doublet.

Figure 7.7: The first three levels of convex-covtree.

add the new element to form the 5-order  , but this 5-order is not a convex-certificate of Γ_4 since it contains  as a 4-suborder. Continuing in this way, we find that it is impossible to form a convex-certificate of Γ_4 by adding an element to . Indeed,  is the unique convex-certificate of Γ_4.

The existence of maximal nodes implies the existence of finite inextendible paths. We can characterise finite inextendible paths as follows:

Lemma 7.3.7. *An inextendible path $\mathcal{P}$ is finite if and only if it contains a singleton $\{C_n\}$, where C_n is not the n-chain or the n-antichain.*

To prove lemma 7.3.7 we will need:

Lemma 7.3.8. *Let C_n be an n-order which is not the n-chain or the n-antichain. Then every convex-certificate of $\{C_n\}$ has cardinality less than n^2.*

Proof. For any (finite or infinite) order C, let $w(C)$ and $h(C)$ denote the width and height of C, respectively. Note that $|C| \leq h(C)w(C)$. Additionally, if C is a convex-certificate of $\{C_n\}$ then $w(C) = w(C_n) < n$. We will show that if C is a convex-certificate of $\{C_n\}$ then $h(C) \leq n$ and the result follows.

Let C be an order with $h(C) > n$ and suppose for contradiction that C is a convex-certificate of $\{C_n\}$. Let D be a chain of length $n+1$ in C and let $\mathcal{H}$ be the convex hull of D. Then $|\mathcal{H}| = n + k$ for some $k > 0$. Note that $\mathcal{H}$ is an interval by construction, *i.e.* it has a single maximal element and a single minimal element. We will now show by induction that $\mathcal{H}$ is a chain and therefore C is not a convex-certificate of $\{C_n\}$.

One way to obtain C_n from $\mathcal{H}$ is to remove the minimal element of $\mathcal{H}$ to form the order $\mathcal{H}_{-1}$, then remove a minimal element of $\mathcal{H}_{-1}$ to form $\mathcal{H}_{-2}$ and so on until $\mathcal{H}_{-k} = C_n$. Since $\mathcal{H}$ has a unique maximal element, $\mathcal{H}_{-k} = C_n$ has a unique maximal element.

Another way to obtain C_n from $\mathcal{H}$ is to remove the maximal element of $\mathcal{H}$ to form the order $\mathcal{H}^{-1}$, then remove a minimal element of $\mathcal{H}^{-1}$ to form $\mathcal{H}^{-1}_{-1}$, then remove a minimal element of $\mathcal{H}^{-1}_{-1}$ to form $\mathcal{H}^{-1}_{-2}$ and continue to remove minimal elements until $\mathcal{H}^{-1}_{-k+1} = C_n$. The top level of $\mathcal{H}^{-1}_{-k+1} = C_n$ is level $h(C) - 1$ of $\mathcal{H}$, and since C_n has a unique maximal element we learn that $\mathcal{H}$ has only one element at level $h(C) - 1$.

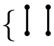
Suppose $\mathcal{H}$ has only one element at each of the levels $h(C), h(C)-1, ..., h(C)-r+1$ for some $r < h(C)$. Then $\mathcal{H}^{-r}_{-k+r} = C_n$ is constructed by removing the top r levels of $\mathcal{H}$ and therefore the top level of $\mathcal{H}^{-r}_{-k+r} = C_n$ is level $h(C)-r$ of $\mathcal{H}$. Since $\mathcal{H}^{-r}_{-k+r} = C_n$ has a unique maximal element we learn that $\mathcal{H}$ has only one element at level $h(C)-r$. Therefore, by induction $\mathcal{H}$ has a single element at each level, *i.e.* $\mathcal{H}$ is a chain. □

Proof to lemma 7.3.7. Let $\{C_n\} \in \mathcal{P}$ and suppose for contradiction that $\mathcal{P}$ is infinite. Then for any $N > n^2$ there exists a node $\Gamma_N \in \mathcal{P}$. Let C denote a convex-certificate of Γ_N and note that $|C| \geq N > n^2$. Since $\Gamma_N \succ \{C_n\}$, C is a convex-certificate of $\{C_n\}$. Contradiction. That the converse is true follows from the fact that every maximal node is a singleton (lemma 7.3.6). □

We can also identify the convex-certificates of the finite inextendible paths:

Lemma 7.3.9. *If $\mathcal{P} = \Gamma_1 \prec \Gamma_2 \prec .. \prec \Gamma_k$ is a finite inextendible path then $C_k \in \Gamma_k$ is the unique convex-certificate of $\mathcal{P}$.*

Proof. Clearly, C_k is a convex-certificate of $\mathcal{P}$ and there are no other convex-certificates of $\mathcal{P}$ with cardinality $\leq k$. Suppose C_l is a convex-certificate of $\mathcal{P}$ with cardinality $l > k$. Then $\{C_l\} \succ \Gamma_k$. Contradiction. □

A corollary is that the corresponding sample space contains spacetimes of finite volume, namely the convex-certificates of the finite inextendible paths. An n-order C_n is an element of the sample space if there is no order $D \neq C_n$ whose only n-suborder is C_n. For example, the sample space contains the 4-order , but it does not contain the 3-order  since $\{$$\} \prec \{$$\}$.

Lemma 7.3.10. *The sample space contains countably many finite orders.*

Proof. Let $Q(n)$ denote the number of singletons $\{C_n\}$ at level n in convex-covtree, where C_n is not the n-chain or the n-antichain. Each of these $Q(n)$ nodes is in at least one finite path and no two are in the same path. Therefore there are at least $\lim_{n\to\infty} Q(n)$ finite inextendible paths. □

It may seem that the sample space is entropically dominated by the infinite configurations, as there are uncountably many of these and only countably many finite configurations. But if one assigns transition probabilities uniformly such that the probabilities to transition from a given node to any of its children are equal, then the event that spacetime has finite volume happens with probability $> \frac{1}{22}$ (since $\frac{1}{22}$ is the probability of reaching a singleton which does not contain a chain or an antichain by level 3). Dynamics which only give rise to infinite universes are exactly those which satisfy $\mathbb{P}(\Gamma) = 0$ whenever Γ is a singleton which does not contain a chain or an antichain.[10]

Finally, we note that every *infinite* order is a convex-certificate of some *infinite* inextendible path. Additionally,

Lemma 7.3.11. *Every infinite path in convex-covtree has a convex-certificate.*

The proof is analogous to that of theorem 4.1.10.

Together, lemmas 7.3.9 and 7.3.11 enable us to interpret a walk on convex-covtree as a growth process—they guarantee that each realisation of the walk will produce some order. A path has more than one convex-certificate if its convex-certificates are convex-rogues and, in this case, which convex-certificate is the growing order is up for interpretation (*e.g.* we can consider all convex-certificates of a given path to be physically equivalent).

