

Wilson–’t Hooft line operators as transfer matrices

Kazunobu Maruyoshi*

Faculty of Science and Technology, Seikei University 3–3–1 Kichijoji-Kitamachi, Musashino-shi, Tokyo 180–8633 Japan

*E-mail: maruyoshi@st.seikei.ac.jp

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We review the relation between half-BPS Wilson–’t Hooft line operators in $\mathcal{N} = 2$ supersymmetric gauge theories on the twisted space-time $S^1 \times_{\epsilon} \mathbb{R}^2 \times \mathbb{R}$ and the transfer matrices constructed from the trigonometric L-operators of an integrable system.
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1. Introduction

The Wilson and ’t Hooft line operators are important objects in quantum field theory to elucidate its vacuum structure [1], e.g., the Wilson line obeys an area law in the confining phase, while the Higgs phase can be characterized by a ’t Hooft line obeying an area law. These have been studied extensively from various viewpoints. We discuss here a recent development in studies of Wilson–’t Hooft line operators in supersymmetric quantum field theory in 4D in relation with mathematical physics.

In supersymmetric quantum field theory, it is often the case that some quantities are related to the integrable systems. A classic example is the realization of the Seiberg–Witten theory [2], which specifies the low energy effective theory of $\mathcal{N} = 2$ supersymmetric theory in 4D, as the classical Hitchin integrable system [3] (see also Refs. [4–7]). This relation is generalized in Ref. [8] to the quantum level such that $\mathcal{N} = 2$ theories on a certain gravitational background corresponds to quantum Hitchin systems. A recent example is the relation [9,10] between the superconformal index of $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric quiver gauge theories and the partition function of the integrable lattice model found in Refs. [11,12]. This latter relation with the lattice model was explained by using the so-called 4D Chern–Simons theory in Refs. [13,14].

In this paper we see the Wilson–’t Hooft line operators play a role in this context, focusing on $\mathcal{N} = 2$ supersymmetric gauge theory. Specifically, we consider certain Wilson–’t Hooft lines in $\mathcal{N} = 2$ gauge theory which preserve a half of the supersymmetries in the twisted space-time $S^1 \times_{\epsilon} \mathbb{R}^2 \times \mathbb{R}$. The line operators wrap on S^1 . The vacuum expectation values (vevs) of the Wilson–’t Hooft lines are shown to be identified [15] with the transfer matrices of trigonometric L-operators which are obtained by a limit of the elliptic ones [16] satisfying the RLL relation with the elliptic R matrix [17–19].

This relation is a variant of the one found in Refs. [20,21] where certain half-BPS surface (codimension-2) defects in $\mathcal{N} = 1, 2$ supersymmetric gauge theories have been identified with the transfer matrices of the L-operators of the elliptic integrable lattice model [16,22]. These results

imply that an insertion of a class of line and surface defects corresponds to an insertion of L-operators in the corresponding integrable system.

The reason why this is the case can be seen from the string theory [15]. Both the half-BPS line and the surface defects in $\mathcal{N} = 2$ supersymmetric gauge theory are realized on the worldvolume of intersecting D-branes, and these are related by duality in string theory with the minor difference that one direction is compactified on S^1 or not, which induces the difference between trigonometric and elliptic integrable models.

The organization of this paper is as follows. In Sect. 2, we introduce the half-BPS Wilson and 't Hooft line operators in $\mathcal{N} = 2$ supersymmetric gauge theory. The insertion of these operators is considered on the space-time geometry $S^1 \times_{\epsilon} \mathbb{R}^2 \times \mathbb{R}$ and their vevs are computed [23] by using the localization method [24–26] (see also Refs. [27–30]). We apply this to the $\mathcal{N} = 2$ circular quiver gauge theory where $SU(N)$ gauge groups are located on a circle and connected by hypermultiplets in the bi-fundamental representation of adjacent gauge groups.

In Sect. 3, L-operators are discussed. We define an L-operator in an abstract manner, satisfying the RLL relation with the corresponding R-matrix. Their transfer matrix constructs an integrable system. An important example of the L-operators is the one introduced in Ref. [16] associated with the elliptic dynamical R-matrix in Refs. [17–19]. We then discuss a trigonometric limit of the elliptic L-operator. Its transfer matrix reproduces the vevs of the Wilson–'t Hooft lines obtained in Sect. 2.

2. Wilson–'t Hooft lines in supersymmetric gauge theories

Let us consider a gauge theory in 4D with gauge group G whose Lie algebra is \mathfrak{g} . Let a maximal torus of G be T with Lie algebra \mathfrak{t} . Also, let $\Lambda_{\text{r}}(\mathfrak{g}) \subset \mathfrak{t}^*$ and $\Lambda_{\text{cr}}(\mathfrak{g}) \subset \mathfrak{t}$ be the root lattice and the coroot lattice of \mathfrak{g} , respectively. Their inner product is denoted by $\langle \cdot, \cdot \rangle$. Their duals are the coweight lattice $\Lambda_{\text{cw}}(\mathfrak{g}) = \Lambda_{\text{r}}(\mathfrak{g})^{\vee} \subset \mathfrak{t}$ and the weight lattice $\Lambda_{\text{w}}(\mathfrak{g}) = \Lambda_{\text{cr}}(\mathfrak{g})^{\vee} \subset \mathfrak{t}^*$.

Let L be a line in the space-time. We consider the following two types of operators defined on a support L and their mixtures:

- A Wilson line operator is the worldline of a very heavy electrically charged particle, which is not dynamical. Let R be a representation of G , then the Wilson line on L is defined by

$$W_R = \text{Tr}_R P \exp\left(i \int_L A\right), \quad (1)$$

where A is the gauge field.

