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PAPER

A Bayesian quantum state tomography along with adaptive frameworks based on linear minimum mean square error criterion

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Abstract

Quantum state tomography (QST) is essential for characterizing unknown quantum states. Several methods of estimating quantum states already exist and can be classified mainly into three broad classes. They are based on the criteria like maximum likelihood, linear inversion, and Bayesian framework. The Bayesian framework for QST gives a better reconstruction performance. However, the existing methods of the Bayesian frameworks are computationally extensive and, most of the time require knowledge about the prior distribution of the quantum state. In this paper, we propose a Bayesian method of QST based on the linear minimum mean square error criterion, where the prior statistics are estimated and the computational complexity is comparable to that of the linear inversion based QST method. We also propose an adaptive version based on the block estimation of parameters. Extensive numerical simulations are conducted to demonstrate its efficacy over the linear inversion-based QST regarding trace distance error metric.

1. Introduction

With the advancement of technology in the last few decades, quantum computing has sparked a candle of hope to solve a set of computational problems which are complexity-wise costly in existing classical computers. However, this needs error-free fault-tolerant quantum computation. Characterizing and reconstructing a classical description of the quantum state is also important to understand the accuracy of such quantum processors. Therefore, quantum state tomography (QST) or quantum state reconstruction technique has become an essential sub-routine in quantum computation [1–3].

Literature survey: several tomography methods for quantum state estimation have already been proposed. Broadly, two prominent methods exist, namely non-Bayesian and Bayesian ones. Non-Bayesian method comes with the likes of maximum likelihood estimation (MLE) [4], linear inversion or least square based methods (LS) [5] and maximum entropy-based approach [6]. A comparative study of different estimation methods has been analyzed in [7]. Though, MLE is non-linear and superior in this category, but it suffers from high time computational complexity and slow convergence compared to the LS one. Thereafter, several improvements over the MLE-based QST framework have been discussed from computational complexity aspect [8–10]. It may also return a point estimate that may not contain any relevant information i.e. the quantum state's region estimate [11, 12]. A faster approach of the MLE-based estimation has been demonstrated in [13]. On the other hand, LS method, though efficient in computational complexity, may not estimate a valid physical density matrix as the estimated density matrix may not be always positive semi-definite [5].

In addition to MLE and LS methods, Bayesian and adaptive models offer an alternative approach to QST, which enhances QST performance. Jones [14], Derka *et al* [15], and Bužek *et al* [16] introduced Bayesian techniques into quantum tomography, which not only incorporate prior knowledge but also bring practical advantages to experimental settings. These advantages include optimality [17], robust region estimates [18], and model selection criteria [19]. Bayesian methods excel in quantum tomography research because they provide a comprehensive representation of an experimenter's knowledge after each data point, making them invaluable tools. However, implementing Bayesian QST in practical scenarios can be challenging due to the scarcity of useful priors and intractable computational complexity [20]. As a response to this challenge, an adaptive Bayesian quantum tomography approach has been developed, focusing on the adaptive determination of optimal tomographic measurements [21–23]. Furthermore, the maximum *a posteriori* (MAP) estimator, which is non-linear, has been considered for both quantum state and process tomography [24], which is, sometimes, computationally intractable. In many real-world scenarios, prior statistics, which is the key input to the Bayesian method, including MAP, may not be readily available. In such cases, it is possible to adaptively discover a prior distribution [25, 26], which is again computationally complex in time. In the above methods, the quantum uncertainty is originated due to device and measurement operations. It does not assume that the quantum state itself will vary following a prior. Recently, performance improvement of machine learning-based QST and Bayesian QST with limited prior distribution has been discussed in [27, 28]. Moreover, experimental realization of adaptive tomography has been demonstrated in [29]. Following this, a self-guided adaptive quantum tomography has been studied in [30].

Summary of gaps in existing works: from the above discussion, it is evident that method with low complexity (LS or linear inversion etc) has poorer tomographic performance, compared to the non-linear and Bayesian method like MAP one having very high complexity. Therefore, it will be a good idea to take a middle path, which gives comparable complexity with respect to the LS method, but gives superior performance as well. This necessitates a linear, but Bayesian approach to fill the gap. As the knowledge of prior distribution is computationally expensive as adopted by previous works, it will be a need to estimate only first or second order statistics.

Contributions:

1. In this work, we propose a model of a linear Bayesian framework to estimate a quantum state based on the linear minimum mean square error criterion (LMMSE). The fundamental requirement of LMMSE-based QST is the prior distribution of the unknown quantum states. With these prior statistics, a Bayesian LMMSE framework of QST can be well developed, promising an advantage over the existing linear inversion-based or MLE-based methods. The need for such QST arises in many practical scenarios. For example, it could be a photon source with an experimental design to emit a selected set of polarized photons. Obtaining the prior statistics is an essential step in the LMMSE system. We perform LS-based QST to obtain the necessary statistical properties for such a scenario if we do not have prior statistical information about the quantum state.
2. We have also proposed two adaptive algorithms that update the statistics based on a block estimation to reduce complexity and comprehend the mild variation in statistics. We first propose a weighted update of the first and second order statistics based on measured data and observations. Then, a matrix inversion based linear equalizer is designed taking into the consideration of updated statistics at each iteration. We name it inversion-based update LMMSE (IBU-LMMSE). In the second adaptive approach, we propose to update the equalizer without any matrix inversion, but with the steepest descent approach. We name it non-inversion-based update LMMSE (NIBU-LMMSE). Furthermore, comparisons between the existing least squares-based method and the proposed method have been discussed with numerical results.
3. We have also analyzed the computational complexity of our proposed algorithms. This shows that the complexity is comparable to the LS method except initial acquisition part of the prior first and second order statistics.

