BUDAPEST UNIVERSITY OF TECHNOLOGY AND ECONOMICS

# Fermionic Entanglement Theory and the Black Hole/Qubit Correspondence

DOCTORAL THESIS

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# *Abstract*

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by Gábor SÁROSI

In the first part of this thesis we study the classification and quantification of quantum entanglement between fermionic particles. We reformulate the classification problem under stochastic local operations and classical communications (SLOCC) in the language of Clifford algebras and show that when extended with Bogoliubov transformations the SLOCC group turns into the so called even Clifford group. This allows us to reinterpret several results in the mathematical literature on the classification of spinors as the solutions to the entanglement classification problem for certain fermionic systems. We discuss how to describe distinguishable systems via embedding them into fermionic ones with particular emphasis put on three and four qubits and three qutrits.

In the second part of this thesis we study classical black hole solutions of ungauged $\mathcal{N} = 2$ supergravity in four dimensions. These black holes are connected to simple entangled qubit and fermion systems via the black hole/qubit correspondence. Extending results on extremal black holes, we show that nonextremal STU black holes correspond to a particular semisimple SLOCC orbit of four qubits. The dictionary between fermionic systems and black holes allows us to write down a new formula for the entropy of general nonextremal STU black holes in terms of U-duality invariants of the asymptotic charges.

# *Kivonat*

Az értekezés első felében a fermionikus részecskék közötti kvantum összefonódás klasszifikálásának és kvantifikálásának lehetőségeit tanulmányozzuk. A sztochasztikus lokális operációkon és klasszikus kommunikáción (SLOCC) alapuló összefonódottsági klasszifikációs problémát újrafogalmazzuk a Clifford algebrák nyelvén és megmutatjuk, hogy ha az invertálható SLOCC protokollokat kombináljuk Bogoliubov transzformációkkal, akkor az úgynevezett páros Clifford csoportot kapjuk. Ez lehetőséget teremt, hogy a matematikai irodalom spinorok klasszifikációjára vonatkozó eredményeit újraértelmezzük, mint bizonyos fermionikus rendszerek összefonódottsági klasszifikációs problémáinak megoldásait. Bemutatjuk hogyan lehet megkülönböztethető rendszerekben az összefonódottságot tárgyalni a fermionikus rendszerekbe való beágyazás segítségével, különös hangsúlyt fektetve a három és négy qubit, illetve a három qutrit rendszerekre.

A dolgozat második felében a négy dimenziós $\mathcal{N} = 2$ szupergravitáció klasszikus fekete lyuk megoldásait vizsgáljuk. Ezeket a fekete lyukakat a fekete lyuk/qubit megfelelés köti össze egyszerű összefonódott rendszerekkel. Az extremális fekete lyukakra vonatkozó megfeleléseket kiegészítjük azzal a megfigyeléssel, hogy az általános nem extremális fekete lyuk töltéseinek megfelelő négy qubit állapot mindig egy bizonyos félegyszerű négy qubit SLOCC osztályban található. A fekete lyuk töltések fermionos leírásának segítségével új formulát adunk az általános nem extremális fekete lyuk entrópiájára, mely az asszimptotikus töltések U-dualitás invariáns függvénye.

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# Contents

# Chapter 1

# Introduction and conclusion

Black holes are interesting and important objects. They have been discovered as classical solutions of Einstein's general relativity and they were quickly recognized to possess regions with a curvature singularity, where general relativity is not meaningful. This, however, does not prevent them from being physical. Today, it is widely accepted that black holes can form in nature from certain collapsing stars.

An important feature of classical black hole solutions that they satisfy thermodynamic laws such as the first law [6, 7]. As such, they have an entropy (proportional to the area of their horizon) and a temperature (proportional to their surface gravity). The fact that they have a temperature suggests that they radiate which is indeed possible via quantum tunneling [8]. Semiclassical considerations predict that, via this radiation, a black hole evaporates into a perfectly thermal gas of Hawking radiation, which implies a loss of unitarity at the semiclassical level [9]. This result is known as the black hole information paradox. It shows that a complete description of the physics of black holes requires a theory of quantum gravity.

The fact that black holes have an entropy suggests that there should be many microstates in this putative theory of quantum gravity corresponding to the same classical black hole. While we do not yet understand microstates of realistic astrophysical black holes, a huge triumph of string theory, as a particular theory of quantum gravity, that it can give account for the number of microstates of certain black holes. Since the seminal result of Strominger and Vafa [10] for supersymmetric black holes in 5d, there have been results along these lines over the last two decades, mostly for near extremal or asymptotically anti de-Sitter black holes. The Schwartzschild black hole remains elusive.

Another important area of research is quantum information theory, specifically the study of quantum entanglement. This "spooky action at a distance"[1] is the resource behind the idea of quantum computing allowing some fascinating applications, like quantum teleportation or quantum cryptography [24]. It is also the key property that distinguishes classical from quantum physics, standing behind many phenomena in strongly correlated quantum systems. It is not so surprising then that it appears to play a key role in the physics of black holes [11, 12], or more broadly speaking even in the emergence of semiclassical spacetime itself [13].

The research presented in this dissertation was initiated by one of the early attempts to connect the physics of black holes to quantum entanglement. This attempt is based on the observation of Duff, that some charged supersymmetric black hole solutions in a variant of four dimensional $\mathcal{N} = 2$

[1] A saying attributed to Einstein.

supergravity have a Bekenstein-Hawking entropy that is the square root of a well-known entanglement measure for three qubits, the three-tangle, provided that we associate a three qubit state to the charges of the black hole [14]. As these black holes can be embedded in string theory in terms of D-branes wrapping cycles of a compact six dimensional internal space, people had hopes that this 3-qubit state can be connected to the physics of D-branes. This turned out to be possible, but not quite in the way people might have hoped for. The three qubit Hilbert space was identified to be the space of harmonic 3-forms on the extra dimensions of the embedding into type IIB string theory [15]. Such a 3-form is Poincaré dual to the homology class that some D3 branes wrap and contains the electric and magnetic charges of the black hole as wrapping numbers. Then, a symmetry of string theory, called U-duality, acts in the same way on these charges as the group of invertible *stochastic local operations and classical communication* (SLOCC) on three qubit states. As we will review, SLOCC protocols are constructed in a way that they cannot increase entanglement between parties and hence a way of constructing measures for entanglement is to look for relative invariant[2] quantities under invertible SLOCC transformations.

Now the key observation is then that this particular group acting on this particular vector space forms a so called *prehomogeneous vector space*. Recall, that a homogeneous vector space is a pair $(G, V)$ (with an action of $G$ on $V$ being implicit) such that every element of $V$ is cyclic: we can just fix a $v_0 \in V$ and reach any other element by the action of $G$: $v = gv_0$. A prehomogeneous vector space is a slight generalization of this definition: it is a pair $(G, V)$ such that $G$ has a dense orbit in $V$. In particular, this guarantees that there is a *single* relative $G$-invariant $f : V \to \mathbb{C}$ which one can form from a given element of $V$. The dense orbit is then the open set satisfying $f(v) \neq 0$.

As both the entropy of a black hole and the 3-tangle of three qubits are relative invariants under U-duality and SLOCC transformations respectively, prehomogenity guarantees that they are a homogeneous function of each other[3]. This might be all the reason behind the black hole/qubit correspondence, as we now undertand it. Nevertheless, it is a fruitful recognition that such a central role is played by the prehomogenity of the pair $(G, V)$, as a complete classification of such prehomogeneous vector spaces is available due to the work of Sato and Kimura [16]. One might hope then to find "new" black hole/qubit correspondences by simply going through the Sato Kimura list. This is indeed the case, prehomogenity is a key aspect of the symmetries of supersymmetric stringy black holes. The reason for this is their connection to Hitchin's functionals – defining form theories of gravity, which we will review in section 4.2 of this thesis – for which prehomogenity turns into the crucial requirement of *stability*.

Examination of the Sato Kimura list allows one to find the SLOCC classification for a handful of systems where the entanglement classification

[2] Suppose that a group $G$ acts on some topological space $X$. We will call relative invariant a continuous function $f : X \to \mathbb{C}$ such that $f(g\phi) = \chi(g)^m f(\phi)$, where $\phi \in X$, $g \in G$, $m$ is some integer and $\chi$ is a one dimensional representation of $G$.

[3] Note that this reasoning only applies to extremal black holes, when the *attractor mechanism* is in action so that the entropy is a function of the charges only and not of the asymptotic values of the scalar fields of the model. Otherwise, because U-dualities act on the scalars as well, the entropy can depend on several other invariants. We will discuss this very explicitly in chapter 4 of this dissertation.

was unknown to the physics literature, along with the respective relative invariants being candidate entanglement measures. Such systems are the ones of three fermions with either seven or eight single particle states. Another interesting thing is that the list contains pairs of the form $(\mathbb{C}^{\times} \times Spin(2d), \wedge^{\text{even/odd}}\mathbb{C}^{d})$, with $d = 6, 7$. These cases already made their debut to physics via generalized Hitchin functionals, but a somewhat simpler idea is that they can be regarded as even or odd parity states in the Fock space of fermions. It turns out that this is indeed a good interpretation, and the physical meaning of the group $G = \mathbb{C}^{\times} \times Spin(2d)$ is that it is the group of SLOCC transformations combined with Bogoliubov transformations. This group turns out to be well know in the study of Clifford algebras. It is the so called even Clifford group, which we will discuss in great detail. This leads us to the mathematical problem known as the classification of spinors which we will show to be naturally connected to the description of entanglement between fermions[4].

Another main concern of this thesis is to understand more about the classification and the structure of the entropy formula of certain *nonextremal* black holes in four dimensions with a stringy origin. There is a good chance that the microscopics of these black holes can be understood in terms of interacting BPS objects [18], and since that this class of nonextremal black holes contains the Schwartzschild and Kerr solutions, the importance of this chance cannot be overstated. A small, but neccessary step in this direction is to write down and analyse the classical Bekenstein-Hawking entropy formula for the general nonextremal black hole, which is the main concern of chapter 4 of this dissertation.

We organize this thesis as follows. The remainder of chapter 1 contains a list of thesis points and a short introduction into the theory of entanglement of three qubits, which will be a recurring theme.

Chapter 2 is devoted to the extension of the fermionic SLOCC classification problem based on the theory of spinors [19]. We set up the notion of fermionic entanglement at the beginning of this chapter. We outline the connection of the classification problem of spinors with fermionic entanglement and review the neccessary mathematical background material in a second quantized fermionic language. The end of the chapter is devoted to a canonical real structure which we introduce for any fermionic Hilbert space and describe how it connects spinor invariants to reduced density matrix elements via so called Fierz identities.

In Chapter 3 we delve into the problem of finding the SLOCC and extended SLOCC classification of certain fermionic systems. This mostly consists of borrowing the classification from the mathematical (and sometimes physical) literature and presenting it in a unified spinor language, but we will describe several new results along the way such as monogamy equations for three fermions with six modes or new invariants for three fermions with nine modes.

Chapter 4 reviews the nonextremal black hole solutions of 4d ungauged $\mathcal{N} = 2$ supergravity coupled to three vector multiplets, from which the most general black holes of maximal $\mathcal{N} = 8$ supergravity can be obtained. We construct the duality invariant entropy formula for these black holes

[4]There were some previous attempts to connect the mathematics of Clifford algebras to entanglement theory, see [17].

as a main result of this chapter. We also discuss how this general black hole fits into the black hole/qubit correspondence via connecting it to a particular semisimple four qubit SLOCC orbit. We discuss several known limits. In the last section of this chapter, section 4.2 we discuss Hitchin functionals as the main bridge between BPS black holes and the fermionic systems described in chapter 2. This last section does not contain any new results, it is included with the hope that it makes the global picture more complete.

## 1.1 Summary of the authors original contribution

In this section, we summarize the results in this dissertation which are the authors own work, by collecting them into thesis points. We indicate the connecting publications at the end of each point and the connecting sections of this dissertation in a superscript.

1. We observe that the even Clifford group acting on spinors is a natural extension of the usual SLOCC group acting on fermionic states with a fixed number of particles[sec. 2.1,2.3]. The additional transformations in the even Clifford group can be interpreted as Bogoliubov transformations. We propose the classification of spinors under the action of the even Clifford group to be a natural generalization of the fermionic SLOCC classification problem. Therefore, we call the even Clifford group, the extended SLOCC group. Pure spinors, which form the simplest orbit under the Clifford group, are shown to generalize the notion of separable states[sec. 2.3.3]. We observe that spinor invariants give possible entanglement measures and we show how various known entanglement measures, like the pure state concurrence for two qubits or the three tangle for three qubits originate from known spinor invariants[sec. 2.4,3.2,3.4]. The connecting publication is [P.2].

2. We identify a real structure which is present on fermionic Fock spaces. We use this to relate SLOCC covariants, used to separate entanglement classes of states, with reduced density matrix elements via spinor Fierz identities[sec. 2.5]. The entanglement monogamy equations for the tangles of three qubits are shown to be a consequence of such a Fierz identity. In fact, we show that these equations are special cases of monogamy equations obeyed by three fermions with six single particle states[sec. 3.4.4]. As a consequence, we confirm that the quartic SLOCC invariant introduced in [20] indeed measures three body entanglement. We identify the fermionic concurrences measuring two body entanglement[sec. 3.4.5]. The connecting publication is [P.4].

3. We study the SLOCC and extended SLOCC classes of certain low dimensional fermionic systems where the solution to the classification problems can be extracted from the mathematical literature. Particular emphasis is put on three fermion systems with up to nine single particle modes. We present invariants and covariants of these systems in the unified language of spinor invariants[ch. 3]. We give a complete set of (noncontinuous) invariants separating the SLOCC orbits in the case of three fermion systems with six, seven and eight

modes[sec. 3.4.2,3.5.2,3.6.2]. In the nine mode case we give new combinations of the four independent continuous invariants which can be used to separate the semisimple families of Vinberg and Elashvili[sec. 3.7]. We discuss how to embed the distinguishable system of three qutrits and present how the fermionic invariants reduce to the invariants of three qutrits. As a consequence, we also obtain invariants separating families of three qutrits. The embedding between families is observed to be injective, a property that previously studied fermionic embeddings of three qubits and four qubits also satisfy. We emphasize the connection of the six and seven mode cases to the theory of $SU(3)$ and $G_2$ special holonomy manifolds via their connection to Hitchin functionals[sec. 4.2]. The connecting publications are [P.1, P.3]

4. We identify the U-duality invariant which is required to express the Bekenstein-Hawking entropy of the most general 4 dimensional, stationary, asymptotically flat, nonextremal STU black holes constructed recently in [21, 22]. This allows us to write the general nonextremal entropy formula entirely in terms of asymptotic charges for the first time[sec. 4.1.7]. The expression also involves the "scalar charges" of these black holes which can in principle be solved in terms of the dyonic charges and the mass. We discuss how the formula reduces to some of the known results as the Klauza-Klein black hole and the dilute gas limit of Cvetič and Larsen[sec. 4.1.10] and how its dependence on the asymptotic scalar moduli can be removed via U-dualities[sec. 4.1.5]. We discuss the identification of the charges of the black hole with a four qubit state and show that the nonextremal black holes of [22] are in a *single* semisimple entanglement class[sec. 4.1.11], in contrary to the complete identification between extremal black holes and nilpotent classes worked out it [23]. The connecting publication is [P.5].

## Papers connecting to thesis points

[P.1] Péter Lévay and Gábor Sárosi. "Hitchin functionals are related to measures of entanglement". In: *Phys. Rev.* D86 (2012), p. 105038. DOI: 10.1103/PhysRevD.86.105038. arXiv: 1206.5066 [hep-th].

[P.2] Gábor Sárosi and Péter Lévay. "Entanglement in fermionic Fock space". In: *Journal of Physics A: Mathematical and Theoretical* 47.11 (2014), p. 115304. DOI: 10.1088/1751-8113/47/11/115304. arXiv: 1309.4300 [quant-ph].

[P.3] Gábor Sárosi and Péter Lévay. "Entanglement classification of three fermions with up to nine single-particle states". In: *Physical Review A* 89.4 (2014), p. 042310. DOI: 10.1103/PhysRevA.89.042310. arXiv: 1312.2786 [quant-ph].

[P.4] Gábor Sárosi and Péter Lévay. "Coffman-Kundu-Wootters inequality for fermions". In: *Physical Review A* 90.5 (2014), p. 052303. DOI: 10.1103/PhysRevA.90.052303. arXiv: 1408.6735 [quant-ph].

[P.5] Gábor Sárosi. "Entropy of nonextremal STU black holes: The F-invariant unveiled". In: *Phys. Rev.* D93.2 (2016), p. 024036. DOI: 10.1103/PhysRevD.93.024036. arXiv: 1508.06667 [hep-th].

## 1.2 A short review on multipartite entanglement for three qubits

Since almost everything in this thesis can in some way or another be connected to entanglement properties of three qubits, here we give a short summary of this relatively well understood system.

### 1.2.1 Bipartite entanglement in two qubit mixed states

Before discussing entanglement between several qubits it is essential to understand entanglement between two qubits. This is a very simple problem in the case of pure states

$$|\psi\rangle = \sum_{ij=0}^{1} \psi_{ij}|ij\rangle. \tag{1.1}$$

This state is separable into a product state if and only if the $2 \times 2$ matrix $\psi_{ij}$ is a dyad. The neccessary and sufficient condition for this is

$$\det \psi \equiv \frac{1}{2}\epsilon^{ij}\epsilon^{i'j'}\psi_{ii'}\psi_{jj'} = 0. \tag{1.2}$$

Let us assume that the above state is normalized i.e. $\sum_{ij}|\psi_{ij}|^2 = 1$ and define its reduced density matrices as

$$(\rho_A)_{ij} = (\psi\psi^\dagger)_{ij}, \quad (\rho_B)_{ij} = (\psi^\dagger\psi)_{ji}, \tag{1.3}$$

where $^\dagger$ denotes the Hermitian conjugate of matrices. It is clear that the condition (1.2) can also be written as

$$\det \rho_A \equiv \det \rho_B = |\det \psi|^2 = 0. \tag{1.4}$$

A quantity to measure the entanglement between qubit $A$ and $B$ with a clear operational meaning[24] is the *entanglement entropy*, which is the von-Neumann entropy of the reduced density matrix, i.e.

$$E_A(|\psi\rangle) \equiv S(\rho_A) = -\mathrm{Tr}\rho_A \log \rho_A. \tag{1.5}$$

For pure states one has $S(\rho_A) = S(\rho_B)$. Now notice that as the eigenvalues $\lambda_1$ and $\lambda_2$ of $\rho_A$ satisfiy $\lambda_1\lambda_2 = \det \rho_A$ and $\lambda_1 + \lambda_2 = 1$ we can express the entanglement entropy via the quantities appearing in (1.4) as

$$E_A(|\psi\rangle) = h\left(\frac{1+\sqrt{1-4\det\rho_A}}{2}\right), \tag{1.6}$$

where $h(x) = -x\log x - (1-x)\log(1-x)$ is the binary entropy function. The upshot of this simple formula is that the operationally meaningful entropic quantity $S(\rho_A)$ is entirely captured by a somewhat algebraically simpler quantity, a homogeneous function of the amplitudes

$$\mathcal{C} = \sqrt{4\det\rho_A}, \tag{1.7}$$

which goes by the name *concurrence*. Notice that this quantity is bounded from both above and below as $0 \leq \det \rho_A \leq \frac{1}{4}$. It will be very useful to recast this formula in terms of a dual two qubit state

$$|\tilde{\psi}\rangle = \sum_{ijj'j'=0}^{1} \epsilon^{ii'} \epsilon^{jj'} \bar{\psi}_{ij} |i'j'\rangle, \tag{1.8}$$

as

$$\mathcal{C} = |\langle \psi | \tilde{\psi} \rangle|. \tag{1.9}$$

While it is very easy to understand bipartite entanglement for pure states, this task is much less trivial for mixed states. The reason is that mixed states contain *classical correlations* between the parties. Indeed, they are the quantum generalization of a probability distribution on a classical phase space which in general encodes nontrivial correlations among subsystems. For a mixed quantum state our aim is then to get rid of the classical part of the correlations and quantify only the quantum part.

To illustrate this, let us consider a general mixed state $\rho$ of a bipartite system $\mathcal{H}_A \otimes \mathcal{H}_B$. In general, $\rho$ can be written as a convex combination of pure states in *several ways*

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \ \ p_i \geq 0, \ \ \sum_i p_i = 1. \tag{1.10}$$

The states $|\psi_i\rangle$ are not required to be orthonormal or to form a basis. We may think of them as the pure states that we can draw from $\rho$ according to the distribution $\{p_i\}$ but again, the important point is that this decomposition is not unique. We will say that the state $\rho$ is *separable* if we can find such a decomposition where all $|\psi_i\rangle$ are product states. Notice that deciding whether $\rho$ is separable or not is a hard problem, so hard that to date no easily computable neccessary and sufficient criteria are known to decide it.

So how do we quantify entanglement in a mixed state? It is clear that the entanglement entropy doesn't work: it is full with classical correlations. We may, however, introduce its *convex roof extension* [24]

$$\mathcal{E}(\rho) = \min_{p_i, |\psi_i\rangle | \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|} \sum_i p_i E_A(|\psi_i\rangle), \tag{1.11}$$

i.e. we take the *minimum* of the *average* of the entanglement entropy over all pure decompositions of $\rho$. This quantity is called the *entanglement of formation* [25]. It is clear that $\mathcal{E}(\rho) = 0$ if and only if $\rho$ is separable. Also, it can be shown that $\mathcal{E}(\rho)$ has a clear operational meaning: it is the asymptotic number of Bell pairs shared between $A$ and $B$ required to construct the state $\rho$.

In general, the entanglement of formation is very hard to calculate as it is defined as a solution of a variational problem. However, in the case of two qubits a simple formula can be obtained for it, reminiscent of (1.6)[26]. It can be written as follows. Define the *spin-flipped dual* state to $\rho$ as

$$\tilde{\rho} = \epsilon \otimes \epsilon \bar{\rho} \epsilon \otimes \epsilon, \tag{1.12}$$

where $\bar{\ }$ denotes complex conjugation and $\epsilon$ is the $2 \times 2$ totally antisymmetric matrix. It is clear that $\tilde{\rho}$ is a positive matrix with positive eigenvalues

$\lambda_1^2, \lambda_2^2, \lambda_3^2, \lambda_4^2$ sorted in decreasing order. Let us define the mixed state concurrence

$$\mathcal{C}(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}. \tag{1.13}$$

A remarkable result is that the entanglement of formation takes the simple form[26]

$$\mathcal{E}(|\psi\rangle) = h\left(\frac{1+\sqrt{1-\mathcal{C}(\rho)}}{2}\right), \tag{1.14}$$

so that the mixed concurrence is really a true measure of quantum entanglement in a mixed state.

