Stability of Schwarzshild-like spacetime in parity violating gravitational theories

Hayato Motohashi[a,b] and Teruaki Suyama[b]

[a] *Department of Physics, Graduate School of Science, The University of Tokyo, Tokyo 113-0033, Japan*
[b] *Research Center for the Early Universe (RESCEU), Graduate School of Science, The University of Tokyo, Tokyo 113-0033, Japan*

Abstract

We study linear perturbations around static, spherically-symmetric spacetimes in $f(R,C)$ gravitational theories whose Lagrangians depend on the Ricci scalar R and the parity violating Chern-Simons term C. By an explicit construction, we show that the Hamiltonian for the perturbation variables is not bounded from below, suggesting that such a background spacetime is unstable against perturbations. This gives a strong limit on a phenomenological gravitational model which violates parity. We also show that either $R=$ const or $\frac{\partial^2 f}{\partial R \partial C}=0$ is a necessary and sufficient condition for the stability. We then implement in detail the perturbation analysis for such theories which satisfy the stability conditions and find that the no-ghost conditions and no-tachyon conditions are the same as those in $f(R)$ theories.

1 Introduction

General relativity (GR) in the weak gravitational field regime has been tested both experimentally and observationally over many decades. In the forthcoming decade, the test of GR in the strong gravitational regime will be also available, for example, by observing the gravitational waves coming from the vicinity of black holes (BHs). These facts have provoked alternative theories of gravity and have led us to understand theoretically what kinds of different phenomena are expected in such theories. In light of this situation, it is interesting to consider gravitational theories which violate parity due to the so-called Chern-Simons(CS) term, or the Pontryagin density, $C \equiv \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}R^{\alpha\beta}{}_{\mu\nu}R^{\gamma\delta\mu\nu}$, where $\epsilon_{\alpha\beta\gamma\delta}$ is the totally antisymmetric tensor. For a recent review on the Chern-Simons gravity, see Ref. [1].

In this paper, we consider the gravitational theories whose Lagrangian is a general function of R and C, $f(R,C)$, and develop linear perturbation theory around the static and spherically symmetric spacetime. Unlike in the case of $f(R)$ theories which can be mapped into equivalent theories where a scalar field having self-interacting potential is minimally coupled to Einstein-Hilbert gravity, $f(R,C)$ theories cannot be mapped into theories where C is coupled solely to a dynamical scalar field due to nontrivial transformation property of the CS term under the conformal transformation. Our aim is to clarify both quantitative and qualitative behaviors of the perturbations. To be more precise, we will derive no-ghost and no-tachyon conditions which are necessary to ensure stability of the background spacetime against perturbation, obtain dispersion relations for the propagating modes and find features that are characteristic to parity violating theories.

2 BH perturbation for $f(R,C)$ theories

We study $f(R,C)$ theory, where the action is described by a general function of Ricci scalar R and the CS term $C \equiv \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}R^{\alpha\beta}{}_{\mu\nu}R^{\gamma\delta\mu\nu}$,

$$S = \frac{M_P^2}{2}\int d^4x\,\sqrt{-g}f(R,C). \tag{2.1}$$

Here, $M_P = 1/\sqrt{8\pi G_N} \simeq 4.34\times10^{-6}$g is the reduced Planck mass. We can rewrite the action (2.1) as

$$S = \frac{M_P^2}{2}\int d^4x\,\sqrt{-g}\left(RF(\lambda,s)+W(\lambda,s)C-V(\lambda,s)\right), \tag{2.2}$$

where λ and s are auxiliary fields and

$$F(\lambda, s) = \frac{\partial f(\lambda, s)}{\partial \lambda}, \quad W(\lambda, s) = \frac{\partial f(\lambda, s)}{\partial s}, \quad V(\lambda, s) = \lambda F(\lambda, s) + sW(\lambda, s) - f(\lambda, s). \tag{2.3}$$

In this section, we calculate the perturbative action around a static and spherically symmetric spacetime whose metric is given by

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{B(r)} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \tag{2.4}$$

There are gauge degrees of freedom for the metric perturbations $h_{\mu\nu}$ [2, 3]. For the odd-type perturbations, we take a gauge in which spherical component $h_{ab}(a, b = \theta, \varphi)$ vanishes. For the even-type perturbations, we impose that both h_{ab} and h_{ta} vanish. As a result of the parity violation term, the odd and even modes do not decouple from each other within the system of equations and we have to deal with it all together. In addition to the metric perturbations, we also need to perturb the other functions λ and s that appear in the action. Just for later convenience, instead of perturbing λ and s as the fundamental fields, we treat δF and δW as perturbation variables.