- A 't Hooft line operator is the worldline of a very heavy magnetically charged monopole [1]. This is a disorder operator, in the sense that the operator is defined by integrating over the fluctuation of the fields around the singular configuration at the location of the monopole in the path integral:

$$A = \frac{\mathbf{m}}{2}(1 - \cos \theta)d\phi + \dots, \quad (2)$$

where θ and ϕ are the polar angle and the azimuthal angle of the spherical coordinate in the direction orthogonal to the worldline, and \dots represents less-singular terms. For simplicity we set the gauge theory theta-angles to zero. The coefficient \mathbf{m} is the magnetic charge of the monopole. Different singular gauge field configurations of the above form describe the same monopole if their magnetic charges are related by gauge transformation. It follows that \mathbf{m} can

be chosen from \mathfrak{t} , and the choice is meaningful only up to the action of the Weyl group $W(G)$ of G .

The above expression of A is valid in a trivialization over a coordinate patch that contains the point $\theta = 0$ of a two-sphere surrounding the monopole. Along $\theta = \pi$, there is a Dirac string singularity which supports an unphysical magnetic flux. For the Dirac string to be invisible (or more precisely, for the gauge transformation by $\exp(i\mathbf{m}\phi)$ which allows us to go to the coordinate patch containing $\theta = \pi$ to be well defined), we must have

$$\langle \mathbf{m}, w \rangle \in \mathbb{Z} \tag{3}$$

for every weight $w \in \mathfrak{t}^*$ of the representation of every field in the theory. The theory always contains fields in the adjoint representation, so \mathbf{m} belongs to the coweight lattice: $\mathbf{m} \in \Lambda_{\text{cw}}(\mathfrak{g})/W(G)$.¹ Equivalently, \mathbf{m} is specified by an irreducible representation of the Langlands dual ${}^L\mathfrak{g}$ of \mathfrak{g} . In general, \mathbf{m} lies in a sublattice of $\Lambda_{\text{cw}}(\mathfrak{g})/W(G)$ determined by the matter content.

- An Wilson–’t Hooft line operator is a worldline of a heavy particle that carries both magnetic and electric charges. In the path integral formalism, a Wilson–’t Hooft line is realized by an insertion of the Wilson line (1) and a singular boundary condition on the support L of the line specified by the magnetic charge. The prescribed singularity (2) breaks the gauge symmetry to the stabilizer $G_{\mathbf{m}}$ of \mathbf{m} , so R is an irreducible representation of $G_{\mathbf{m}}$.² The data specifying such a pair (\mathbf{m}, R) is actually the same as for a pair (\mathbf{m}, \mathbf{e}) of coweight \mathbf{m} and weight \mathbf{e} modulo the Weyl group action:

$$(\mathbf{m}, \mathbf{e}) \in (\Lambda_{\text{cw}}(\mathfrak{g}) \times \Lambda_{\text{w}}(\mathfrak{g}))/W(G). \tag{4}$$

As emphasized in Ref. [31], this data has more information than a pair of irreducible representations of \mathfrak{g} and ${}^L\mathfrak{g}$.

In what follows, we will label line operators by (\mathbf{m}, \mathbf{e}) . In this notation, $(0, \mathbf{e})$ and $(\mathbf{m}, 0)$ are purely a Wilson line and a ’t Hooft line, respectively.

2.1. Wilson–’t Hooft lines in $\mathcal{N} = 2$ gauge theories on $S^1 \times_{\epsilon} \mathbb{R}^2 \times \mathbb{R}$

These Wilson–’t Hooft line operators were generalized to the ones preserving a half of supersymmetries in $\mathcal{N} = 2$ gauge theories in Ref. [31]. We here consider these lines on the space-time geometry $S^1 \times_{\epsilon} \mathbb{R}^2 \times \mathbb{R}$. Here, $S^1 \times_{\epsilon} \mathbb{R}^2$ is the twisted product defined by the identification $(2\pi\beta, z) \sim (0, e^{2\pi i\epsilon} z)$, where z is the complex coordinate of $\mathbb{R}^2 \simeq \mathbb{C}$ and β is the radius of S^1 . We always choose that the Wilson–’t Hooft line winds around S^1 and is located at the origin of \mathbb{R}^3 .

To preserve a half of the supersymmetries, the gauge field in Eq. (1) must be paired up with the scalar field ϕ in the vector multiplet

$$L_{(0,\mathbf{e})} := W_{R_{\mathbf{e}}} := \text{Tr}_{R_{\mathbf{e}}} P \exp\left(\int (iA + \text{Re } \phi d\tau)\right), \tag{5}$$

¹ Further, \mathbf{m} belongs to the cocharacter lattice $\{v \in \mathfrak{t} \mid \exp(2\pi i v) = \text{id}_G\}$, which is a sublattice of $\Lambda_{\text{cw}}(\mathfrak{g})$. If we take G to be the adjoint group, the cocharacter lattice coincides with $\Lambda_{\text{cw}}(\mathfrak{g})$.

² More precisely, R is an irreducible representation of the stabilizer of \mathbf{m} in the universal cover \tilde{G} of G [31].

where R_e is a representation specified by weight \mathbf{e} and $d\tau$ is the line element along L . To define the 't Hooft loop $L_{(\mathbf{m},0)}$, we carry out a path-integral with the singular configuration of A (2) and the scalar field ϕ as well.