It might not be immediately obvious that the mixed state concurrence agrees with the one (1.4) for pure states. To see this, notice that with the use of (1.8) we may write

$$\rho\tilde{\rho} = \langle\psi|\tilde{\psi}\rangle|\psi\rangle\langle\tilde{\psi}|, \tag{1.15}$$

which is a single dyad so indeed its only non-vanishing eigenvalue is $\lambda_1^2 = |\langle\psi|\tilde{\psi}\rangle|^2$.

### 1.2.2 Three qubits

The system of three qubits is the smallest possible entangled setup posessing genuine *multipartite entanglement*. For pure states, this multipartite entanglement can in fact be explicitly quantified through a quantity, called the 3-tangle, due to Coffman, Kundu and Wootters [27].

Let us consider a normalized pure three qubit state

$$|\psi\rangle = \sum_{ijk=0}^{1} \psi_{ijk}|ijk\rangle, \tag{1.16}$$

and define its marginals as

$$\begin{aligned}(\rho_{AB})_{ij,kl} &= \sum_n \psi_{ijn}\bar{\psi}_{kln}, \quad (\rho_{AC})_{ij,kl} = \sum_n \psi_{inj}\bar{\psi}_{knl}, \quad (\rho_{BC})_{ij,kl} = \sum_n \psi_{nij}\bar{\psi}_{nkl},\\ (\rho_A)_{ik} &= \sum_{n,m} \psi_{inm}\bar{\psi}_{knm}, \quad (\rho_B)_{ik} = \sum_{n,m} \psi_{nim}\bar{\psi}_{nkm}, \quad (\rho_C)_{ik} = \sum_{n,m} \psi_{nmi}\bar{\psi}_{nmk}.\end{aligned} \tag{1.17}$$

A natural question is: how much genuine quantum entanglement is there between a pair of qubits? We can obtain the answer by calculating the two qubit concurrence (1.13) of the corresponding two qubit reduced density matrix. Because these RDMs come from a pure three qubit state, the calculation simplifies. Indeed, we may think of $\psi_{ijk}$ as a map $\mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^2$ by grouping two of its indices into in and the remaining one to an out index and so we easily see that the two body RDMs correspond to maps $\mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^2 \times \mathbb{C}^2$ of the form $\psi^\dagger \circ \psi$ and that the one body RDMs correspond to maps $\mathbb{C}^2 \to \mathbb{C}^2$ of the form $\psi \circ \psi^\dagger$. This way, it is easy to see that nonzero eigenvalues of $\rho_A$ and $\rho_{BC}$ must agree, and similar relations hold for the other two pairs. It follows, that $\rho_{AB}\tilde{\rho}_{AB}$ can have at most two nonzero eigenvalues and we may write

$$\begin{aligned}\mathcal{C}(\rho_{AB})^2 \equiv \tau_{AB} &= (\lambda_1 - \lambda_2)^2 \\ &= \mathrm{Tr}(\rho_{AB}\tilde{\rho}_{AB}) - 2\lambda_1\lambda_2 \\ &\leq \mathrm{Tr}(\rho_{AB}\tilde{\rho}_{AB}),\end{aligned} \tag{1.18}$$

where we have defined the *2-tangle* $\tau_{AB}$ and a similar upper bound holds for $\tau_{BC}$ and $\tau_{AC}$. A short calculation shows that for normalized states one has

$$\mathrm{Tr}(\rho_{AB}\tilde{\rho}_{AB}) = 2(\det \rho_A + \det \rho_B - \det \rho_C), \tag{1.19}$$

and hence

$$\mathrm{Tr}(\rho_{AB}\tilde{\rho}_{AB}) + \mathrm{Tr}(\rho_{AC}\tilde{\rho}_{AC}) = 4\det \rho_A. \tag{1.20}$$

Combine this with the bound (1.18) to obtain the seminal result of Coffman, Kundu and Wootters [27]

$$\tau_{AB} + \tau_{AC} \leq 4\det \rho_A \equiv \mathcal{C}(\rho_A)^2 \equiv \tau_{A(BC)}. \tag{1.21}$$

This inequality expresses the *monogamy* of entanglement. The tangle $\tau_{A(BC)}$ measures the entanglement of qubit $A$ with the joint system $BC$ while $\tau_{AB}$ and $\tau_{AC}$ measure the entanglement between $A$ and $B$ and $A$ and $C$ respectively. The sum of the latter two must be less than the former. The fact that there is an inequality instead of an equality signals the existence of genuine tripartite entanglement, i.e. entanglement which is not shared by any pair of the tree parties. One can quantify this with the positive quantity

$$\tau_{ABC} = \tau_{A(BC)} - \tau_{AB} - \tau_{AC}, \tag{1.22}$$

which we call 3-tangle. Note the crucial fact that the value of this quantity is independent of which subsystem we select at the first place i.e. $\tau_{ABC} = \tau_{B(AC)} - \tau_{AB} - \tau_{BC}$ and a similar relation holds with $\tau_{C(AB)}$. An explicit calculation reveals that the 3-tangle is expressed with the amplitudes as

$$\tau_{ABC} = 4|\mathrm{HDet}(\psi)|, \tag{1.23}$$

where

$$\begin{aligned}\mathrm{HDet}(\psi) = {} & \psi_{000}^2\psi_{111}^2 + \psi_{001}^2\psi_{110}^2 + \psi_{010}^2\psi_{101}^2 + \psi_{100}^2\psi_{011}^2 \\ & - 2(\psi_{000}\psi_{001}\psi_{100}\psi_{111} + \psi_{000}\psi_{010}\psi_{101}\psi_{111} \\ & + \psi_{000}\psi_{100}\psi_{011}\psi_{111} + \psi_{001}\psi_{010}\psi_{101}\psi_{110} \\ & + \psi_{001}\psi_{100}\psi_{011}\psi_{110} + \psi_{010}\psi_{100}\psi_{011}\psi_{101}) \\ & + 4(\psi_{000}\psi_{011}\psi_{101}\psi_{110} + \psi_{001}\psi_{010}\psi_{100}\psi_{111}).\end{aligned} \tag{1.24}$$

is Cayley's hyperdeterminant for $2 \times 2 \times 2$ arrays.

We will rederive these results in 3.4.4 as a special case of the monogamy inequality for three fermions with six single particle states. The somewhat arbitrary form of the spin-flip (1.12) required to compute the 2-tangle will be understood in terms of a canonical spinor conjugation (see sec. 2.5) and the origin of the CKW equations

$$\tau_{AB} + \tau_{AC} + \tau_{ABC} = \tau_{A(BC)}, \tag{1.25}$$

will be tracked back to follow from *Fierz identities* for spinors.

### 1.2.3 Stochastic local operations and classical communication

The appearance of a quantity HDet($\psi$) in (1.24), well known in invariant theory, as an entanglement measure is not a coincidence. Let us briefly review an old idea for a systematic way of classifying different types of entanglement: the classification based on equivalence under stochastic local operations and classical communication (SLOCC).

The basic idea is that when we have a multipartite composite quantum system, we cannot create or increase any kind of entanglement by allowing the parties to perform arbitrary quantum operations (e.g. time evolution or measurement) on their part of the subsystem and share the results with each other via classical channels. These operations are rougly make up what we call local operations and classical communication (LOCC). As these operations do not create entanglement we may say that the states $|\psi_1\rangle$ and $|\psi_2\rangle$ are equally entangled when they can be mutually transformed into each other by means of LOCC. It follows that finding classes of equally entangled states requires finding the orbits of the group of invertible LOCC transformations on the Hilbert space of the system. This group turns out to be the group of local unitaries, for the case of three qubits, it is $U(2) \times U(2) \times U(2)$. The action of an element $(u_1, u_2, u_3) \in U(2) \times U(2) \times U(2)$ on the amplitudes of the state is just

$$\psi_{ijk} \mapsto (u_1)_i{}^{i'} (u_2)_j{}^{j'} (u_3)_k{}^{k'} \psi_{i'j'k'}, \tag{1.26}$$

where we are beginning to use Einstein convention for the summation of upper and lower indices.

It is not difficult to see that this classification problem gives rise to a continuum family of orbits, even in the simplest cases. Indeed, clearly all the eigenvalues of the reduced density matrices are invariant under this action, so that whenever they have different values, the state is automatically from a different orbit. An attempt to coarse grain this classification problem is the idea of classification under SLOCC. Here, we only require that we can turn $|\psi_1\rangle$ and $|\psi_2\rangle$ into each other by means of LOCC with *probability greater than zero*. This amounts to finding the orbits of the group of invertible local operators, which we will call the SLOCC group. For the case of three qubits, this is $GL(2, \mathbb{C})^{\times 3}$ and an element $(g_1, g_2, g_3) \in GL(2, \mathbb{C})^{\times 3}$ acts as

$$\psi_{ijk} \mapsto (g_1)_i{}^{i'} (g_2)_j{}^{j'} (g_3)_k{}^{k'} \psi_{i'j'k'}. \tag{1.27}$$

We will see later very clearly, that the hyperdeterminant (1.24), giving rise to the 3-tangle, is a *relative invariant* under this group action, transforming as

$$\text{HDet}(g_1 \otimes g_2 \otimes g_3 \psi) = (\det g_1 \det g_1 \det g_1)^2 \text{HDet}(\psi), \tag{1.28}$$

hence states which contain genuine tripartite entanglement are in a different SLOCC class than those of which do not.

The SLOCC classification problem can be solved for three qubits[28, 29]. It turns out that there are six SLOCC classes, and the one with HDet$(\psi) \neq 0$ is *dense* in the space of all three qubit states. This is because the single relative invariant of the action of the SLOCC group is HDet($\psi$) which makes the pair $(GL(2,\mathbb{C})^{\times 3}, \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$ a prehomogeneous vector space. We illustrate these classes on figure 1.1. An arrow points from class $A$ to class $B$ if there is a nonzero probability of turning a state from $A$ into a state from $B$ by means of LOCC but states from $B$ cannot be turned into states from $A$. Now let us list some representative states for these classes:

$$\begin{aligned} \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) &\text{ is in GHZ,} \\ \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle) &\text{ is in W,} \\ \frac{1}{\sqrt{2}}|0\rangle \otimes (|10\rangle + |01\rangle) &\text{ and its permutations} \\ &\text{are in A(BC), B(AC), C(AB) respectively,} \\ |000\rangle &\text{ is separable.} \end{aligned} \tag{1.29}$$

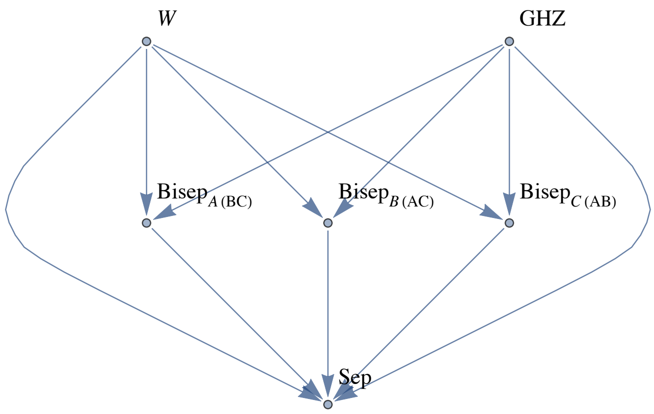


FIGURE 1.1: SLOCC classes for three qubits. An arrow points from class $A$ to class $B$ if there is a degenerate SLOCC transformation turning a state in $A$ to a state in $B$.

Notice that the class W contains states which cannot be separated (in particular can have maximally mixed one qubit reduced density matrices) but still do not contain any tripartite entanglement. We will review this classification in more detail as a special case of a similar classification of a fermionic system in section 3.4. We will see that the classes can be separated in a very natural way with the use of a fermionic quantity.

It turns out that for general quantum system the concept of SLOCC classification becomes untractable and needs further coarse graining. In fact, precisely the small quantum systems corresponding to prehomogeneous vector spaces are the ones where one can hope for a finite number of SLOCC classes. Indeed, let us assume that we happen to have more then one relative invariants of the SLOCC group, say $I_1$ and $I_2$, and let us assume that they have different homogeneous degree, say $k_1$ and $k_2$. This means that for states $|\psi_1\rangle$ and $|\psi_2\rangle$ to be SLOCC equivalent there has to be a nonzero complex number $\lambda$ such that

$$(I_1, I_2)|_{|\psi_1\rangle} = (\lambda^{k_1} I_1, \lambda^{k_2} I_2)|_{|\psi_2\rangle}, \tag{1.30}$$

i.e. different values of $\frac{I_1^{1/k_1}}{I_2^{1/k_2}}$ correspond to different SLOCC orbits so that we have a continuum number of orbits. In this case there are different ways to coarse grain the SLOCC classification problem. For example, for four qubits[30], for fermions with eight single particle states[31], three qutrits [32] or three fermions with nine single particle states [P.3] one can group these orbits into a finite number of families. This, however, is strongly based on another nonuniversal feature: the SLOCC problem for these systems can be turned into a problem in the theory of semisimple Lie algebras. We will briefly return to this question of coarse graining the SLOCC classification for fermionic systems in section 2.4.3.

# Chapter 2

# Entanglement between fermions

## 2.1 Fermionic states and Clifford algebras

We are interested in the algebra of fermionic observables $\mathcal{A}$ which is associated to the vector space $\mathcal{H}$ of some single particle modes[1]. We select some basis to this space and associate fermionic creation operators $p^a$, $a = 1, ..., \dim \mathcal{H} = d$ to them. Let us denote the vector space spanned by these operators by $W$. We introduce annihilation operators $n_a$ as the basis of the dual vector space $W^*$ through $n^a(p_b) = \delta^a{}_b$. We generate $\mathcal{A}$ through the canonical anticommutation relations (CAR):

$$\{p^a, n_b\} = \delta^a{}_b I. \tag{2.1}$$

Equivalently for vectors $x = x_a p^a + x^a n_a$ and $y = y_a p^a + y^a n_a$ taken from $V \equiv W \oplus W^*$ this reads as

$$\{x, y\} = 2(x, y)I, \tag{2.2}$$

where the symmetric inner product is $(x, y) = \frac{1}{2}(x^a y_a + x_b y^b)$. The algebra generated by such a relation is by definition a Clifford algebra, hence $\mathcal{A} = Cliff(V)$. The (up to isomorphism) unique representation of this algebra is built as follows. One chooses a vacuum state $|0\rangle$ which is annihilated by half of the space $V$ e.g. $n_a|0\rangle = 0$, $a = 1, ..., d$. Then one acts in all the possible ways with the creation operators. The representation space is then spanned by basis vectors $p^{a_1}...p^{a_k}|0\rangle$, $k = 0, ..., d$. The resulting $2^d$ dimensional space is called the fermionic Fock space $\mathcal{F}$ and is in fact identical[2] to the vector space underlying the exterior algebra of the single particle space[3]: $\mathcal{F} = \wedge^\bullet \mathcal{H}$. The states in $\mathcal{F}$ can be rightfully called *spinors* as they form a representation of a Clifford algebra.

[1]Notice that we do not assume that there is an inner product on $\mathcal{H}$ for now. We will introduce it later to emphasize what role it does and does not play. As a consequence, the adjoint of operators is not defined for the time being.

[2]Some authors add an extra one dimensional factor so that $\mathcal{F} = \wedge^\bullet \mathcal{H} \otimes (\wedge^d \mathcal{H})^{-\frac{1}{2}}$. This is to obtain the correct action of $GL(\mathcal{H})$ acting as a subgroup of $Spin(V)$ but it clearly does not change the linear structure. We will comment on this in more detail later, in sec. 2.3.2

[3] As such, by writing a single particle basis element as $e^a = p^a|0\rangle$ we may write the basis elements of the Fock space in the exterior algebra notation $p^{a_1}...p^{a_k}|0\rangle = e^{a_1} \wedge ... \wedge e^{a_k}$. The action of the space $W \oplus W^*$ of creation and annihilation operators turns into the usual exterior $\wedge$ and interior $\iota$ products of vectors and dual vectors respectively with exterior forms. Indeed, one has

$$\{x\wedge, \iota_y\} = y(x)1, \quad x \in \mathcal{H}, \quad y \in \mathcal{H}^*, \tag{2.3}$$

There are two canonical maps that come with every Clifford algebra:

- *Parity or main automorphism*. This is generated by the map $x \mapsto -x$ on $V$ which leaves (2.2) invariant. It follows that we can lift this up to $Cliff(V)$ by extending linearly the action on monomials:

$$P : x_1 x_2 ... x_k \mapsto (-x_1)(-x_2)...(-x_k) = (-1)^k x_1 x_2 ... x_k. \tag{2.4}$$

  It is clear that since $P^2 = I$ and $P$ is an automorphism it introduces a $\mathbb{Z}_2$ grading of $Cliff(V)$. This is just the grading into even and odd elements.

- *Transposition or main antiautomorphism* We define on monomials the map

$$^t : x_1 x_2 ... x_k \mapsto x_k ... x_2 x_1, \tag{2.5}$$

  and extend it linearly on $Cliff(V)$. Again, this definition is independent of the blocking of the Clifford algebra elements into monomials as $^t$ leaves (2.2) invariant. It is clear that $^t$ is an antiautomorphism:

$$(AB)^t = B^t A^t, \;\; A, B \in Cliff(V). \tag{2.6}$$

The orthogonal group of the inner product (2.2) on $V$ is defined as

$$(\mathcal{O}x, \mathcal{O}y) = (x, y), \;\; \mathcal{O} \in O(V). \tag{2.7}$$

In writing $O(V)$ we mean $O(V, (.,.))$ and keep the inner product implicit. These transformations do not change the anticommutator, so they posess a clear physical meaning: they are (not necessarily unitary) *Bogoliubov transformations*. We define the *Clifford group* $\Gamma$ as the group of invertible elements $g$ of $Cliff(V)$ which fix $V$ in the sense that $gxg^{-1} \in V$ for every $x \in V$. The even subgroup $\Gamma_0$ consists of even elements $P(g) = g$ with respect to (2.4). Now notice that

$$(gxg^{-1}, gxg^{-1})I = gxg^{-1}gxg^{-1} = g(x,x)Ig^{-1} = (x,x)I, \tag{2.8}$$

so the transformation

$$x \mapsto \mathcal{O}_g x = gxg^{-1}, \tag{2.9}$$

is in $O(V)$. The map $g \mapsto \mathcal{O}_g$ is a homomorphism between $\Gamma$ and $O(V)$ with kernel $\mathbb{C}^\times$. Notice that since for every $x \in V$ we have $x^t = x$ and it follows that

$$\begin{aligned} gxg^{-1} &= (gxg^{-1})^t \\ &= (g^t)^{-1} x g^t \end{aligned} \tag{2.10}$$

so that $g^t g x = x g^t g$ and hence consistency requires

$$g^{-1} = \alpha_g g^t, \;\; g \in \Gamma, \; \alpha_g \in \mathbb{C}^\times. \tag{2.11}$$

We call the number $\alpha_g^{-1}$ the Clifford norm of $g$. It is easy to see that $g \mapsto \alpha_g^{-1}$ is a group homomorphism from $\Gamma$ to $\mathbb{C}^\times$. We define the double cover of

which represents the canonical anticommutation relation. Throughout this chapter, we will stick to the creation/annihilation operator formalism.

$O(V)$, the $pin(V)$ group as elements in $\Gamma$ with norm plus or minus one:

$$OO^t = \pm I, \quad O \in pin(V). \tag{2.12}$$

The subgroup $Spin(V)$ is the one which is even under $P$ i.e. the $\pm 1$ norm subgroup of $\Gamma_0$. The component $Spin_0(V)$ connected to the identity clearly has $OO^t = I$. It is clear that all these definitions define subgroups of $\Gamma$ and examination of the homomorphism $\mathcal{O}_g$ defined through (2.9) reveals that $pin(V)$ and $Spin(V)$ are indeed the double covers of $O(V)$ and $SO(V)$ respectively.

All these groups are naturally represented on the fermionic Fock space $\wedge^\bullet \mathcal{H}$ by their action as Clifford algebra elements:

$$|\psi\rangle \mapsto g|\psi\rangle, \quad |\psi\rangle \in \wedge^\bullet \mathcal{H}, \; g \in \Gamma. \tag{2.13}$$

Now let us examine more closely the even Clifford group $\Gamma_0$. It is clear that $\Gamma_0 \cong \mathbb{C}^\times \times Spin_0(V)$ and hence we may write every element of $g \in \Gamma_0$ as

$$g = \lambda e^s, \tag{2.14}$$

where $\lambda$ is a nonzero complex number and $s$ is a generator of $Spin_0(V)$, i.e. an element of the Lie-algebra $\mathfrak{so}(V)$. We claim that any element $s$ of this Lie algebra can be written with the creation and annihilation operators as the bilinear expression

$$s = \frac{1}{2} A_a{}^b [p^a, n_b] + \frac{1}{2} B_{ab} p^a p^b + \frac{1}{2} C^{ab} n_a n_b \tag{2.15}$$

Indeed, putting $g = e^s$ in the double cover equation (2.9) and expanding to first order in $s$ yields the generator $\mathcal{T}_s$

$$\mathcal{T}_s x = [s, x], \tag{2.16}$$

acting on $V$. We see that in order for the right hand side to be in $V$ the generator $s$ must be at most a bilinear expression of creation and annihilation operators. Such a bilinear is indeed in $\mathfrak{so}(V)$ as it satisfies

$$(\mathcal{T}_s x, y)I + (x, \mathcal{T}_s y)I \equiv \{\mathcal{T}_s x, y\} + \{x, \mathcal{T}_s y\} = [s, \{x, y\}] = 0, \tag{2.17}$$

where we have used that $\{x, y\}$ is proportional to the identity. It will later be rewarding to write out (2.16) explicitly in the basis $p^a, n_a$:

$$[s, \begin{pmatrix} p^a \\ n_b \end{pmatrix}] = \begin{pmatrix} A_{a'}{}^a & C^{ab'} \\ B_{ba'} & -A_b{}^{b'} \end{pmatrix} \begin{pmatrix} p^{a'} \\ n_{b'} \end{pmatrix}, \tag{2.18}$$

so that it is really straightforward to see that the block matrix is in $\mathfrak{so}(V)$ with respect to the inner product matrix

$$g = \left( \begin{array}{c|c} 0 & I \\ \hline I & 0 \end{array} \right), \tag{2.19}$$

where $I$ is now the $d \times d$ identity matrix. The Lie algebra of the even Clifford group $\Gamma_0$ is then obtained by setting $\lambda = e^\epsilon$ and expanding in $\epsilon$. The generators are of the form $s + \epsilon I$, where $I$ is the identity of $Cliff(V)$. Adding this to (2.15) we see that the Lie algebra of $\Gamma_0$ consists of the most general bilinears formed by creation and annihilation operators. Notice that the Lie algebra of $Spin(V)$ is recovered inside this as the $-1$ eigenspace of the main antiautomorphism (2.5).