With these perturbation variables, expanding the action (2.2) to second order yields the quadratic action for the perturbation variables. It then turns out that not all of the variables are dynamical. Actually, we find that H_0, H_1 and δW are auxiliary fields. Therefore, they can be eliminated from the action by using their equations of motion. After substituting the constraints and many integration by parts, we end up with the following Lagrangian density(for the detailed calculations, see Ref. [4]):

$$\begin{aligned}\mathcal{L} \;=\;& p_1\ddot{h}_1^2 + p_2\ddot{h}_1(r\dot{h}_0' - 2\dot{h}_0) + p_3\dot{h}_0'^2 + p_4\dot{h}_0^2 + p_5\dot{h}_1^2 + p_6\delta\dot{F}^2 + p_7\dot{\beta}^2 + p_8\dot{h}_0\delta\dot{F} + p_9\dot{h}_0\dot{\beta} + p_{10}\dot{\beta}\delta\dot{F} + p_{11}h_0'^2 \\ &+ p_{12}\delta F'^2 + p_{13}\beta'^2 + p_{14}h_0'\delta F' + p_{15}h_0'\beta' + p_{16}\beta'\delta F' + p_{17}h_0'\dot{h}_1 + p_{18}\dot{h}_0h_1 + p_{19}h_0'\delta F + p_{20}h_0'\beta \\ &+ p_{21}\dot{h}_1\delta F + p_{22}\dot{h}_1\beta + p_{23}\delta F\beta' + p_{24}h_0^2 + h_0(p_{25}\delta F + p_{26}\beta) + p_{27}h_1^2 + p_{28}\delta F^2 + p_{29}\delta F\beta + p_{30}\beta^2\end{aligned} \tag{2.5}$$

Since all the fields (h_0, h_1, β, δF) have time derivatives that are not removed by any integration by parts, all of them are dynamical fields. Hence, this is our final Lagrangian. Explicit expressions of the background dependent coefficients $p_1, \cdots$ are given in the appendix of Ref. [4].

Contrary to the corresponding Lagrangian in $f(R)$ gravity, the above Lagrangian contains a term $\ddot{h}_1^2$. This term results in fourth order differential equations for h_1 with respect to time. We can confirm this fact by looking at the explicit expressions for p_1, p_2 and p_3, which are given by

$$p_1 = -\frac{32\pi\ell(\ell+1)M_P^2W'^2}{(2\ell+1)F\left(\frac{A}{B}\right)^{3/2}}, \quad p_2 = -\frac{2p_1}{r}, \quad p_3 = p_1.$$

The presence of the $\ddot{h}_1$ term in the general $f(R, C)$ gravity is a signal that the theory is plagued by instability [5]. In particular, the Hamiltonian can take arbitrary negative values and hence it is not bounded from below. This result shows that the general $f(R, C)$ gravity has the problem of having a ghost around the static and spherically symmetric background and provides a severe condition on the functional form of $f(R, C)$.

There are two possible cases where the presence of the ghost does not become problematic. The first one is to assume that the $f(R, C)$ theory under consideration is an effective theory which is valid only on length scales larger than a certain length d_c. From this point of view, the presence of the ghost does not matter if its mass is larger than the energy scale d_c^{-1} since the dynamics of the ghost cannot be described by the low energy $f(R, C)$ theory. The more fundamental theory which is valid above d_c^{-1} may cure the problem. To see this, let us pick up terms in the Hamiltonian relevant to the ghost mode,

$$\begin{aligned}\mathcal{H}_{\rm sub} \;=\;& -\cos\delta\left(K_{11}\bar{M}_{11}\cos\delta + 2K_{13}\sqrt{\bar{M}_{11}\bar{M}_{33}}\sin\delta\right)\bar{P}_1^2 \\ &-\sin\delta\left(K_{11}\bar{M}_{11}\sin\delta - 2K_{13}\sqrt{\bar{M}_{11}\bar{M}_{33}}\cos\delta\right)\bar{P}_3^2 - \bar{Q}_1^2 - \bar{Q}_3^2,\end{aligned}$$

where δ is determined from the equation

$$\tan 2\delta = -\frac{2K_{13}}{K_{11}}\sqrt{\frac{\bar{M}_{33}}{\bar{M}_{11}}} = \sqrt{\frac{3}{\ell^2+\ell-2}}\;\mathrm{sgn}\left(r^4F_\lambda - 64\ell(\ell+1)F_s^2F'^2\right).$$