2. LMMSE based tomography

Quantum tomography requires an optimal set of measurements over multiple identical copies of the same quantum state. However, quantum density operators can be parameterized by traceless orthonormal basis matrix ($\boldsymbol{\sigma}_i$) of the Hilbert space as follows

$$\boldsymbol{\rho} = \frac{\mathbf{I}}{d} + \sqrt{\frac{(d-1)}{2d}} \sum_{i=1}^{d^2-1} \theta[i] \boldsymbol{\sigma}_i, \quad (1)$$

where $\theta[i]$ is the coefficient of the ith basis, d is the dimension of the Hilbert space, and $\mathbf{I}$ is the $d \times d$ identity matrix. The vector $\boldsymbol{\theta}$ is called the generalized coherence vector. For our informationally complete QST, we perform positive operator valued measure (POVM) over identical copies of the quantum state. The observed probability vector $\mathbf{p}$ depends on the set of linear equations as follows

$$\mathbf{y} = \mathbf{H}\boldsymbol{\theta} + \mathbf{z}, \tag{2}$$

where $\mathbf{y} = \frac{d(\mathbf{p} - 1/d)}{(d-1)}$, $\mathbf{1}$ is a vector with all ones, $\boldsymbol{\theta}$ is the row vector containing coefficients of the traceless orthonormal basis of the Hilbert space, the matrix $\mathbf{H}$ depends on POVM operators, $\mathbf{z}$ represents the measurement uncertainty. We consider $\boldsymbol{\theta}$ as the coefficient of the generalized Pauli operators. The linear inversion-based QST finds out the optimum $\boldsymbol{\theta}$ for which the cost function $||\mathbf{y} - \mathbf{H}\boldsymbol{\theta}||_2^2$ is minimized. The optimum solution for $\boldsymbol{\theta}$ is $\hat{\boldsymbol{\theta}} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{y}$. Following (1), we can reconstruct the unknown density matrix. However, this model of QST does not exploit the prior statistical information about quantum states. The Bayesian framework of QST is useful in such a scenario. In the following section, we propose a linear Bayesian model of QST to obtain a state estimate with prior knowledge about the state's prior distribution and noise characteristics.

2.1. LMMSE based QST model

Considering prior knowledge about the unknown quantum state, Bayesian framework-based QST is more suitable than the existing LS-based QST scenario. One example of prior knowledge can correspond to the distribution of mixed quantum states with a specific purity bias [27]. Our work utilizes the statistical properties of $\boldsymbol{\theta}$ instead of the prior probability density over the Hilbert space. In our proposal, we consider the statistics of the unknown quantum state and the noise due to measurement uncertainty. With the above background, we propose the LMMSE-based QST algorithm, which minimizes the cost function as follows

$$\min_{\mathbf{W}} \mathbb{E}||\mathbf{W}\mathbf{y} - \boldsymbol{\theta}||_2^2, \tag{3}$$

where $\mathbf{W}$ is the LMMSE-based equalizer. Solving for (3), we obtain the closed form solution for optimal $\mathbf{W}$ as follows

$$\begin{aligned}\mathbf{W} &= \mathbb{E}\left(\boldsymbol{\theta}\mathbf{y}^T\right)\left(\mathbb{E}\left[\mathbf{y}\mathbf{y}^T\right]\right)^{-1} \\ &= \mathbb{R}_{\theta y}\mathbb{R}_{yy}^{-1},\end{aligned} \tag{4}$$

where $\mathbb{R}_{\theta y}$ is the cross-covariance between $\boldsymbol{\theta}$ and $\mathbf{y}$, and $\mathbb{R}_{yy}$ is the auto correlation of $\mathbf{y}$. However, in practice, we do not have the exact matrices $\mathbb{R}_{\theta y}$ and $\mathbb{R}_{yy}$. We propose that LS-based estimation of $\boldsymbol{\theta}$ is performed first with multiple $\mathbf{y}$ and then obtain the sample correlation matrices $\mathbb{R}_{\theta y}$ and $\mathbb{R}_{yy}$. In our experimental setup, we assume a quantum source that can emit different unknown quantum state $\boldsymbol{\rho}$ among N_ρ different representation $[\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \ldots, \boldsymbol{\rho}_{N_\rho}]$. However, it can emit only one of them in a particular instance. Suppose we observe the scenario for T different instances, and for each instance, we estimate the $\boldsymbol{\theta}$ with an LS-based estimator from various instances of $\mathbf{y}$ vectors. Finally, $\mathbb{R}_{\theta y}$ and $\mathbb{R}_{yy}$ are obtained as follows

$$\mathbb{R}_{\theta y} = \frac{1}{T}\sum_{m=1}^{T}\left(\hat{\boldsymbol{\theta}}_{ls}^m - \mathbb{E}[\boldsymbol{\theta}]\right)\left(\mathbf{y}^m - \mathbb{E}[\mathbf{y}]\right)^T \tag{5}$$

$$\mathbb{R}_{yy} = \frac{1}{T}\sum_{m=1}^{T}\left(\mathbf{y}^m - \mathbb{E}[\mathbf{y}]\right)\left(\mathbf{y}^m - \mathbb{E}[\mathbf{y}]\right)^T, \tag{6}$$

where $\hat{\boldsymbol{\theta}}_{ls}^m$ is the estimated $\boldsymbol{\theta}$ from the LS-based estimator at the mth observation vector, $\mathbf{y}^m$ is the mth observation vector. The sample means $\mathbb{E}[\mathbf{y}]$ and $\mathbb{E}[\boldsymbol{\theta}]$ in (6) are calculated as follows

$$\mathbb{E}[\mathbf{y}] = \frac{1}{T}\sum_{m=1}^{T}\mathbf{y}^m \tag{7}$$

$$\mathbb{E}[\boldsymbol{\theta}] = \frac{1}{T}\sum_{m=1}^{T}\hat{\boldsymbol{\theta}}_{ls}^m. \tag{8}$$