By regarding S^1 as a time direction, the line operator defines a Hilbert space \mathcal{H}_L . Therefore the vev of a line operator is basically given by a trace over \mathcal{H}_L . The twisted space $S^1 \times_{\epsilon} \mathbb{R}^2$ leads to the insertion of $e^{2\pi i \epsilon J}$, where J is the generator of the rotation of \mathbb{R}^2 . To preserve a supersymmetry, this should be paired up with $e^{2\pi i \epsilon I_3}$ where I_3 is the Cartan of $SU(2)_R$. Then we have

$$\langle L_{(\mathbf{m},\mathbf{e})} \rangle = \text{Tr}_{\mathcal{H}_L(\mathbf{m},\mathbf{e})} (-1)^F e^{-2\pi\beta H} e^{2\pi i \epsilon (J+I_3)} e^{-2\pi i m_f F_f}, \tag{6}$$

where m_f are the mass parameters of the matter hypermultiplets associated with the Cartans of the flavor symmetry F_f .

These vevs depend holomorphically on the parameters [23]

$$a \in \mathfrak{t}_{\mathbb{C}}, \quad b \in \mathfrak{t}_{\mathbb{C}}^*. \tag{7}$$

These have a semi-classical expansion in terms of the electric and magnetic holonomies, $e^{i\theta_e}$ and $e^{i\theta_m}$, and the scalar field as [23,28]

$$a = \frac{\theta_e}{2\pi} + i\beta \text{Re } \phi + \dots, \quad b = \frac{\theta_m}{2\pi} - \frac{4\pi i\beta}{g^2} \text{Im } \phi + i\frac{\vartheta}{2\pi} \beta \text{Re } \phi + \dots, \tag{8}$$

where ϑ is the theta angle of the gauge theory. The vev of a Wilson line $L_{(0,\mathbf{e})}$ is simply given by the classical value of the holonomy:

$$\langle L_{(0,\mathbf{e})} \rangle = \text{Tr}_{R_e} e^{2\pi i a}. \tag{9}$$

For an 't Hooft line $L_{(\mathbf{m},0)}$ with magnetic charge \mathbf{m} , the vev takes the form

$$\langle L_{(\mathbf{m},0)} \rangle = \sum_{\substack{v \in \Lambda_{\text{cr}}(\mathfrak{g}) + \mathbf{m} \\ \|v\| \leq \|\mathbf{m}\|}} e^{2\pi i \langle v, b \rangle} Z_{1\text{-loop}}(a, m, \epsilon; v) Z_{\text{mono}}(a, m, \epsilon; \mathbf{m}, v), \tag{10}$$

where m collectively denotes complex mass parameters. The summation over the coweights v in the shifted coroot lattice $\Lambda_{\text{cr}}(\mathfrak{g}) + \mathbf{m}$ accounts for the so-called monopole bubbling, a phenomenon in which smooth monopoles are absorbed by the 't Hooft line and screen the magnetic charge. The norm $\|v\|$ with respect to a Killing form is bounded by $\|\mathbf{m}\|$, so this is a finite sum. The first two factors in the summand are the classical action and the one-loop determinant in the screened monopole background, respectively. The last factor is the nonperturbative contributions coming from degrees of freedom trapped on the 't Hooft line due to monopole bubbling.

The one-loop contribution $Z_{1\text{-loop}}$ has been computed in Ref. [23]. The one-loop contribution of the vector multiplet with gauge group G and the hypermultiplet transforming in the representation R with complex mass m are respectively given by

$$Z_{1\text{-loop}}^{\text{vm}}(a, \epsilon; v) = \prod_{\alpha \in \Phi(\mathfrak{g})} \prod_{k=0}^{|\langle v, \alpha \rangle| - 1} \sin^{-1/2} \left(\pi \langle a, \alpha \rangle + \pi \left(\frac{1}{2} |\langle v, \alpha \rangle| - k \right) \epsilon \right), \tag{11}$$

$$Z_{1\text{-loop}}^{\text{hm},R}(a, m, \epsilon; v) = \prod_{w \in P(R)} \prod_{k=0}^{|\langle v, w \rangle| - 1} \sin^{1/2} \left(\pi \langle a, w \rangle - \pi m + \pi \left(\frac{1}{2} |\langle v, w \rangle| - \frac{1}{2} - k \right) \epsilon \right), \tag{12}$$

where $\Phi(\mathfrak{g})$ is the set of roots of \mathfrak{g} and $P(R)$ is the set of weights of R . In the following, we will discuss gauge theories with multiple gauge groups and hypermultiplets transforming in the bi-fundamental representation $(\square_1, \bar{\square}_2) \in (G_1, G_2)$. The one-loop contribution of the bi-fundamental hypermultiplet is

$$Z_{1\text{-loop}}^{\text{hm},(\square_1, \bar{\square}_2)}(a, m, \epsilon; v^1, v^2) = \prod_{(w^1, w^2) \in (P(\square_1), P(\bar{\square}_2))} \prod_{k=0}^{|\langle v, w \rangle| - 1} \sin^{1/2} \left(\pi \langle a, w \rangle - \pi m + \pi \left(\frac{1}{2} |\langle v, w \rangle| - \frac{1}{2} - k \right) \epsilon \right), \tag{13}$$

where $\langle v, w \rangle = \langle v^1, w^1 \rangle + \langle v^2, w^2 \rangle$ and $\langle a, w \rangle = \langle a^1, w^1 \rangle + \langle a^2, w^2 \rangle$. Here a^1 and a^2 are parameters in Eq. (8) for two gauge groups, G_1 and G_2 .

The factor Z_{mono} is subtle. The original computation in Ref. [23] did not give an answer that completely matches predictions from the Alday-Gaiotto-Tachikawa (AGT) correspondence. The subtleties have been addressed in subsequent works [27,32–34] but not resolved in full generality. For the Wilson–’t Hooft lines that we will discuss later, the screened magnetic charges are in the same $W(G)$ -orbit as \mathbf{m} . The corresponding contributions are therefore obtained by the $W(G)$ -action from the perturbative term, for which $v = \mathbf{m}$ and $Z_{\text{mono}} = 1$.