7.3.3 $\mathbb{Z}$-covtree

Since our motivation in this chapter is to find a covariant analogue to alternating growth, it is natural for us to ask whether convex-covtree can be truncated into a tree whose sample space corresponds to $\Omega_{\mathbb{Z}}$. For a start, we can consider the subtree of convex-covtree which contains only the nodes which have *infinite* convex-certificates or equivalently the subtree of convex-covtree which contains exactly all infinite paths. By truncating the finite inextendible paths we remove the finite orders from the sample space and lemma 7.3.11 guarantees that each inextendible path in this truncated covtree has a convex-certificate in Ω. However, there is no guarantee that every path has a convex-certificate

[10] For any $n > 1$, if Γ_n is a singleton which contains a chain then it is contained in a unique inextendible path, $\{\,\cdot\,\} \prec \{:\} \prec \{\vdots\} \prec ...$. Similarly, if Γ_n is a singleton which contains an antichain then it is contained in a unique inextendible path, $\{\cdot\} \prec \{\cdot\cdot\} \prec \{\cdot\ \cdot\ \cdot\} \prec ...$.

in $\Omega_{\mathbb{Z}}$. Indeed, there exist infinite paths which only have convex-certificates in $\Omega_{\mathbb{N}}$ and others which only have convex-certificates in $\Omega_{\mathbb{Z}^-}$. The following theorem identifies the paths which have convex-certificates in $\Omega_{\mathbb{Z}}$ and which are therefore of interest to us,

Theorem 7.3.12. *An infinite path $\mathcal{P}$ has a convex-certificate in $\Omega_{\mathbb{Z}}$ if and only if every node in $\mathcal{P}$ has a convex-certificate in $\Omega_{\mathbb{Z}}$.*

Proof. Given an infinite path $\mathcal{P} = \Gamma_1 \prec \Gamma_2 \prec ...$ each of whose nodes has a convex-certificate in $\Omega_{\mathbb{Z}}$, the following inductive algorithm generates an infinite nested sequence of causal sets, $\tilde{C}_{t_1} \subset \tilde{C}_{t_2} \subset ...$, whose ground-sets $[r_1, s_1], [r_2, s_2], ...$ respectively, satisfy $r_1 > r_2 > ....$ and $s_1 < s_2 < ...$:

Step 1:

1.0) Pick some natural number $m_0 > 0$ and consider $\Gamma_{m_0} \in \mathcal{P}$.

1.1) Let $\Gamma_{m_1} \in \mathcal{P}$ contain some convex-certificate C_{m_1} of Γ_{m_0} (cf. lemma 4.1.12). Pick a representative $\tilde{C}_{m_1}$ of C_{m_1} and set $\tilde{C}_{t_1} := \tilde{C}_{m_1}$.

1.2) Go to step 2.

Step $k > 1$:

k.1) Let $\Gamma_{m_k} \in \mathcal{P}$ contain some convex-certificate C_{m_k} of $\Gamma_{t_{k-1}} \in \mathcal{P}$ (cf. lemma 4.1.12). Additionally, there exists a representative $\tilde{C}_{m_k}$ of C_{m_k} with ground-set $[p_k, q_k]$ which contains $\tilde{C}_{t_{k-1}}$ as a sub-causet and satisfies at least one of (a) $p_k < r_{k-1}$ or (b) $q_k > s_{k-1}$. If there exists some $\tilde{C}_{m_k}$ which satisfies both (a) and (b), set $\tilde{C}_{t_k} := \tilde{C}_{m_k}$. Otherwise, pick a representative $\tilde{C}_{m_k}$ that satisfies (a) or (b). Go up one node along the path to $\Gamma_{1+m_k} \in \mathcal{P}$. Let $\tilde{C} \in \tilde{\Omega}_{\mathbb{Z}}$ be an infinite convex-certificate of Γ_{1+m_k} which contains $\tilde{C}_{m_k}$ as a subcauset. Set $\tilde{C}_{t_k} := \tilde{C}|_{[p_k, q_k+1]}$ if $\tilde{C}_{m_k}$ satisfies (a) or $\tilde{C}_{t_k} := \tilde{C}|_{[p_k-1, q_k]}$ if $\tilde{C}_{m_k}$ satisfies (b).

k.2) Go to step $k+1$.

By construction, the union $\tilde{C} := \bigcup_{i=1}^{\infty} \tilde{C}_{t_i} \in \tilde{\Omega}_{\mathbb{Z}}$ is a labeled convex-certificate of $\mathcal{P}$. Therefore, if every node in $\mathcal{P}$ has a convex-certificate in $\Omega_{\mathbb{Z}}$ then $\mathcal{P}$ has a convex-certificate in $\Omega_{\mathbb{Z}}$. That the converse is true follows from definition 7.3.5. □

Finally, we can define:

Definition 7.3.13 ($\mathbb{Z}$-covtree)**.** *$\mathbb{Z}$-covtree is the subtree of convex-covtree which contains* *exactly all nodes which have a convex-certificate in* $\Omega_{\mathbb{Z}}$.

$\mathbb{Z}$-covtree is the two-way infinite analogue of covtree which we have set out to build. Theorem 7.3.12 guarantees that every inextendible path in $\mathbb{Z}$-covtree has at least one convex-certificate in $\Omega_{\mathbb{Z}}$ and thus allows for every random walk on $\mathbb{Z}$-covtree to be interpreted as a dynamics with sample space $\Omega_{\mathbb{Z}}$.

To see the relationship between a walk on $\mathbb{Z}$-covtree and the corresponding dynamics, for each Γ_n in $\mathbb{Z}$-covtree let $cert_{\mathbb{Z}}(\Gamma_n) \subset \tilde{\Omega}_{\mathbb{Z}}$ denote the set of labeled convex-certificates of Γ_n whose ground-set is $\mathbb{Z}$. A dynamics is given by a measure μ on the σ-algebra generated by the $cert_{\mathbb{Z}}(\Gamma_n)$'s, where $\mu(cert_{\mathbb{Z}}(\Gamma_n)) = \mathbb{P}(\Gamma_n)$. The observables in these dynamics are exactly the convex-events since,

Lemma 7.3.14. *The* σ*-algebra generated by the* $cert_{\mathbb{Z}}(\Gamma_n)$ *is equal to the* σ*-algebra generated by the* $convex(C_n)$.

The proof is analogous to that of lemma 4.1.15.

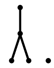
Lemma 7.3.14 strengthens the analogy between covtree and $\mathbb{Z}$-covtree—the observables of covtree are the stem-events while the observables of $\mathbb{Z}$-covtree are the convex-events. $\mathbb{Z}$-covtree is to alternating poscau what covtree is to labeled poscau. Convex suborders are to two-way infinite dynamics what stems are to past-finite dynamics.

7.3.4 The non-existence of $\mathbb{N}$-covtree

This section highlights the technicalities which enabled us to define $\mathbb{Z}$-covtree in the previous section. For comparison, we show that one cannot truncate convex-covtree into a tree whose sample space is $\Omega_{\mathbb{N}}$.

Why do some infinite paths in convex-covtree have no convex-certificates in $\Omega_{\mathbb{Z}}$? Theorem 7.3.12 tells us that an infinite path in convex-covtree has a convex-certificate in $\Omega_{\mathbb{Z}}$ if and only if each of its nodes has a convex-certificate in $\Omega_{\mathbb{Z}}$, and therefore the reason that some paths have no convex-certificate in $\Omega_{\mathbb{Z}}$ is because they contain nodes which have no convex-certificate in $\Omega_{\mathbb{Z}}$.