## 2.2 Entanglement between particles and entanglement between modes

In order to talk about entanglement one needs to split the system in question into smaller subsystems. This is an easy task when there is a clear physical guide how to factor a Hilbert space into the tensor product of smaller spaces. There are situations, however, when physics does not provide us such a naive factorization. There are far from trivial examples of such ambiguities, like entanglement in gauge theories [33], but one has to be careful describing free fermions as well.

There are two physically different ways to study entanglement in fermionic systems. One can consider entanglement between *particles*, say two electrons, or entanglement between *modes* say modes corresponding to a subregion of space and its complement. It is immediately clear that the local operations can be wildly different in these two cases. Consider for example two fermions on a lattice, each being in a plane wave state

$$|\psi\rangle = p^{k_1} p^{k_2} |0\rangle, \tag{2.20}$$

with

$$p^k = \frac{1}{\sqrt{V}} \sum_R e^{ikR} p^R, \tag{2.21}$$

where $R$ are some lattice vectors and $p^R$ creates an electron localized on a given site. It is clear that for any bipartition of the lattice $|\psi\rangle$ is an entangled state when considering entanglement between spatial modes[4],[5]. On the other hand, the two electrons are in a Slater determinant state and hence the state contains only statistical correlations. It is unentangled, a "product state" when the entanglement between particles is considered. We will see later that despite the significant physical difference between these two concepts, they are not entirely unrelated.

Let us be a little more explicit what we mean by entanglement between modes and entanglement between particles.

[4]Otherwise stated, (2.21) is a nonlocal unitary transformation from the point of view of a bipartitioning of the lattice modes $p^R$.

[5]In relativistic quantum field theories even the vacuum is spatially entangled. Indeed, as an example consider the local modes of a free field theory (say a Majorana field $\psi(x)$) and the modes diagonalizing the Hamiltonian (say $c_k^\dagger$ with momentum $k$). These modes are already related via a Bogoliubov transformation $\psi(x) \sim \int \mathrm{d}k(e^{ikx}c_k + e^{-ikx}c_k^\dagger)$ due to the requirement that the anticommutator of two (not equal time) $\psi(x)$ should vanish outside the light cone. This results in the Fock vacuum of the modes $c_k$ being entangled with respect to a spatial bipartitioning of the modes $\psi(x)$. On the contrary to this, in a lattice model where the local modes are related to energy eigenstates via (2.21), the vacuum is not entangled.

### 2.2.1 Entanglement between fermionic modes

Consider dividing the space of single particle modes into two pieces as

$$W = W_1 \oplus W_2. \tag{2.22}$$

Notice that we use a direct *sum* and not a product. We are interested in entanglement between modes from $W_1$ and $W_2$. The generating vector space of the Clifford algebra decomposes as

$$V = (W_1 \oplus W_2) \oplus (W_1 \oplus W_2)^* = (W_1 \oplus W_1^*) \oplus (W_2 \oplus W_2^*) = V_1 \oplus V_2, \tag{2.23}$$

where $V_1$ and $V_2$ generate the Clifford algebra of the modes from the first set and the second set respectively. Notice that

$$\{V_1, V_2\} = 0. \tag{2.24}$$

Actually this is the only requirement we need to divide the generating vector space $V$ into two pieces without a priori deciding how to split it into creation and annihilation operators. Say that the split is defined by a projection $\pi : V \to V_1$, $\pi^2 = \pi$. Then (2.24) is equivalent with demanding

$$0 = \{(I - \pi)x, \pi y\} = 2(x, \pi y) - 2(\pi x, \pi y), \tag{2.25}$$

for all $x, y \in V$. Since $\pi^2 = \pi$ we see that this is satisfied if and only if $(\pi x, y) = (x, \pi y)$ i.e. $\pi$ is an orthogonal projection with respect to the inner product of the Clifford algebra. Note that one always has $Cliff(V_1 \oplus V_2) = Cliff(V_1) \otimes Cliff(V_2)$ as vector spaces but certainly not as algebras. However if (2.24) is satisfied we have a graded tensor product of graded algebras

$$Cliff(V_1 \oplus V_2) \cong Cliff(V_1) \hat{\otimes} Cliff(V_2), \tag{2.26}$$

where the $\mathbb{Z}_2$ grading is the one introduced by the parity (2.4). The symbol $\hat{\otimes}$ denotes that one defines the algebra product on the tensor product vector space via extending bilinearly the product rule

$$(A_1 \otimes B_1)(A_2 \otimes B_2) = (-1)^{|A_2||B_1|}(A_1 A_2) \otimes (B_1 B_2), \tag{2.27}$$

where $|A|$ denotes the parity of $A$ and $A_{1,2}$ are composed of $V_1$ modes while $B_{1,2}$ are composed of $V_2$ modes. This rule is indeed valid, as if (2.24) is true, we can just (anti)commute $A_2$ over $B_1$ and we only need to compensate for the introduced sign. A more formal way of saying the same thing is to note that the linear map

$$\begin{aligned} f : V_1 \oplus V_2 &\to Cliff(V_1) \hat{\otimes} Cliff(V_2) \\ (x_1, x_2) &\mapsto x_1 \otimes I + I \otimes x_2, \end{aligned} \tag{2.28}$$

is Clifford, i.e. $f(x_1, x_2)^2 = x_1^2 \otimes I + I \otimes x_2^2 \equiv 2((x_1, x_1) + (x_2, x_2)) I \otimes I$ provided we use (2.27) as the multiplication rule. This ensures that $f$ can be extended uniquely into an isomorphism $Cliff(V_1 \oplus V_2) \to Cliff(V_1) \hat{\otimes} Cliff(V_2)$.

This is the splitting of the algebra of observables that we are looking for when we wish to describe mode entanglement: local operations acting on the first and second subsystem are the ones built out from $Cliff(V_1)$ and $Cliff(V_2)$ respectively. It is important to keep in mind that such a splitting is possible because in the case of mode entanglement, the subsystems in question – the modes – are *distinguishable*, despite the Hilbert space being fermionic.

Mode entanglement is a vast field and can be argueably more useful than entanglement between particles when one wishes to learn about important topics like the entanglement structure of quantum field theories. Yet, in the following, we will almost exclusively focus on entanglement between particles as this is the concept which emerges in generalizations of the black hole/qubit correspondence.

### 2.2.2 Entanglement between fermionic particles

Suppose we have an $n$ particle state. When we wish to describe entanglement between particles the first thing that we want is to somehow define a split into a system of $k$ and $n-k$ particles. As we have seen in the previous case, this should naturally be done by splitting the algebra of observables into two commuting subalgebras. Now our observables are $k$-particle operators. Recall, that the set of single particle operators is

$$O^{(1)} = \{I\} \cup \{A_a{}^b p^a n_b | A_a{}^b \in \mathbb{C}\}, \tag{2.29}$$

while the set of $k$ particle operators is

$$O^{(k)} = \bigcup_{l=0}^{k} \{A_{a_1...a_l}{}^{b_1...b_l} p^{a_1}...p^{a_l} n_{b_1}...n_{b_l} | A_{a_1...a_l}{}^{b_1...b_l} \in \mathbb{C}\}. \tag{2.30}$$

We immediately meet a serious obstacle: these sets, appart from the trivial cases $O^{(0)}$ and $O^{(d)}$, do not form closed subalgebras. This, of course, prevents us from splitting the observables into subsystems. However, there are some crucial notions which are still well defined. Such is the $k$-particle reduced density matrix $\rho^{(k)} : \wedge^k\mathcal{H} \to \wedge^k\mathcal{H}$ of a state $|\psi\rangle$, which has the standard definition[6]

$$\langle\psi|A\psi\rangle = \mathrm{Tr}(\rho^{(k)}A), \quad \forall A \in O^{(k)}, \tag{2.31}$$

which means that the expectation value of a $k$ particle operator can still be expressed if we keep only a reduced amount of information about the state $|\psi\rangle$ encoded in a mixed state on $\wedge^k\mathcal{H}$. Expanding $A$ shows that the matrix elements of $\rho^{(k)}$ are

$$(\rho^{(k)})_{j_1...j_k}{}^{i_1...i_k} = \langle\psi|p^{i_1}...p^{i_k} n_{j_1}...n_{j_k}\psi\rangle. \tag{2.32}$$

For example, for an $n$ particle state

$$|P\rangle = \frac{1}{n!} P_{i_1...i_n} p^{i_1}...p^{i_n}|0\rangle, \tag{2.33}$$

[6]In this subsection we temporarly reintroduce the Hermitian inner product $\langle.|.\rangle$ which we need to define expectation values and hence density matrices. We will describe the consequences of introducing this extra structure in more detail in section 2.5.

we have

$$(\rho^{(k)})_{j_1\ldots j_k}{}^{i_1\ldots i_k} = \frac{1}{(n-k)!} P_{j_1\ldots j_k i_{k+1}\ldots i_n} \bar{P}^{i_1\ldots i_k i_{k+1}\ldots i_n}, \tag{2.34}$$

which satisfies Löwdin normalization

$$\mathrm{Tr}\rho^{(k)} = \binom{n}{k} \langle P|P\rangle. \tag{2.35}$$

Note that we have adopted the convention that a complex conjugation raises an index. Spectral data of $\rho^{(k)}$ contains information about how entangled a set of $k$ particles are with the rest. For example, the one particle RDM $\rho^{(1)}$ is a projection to an $n$ dimensional subspace, i.e. $n$ of its eigenvalues are 1 and $d-n$ of them are 0, if and only if the state $|P\rangle$ is a single Slater determinant, i.e. it is *separable*. The fermionic single particle *local unitary group* is defined to be $U(d)$ and it acts on the amplitudes of (2.33) as

$$P_{i_1\ldots i_n} \mapsto U_{i_1}{}^{i'_1} \ldots U_{i_n}{}^{i'_n} P_{i'_1\ldots i'_n}, \tag{2.36}$$

where the matrix $U_i{}^{i'}$ is unitary. It is worth to compare this to the qubit case (1.26). We see that we must act with the same $U$ on each slot due to the indistinguishability of the parties. The spectral information contained in the reduced density matrices $\rho^{(k)}$ is clearly invariant under the action of the fermionic local unitary group.

There is a nice way of connecting these reduced density matrices to the better understood case of mode entanglement [34]. To see this, consider for example the splitting of the mode space $W = \mathrm{span}(p^1, \ldots, p^d)$ into the pair of a lone single particle mode and the rest, $W_j = \mathrm{span}(p^j)$ and $W_{\bar{j}} = \mathrm{span}(p^1, \ldots, p^{j-1}, p^{j+1}, \ldots, p^d)$. One can find the reduced density matrix of a state $|\psi\rangle$ to the algebra $Cliff(W_j \oplus W_j^*)$ to be[7]

$$\rho_j = \langle\psi|p^j n_j \psi\rangle p^j |0\rangle\langle 0| n_j + (1 - \langle\psi|p^j n_j \psi\rangle)|0\rangle\langle 0| \quad \text{(no sum)}. \tag{2.37}$$

For bipartite entanglement of pure states, a standard measure of entanglement is the entanglement entropy that we have already met in (1.5) for qubits. It reads in this case as

$$E_j(\psi) = -\mathrm{Tr}\rho_j \log \rho_j = h(\langle\psi|p^j n_j \psi\rangle), \tag{2.38}$$

where $h$ is the same binary entropy function as in (1.6). It vanishes if and only if we can separate the mode $j$ in $|\psi\rangle$. As noticed in [34], a natural way of obtaining a quantity measuring the separability of a single particle is to eliminate the dependence from $E_j(\psi)$ on the basis picked on the space of single particle modes. This is done in two steps. First we sum up $j$ and then we take the minimum of the resulting quantity over all possible choice of basis vectors on $W$, the space of creation operators. This clearly gives an invariant of the group action (2.36). The minimum is reached in the basis which diagonalizes the single particle reduced density matrix $\rho^{(1)}$ (not to be confused with $\rho_j$!). This is sometimes called the basis of *natural orbitals*.

[7] By $|0\rangle\langle 0|$ in this formula we mean $n_j p^j$ (no sum), i.e. the vacuum projector of a single mode. The point here is that $\rho_j$ acts on the two dimensional Fock space spanned by $|0\rangle$ and $p^j|0\rangle$.

Indeed, for any other basis there is a diagonalizing unitary $U^a{}_b$ such that

$$\langle\psi|p^a n_b\psi\rangle = U^a{}_{a'}\bar{U}_b{}^{b'}\delta^{a'}_{b'}\lambda_{b'}, \tag{2.39}$$

and therefore using the concavity of $h$ we have

$$\sum_j h(\langle\psi|p^j n_j\psi\rangle) \geq \sum_{j,b} U^j{}_b\bar{U}_j{}^b h(\lambda_b) = \sum_b h(\lambda_b). \tag{2.40}$$

On the right hand side we see appearing the von-Neumann entropy of the single particle RDM $\rho^{(1)}$ which describes measurements of single particle operators. This way we may reinterpret entanglement between particles as a state dependent version of entanglement between modes: the entanglement of a single particle with the rest is the average mode entanglement of a single natural orbital with the rest.

## 2.3 SLOCC and extended SLOCC

### 2.3.1 SLOCC

Now consider an $n$ fermion state with a $d \geq n$ dimensional single particle Hilbert space $\mathcal{H}$

$$|\psi\rangle = \frac{1}{n!}\psi_{a_1,\dots,a_n}p^{a_1}\dots p^{a_n}|0\rangle \in \wedge^n\mathcal{H}. \tag{2.41}$$

The ordinary fermionic SLOCC group is $GL(\mathcal{H}) \cong GL(d,\mathbb{C})$ and it acts on the amplitudes as

$$\psi_{a_1,\dots,a_n} \mapsto G_{a_1}{}^{b_1}\dots G_{a_1}{}^{b_1}\psi_{b_1,\dots,b_n}, \quad G \in GL(\mathcal{H}). \tag{2.42}$$

This corresponds to an invertible protocol acting locally on a single particle, but as the particles are indistinguishable, we can only apply the same protocol to all of the particles. It is the same short of coarse graining of the local unitary group (2.36) as (1.27) to (1.26) in the case of qubits.

Finding the orbits of the above defined group action is what we call the *SLOCC classification problem for fermions*. States on the same orbit are called SLOCC equivalent and we think of them as being equally entangled in the sence that they can be turned into each other by a SLOCC protocol with probability greater than zero. We may say that a state $|\psi_1\rangle$ is less entangled than $|\psi_2\rangle$ if there is a degenerate SLOCC protocol transforming $|\psi_2\rangle$ into $|\psi_1\rangle$.

### 2.3.2 Extended SLOCC

One can ask the question if there is a natural way to describe these transformations, originaly introduced on a physical basis, in the formalism of section 2.1. where the fermionic state space is regarded to be the spinor representation of the Clifford algebra generated by the single particle modes. It is easy to see that the answer to this question is affirmative. The SLOCC transformation (2.42) is implemented by an invertible transformation $p^a \mapsto G_{a'}{}^a p^{a'}$ on the space $W$ of creation operators which is clearly just a special case of the Bogoliubov transformation introduced in e.q. (2.7). Let us find the elements of the even Clifford group $\Gamma_0$ which implement this transformation through (2.9). From (2.14) and (2.15) we see that the elements

$$g = \lambda e^{\frac{1}{2}A_a{}^b[p^a, n_b]} \tag{2.43}$$

act on $V$ as

$$gp^a g^{-1} = G_b{}^a p^b, \quad gn_a g^{-1} = (G^{-1})_a{}^b n_b, \tag{2.44}$$

where the matrix $G_a{}^b$ is the exponential of the matrix $A_a{}^b$. Using the identity $e^{\mathrm{Tr}A} = \det G$ we easily find that the action of $g$ on the vaccum is

$$g|0\rangle = \lambda(\det G)^{-\frac{1}{2}}|0\rangle, \tag{2.45}$$

and hence we may fix $\lambda = (\det G)^{\frac{1}{2}}$ to keep the vacuum invariant[8]. The resulting action on $n$-particle states (2.41) is

$$\begin{aligned} g|\psi\rangle &= \frac{1}{n!}\psi_{a_1,\dots,a_n}(gp^{a_1}g^{-1})\dots(gp^{a_n}g^{-1})g|0\rangle \\ &= \frac{1}{n!}(G_{a_1}{}^{b_1}\dots G_{a_n}{}^{b_n}\psi_{b_1,\dots,b_n})p^{a_1}\dots p^{a_n}|0\rangle, \end{aligned} \tag{2.46}$$

which is indeed the same as (2.42). In fact, elements of the form (2.43) in $\Gamma_0$ are precisely the ones which conserve particle number, i.e. keep the $\mathbb{Z}_d$ grading of the exterior algebra underlying the fermionic Fock space.

We have seen in section 2.1 that the groups $\Gamma, \Gamma_0, Spin(V)$, etc. and their action arise naturally only by prescribing canonical anticommutation relations between fermionic modes, without fixing anything additionally. Finding the orbits of these groups on $\mathcal{F}$ is then a natural problem, known to mathematicians as the classification problem of spinors [19, 35]. It is satisfying to see that the the SLOCC classification problem for fermions is recovered in this framework as a special case. Now it is tempting to consider coarse grainings of the SLOCC classification problem by replacing the SLOCC group by a bigger one. The natural candidate is the even Clifford group $\Gamma_0$ because of the following physical reason. We do not want to consider entanglement between states with different overall statistics i.e. even and odd particle number. This is sometimes refered to as the parity superselection rule. Indeed, by definition, the action of the even Clifford group does not change the parity (see (2.4)) of the states: it acts reducibly on $\mathcal{F}$ with irreducible invariant subspaces $\mathcal{F}^+$ and $\mathcal{F}^-$ corresponding to *left and right handed Weyl spinors*.

This motivates us to call the even Clifford group $\Gamma_0 \cong \mathbb{C}^\times \times Spin_0(V)$ the *generalized*, or more appropriately, the *extended* SLOCC group.

A recent result [34] shows that there is a way of thinking about entanglement classification under the extended SLOCC group as a type of average mode entanglement, by a suitable generalization of the argument given at the end of section 2.2.2. We will give a review of this at the end of section 2.5.3.

[8]Recall that the spin group is recovered with $\lambda = 1$. The determinant factor then justifies writing the Fock space as $\mathcal{F} = \wedge^\bullet\mathcal{H} \otimes (\wedge^d\mathcal{H})^{-\frac{1}{2}}$ as this is how $GL(\mathcal{H})$ acts as a subgroup of $Spin(V)$.

### 2.3.3 Pure spinors and Slater determinants

The first, somewhat obious step in solving the SLOCC classification problem is to notice that states that are just exterior products of vectors from $W$ always form a single SLOCC orbit. Indeed, using the definition (2.42), we have

$$(v^1_{a_1} p^{a_1})...(v^n_{a_n} p^{a_n})|0\rangle \mapsto (v^1_{a_1} G_{b_1}{}^{a_1} p^{b_1})...(v^n_{a_n} G_{b_n}{}^{a_n} p^{b_n})|0\rangle. \tag{2.47}$$

We call these states *separable* or more traditionally *single Slater determinant* as they only contain statistical correlations between the parties which cannot be removed due to the indistinguishable nature of the particles.

One may wonder what happens to this orbit when we extend the classification to the extended SLOCC group $\Gamma_0$. The answer is that it will be a part of the orbit of so called *pure spinors*. A state $|\psi\rangle \in \mathcal{F}$ is called a pure spinor if its *annihilator subspace* $E_\psi = \{x \in V | x|\psi\rangle = 0\}$ is of maximal dimension. Note that every element $x$ of the annihilator subspace have zero inner product with itself since $0 = x^2|\psi\rangle = (x,x)|\psi\rangle$. As a consequence, the maximal dimension of $E_\psi$ is $d$ and if $\psi$ is pure, we may decompose $V$ into new creation and annihilation operators as $V = E_\psi \oplus E^*_\psi$. This makes it clear why these states are sometimes called *quasiparticle vacuum states* or *quasifree states* in the physics literature. We call a *B-transformation* an element of $\Gamma_0$ written in the form (2.14), (2.15) with the matrices $A$ and $C$ set to zero. It can be shown that every pure spinor can be written as

$$|\psi\rangle = e^{\frac{1}{2}B_{ab}p^a p^b}(v^1_{a_1} p^{a_1})...(v^n_{a_n} p^{a_n})|0\rangle, \tag{2.48}$$

i.e. a Slater determinant transformed by a $B$-transformation.

