Prog. Theor. Exp. Phys. **2021**, 12C104 (7 pages)
DOI: 10.1093/ptep/ptab064

Note on a description of a perfect fluid by the Kalb–Ramond field

Yutaka Matsuo[1,*] and Akio Sugamoto[2]

[1]*Department of Physics & Trans-scale Quantum Science Institute & Mathematics and Informatics Center, University of Tokyo, Hongo 7-3-1, Bunkyo-ku, Tokyo 113-0033, Japan*
[2]*Department of Physics, Graduate School of Humanities and Sciences, Ochanomizu University, 2-1-1 Otsuka, Bunkyo-ku, Tokyo 112-8610, Japan*
*E-mail: matsuo@phys.s.u-tokyo.ac.jp

Received February 24, 2021; Accepted April 11, 2021; Published May 21, 2021

We explore a formulation of a perfect fluid in $3 + 1$ dimensions in terms of the Kalb–Ramond field. This was proposed long ago by Nambu and one of the present authors. In this note, we refine the statements in a more explicit form. We also comment on the duality with the Gross–Pitaevsky formulation written by a complex scalar field.

Subject Index J18

1. Introduction

The purpose of this note is to summarize the explicit relation between two field theoretical descriptions of fluid dynamics. We are going to treat the non-relativistic perfect fluid, where the dynamics is described by the Euler equation and the continuity relation:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u} = -\frac{1}{\rho}\vec{\nabla}p = -\vec{\nabla}P \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho\vec{u}) = 0. \tag{2}$$

Here $\vec{u}(\vec{x}, t)$ is the velocity and ρ is the density.

The equation for the vorticity $\vec{\omega} = \vec{\nabla} \times \vec{u}$ is

$$\frac{\partial \vec{\omega}}{\partial t} = -\vec{\nabla} \times \vec{G}, \qquad \vec{G} = \vec{\omega} \times \vec{u}. \tag{3}$$

The two descriptions that we compare are

- Description by the Kalb–Ramond gauge field. This was noticed by Nambu [1] long ago, and one of the present authors developed the idea in Ref. [2].
- Gross–Pitaevsky: description in terms of a complex scalar field (see, e.g., Ref. [3] for a review).

In four dimensions, a scalar field is dual to a two-form tensor field (Kalb–Ramond). In this sense, these two descriptions should be dual to each other.

© The Author(s) 2021. Published by Oxford University Press on behalf of the Physical Society of Japan.
This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted reuse, distribution, and reproduction in any medium, provided the original work is properly cited.

Downloaded from https://academic.oup.com/ptep/article/2021/12/12C104/6279749 by DESY-Zentralbibliothek user on 23 January 2022

2. Gross–Pitaevsky description

The Gross–Pitaevsky equation in $3+1$ dimensions is given by[1]

$$i\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2}{2M}\nabla^2 - \mu + g_0|\psi|^2\right)\psi. \tag{4}$$

We write the complex scalar field as

$$\psi = \sqrt{\rho}e^{i\phi}. \tag{5}$$

Plugging this formula into Eq. (4), one obtains

$$\dot{\rho} = -\nabla(\rho\vec{u}) \tag{6}$$

$$-\hbar\dot{\phi} = -\frac{\hbar^2}{4M}\left(-\frac{(\nabla\rho)^2}{4\rho^2} + \frac{\nabla^2\rho}{2\rho} - (\nabla\phi)^2\right) - \mu + g_0\rho. \tag{7}$$

Here we define the velocity field $\vec{u}$ as

$$\vec{u} = -\frac{\hbar}{M}\nabla\phi. \tag{8}$$

The first equation (6) is the continuity equation of the fluid. The second equation (7), after taking divergence, is

$$\frac{\partial\vec{u}}{\partial t} + \frac{1}{2}\nabla(\vec{u}^2) = \nabla\left(-g_0\rho + \frac{\hbar^2}{2M}\left(-\left(\frac{(\nabla\rho)^2}{2\rho} + \frac{\nabla^2\rho}{2\rho}\right)\right)\right). \tag{9}$$

This describes the quantized Euler equation. The first term on the right-hand side of Eq. (9) gives the barotropic pressure. The second term gives the “quantum pressure”. The existence of the quantum correction helps the Gross–Pitaevsky equation to describe the superfluidity.