In other words, there exist nodes all of whose infinite convex-certificates are contained in $\Omega_{\mathbb{N}}$, and an infinite path which contains such nodes can only have convex-certificates in

$\Omega_{\mathbb{N}}$. For example, consider the node $\Gamma_3 = \{$$\}$ whose unique minimal convex-certificate is . We can construct any convex-certificate of Γ_3 by starting with its minimal convex-certificate and then adding elements to it. In particular, if Γ_3 has a convex-certificate in $\Omega_{\mathbb{Z}}$ or $\Omega_{\mathbb{Z}^-}$ then we should be able to grow a convex-certificate of Γ_3 by adding an element which is below or unrelated to every element in . There are 5 ways to add such an element, but none produces a convex-certificate of Γ_3 (*e.g.*  contains the 3-antichain as a convex-suborder). Therefore, Γ_3 has no convex-certificates in $\Omega_{\mathbb{Z}}$ or in $\Omega_{\mathbb{Z}^-}$. Finally, note that Γ_3 does have a convex-certificate in $\Omega_{\mathbb{N}}$, namely the order which contains  topped with an infinite chain. Therefore the infinite path containing Γ_3 only has convex-certificates in $\Omega_{\mathbb{N}}$. Similarly, there exist nodes all of whose infinite convex-certificates are contained in $\Omega_{\mathbb{Z}^-}$, and an infinite path which contains such nodes can only have convex-certificates in $\Omega_{\mathbb{Z}^-}$. The node $\{$$\}$ is an example.

In contrast, there exists no node in convex-covtree all of whose infinite convex-certificates are contained in $\Omega_{\mathbb{Z}}$, since if a node has a convex-certificate in $\Omega_{\mathbb{Z}}$ then it has a convex-certificate in $\Omega_{\mathbb{N}}$ and in $\Omega_{\mathbb{Z}^-}$.[11] Despite this, there exist infinite paths in convex-covtree whose infinite convex-certificates are only contained in $\Omega_{\mathbb{Z}}$. For example, consider the path,

$\mathcal{P} =$  $\prec \dots ,$ (7.12)

whose convex-certificate is the order D shown on the right of figure 7.8. Each node in $\mathcal{P}$ has a convex-certificate in $\Omega_{\mathbb{N}}$ (as illustrated in figure 7.8) but there is no one order in $\Omega_{\mathbb{N}}$ which is a convex-certificate of *every* node in $\mathcal{P}$.[12]

We find that for every node in an infinite path to have a convex-certificate in $\Omega_{\mathbb{N}}$ is

[11] To see this, let $\tilde{C} \in \tilde{\Omega}_{\mathbb{Z}}$ be a labeled convex-certificate of some Γ_n and let $\tilde{C}|_{[k,l]}$ be a finite convex-certificate of Γ_n. Then $\tilde{C}|_{[k,\infty)}$ is order-isomorphic to some $\tilde{D} \in \tilde{\Omega}_{\mathbb{N}}$ and $\tilde{D}$ is a convex-certificate of Γ_n. Similarly, $\tilde{C}|_{(\infty,l]}$ is order-isomorphic to some $\tilde{E} \in \tilde{\Omega}_{\mathbb{Z}^-}$ and $\tilde{E}$ is a convex-certificate of Γ_n.

[12] One way to see this is to notice that for every $n > 3$, $\Gamma_n \in \mathcal{P}$ has a unique minimal convex-certificate, namely the diamond sandwiched between two $(n-3)$-chains. Now, pick some $n > 3$ and without loss of generality pick a representative of its minimal convex-certificate, $\tilde{C}_{2n-6}$, with ground-set $[0, 2n-6]$. We seek a labeled minimal convex-certificate $\tilde{C}_{2n-4}$ of Γ_{n+1} which contains $\tilde{C}_{2n-6}$ as a subcauset, and find that $\tilde{C}_{2n-4}$ must have ground-set $[-1, 2n-5]$. Next we seek a labeled minimal convex-certificate $\tilde{C}_{2n-2}$ of Γ_{n+2} which contains $\tilde{C}_{2n-4}$ as a subcauset, and find that $\tilde{C}_{2n-2}$ must have ground-set $[-2, 2n-4]$ *etc.* Since at each stage we add a positive and a negative integer to the ground-set, in the infinite limit the labeled convex-certificate must have ground-set $\mathbb{Z}$.

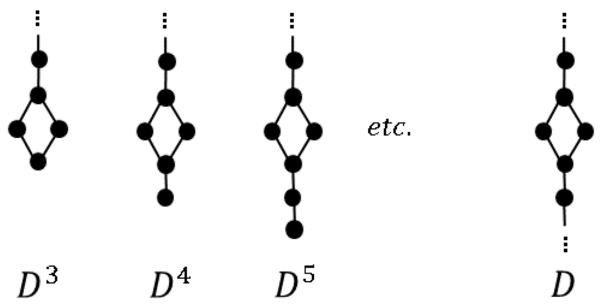


Figure 7.8: The order $D \in \Omega_{\mathbb{Z}}$ shown on the right is a convex-certificate of the path $\mathcal{P}$. Every node in $\mathcal{P}$ has a convex-certificate in $\Omega_{\mathbb{N}}$: D^3 is a convex-certificate of $\Gamma_n \in \mathcal{P}$ only for $n \leq 3$, D^4 is a convex-certificate of $\Gamma_n \in \mathcal{P}$ only for $n \leq 4$, D^5 is a convex-certificate of $\Gamma_n \in \mathcal{P}$ only for $n \leq 5$, *etc.* There is no order in $\Omega_{\mathbb{N}}$ which is a convex-certificate of every node in $\mathcal{P}$.

necessary but not sufficient for the path to have a convex-certificate in $\Omega_{\mathbb{N}}$. There is no analogue to theorem 7.3.12 in the past-finite case.

Altogether, this reveals that the subset of convex-covtree which contains exactly the paths whose convex-certificates are in $\Omega_{\mathbb{N}}$ is ill-defined. This is because, if it exists then it must contain every node which has a convex-certificate in $\Omega_{\mathbb{N}}$, and therefore it must contain paths which do not have a convex-certificate in $\Omega_{\mathbb{N}}$ (*e.g.* $\mathcal{P}$ of (7.12)), which is a contradiction. There is no $\mathbb{N}$-covtree. This suggests that convex-events are not rich enough to exhaust the set of observables in past-finite dynamics. This is also true in the future-finite case.

One can understand this difference between $\Omega_{\mathbb{N}}$ and $\Omega_{\mathbb{Z}}$ using the language of metric spaces. For any two orders C and D, let $C \sim D$ if and only if C and D are a convex-rogue pair, *i.e.* if they share the same n-suborders for all n. Let $\Omega_{\mathbb{N}}/\sim$ and $\Omega_{\mathbb{Z}}/\sim$ be quotient spaces under the convex-rogue equivalence relation, so that their elements are equivalence classes of orders denoted by $[C]$ *etc.* We can consider these quotient spaces as metric spaces with metric $d([C],[D]) = \frac{1}{2^n}$, where n is the largest integer for which representatives of $[C]$ and $[D]$ have the same sets of n-suborders. Given a node Γ_n in convex-covtree we can associate with it a subset $[cert_{\mathbb{N}}(\Gamma_n)] \subseteq \Omega_{\mathbb{N}}/\sim$, namely the set of elements of $\Omega_{\mathbb{N}}/\sim$ whose representatives are convex-certificates of Γ_n, and similiarly $[cert_{\mathbb{Z}}(\Gamma_n)] \subseteq \Omega_{\mathbb{Z}}/\sim$. Given a path $\mathcal{P} = \Gamma_1 \prec \Gamma_2 \prec ...$, we can associate with it the sets $[cert_{\mathbb{N}}(\mathcal{P})] = \bigcap_{\Gamma_n \in \mathcal{P}} [cert_{\mathbb{N}}(\Gamma_n)]$ and $[cert_{\mathbb{Z}}(\mathcal{P})] = \bigcap_{\Gamma_n \in \mathcal{P}} [cert_{\mathbb{Z}}(\Gamma_n)]$. The metric space $(\Omega_{\mathbb{Z}}/\sim, d)$ is complete, and therefore by Cantor's lemma $[cert_{\mathbb{Z}}(\mathcal{P})]$ is non-empty whenever all the $[cert_{\mathbb{Z}}(\Gamma_n)]$'s are non-empty (cf. theorem 7.3.12). On the other hand, the metric space $(\Omega_{\mathbb{N}}/\sim, d)$ is not complete (as shown by example in figure 7.8) and therefore $[cert_{\mathbb{N}}(\mathcal{P})]$ can be empty when all the $[cert_{\mathbb{Z}}(\Gamma_n)]$'s are non-empty.