2.2. $\mathcal{N} = 2$ circular quiver theory

We now consider the $\mathcal{N} = 2$ gauge theory described by the quiver diagram



Each node represents a vector multiplet for an $SU(N)$ gauge group, and each edge a hypermultiplet that transforms in the bi-fundamental representation under the gauge groups of the nodes it connects.³ Let the total number of nodes be n , and the mass parameters of the bi-fundamental hypermultiplets be $m^r, r = 1, \dots, n$. We will refer to the gauge theory described by the above quiver diagram as circular quiver theory.

Let us set up the notation of $\mathfrak{g} = \mathfrak{su}_N$ here. We denote by E_{ij} a matrix that has 1 in the (i, j) th entry and 0 elsewhere, and by E_{ij}^* such that $\langle E_{ij}, E_{kl}^* \rangle = \delta_{ik} \delta_{jl}$. The roots of \mathfrak{su}_N are $\alpha_{ij} = E_{ii}^* - E_{jj}^*, i \neq j$. The positive roots are $\alpha_{ij}, i < j$, and the simple roots are $\alpha_i := \alpha_{i, i+1}, i = 1, \dots, N - 1$. The simple coroots are denoted by $\alpha_i^\vee = E_{ii} - E_{i+1, i+1}, i = 1, \dots, N - 1$.

The fundamental weights are dual of the coroots: $\omega_i = (\alpha_i^\vee)^* = \sum_{j=1}^i h_j$, while the fundamental coweights are $\omega_i^\vee = (\alpha_i)^* = \sum_{j=1}^i h_j^\vee$, where

$$h_i = E_{ii}^* - \frac{1}{N} \sum_{j=1}^N E_{jj}^*, \quad h_i^\vee = E_{ii} - \frac{1}{N} \sum_{j=1}^N E_{jj}. \tag{15}$$

³ More precisely, the gauge group is a product of $PSU(N)$.

The various lattices are given by

$$\Lambda_r = \bigoplus_{i=1}^{N-1} \mathbb{Z}\alpha_i, \quad \Lambda_{cr} = \bigoplus_{i=1}^{N-1} \mathbb{Z}\alpha_i^\vee, \quad \Lambda_w = \bigoplus_{i=1}^{N-1} \mathbb{Z}\omega_i, \quad \Lambda_{cw} = \bigoplus_{i=1}^{N-1} \mathbb{Z}\omega_i^\vee. \quad (16)$$

For the circular quiver theory with $G = \text{SU}(N)^n$, the lattices are simply the direct product, e.g., the coweight lattice is $\Lambda_{cw}(\mathfrak{g}) = \Lambda_{cw}(\mathfrak{su}_N)^{\oplus n}$.

We now consider the 't Hooft line with the magnetic charge

$$\mathbf{m} = \omega_1^\vee \oplus \cdots \oplus \omega_1^\vee = h_1^\vee \oplus \cdots \oplus h_1^\vee, \quad (17)$$

the highest weight of the fundamental representation of each \mathfrak{su}_N . The vev of the 't Hooft line (10) is obtained by summing over all coweights of the form $\nu = h_{i^1}^\vee \oplus \cdots \oplus h_{i^n}^\vee$. The classical and the one-loop contributions, for which $i^1 = \cdots = i^n = 1$, is given by

$$\prod_{r=1}^n e^{2\pi i b'_1} \prod_{j=2}^N \sin^{-1/2} \left(\pi a_{1j}^r + \frac{1}{2} \pi \epsilon \right) \sin^{-1/2} \left(\pi a_{j1}^r + \frac{1}{2} \pi \epsilon \right) \times \sin^{1/2} (\pi (a_j^r - a_1^{r+1}) - \pi m^r) \sin^{1/2} (\pi (a_1^r - a_j^{r+1}) - \pi m^r). \quad (18)$$

The superscript r refers to the r th $\text{SU}(N)$ factor of G , with $a^{n+1} = a^1$. By collecting the contributions from the other coweights, we get

$$\langle L_{(\mathbf{m},0)} \rangle = \sum_{i^1, \dots, i^n} \prod_{r=1}^n e^{2\pi i b'_{i^r}} \ell_{m^r}(a^r, a^{r+1})_{i^r}^{r+1}, \quad (19)$$

where

$$\ell_m(a^1, a^2)_{i^1}^j = \left(\frac{\prod_{k(\neq i)} \sin \pi (a_k^1 - a_j^2 - m) \prod_{l(\neq j)} \sin \pi (a_i^1 - a_l^2 - m)}{\prod_{k(\neq i)} \sin \pi (a_{ki}^1 - \frac{1}{2} \epsilon) \sin \pi (a_{ik}^1 - \frac{1}{2} \epsilon)} \right)^{1/2}. \quad (20)$$

We add the following electric charge

$$\mathbf{e} = \sum_{r=1}^n \sigma^r \frac{1}{2} (h_1^{r+1} - h_1^r) = \sum_{r=1}^n (\sigma^r 1 - \sigma^{r+1} 1) \frac{1}{2} h_1^{r+1}. \quad (21)$$