We can actually identify what the orbit of pure spinors is in the general case. It is clear that the orbit of a vector $|\psi\rangle$ is homeomorphic to the coset space $\Gamma_0/\Gamma_\psi$ where $\Gamma_\psi$ is the stabilizer of $|\psi\rangle$. It is not too difficult to determine $\Gamma_\psi$ for pure spinors, as we will now show. We can write a general element of $\Gamma_0$ in the form described by equations (2.14) and (2.15). We may rewrite this as

$$g = \lambda e^{-\frac{1}{2}\mathrm{Tr}A} e^{A_a{}^b p^a n_b + \frac{1}{2}B_{ab}p^a p^b + \frac{1}{2}C^{ab}n_a n_b}. \tag{2.49}$$

Now without the loss of generality, we can take the $n_a$ to form a basis of annihilation operators corresponding to the pure spinor $|\psi\rangle$, i.e. a basis of $E_\psi$. Then, the $p^a$ operators form a basis of $E^*_\psi$. It should be clear, that in this case the elements fixing $|\psi\rangle$ must have

$$B_{ab} = 0 \text{ and } \lambda = e^{\frac{1}{2}\mathrm{Tr}A}. \tag{2.50}$$

Under the image of (2.9) we obtain matrices in $O(V)$ of the form

$$\left(\begin{array}{c|c} G & 0 \\ \hline 0 & (G^{-1})^T \end{array}\right)\left(\begin{array}{c|c} I & C \\ \hline 0 & I \end{array}\right), \tag{2.51}$$

where $G \in GL(E_\psi) \cong GL(d,\mathbb{C})$ and $C$ is antisymmetric. As $\lambda$ is fixed to a unique value in (2.50), the group formed by these matrices is actually isomorphic to $\Gamma_\psi$. This group is just the semidirect product $\Gamma_\psi = GL(d,\mathbb{C}) \ltimes H(d)$, where $H(d)$ is the $\frac{1}{2}d(d-1)$ dimensional Abelian group of upper triangular matrices of the form

$$\left(\begin{array}{c|c} I & C \\ \hline 0 & I \end{array}\right), \tag{2.52}$$

where $C$ is a $d \times d$ antisymmetric matrix. It follows, that the pure spinor orbit is nothing else than

$$\mathbb{C}^{\times} \times [Spin_0(V)/(GL(d,\mathbb{C}) \ltimes H(d))] \cong \mathbb{C}^{\times} \times [SO(2d,\mathbb{C})/(GL(d,\mathbb{C}) \ltimes H(d))]\,. \tag{2.53}$$

and has dimension $1+\frac{1}{2}d(d-1)$. We note that it is possible to write the orbit of pure spinors in a different form as $\mathbb{C}^{\times} \times SO(2d,\mathbb{R})/U(d)$ by noticing that it is just the quadratic Grassmanian $Q(2d,d)$.

## 2.4 Covariants and invariants of the extended SLOCC group

We proceed by introducing the basic building blocks of covariants and invariants under the extended SLOCC group. These are necessary to describe the orbits of this group and to construct entanglement measures and monotones.

### 2.4.1 The invariant bilinear product

Recall that so far we have not used the usual Hermitian inner product $\langle .|. \rangle$ associating probabilities to states and observables. This is an extra structure on the single particle space $\mathcal{H}$ which we did not need to build up the group $\Gamma_0$ and its action on $\mathcal{F}$. Here, still without fixing the Hermitian inner product, we introduce a natural non-degenerate bilinear form, called the *Mukai pairing* [36], which is very different from the usual Hermitian product, and is canonically defined (up to a constant multiplier) once the Clifford algebra and its vacuum state $|0\rangle$ are fixed. This product has the nice property that it is invariant under the action of the connected component of the spin group $Spin_0(V)$ and *relative invariant* under the action of $\Gamma_0$.

To define this product we use the transposition defined in (2.5). Usually, transposition of operators is defined with respect to a bilinear product. Now we go the other way around. We are looking for a product satisfying[9]

$$(\phi, \gamma\psi) = (\gamma^t\phi, \psi), \tag{2.54}$$

for all $|\phi\rangle, |\psi\rangle \in \mathcal{F}$ and $\gamma \in Cliff(V)$. Before proceeding, recall that for all $O \in Spin_0(V)$ we have $OO^t = I$ (see comments around (2.12)) and hence any such product automatically satisfies

$$(O\phi, O\psi) = (\phi, \psi), \quad \forall O \in Spin_0(V), \tag{2.55}$$

so that it is indeed invariant. Now we show that the condition (2.54) defines the bilinear product uniquely, up to a constant multiplier. To see this, consider the one dimensional $d$-particle subspace $\wedge^d\mathcal{H}$ and pick a vector in

[9] Note that we abuse notation here by using $(.,.)$ which is the same as we used for the inner product (2.2) of $V$. We hope that it will be clear from the context which one we mean.

it, say $|top\rangle = p^1...p^d|0\rangle$. This is unique up to a constant. It is clear that the "$k$-hole" Slater determinants

$$n_{a_1}...n_{a_k}|top\rangle, \quad k = 0, 1, ..., d, \tag{2.56}$$

form a basis of $\mathcal{F}$ in the same way as the $k$-particle states do. Using the requirement (2.54) the pairing of any basis vector with the vacuum reads as

$$(|0\rangle, n_{a_1}...n_{a_k}|top\rangle) = (n_{a_1}|0\rangle, n_{a_2}...n_{a_k}|top\rangle) = 0, \tag{2.57}$$

unless $k = 0$ and hence, by bilinearity, the only state that can have a non-vanishing pairing with the vacuum must be proportional to $|top\rangle$. By the assumption of nondegeneracy, $(|0\rangle, |top\rangle)$ must be a nonzero number. This number is all our freedom in $(.,.)$. To see this it is convenient to expand the states on a Slater basis

$$|\psi\rangle = \Psi|0\rangle, \quad \Psi \equiv \sum_{k=0}^{d} \frac{1}{k!}\psi^{(k)}_{a_1...a_k} p^{a_1}...p^{a_k} \in Cliff(V). \tag{2.58}$$

Using this, we may write

$$\begin{aligned}(\phi, \psi) &= (\Phi|0\rangle, \Psi|0\rangle) \\ &= (|0\rangle, \Phi^t\Psi|0\rangle) \\ &= (|0\rangle, |top\rangle)(\Phi^t\Psi|0\rangle)_{top},\end{aligned} \tag{2.59}$$

where $()_{top}$ denotes the top component of the vector, i.e. the coefficient multiplying $|top\rangle$. This shows that the product between any two states is determined once $(|0\rangle, |top\rangle)$ is fixed.

Notice that the symmetry properties of the pairing under exchange of the arguments are determined by the symmetry of $(|0\rangle, |top\rangle)$. Indeed, running (2.59) the other way around we have

$$(\psi, \phi) = (\Phi^t\Psi|0\rangle, |0\rangle) = (|top\rangle, |0\rangle)(\Phi^t\Psi|0\rangle)_{top}. \tag{2.60}$$

But it is clear that

$$\begin{aligned}(|0\rangle, |top\rangle) &= (|0\rangle, p^1...p^d|0\rangle) \\ &= (p^d...p^1|0\rangle, |0\rangle) \\ &= (-1)^{\frac{d(d-1)}{2}}(p^1...p^d|0\rangle, |0\rangle) \\ &= (-1)^{\frac{d(d-1)}{2}}(|top\rangle, |0\rangle),\end{aligned} \tag{2.61}$$

and hence

$$(\phi, \psi) = (-1)^{\frac{d(d-1)}{2}}(\psi, \phi). \tag{2.62}$$

We see that the product is *symmetric* when $d = 0, 1(\mod 4)$ while *antisymmetric* when $d = 2, 3(\mod 4)$. In the following, we will set $(|0\rangle, |top\rangle) = 1$ for convenience. It is worth to comment on the value of the pairing between states of definite parity. We have seen that $(\psi, \phi)$ is proportional to the top component of $|\psi\rangle^t \wedge |\phi\rangle$ so that depending on the dimension $d$ of the single particle space $\mathcal{H}$ we have

$$\begin{aligned}(\mathcal{F}^+,\mathcal{F}^-) = 0, &\quad \text{when } d \text{ is even,}\\ (\mathcal{F}^+,\mathcal{F}^+) = (\mathcal{F}^-,\mathcal{F}^-) = 0, &\quad \text{when } d \text{ is odd.}\end{aligned} \tag{2.63}$$

To conclude this section, we give an alternative expression for $(.,.)$ which will be useful to perform some manipulations later in the thesis. Define $\Omega = n_1...n_d$. It is easy to see that every state is annihilated by $\Omega$ except the subspace of $|top\rangle$:

$$\Omega|top\rangle = (-1)^{\frac{d(d-1)}{2}}|0\rangle, \tag{2.64}$$

Using this, we easily see that

$$(\phi,\psi)|0\rangle = (-1)^{\frac{d(d-1)}{2}}\Omega\Phi^t\Psi|0\rangle. \tag{2.65}$$

### 2.4.2 The basic covariants

It turns out, that all covariants of the action of the Spin group on a state $|\psi\rangle \in \mathcal{F}$ can be built out from a set of basic covariants bilinear[10] in $|\psi\rangle$.

Let us introduce a general basis of $V$, and denote it with $\{e_I\}_{I=1}^{2d}$. The inner product of $V$ defined by the CAR (2.2) will be

$$\{e_I, e_J\} = 2(e_I, e_J)I \equiv 2g_{IJ}I, \tag{2.66}$$

where the equation above defines the symmetric, non-degenerate matrix $g_{IJ}$. In this basis, the matrices of the orthogonal group $O(V)$ of this inner product satisfy

$$\mathcal{O}^I{}_{I'}\mathcal{O}^J{}_{J'}g_{IJ} = g_{I'J'}, \quad \mathcal{O} \in O(V). \tag{2.67}$$

Specially, de action of the homomorphism (2.9) will be

$$g e_I g^{-1} = (\mathcal{O}_g)^J{}_I e_J, \quad g \in \Gamma. \tag{2.68}$$

As the simplest example of a bilinear, let us consider the vector

$$K(\psi)_I = (\psi, e_I\psi). \tag{2.69}$$

It is clear that

$$\begin{aligned}K(g\psi)_I &= (\psi, g^t e_I g\psi)\\ &= \alpha_g^{-1}(\psi, g^{-1}e_I g\psi)\\ &= \alpha_g^{-1}(\mathcal{O}_{g^{-1}})^J{}_I (K_\psi)_J,\end{aligned} \tag{2.70}$$

where we have used the definition (2.11) of the Clifford norm $\alpha_g$. The map $K : \mathcal{F} \to V$ is an example of an equivariant function. It turns the action of the Clifford group on $\mathcal{F}$ into its action on $V$. This latter is basically the same as the action of $O(V)$ on $V$ which admits a single invariant

$$I_0(\psi) = K(\psi)_I K(\psi)^I, \tag{2.71}$$

[10] When $d = 2$ this is closely related to the result that all covariants of $Spin(3,1)$ are built out of from Dirac bilinears. Note that here we will not have a conjugation in the bilinears.

where the index is raised by the inverse of $g_{IJ}$, which we will denote as $g^{IJ}$. Under the Clifford group this defines a relative invariant:

$$I_0(g\psi) = \alpha_g^{-2} I_0(\psi). \tag{2.72}$$

After discussing this simple example we define a general bilinear to be

$$K(\psi)_{I_1...I_m} = (\psi, e_{I_1}...e_{I_m}\psi), \tag{2.73}$$

which satisfies

$$K(g\psi)_{I_1...I_m} = \alpha_g^{-1}(\mathcal{O}_{g^{-1}})^{J_1}{}_{I_1}...(\mathcal{O}_{g^{-1}})^{J_m}{}_{I_m}K(\psi)_{J_1...J_m}. \tag{2.74}$$

These are the basic covariants that we will work with. Continous relative invariants can be created by multiplying a certain number of $K(\psi)$s and contracting all indices with the invariant tensors $g_{IJ}$ and $\epsilon_{I_1...I_{2d}}$. In addition we can form non-continous invariants as well. By grouping, say $p$ indices of $K(\psi)$ and raising them with $g^{IJ}$ to obtain $K(\psi)^{I_1...I_p}{}_{I_{p+1}...I_m}$ we can form linear maps $(W\oplus W^*)^m \to (W\oplus W^*)^p$, which have a *rank* which is invariant under the action of $\Gamma_0$.

An extremely important subcase is when our state has a fixed parity i.e. when $|\psi\rangle \in \mathcal{F}^{\pm}$. According to (2.63) in this case we have

$$\begin{aligned} K(\psi)_{I_1...I_{2k+1}} &= 0, \quad \text{when } d = \dim\mathcal{H} \text{ is even,} \\ K(\psi)_{I_1...I_{2k}} &= 0, \quad \text{when } d = \dim\mathcal{H} \text{ is odd.} \end{aligned} \tag{2.75}$$

It is also convenient to organize these covariants in the following way. We call an orthogonal basis of $V$ a *Majorana basis*. Denoting the basis elements with $\gamma_I$ to distinguish from the general basis elements $e_I$, the Majorana basis satisfies

$$\{\gamma_I, \gamma_J\} = 2\delta_{IJ}I, \tag{2.76}$$

i.e. $g_{IJ} = \delta_{IJ}$. We can enlist all bilinears as $\{(\psi, \gamma_{I_1}...\gamma_{I_m}\psi)|I_1 < ... < I_m,\ m = 0, ..., 2d\}$ as $\gamma_I^2 = I$. Moreover, in this basis all the covariants are totaly antisymmetric in their indices. This observation makes it possible to organize these covariants into an element of the exterior algebra $\wedge^\bullet V$ underlying the Clifford algebra $Cliff(V)$. Then, the map

$$\begin{aligned} \mathcal{K}: \quad & \mathcal{F} \to \wedge^\bullet V, \\ & |\psi\rangle \mapsto \sum_m \frac{1}{m!}(\psi, \gamma_{I_1}...\gamma_{I_m}\psi)e^{I_1}\wedge ... \wedge e^{I_m}, \end{aligned} \tag{2.77}$$

turns the action of the Clifford group on $\mathcal{F}$ into the (reducible) action of $O(V)$ on $\wedge^\bullet V$. Note that due to the symmetry property (2.62) we have

$$\begin{aligned} (\psi, \gamma_{I_1}...\gamma_{I_m}\psi) &= (-1)^{\frac{d(d-1)}{2}}(\gamma_{I_1}...\gamma_{I_m}\psi, \psi) \\ &= (-1)^{\frac{d(d-1)+m(m-1)}{2}}(\psi, \gamma_{I_1}...\gamma_{I_m}\psi), \quad I_1 < ... < I_m, \end{aligned} \tag{2.78}$$

and hence in a Majorana basis we have $K(\psi)_{I_1...I_m} = (-1)^{\frac{d(d-1)+m(m-1)}{2}}K(\psi)_{I_1...I_m}$. This shows that either $K(\psi)_{I_1...I_m}$ vanishes or $(-1)^{\frac{d(d-1)+m(m-1)}{2}} = 1$. It follows, that when the number of single particle modes is $d = 0, 1 (\mod 4)$ only covariants with $m = 0, 1 (\mod 4)$ are nonzero, while when $d = 2, 3($

$\mod 4)$ only covariants with $m = 2, 3(\mod 4)$ are nonzero. For states with a definite parity, this means that an $m$ index bilinear in the Majorana basis must have $m = d(\mod 4)$ in order to be nonzero.

**Pure spinor conditions**

As a first application of these covariants, let us write down the condition for $|\psi\rangle$ to be a pure spinor. We have seen that any pure spinor can be regarded as a vacuum generating the Fock space $\mathcal{F}$ so we can use identical arguments as in sec. 2.4.1, leading up to the uniqueness of the Mukai pairing, to conclude that

$$(\psi, p^{a_1}...p^{a_m}\psi) = 0, \text{ unless } m = d, \tag{2.79}$$

where $\{p^m\}_{m=1}^d \subset E_\psi^*$ is a basis of creation operators with respect to the quasi-particle vacuum $|\psi\rangle$. It turns out that these are actually sufficient conditions for $|\psi\rangle$ to be pure. It can be shown [19] that the condition

$$K(\psi)_{I_1...I_m} = (\psi, e_{I_1}...e_{I_m}\psi) = 0, \quad m = 1, 2, ..., d-1 \tag{2.80}$$

implies that $|\psi\rangle$ is pure. For $k$-particle states $\frac{1}{k!}\psi_{a_1...a_k}p^{a_1}...p^{a_l}|0\rangle$ one can show that these conditions reduce to the Plücker relations [37]

$$\psi_{a_1[a_2...a_k}\psi_{b_1...b_k]} = 0, \tag{2.81}$$

which are the neccessary and sufficient criteria for a $k$-particle state to be a Slater determinant [38].

### 2.4.3 Coarse graining: nullity and pure index

Though extending the fermionic SLOCC group to $\Gamma_0$ is a good coarse graining of the orbit structure, it is not good enough in the sense that finding the extended SLOCC classes still becomes untractable for large dimensions. To overcome this difficulty, it is a common method to try grouping SLOCC orbits into families based on some SLOCC invariant property that we think is important. We will see some examples of this in chapter 3. Here we would like to mention a way one can coarse grain the classification problem of spinors when the number of dimensions is large, see e.g. [39, 40]. First, we define two invariant quantities.

- We have already met the notion of the annihilator subspace $E_\psi \subset W \oplus W^*$ of a state $|\psi\rangle$ in section 2.3.3. It is just the subspace of creation/annihilation operators which annihilates the state $|\psi\rangle$. We have seen that $\dim E_\psi \leq d$ where we have equality only for pure spinors. This is the reason why pure spinors are called quasi particle vacuum states: they are annihilated by exactly half of the mode operators. Now for a general state the quantity

$$\text{Null}(\psi) = \dim E_\psi, \tag{2.82}$$

  called *nullity*, is clearly an invariant under the action of $\Gamma$. Indeed, we easily see that for $g \in \Gamma$ we have $E_{g\psi} = \mathcal{O}_g E_\psi$, see (2.9). Note that the nullity is easy, though for large dimension computationaly costy to compute. It is just the dimension of the kernel of the $\Gamma$-covariant linear map $W \oplus W^* \to \mathcal{F}^\pm$, $x \mapsto x|\psi\rangle$.

- The other quantity is called *pure index*. It is defined to be the minimal number of pure spinors we need to add up to obtain the state $|\psi\rangle$, i.e.

$$\mathcal{P}(\psi) = \min\{m|\ |\psi\rangle = \sum_{i=1}^{m} |\varphi_i\rangle \text{ where } |\varphi_i\rangle \text{ are pure spinors}\}. \quad (2.83)$$

  To see that this is indeed a $\Gamma$-invariant quantity note that the image of a pure spinor under $g \in \Gamma$ is clearly a pure spinor, so that acting with $g$ on a minimal decomposition of $|\psi\rangle$ shows that $\mathcal{P}(g\psi) \leq \mathcal{P}(\psi)$. Conversely, acting on a minimal decomposition of $g|\psi\rangle$ with $g^{-1}$ shows that $\mathcal{P}(\psi) \leq \mathcal{P}(g\psi)$. It follows that $\mathcal{P}(g\psi) = \mathcal{P}(\psi)$. It is clear that the pure index is a Fock space generalization of the *Slater rank* [41, 42], the minimal number of Slater determinants required to form a fixed particle state through linear combination. The pure index of such a state is clearly less than or equal to its Slater rank. We note that the pure index is hard to compute: to the authors knowledge there is no algorithm to date to compute the pure index of a general state.

The dimensions of the collections of extended SLOCC orbits with a given nullity was computed in [39]. Let us consider the even or odd particle subspace $\mathcal{F}^{\pm}$ of the fermionic Fock space of a $d$ dimensional single particle state space. These have dimension $2^{d-1}$. Let us define the sets

$$\Sigma_d^m = \{|\psi\rangle \in \mathcal{F}^{\pm} | \mathrm{Null}(\psi) = m\}, \quad (2.84)$$

which are clearly a collection of some number of extended SLOCC orbits. The key result of [39] is that

$$\begin{aligned} \dim \Sigma_d^d &= 1 + \frac{1}{2} d(d-1), \\ \Sigma_d^{d-1}, \Sigma_d^{d-2}, \Sigma_d^{d-3}, \Sigma_d^{d-5} &\text{ are empty}, \\ \dim \Sigma_d^m &= m(2d - \frac{1}{2}(3m+1)) + 2^{d-m-1} \text{ for } m = d-4 \text{ and } m < d-5. \end{aligned} \quad (2.85)$$

The first number $\dim \Sigma_d^d$ is the dimension of the pure spinor orbit. This agrees with the dimension coming from the explicit form of the orbit (2.53). We also have $\dim \Sigma_d^0 = 2^{d-1}$ which shows that $\Sigma_d^0$ is *dense* in $\mathcal{F}^{\pm}$: a generic state has nullity 0.

This classification can be refined in principle by adding the pure index which allows one to group the states into families $\Sigma_d^{m,p}$ where $p$ is the pure index of states in $\Sigma_d^{m,p}$. This gives a finite number of families as we clearly have $\mathcal{P}(\psi) \leq 2^{d-1}$, since $\mathcal{F}^{\pm}$ admits bases consisting of pure spinors. However, for this grouping to be useful, we would need a method to calculate $\mathcal{P}(\psi)$.

### 2.4.4 Isomorphisms between $\mathcal{F}^-$ and $\mathcal{F}^+$

When the number of single particle modes $d$ is odd, we have a natural way of identifying the odd and even particle subspaces $\mathcal{F}^- = \wedge^{odd} W$ and $\mathcal{F}^+ = \wedge^{even} W$. This is via particle-hole duality $\wedge^k W \cong \wedge^{d-k} W^*$. Instead of this, in this subsection we would like to consider the linear invertible maps between $\mathcal{F}^-$ and $\mathcal{F}^+$, for general $d$, which are generated by odd elements of the Clifford group $\Gamma$. These maps act as *intertwiners* between the action of the extended SLOCC group $\Gamma_0$ on $\mathcal{F}^-$ and $\mathcal{F}^+$. We may acually write any odd element $g \in \Gamma$ as $g = vg_0$, where $g_0$ is an extended SLOCC transformation $g_0 \in \Gamma_0$ and $v \in V$ is a vector of creation and annihilation operators. It is then sufficient to take $g_0 = I$ and consider the map

$$|\phi\rangle \in \wedge^{odd} W \mapsto v|\phi\rangle \in \wedge^{even} W. \tag{2.86}$$

Now we would like to work out the properties of this intertwiner. In order to do this define the following linear map for each $v \in V$, $v^2 = (v, v)I \neq 0$:

$$\begin{aligned} \mathcal{V}_v &: V \to V, \\ &x \mapsto \mathcal{V}_v(x) = \frac{1}{(v,v)} vxv \end{aligned} \tag{2.87}$$

It is not difficult to see that $\frac{1}{(v,v)} vxv$ is indeed an element of $V$:

$$\frac{1}{(v,v)} vxv = \frac{1}{(v,v)} (\{v,x\}v - xv^2) = \frac{2(v,x)}{(v,v)} v - x. \tag{2.88}$$

Obviously, $\mathcal{V}_v$ is just a reflection. Let us define its matrix as $\mathcal{V}_v(e_I) = \mathcal{V}^J{}_I e_J$ which is then an element of $O(V)$ and squares to the identity. Using this matrix, it is easy to write down how a general covariant transforms under the action of the intertwiner (2.86), given that $v^2 \neq 0$.

$$\begin{aligned} (K(v\phi))_{I_1 \dots I_m} &= (v\phi, e_{I_1} \dots e_{I_m} v\phi) \\ &= \frac{1}{(v,v)^{m-1}} (\phi, ve_{I_1} v \dots ve_{I_m} v\phi) \\ &= (v,v) \mathcal{V}^{I'_1}{}_{I_1} \dots \mathcal{V}^{I'_m}{}_{I_m} (\phi, e_{I'_1} \dots e_{I'_m} \phi) \\ &= (v,v) \mathcal{V}^{I'_1}{}_{I_1} \dots \mathcal{V}^{I'_m}{}_{I_m} (K(\phi))_{I'_1 \dots I'_m}. \end{aligned} \tag{2.89}$$

In particular, all realtive invariants agree up to a rescaling by a power of $(v, v)$. These maps are indeed intertwiners for the acion of the extended SLOCC group between $\mathcal{F}^+$ and $\mathcal{F}^-$

$$vg|\phi\rangle = (vgv^{-1})v|\phi\rangle, \quad g \in \Gamma_0. \tag{2.90}$$

Since vectors with $v^2 \neq 0$ exist in any dimensions (e.g. elements of the Majorana basis), the orbit structure of $\mathcal{F}^+$ and $\mathcal{F}^-$ agrees. We note that these intertwiners are closely related to T-dualities for topological strings, see sec. 4.2.4.