Chapter 8

Conclusion

This thesis presents a contribution to the growth dynamics program whose aim is to interpret the path integral for quantum gravity as a generalised stochastic process of spacetime growth. Our work was motivated by two pillars of modern theoretical physics: general covariance and quantum interference.

General covariance, the gauge invariance of General Relativity, must emerge from quantum gravity and as such can be used to formulate guiding precepts for each quantum gravity approach. Additionally, its importance in the development of General Relativity suggests that it also has an important part to play in the development of a theory of quantum gravity. We focused on one particular facet of general covariance by asking whether one can formulate the laws of physics in a way which makes reference only to physical (and not to gauge) degrees of freedom. Physics has thus far favoured gauge theories, but we were encouraged by the thought that the unified nature of quantum gravity combined with the discreteness of causal sets may offer new possibilities. Indeed, we were able to make progress. Building on the understanding that in causal set theory labels play the role of spacetime coordinates, we set out to find a formulation of growth dynamics which made no reference to labels and to understand its relationship to the existing label-dependent (gauge) formulation. We have been successful in building a label-independent framework (dubbed "covtree") and in making a precise mathematical statement to the effect that the set of label-independent dynamics and the set of label-dependent dynamics are equal. But at the physical level the relationship between the two remains far from clear. In particular, understanding the relationship between label-independent models and those label-dependent models which satisfy the so-called "discrete general covariance" condition remains an open problem.

We had another purpose in seeking a label-independent formulation for growth dynamics: the label-dependent dynamics have thus far resisted quantization and this new formulation may offer a new route to quantum dynamics. Indeed, a label-independent formulation may prove necessary since concepts unrelated to each other in our current theories, such as general covariance and quantum interference, may prove inseparable in quantum gravity. Building on an existing—but never before realised—proposal for deriving a decoherence functional from growth dynamics, we proved theorems which determine when the proposal is valid and applied it to obtain the first examples of (label-dependent) quantum dynamics for causal sets. We generalised the construction to a wide class of growth dynamics, including our new label-independent models. Our results are an important proof of principle that quantal generalisations of growth dynamics can be obtained, but the examples we provided are unphysical toy models and obtaining a physical quantum dynamics remains a challenge.

8.1 Summary of main results

(Chapter 2). Building on the existing literature, we established a coherent language which enabled us to handle unlabeled objects directly.

(Chapter 4). We defined covtree, a tree whose nodes are certain sets of n-orders, and proved that in a well-defined sense a random walk on covtree is a label-independent growth dynamics. We proved that the stem algebra, $\mathcal{R}(\mathcal{S})$, is the algebra of observables for these dynamics and that therefore every walk on labeled poscau induces a walk on covtree. Additionally, we proved that every covtree walk can be extended to a walk on labeled poscau, revealing the complex nature of the relationship between the label-independent models and the "covariant" label-dependent models.

We identified the covtree paths which correspond to orders with posts and breaks, and proved that covtree is self-similar. We used these results to obtain the label-independent analogue of cosmic renormalisation and to provide a classification of self-similar dynamics. This led to speculative proposals regarding a causality condition and the relationship between covtree dynamics and the CSG models.

We provided further results about the structure of covtree and a toy example of how a better understanding of this structure can be used to obtain physically interesting dynamics.

(Chapter 5). Building on an existing—but never before realised—proposal for deriving a decoherence functional from growth dynamics, we proved theorems which determine when the proposal is valid, applied it to obtain the first examples of quantum dynamics for causal sets and generalised the construction to a wide class of growth dynamics.

(Chapter 6). We obtained a functional form of the probability that the element n is a post in any CSG dynamics.

We identified the physical observables in two classes of (labeled) cyclic dynamics as those measurable events which cannot distinguish between non-cyclic cosmologies. We proved that the σ-algebras containing these observables are countably generated by certain subsets of cylinder sets. This enabled us to conceive of these cyclic dynamics as random walks on the associated trees and we applied our generalised theorem from chapter 5 to consider their quantization.

(Chapter 7). We considered whether the growth dynamics paradigm can accommodate cosmologies in which time has no beginning. We modified the CSG models to allow for two-way growth, leading to the identification of the convex-events as the observables in two-way growth dynamics. Building on this, we constructed a variation of covtree (dubbed $\mathbb{Z}$-covtree) and proved that it provides a framework for two-way growth dynamics.

8.2 Outlook

We were successful in building frameworks for growth dynamics by identifying σ-algebras of observables and establishing new methods for defining measures on them. But these frameworks accommodate vast classes of models and identifying which of these models are physically interesting remains an open problem.

Which physically meaningful constraints can be imposed on the covtree transition probabilities to obtain physical dynamics? We can no longer impose a label-invariance condition, since covtree makes no reference to labels, while the tension between the global nature of covtree and the local nature of causality poses a challenge in identifying a suitable causality condition. It is the opinion of the author that solving for cyclic covtree dynamics (namely, dynamics which give rise to infinitely many posts or breaks with unit probability) would be a worthwhile pursuit in which progress can be made by building on the results of section 4.2. Solving for those covtree dynamics which correspond to CSG models and/or in which the set of rogues is null may also be worthwhile, but considerably more ambitious.

Cyclic dynamics could also play a key role in the development of quantum growth models. Transitive Percolation is a cyclic dynamics which resisted our attempts at quantization and it is natural to wonder whether this is a feature shared by all cyclic models or whether there exist cyclic models which can be quantized through the prescription which we considered. This problem could be approached by building on the work in [Ash and McDonald, 2003, 2005] which says that a CSG model is cyclic whenever its sequence of couplings $(t_0, t_1, t_2, ...)$ is the sequence of moments of a probability distribution satisfying certain criteria. It would be interesting to understand whether an analogous statement can be made when the couplings take complex values since this will enable a systematic study of whether such models are amenable to quantization by applying the theorems we proved in section 5.2. Another avenue which one can explore in the search for quantum dynamics is, given a pair of σ-algebras $\Sigma_1 \subset \Sigma_2$, under which conditions does a complex measure on Σ_1 extend to Σ_2? This would shed light on whether, when the complex measure fails to extend to the σ-algebra, it is worthwhile to seek a sub-σ-algebra of observables, as we did in section 6.2.3. Seeking alternative prescriptions for quantization is also worthwhile. One route to such an alternative has already been suggested by Rafael Sorkin and Jason Wien in unpublished work where they provided a new derivation of the CSG models in which the causality condition is replaced by an ansatz for the transition probabilities. The hope is that generalising the ansatz to the quantum case would be an easier task than formulating a quantum causality condition, but this remains an open problem to date.