This electric charge is in a sense a minimal one that is compatible with the choice of the magnetic charge (17) for the Dirac–Schwinger–Zwanziger quantization condition: the charges (\mathbf{m}, \mathbf{e}) and $(\mathbf{m}', \mathbf{e}')$ of two dyons must satisfy $\langle \mathbf{m}, \mathbf{e}' \rangle - \langle \mathbf{m}', \mathbf{e} \rangle \in \mathbb{Z}$. The magnetic charge (17) breaks the gauge group to $\text{S}(\text{U}(1) \times \text{U}(N - 1))^n$, and we are turning on a Wilson line that is charged under the $\text{U}(1)$ factors with charges proportional to $(\sigma^r 1 - \sigma^{r+1} 1)/2$. The Wilson line multiplies the one-loop term (18) by the phase factor

$$\prod_{r=1}^n e^{\sigma^r \pi i (a_1^{r+1} - h_1^r)} = \prod_{r=1}^n e^{\sigma^r \pi i (a_1^{r+1} - a_1^r)}. \quad (22)$$

Hence, the term with $v = h_{i^1}^\vee \oplus \cdots \oplus h_{i^n}^\vee$ gets the phase factor $e^{\sigma^r \pi i (a_{i^r+1}^{r+1} - a_{i^r}^r)}$, and the vev of this Wilson–t Hooft line is

$$\langle L_{(\mathbf{m}, \mathbf{e})} \rangle = \sum_{i^1, \dots, i^n} \prod_{r=1}^n e^{2\pi i b_{i^r}^r} e^{\sigma^r \pi i (a_{i^r+1}^{r+1} - a_{i^r}^r)} \ell_{m^r}(a^r, a^{r+1})_{i^r}^{r+1}. \tag{23}$$

This will later be compared with the transfer matrix of the L-operators.

3. L-operators in an integrable model

3.1. Definition of L-operators

Let \mathfrak{h} be a finite-dimensional commutative complex Lie algebra and V a finite-dimensional diagonalizable \mathfrak{h} -module. Choosing a basis $\{v_i\}$ of V that is homogeneous with respect to weight decomposition, we denote the weight of v_i by h_i and the (i, j) th entry of a matrix $M \in \text{End}(V)$ by M_j^i . We write $\mathcal{M}_{\mathfrak{h}^*}$ for the field of meromorphic functions on the dual space \mathfrak{h}^* of \mathfrak{h} .

Let $R: \mathbb{C} \times \mathfrak{h}^* \rightarrow \text{End}(V \otimes V)$ be an $\text{End}(V \otimes V)$ -valued meromorphic function on $\mathbb{C} \times \mathfrak{h}^*$ that is invertible at a generic point $(z, a) \in \mathbb{C} \times \mathfrak{h}^*$. The coordinate z is called the spectral parameter and a is called the dynamical parameter. Associated with R , let $L: \mathbb{C} \rightarrow \text{End}(V \otimes \mathcal{M}_{\mathfrak{h}^*} \otimes \mathcal{M}_{\mathfrak{h}^*})$, which we think of as a matrix whose entries are linear operators on meromorphic functions on $\mathfrak{h}^* \times \mathfrak{h}^*$. We refer to this L as an L-operator for R . This definition of L-operators depends on two independent dynamical parameters, thus is more general than the one given in Ref. [19]. This generalization appeared in Ref. [16] in the formalism of a dynamical R-matrix.

The L-operator satisfies the following two conditions:

- Its matrix elements act on $f \in \mathcal{M}_{\mathfrak{h}^*} \otimes \mathcal{M}_{\mathfrak{h}^*}$ as

$$L(z)_i^j f(a^1, a^2) = L(z; a^1, a^2)_i^j \Delta_i^1 \Delta_j^2 f(a^1, a^2), \tag{24}$$

where $L(z; a^1, a^2)_i^j$ is a meromorphic function on $\mathbb{C} \times \mathfrak{h}^* \times \mathfrak{h}^*$ and Δ_i^1, Δ_j^2 are difference operators, $\Delta_i^1 f(a^1, a^2) = f(a^1 - \epsilon h_i, a^2)$ and $\Delta_j^2 f(a^1, a^2) = f(a^1, a^2 - \epsilon h_j)$. Here ϵ is a fixed complex parameter.

- The L-operator satisfies the RLL relation

$$\begin{aligned} \sum_{k,l} R(z - z', a^2)_{kl}^{mn} L(z; a^1, a^2)_i^k L(z'; a^1 - \epsilon h_i, a^2 - \epsilon h_k)_j^l \\ = \sum_{k,l} L(z'; a^1, a^2)_l^m L(z; a^1 - \epsilon h_l, a^2 - \epsilon h_n)_k^m R(z - z', a^1)_{ij}^{kl}. \end{aligned} \tag{25}$$

Equivalently, the operator relation

$$\sum_{k,l} R(z - z', a^2)_{kl}^{mn} L(z)_i^k L(z')_j^l = \sum_{k,l} R(z - z', a^1)_{ij}^{kl} L(z')_l^m L(z)_k^m \tag{26}$$

holds on any meromorphic function $f(a^1, a^2)$.

We see that for R to satisfy the RLL relation with some L-operators, generally it must commute with $h \otimes 1 + 1 \otimes h$ for all $h \in \mathfrak{h}$; in other words, $R(z, a)_{ij}^{kl} = 0$ unless $h_i + h_j = h_k + h_l$.

3.2. L-operator and quantum integrable system

Associated with an L-operator, there is an integrable quantum mechanical system consisting of particles moving in the space \mathfrak{h}^* .⁴ The Hilbert space of each particle is $\mathcal{M}_{\mathfrak{h}^*}$, and that of the system is $\mathcal{M}_{\mathfrak{h}^*}^{\otimes n}$ if n is the number of particles.