3. Gauge theory description

We use a convention to describe $3+1$D space-time:

$$g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1), \tag{10}$$

$$x^0 = ct, \tag{11}$$

where c (>0) is a constant.

3.1. Kalb–Ramond

The Kalb–Ramond field is an antisymmetric tensor in $3+1$ dimensions. We parameterize it as

$$B_{\mu\nu} = \begin{pmatrix} 0 & e_1/c & e_2/c & e_3/c \\ -e_1/c & 0 & b_3 & -b_2 \\ -e_2/c & -b_3 & 0 & b_1 \\ -e_3/c & b_2 & -b_1 & 0 \end{pmatrix}. \tag{12}$$

[1] This section is taken from a talk by M. Kobayashi during the workshop [4].

Downloaded from https://academic.oup.com/ptep/article/2021/12/12C104/6279749 by DESY-Zentralbibliothek user on 23 January 2022

The curvature associated with $B_{\mu\nu}$ is

$$C_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}, \tag{13}$$

$$C_{0ij} = c^{-1}\left(\epsilon_{ijk}\partial_t b_k - (\partial_i e_j - \partial_j e_i)\right), \tag{14}$$

$$C_{ijk} = \epsilon_{ijk}\vec{\nabla}\cdot\vec{b}. \tag{15}$$

The action for $B_{\mu\nu}$ is given by

$$S^{(KR)} = \frac{1}{12}\int d^4x\, C_{\mu\nu\rho}C^{\mu\nu\rho} = \int d^4x \left(\frac{1}{2c^2}\left(\dot{\vec{b}} - \vec{\nabla}\times\vec{e}\right)^2 - \frac{1}{2}(\vec{\nabla}\cdot\vec{b})^2\right). \tag{16}$$

The action has a gauge invariance for $B_{\mu\nu}$ with a vector gauge parameter $\lambda_\mu = (\epsilon/c, \vec{\lambda})$:

$$\delta B_{\mu\nu} = \partial_\mu\lambda_\nu - \partial_\nu\lambda_\mu, \tag{17}$$

$$\delta\vec{e} = \partial_t\vec{\lambda} - \vec{\nabla}\epsilon, \quad \delta\vec{b} = \vec{\nabla}\times\vec{\lambda}. \tag{18}$$

It has a gauge symmetry for the gauge parameters:

$$\delta\lambda_\mu = \partial_\mu\kappa. \tag{19}$$

By this transformation, the gauge transformation (17) vanishes. The effective degree of freedom for the gauge transformation becomes three.

3.2. *Source term*

We introduce a source term $K_{\mu\nu}$ that couples to $B_{\mu\nu}$. We denote $K_{0i} = c\,\omega_i$ and $K_{ij} = \epsilon_{ijk}\chi_k$. The action that describes the coupling is

$$S^{(S)} = \int d^4x \frac{1}{2}B_{\mu\nu}K^{\mu\nu} = \int d^4x\left(-\vec{e}\cdot\vec{\omega} + \vec{b}\cdot\vec{\chi}\right). \tag{20}$$

The gauge invariance implies the conservation law for the source term $\partial_\mu K^{\mu\nu} = 0$. In terms of the components,

$$\partial_t\vec{\omega} + \vec{\nabla}\times\vec{\chi} = 0, \quad \vec{\nabla}\cdot\vec{\omega} = 0. \tag{21}$$

We would like to identify $\vec{\omega}$ with the vorticity $\nabla\times\vec{u}$. With this choice, the second constraint is automatically satisfied. The Euler equation for the vorticity (3) implies that we may identify $\vec{\chi} = \vec{G} + \vec{\nabla}f = \vec{\omega}\times\vec{u} + \vec{\nabla}f$. Here f is an arbitrary function. For the convenience of later discussions, we choose it as $f = \frac{1}{2}\vec{\nabla}(\vec{u}^2)$ such that

$$\vec{\chi} = (\vec{u}\cdot\vec{\nabla})\vec{u}. \tag{22}$$

We emphasize that the relevance of the Kalb–Ramond field for the description of the perfect fluid is that the constraints for the source term are identical to the Euler equation. On the other hand, the choice of the source term breaks the Lorentz covariance since $\vec{\omega}$ and $\vec{\chi}$ do not transform properly under the Lorentz transformation. In this sense, the interpretation of the parameter c as the light velocity may be optional.