We now design the LMMSE-based equalizer as per (4). Based on the equalizer $\mathbf{W}$, the LMMSE-based estimate of $\boldsymbol{\theta}$ for an unknown state can be obtained as follows

$$\hat{\boldsymbol{\theta}}_{\text{lmmse}} = \mathbf{W}\left(\mathbf{y}_{\text{tar}} - \mathbb{E}[\mathbf{y}]\right) + \mathbb{E}[\boldsymbol{\theta}], \tag{9}$$

where $\mathbf{y}_{\text{tar}}$ is any target $\mathbf{y}$. The overall algorithm is given in algorithm 1.

Algorithm 1. Proposed method for obtaining prior statistics and LMMSE-based estimation.

Input: $\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \dots \boldsymbol{\rho}_{N_\rho}$, $T(T >> N_\rho)$, $l = 1$.
$\mathbf{y}^l = \mathbf{0}_{d^2\times 1}$, $\boldsymbol{\theta}^l = \mathbf{0}_{d^2-1\times 1}$, $\mathbf{R}_1^0 = \mathbf{0}_{d^2-1\times d^2}$, $\mathbf{R}_2^0 = \mathbf{0}_{d^2\times d^2}$.
for $m = 1$ to T **do**
    mth instance, $\boldsymbol{\rho}_m \in \left[\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \dots \boldsymbol{\rho}_{N_\rho}\right]$.
    SIC-POVM based measurement on $\boldsymbol{\rho}_m$.
    Obtain $\mathbf{p}^m$, and calculate $\mathbf{y}^m$, $\hat{\boldsymbol{\theta}}_{ls}^m$ for mth instance.
    $\mathbf{y}^l = \mathbf{y}^{l-1} + \mathbf{y}^m$.
    $\boldsymbol{\theta}^l = \boldsymbol{\theta}^{l-1} + \hat{\boldsymbol{\theta}}_{ls}^m$.
end for
$\mathbb{E}[\mathbf{y}] = \frac{\mathbf{y}^l}{T}$, $\mathbb{E}[\boldsymbol{\theta}] = \frac{\boldsymbol{\theta}^l}{T}$.
for $m = 1$ to T **do**
    $\mathbf{R}_1^m = (\hat{\boldsymbol{\theta}}_{ls}^m - \mathbb{E}[\boldsymbol{\theta}])(\mathbf{y}^m - \mathbb{E}[\mathbf{y}])^T + \mathbf{R}_1^{m-1}$.
    $\mathbf{R}_2^m = (\mathbf{y}^m - \mathbb{E}[\mathbf{y}])(\mathbf{y}^m - \mathbb{E}[\mathbf{y}])^T + \mathbf{R}_2^{m-1}$.
end for
$\mathbb{R}_{\theta y} = \mathbf{R}_1/T$, $\mathbb{R}_{yy} = \mathbf{R}_2/T$.
$\mathbf{W} = \mathbb{R}_{\theta y}\mathbb{R}_{yy}^{-1}$.
Measure unknown $\boldsymbol{\rho}_{tar}$ to obtain $\mathbf{y}_{tar}$.
$\hat{\boldsymbol{\theta}}_{lmmse} = \mathbf{W}(\mathbf{y}_{tar} - \mathbb{E}[\mathbf{y}]) + \mathbb{E}[\boldsymbol{\theta}]$.
Output: $\hat{\boldsymbol{\rho}}_{tar} = \frac{\mathbf{I}}{d} + \sqrt{\frac{(d-1)}{2d}}\sum_{i=1}^{d^2-1}\hat{\theta}_{lmmse}[i]\boldsymbol{\sigma}_i$.

In scenarios where no prior statistical information is available, the LMMSE-based equalizer can be effectively substituted with the LS-based estimator, represented as the pseudo-inverse $(\mathbf{H}^{\mathbf{T}}\mathbf{H})^{-1}\mathbf{H}^{\mathbf{T}}$. In such non-Bayesian scenarios, it is important to note that LMMSE-based estimation is equivalent to the least-squares-based estimation.

2.2. Adaptive linear mean square error based QST model

In a practical scenario, accurate estimation of statistical properties $\mathbb{E}[\mathbf{y}]$, $\mathbb{E}[\boldsymbol{\theta}]$, $\mathbb{R}_{\theta y}$, and $\mathbb{R}_{yy}$ requires a large number of initial sample data. Following the above restriction, we propose to update the estimated statistics after a time block containing b_l different quantum states. In contrast to the previous approach, we improve the statistics after each time block. However, we update the statistics through a dynamically weighted average between the old and new statistics. The weight (r) is calculated based on the sample complexity ($N_{b_l} \times b_l$) used to estimate the newer statistics, where N_{b_l} is the average sample complexity in one b_l. The value of r is updated iteratively after each small time block as follows