Here, the sign function is defined as $\mathrm{sgn}(x) = +1, 0, -1$ for $x > 0, x = 0, x < 0$, respectively. The coefficients in front of $\bar{P}_1$ and $\bar{P}_3$ must be larger than d_c^{-2} in order for those fields to be in the high energy regime where the effective $f(R, C)$ theory does not work. This leads to a condition,

$$|K_{11}M_{11}| \simeq \left| \frac{r^4 F_\lambda - 64\ell(\ell+1)F_s^2 F'^2}{64 r^4 F_\lambda W'^2} \right| \gtrsim d_c^{-2}.$$

In particular, when the second term in the numerator is negligible, we obtain the very simple condition for W',

$$|W'| \lesssim d_c.$$

Since W', which has dimensions of length, represents how large the effects of the Chern-Simons term are, this condition says these effects are suppressed on distances larger than d_c.

The second possibility where the presence of the ghost does not become problematic is that $f(R, C)$ belongs to the special class in which $W' = 0$ is satisfied identically. Using the background metric, W' can be written as

$$W' = F_s R' + W_s C' = F_s R', \tag{2.6}$$

where we have used an identity $C = 0$ for the background metric. Therefore, if $f(R, C)$ satisfies either $F_s = 0$ or $R = \text{const}$, we have $W' = 0$ identically. For example, $F_s = 0$ is trivially satisfied if $f(R, C)$ takes a separable form, *i.e.*, $f(R, C) = f_1(R) + f_2(C)$, where f_1 and f_2 are arbitrary functions of R and C, respectively. $f(R)$ gravity is included in this case. The second case $R = \text{const}$ is satisfied, for example, if the Schwarzschild metric is a solution of the model. In either case, we have still many $f(R, C)$ theories. We deal with this class of $f(R, C)$ theories in the next section.

3 Study of special cases with $W' = 0$.

As we have shown in the previous section, the general $f(R, C)$ theories with nonvanishing W' have the problem of instability. Thus, the cases with $W' = 0$ are more phenomenologically interesting and deserve further investigation. The Lagrangian for the general $f(R, C)$ theories (2.5) can be also used for the special case $W' = 0$ as well. We find that even for the special cases, the odd and even modes are still coupled as long as $F_s = 0$.

By introducing a new variable q, we can rewrite the Lagrangian so that it contains only q, δF and β,

$$\mathcal{L} = k_{ij}\dot{q}_i\dot{q}_j - d_{ij}q_i'q_j' - e_{ij}q_i'q_j - m_{ij}q_iq_j, \tag{3.1}$$

where we have defined $(q_1, q_2, q_3) = (\delta F, \beta, q)$. A determinant of the kinetic matrix k_{ij} is found to be

$$\det(k_{ij}) = \frac{384\pi^3\ell^2(\ell+1)^2 M_P^6 r^4 Y^2}{(2\ell+1)^3 A^3 F^3 \left(\frac{A}{B}\right)^{3/2} \left(rBA'\left(rF' + 2F\right) + 2A\left(F\left(-2B + \ell^2 + \ell\right) - rBF'\right)\right)^2}, \tag{3.2}$$

where the definition of Y is given in Ref. [4]. This is not zero in general. Therefore, all the variables are dynamical and there are three propagating modes, one of which is odd (*i.e.*, q) and the remaining two are even (*i.e.*, δF and β). This structure is the same as that of the $f(R)$ gravity theories where there is one propagating odd mode and two propagating even modes. This result shows that the condition $W' = 0$ kills all the pathological modes which exist in the general $f(R, C)$ theories. By evaluating k_{33} and $k_{22}k_{33} - k_{23}^2$, we find that $F > 0$ is needed to ensure their positivity. Therefore, as is the case with $f(R)$ theories, $F > 0$ is the no-ghost condition for $f(R, C)$ theories that satisfy $W' = 0$.