Downloaded from https://academic.oup.com/ptep/article/2021/12/12C104/6279749 by DESY-Zentralbibliothek user on 23 January 2022

3.3. *Solving the equations of motion*

The equations of motion from $S^{(KR)} + S^{(S)}$ are

$$- c^{-2}\partial_t(\partial_t\vec{b} - \vec{\nabla}\times\vec{e}) + \vec{\nabla}(\vec{\nabla}\cdot\vec{b}) + \vec{\chi} = 0, \tag{23}$$

$$c^{-2}\vec{\nabla}\times(\vec{\nabla}\times\vec{e} - \partial_t\vec{b}) - \vec{\omega} = 0. \tag{24}$$

The Bianchi identity becomes

$$\partial_\mu V^\mu = 0, \quad V^\mu = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\partial_\nu B_{\lambda\rho}. \tag{25}$$

In the component form, if we denote $V^\mu = (\varphi, c^{-1}\vec{v})$ with $\varphi = \vec{\nabla}\cdot\vec{b}$, $\vec{v} = \vec{\nabla}\times\vec{e} - \partial_t\vec{b}$,

$$\partial_t\varphi + \vec{\nabla}\cdot\vec{v} = 0. \tag{26}$$

Equations (23), (24), and (26) are an analog of the Maxwell equation in four dimensions.

In the following, we use a set of gauge-fixing conditions that are an analog of the “Coulomb gauge”:

$$\vec{\nabla}\times\vec{b} = 0, \quad \vec{\nabla}\cdot\vec{e} = 0. \tag{27}$$

As we have noted, the effective degree of freedom for the gauge transformation is three. One can fix these degrees of freedom entirely by the constraint. The number of massless degrees of freedom for the antisymmetric tensor is one. It is hidden as a component of $\vec{b}$ and may be identified with φ.

With these gauge-fixing conditions, Eq. (24) becomes

$$\vec{\omega} = c^{-2}\vec{\nabla}\times(\vec{\nabla}\times\vec{e}) = -c^{-2}\Delta\vec{e}. \tag{28}$$

Thus one may solve $\vec{e}$ by using the Coulomb potential:

$$\vec{e}(\vec{x}) = \frac{c^2}{4\pi}\int d^3y\,\frac{1}{|\vec{x}-\vec{y}|}\,\vec{\omega}(\vec{y}). \tag{29}$$

One may also set

$$\vec{u} = c^{-2}\vec{v}, \tag{30}$$

which satisfies $\vec{\nabla}\times u = \vec{\omega}$ and an analog of the continuity equation (26) [2].

Plugging these equations into Eq. (23), one obtains the Euler equation:

$$\partial_t\vec{u} + (\vec{u}\cdot\vec{\nabla})\,\vec{u} = -\vec{\nabla}\varphi. \tag{31}$$

3.4. *Comparison of the two approaches*

In three dimensions, the scalar field is dual to a one-form. In this sense, the Gross–Pitaevsky and the gauge theory descriptions are dual (see Eqs. (8) and (30)):

$$\vec{u} = -\frac{\hbar}{M}\vec{\nabla}\phi = c^{-2}(\vec{\nabla}\times\vec{e} - \partial_t\vec{b}). \tag{32}$$

The relation between ϕ and $B_{\mu\nu}$ is analogous to that between electricity and magnetism, namely, the duality between two one-form gauge potentials $dA = \star dA_D$, where $\star$ is the Hodge dual.