$$\begin{aligned} N_{\mathrm{new}} &= N_{\mathrm{old}} + N_{b_l} \times b_l \\ r &= 1 - \frac{N_{b_l} \times b_l}{N_{\mathrm{new}}}, \end{aligned} \tag{10}$$

where the initial value of $N_{\mathrm{new}} = 0$ for the first b_l. After each block, N_{new} iteratively increases following (10), which implies that the weight r depends on the number of copies utilized to estimate the statistics in a particular b_l. After each block iteration, $\mathbb{E}[\mathbf{y}]$, $\mathbb{E}[\boldsymbol{\theta}]$, $\mathbb{R}_{\theta y}$, and $\mathbb{R}_{yy}$ are updated based on the weight of r as follows

$$\begin{aligned} \mathbb{E}[\mathbf{y}] &= r\mathbb{E}[\mathbf{y}] + (1-r)\frac{1}{N_{b_l}}\sum_{j=1}^{N_{b_l}}\mathbf{y}^j, \\ \mathbb{E}[\boldsymbol{\theta}] &= r\mathbb{E}[\boldsymbol{\theta}] + (1-r)\frac{1}{N_{b_l}}\sum_{j=1}^{N_{b_l}}\hat{\boldsymbol{\theta}}_{ls}^j \\ \mathbb{R}_{\theta y} &= r\mathbb{R}_{\theta y} + \frac{1-r}{N_{b_l}}\sum_{j=1}^{N_{b_l}}\left(\hat{\boldsymbol{\theta}}_{ls}^j - \mathbb{E}[\boldsymbol{\theta}]\right)\left(\mathbf{y}^j - \mathbb{E}[\mathbf{y}]\right) \\ \mathbb{R}_{yy} &= r\mathbb{R}_{yy} + \frac{1-r}{N_{b_l}}\sum_{j=1}^{N_{b_l}}\left(\mathbf{y}^j - \mathbb{E}[\mathbf{y}]\right)\left(\left(\mathbf{y}^j - \mathbb{E}[\mathbf{y}]\right)\right)^T. \end{aligned} \tag{11}$$

After the first time block, we can update the LMMSE-based equalizer as follows

$$\mathbf{W}_{ada}^1 = \mathbb{R}_{\theta y}^1\left(\mathbb{R}_{yy^1}^1\right)^{-1}. \tag{12}$$

We propose to update the LMMSE-based equalizer from the second time block in two ways. The first is an inversion-based update (IBU), and the second adaptive approach is a non-inversion-based update (NIBU), as depicted below.

1. **IBU-LMMSE**: the first proposal is to update the equalizer with the updated statistics following (11) after each time block which includes the matrix inversion of $\mathbb{R}_{yy}$. The equalizer at the ith iteration is as follows

$$\mathbf{W}^{i}_{ada} = \mathbb{R}^{i}_{\theta y}\left(\mathbb{R}^{i}_{yy}\right)^{-1}, \tag{13}$$

2. **NIBU-LMMSE**: our second proposal is to update it recursively after each time block without performing the matrix inversion. The equalizer after ith time block is updated as follows

$$\mathbf{W}^{i}_{ada} = \mathbf{W}^{i-1}_{ada} + (1-r)\left(\mathbb{R}^{i}_{\theta y} - \mathbf{W}^{i-1}_{ada}\mathbb{R}^{i}_{yy}\right). \tag{14}$$

where $\mathbb{R}^{i}_{\theta y}$ and $\mathbb{R}^{i}_{yy}$ is the updated statistics after ith block as illustrated in (11).

The performance of the adaptive algorithm can be quantified after each of the ith time blocks. For such purpose, we may perform QST based on the proposed adaptive algorithm on an unknown quantum state generated by the quantum source to observe our adaptive algorithm's estimation performance after each time block. The adaptive algorithm is described in the following pseudo-code.

Algorithm 2. Proposed adaptive LMMSE framework.