To construct this system, we define the monodromy matrix $M: \mathbb{C} \rightarrow \text{End}(V \otimes \mathcal{M}_{\mathfrak{h}^*}^{\otimes n+1})$ by the product of n copies of the L-operator. The matrix elements are given by

$$M(z)_{i^1}^{i^{n+1}} = \sum_{i^2, \dots, i^n} \prod_{r=1}^n L(z; a^r, a^{r+1})_{i^r}^{i^{r+1}} \prod_{s=1}^{n+1} \Delta_{i^s}^s, \tag{27}$$

acting on any meromorphic function $f(a^1, \dots, a^{n+1})$. By identifying $a^{n+1} = a^1$ and taking the trace, one obtains the transfer matrix $T: \mathbb{C} \rightarrow \text{End}(\mathcal{M}_{\mathfrak{h}^*}^{\otimes n})$:

$$T(z) = \sum_{i^1, \dots, i^n} \prod_{r=1}^n L(z; a^r, a^{r+1})_{i^r}^{i^{r+1}} \prod_{s=1}^n \Delta_{i^s}^s, \quad i^{n+1} = i^1. \tag{28}$$

Since T is an $\text{End}(\mathcal{M}_{\mathfrak{h}^*}^{\otimes n})$ -valued meromorphic function, each coefficient T_m in the Laurent expansion $T(z) = \sum_{m \in \mathbb{Z}} T_m z^m$ is an operator acting on the Hilbert space $\mathcal{M}_{\mathfrak{h}^*}^{\otimes n}$. Then, one may pick a particular linear combination of these coefficients and declare that it is the Hamiltonian of the quantum mechanical system. The Hamiltonian thus obtained is a difference operator, which is typical of relativistic systems.

The integrability of the system is a consequence of the RLL relation. By repeated use of the RLL relation, one deduces that the monodromy matrix satisfies a similar relation:

$$\sum_{k,l} R(z - z', a^{n+1})_{kl}^{mn} M(z)_i^k M(z')_j^l = \sum_{k,l} R(z - z', a^1)_{ij}^{kl} M(z')_i^n M(z)_k^m. \tag{29}$$

By multiplying both sides by $R^{-1}(z - z', a^1)_{mn}^{ij}$, setting $a^{n+1} = a^1$ and summing over i, j, m, n , one finds

$$T(z)T(z') = T(z')T(z). \tag{30}$$

In other words, transfer matrices at different values of the spectral parameter commute. It follows that the Laurent coefficients $\{T_m\}$ mutually commute and, in particular, commute with the Hamiltonian. Therefore the system has a series of commuting conserved charges.

3.3. Elliptic L-operator

An important example of an L-operator is one for the elliptic dynamical R-matrix [35–37], which is a representation of the elliptic quantum group for \mathfrak{sl}_N . In this example, \mathfrak{h} is the Cartan subalgebra of \mathfrak{sl}_N and $V = \mathbb{C}^N$ is the vector representation of \mathfrak{sl}_N .

The Lie algebra \mathfrak{sl}_N consists of the traceless complex $N \times N$ matrices. Let us define $E_{ij} \in \mathfrak{gl}_N$ as the same matrix defined in Sect. 2, and their dual E_{ij}^* the element of $\mathfrak{gl}_N^* = \text{Hom}(\mathfrak{gl}_N, \mathbb{C})$ such that $\langle E_{ij}, E_{kl}^* \rangle = \delta_{ik} \delta_{jl}$, where the bilinear map $\langle -, - \rangle: \mathfrak{gl}_N \times \mathfrak{gl}_N^* \rightarrow \mathbb{C}$ is the natural pairing. The elements of \mathfrak{h} are matrices of the form $\sum_{i=1}^N b_i E_{ii}$, with $\sum_{i=1}^N b_i = 0$. Since \mathfrak{h} is isomorphic to the

⁴ This is quantum mechanics in which real variables are analytically continued to complex ones.

quotient of the subspace of \mathfrak{f}_N consisting of the diagonal matrices by the subspace spanned by the identity matrix $I = \sum_{i=1}^N E_{ii}$, the dual space \mathfrak{h}^* is isomorphic to the subspace of \mathfrak{f}_N^* consisting of elements of the form $\sum_{i=1}^N a_i E_{ii}^*$ such that $\langle I, \sum_{i=1}^N a_i E_{ii}^* \rangle = \sum_{i=1}^N a_i = 0$. Thus, \mathfrak{h}^* may also be identified with the space of traceless diagonal matrices.

The natural action of \mathfrak{sl}_N on \mathbb{C}^N defines the vector representation of \mathfrak{sl}_N . In terms of the standard basis $\{e_1, \dots, e_N\}$ of \mathbb{C}^N , we have $\sum_{j=1}^N a_j E_{jj} e_i = a_i e_i$. The weight of e_i is $h_i = E_{ii}^* - \frac{1}{N} \sum_{j=1}^N E_{jj}^*$. For $a \in \mathfrak{h}^*$, we write $a_i = \langle E_{ii}, a \rangle$. Then, $\sum_{i=1}^N a_i = 0$ and $a = \sum_{i=1}^N a_i E_{ii}^* = \sum_{i=1}^N a_i h_i$.