Downloaded from https://academic.oup.com/ptep/article/2021/12/12C104/6279749 by DESY-Zentralbibliothek user on 23 January 2022

4. Description by vortex filaments

The perfect fluid in three dimensions has an alternative description in terms of the vortex filaments, which are described by the string variable $\vec{X}_i(\sigma_i, t)$:

$$\vec{\omega}(x) = \sum_{i=1}^{N} \Gamma_i \int d\sigma_i \frac{\partial \vec{X}_i(\sigma_i, t)}{\partial \sigma_i} \delta^{(3)}\left(\vec{x} - \vec{X}_i(\sigma_i, t)\right), \tag{33}$$

$$\vec{u}(x) = \sum_{i=1}^{N} \frac{\Gamma_i}{4\pi} \int d\sigma_i \frac{\partial \vec{X}_i}{\partial \sigma_i} \times \frac{\vec{x} - \vec{X}_i}{|\vec{x} - \vec{X}_i|^3}. \tag{34}$$

One puts these formulae into the Euler equation. After some computations, one finds the time evolution of the string variables, $\vec{X}(\sigma, t)$:

$$\frac{\partial \vec{X}_j}{\partial \sigma_j} \times \frac{\partial \vec{X}_j}{\partial t} = \frac{\partial \vec{X}_j}{\partial \sigma_j} \times \left(\sum_{k}^{N} \frac{\Gamma_k}{4\pi} \int d\sigma_k \frac{\partial \vec{X}_k}{\partial \sigma_k} \times \frac{\vec{X}_j - \vec{X}_k}{|\vec{X}_j - \vec{X}_k|^3} \right). \tag{35}$$

This is equivalent to

$$\begin{aligned} \left(\frac{\partial \vec{X}_j}{\partial t} \right)^{\perp} &= \sum_{k}^{N} \frac{\Gamma_k}{4\pi} \int d\sigma_k \frac{\partial \vec{X}_k}{\partial \sigma_k} \times \frac{\vec{X}_j - \vec{X}_k}{|\vec{X}_j - \vec{X}_k|^3} \\ &= \vec{u}(\vec{X}_j). \end{aligned} \tag{36}$$

The notation $\perp$ implies a projection in the perpendicular direction to $\frac{\partial \vec{X}_j}{\partial \sigma_j}$ on the left-hand side. We note that one can absorb the velocity in the longitudinal direction, which contains some divergences. The expression of the velocity field coincides with the expression by the Biot–Savart law.

The action that produces the vortex filament is given by $S^{(T)} + S^{(U)}$ with

$$S^{(T)} = \sum_{j=1}^{N} \frac{\Gamma_j}{3} \int dt d\sigma_j \vec{X}_j \cdot \left(\frac{\partial \vec{X}_j}{\partial \sigma_j} \times \frac{\partial \vec{X}_j}{\partial t} \right) \tag{37}$$

$$\begin{aligned} S^{(U)} &= -\frac{1}{2} \int d^4x |\vec{u}(x)|^2 \\ &= -\frac{1}{8\pi} \sum_{jk} \Gamma_j \Gamma_k \int d\sigma_j \int d\sigma_k \left(\frac{\partial \vec{X}_j}{\partial \sigma_j} \cdot \frac{\partial \vec{X}_k}{\partial \sigma_k} \right) \frac{1}{|\vec{X}_j - \vec{X}_k|}. \end{aligned} \tag{38}$$

The kinetic action $S^{(T)}$ was proposed by Takhtajan [5] to produce a Nambu bracket [6]. The appearance of the Nambu bracket here is natural since the Euler equation is associated with volume-preserving diffeomorphism. The second part of the action is identical to the energy of the fluid. This type of action was proposed long ago in Ref. [7][2]. Together with the variation formula,

$$\delta H = \frac{1}{4\pi} \sum_{jk} \Gamma_j \Gamma_k \int d\sigma_j \int d\sigma_k \delta \vec{X}_j \cdot \left(\frac{\partial \vec{X}_j}{\partial \sigma_j} \times \left(\frac{\partial \vec{X}_k}{\partial \sigma_k} \times \frac{\vec{X}_j - \vec{X}_k}{|\vec{X}_j - \vec{X}_k|^3} \right) \right), \tag{39}$$

and the variation of the first term, Eq. (37) gives an equation of motion in the form (35).

[2] See also Refs. [8,9] for a proposal of quantization and generalization to higher dimensions.