Finally, we must acknowledge that real progress requires a better understanding of quantum foundations, in particular within an irreproducible cosmological setting which hosts no observers. This will guide us towards more realistic formulations of quantum dynamics (*e.g.* ones in which the measure is valued in a higher dimensional vector space) and will enable extracting meaningful predictions from our models. Advances in this direction will enable the heuristic of becoming to be realised as a physical quantum dynamics for discrete spacetimes. In the meantime, growth dynamics remain a rich and fruitful research area in which to explore the challenges which arise en route to a theory of quantum gravity.

Appendix A

Glossaries

A.1 Causal sets

A **partial order** (or "**partially ordered set**" or "**poset**") is a pair, $(\Pi, \prec)$, where $\prec$ is a transitive, irreflexive relation on the ground-set Π. We may use the ground-set to denote the partial order (*e.g.* we may write "the partial order Π" instead of "the partial order $(\Pi, \prec)$"). The meaning should be clear from the context.

Let $x, y \in \Pi$. For any relation $x \prec y$, the **interval** $int(x, y)$ is defined to be the set of elements which lie between x and y in the order, namely, $int(x, y) := \{z | x \prec z \prec y\}$.

A partial order is **locally finite** if all of the intervals it contains are finite.

A **causal set** (or "**causet**") is a locally finite partial order.

An **order-isomorphism** is an order-preserving bijection between two causets. If there exists an order-isomorphism between Π and Ψ we say that Π and Ψ are **order-isomorphic** and write $\Pi \cong \Psi$.

The **restriction** of a causet $(\Pi, \prec)$ to $\Psi \subset \Pi$ is the partial order $(\Psi, \prec_\Psi)$ where $x \prec_\Psi y \iff x \prec y \; \forall x, y \in \Psi$. We say that $(\Psi, \prec_\Psi)$ is a **subcauset** in Π.

Let Ψ be a subcauset in Π. If $\{z | z \in \Pi \text{ and } x \prec z \prec y\} \subset \Psi \; \forall \; x, y \in \Psi$ then Ψ is a **convex subcauset** in Π.
The **convex hull** of Ψ is the smallest convex subcauset of Π which contains Ψ.

We say that Π contains a **copy** of Φ if there exists a convex subcauset $\Phi' \subseteq \Pi$ such that

$\Phi \cong \Phi'$.

If $x \prec y$ is a relation in Π, we say that x is **below** y or that y is **above** x or that x is an **ancestor** of y or that y is a **descendant** of x.

A relation $x \prec y$ is called a **link** if $int(x, y) = 0$. If $x \prec y$ is a link, we say that x is **directly below** y or that y is **directly above** x or that x is a **direct ancestor** of y or that y is a **direct descendant** of x.

The **valency** of x is the number of direct descendants of x.

The **past** of $x \in \Pi$ is the subcauset $past(x) := \{y \in \Pi | y \prec x\}$. This is the *non-inclusive* past, *i.e* $x \notin past(x)$. The **future** of $x \in \Pi$ is the subcauset $future(x) := \{y \in \Pi | y \succ x\}$. This is the *non-inclusive* future.

The **inclusive past** of $x \in \Pi$ is the subcauset $\overline{past(x)} := past(x) \cup \{x\}$. The **inclusive future** of $x \in \Pi$ is the subcauset $\overline{future(x)} := future(x) \cup \{x\}$.

A finite subcauset Ψ of Π is a **stem** in Π if $x \in \Psi \implies past(x) \subseteq \Psi$.
A causet Π is a **rogue** if there exists another causet $\Phi \ncong \Pi$ such that Ψ is a stem in Π if and only if Φ contains a copy of Ψ as stem.

$x \in \Pi$ is **minimal** if $past(x) = \emptyset$. Similarly, $x \in \Pi$ is **maximal** if $future(x) = \emptyset$.

Π is **originary** if it contains a unique minimal element.

Π is a **tree** if it is originary and every element in Π is directly above at most one other element. We call the minimal element of the tree the **root**.

Π is **past-finite** if $|past(x)| < \infty \ \forall \ x \in \Pi$. Similarly, Π is **future-finite** if $|future(x)| < \infty \ \forall \ x \in \Pi$. Π is **past-infinite** (future-infinite) if it is not past-finite (future-finite). Π is **two-way infinite** if it is both past-infinite and future-infinite.

An **antichain** is a causet all of whose elements are unrelated to each other. An n-antichain is an antichain with n elements.

A **chain** is a causet all of whose elements are related to each other. An n-chain is a chain with n elements.

A **path** in Π is a convex subcauset of Π which is a chain.

An element $x \in \Pi$ is in **level** L in Π if the longest chain of which x is the maximal element has cardinality L, *e.g.* level 1 comprises the minimal elements. For any $x \in \Pi$, $L(x)$ is an integer which denotes the level of x, *e.g.* $L(x) = 1$ if x is minimal.

The **width** of Π, $w(\Pi)$, is the cardinality of the largest antichain in Π. The **height** of Π, $h(\Pi)$, is the cardinality of the longest chain in Π, or equivalently the number of levels in Π.

A **post** is an element $x \in \Pi$ which is related to every other element in Π.

A **break** in Π is an ordered pair, (Ψ, Φ), of nonempty subsets of Π such that

(i) $\psi \in \Psi, \phi \in \Phi \implies \psi \prec \phi$, and

(ii) $\{\Psi, \Phi\}$ is a partition of Π.

We call Ψ and Φ the past and future of the break, respectively. If Π contains a break with past Ψ we say that Π contains a Ψ-break. If Π contains a post x with $past(x) = \Psi$ we say that Π contains a Ψ-post.

We will often represent a causet using a **Hasse diagram**, a graph in which elements are represented by nodes and links are represented by upward-going edges, *i.e.* there is an upward-going edge from the node x to the node y if and only if $x \prec y$ is a link.

A.2 Measure theory

For the benefit of the reader, we present some terminology of measure theory used in the text. The following is adapted from [Diestel and Uhl, 1977; Kolmogorov and Fomin, 1975].

Let X denote a fixed set. We call X the **sample space**, **history space** or **space of histories**. An element of X is a **history** or an **outcome**.

A **system of sets** is a set whose elements are themselves sets. In the following, the elements of a given system of sets are assumed to be certain subsets of X, unless otherwise stated.

A nonempty system of sets R is called a **ring** if $A \triangle B \in R$ and $A \cap B \in R$ whenever $A, B \in R$. Equivalently, a ring is a system of sets closed under finite unions, intersections, differences and symmetric differences.

A ring of sets is called a **σ-ring** if it contains the union $\bigcup_{n=1}^{\infty} A_n$ whenever it contains the sets $A_1, A_2, ..., A_n, ...$ It follows that a σ-ring is also closed under countable intersections and under complements.

A set E is called the **unit** of a system of sets S if $E \in S$ and $A \cap E = A$ for every $A \in S$.

A ring with a unit E is called an **event algebra** or **algebra**. An element of an algebra is called a **measurable event** or **event**.

A σ-ring with a unit E is called a **σ-algebra**.

Given any nonempty system of sets S, there is a unique σ-algebra $\sigma(S)$ containing S and contained in every σ-algebra containing S. $\sigma(S)$ is called the **σ-algebra generated by S**.