Input: $\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \ldots \boldsymbol{\rho}_{N_\rho}$ with dimension d, b_l, c_l.
Set $N=0, \mathbb{R}_{\theta y} = \mathbf{0}_{d^2-1\times d^2}, \mathbb{R}_{yy} = \mathbf{0}_{d^2\times d^2}, r=0$.
for $i=1$ to b_l **do**
    Set $\mathbf{y}^{b_l} = \mathbf{0}_{d^2\times 1}, \boldsymbol{\theta}^{b_l} = \mathbf{0}_{d^2-1\times 1}$.
    for $j=1$ to c_l **do**
        jth instance, $\boldsymbol{\rho}_j \in [\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \ldots \boldsymbol{\rho}_{N_\rho}]$.
        SIC-POVM based measurement on $\boldsymbol{\rho}_j$.
        Sample complexity for each jth instance N_{c_l}.
        Obtain $\mathbf{p}^j$, and calculate $\mathbf{y}_{ls}{}^j$, $\boldsymbol{\theta}^j_{ls}$.
    $\mathbf{y}^{b_l} = \mathbf{y}^{b_l} + \mathbf{y}^j_{ls}$.
    $\boldsymbol{\theta}^{b_l} = \boldsymbol{\theta}^{b_l} + \boldsymbol{\theta}^j_{ls}$.
  end for
  $\mathbb{E}[\mathbf{y}]^i = \frac{\mathbf{y}^{b_l}}{c_l}, \mathbb{E}[\boldsymbol{\theta}]^i = \frac{\boldsymbol{\theta}^{b_l}}{c_l}$.
  Set $\mathbb{R}^{b_l}_{yy} = \mathbf{0}_{d^2\times d^2}, \mathbb{R}^{b_l}_{\theta y} = \mathbf{0}_{d^2-1\times d^2}$.
  for $m=1$ to c_l **do**
    $\mathbb{R}^{b_l}_{\theta y} = (\boldsymbol{\theta}^m_{ls} - \mathbb{E}[\boldsymbol{\theta}]^i)(\mathbf{y}^m_{ls} - \mathbb{E}[\mathbf{y}]^i)^T + \mathbb{R}^{b_l}_{\theta y}$.
    $\mathbb{R}^{b_l}_{yy} = (\mathbf{y}^m_{ls} - \mathbb{E}[\mathbf{y}]^i)(\mathbf{y}^m_{ls} - \mathbb{E}[\mathbf{y}]^i)^T + \mathbb{R}^{b_l}_{yy}$.
  end for
  $\mathbb{R}^i_{\theta y} = \frac{\mathbb{R}^{b_l}_{\theta y}}{c_l}, \mathbb{R}^i_{yy} = \frac{\mathbb{R}^{b_l}_{yy}}{c_l}$.
  $\mathbb{E}[\mathbf{y}]^i = r\mathbb{E}[\mathbf{y}]^{i-1} + (1-r)\mathbb{E}[\mathbf{y}]^i$.
  $\mathbb{E}[\boldsymbol{\theta}]^i = r\mathbb{E}[\boldsymbol{\theta}]^{i-1} + (1-r)\mathbb{E}[\boldsymbol{\theta}]^i$.
  $\mathbb{R}^i_{\theta y} = r\mathbb{R}^{i-1}_{\theta y} + (1-r)\mathbb{R}^i_{\theta y}$.
  $\mathbb{R}^i_{yy} = r\mathbb{R}^{i-1}_{yy} + (1-r)\mathbb{R}^i_{yy}$.
  $N = N + N_{c_l}\times c_l$.
  $r = 1 - \frac{N_{c_l}\times c_l}{N}$.
  if i = 1 **then**
    $\mathbf{W}^i_{ada} = \mathbb{R}^i_{\theta y}(\mathbb{R}^i_{yy})^{-1}$.
  else
    if NIBU LMMSE-based equalizer **then**
      $\mathbf{W}^i_{ada} = \mathbf{W}^{i-1}_{ada} + (1-r)(\mathbb{R}^i_{\theta y} - \mathbf{W}^{i-1}_{ada}\mathbb{R}^i_{yy})$.
    else for IBU LMMSE-based equalizer
      $\mathbf{W}^i_{ada} = \mathbb{R}_{\theta y}(\mathbb{R}^i_{yy})^{-1}$
    end if
  end if
  SIC-POVM based measurement on $\boldsymbol{\rho}_{tar}$.
  Compute $\hat{\boldsymbol{\theta}}^i_{ada} = \mathbf{W}_{ada}(\mathbf{y}_{tar} - \mathbb{E}[\mathbf{y}]^i) + \mathbb{E}[\boldsymbol{\theta}]^i$.
Output: Reconstruct $\hat{\boldsymbol{\rho}}^i = \frac{\mathbf{I}}{d} + \sqrt{\frac{(d-1)}{2d}}\sum_{k=1}^{d^2-1}\hat{\theta}^i_{ada}[k]\boldsymbol{\sigma}_k$
end for

Table 1. Comparison of computational complexity between the existing method and our proposed Algorithms.

QST Methods	Complexity
LS	$\mathcal{O}(k_1 d^3)$
LMMSE	$\mathcal{O}(k_1 d^3)$
Adaptive IBU-LMMSE	$\mathcal{O}(b_l c_l d^2) + \mathcal{O}(b_l k_1 d^3)$
Adaptive NIBU-LMMSE	$\mathcal{O}(b_l c_l d^2) + \mathcal{O}(b_l d^{\gamma_1})$, $\gamma_1 \leqslant 3$ [31]

3. Complexity analysis of proposed methods

The complexity of the proposed algorithms with respect to LS counterparts is described in this section and also detailed in Table 1.

1. **Complexity of LS:** for a d-dimensional Hilbert space, the post-processing complexity of the LS-based QST depends on the pseudo-inverse of the $\mathbf{H}$ matrix, which is in $\mathcal{O}(k_1 d^3)$, where k_1 is a constant as matrix inversion is an iterative operation.
2. **Complexity of LMMSE:** the complexity of the LMMSE algorithm, as illustrated by (4) and (9), is equal to the complexity of existing LS-based QST when known prior statistics are considered.
3. **Complexity of adaptive IBU-LMMSE:** in the adaptive algorithm, we update the prior statistics after each time block. The process of building these prior statistics consecutively incurs a computational complexity of $\mathcal{O}(c_l d^2)$. This is due to the computational complexity associated with computing vector outer products c_l times in lines 15–17 of algorithm 2, which itself is of the order $\mathcal{O}(d^2)$. The complexity of reconstructing the state is in $\mathcal{O}(b_l d^3)$ as this requires the reconstruction of W_{ada} in each iteration of time blocks. So, the total complexity after b_l block is in $\mathcal{O}(c_l b_l d^2) + \mathcal{O}(b_l k_1 d^3)$.
4. **Complexity of adaptive NIBU-LMMSE:** the computational complexity of estimating prior statistics is comparable to that of IBU LMMSE. However, in the case of non-inversion-based adaptive algorithms, there is no need to perform matrix inversion after each time block; only a single matrix inversion operation is required. The time complexity of matrix multiplication is $\mathcal{O}(d^3)$. Therefore, the total complexity of NIBU-LMMSE is $\mathcal{O}(b_l c_l d^2) + \mathcal{O}(b_l d^{\gamma_1})$, $\gamma_1 \leqslant 3$ [31]. It is important to note that the constant k_1, which is necessary for matrix inversion, is not required for NIBU-LMMSE.