Fix a point τ in the upper half plane, $\text{Im } \tau > 0$, and let

$$\theta_1(z) = - \sum_{j \in \mathbb{Z} + 1/2} e^{\pi i j^2 \tau + 2\pi i j(z + 1/2)} \tag{31}$$

be Jacobi’s first theta function. The elliptic dynamical R-matrix R^{ell} is defined by [17–19]

$$R^{\text{ell}}(z, a) = \sum_{i=1}^N E_{ii} \otimes E_{ii} + \sum_{i \neq j} \alpha(z, a_{ij}) E_{ii} \otimes E_{jj} + \sum_{i \neq j} \beta(z, a_{ij}) E_{ji} \otimes E_{ij}, \tag{32}$$

where $a_{ij} = a_i - a_j$ and

$$\alpha(z, a) = \frac{\theta_1(a + \epsilon)\theta_1(-z)}{\theta_1(a)\theta_1(\epsilon - z)}, \quad \beta(z, a) = \frac{\theta_1(a - z)\theta_1(\epsilon)}{\theta_1(a)\theta_1(\epsilon - z)}. \tag{33}$$

The elliptic L-operator L^{ell} , which satisfies the RLL relation with R^{ell} , has the matrix elements given by [16]

$$L_{w,y}^{\text{ell}}(z; a^1, a^2)_i^j = \frac{\theta_1(z - w + a_j^2 - a_i^1)}{\theta_1(z - w)} \prod_{k(\neq i)} \frac{\theta_1(a_k^1 - a_j^2 - y)}{\theta_1(a_{ki}^1)}. \tag{34}$$

The complex numbers w, y may be thought of as spectral parameters. The presence of the two parameters is due to the fact that $R^{\text{ell}}(z, a)$ is invariant under shift of a by a multiple of the identity matrix I , and in the RLL relation (25) the spectral parameters z, z' enter the R-matrix only through the difference $z - z'$; note also that the L-operator can be multiplied by any function of the spectral parameter.

The elliptic dynamical R-matrix and the elliptic L-operator have many more properties than just that they satisfy the RLL relation. Most importantly, the R-matrix is a solution of the dynamical Yang–Baxter equation [17,18,38] and encodes the Boltzmann weights for a two-dimensional integrable lattice model [35–37]. This model is equivalent to the eight-vertex model [39,40] and the Belavin model [41], an \mathfrak{sl}_N generalization of the eight-vertex model, in the sense that the transfer matrices of the two models are related by a similarity transformation. The elliptic L-operator, on the other hand, satisfies the RLL relation with another R-matrix which describes an integrable lattice model called the Bazhanov–Sergeev model [11,12], whose spin variables take values in \mathfrak{h}^* .

3.4. Trigonometric L-operators

The L-operators that appear in the correspondence with Wilson–’t Hooft lines are obtained from the elliptic L-operator L^{ell} via the trigonometric limit $\tau \rightarrow i\infty$. For comparison with gauge theory results, we actually need to express these L-operators in somewhat different forms.

We firstly describe L-operators in a quantum mechanical language. Consider quantum mechanics of a particle living in $\mathfrak{h}^* \times \mathfrak{h}^*$, with Planck constant

$$\hbar = -\frac{\epsilon}{2\pi}. \tag{35}$$

If $(a^1, a^2) \in \mathfrak{h}^* \times \mathfrak{h}^*$ is the position of the particle, we write $a^r = \sum_{i=1}^{N-1} q_i^r \omega_i$, $r = 1, 2$. Similarly, we write the momenta $(b^1, b^2) \in \mathfrak{h} \times \mathfrak{h}$ of the particle as $b^r = \sum_{i=1}^{N-1} p_i^r \alpha_i^\vee$. The corresponding position and momentum operators \hat{q}_i^r, \hat{p}_i^s satisfy the canonical commutation relations:

$$[\hat{q}_i^r, \hat{p}_j^s] = i\hbar \delta^{rs} \delta_{ij}, \quad i, j = 1, \dots, N-1. \tag{36}$$

(As before, we are treating q_i^r, p_i^r as analytically continued variables.)

To rewrite the commutation relations in a form that is invariant under the action of the Weyl group, we make a change of basis

$$a^r = \sum_{i=1}^N a_i^r E_{ii}^*, \quad b^r = \sum_{i=1}^N b_i^r E_{ii}. \tag{37}$$

Then, the corresponding observables \hat{a}_i^r, \hat{b}_i^r obey the traceless condition, $\sum_{i=1}^N \hat{a}_i^r = \sum_{i=1}^N \hat{b}_i^r = 0$, and satisfy the commutation relations

$$[\hat{a}_i^r, \hat{b}_j^s] = i\hbar \delta^{rs} \left(\delta_{ij} - \frac{1}{N} \right), \quad i, j = 1, \dots, N. \tag{38}$$

By using these observables we can identify the matrix elements of an L-operator L with an operator in the Hilbert space of this quantum mechanical system:

$$L(z)_i^j = L(z; \hat{a}^1, \hat{a}^2)_i^j e^{2\pi i(\hat{b}_i^1 + \hat{b}_j^2)} = e^{\pi i(\hat{b}_i^1 + \hat{b}_j^2)} \tilde{L}(z; \hat{a}^1, \hat{a}^2)_i^j e^{\pi i(\hat{b}_i^1 + \hat{b}_j^2)}. \tag{39}$$

In quantum mechanics, there is an invertible map from functions on the classical phase space to operators in the Hilbert space, known as the Weyl transform: if q and p are canonically conjugate variables, it maps

$$f(q, p) \mapsto \hat{f}(\hat{q}, \hat{p}) = \int_{\mathbb{R}^4} dx dy dp dq f(q, p) e^{i(x(\hat{q}-q)+y(\hat{p}-p))}. \tag{40}$$

The inverse map is the *Wigner transform*, which we denote by $\langle - \rangle$:

$$f(\hat{q}, \hat{p}) \mapsto \langle f(\hat{q}, \hat{p}) \rangle = \int_{\mathbb{R}} dx e^{ipx/\hbar} \left\langle q + \frac{1}{2}x \left| \hat{f}(\hat{q}, \hat{p}) \right| q - \frac{1}{2}x \right\rangle. \tag{41}$$

In the situation at hand (39), we have

$$\langle L(z)_i^j \rangle = e^{2\pi i(b_i^1 + b_j^2)} \tilde{L}(z; a^1, a^2)_i^j. \tag{42}$$