A system of sets S is called a **semiring** if

1. S contains the empty set;
2. $A \cap B \in S$ whenever $A \in S, B \in S$;
3. if S contains the sets A and $A_1 \subset A$, then A can be represented as a finite union $A = \bigcup_{k=1}^{n} A_k$ of pairwise disjoint sets, with the given A_1 as its first term.

A set function μ is called a **(real probability) measure** if

1. The domain of definition S_μ of μ is a semiring;
2. μ takes values in $[0, 1]$;
3. μ is finitely additive in the sense that if A is a set in S_μ such that $A = \bigcup_{k=1}^{n} A_k$, where $A_1, ..., A_n$ are pairwise disjoint sets in S_μ then $\mu(A) = \sum_{k=1}^{n} \mu(A_k)$.

A measure μ with domain of definition S_μ is said to be **countably additive** if whenever A is a set in S_μ such that $A = \bigcup_{k=1}^{\infty} A_k$, where $A_1, ..., A_n, ...$ are pairwise disjoint sets in S_μ then $\mu(A) = \sum_{k=1}^{\infty} \mu(A_k)$.

A measure μ is called an **extension** of a measure m if $S_m \subset S_\mu$ and $\mu(A) = m(A)$ for every $A \in S_m$. Given a measure m on $S_m \subset S_\mu$, if there exists an extension μ we say that m **possesses an extension** or m **extends** to S_μ.

A measure μ is called a **restriction** of a measure m if $S_m \supset S_\mu$ and $\mu(A) = m(A)$ for every $A \in S_m$.

Let $\mathfrak{A}$ be an algebra of subsets of X, and let B be a Banach space. Let B^* denote the dual (or conjugate) space of B whose elements b^* are continuous linear functions on X (with respect to the norm topology). A **vector measure** F is a finitely additive function $F : \mathfrak{A} \to B$. If $B = \mathbb{C}$ we say that F is a **complex measure**.

Given a vector measure F, its **semi-variation** $||F||$ is defined by

$$||F||(E) := \sup\{|b^*F|(E) : ||b^*|| \leq 1\}, \tag{A.1}$$

where $|b^*F|$ is the variation (definition 5.1.2). If $||F||(X) < \infty$ then F is **bounded**.

Let (E_n) denote a sequence of pairwise disjoint members of $\mathfrak{A}$. F is **strongly additive** if, for any (E_n), the series $\sum_{n=1}^{\infty} F(E_n)$ converges in the norm topology. F is **countably additive** if, for any (E_n) such that $\bigcup_{n=1}^{\infty} E_n \in \mathfrak{A}$, $F(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} F(E_n)$ in the norm topology. F is **weakly countably additive** if, for any (E_n) such that $\bigcup_{n=1}^{\infty} E_n \in \mathfrak{A}$, $F(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} F(E_n)$ in the weak topology (where the weak topology is the coarsest topology on B such that each element of B^* remains a continuous function).

Bibliography

N. Afshordi, S. Aslanbeigi, and R. D. Sorkin. A Distinguished Vacuum State for a Quantum Field in a Curved Spacetime: Formalism, Features, and Cosmology. *JHEP*, 08:137, 2012. doi: 10.1007/JHEP08(2012)137.

M. Ahmed and D. Rideout. Indications of de Sitter Spacetime from Classical Sequential Growth Dynamics of Causal Sets. *Phys. Rev. D*, 81:083528, 2010. doi: 10.1103/PhysRevD.81.083528.

M. Ahmed, S. Dodelson, P. B. Greene, and R. Sorkin. Everpresent Λ. *Phys. Rev. D*, 69:103523, 2004. doi: 10.1103/PhysRevD.69.103523.

N. Alon, B. Bollobás, G. Brightwell, and S. Janson. Linear extensions of a random partial order. *Ann. Applied Prob.*, 4(1):108–123, February 1994.

A. Anderson and B. S. DeWitt. Does the Topology of Space Fluctuate? *Found. Phys.*, 16: 91–105, 1986. doi: 10.1007/BF01889374.

A. Ash and P. McDonald. Moment problems and the causal set approach to quantum gravity. *J. Math. Phys.*, 44:1666–1678, 2003. doi: 10.1063/1.1519668.

A. Ash and P. McDonald. Random partial orders, posts, and the causal set approach to discrete quantum gravity. *J. Math. Phys.*, 46:062502, 2005. doi: 10.1063/1.1922070.

J. Bell. *Speakable and Unspeakable in Quantum Mechanics*. Cambridge University Press, second edition, 2004.

D. M. Benincasa and F. Dowker. The Scalar Curvature of a Causal Set. *Phys. Rev. Lett.*, 104: 181301, 2010. doi: 10.1103/PhysRevLett.104.181301.

B. V. Bento. Causal set dynamics and the problem of time. Master's thesis, Imperial College London, 2018.

N. Berkovits. Super Poincare covariant quantization of the superstring. *JHEP*, 04:018, 2000. doi: 10.1088/1126-6708/2000/04/018.

J. Bičák. The art of science: interview with Professor John Archibald Wheeler. *General Relativity and Gravitation*, 41(4):679–689, April 2009.

L. Bombelli and D. A. Meyer. The Origin of Lorentzian Geometry. *Phys. Lett. A*, 141:226–228, 1989. doi: 10.1016/0375-9601(89)90474-X.

L. Bombelli, J. Lee, D. Meyer, and R. Sorkin. Space-Time as a Causal Set. *Phys. Rev. Lett.*, 59:521–524, 1987. doi: 10.1103/PhysRevLett.59.521.

L. Bombelli, I. Seggev, and S. Watson. A Computation of the Expected Number of Posts in a Finite Random Graph Order. 9 2008.

L. Bombelli, A. Corichi, and O. Winkler. Semiclassical Quantum Gravity: Obtaining Manifolds from Graphs. *Class. Quant. Grav.*, 26:245012, 2009a. doi: 10.1088/0264-9381/26/24/245012.

L. Bombelli, J. Henson, and R. D. Sorkin. Discreteness without symmetry breaking: A Theorem. *Mod. Phys. Lett. A*, 24:2579–2587, 2009b. doi: 10.1142/S0217732309031958.

G. Brightwell. Linear extensions of random orders. *Discrete Mathematics*, 125(1):87–96, 1994. ISSN 0012-365X. doi: https://doi.org/10.1016/0012-365X(94)90147-3. URL `https://www.sciencedirect.com/science/article/pii/0012365X94901473`.

G. Brightwell and N. Georgiou. Continuum limits for classical sequential growth models. *Random Structures & Algorithms*, 36(2):218–250, 2010. doi: 10.1002/rsa.20278.

G. Brightwell and M. Luczak. Order-invariant measures on causal sets. *The Annals of Applied Probability*, 21(4):1493–1536, 2011. doi: 10.1214/10-AAP736.

G. Brightwell and M. Luczak. Order-invariant measures on fixed causal sets. *Combinatorics, Probability and Computing*, 21:330–357, 2012. doi: 10.1017/S0963548311000721.

G. Brightwell and M. Luczak. The mathematics of causal sets. In *Recent Trends in Combinatorics*, volume 159 of *The IMA Volumes in Mathematics and its Applications*. Springer, Cham, 2016.

G. Brightwell and P. Winkler. Counting linear extensions. *Order*, 8:225–242, 1991. doi: https://doi.org/10.1007/BF00383444.