We can derive the dispersion relations for the three modes from an equation,

$$\det(-\omega^2 k_{ij} + k^2 d_{ij}) = \frac{768\pi^3\ell^2\left(-\ell^3 - 2\ell^2 + \ell + 2\right)^2 M_P^6 r B F^3 q_1^2 \sqrt{\frac{A}{B}}\left(\omega^2 - k^2 AB\right)^3}{(2\ell+1)^3 q_{13}\left(2rq_{11} - rq_8' - 2q_8\right)\left(rBA'\left(rF' + 2F\right) + 2A\left(F\left(-2B + \ell^2 + \ell\right) - rBF'\right)\right)^2}.$$

We see all the modes obey a dispersion relation $\omega^2 = ABk^2$. The appearance of the factor AB is due to the fact that t and r are coordinate time and distance. In terms of the physical time and distance, the dispersion relation says all the modes propagate at the speed of light, which is exactly the same as in the case of $f(R)$ theories. Although there are no new contributions to the propagation speeds due to the Chern-Simons term, the coupling between δF and q means that we cannot consider the propagation of the odd and even modes separately as we can do in the case of $f(R)$ theories, which is a clear difference from $f(R)$ theories. This shows the potential usefulness of using the distinct nature of mode propagation in $f(R,C)$ theories for putting constraints on $f(R,C)$ models using observations of gravitational waves from compact astrophysical objects.

We can also evaluate the mass for each eigenmode. However, since each matrix element of m_{ij} is too lengthy to obtain analytic expressions for the mass eigenvalues, we will make an assumption that the background is very close to GR, *i.e.*, $F=1$, $A=B=1-\frac{r_g}{r}$ and also expand the eigenvalues in $\varepsilon\equiv\frac{r_g}{r}$ (weak field approximation). Under these assumptions, three eigenvalues are given by

$$\begin{aligned}
m_1^2 &= \frac{1}{3F_\lambda}-\frac{\ell^2(\ell+1)^2 r_g-2\left(\ell^2+\ell-2\right)^2\left(\ell^2+\ell+2\right)r}{3\left(\ell^2+\ell-2\right)^2 r^3}+\mathcal{O}\left(\varepsilon^2\right),\\
m_2^2 &= \frac{\ell(\ell+1)}{r^2}-\frac{\ell^2(\ell+1)^2 r_g}{\left(\ell^2+\ell-2\right)^2 r^3}+\mathcal{O}(\varepsilon^3),\\
m_3^2 &= \frac{\ell^2+\ell+4}{r^2}-\frac{192\ell\left(\ell^5+3\ell^4+7\ell^3+9\ell^2-4\right)r_g^2F_s^2}{\left(\ell^2+\ell-2\right)^2 r^8F_\lambda^2}+\mathcal{O}(\varepsilon^3).
\end{aligned}$$

Since m_1^2 is inversely proportional to F_λ at leading order, this mode corresponds to the scalar graviton that exists in the general $f(R)$ theories. To avoid the tachyonic mode, we need to impose a condition $F_\lambda>0$. The Chern-Simons corrections, F_s, appear in m_1^2 and m_3^2, but only in a combination with r_g. This means those corrections are important only in the vicinity of the BH and are suppressed compared to the standard terms that exist in GR far from the BH.

4 Conclusion

We have studied linear perturbations around the static, spherically-symmetric spacetime for general $f(R,C)$ theories, where C is the parity violating Chern-Simons term. By explicitly constructing the second order action, we showed that one odd mode appears in the action as a quadratic in its second time derivative. Irrespective of its sign, this results in an Hamiltonian that is not bounded from below. Therefore, the static and spherically symmetric spacetime is unstable in general $f(R,C)$ theories. This gives a strong limit on any phenomenological gravitational model which violates parity.

We also showed that either $R=\text{const}$ or $\frac{\partial^2 f}{\partial R\partial C}=0$ for the background metric is a necessary and sufficient condition to avoid the instability mentioned above. For such theories, the number of propagating modes for $\ell\geq 2$ is three, one from the odd and the other two from the even. Unlike in the case of $f(R)$ theories, those modes are coupled, which can be used as a distinctive feature to test the parity violating theories from observations. All the modes propagate at the speed of light. The no-ghost condition is $\frac{\partial f}{\partial R}>0$ and the no-tachyon condition is $\frac{\partial^2 f}{\partial R^2}>0$, which are the same as in the case of $f(R)$ theories.

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