### 2.4.5 Fermions and qudits

An important feature of the introduced notion and formalism of the fermionic extended SLOCC group is that it can be used to learn things about entanglement between distinguishable systems. In order to see this, consider the composite system of $n$ distinguishable subsystems with Hilbert spaces of dimensions $d_1$,...,$d_n$ respectively. The whole system has Hilbert space

$\mathcal{H} = \mathbb{C}^{d_1} \otimes ... \otimes \mathbb{C}^{d_n}$. A pure state of this system is described by the vector

$$|\psi\rangle = \sum_{\mu_1=1}^{d_1} ... \sum_{\mu_n=1}^{d_n} \psi_{\mu_1...\mu_n} |\mu_1\rangle \otimes ... \otimes |\mu_n\rangle \in \mathcal{H}. \tag{2.91}$$

Now consider a system of $n$ fermions with single particle Hilbert space $\mathcal{H}' = \mathbb{C}^{d_1} \oplus ... \oplus \mathbb{C}^{d_n}$ where there is a *sum* between the original Hilbert spaces instead of a product. The fermionic Hilbert space is now $\wedge^n \mathcal{H}'$. Any $|\psi\rangle \in \mathcal{H}$ can be embedded[43] in this space using the map

$$|\psi\rangle \mapsto |P_\psi\rangle = \sum_{\mu_1=1}^{d_1} ... \sum_{\mu_n=1}^{d_n} \psi_{\mu_1...\mu_n} p^{\mu_1} p^{d_1+\mu_2} ... p^{d_1+...+d_{n-1}+\mu_n} |0\rangle. \tag{2.92}$$

This embedding has the nice propierty that it also embeds the SLOCC group of the distinguishable system into the *ordinary* fermionic SLOCC group in a nice way. The SLOCC group of the distinguishable system is $GL(d_1, \mathbb{C}) \otimes ... \otimes GL(d_n, \mathbb{C})$ acting locally as

$$\begin{aligned} \psi_{\mu_1...\mu_n} &\mapsto (g_1)_{\mu_1}{}^{\nu_1} ... (g_1)_{\mu_n}{}^{\nu_n} \psi_{\nu_1...\nu_n}, \\ g_1 \otimes ... \otimes g_n &\in GL(d_1, \mathbb{C}) \otimes ... \otimes GL(d_n, \mathbb{C}). \end{aligned} \tag{2.93}$$

These transformations are implemented via the fermionic SLOCC transformations of (2.42) as

$$G = \begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_n \end{pmatrix} \in GL(\mathcal{H}') \tag{2.94}$$

This shows that states of the embedded system on the same $GL(d_1, \mathbb{C}) \otimes ... \otimes GL(d_n, \mathbb{C})$ orbit are on the same $GL(\mathcal{H}')$ orbit when considered as fermionic states. Hence the entanglement classes of the fermionic system give a coarse graining of the classes of the distinguishable system.

Note that the embedding (2.92) has a nice physical interpretation. Consider $n$ nodes where fermions can be localized and on the $k$th node a fermion can have $d_k$ internal states. These nodes can be energy levels of atoms or nodes in a lattice or anything alike. The subspace of the $n$ fermion Hilbert space $\wedge^n \mathcal{H}'$ defined by (2.92) is just the single occupancy subspace of this node interpretation: where we allow only states where on each node there is exactly one fermion. If we prescribe this condition it is clear that the fermions suddenly become distinguishable and the resulting Hilbert space is just $\mathcal{H}$.

For a further coarse graining we may regard (2.94) as an element of the extended SLOCC group. This allows one to realize more exotic ways of describing qudit systems as part of a fermionic system[P.2, 44]. Indeed, for example take an arbitrary element $h$ of the Clifford group $\Gamma$ and consider the embedding

$$|\psi\rangle \mapsto h|P_\psi\rangle, \tag{2.95}$$

where $|P_\psi\rangle$ is defined in (2.92). The states $h|P_\psi\rangle$ can be rather exotic compared to the previous simple node picture, e.g. they can be superpositions of states with different fermion number. Nevertheless, the original SLOCC transformations of the distinguishable system are clearly implemented as extended SLOCC transformations $hgh^{-1}$, where $g$ is the ordinary fermionic SLOCC transformation given by (2.94). We will see examples of such exotic embeddings in section 3.4.3. For a recent extended discussion, we refer the reader to [44].

## 2.5 The Hermitian inner product

### 2.5.1 Canonical real structure

We have seen that there are many useful structures on the fermionic Fock space which exist solely because of the requirement that the Fock space should represent the canonical anticommutation relations. In addition to these, a physical fermion Hilbert space also posesses a Hermitian inner product[11] $\langle . | . \rangle$ associating probability distributions to states and observables. This inner product originates from an inner product of the single particle state space $\mathcal{H}$, which is from now on rightfully called a Hilbert space. This introduces a real structure on the space $W \oplus W^*$ of creation and annihilation operators: for an orthonormal basis $\langle e^i | e^j \rangle = \delta^i{}_j$ one has $p^i = (n_i)^\dagger$: the creation operators are the adjoints of the annihilation operators. Stated in a more formal way, for every state $v_i | e^i \rangle \in \mathcal{H}$ one can associate a creation operator $p_v = v_i p^i$ and for every dual state $u^i | e_i \rangle \in \mathcal{H}^*$ one can associate an annihilation operator $n_u = u^i n_i$. The inner product introduces an antilinear map $A : \mathcal{H} \to \mathcal{H}^*$ defined as $A(v)(w) = \langle v | w \rangle$. Then the adjoint defined from this inner product satisfies $(n_{A(v)})^\dagger = p_v$.

Recall that on every fermionic Fock space there is a canonical invariant bilinear pairing, which we have described in section 2.4.1. When we have an additional Hermitian inner product, the two can be related giving rise to an antilinear map of $\mathcal{F}$ [P.4]. We define this antilinear automorphism $\chi : \mathcal{F} \to \mathcal{F}$ of the Fock space as[12]

$$\begin{aligned} \chi : & |\phi\rangle \mapsto \chi(|\phi\rangle) \equiv |\tilde{\phi}\rangle, \\ & \langle \tilde{\phi} | \psi \rangle = (\phi, \psi), \quad \forall |\phi\rangle, |\psi\rangle \in \mathcal{F}. \end{aligned} \tag{2.96}$$

We will denote this conjugate state of $|\phi\rangle$ with $\chi(|\phi\rangle) \equiv |\tilde{\phi}\rangle$. Using the form (2.65) with the assumption that we chose a normalized vacuum we can write this as

$$\langle 0 | (\tilde{\Phi})^\dagger \Psi | 0 \rangle = (-1)^{\frac{d(d-1)}{2}} \langle 0 | \Omega \Phi^t \Psi | 0 \rangle, \tag{2.97}$$

from where we can read off the action of $\chi$ on an arbitary state:

$$|\tilde{\phi}\rangle = (-1)^{\frac{d(d-1)}{2}} (\Omega \Phi^t)^\dagger | 0 \rangle. \tag{2.98}$$

[11] We have used the Dirac ket notation for states so far. We will use the bra-ket notation for the Hermitian inner product from now on. We can think about this as now we are allowed to flip ket vectors to obtain bra vectors.

[12] We note that this conjugation operation was first considered in [45, 41] in the special case of two fermions with four single particle states (see sec. 3.2). Here we give its definition for arbitrary fermionic states. We now see that a self dual basis, called a magic basis in [41], is nothing else but a basis consisting of *Majorana spinors*.

Now it is easy to see that $\Omega^\dagger = n_d^\dagger ... n_1^\dagger = (-1)^{\frac{d(d-1)}{2}} n_1^\dagger ... n_d^\dagger$ and hence $(-1)^{\frac{d(d-1)}{2}} \Omega^\dagger |0\rangle = |top\rangle$. One is left with

$$|\tilde{\phi}\rangle = (\Phi^t)^\dagger |top\rangle. \tag{2.99}$$

The adjoint conjugates every amplitude in $\Phi$ and changes creation operators to annihilation ones with reversing the order. The transpose restores the original order i.e. for

$$\Phi = \phi^{(0)} + \phi_i^{(1)} n_i^\dagger + \frac{1}{2} \phi_{ij}^{(2)} n_i^\dagger n_j^\dagger + ... \tag{2.100}$$

one has

$$(\Phi^t)^\dagger = \bar{\phi}^{(0)} + (\bar{\phi}^{(1)})^i n_i + \frac{1}{2} (\bar{\phi}^{(2)})^{ij} n_i n_j + ... \,. \tag{2.101}$$

This shows that $|\tilde{\phi}\rangle$ is short of a particle-hole dual of $|\phi\rangle$ with an additional complex conjugation: the complex conjugate state is annihilated out of the fully filled state. This picture is reassured if one calculates the action on Slater determinant states:

$$\chi(n_{i_1}^\dagger ... n_{i_k}^\dagger |0\rangle) = (-1)^{\frac{k(k-1)}{2}} \frac{1}{(d-k)!} \epsilon_{i_1 ... i_k j_{k+1} ... j_d} n_{j_{k+1}}^\dagger ... n_{j_d}^\dagger |0\rangle. \tag{2.102}$$

From e.q. (2.98) it is easy to see that $\chi^2 = (-1)^{\frac{d(d-1)}{2}}$. It follows that either $\chi$ or $i\chi$ is an antilinear involution and hence a real structure. Since $\chi$ is uniquely defined by the Hermitian inner product it follows that fixing a Hermitian inner product on $\mathcal{H}$ is equivalent with fixing a real structure on $\mathcal{F}$.

The particle-hole picture for this conjugation can also be seen from the matrix elements of the one particle reduced density matrix, see (2.32). Using that $\chi^{-1} = (-1)^{\frac{d(d-1)}{2}} \chi$ and e.q. (2.62) it is easy to see that $\chi$ is antiunitary: $\langle \tilde{\phi} | \tilde{\psi} \rangle = \langle \psi | \phi \rangle$. Using this the following result is straightforward

$$\begin{aligned} \langle \tilde{\psi}_1 | A \tilde{\psi}_2 \rangle &= (\psi_1, A \tilde{\psi}_2) \\ &= (A^t \psi_1, \tilde{\psi}_2) \\ &= \langle \chi(A^t \psi_1) | \chi(\psi_2) \rangle \\ &= \langle \psi_2 | A^t \psi_1 \rangle, \quad \forall A \in Cliff(V), \end{aligned} \tag{2.103}$$

hence the expectation values of operators in the conjugated state are just the expectation values of the Clifford transpositions of (2.5). Since $(n_i^\dagger n_j)^t = n_j n_i^\dagger = \delta^i{}_j - n_i^\dagger n_j$ we indeed have a particle-hole relation

$$\tilde{\rho}^i{}_j = \langle \tilde{\psi} | n_i{}^\dagger n_j \tilde{\psi} \rangle = \delta^i{}_j - \langle \psi | n_i{}^\dagger n_j \psi \rangle = \delta^i{}_j - \rho^i{}_j, \tag{2.104}$$

between the matrix elements of the one particle RDM of the state and its conjugate.

### 2.5.2 Unitary Bogoliubov transformations

Now let us consider unitary elements of the Clifford group with respect to the Hermitian inner product, i.e. $u \in \Gamma$ such that

$$\langle u\psi | u\phi \rangle = \langle \psi | \phi \rangle. \tag{2.105}$$

A consequence of this relation is that that $(u n_a u^{-1})^\dagger = u(n_a^\dagger)u^{-1}$ i.e. the Bogoliubov transformed creation operators remain the adjoints of the transformed annihilation operators. Let us restrict our attention to the identity component of the spin group $Spin_0(V)$. Unitarity of $u = e^s \in Spin_0(V)$ translates to anti-Hermicity of the generator $s$ given explicitly in (2.15) One can check that this means that the matrix $A^a{}_b$ must be anti-Hermitian and the matrices $B_{ab}$, $C^{ab}$ must satisfy $(B^\dagger)_{ab} = -C^{ab}$, thus the group of unitary Bogoliubov transformations is restricted to $SO(W \oplus W^*) \bigcap SU(W \oplus W^*)$. Let us identify what this group is. With the inner product matrix $g$ of (2.66) the $SO$ property of the matrix $\mathcal{O}_u$ defined via (2.9) reads as $\mathcal{O}_u g \mathcal{O}_u^T = g$. On the other hand, clearly $\mathcal{O}_u^\dagger = \mathcal{O}_u^{-1}$. Combine the two to get the following reality condition

$$\bar{\mathcal{O}}_u = g \mathcal{O}_u g. \tag{2.106}$$

Now we shall prove that the subgroup of $SO(W \oplus W^*) = SO(2d, \mathbb{C})$ satisfying the above reality condition is just the compact real form $SO(2d, \mathbb{R})$. The inner product matrix $g_{IJ}$ in the basis of orthonormal creation/annihilation operators $\{p^a, n_b\} = \delta^a_b$ reads as

$$g = \left( \begin{array}{c|c} 0 & I \\ \hline I & 0 \end{array} \right), \tag{2.107}$$

where $I$ is now the $d \times d$ identity matrix. Define the $2d \times 2d$ unitary matrix

$$N = \frac{1}{\sqrt{2}} \left( \begin{array}{c|c} I & iI \\ \hline I & -iI \end{array} \right). \tag{2.108}$$

which diagonalizes $g$

$$gN = N g_0, \tag{2.109}$$

where we have defined

$$g_0 = \left( \begin{array}{c|c} I & 0 \\ \hline 0 & -I \end{array} \right). \tag{2.110}$$

Now we show that the map $\mathcal{O} \mapsto S = N^\dagger \mathcal{O} N$ is a group isomorphism from the subgroup $\{\mathcal{O} \in SO(2d, \mathbb{C}) | \mathcal{O}^* = g \mathcal{O} g\}$ to the group $SO(2d, \mathbb{R})$. Indeed, it is very easy to directly check that $gN^* = N$ and hence $N^T = N^\dagger g$. We have

$$SS^T = N^\dagger \mathcal{O} N N^T \mathcal{O}^T N^* = N^\dagger g N^* = I, \tag{2.111}$$

hence for every unitary $\mathcal{O}$ satisfying $\mathcal{O} g \mathcal{O}^T = g$, $S$ is an element of $SO(2d, \mathbb{R})$. For the converse, we have to check whether $\mathcal{O} = NSN^\dagger$ is a unitary matrix with the condition $\mathcal{O} g \mathcal{O}^T = g$ satisfied for all $S \in SO(2d, \mathbb{R})$. It is obvious that $\mathcal{O}$ is unitary. For the other write

$$\mathcal{O} g \mathcal{O}^T = NSN^\dagger g N^* S^T N^T, \tag{2.112}$$

but again $gN^* = N$ so

$$\mathcal{O}g\mathcal{O}^T = NSN^\dagger NS^TN^T = NSS^TN^T = NN^T = g. \tag{2.113}$$

Notice that by restricting our attention to particle number conserving transformations i.e. setting $B_{ab} = C^{ab} = 0$ in (2.15) we end up with the usual local unitary transformations (LU) corresponding to invertible LOCC transformations. Indeed, an unitary element $u = e^{\frac{1}{2}A_a{}^b[p^a,n_b]}$ of the spin group has the following action on creation operators

$$up^au^{-1} = (e^A)_b{}^a p^b. \tag{2.114}$$

As $A_b{}^a$ is an anti-Hermitian, $(e^A)_b{}^a$ is an unitary matrix.

### 2.5.3 Extended density matrix

Now recall from secion 2.2.2 that the matrix elements of the $k$ particle reduced density matrix of a state $|\psi\rangle$ are

$$(\rho^{(k)})_{j_1\dots j_k}{}^{i_1\dots i_k} = \langle\psi|p^{i_1}\dots p^{i_k}n_{j_1}\dots n_{j_k}\psi\rangle. \tag{2.115}$$

For a state with a fixed number of particles only expectation values of the product of the same number of creation and annihilation operators is non-vanishing. However, if $|\psi\rangle$ is a general element of $\mathcal{F}^\pm$ this set of matrix elements misses information contained in expectation values like $\langle\psi|n_a n_b\psi\rangle$. It is clear how to add this information to $\rho$: we need to extend our definitions of $k$-particle operators of section 2.2.2 to incorporate all operators which are combinations of at most $k$-tuples of arbitrary single particle mode operators:

$$\mathcal{O}^{(k)} = \{A_{I_1\dots I_{2k}}e^{I_1}\dots e^{I_{2k}}\}. \tag{2.116}$$

Notice that there is no longer a need to take the union of bilinears with $2l \leq 2k$ operators as with an appropriate choice of the $A_{I_1\dots I_{2k}}$ and the use of $\{e_I, e_J\} = 2g_{IJ}I$ we can produce operators which are actually tuples of $2l \leq 2k$ number of mode operators. The reason for allowing only an even number of modes comes from the requirement that physical operators should not change the parity (statistics) of a state.

Defining the sets $\mathcal{O}^{(k)}$ as $k$-particle operators is in fact even more natural in the Clifford algebra setting. This is because a Clifford algebra does not have a $\mathbb{Z}_{2d}$ grading into rank $2d$ objects and hence the sets of $k$-particle operators defined in section 2.2.2 depend heavily on the choice of basis. Instead of this, all Clifford algebras have a natural *filtration*

$$\mathcal{O}^{(0)} \subset \mathcal{O}^{(1/2)} \subset \dots \subset \mathcal{O}^{(d-\frac{1}{2})} \subset \mathcal{O}^{(d)} \equiv Cliff(V), \tag{2.117}$$

into elements containing linear combination of at most $2k$ modes. This is independent of the choice of basis.

Now the requirement

$$\langle\psi|A\psi\rangle = \mathrm{Tr}(\rho^{(k)}A), \quad \forall A \in \mathcal{O}^{(k)}, \tag{2.118}$$

fixes $\rho^{(k)}$ to live in $\mathcal{O}^{(k)}$ with coefficients

$$(\rho^{(k)})_{I_1...I_{2k}} = \langle\psi|e_{I_1}...e_{I_{2k}}\psi\rangle, \tag{2.119}$$

which contains the ordinary $k$ particle density matrix elements and any additional anomalous pairing with $2k$ modes. Following [34], we will call this object an *extended reduced density matrix*[13]. For example, the single particle extended reduced density matrix is

$$\rho^{(1)} = \begin{pmatrix} \langle\psi|p^a n_{a'}\psi\rangle & \langle\psi|p^a p^{b'}\psi\rangle \\ \langle\psi|n_b n_{a'}\psi\rangle & \langle\psi|n_b p^{b'}\psi\rangle \end{pmatrix}. \tag{2.120}$$

Now there is a way of relating this extended density matrix to mode entanglement by a slight modification of the argument at the end of section 2.2.2, due to [34]. We consider the same quantity as there, the average entanglement entropy of a single mode with the rest

$$\sum_{j=1}^{d} E_j(\psi) = \sum_{j=1}^{d} h(\langle\psi|p^j n_j\psi\rangle), \tag{2.121}$$

and minimize it again, but instead of minimizing only over different choice of basis vectors in $W$, now we minimize it over different choice of basis vectors in $V = W \oplus W^*$, i.e. we allow Bogoliubov transformations to reduce the value. It is clear that the minimum is reached at the basis diagonalizing the extended reduced density matrix $\rho^{(1)}$. Indeed, we may rewrite (2.121) as

$$\sum_{j=1}^{d} E_j(\psi) = \sum_{I}^{2d} f(\langle\psi|e^I e_I\psi\rangle), \tag{2.122}$$

where $f(x) = -x\log x$, $e^I = (p^a, n_b)$ and the index is lowered with the inverse of the inner product matrix $g^{IJ}$ satisfying $\{e^I, e^J\} = g^{IJ} I$ as usual. Using concavity of $f$ it is then manifest that

$$\sum_{I=1}^{2d} f(\langle\psi|e^I e_I\psi\rangle) \geq \sum_{I,K,L=1}^{2d} (\mathcal{O}^\dagger)^I{}_K \mathcal{O}^L{}_I f(\delta_L^K \Lambda_K) \equiv \sum_{K=1}^{2d} f(\Lambda_K), \tag{2.123}$$

where $\mathcal{O}$ is a unitary Bogoliubov transformation, described in section 2.5.2, diagonalizing $(\rho^{(1)})^I{}_J$, while $\Lambda_K$ are the eigenvalues of $\rho^{(k)}$. It is clear, that these latter come in pairs: $\Lambda_k + \Lambda_{k+d} = 1$, $k = 1, ..., d$. This again gives a state dependent way of connecting mode entanglement to entanglement based on classification under the extended SLOCC group: the latter can be regarded as describing the average mode entanglement between a single eigenmode of the extended density matrix with the rest of the eigenmodes.

We note that by studying the extended density matrix $\rho^{(2)}$, the authors of [34] recover some of the extended SLOCC invariants, like the ones that will be discussed in section 3.2. This is not a coincidence, as we describe in the following section.

[13] We could annihilate the words extended and reduced with each other and call this object a density matrix, but that would only increase confusion.