4. Results and discussions

The novelty of our proposed LMMSE algorithm is explored through numerical simulation. This includes exhaustive numerical simulations with different prior statistics. For example, the prior statistics of quantum states may belong to the nearly maximally mixed, pure, or random quantum states. For simulation purposes, we initiated the generation of prior states (N_ρ) using algorithm 3. These prior states exclusively constitute the set from which the quantum source can emit quantum states in our simulation model. Line 5 of algorithm 3 plays a key role in deciding the characteristics of these priors, which have been described below. During the random prior generation process, we assigned each element of $\mathbf{e}$ by drawing from an identical uniform distribution. In scenarios involving pure states, the density matrix has one distinct eigenvalue equal to 1, with all others being zero. In this case, we simply set one element of the vector $\mathbf{e}$ to 1 while keeping the rest at zero instead of choosing every element from a random uniform distribution. Conversely, in the case of a nearly maximally mixed state, all eigenvalues are approximately equal. To generate such a state, we set the ith element $e[i]$ as $\frac{1}{d} + \epsilon[i]$, where $\epsilon[i]$ represents values that are reasonably small in comparison to $e[i]$.

Algorithm 3. MATLAB microcode to generate random quantum states.

1: **Input:** $\mathbf{X}_{d\times d}, \mathbf{Y}_{d\times d} = rand(d,d)$, $\mathbf{e}_{1\times d} = rand(1,d)$.
2: $\mathbf{Z} = \mathbf{X} + 1j * \mathbf{Y}$, $1j$ denotes unit imaginary number.
3: Matrix QR decomposition : $[\mathbf{Q}, \mathbf{R}] = qr(\mathbf{Z})$.
4: Generate vector $\mathbf{e}_{d\times 1}$, where $e[i] \in \mathcal{U}[0,1]$
5: $\mathbf{e} = \mathbf{e} / \sum_{i=1}^{d} e[i]$.
6: **Output:** $\boldsymbol{\rho} = \mathbf{Q}^* \left(diag(\mathbf{e})\right) \mathbf{Q}$.

The accuracy of any QST algorithm is generally appraised through error metrics such as the Frobenius norm, trace distance metric, fidelity, and many more. Our simulation model quantifies the accuracy with trace distance error metrics. The trace distance metric between the unknown and estimated density matrix can be defined as follows

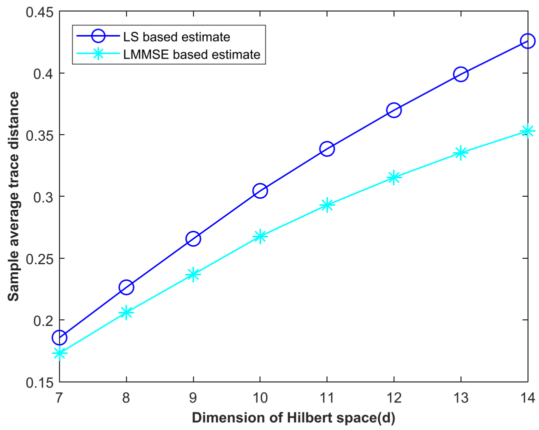

Figure 1. Comparison of the sample average trace distance between the proposed LMMSE-based method and the existing LS-based method. This comparison was performed across various Hilbert space dimensions, all with nearly maximally mixed prior statistics. The sample complexity for these comparisons was 2000.

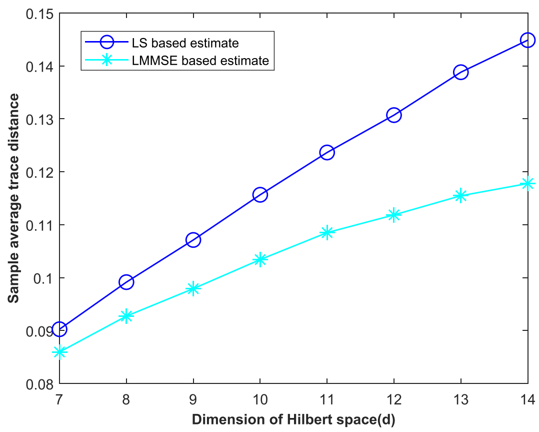

Figure 2. Comparison of the sample average trace distance between the proposed LMMSE-based method and the existing LS-based method across different Hilbert space dimensions. In this comparison, we focused on pure state prior statistics, and the sample complexity for each scenario was set at 2000.

$$T(\boldsymbol{\rho},\hat{\boldsymbol{\rho}}) = \sum_{i=1}^{d^2-1} |\nu^i|, \tag{15}$$

where ν^i is the ith eigenvalue of the matrix $(\boldsymbol{\rho}-\hat{\boldsymbol{\rho}})$.

In the simulation platform, we study the accuracy of our proposed algorithm by generating and estimating the unknown quantum states from the quantum source. We quantify the performance of the proposed algorithm with the sample average trace distance (S_a) across different samples taken at different time instances. In our simulations, we investigated the performance of the proposed algorithm across a range of Hilbert space dimensions, specifically spanning from $d = 7$ to $d = 14$, while considering different prior statistics scenarios. For our measurement operations, we utilized SIC-POVM-based operators, which include d^2 elements suitable for a d-dimensional Hilbert space. The QST process was carried out with a consistent total sample complexity of $N = 2000$ across all Hilbert space dimensions and different prior statistics scenarios.