Downloaded from https://academic.oup.com/ptep/article/2021/12/12C104/6279749 by DESY-Zentralbibliothek user on 23 January 2022

4.1. Coupling to the Kalb–Ramond field

There is a standard coupling of the string worldsheet with the Kalb–Ramond field:

$$S^{(S2)} = \sum_i \frac{\Gamma_i}{2} \int d\tau d\sigma_i B_{\mu\nu}(X_i(\tau,\sigma)) \frac{\partial X_i^\mu}{\partial \sigma_i} \frac{\partial X_i^\nu}{\partial \tau}. \tag{40}$$

One may relate $S^{(S2)}$ with $S^{(S)}$ by taking a static gauge-fixing condition:

$$c\tau = ct = X_i^0. \tag{41}$$

The action becomes

$$S^{(S2)} = \sum_i \Gamma_i \int dtd\sigma_i \left(-\vec{e}(t,\vec{X}_i) \frac{\partial \vec{X}_i}{\partial \sigma_i} + \vec{b}(\vec{X}_i) \cdot \frac{\partial \vec{X}_i}{\partial \sigma_i} \times \frac{\partial \vec{X}_i}{\partial t} \right). \tag{42}$$

We note that Eq. (33) implies

$$-\int d^4x\, \vec{e}(x) \cdot \vec{\omega}(x) = -\sum_i \Gamma_i \int dtd\sigma_i\, \vec{e}(t,\vec{X}_i) \cdot \frac{\partial \vec{X}_i}{\partial \sigma_i}. \tag{43}$$

The second term takes the form $\int d^4x\, \vec{b}(x) \cdot \vec{\chi}(x)$ with

$$\vec{\chi}(x) = \sum_i \Gamma_i \int d\sigma_i \frac{\partial \vec{X}_i(\sigma_i,t)}{\partial \sigma_i} \times \frac{\partial \vec{X}_i(\sigma_i,t)}{\partial t} \delta^{(3)}\left(\vec{x} - \vec{X}_i(\sigma_i,t)\right). \tag{44}$$

Unlike Eq. (22), the coupling to $B_{\mu\nu}$ becomes Lorentz covariant. It reduces to Eq. (22) if we use the equation of motion (36). In the vortex filament description, the action is given by $S^{(KR)} + S^{(S2)} + S^{(T)}$. The Lorentz invariance is broken by the Takhtajan action (37). After taking the Coulomb gauge (29), $\vec{e}$ is given by the Coulomb potential (29). The coupling to $\vec{e}$ in the source term $S^{(S2)}$ gives the energy of the fluid (38).

We note that the Nambu–Goto term for the vortex filament is absent. In this sense, it may be regarded as a "tensionless string".

4.2. Comparison with Gross–Pitaevsky

In the Gross–Pitaevsky description, the vortex filament appears as a singularity of the scalar field. Indeed, the circulation around the vortex filament is described as

$$\Gamma_i = \int_{C_i} d\vec{\ell} \cdot \vec{v} = \begin{cases} \int_{D_i} d\vec{S} \cdot \vec{\omega}(x) & \text{Vortex description} \\ \int_{C_i} d\vec{\ell} \cdot \frac{\hbar}{M} \nabla \phi & \text{GP description} \end{cases}, \tag{45}$$

where C_i is a small closed path around the ith vortex filament and D_i is the 2D domain whose boundary is described by C_i. This implies that the phase ϕ is multi-valued around the vortex. Since the complex scalar ψ should be single-valued, one has to impose the quantization condition:

$$\frac{M\Gamma_i}{2\pi\hbar} = \frac{M\Gamma_i}{h} \in \mathbb{Z}. \tag{46}$$

This implies a Dirac quantization condition for the circulation, which is the coupling of the Kalb–Ramond field with the vortex.

Downloaded from https://academic.oup.com/ptep/article/2021/12/12C104/6279749 by DESY-Zentralbibliothek user on 23 January 2022

5. Future directions

There are several directions to be explored from this note.

Perhaps the most interesting one is to find the scaling behavior of the turbulence (Kolmogolov's law) from the Kalb–Ramond description. We note a numerical simulation in Gross–Pitaevsky's description (see, e.g., Ref. [10]) that supports this idea. We may find a similar numerical and analytical result from the Kalb–Ramond approach.