### 2.5.4 SLOCC covariants and reduced density matrices

Notice the following duality. Invariants and covariants of the local unitary group are calculated from the Hermitian inner product. These in particular involve elements of the reduced density matrices, extended reduced density matrices and also their Rényi entropies. On the other hand, SLOCC covariants and invariants are calculated from the invariant bilinear pairing of (2.65). One then sees that the role of the conjugation (2.96) is to allow one the calculation of SLOCC invariants with the use of the Hermitian inner product. We now make this more explicit and relate reduced density matrix elements to extended SLOCC covariants[P.4]. This is done via using Fierz identities for spinors. We will later see in secion 3.4.4 that the entanglement monogamy equations of [27] for three qubits are direct consequences of such a Fierz identity.

To derive the general relations between reduced density matrix elements and SLOCC covariants, first note that there are two type of projections defined from the two inner products. The first is the usual one defining density matrices from pure states:

$$\begin{aligned} P_\psi &: \mathcal{F} \to \mathcal{F}, \\ &|\phi\rangle \mapsto \langle\psi|\phi\rangle|\psi\rangle. \end{aligned} \tag{2.124}$$

We can write this as $P_\psi = |\psi\rangle\langle\psi|$. The other one is defined from the invariant bilinear product:

$$\begin{aligned} \tilde{P}_\psi &: \mathcal{F} \to \mathcal{F}, \\ &|\phi\rangle \mapsto (\psi,\phi)|\psi\rangle. \end{aligned} \tag{2.125}$$

From the definition (2.96) one sees that this can be written as $\tilde{P}_\psi = |\psi\rangle\langle\tilde{\psi}|$. On the other hand from (2.65) we have $\tilde{P}_\psi = (-1)^{\frac{d(d-1)}{2}}\Psi\Omega\Psi^t$. We would now like an expansion of $P_\psi$ and $P'_\psi$ in terms of density matrix elements and SLOCC covariants respectively. To obtain this we employ that since $Cliff(W\oplus W^*) \cong End(\mathcal{F})$ the trace of an arbitary Clifford algebra element is well-defined. Let $\{|\vartheta_i\rangle\}_{i=1}^{2^d}$ be a basis of $\mathcal{F}$ and $|\vartheta^*{}_i\rangle$ be the dual basis with respect to the product $\langle.|.\rangle$ while $|\tilde{\vartheta}_i^*\rangle$ is clearly the dual basis with respect to the product $(.,.)$. We have

$$\mathrm{Tr}A = \sum_i \langle\vartheta^*{}_i|A\vartheta_i\rangle = \sum_i (\tilde{\vartheta}_i^*, A\vartheta_i), \quad A \in Cliff(W\oplus W^*). \tag{2.126}$$

It is obvious that this trace with the previously introduced projections satisfies

$$\mathrm{Tr}(AP_\psi) = \langle\psi|A\psi\rangle, \tag{2.127}$$

and

$$\mathrm{Tr}(A\tilde{P}_\psi) = (\psi, A\psi). \tag{2.128}$$

Now choose a basis $\{\theta_i\}_{i=1}^{2^{2d}}$ of $Cliff(W\oplus W^*)$ and denote its trace-dual with $\theta^i$ i.e. $\mathrm{tr}\theta^i\theta_j = \delta^i{}_j$. A self-dual basis for example can be obtained from a Majorana basis (see (3.42)) of $W\oplus W^*$ as $\{\frac{1}{k!}\gamma_{[I_1}\ldots\gamma_{I_k]}|\{I_1,\ldots,I_k\} \subseteq \{1,\ldots 2d\},\ 0 \le k \le 2d\}$. Using such a basis to expand $P_\psi$ and $\tilde{P}_\psi$ and e.q. (2.127) along with (2.128) with $A = \theta_i$ it is straightforward to get the expansions

$$\begin{aligned} P_\psi &= |\psi\rangle\langle\psi| = \sum_i \langle\psi|\theta_i\psi\rangle\theta^i, \\ \tilde{P}_\psi &= |\psi\rangle\langle\tilde{\psi}| = \sum_i (\psi, \theta_i\psi)\theta^i. \end{aligned} \tag{2.129}$$

Notice that the coefficients of $P_\psi$ are nothing else than matrix elements of the extended reduced density matrices, see (2.119), while the coefficients of $\tilde{P}_\psi$ are the extended SLOCC covariant bilinears of section 2.4.2. Using these expansions we can easily derive our final results

$$(\psi, A\psi)\overline{(\psi, B\psi)} = \langle B\psi|\tilde{\psi}\rangle\langle\tilde{\psi}|A\psi\rangle = \sum_i \langle\psi|B^\dagger\theta^i A\psi\rangle\langle\tilde{\psi}|\theta_i\tilde{\psi}\rangle, \tag{2.130}$$

and

$$\langle\psi|A\psi\rangle\langle\tilde{\psi}|B\tilde{\psi}\rangle = \sum_i (\psi, \theta_i\psi)\overline{(\psi, (A\theta^i B)^\dagger\psi)}, \tag{2.131}$$

valid for all $A, B \in Cliff(W \oplus W^*)$. Physically speaking the first relation expresses that a SLOCC covariant multiplied by its conjugate can be expanded with the use of reduced density matrix elements of the state and its conjugate or "spin-flipped" dual, while the second relation is basically the inverse of the first. Mathematically speaking these equations relate spinor bilinears between the spinor $|\psi\rangle$ and its conjugate $|\tilde{\psi}\rangle$ with the bilinears of these spinors with themselves. These kinds of relations between two spinors are called Fierz identities[46] in the theory of spinors.

# Chapter 3

# Results on SLOCC classification of certain fermionic systems

The previously introduced classification problems are in general very hard. In this chapter, we will review the available results, which are for $d \leq 9$ single particle modes, and comment on some of their physical interpretations.

There is an interesting common point in all of the mode numbers discussed bellow. For $d \leq 8$ the orbit classification is entirely governed by the covariant $K(\psi)_{I_1 \ldots I_{d-4}}$ with $d-4$ indices.

## 3.1 Two and three modes

Let us discuss briefly the $d=2$ and $d=3$ cases, which both turn out to be trivial. In the case of $d=2$ the single particle space is $\mathcal{H}=\mathbb{C}^2$. The full Fock space is $\mathbb{C} \oplus \mathbb{C}^2 \oplus \wedge^2\mathbb{C}^2$ with even and odd components:

$$\begin{aligned} \mathcal{F}^+ &= \mathbb{C} \oplus \wedge^2\mathbb{C}^2, \\ \mathcal{F}^- &= \mathbb{C}^2, \end{aligned} \tag{3.1}$$

which are simply one qubit Hilbert spaces. Take

$$\begin{aligned} |\phi\rangle &= \phi_0|0\rangle + \phi_t p^1 p^2|0\rangle \in \mathcal{F}^+, \\ |\psi\rangle &= \psi_i p^i|0\rangle \in \mathcal{F}^-. \end{aligned} \tag{3.2}$$

In the case of $|\psi\rangle$ the generators (2.15) act as $C|\psi\rangle = B|\psi\rangle = 0$ thus only the $A$ generators have nontrivial action on $\mathcal{F}^-$. This means that we only have to consider the action of $GL(2,\mathbb{C}) \subset Spin(4,\mathbb{C})$ which is just the SLOCC group of the trivial one-qubit system.

Note that both $|\phi\rangle$ and $|\psi\rangle$ are pure spinors. Indeed, we see that $|\psi\rangle$ is manifestly of the form (2.48), while $|\phi\rangle$ can be written as

$$|\phi\rangle = \phi_0 \text{exp}\left(\frac{\phi_t}{\phi_0}p^1p^2\right)|0\rangle, \tag{3.3}$$

hence both $\mathcal{F}^+$ and $\mathcal{F}^-$ forms a single orbit under the extended SLOCC group.

Now let us briefly consider the three state system $\mathcal{H}=\mathbb{C}^3$. It is clear that we only need to consider one parity due to particle-hole duality. This is always true when $d$ is odd. Let us pick $\mathcal{F}^+$ and write a general vector as

$$|\phi\rangle = (\phi_0 + \phi_1 p^2 p^3 + \phi_2 p^1 p^3 + \phi_3 p^1 p^2)|0\rangle. \tag{3.4}$$

It is again clear that this can be written as

$$|\phi\rangle = \phi_0 \exp\left(\frac{1}{\phi_0}(\phi_1 p^2 p^3 + \phi_2 p^1 p^3 + \phi_3 p^1 p^2)\right)|0\rangle, \tag{3.5}$$

and hence every state is a pure spinor.

Moving to ordinary SLOCC classification the only case not immediately trivial is the case of the two particle subspace, where a general state reads as $\frac{1}{2!}\phi_{ij}p^i p^j|0\rangle$. But all $3\times 3$ antisymmetric matrices have a degenerate eigenvector and hence using the canonical form of the $3\times 3$ array $\phi_{ij}$ we see that $|\phi\rangle$ can be written as a wedge product of two vectors and hence it is always a Slater determinant.

## 3.2 Four modes

### 3.2.1 Extended SLOCC classification

Here the single particle space is $\mathcal{H} = \mathbb{C}^4$. In the case of even particle number $\mathcal{F}^+$ we parametrize a state as

$$|\phi\rangle = \eta|0\rangle + \frac{1}{2}\xi_{ab}p^a p^b|0\rangle + \frac{1}{4!}\rho\epsilon_{abcd}p^a p^b p^c p^d|0\rangle. \tag{3.6}$$

The covariant, relevant for the classification, is the one with $d-4=0$ indices, i.e. the quadratic invariant

$$(\phi,\phi) = 2\eta\rho - 2\mathrm{Pf}(\xi), \tag{3.7}$$

where the Pfaffian of the antisymmetrix matrix $\xi$ is $\mathrm{Pf}(\xi) = \frac{1}{2^2 2!}\epsilon^{ijkl}\xi_{ij}\xi_{kl}$. The orbit of pure spinors has $(\phi,\phi)=0$. Indeed, the additional bilinears in the pure spinor conditions (2.80) in this case are

$$\begin{aligned} K(\phi)_{IJ} &= (\phi, [\frac{1}{2}[e_I,e_J] + \frac{1}{2}\{e_I,e_J\}]\phi) \\ &= g_{IJ}(\phi,\phi), \end{aligned} \tag{3.8}$$

as the commutator piece gives zero due to the symmetry of the product. We see that $(\phi,\phi)=0$ implies the pure spinor conditions on $|\phi\rangle$.

Now let us move on to $\mathcal{F}^-$. Take an element $|\psi\rangle$ of $\mathcal{F}^-$ parametrized as

$$|\psi\rangle = v_a p^a|0\rangle + \frac{1}{3!}P_{abc}p^a p^b p^c|0\rangle. \tag{3.9}$$

For this we have

$$(\psi,\psi) = \frac{1}{3}v_a P_{bcd}\epsilon^{abcd}, \tag{3.10}$$

showing us in particular that for a three fermion state with $v_i = 0$ no entanglement can occur. This can also be understood to be a consequence of the duality $\wedge^3\mathbb{C}^4 \cong \wedge^1\mathbb{C}$. The spaces $\mathcal{F}^\pm$ contain two Spin orbits respectively (other than the zero vector) [35] one with $(\psi,\psi) = 0$ (the orbit of pure spinors) and one with $(\psi,\psi) \neq 0$.

### 3.2.2 SLOCC classification

Consider two fermion states which are of the form (3.6) with $\eta = \rho = 0$. The pure spinor condition $(\phi,\phi) = -2\mathrm{Pf}(\xi) = 0$ requires the antisymmetric matrix $\xi_{ab}$ to be degenerate which in four dimensions is neccesary and sufficient for the state to be separable. Again, there are two orbits under SLOCC, one with $\mathrm{Pf}(\xi) = 0$ and one with $\mathrm{Pf}(\xi) \neq 0$. Note that for normalized states the quantity $0 \leq 64|\mathrm{Pf}(\xi)| \leq 1$ is just the canonical entanglement measure used for two fermions with four single particle states[42].

### 3.2.3 Two qubits and the Wootters spin-flip

An arbitrary two-qubit state

$$|\chi\rangle = \sum_{i,j\in\{0,1\}} \chi_{ij}|i\rangle \otimes |j\rangle \tag{3.11}$$

can be embedded into this fermionic system via (2.92) as[43, 31]

$$|\xi_\chi\rangle = \sum_{i,j\in\{0,1\}} \chi_{ij} p^{i+1} p^{j+3}|0\rangle. \tag{3.12}$$

Under this embedding the entanglement measure $64|\mathrm{Pf}(\xi)| = 2|\det\chi|$ is just the usual pure state concurrence (1.4). We can also express the somewhat arbitrary spin-flipp operation of (1.12), required to calculate the two qubit concurrence, in the fermionic language. The two body density matrix $\rho : \wedge^2\mathbb{C}^4 \to \wedge^2\mathbb{C}^4$ defines an element of the Clifford algebra of four modes as

$$\rho = \rho_{ab}{}^{cd} n_a^\dagger n_b^\dagger |0\rangle\langle 0| n_c n_d, \tag{3.13}$$

where $|0\rangle\langle 0| = \prod_{a=1}^4 n_a n_a^\dagger$ Now let us determine what happens with the matrix elements of $\rho$ under the Clifford transposition (2.5). To do this, we use (2.103) and (2.102) to arrive at

$$\begin{aligned} \rho^t &= \tilde{\rho}_{ab}{}^{cd} n_a^\dagger n_b^\dagger |0\rangle\langle 0| n_c n_d, \\ \tilde{\rho}_{ab}{}^{cd} &= \frac{1}{4}\epsilon_{aba'b'}\bar{\rho}^{a'b'}{}_{c'd'}\epsilon^{cdc'd'}. \end{aligned} \tag{3.14}$$

This dual fermionic density matrix was first considered in [41] in the context of entanglement[1]. Here we have shown that it is nothing else but the Clifford transposition of the original density matrix. The two qubit density matrices are recovered by considering fermionic density matrices restricted to the single occupancy subspace. Then this expression recovers (1.12). This means that the spin-flip operation naturally descends from the

[1] The entanglement of formation for mixed states of this system can be calculated from the eigenvalues of $\rho\rho^t$ in a similar way as it is done for two qubits, see [41]. We point out here that $\rho\rho^t$ is a covariant object under the action of $\Gamma_0$ while $\rho$ and $\rho^t$ alone are not.

Clifford transposition of (2.5). For pure states one has

$$\rho = |\psi\rangle\langle\psi|, \quad \rho^t = |\tilde{\psi}\rangle\langle\tilde{\psi}|, \tag{3.15}$$

where $|\psi\rangle$ is the conjugate state of (2.96). We have

$$|(\psi, \psi)|^2 = \mathrm{Tr}\rho\rho^t, \tag{3.16}$$

which is a special case of the Fierz identity (2.130) relating density matrix elements to SLOCC covariants, with $A = B = I$.

## 3.3 Five modes

### 3.3.1 Extended SLOCC classification

We note again, that in the case of an odd dimensional single particle space $\mathcal{F}^+$ is dual to $\mathcal{F}^-$ so one only has to consider one of them. We parametrize a state from $\mathcal{F}^+$ as

$$|\phi\rangle = \eta|0\rangle + \frac{1}{2!}\xi_{ab}p^a p^b|0\rangle + \frac{1}{4!}\chi^e\epsilon_{eabcd}p^a p^b p^c p^d|0\rangle. \tag{3.17}$$

The relevant covariant is the one with $d - 4 = 1$ indices, i.e. $K(\phi)_I = (\phi, e_I\phi)$. A short calculation shows that the components of $K(\phi)$ are

$$K(\phi)_a = (\phi, n_a\phi) = 2\xi_{ab}\chi^b, \quad K(\phi)^a = (\phi, p^a\phi) = \frac{1}{12}\eta\chi^a - \frac{1}{4}\xi_{bc}\xi_{de}\epsilon^{abcde}, \tag{3.18}$$

and by construction it transforms as an $SO(10, \mathbb{C})$ vector under $Spin(10, \mathbb{C})$. A quartic invariant can be constructed as $K(\phi)_I K(\phi)^I$ but this turns out to be identically zero. However, $\mathcal{F}^+$ consists of two orbits[35] one with $K(\phi)_I = 0$ (the orbit of pure spinors) and one with $K(\phi)_I \neq 0$.

### 3.3.2 SLOCC classification

The only nontrivial case is the one of two fermions with five modes. The case of three fermions is obviously dual to this. We set $\eta = 0$ and $\chi^e = 0$ in (3.17). These states are then single Slater determinants if $K(\phi)^a = \frac{1}{4}\xi_{bc}\xi_{de}\epsilon^{abcde} = 0$. The antisymmetric coefficient matrix $\xi_{ab}$ can have rank at most 4 which is decreased to 2 if this quantity vanishes. These are the only two SLOCC orbits.

## 3.4 Six modes

### 3.4.1 Extended SLOCC classification

Here $\mathcal{H} = \mathbb{C}^6$ and the Fock space is of dimension 64. Let us begin with the 32 dimensional even particle subspace $\mathcal{F}^+$.

We parametrize a general state with two complex scalars $\eta$, $\xi$ and two antisymmetric $6\times 6$ complex matrices $y$ and $x$ in the following way

$$\begin{aligned}|\phi\rangle =&\eta|0\rangle + \frac{1}{2!}y_{ab}p^{ab}|0\rangle + \frac{1}{2!4!}x^{ab}\epsilon_{abcdef}p^{cdef}|0\rangle \\ &+ \frac{1}{6!}\xi\epsilon_{abcdef}p^{abcdef}|0\rangle,\end{aligned} \tag{3.19}$$

where we have introduced the shorthand notation $p^{ab\ldots z} = p^a p^b \ldots p^z$. The covariant governing the classification is the one with $d-4=2$ indices, i.e. $K(\phi)^I{}_J$. It has components

$$K(\phi)^I{}_J = \begin{pmatrix} [A_\phi]^i{}_j & [B_\phi]_{i'j} \\ [C_\phi]^{ij'} & -[A_\phi]^{j'}{}_{i'} \end{pmatrix}, \tag{3.20}$$

where

$$\begin{aligned}[A_\phi]^i{}_k &= (\phi, p^i n_k \phi) = 2x^{ia}y_{ak} - \left(\frac{1}{2}\mathrm{Tr}(xy) + \eta\xi\right)\delta^i{}_k, \\ [B_\phi]_{jk} &= (\phi, n_j n_k \phi) = \frac{1}{4}x^{ab}x^{cd}\epsilon_{abcdjk} - 2\xi y_{jk}, \\ [C_\phi]^{il} &= (\phi, p^i p^l \phi) = \frac{1}{4}y_{ab}y_{cd}\epsilon^{abcdil} - 2\eta x^{il}.\end{aligned} \tag{3.21}$$

Notice that $K(\phi)_{IJ}$, being a two index antisymmetric covariant (recall that $(\phi,\phi)=0$ due to the antisymmetry of the product in $d=6$), is an element of the Lie algebra of $Spin(V)$. For this reason, the map $|\phi\rangle \mapsto K(\phi)_{IJ}$ is sometimes refered to as the *moment map*. An easy calculation shows that we have a non-trivial quartic invariant

$$\begin{aligned}q_{\text{even}}(\phi) &= \frac{1}{12}K(\phi)^I{}_J K(\phi)^J{}_I \\ &= \left(\eta\xi + \frac{1}{2}\mathrm{Tr}(yx)\right)^2 + 4\eta\mathrm{Pf}(x) + 4\xi\mathrm{Pf}(y) - \frac{1}{2}\left((\mathrm{Tr}(yx))^2 - 2\mathrm{Tr}(yxyx)\right),\end{aligned} \tag{3.22}$$

where we have introduced the Pfaffian of an antisymmetric matrix $\mathrm{Pf}(x) = \frac{1}{3!2^3}\epsilon_{abcdef}x^{ab}x^{cd}x^{ef}$.

It turns out [35] that there are four distinct orbits of the extended SLOCC group $\Gamma_0$. These are closely related to SLOCC classes of three qubits, as we will soon see. Because of this reason, we call the classes separable, biseparable, W and GHZ. Compared to three qubits, the difference is that there is only one orbit in the biseparable class[2]. We can list representatives from all of the classes. Consider a state parametrized by four complex numbers $a, b, c, d$ defined by

$$\eta = 0, \quad y = \begin{pmatrix} 0 & a & 0 & 0 & 0 & 0 \\ -a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & -b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & -c & 0 \end{pmatrix}, \quad x = 0, \quad \xi = d. \tag{3.23}$$

[2]We will later see that the reason for this is that we can embedd three qubits in a system of three fermions. The SLOCC group of this system contains the permutation of qubits as well, hence there is no distinction between the biseparable classes.

For these states we have $\frac{1}{12}K(\phi)^I{}_J K(\phi)^J{}_I = 4\xi \mathrm{Pf}(x) = 4abcd$. The values of the four parameters for the different classes can be found in table 3.1. The structure of the classes can be seen on figure 3.1.

| Class | rank $K(\phi)^I{}_J$ | $K(\phi)^I{}_J K(\phi)^J{}_I$ | Null($\psi$) | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|---|---|---|
| GHZ | 12 | $\neq 0$ | 0 | 1 | 1 | 1 | 1 |
| W | 6 | 0 | 0 | 1 | 1 | 1 | 0 |
| Bisep | 2 | 0 | 2 | 1 | 1 | 0 | 0 |
| Sep | 0 | 0 | 6 | 1 | 0 | 0 | 0 |
| Null | 0 | 0 | 12 | 0 | 0 | 0 | 0 |

TABLE 3.1: Separation of the orbits of $\mathcal{F}^{\pm}$ using the covariant $K(\phi)^I{}_J$, canonical forms for $\mathcal{F}^+$ and nullity (2.82) in six dimensions. The parametrization is given in (3.23).

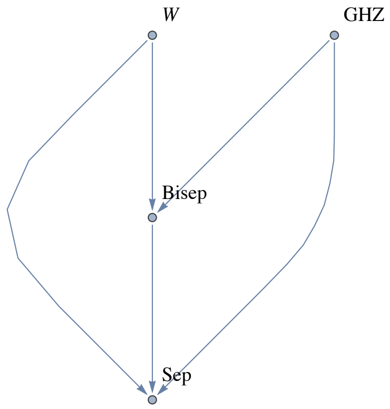


FIGURE 3.1: Extended SLOCC classes for fermions with six single particle modes. An arrow points from class $A$ to class $B$ if there is a degenerate SLOCC transformation turning a state in $A$ to a state in $B$.