Next, we apply a similarity transformation to the elliptic L-operator. Assume $\text{Im } \epsilon > 0$ and let

$$\Gamma(z, \tau, \epsilon) = \prod_{m,n=0}^{\infty} \frac{1 - e^{2\pi i((m+1)\tau + (n+1)\epsilon - z)}}{1 - e^{2\pi i(m\tau + n\epsilon + z)}} \tag{43}$$

be the elliptic gamma function. Then, $\bar{\Gamma}(z) = e^{\pi iz^2/2\epsilon} \Gamma(z, \tau, \epsilon)$ has the property that $\bar{\Gamma}(z + \epsilon, \tau, \epsilon) = g(\tau, \epsilon) \theta_1(z) \bar{\Gamma}(z, \tau, \epsilon)$ for some function $g(\tau, \epsilon)$. We define the conjugated L-operator $\mathcal{L}_{w,m}^{\text{ell}}(z)$ by

$$\mathcal{L}_{w,m}^{\text{ell}}(z)_i^j = \Phi_{m-1/2\epsilon} L_{w,m-1/2\epsilon}^{\text{ell}}(z)_i^j \Phi_{m-1/2\epsilon}^{-1}, \tag{44}$$

where

$$\Phi_y = \prod_{k,l=1}^N \bar{\Gamma}(\hat{a}_k^1 - \hat{a}_l^2 - y)^{1/2} \prod_{k \neq l} \bar{\Gamma}(\hat{a}_{kl}^1)^{-1/2}. \tag{45}$$

It has the Wigner transform

$$\langle \mathcal{L}_{w,m}^{\text{ell}}(z)_i^j \rangle = e^{2\pi i(b_i^1 + b_j^2)} \frac{\theta_1(z - w + a_j^2 - a_i^1)}{\theta_1(z - w)} \times \left(\frac{\prod_{k(\neq i)} \theta_1(a_k^1 - a_j^2 - m) \prod_{l(\neq j)} \theta_1(a_i^1 - a_l^2 - m)}{\prod_{k(\neq i)} \theta_1(a_{ki}^1 - \frac{1}{2}\epsilon) \theta_1(a_{ik}^1 - \frac{1}{2}\epsilon)} \right)^{1/2}. \tag{46}$$

With these preparations, let us finally take the trigonometric limit to define the trigonometric L-operator:

$$\mathcal{L}_{w,m} = \lim_{\tau \rightarrow i\infty} \mathcal{L}_{w,m}^{\text{ell}}. \tag{47}$$

The trigonometric L-operator satisfies the RLL relation with the trigonometric limit R^{trig} of the elliptic R-matrix R^{ell} . Concretely, $\mathcal{L}_{w,m}$ and R^{trig} are obtained from $\mathcal{L}_{w,m}^{\text{ell}}$ and R^{ell} by the replacement $\theta_1(z) \rightarrow \sin(\pi z)$.

Once we are in the trigonometric setup, the quasi-periodicity in $z \rightarrow z + \tau$ is lost and we can further take the limits $w \rightarrow \pm i\infty$. This allows us to introduce more fundamental L-operators:

$$\mathcal{L}_{\pm,m} = \lim_{w \rightarrow \pm i\infty} \mathcal{L}_{w,m}. \tag{48}$$

These L-operators do not depend on the spectral parameters z, w , and their matrix elements have the Wigner transforms

$$\langle (\mathcal{L}_{\pm,m})_i^j \rangle = e^{2\pi i(b_i^1 + b_j^2)} e^{\pm \pi i(a_j^2 - a_i^1)} \ell_m(a^1, a^2)_i^j, \tag{49}$$

where $\ell_m(a^1, a^2)_i^j$ is exactly the same as (20). The L-operator for arbitrary parameters z, w can be realized as a linear combination of $\mathcal{L}_{\pm,m}$:

$$\mathcal{L}_{w,m}(z) = \frac{e^{\pi i(z-w)} \mathcal{L}_{+,m} - e^{-\pi i(z-w)} \mathcal{L}_{-,m}}{\sin \pi(z-w)}. \tag{50}$$

The monodromy matrix $\mathcal{M}_{\sigma,m}$ constructed from $\mathcal{L}_{\pm,m}$ is labeled by an n -tuple of signs $\sigma = (\sigma^1, \dots, \sigma^n) \in \{\pm\}^n$ and an n -tuple of complex numbers $m = (m^1, \dots, m^n)$:

$$\langle (\mathcal{M}_{\sigma,m})_i^{n+1} \rangle = \sum_{i^2, \dots, i^n} \prod_{s=1}^{n+1} e^{2\pi i b_{i^s}^s} \prod_{r=1}^n e^{\sigma^r \pi i (a_{i^{r+1}}^{r+1} - a_{i^r}^r)} \ell_{m^r}(a^r, a^{r+1})_{i^r}^{i^{r+1}}. \tag{51}$$

The corresponding transfer matrix $\mathcal{T}_{\sigma,m}$ has the Wigner transform

$$\langle \mathcal{T}_{\sigma,m} \rangle = \sum_{i^1, \dots, i^n} \prod_{r=1}^n e^{2\pi i b_{i^r}^r} e^{\sigma^r \pi i (a_{i^{r+1}}^{r+1} - a_{i^r}^r)} \ell_{m^r}(a^r, a^{r+1})_{i^r}^{i^{r+1}}, \tag{52}$$

with $a^{n+1} = a^1, i^{n+1} = i^1$. These quantities equal the vevs of Wilson–’t Hooft lines in $\mathcal{N} = 2$ supersymmetric gauge theories (23).

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