The other directions may be, for instance,

- Description of the relativistic vortex: there is some literature (see, for instance, Ref. [11]) in which one can find the Kalb–Ramond description for the relativistic perfect fluid. It may be possible to find a formulation for the relativistic vortex filaments.
- Non-Abelian generalization: there is a non-Abelian analog for the Gross–Pitaevsky description [12]. One may find the dual formulation by Kalb–Ramond (see, for instance, Refs. [13–15]).
- Quantization of the motion of vortex filaments: in Refs. [8,9], one of the present authors studied a Dirac quantization of Lund–Regge action [7]. It may be useful to compare this with the quantization of the circulation given by the Gross–Pitaevsky approach.
- String field theory: in the vortex filament description, the number of the vortex is conserved during the time development. In an actual superfluid, there occur recombinations of vortices. One has to introduce a framework of string field theory to describe such processes [16].

Acknowledgements

This article was prepared for the workshop on "Space-time topology behind formation of micro–macro magneto-vortical structure manifested by Nambu mechanics" held at Osaka City University, 28 September–1 October 2020. YM's presentation was based on Ref. [17]. Since it has already been published, we have prepared this note to explain the ideas that appeared after the workshop. We would like to thank the organizers of the workshop (organized by NITEP, Osaka City University), which triggered us to develop the idea in this note. We also acknowledge the communications with Y. Fukumoto, H. Itoyama, M. Kobayashi, and T. Yoneya during the workshop. YM is supported in part by Japan Society for the Promotion of Science (JSPS) KAKENHI Grant numbers 18K03610 and 18H01141.

References

[1] Y. Nambu, in *Quark Confinement and Field Theory: Proceedings of a Conference at the University of Rochester, Rochester, New York, 14–18 June 1976*, eds. D. R. Stump and D. H. Weingarten (Wiley, New York, 1977).
[2] A. Sugamoto, Phys. Rev. D **19**, 1820 (1979).
[3] M. Tsubota, M. Kobayashi, and H. Takeuchi, Phys. Rep. **522**, 191 (2013).
[4] M. Kobayashi, "Dual cascade of energy and helicity accompanying mass and spin in non-Abelian quantum turbulence of spinor Bose–Einstein condensates", a talk during the workshop.
[5] L. Takhtajan, Commun. Math. Phys. **160**, 295 (1994).
[6] Y. Nambu, Phys. Rev. D **7**, 2405 (1973).
[7] F. Lund and T. Regge, Phys. Rev. D **14**, 1524 (1976).
[8] Y. Matsuo, Mod. Phys. Lett. A **8**, 2677 (1993) [arXiv:hep-th/9305151] [Search INSPIRE].
[9] Y. Matsuo and Y. Shibusa, J. High Energy Phys. **0102**, 006 (2001).
[10] M. Kobayashi and M. Tsubota, Phys. Rev. Lett. **94**, 065302 (2005).
[11] B. Carter and D. Langlois, Nucl. Phys. B **454**, 402 (1995).
[12] M. Kobayashi, Y. Kawaguchi, M. Nitta, and M. Ueda, Phys. Rev. Lett. **103**, 115301 (2009).
[13] K. Seo, M. Okawa, and A. Sugamoto, Phys. Rev. D **19**, 3744 (1979).
[14] P.-M. Ho, K.-W. Huang, and Y. Matsuo, J. High Energy Phys. **1107**, 021 (2011).
[15] P.-M. Ho and Y. Matsuo, J. High Energy Phys. **1209**, 075 (2012).
[16] A. Sugamoto, K. Bamba, T. Kawamura, A. Kuwana, Y. Nagata, and M. Saitou, "Nambu dynamics and hydrodynamics of granular material", a talk presented by A. Sugamoto during the workshop.
[17] P.-M. Ho and Y. Matsuo, Prog. Theor. Exp. Phys. **2016**, 06A104 (2016).

Downloaded from https://academic.oup.com/ptep/article/2021/12/12C104/6279749 by DESY-Zentralbibliothek user on 23 January 2022