Let us move on to the odd particle subspace, $\mathcal{F}^-$. A general state can be parametrized as

$$|\psi\rangle = u_a p^a|0\rangle + \frac{1}{3!}P_{abc}p^{abc}|0\rangle + \frac{1}{5!}w^l \epsilon_{labcde}p^{abcde}|0\rangle, \tag{3.24}$$

where $u$ and $w$ are six dimensional complex vectors and $P$ is a rank 3 antisymmetric tensor. The components of the covariant (3.20) read as

$$\begin{aligned} [A_\phi]^i{}_k &= 2w^i u_k - (K_P)^i{}_k - w^a u_a \delta^i{}_k, \\ [B_\phi]_{jk} &= 2P_{akj}w^a, \\ [C_\phi]^{il} &= \frac{2}{3!}u_a P_{bcd}\epsilon^{ilbcda}, \end{aligned} \tag{3.25}$$

where we have defined the matrix

$$(K_P)^i{}_k = \frac{1}{3!2!}P_{kab}P_{cde}\epsilon^{iabcde}, \tag{3.26}$$

which, as we will see, is important in the ordinary SLOCC classification of the three particle subspace. The nontrivial quartic invariant is

$$q_{\text{odd}}(\psi) = \frac{1}{12}K(\psi)^I{}_J K(\psi)^J{}_I = (w^i u_i)^2 - \frac{2}{3}w^i u_j (K_P)^j_i + \frac{1}{6}\text{Tr}K_P^2. \quad (3.27)$$

The orbits under $\Gamma_0$ are in one to one correspondence with the case of $\mathcal{F}^+$ and the ranks of $K(\psi)^I{}_J$ are the same as well. Hence, table 3.1 and figure 3.1 applies to this case as well.

It is clear that $K(\phi)^I{}_J$ defines an element of the Lie algebra $\mathfrak{so}(12, \mathbb{C})$ as $\text{Tr}K(\phi) = 0$ both in the even and the odd particle case. The corresponding generator can be represented on the Fock space as $(K_\phi)_* = \frac{1}{2}(K(\phi))^I{}_J[e^J, e_I]$, see (2.15). One can prove with explicit calculations that the states

$$\begin{aligned} |\varphi\rangle &= \left(1 + \frac{1}{\sqrt{q_{\text{even/odd}}(\phi)}}(K_\phi)_*\right)|\phi\rangle, \\ |\bar{\varphi}\rangle &= \left(1 - \frac{1}{\sqrt{q_{\text{even/odd}}(\phi)}}(K_\phi)_*\right)|\phi\rangle, \end{aligned} \quad (3.28)$$

are both *pure spinors* for all $|\phi\rangle \in \mathcal{F}^\pm$ in the respective GHZ class. Moreover, they have a vanishing Mukai pairing: $(\varphi, \bar{\varphi}) = 0$. This gives a canonical decomposition of a GHZ state $|\phi\rangle$ as a sum of two pure spinors:

$$|\phi\rangle = \frac{|\varphi\rangle + |\bar{\varphi}\rangle}{2}, \quad (3.29)$$

so we see, as a byproduct, that the pure index of this class, defined in (2.83), is 2. Finally, notice that the *dual state*

$$|\hat{\phi}\rangle = \frac{|\varphi\rangle - |\bar{\varphi}\rangle}{2i} = -i\frac{1}{\sqrt{q_{\text{even/odd}}(\phi)}}(K_\phi)_*|\phi\rangle, \quad (3.30)$$

transforms exactly the same way under the extended SLOCC group $\mathbb{C}^\times \times Spin(12, \mathbb{C})$ as $|\phi\rangle$. Indeed, the factors of the Clifford norm $\alpha_g$ coming from $(K_\phi)_*$ and $\frac{1}{\sqrt{q_{\text{even/odd}}(\phi)}}$ cancel each other, see sec. 2.4.2. The extra factor of $-i$ is just a convention for now, we have inserted it in for later convencience.

### 3.4.2 SLOCC classification

There are two inequivalent nontrivial setups for six modes. One of them is the case of two particles, this is identical through particle-hole duality to the four particle case. The other is the case of three particles, which is the smallest setup with genuine tripartite entanglement. There are many interesting features of this latter case which is lately dubbed in the literature as the Borland-Dennis setup.

Let us briefly discuss the simpler case of two particles. In the notation of (3.19), we have $\xi = \eta = 0$ along with $x^{ab} = 0$. The ordinary SLOCC classes are resolved by the rank of the coefficient matrix $y_{ab}$, which can be 2 (separable), 4 or 6. The single SLOCC invariant, as always in the case of two particles with an even number of modes, is the Pfaffian of $y$ i.e.

$$\mathrm{Pf}(y) = \frac{1}{3!2^3}\epsilon^{abcdef} y_{ab} y_{cd} y_{ef}. \tag{3.31}$$

Now we move on to the case of three fermions[20]. The states of interest are obtained by setting $u = v = 0$ in (3.24)

$$|P\rangle = \frac{1}{3!} P_{abc} p^{abc} |0\rangle. \tag{3.32}$$

The $12 \times 12$ matrix $K(P)^I{}_J$ reduces to the direct sum of the $6 \times 6$ matrix $(K_P)^i{}_j$ (see (3.26)) with minus its transpose. Its rank is therefore $\mathrm{rank}K(P)^I{}_J = 2 \cdot \mathrm{rank}K_P$ and the claim is that $\mathrm{rank}K_P$ is enought to resolve all the SLOCC classes, namely if $\mathrm{rank}K_P = 6, 3, 1,$ or $0$ then $|\psi_0\rangle$ belongs to the GHZ, W, biseparable or separable class respectively[20, P.3], see table 3.2. We may give representatives of each orbit in the following way

$$|P_0\rangle = \alpha p^{123}|0\rangle + \beta p^{145}|0\rangle + \gamma p^{246}|0\rangle + \delta p^{356}|0\rangle. \tag{3.33}$$

It follows that the SLOCC classes of three fermions with six single particle states and the extended SLOCC classes of fermions with six modes are in one to one correspondence with each other. The quartic invariant reads as

$$\mathcal{D}(P) = \frac{1}{12} K(P)^I{}_J K(P)^J{}_I = \frac{1}{6}\mathrm{Tr} K_P^2, \tag{3.34}$$

which is just the usual quartic invariant of three fermions with six single particle states[20].

| Class | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ | rank $K_P$ | $\mathrm{Tr}K_P^2$ |
|---|---|---|---|---|---|---|
| GHZ | $\neq 0$ | $\neq 0$ | $\neq 0$ | $\neq 0$ | 6 | $\neq 0$ |
| W | $\neq 0$ | $\neq 0$ | $\neq 0$ | $= 0$ | 3 | 0 |
| Bisep | $\neq 0$ | $\neq 0$ | $= 0$ | $= 0$ | 1 | 0 |
| Sep | $\neq 0$ | $= 0$ | $= 0$ | $= 0$ | 0 | 0 |
| Null | $= 0$ | $= 0$ | $= 0$ | $= 0$ | 0 | 0 |

TABLE 3.2: Separation of the orbits of $\wedge^3\mathbb{C}^6$ under SLOCC using the matrix $K_P$. The parameters of the canonical form (3.33) are listed as well.

We note one additional feature of the matrix $K_P$ which will be useful later. By using the canonical form (3.33) it is easy to see that

$$K_P^2 = \mathcal{D}(P) I, \tag{3.35}$$

which, by covariance, is true for all states. It follows that $K_P$ is nilpotent when $|P\rangle$ is not from the GHZ orbit. It has three Jordan blocks for W class states, and two for biseparable states. For GHZ states, it has two eigenspaces, each of dimension 3, with eigenvalues $\pm\sqrt{\mathcal{D}(P)}$. Let us denote the single particle creation/annihilation operators diagonalizing $K_P$ with $p_\pm^\alpha$ and $n_{\alpha,\pm}$, $\alpha = 1, 2, 3$, i.e.

$$(P, p_\pm^\alpha n_{\beta,\pm} P) = \mp\sqrt{\mathcal{D}(P)}\delta_\beta^\alpha. \tag{3.36}$$

Now consider the product states

$$|\Omega_{abc}\rangle = p^1_a p^2_b p^3_c|0\rangle, \quad a, b, c = \pm, \tag{3.37}$$

which form a natural basis of three particle states corresponding to the eigenmodes of $K_P$ for a GHZ state $|P\rangle$. They are clearly orthogonal with respect to the Mukai pairing: $(\Omega_{+++}, \Omega_{---}) = 0$. Now consider the generator $(K_P)_* = (K_P)^a{}_b p^b n_a$ which is a special case of (2.15). We have the identities

$$\begin{aligned}\left(1 + \frac{1}{\sqrt{\mathcal{D}(P)}}(K_P)_*\right)|P\rangle &= |\Omega_{+++}\rangle,\\ \left(1 - \frac{1}{\sqrt{\mathcal{D}(P)}}(K_P)_*\right)|P\rangle &= |\Omega_{---}\rangle,\end{aligned} \tag{3.38}$$

which are then the special cases of (3.28) for three particle states as the $|\Omega_{abc}\rangle$ are all separable. The easiest way to directly prove these identities is to explicitly prove them for the GHZ state $|P\rangle = (p^{123} + p^{456})|0\rangle$ and then use that every GHZ state is on the same $GL(6, \mathbb{C})$ orbit. This is then also a proof of (3.28) by the same reasoning, just with the group $\Gamma_0$ replacing $GL(6, \mathbb{C})$. The projectors $(1 \pm \frac{1}{\sqrt{\mathcal{D}(P)}}(K_P)_*)$ give the canonical decomposition of GHZ states in terms of a pair of Slater determinants.

### 3.4.3 Ways of embedding three qubits

Using the construction described in section 2.4.5 we may embed three qubit states $\sum_{i,j,k=0}^{1} \psi_{ijk}|ijk\rangle$ into the system of three fermions with six modes as

$$|\psi\rangle \mapsto |P_\psi\rangle = \sum_{i,j,k=0}^{1} \psi_{ijk} p^{3i+1} p^{3j+2} p^{3k+3}|0\rangle. \tag{3.39}$$

Note that this map differs from (2.92) by a permutation of the fermionic indices $(1, 2, 3, 4, 5, 6) \mapsto (1, 4, 2, 5, 3, 6)$. This difference has no physical significance but it gives an intuitively more clear presentation after we relabel the indices $(4, 5, 6)$ to $(\bar{1}, \bar{2}, \bar{3})$ respectively. The mapping between the first few basis states is

$$\begin{aligned}&|000\rangle \leftrightarrow p^1 p^2 p^3|0\rangle, \quad |001\rangle \leftrightarrow p^1 p^2 p^{\bar{3}}|0\rangle,\\ &|011\rangle \leftrightarrow p^1 p^{\bar{2}} p^{\bar{3}}|0\rangle, \quad \text{etc.,}\end{aligned} \tag{3.40}$$

i.e. indices $1$ and $\bar{1}$ correspond to the first qubit being $0$ or $1$. This embedding has the remarkable property that it intersects all the fermionic SLOCC orbits and the three qubit SLOCC classes are basically identical[3] with the fermionic ones of table 3.1. The quartic invariant $\mathcal{D}(P_\psi)$ reduces to the hyperdeterminant (1.24) of the $2 \times 2 \times 2$ array $\psi_{ijk}$:

$$\mathcal{D}(P_\psi) = \text{HDet}(\psi), \tag{3.41}$$

[3] The single biseparable class of the fermionic system splits into three different classes of the qubit system: the $A(BC)$, $B(AC)$ and $C(AB)$ biseparable classes. These are identified under the permutation of qubits. Precisely, there is a one-to-one map between the fermionic SLOCC classes and the orbits of $GL(2, \mathbb{C})^{\times 3} \rtimes S_3$ on three qubits, where $S_3$ is the permutation group of three elements.

This embedding posesses the simple single occupancy node picture of section 2.4.5. However, we have also seen there that it is possible to construct more exotic embeddings. As an example let us consider the embedding into the 8 dimensional subspace of double occupancy states[P.2, 44]. This can be obtained from the single occupancy embedding by composing it with an intertwiner between $\mathcal{F}^+$ and $\mathcal{F}^-$ as discussed in 2.4.4. The relevant intertwiner can be constructed from the Majorana basis

$$\gamma_j = p^j + n_j, \quad \gamma_{j+d} = \frac{p^j - n_j}{i}, \tag{3.42}$$

of $V$. Consider the state

$$\begin{aligned}\gamma_1\gamma_2\gamma_3|P_\psi\rangle &= \eta_\psi|0\rangle + \frac{1}{2!}(y_\psi)_{ab}p^{ab}|0\rangle + \frac{1}{2!4!}x_\psi^{ab}\epsilon_{abcdef}p^{cdef}|0\rangle \\ &+ \frac{1}{6!}\xi_\psi\epsilon_{abcdef}p^{abcdef}|0\rangle \in \mathcal{F}^+.\end{aligned} \tag{3.43}$$

with the qubit amplitudes sitting in the coefficients as

$$\begin{aligned}\eta_\psi = \psi_{000}, \quad y_\psi &= \begin{pmatrix} 0 & 0 & 0 & \psi_{100} & 0 & 0 \\ 0 & 0 & 0 & 0 & \psi_{010} & 0 \\ 0 & 0 & 0 & 0 & 0 & \psi_{001} \\ -\psi_{100} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\psi_{010} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\psi_{001} & 0 & 0 & 0 \end{pmatrix}, \\ \xi_\psi = -\psi_{111}, \quad x_\psi &= \begin{pmatrix} 0 & 0 & 0 & -\psi_{011} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\psi_{101} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\psi_{110} \\ \psi_{011} & 0 & 0 & 0 & 0 & 0 \\ 0 & \psi_{101} & 0 & 0 & 0 & 0 \\ 0 & 0 & \psi_{110} & 0 & 0 & 0 \end{pmatrix}.\end{aligned} \tag{3.44}$$

As we have described in 2.4.5, the conventional embedding of three qubits (3.39) gives the single occupancy states in a fermions on a lattice picture, where single particle states $(1, \bar{1})$ are the internal states associated to the first node, $(2, \bar{2})$ are associated to the second node and finally $(3, \bar{3})$ are associated to the third node. On the other hand when the embedding is made into the even particle Fock space as in (3.43) the correspondence between states reads as

$$\begin{aligned}&|000\rangle \leftrightarrow |0\rangle, && |001\rangle \leftrightarrow [p^3p^{\bar{3}}]|0\rangle, \\ &|011\rangle \leftrightarrow [p^2p^{\bar{2}}][p^3p^{\bar{3}}]|0\rangle, && \text{etc.}\end{aligned} \tag{3.45}$$

This shows that the three qubit states are mapped to the *double occupancy* states of the even particle number Fock space. Schematically, the states correspond to each other in the following way

$$\begin{array}{ccc} |000\rangle & \longleftrightarrow & \left|\begin{array}{c|c|c} \uparrow & \uparrow & \uparrow \\ \uparrow & \uparrow & \downarrow \\ \uparrow & \downarrow & \downarrow \end{array}\right| \begin{array}{c} \longleftrightarrow \\ \longleftrightarrow \\ \longleftrightarrow \end{array} \left|\begin{array}{c|c|c} & & \\ & & \uparrow\downarrow \\ & \uparrow\downarrow & \uparrow\downarrow \end{array}\right| . \\ |001\rangle & \longleftrightarrow & \\ |011\rangle & \longleftrightarrow & \end{array} \tag{3.46}$$

There are also other, mixed occupancy ways to embedd a three qubit system in $\mathcal{F}$. Let us denote with $\mathcal{F}^{000}$ the eight dimensional subspace of $\mathcal{F}^-$ spanned by the single occupancy state (3.39). Now let us define seven other eight dimensional subspaces by intertwining with half of the Majorana operators of (3.42)

$$\mathcal{F}^{\mu_1\mu_2\mu_3} = \gamma_1^{\mu_1}\gamma_2^{\mu_2}\gamma_3^{\mu_3}\mathcal{F}^{000}, \quad \mu_1,\mu_2,\mu_3 \in \{0,1\}. \tag{3.47}$$

It is clear that the previously discussed double occupancy state spans $\mathcal{F}^{111}$. The other spaces correspond to states where a node having 0 is in a single occupancy, while a node having 1 is in a double occupancy state, i.e. $\mathcal{F}^{001}$ means that the first two nodes have single and the last one has double occupancy. It can be shown[44] that this way the whole Fock space can be generated:

$$\mathcal{F} = \mathcal{F}^- \oplus \mathcal{F}^+ = (\mathcal{F}^{000}\oplus\mathcal{F}^{011}\oplus\mathcal{F}^{101}\oplus\mathcal{F}^{110})\oplus(\mathcal{F}^{001}\oplus\mathcal{F}^{010}\oplus\mathcal{F}^{100}\oplus\mathcal{F}^{111}), \tag{3.48}$$

i.e. one may think about the fermionic Fock space $\mathcal{F}$ as describing the *tripartite entanglement of six qubits*, a term first appeared in the context of the black hole/qubit correspondence. Indeed, we have two types of qubits, a single and a double occupancy for each node, which adds up to six.

### 3.4.4 Monogamy and three tangle for three fermions

As we have already mentioned, the system of three fermions with six single particle states mimics very closely the system of three qubits. In particular, measures for genuine two- and threebody entanglement can be identified and they satisfy the same monogamy equations as in the case of three qubits[P.4].

At the heart of this seemingly surprising analogy is the following observation. The one particle reduced density matrix of (3.32) reads as (see (2.34))

$$\rho_i{}^j = \frac{1}{2}P_{inm}\bar{P}^{jnm}. \tag{3.49}$$

This is a covariant quantity under local unitary transformations (corresponding to invertible LOCC) but not under local invertible transformations (corresponding to invertible SLOCC). Note that we have adopted the convention that a complex conjugation raises an index, however indices contracted this way only stay invariant under unitary transformations and not SLOCC transformations. Note also that we have adopted Löwdin normalization[47]: $\mathrm{Tr}\rho = 3||P||^2$. The eigenvalues $\lambda_i$, $i = 1...6$ of $\rho$ are unitary invariants and they are not all independent, even for a *generic* normalized state. The classical result of Borland and Dennis[48] is that these eigenvalues satisfy the (in)equalities:

$$\begin{aligned} \lambda_1 + \lambda_6 = 1, \quad \lambda_2 + \lambda_5 = 1, \\ \lambda_3 + \lambda_4 = 1, \quad \lambda_5 + \lambda_6 \geq \lambda_4, \end{aligned} \tag{3.50}$$

where the eigenvalues are ordered as $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_6$. The first three equations are relevant with respect to the relationship of the system to the one of three qubits. They show that occupancy of natural orbitals[4] is pairwise linked: the largest and the smallest occupancy always add up to one etc. When the eigenvalues of the one particle RDM satisfy extra equations in addition to $\mathrm{Tr}\rho = 3$, we say that the state in question is *pinned*. As we will later explain in section 3.8.1, pinning of eigenvalues leads to selection rules for the amplitudes of the state in the basis of natural orbitals. In the present case, this implies that any state in this basis can have at most 8 nonzero amplitudes and hence all states are local unitary equivalent to three qubit states of the form (3.39).

A remarkable consequence is that it is possible to express the (tangles) of three qubits with local unitary invariants of the fermionic system. Therefore, two and three body entanglement can be quantified for any 3 fermion state.

As a first step, recall that the matrix $K_P$ transforms under an invertible SLOCC transformation $G \in GL(6, \mathbb{C})$ (see (2.42)) as

$$K_P \mapsto (\det G)(G^T)^{-1} K_P G^T. \tag{3.51}$$

If we restrict ourselves to local unitary transformations we have $G \in U(6)$ and then both $K_P$ and $K_P^\dagger$ transform the same way: they can be multiplied in a local unitary invariant way. Now one can confirm with explicit calculation the identity

$$4\rho(I - \rho) = (\mathrm{Tr} K_P K_P^\dagger) I - \{K_P, K_P^\dagger\}. \tag{3.52}$$

The eigenvalues $\mu_i$ of the l.h.s. are $\mu_i = 4\lambda_i(1 - \lambda_i)$ and they are pairwise degenerate as a consequence of the Borland-Dennis relations (3.50). Now define the *concurrence matrix*

$$C = i(e^{-i\varphi/2} K_P - e^{i\varphi/2} K_P^\dagger), \tag{3.53}$$

where $\varphi$ is the phase of $\mathcal{D}(P) = \frac{1}{6}\mathrm{Tr} K_P^2$. We see that $C^\dagger = C$ and hence $C$ is a diagonalizable matrix with real eigenvalues. It is straightforward to calculate

$$C^2 = \{K_P, K_P^\dagger\} - 2|\mathcal{D}(P)| I = (\mathrm{Tr} K_P K_P^\dagger - 2|\mathcal{D}(P)|) I - 4\rho(I - \rho), \tag{3.54}$$

where we have made use of (3.52). Hence the eigenvalues $\xi_i$ of $C$ satisfy

$$\xi_i^2 = \frac{1}{2}\sum_{j=1}^{3} \mu_j - 2|\mathcal{D}(P)| - \mu_i, \tag{3.55}$$

or written out

$$\begin{aligned} \xi_1^2 &= \frac{1}{2}(-\mu_1 + \mu_2 + \mu_3) - 2|\mathcal{D}(P)|, \\ \xi_2^2 &= \frac{1}{2}(\mu_1 - \mu_2 + \mu_3) - 2|\mathcal{D}(P)|, \\ \xi_3^2 &= \frac{1}{2}(\mu_1 + \mu_2 - \mu_3) - 2|\mathcal{D}(P)|, \end{aligned} \tag{3.56}$$

[4] Natural orbitals are the eigenstates of the one particle RDM.

Solving for $\mu_i$ gives

$$\begin{aligned}\mu_1 &\equiv 4\lambda_1(1-\lambda_1) = \xi_2^2 + \xi_3^2 + 4|\mathcal{D}(P)|,\\ \mu_2 &\equiv 4\lambda_2(1-\lambda_2) = \xi_1^2 + \xi_3^2 + 4|\mathcal{D}(P)|,\\ \mu_3 &\equiv 4\lambda_3(1-\lambda_3) = \xi_1^2 + \xi_2^2 + 4|\mathcal{D}(P)|.\end{aligned} \tag{3.57}$$

These are precisely the CKW equations for three qubits corresponding to the Borland-Dennis tripartitioning of the natural orbitals with the identification with the tangles being $\tau_{ABC} = 4|\mathcal{D}(P)|$, $\tau_{A(BC)} = \mu_1$, $\tau_{AB} = \xi_2^2$, $\tau_{AC} = \xi_3^2$ and so on. One can check this explicitly for the embedding (3.39). On the other hand, $\mu_i$ and $\xi_i$ are well defined, *local unitary invariants* for any fermionic pure state and they quantify the entanglement of a single particle with the rest of the system and the amount of two-body entanglement respectively, as we will describe in the next subsection. Before doing this, let us emphasize how these fermionic CKW equations, and hence the three qubit CKW equations which directly follow from them, fit into the general spinor picture. Notice that equation (3.52) is just a special case of (2.130) with $A = p^a n_b$ and $B = p^b n_c$ and a sum over $b$. It follows that (3.52) is nothing else than a spinorial *Fierz identity* between bilinears of the state and its conjugate.