G. Brightwell, H. F. Dowker, R. S. Garcia, J. Henson, and R. D. Sorkin. General covariance and the 'Problem of time' in a discrete cosmology. In *Alternative Natural Philosophy Association Meeting Cambridge, England, August 16-21, 2001*, 2002.

G. Brightwell, H. F. Dowker, R. S. Garcia, J. Henson, and R. D. Sorkin. 'Observables' in causal set cosmology. *Phys. Rev.*, D67:084031, 2003. doi: 10.1103/PhysRevD.67.084031.

G. R. Brightwell. Linear extensions of infinite posets. *Discrete Mathematics*, 70(2):113–136, 1988. ISSN 0012-365X. doi: https://doi.org/10.1016/0012-365X(88)90087-8. URL `https://www.sciencedirect.com/science/article/pii/0012365X88900878`.

M. Buck, F. Dowker, I. Jubb, and S. Surya. Boundary Terms for Causal Sets. *Class. Quant. Grav.*, 32(20):205004, 2015. doi: 10.1088/0264-9381/32/20/205004.

M. Buck, F. Dowker, I. Jubb, and R. Sorkin. The Sorkin–Johnston state in a patch of the trousers spacetime. *Class. Quant. Grav.*, 34(5):055002, 2017. doi: 10.1088/1361-6382/aa589c.

S. Carlip. Observables, Gauge Invariance, and Time in (2+1)-dimensional Quantum Gravity. *Phys. Rev. D*, 42:2647–2654, 1990. doi: 10.1103/PhysRevD.42.2647.

E. Dable-Heath, C. J. Fewster, K. Rejzner, and N. Woods. Algebraic Classical and Quantum Field Theory on Causal Sets. *Phys. Rev. D*, 101(6):065013, 2020. doi: 10.1103/PhysRevD.101.065013.

J. Diestel and J. J. Uhl, Jr. *Vector Measures*, volume 15 of *Mathematical Surveys and Monographs*. American Mathematical Society, 1977.

F. Dowker. Spacetime discreteness, Lorentz invariance and locality. *J. Phys. Conf. Ser.*, 306:012016, 2011. doi: 10.1088/1742-6596/306/1/012016.

F. Dowker. Introduction to causal sets and their phenomenology. *Gen. Rel. Grav.*, 45(9):1651–1667, 2013. doi: 10.1007/s10714-013-1569-y.

F. Dowker. The birth of spacetime atoms as the passage of time. In *Do We Need a Physics of 'Passage'? Cape Town, South Africa, December 10-14, 2012*, 2014. doi: 10.1111/nyas.12542.

F. Dowker. *Being and Becoming on the Road to Quantum Gravity; or, the Birth of a Baby Is Not a Baby.* 4 2020. doi: 10.1017/9781108655705.009.

F. Dowker and L. Glaser. Causal set d'Alembertians for various dimensions. *Class. Quant. Grav.*, 30:195016, 2013. doi: 10.1088/0264-9381/30/19/195016.

F. Dowker and S. Surya. Observables in extended percolation models of causal set cosmology. *Class. Quant. Grav.*, 23:1381–1390, 2006. doi: 10.1088/0264-9381/23/4/018.

F. Dowker and S. Zalel. Evolution of Universes in Causal Set Cosmology. *Comptes Rendus Physique*, 18:246–253, 2017. doi: 10.1016/j.crhy.2017.03.002.

F. Dowker, J. Henson, and R. Sorkin. Discreteness and the transmission of light from distant sources. *Phys. Rev. D*, 82:104048, 2010a. doi: 10.1103/PhysRevD.82.104048.

F. Dowker, S. Johnston, and R. D. Sorkin. Hilbert Spaces from Path Integrals. *J. Phys.*, A43: 275302, 2010b. doi: 10.1088/1751-8113/43/27/275302.

F. Dowker, S. Johnston, and S. Surya. On extending the Quantum Measure. *J. Phys.*, A43: 505305, 2010c. doi: 10.1088/1751-8113/43/50/505305.

A. Eichhorn, S. Mizera, and S. Surya. Echoes of Asymptotic Silence in Causal Set Quantum Gravity. *Class. Quant. Grav.*, 34(16):16LT01, 2017. doi: 10.1088/1361-6382/aa7d1b.

A. Eichhorn, S. Surya, and F. Versteegen. Induced Spatial Geometry from Causal Structure. *Class. Quant. Grav.*, 36(10):105005, 2019a. doi: 10.1088/1361-6382/ab114b.

A. Eichhorn, S. Surya, and F. Versteegen. Spectral dimension on spatial hypersurfaces in causal set quantum gravity. *Class. Quant. Grav.*, 36(23):235013, 2019b. doi: 10.1088/1361-6382/ab47cd.

N. Georgiou. The random binary growth model. *Random structures and algorithms*, 27:520–552, 2005.

G. Gibbons, S. Hawking, and M. Perry. Path integrals and the indefiniteness of the gravitational action. *Nuclear Physics B*, 138(1):141–150, 1978. ISSN 0550-3213. doi: https://doi.org/10.1016/0550-3213(78)90161-X. URL `https://www.sciencedirect.com/science/article/pii/055032137890161X`.

G. Gibbons, S. Hawking, and J. Stewart. A natural measure on the set of all universes. *Nuclear Physics B*, 281(3):736–751, 1987. ISSN 0550-3213. doi: https://doi.org/10.1016/0550-3213(87)90425-1. URL `https://www.sciencedirect.com/science/article/pii/0550321387904251`.

G. W. Gibbons and N. Turok. The Measure Problem in Cosmology. *Phys. Rev. D*, 77:063516, 2008. doi: 10.1103/PhysRevD.77.063516.

L. Glaser and S. Surya. The Hartle–Hawking wave function in 2D causal set quantum gravity. *Class. Quant. Grav.*, 33(6):065003, 2016. doi: 10.1088/0264-9381/33/6/065003.

A. S. Gupta. Realising charge-parity-time symmetry in causal set dynamics. Master's thesis, Imperial College London, 2018.

J. J. Halliwell. A Review of the decoherent histories approach to quantum mechanics. *Annals N. Y. Acad. Sci.*, 755:726–740, 1995. doi: 10.1111/j.1749-6632.1995.tb39014.x.

F. Harary and E. M. Palmer. *Graphical Enumeration.* Academic Press, 1973. ISBN 978-0-12-324245-7.

J. B. Hartle. Space-time quantum mechanics and the quantum mechanics of space-time. In *Les Houches Summer School on Gravitation and Quantizations, Session 57*, pages 0285–480, 7 1992.

S. Hawking. Singularities and the geometry of spacetime. *EPJ H*, 39:413, 2014. doi: 10.1140/epjh/e2014-50013-6.

S. W. Hawking. Quantum gravity and path integrals. *Phys. Rev. D*, 18:1747–1753, Sep 1978. doi: 10.1103/PhysRevD.18.1747. URL `https://link.aps.org/doi/10.1103/PhysRevD.18.1747`.

B. Honan. Causal set dynamics: Interpretations of CPT invariance and defining a new dynamics. Master's thesis, Imperial College London, 2018.

C. J. Isham. Canonical quantum gravity and the problem of time. In *19th International Colloquium on Group Theoretical Methods in Physics (GROUP 19) Salamanca, Spain, June 29-July 5, 1992*, pages 0157–288, 1992. [NATO Sci. Ser. C409,157(1993)].