First, we have considered nearly maximally mixed prior statistics scenario with $N_\rho = 5$. The term maximally mixed prior statics implies that each of the representations in $[\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \ldots, \boldsymbol{\rho}_{N_\rho}]$ is nearly a maximally mixed state. In figure 1, we observe that our proposed method obtains better measurement than the existing LS-based method for different Hilbert space dimensions, keeping the sample complexity constant. A similar experiment is performed where the prior statistics correspond to the pure states. We can conclude that the proposed LMMSE also achieves better estimation as depicted in figure 2. We also achieve an accuracy advantage for random priors as illustrated in figure 3.

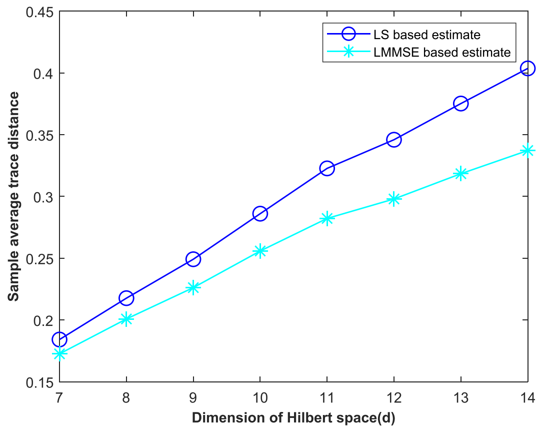

Figure 3. Comparison of the sample average trace distance error metric between the proposed LMMSE-based method and the existing LS-based method across various Hilbert space dimensions. In this comparison, we used random quantum states for prior statistics, and the sample complexity for each scenario was set at 2000.

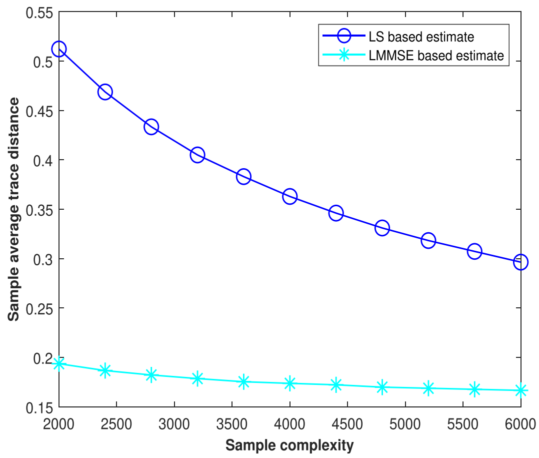

Figure 4. Comparison of sample average trace distance error metric between the proposed LMMSE-based method and the existing LS-based for different sample complexity for $d = 14$.

In conclusion, the proposed LMMSE obtains an accuracy advantage over the existing methods for mixed, pure, and random prior statistics of quantum states. Another key observation is that our proposed algorithm obtains a better estimate with higher dimension d.

Next, the numerical simulation demonstrates the variation of sample average trace distance concerning different sample complexity for $d = 14$. Sample complexity can be quantified as the number of identical copies of quantum states required for performing QST. Our numerical simulation suggests that the proposed LMMSE method acquires an efficient state estimation in a lesser sample complexity scenario in contrast to the existing LS-based method. A lower sample complexity suggests a range of 1000 to 1800 samples, while a higher sample complexity might correspond to a sample complexity of 2400 or greater in figure 4. In a lesser sample complexity scenario, measurement statistics are perturbed by the statistical noise due to the uncertainty of quantum mechanics. However, The proposed algorithm obtains a better estimate in lesser sample complexity scenarios with the help of prior statistics, even if the measurement statistics are insufficient. In contrast, measurement statistics are less vulnerable to measurement uncertainty in higher sample complexity scenarios. The LS-based method obtains a reasonable good state estimation for such a scenario. Despite this, the proposed method achieves the advantage over the existing algorithm by taking a cue from the estimated prior statistics.

In the previous section, we discussed the novelty of the LMMSE-based QST framework over the existing LS-based estimate. However, we also proposed adaptive versions of the LMMSE-based QST framework. The first proposal is to find the LMMSE-based equalizer after every block iteration with matrix inversion. In contrast, the second one follows a low-complexity approach of the adaptive LMMSE by avoiding the covariance matrix inversion after each block. In the simulation, we have considered an unknown quantum source with $d = 9$ that can produce $N_\rho = 7$ different quantum states. We evaluate the estimation accuracy of

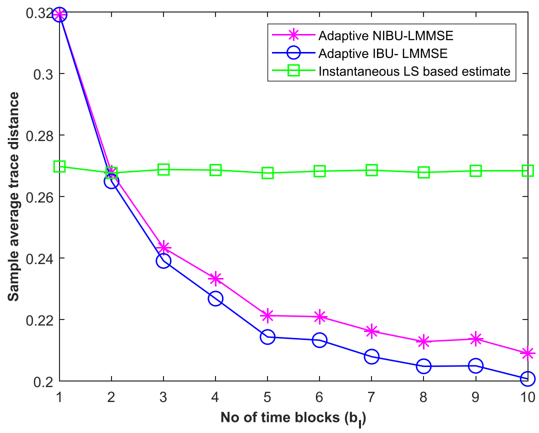

Figure 5. Comparison of average trace distance error metric between the proposed LMMSE-based methods and the existing LS-based for $d = 9$. The prior statistics correspond to random quantum states, and each time block contains six clusters.