S. H. Jeffreys and B. Swirles. *Methods of Mathematical Physics*, page 52. Cambridge University Press, 1966.

S. P. Johnston. Quantum Fields on Causal Sets. Other thesis, 10 2010.

A. N. Kolmogorov and S. V. Fomin. *Introductory Real Analysis.* Dover Publications, New York, 1975.

E. H. Kronheimer and R. Penrose. On the structure of causal spaces. *Mathematical Proceedings of the Cambridge Philosophical Society*, 63:481–501, 4 1967. ISSN 1469-8064. doi: 10.1017/S030500410004144X. URL `http://journals.cambridge.org/article_S030500410004144X`.

D. Landers and L. Z. Rogge. On the extension problem for measures. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 30:167–169, 1974. doi: https://doi.org/10.1007/BF00532268.

T. L. Lindstrøm. *Spaces: An Introduction to Real Analysis.* Pure and Applied Undergraduate Texts. American Mathematical Society, 2017.

L. Machet and J. Wang. On the continuum limit of Benincasa–Dowker–Glaser causal set action. *Class. Quant. Grav.*, 38(1):015010, 2021. doi: 10.1088/1361-6382/abc274.

S. Major, D. Rideout, and S. Surya. On Recovering continuum topology from a causal set. *J. Math. Phys.*, 48:032501, 2007. doi: 10.1063/1.2435599.

S. Major, D. Rideout, and S. Surya. Stable Homology as an Indicator of Manifoldlikeness in Causal Set Theory. *Class. Quant. Grav.*, 26:175008, 2009. doi: 10.1088/0264-9381/26/17/175008.

D. B. Malament. The class of continuous timelike curves determines the topology of spacetime. *J. Math. Phys.*, 18:1399–1404, 1977.

X. Martin, D. O'Connor, D. P. Rideout, and R. D. Sorkin. On the 'renormalization' transformations induced by cycles of expansion and contraction in causal set cosmology. *Phys. Rev.*, D63:084026, 2001. doi: 10.1103/PhysRevD.63.084026.

J. Norton. Time really passes. *HUMANA.MENTE Journal of Philosophical Studies*, 4(13): 23–34, Jan. 2018. URL `http://www.humanamente.eu/index.php/HM/article/view/228`.

M. Rees, R. Fuffini, and J. A. Wheeler. *Black Holes, Gravitational Waves and Cosmology: An introduction to current research*, volume 10 of *Topics in Astrophysics and Space Physics*. Gordon and Breach, Cambridge, 1974.

D. P. Rideout and R. D. Sorkin. A Classical sequential growth dynamics for causal sets. *Phys. Rev.*, D61:024002, 2000. doi: 10.1103/PhysRevD.61.024002.

D. P. Rideout and R. D. Sorkin. Evidence for a continuum limit in causal set dynamics. *Physical Review D*, 63, 2001.

C. Rovelli. What Is Observable in Classical and Quantum Gravity? *Class. Quant. Grav.*, 8: 297–316, 1991. doi: 10.1088/0264-9381/8/2/011.

W. Rudin. *Real and Complex Analysis*. McGraw-Hill, New York, 1987.

P. A. Schilpp, editor. *Albert Einstein: Philospher-Scientist*. Library of Living Philosophers. Open Court Publishing, Evanston, Ill., 1949.

L. Schwartz. *Radon Measures on Arbitrary Topological Spaces and Cylindrical Measures*. Oxford University Press, London, New York, 1973.

S. Shirali and H. L. Vasudeva. *Metric Spaces*. Springer, London, 2006.

L. Smolin. Did the universe evolve? *Class. Quantum Grav.*, 9(1):173–191, 1992.

L. Smolin. The status of cosmological natural selection. 2008. `arXiv:hep-th/0612185`.

R. Sorkin. *Foundations of Space and Time Reflections on Quantum Gravity*, chapter Logic is to the quantum as geometry is to gravity, pages 363–384. Cambridge University Press, 2012.

R. D. Sorkin. Space-time and causal sets. In *7th Latin American International Symposium on General Relativity and Gravitation*, 12 1990.

R. D. Sorkin. Quantum mechanics as quantum measure theory. *Mod. Phys. Lett.*, A9:3119–3128, 1994. doi: 10.1142/S021773239400294X.

R. D. Sorkin. Forks in the road, on the way to quantum gravity. *Int. J. Theor. Phys.*, 36: 2759–2781, 1997. doi: 10.1007/BF02435709.

R. D. Sorkin. Indications of causal set cosmology. *Int. J. Theor. Phys.*, 39:1731–1736, 2000. doi: 10.1023/A:1003629312096.

R. D. Sorkin. Relativity Theory Does Not Imply that the Future Already Exists: A Counterexample. *Fundam. Theor. Phys.*, 153:153–161, 2007. doi: 10.1007/978-1-4020-6318-3_9.

R. D. Sorkin. Scalar Field Theory on a Causal Set in Histories Form. *J. Phys. Conf. Ser.*, 306: 012017, 2011a. doi: 10.1088/1742-6596/306/1/012017.

R. D. Sorkin. Toward a 'fundamental theorem of quantal measure theory'. 4 2011b.

R. D. Sorkin. Expressing entropy globally in terms of (4D) field-correlations. *J. Phys. Conf. Ser.*, 484:012004, 2014. doi: 10.1088/1742-6596/484/1/012004.

R. D. Sorkin. From Green Function to Quantum Field. *Int. J. Geom. Meth. Mod. Phys.*, 14 (08):1740007, 2017. doi: 10.1142/S0219887817400072.

R. D. Sorkin and Y. K. Yazdi. Entanglement Entropy in Causal Set Theory. *Class. Quant. Grav.*, 35(7):074004, 2018. doi: 10.1088/1361-6382/aab06f.

J. Stachel. The Hole Argument and Some Physical and Philosophical Implications. *J. Living Rev. Relativity*, 17:1, 2014. doi: 10.12942/lrr-2014-1.

S. Surya. The causal set approach to quantum gravity. *Living Rev. Rel.*, 22(1):5, 2019. doi: 10.1007/s41114-019-0023-1.

S. Surya, N. X, and Y. K. Yazdi. Studies on the SJ Vacuum in de Sitter Spacetime. *JHEP*, 07: 009, 2019. doi: 10.1007/JHEP07(2019)009.

The On-Line Encyclopedia of Integer Sequences. A000112 Number of partially ordered sets ("posets") with n unlabeled elements. oeis.org/A000112.

M. Varadarajan and D. Rideout. A General solution for classical sequential growth dynamics of causal sets. *Phys. Rev. D*, 73:104021, 2006. doi: 10.1103/PhysRevD.73.104021.

E. Witten. Quantum Field Theory and the Jones Polynomial. *Commun. Math. Phys.*, 121: 351–399, 1989. doi: 10.1007/BF01217730.

C. Wuthrich and C. Callender. What Becomes of a Causal Set? *Brit. J. Phil. Sci.*, 68(3): 907–925, 2017. doi: 10.1093/bjps/axv040.

N. X, F. Dowker, and S. Surya. Scalar Field Green Functions on Causal Sets. *Class. Quant. Grav.*, 34(12):124002, 2017. doi: 10.1088/1361-6382/aa6bc7.

N. Zwane, N. Afshordi, and R. D. Sorkin. Cosmological tests of Everpresent Λ. *Class. Quant. Grav.*, 35(19):194002, 2018. doi: 10.1088/1361-6382/aadc36.