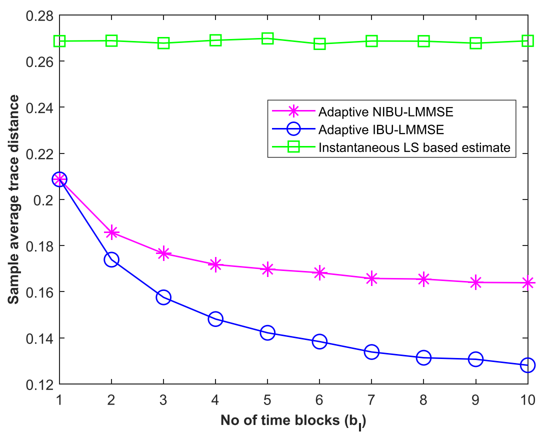

Figure 6. Comparison of average trace distance error metric between the proposed LMMSE-based methods and the existing LS-based for $d = 9$. The prior statistics correspond to random quantum states; each time block contains ten clusters.

the proposed and existing QST with a sample complexity of $N = 2000$ after estimating the statistics after each block iteration. However, in each time block, the statistics are updated recursively. The whole simulation is repeated 500 times to obtain a reasonable average accuracy estimation. In figure 5, we observe that the proposed adaptive methods promise to obtain a better estimate after the second iteration where each block iteration contains six clusters. A similar experiment is repeated in figure 6 with 10 clusters in each block. Our proposed methods exhibit the estimation accuracy from the first iteration with an increased cluster number in a block. From the observation, we conclude that the estimated prior statistics become more accurate with the increase in iteration. Our proposed adaptive methods obtain better precision than the existing LS-based methods. However, the proposed IBU-LMMSE gets a better accuracy than the proposed NIBU-LMMSE, which comes at the cost of the complexity of performing matrix inversion of the covariance matrix. Furthermore, we have performed the simulation with the pure state prior statistics, and the corresponding simulation result is depicted in figure 7. The proposed adaptive algorithms obtain a better accuracy concerning the current LS-based estimate. In our study, we conducted a study of the performance of both IBU-LMMSE and NIBU-LMMSE in relation to the Hilbert space dimension. During the simulation, we performed state estimation across 8 time blocks. In each time block, we considered a total of 15 clusters for estimating the prior statistics. The results of the estimation accuracy after each time block are depicted in figure 8.

Notably, the sample average trace distance error metric displayed similar characteristics as a function of dimension for both IBU-LMMSE and NIBU-LMMSE, with an increase in the time block. However, it remained consistently evident that IBU-LMMSE outperformed NIBU-LMMSE in terms of estimation accuracy despite the higher computational complexity associated with IBU-LMMSE.

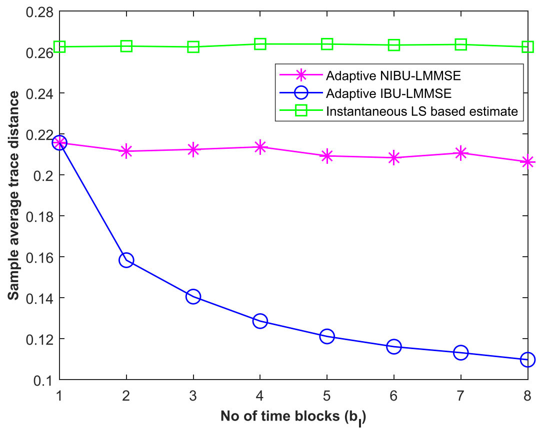

Figure 7. Comparison of average trace distance error metric between the proposed LMMSE-based methods and the existing LS-based for $d = 9$. The prior statistics correspond to pure quantum states; each time block contains 30 clusters.

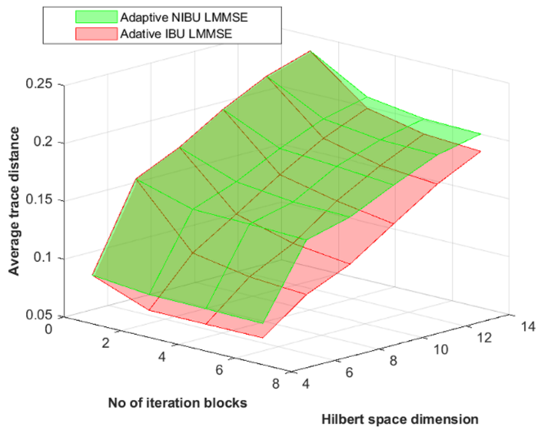

Figure 8. Comparison of sample average trace distance between the NIBU-LMMSE and IBU-LMMSE as a function of Hilbert space dimension.

5. Conclusion

In this work, we have proposed a linear Bayesian tomography based on the LMMSE criterion. The complexity is comparable to that of the non-Bayesian LS approach, but with superior performance. The prior statistics of the states are not assumed to be known. In fact, we propose to estimate them, specifically only the first and second order statistics, and proceed with a QST technique. We have also proposed adaptive versions of the algorithm with better performance than the LS-based QST one. A weighted update of the statistical parameters is also proposed. Overall, the proposed algorithms can make a good choice for QST, where the prior is initially unknown. This approach can be taken as middle path between low cost LS method and the high complex traditional Bayesian approaches.

Data availability statement

The data generated and/or analysed during the current study are not publicly available for legal/ethical reasons but are available from the corresponding author on reasonable request.

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