### 3.4.5 Entanglement entropy and entanglement of formation

First of all, it is clear that the invariants $\mu_i = 4\lambda_i(1-\lambda_i)$ are sufficient to express the entanglement entropy of the single particle RDM $\rho$. Indeed, from (3.50) and (3.52) one has

$$S(\rho) \equiv -\mathrm{Tr}\rho\log\rho = \sum_{i=1}^{3} h\left(\frac{1}{2}\left(1+\sqrt{1-\mu_i}\right)\right), \tag{3.58}$$

where the binary entropy function $h$ is defined by[5]

$$h(x) = -\left(x\log x + (1-x)\log(1-x)\right), \tag{3.59}$$

where the $1/3$ factors appear to restore the probability normalization of the eigenvalues from the $\mathrm{Tr}\rho = 3$ normalization. This shows that the $\mu_i$ invariants quantify the entanglement of a single particle with the rest of the system.

Now we move on to quantify two-body entanglement, which requires a measure of entanglement for mixed states. We will rely on the previously introduced entanglement of formation, as it has a clear operational meaning. We will give an upper bound for this quantity in terms of the 3 qubit concurrences $\xi_i$ and discuss its consequences. Recall, that the entanglement of formation is the convex roof of the entanglement entropy

$$\mathcal{E}(\rho^{(2)}) = \min_{\rho^{(2)}=\sum_i p_i|E_i\rangle\langle E_i|} \sum_i p_i S(|E_i\rangle\langle E_i|), \tag{3.60}$$

[5]Note that we use the $\mathrm{Tr}\rho = 3$ normalization so that this quantity is not the entropy of a probability distribution. The properly normalized quantity would be $-\mathrm{Tr}\frac{\rho}{3}\log\frac{\rho}{3}$. However, the minimal value of this quantity is $\log 3$ due to the fermionic nature of the system. It is common to remove this shift of $\log 3$ and multiply by 3 which results in the $S(\rho)$ we use.

where $\rho^{(2)}$ is the two-body RDM, in this case it has matrix elements

$$(\rho^{(2)})_{ij}{}^{kl} = \frac{1}{2} P_{ijm} \bar{P}^{klm}. \tag{3.61}$$

Before proceeding to discuss the entanglement of formation, let us mention some properties of $\rho^{(2)}$.

- It is clear that the map $\rho^{(2)} : \wedge^2\mathbb{C}^6 \to \wedge^2\mathbb{C}^6$ can be written as $\frac{1}{2}P \circ P^\dagger$, where $P : \mathbb{C}^6 \to \wedge^2\mathbb{C}^6$ is the map taking the vector $v^i$ to $P_{ijk}v^k$. From this, one readily sees that the maximal rank of $\rho^{(2)}$ is 6. The reason why this is interesting is that $\rho^{(2)}$ effectively works on a $\binom{4}{2} = 6$ dimensional subspace, i.e. a mixed state of two fermions with four single particle states. As we have seen, this is the natural fermionic system to incorporate two qubit entanglement.
- The six eigenvectors (corresponding to nonzero eigenvalues) of $\rho^{(2)}$ can be written as $P_{ijk}\bar{e}^k_{(n)}$, where $\{\bar{e}^k_{(n)}\}_{n=1}^6$ are the eigenvectors of the conjugate of the one particle RDM $\rho_i{}^j$. They have eigenvalues $\lambda_n$ which are the same as the eigenvalues of $\rho_i{}^j$. It follows that the (nonzero part of the) spectrum of the one body and the two body density matrices agree. Recall a similar well-known fact: for a bipartite system the reduced density matrices of the two subsystems have the same non-zero eigenvalues[24]. This is also well-known to be true for a composite sytem of two fermions[42]. In our case one can think of dividing the system of three fermions into the "bipartite" system of one fermion entangled with two fermions. In this case the non-zero eigenvalues of the reduced density matrices agree as one expects. This observation trivialy generalizes to $N$ fermions: the non-zero eigenvalues of the $k$ particle and the $N-k$ particle RDMs agree in general.

Now let us move on to give the promised upper bound to the entanglement of formation $E(\rho^{(2)})$. It is clear that $E(\rho^{(2)})$ is invariant under local unitary transformations. But the selection rules implied by (3.50) ensure that all states can be brought to the form (3.39) by means of a local unitary transformation. The two particle reduced density matrix of this state has matrix elements

$$\begin{aligned}
\rho^{(2)}{}_{12}^{12} &= \frac{1}{2}\rho^{AB}_{00|00}, \quad \rho^{(2)}{}_{12}^{1\bar{2}} = \frac{1}{2}\rho^{AB}_{00|01}, \quad \rho^{(2)}{}_{12}^{\bar{1}\bar{2}} = \frac{1}{2}\rho^{AB}_{00|11}, \\
\rho^{(2)}{}_{1\bar{2}}^{12} &= \frac{1}{2}\rho^{AB}_{01|00}, \qquad\qquad \cdots, \quad \rho^{(2)}{}_{\bar{1}\bar{2}}^{\bar{1}\bar{2}} = \frac{1}{2}\rho^{AB}_{11|11}, \\
\rho^{(2)}{}_{13}^{13} &= \frac{1}{2}\rho^{AC}_{00|00}, \qquad\qquad \cdots, \quad \rho^{(2)}{}_{\bar{1}\bar{3}}^{\bar{1}\bar{3}} = \frac{1}{2}\rho^{AC}_{11|11},
\end{aligned} \tag{3.62}$$

and so on. Here, $\rho^{AB}_{ij|kl}$, etc. are the two qubit reduced density matrixes of the state $\sum_{i,j,k=0}^{1}\psi_{ijk}|ijk\rangle$. One observes the scheme that the indices $1,2,3$ correspond to qubits $A,B,C$ respectively. Cases where index pairs contain the same number multiple times like $\rho^{(2)}{}_{2\bar{2}}^{12}$ or where the lower and upper indices contain different numbers like $\rho^{(2)}{}_{12}^{13}$ give zero. Indices without a bar correspond to the corresponding qubit in $|0\rangle$ state while with a bar to the qubit in $|1\rangle$ state. Therefore, $\rho^{(2)}$ for the state (3.39) is just the diagonal direct sum of the two qubit RDMs. As such, we may consider its convex decomposition into pure states which consists of the direct sum of the *optimal decompositions* of the two qubit density matrices. In this decomposition, its average entanglement entropy is just $\mathcal{E}(\rho^{AB}) + \mathcal{E}(\rho^{AC}) + \mathcal{E}(\rho^{BC})$, i.e. the sum of the entanglement of formations for each two qubit RDM. The optimal decomposition for the fermionic RDM must be at least this good, hence we have

$$\mathcal{E}(\rho^{(2)}) \leq \mathcal{E}(\rho^{AB}) + \mathcal{E}(\rho^{AC}) + \mathcal{E}(\rho^{BC}). \tag{3.63}$$

The right hand side can be expressed with the qubit squared concurrences $\xi_i$ and hence

$$\mathcal{E}(\rho^{(2)}) \leq \sum_{i=1}^{3} h(\frac{1}{2}\left(1 + \sqrt{1 - \xi_i^2}\right)), \tag{3.64}$$

where we remind the reader that $\xi_i$ are eigenvalues of the matrix (3.53) and hence are well defined for any 3-fermion state. Although a proof is not yet available, the author believes that this bound is actually an equality. Regardless of weather this is true, an immediate consequence of this bound is that when all $\xi_i = 0$, i.e. the concurrence matrix $C$ of (3.53) is zero, the entanglement of formation vanishes and hence the three fermion state $|P\rangle$ does not contain bipartite entanglement. In this case the two body RDM $\rho^{(2)}$ is a convex combination of separable states. A remarkable property of multipartite systems is that separability of the whole state does not follow from this. Indeed, we may ensure $\xi_i = 0$ by demanding the vanishing of the quantity

$$\mathrm{Tr}C^2 = 2\mathrm{Tr}K_P K_P^\dagger - 6|\mathcal{D}(P)|, \tag{3.65}$$

which can happen in the GHZ class! We will call states for which $2\mathrm{Tr}K_P K_P^\dagger = 6|\mathcal{D}(P)|$ states with vanishing concurrence. In this case, all entanglement is in the form of *tripartite* entanglement, which is measured by the quantity $4|\mathcal{D}(P)|$, as readily seen from the monogamy equations (3.57).

### 3.4.6 Conjugation and the Wootters spin-flip

Now that we understand how to describe three qubits as the single occupancy states of fermions with six modes, we are in position to discuss again the somewhat arbitrary quantity, the spin-flipped density matrix of (1.12) required to calculate the two qubit concurrence. We have seen in section 3.2.3. that this quantity is naturally related to the transposition (2.5) of the Clifford algebra with four modes. In the present situation, we are required to take a partial trace of the fermionic density matrix $|P\rangle\langle P|$. It is not difficult to see that the spin-flip can be obtained by taking the partial trace of the conjugate state $|\tilde{P}\rangle$ of $|P\rangle$ under the conjugation introducied in (2.96). Supposing that the amplitudes $P_{abc}$ correspond to a $\langle . | . \rangle$-orthonormal basis, the amplitudes of $|\tilde{P}\rangle$ are easily found to be

$$\tilde{P}_{ijk} = \frac{1}{3!}\epsilon_{ijklmn}\bar{P}^{lmn}. \tag{3.66}$$

For the embedded three qubit state $|P_\psi\rangle$, the dual $|\tilde{P}_\psi\rangle$ can be obtained with the embedding (3.39) from the dual three qubit state $|\tilde{\psi}\rangle$ with coefficients:

$$\tilde{\psi}_{ijk} = \sum_{i'j'k'=0}^{1} \varepsilon_{ii'}\varepsilon_{jj'}\varepsilon_{kk'}\bar{\psi}_{i'j'k'}, \tag{3.67}$$

It is straightforward to see that the spin-flipped density matrix (1.12) for pure three qubit states is just the two particle RDM of qubits $A$ and $B$ for the dual three qubit state $|\tilde{\psi}\rangle$.

## 3.5 Seven modes

### 3.5.1 Extended SLOCC classification

The single particle space is now $\mathcal{H} = \mathbb{C}^7$ and the spaces $\mathcal{F}^\pm$, with fixed parity, are 64 dimensional. The relevant bilinear is the one with $d-4=3$ indices

$$K(\psi)_{IJK} = (\psi, e_I e_J e_K \psi). \tag{3.68}$$

Notice that as $e_I e_J = -e_J e_I + g_{IJ} I$, $K(\psi)_{IJK}$ is totally antisymmetric in its indices in *any basis* not just the Majorana basis. It is useful to think about these covariants giving rise to the linear map

$$\begin{aligned} &M_\psi : V \to \mathfrak{so}(V) \\ &v = v^I e_I \mapsto T_\psi(v) = \big(K(\psi)_{IJ}{}^K v_K\big)\frac{1}{4}[e^I, e^J], \end{aligned} \tag{3.69}$$

where, as usual, indices are raised by $g^{IJ}$. It is clear that this linear map has an invariant rank under the action of the extended SLOCC group as $T_{g\psi}(v) = \alpha_g^{-1} g^{-1} T_\psi(\mathcal{O}_g v) g$ for every $g \in \Gamma_0$. It is then natural to introduce and endomorphism of $V$ which is just the map composed with its transposition[6]

$$L_\psi = M_\psi^T \circ M_\psi. \tag{3.70}$$

This also has a $\Gamma_0$ invariant rank. The matrix of $L_\psi$ reads as

$$(L_\psi)^I{}_J = \frac{1}{4}(\psi, e^I e_K e_L \psi)(\psi, e_J e^K e^L \psi), \tag{3.71}$$

Moreover, we have a relative invariant

$$\mathcal{I}_8(\psi) = \mathrm{Tr} L_\psi^2, \tag{3.72}$$

which is a homogeneous function of order 8 of $|\psi\rangle$.

Let us be more explicit, and work out the bilinears $K(\psi)_{IJK}$ for the odd particle subspace $\mathcal{F}^-$. Formulae for $\mathcal{F}^+$ can be obtained from these in a trivial manner by dualizing every odd index amplitude with $\epsilon_{a_1 \dots a_7}$. We

[6] The transposition here is with respect to the inner product (2.2) on $V$ and the Killing form on $\mathfrak{so}(V)$.

parametrize a state as

$$|\psi\rangle = u_a p^a|0\rangle + \frac{1}{3!}\mathcal{P}_{abc}p^{abc}|0\rangle + \frac{1}{2!5!}Q^{ab}\epsilon_{abcdefg}p^{cdefg}|0\rangle \\ + \frac{1}{7!}\xi\epsilon_{abcdefg}p^{abcdefg}|0\rangle. \tag{3.73}$$

Then the explicit form of the bilinears $(\psi, e_I e_J e_K\psi)$ are:

$$\begin{aligned}(\psi, n_a n_b n_c\psi) &= \frac{1}{4}Q^{pq}Q^{nm}\epsilon_{pqnmcba} - 2\xi\mathcal{P}_{cba},\\ (\psi, p^a n_b n_c\psi) &= \xi(u_c\delta^a_b - u_b\delta^a_c) - \frac{1}{2}Q^{pq}(\mathcal{P}_{pqc}\delta^a_b - \mathcal{P}_{pqb}\delta^a_c) - 2Q^{ap}\mathcal{P}_{bcp},\\ (\psi, p^a p^b n_c\psi) &= u_d(\delta^b_c Q^{da} - \delta^a_c Q^{bd}) - (\mathcal{M}_{\mathcal{P}})^{ab}{}_c + Q^{ab}u_c,\\ (\psi, p^a p^b p^c\psi) &= \frac{2}{3!}\mathcal{P}_{def}u_p\epsilon^{abcdefp}.\end{aligned} \tag{3.74}$$

Here we introduced the notation

$$(\mathcal{M}_{\mathcal{P}})^{ab}{}_c = \frac{1}{2!3!}\epsilon^{abpqijk}\mathcal{P}_{cpq}\mathcal{P}_{ijk}. \tag{3.75}$$

This is a covariant important for the ordinary SLOCC classification of the three particle subspace, as we will see.

It is useful to introduce the seven parameter family of states

$$|\psi_0\rangle = \big(\gamma_1 p^{123} + \gamma_2 p^{145} + \gamma_3 p^{167} + \gamma_4 p^{246} + \gamma_5 p^{257} + \gamma_6 p^{347} + \gamma_7 p^{356} \\ + p^{1234567}\big)|0\rangle. \tag{3.76}$$

which intersects every $\Gamma_0$ orbit. The orbit structure was found by Popov[49] and is listed in table 3.3 along with the invariants sufficient to resolve them. Notice that in addition to the bilinears $K(\psi)_{IJK}$ we need the nullity of (2.82) to resolve the classes. There is a lone orbit for which $\mathcal{I}_8(\psi)$ is nonzero: the orbit IX in 3.3. For the state (3.76) we have

$$\mathcal{I}_8(\psi_0) = 2^9 3^2 7 \cdot \gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6\gamma_7. \tag{3.77}$$

| Name | $\gamma_1$ | $\gamma_2$ | $\gamma_3$ | $\gamma_4$ | $\gamma_5$ | $\gamma_6$ | $\gamma_7$ | rank $L_{\psi_0}$ | rank $M_{\psi_0}$ | Null($\psi_0$) |
|---|---|---|---|---|---|---|---|---|---|---|
| IX | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 14 | 14 | 0 |
| VIII | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 7 | 14 | 0 |
| VII | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 3 | 14 | 0 |
| VI | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 13 | 0 |
| V | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 13 | 1 |
| IV | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 10 | 0 |
| III | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 7 | 1 |
| II | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 | 3 |
| I | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 7 |

TABLE 3.3: Extended SLOCC classes for $d = 7$, their canonical forms and the invariants resolving them.

The hierarchy of the classes can be seen on figure 3.2.

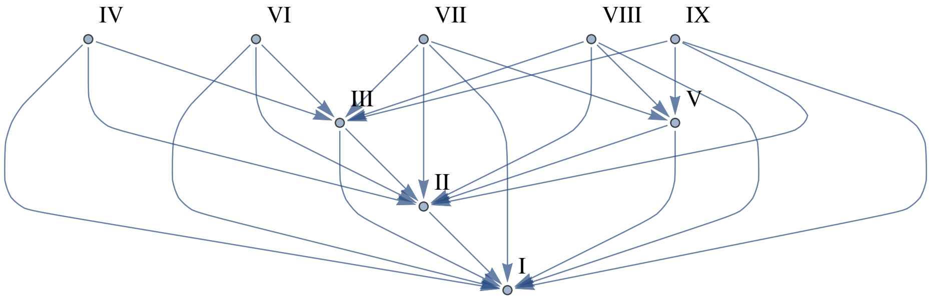


FIGURE 3.2: Extended SLOCC classes for fermions with seven single particle modes. An arrow points from class $A$ to class $B$ if all the ranks of class $B$ are smaller than those of class $A$.

### 3.5.2 SLOCC classification

Again, there are two cases when nontrivial entanglement can occur for fixed particle number. We can consider two particles (which is dual to five) and three particles (which is dual to four). The two particle SLOCC classes[7], as usual, are given by the rank of the antisymmetric coefficient matrix, which can be 0,2,4 and 6.

The interesting case is again the one of three particles[P.1, P.3]. As in the case of six modes, we can deduce the relevant covariants from the ones introduced to resolve the extended SLOCC classes. In order to do this, first consider the state

$$|\psi\rangle = \frac{1}{3!}\mathcal{P}_{abc}p^{abc}|0\rangle + \frac{1}{7!}\xi\epsilon_{abcdefg}p^{abcdefg}|0\rangle, \tag{3.78}$$

which is the three particle state we wish to consider, deformed by the top form. From (3.74) it is easy to see that the only non-vanishing component of the bilinear $K(\psi)_{IJK}$ is $(\psi, p^a p^b n_c \psi) = -(\mathcal{M}_\mathcal{P})^{ab}{}_c$, given by (3.75), and $(\psi, n_a n_b n_c \psi) = -2\xi\mathcal{P}_{abc}$. It is clear that these are covariants themselves when only the action of the ordinary SLOCC group $GL(7,\mathbb{C})$ is considered. The other important quantity can be extracted from $L_\psi$, which for this state has components

$$\begin{aligned}(L_\psi)^{ab} &= -\frac{1}{2}\mathcal{L}_\mathcal{P}{}^{ab},\\ (L_\psi)_{ab} &= -\xi\mathcal{N}_{\mathcal{P}ab},\\ (L_\psi)^a{}_b &= (L_\psi)_b{}^a = 0,\end{aligned} \tag{3.79}$$

where we have introduced the $7\times 7$ matrices

$$\begin{aligned}\mathcal{L}_\mathcal{P}{}^{ab} &= \mathcal{M}_\mathcal{P}{}^{ac}{}_d\mathcal{M}_\mathcal{P}{}^{bd}{}_c,\\ \mathcal{N}_{\mathcal{P}ab} &= \frac{1}{2}\mathcal{M}_\mathcal{P}{}^{pq}{}_b\mathcal{P}_{apq},\end{aligned} \tag{3.80}$$

which are themselves covariants under the ordinary SLOCC group $GL(7,\mathbb{C})$. The covariants $(\mathcal{M}_\mathcal{P})^{ab}{}_c$, $\mathcal{L}_\mathcal{P}{}^{ab}$ and $\mathcal{N}_{\mathcal{P}ab}$ depend only on the amplitudes $\mathcal{P}_{abc}$ of the three particle part and hence we may use them to characterize

[7] After dualization, this corresponds to keeping only the matrix $Q^{ab}$ in (3.73).

the SLOCC orbits of the state

$$|\mathcal{P}\rangle = \frac{1}{3!}\mathcal{P}_{abc}p^{abc}|0\rangle. \tag{3.81}$$

The single relative invariant of the action of the SLOCC group can be written as

$$\mathcal{J}_7(\mathcal{P}) = \frac{1}{2^4 3^2 7}\mathrm{Tr}(\mathcal{L}_\mathcal{P}\mathcal{N}_\mathcal{P}) = (\det \mathcal{N}_\mathcal{P})^{\frac{1}{3}}, \tag{3.82}$$

which is a homogeneous function of degree 7 of the amplitudes. We also have the relation

$$\mathcal{L}_\mathcal{P}{}^{ac}\mathcal{N}_{\mathcal{P}cb} = \frac{1}{2^4 3^2 7^2}\mathcal{J}_7(\mathcal{P})\delta^a{}_b. \tag{3.83}$$

Note that the degree 8 extended SLOCC invariant for the state (3.78) can be expressed with this as

$$\mathcal{I}_8(\psi) = 2(L_\psi)^{ab}(L_\psi)_{ab} = \xi\mathrm{Tr}(\mathcal{L}_\mathcal{P}\mathcal{N}_\mathcal{P}) = 2^4 3^2 7 \cdot \xi\mathcal{J}_7(\mathcal{P}). \tag{3.84}$$

The orbit structure of $\wedge^3\mathbb{C}^7$ can be found in the mathematical literature[50, 51]. There are a couple of intuitive ways to present representatives in each class. For the first one, it is convenient to introduce a complex basis in the one particle Hilbert space[8]

$$q^{1,2,3} = p^{1,2,3} + ip^{4,5,6}, \quad q^{\bar{1},\bar{2},\bar{3}} = p^{1,2,3} - ip^{4,5,6}, \quad q^7 = ip^7 \tag{3.85}$$

and let us denote the three particle Slater determinants with the shorthand

$$|E^{abc}\rangle \equiv q^a q^b q^c|0\rangle. \tag{3.86}$$

Now consider the states

$$\begin{aligned}
|\mathrm{SEP}\rangle &= |E^{123}\rangle \\
|\mathrm{BISEP}\rangle &= |E^{1}\rangle \wedge (|E^{23}\rangle + |E^{\bar{2}\bar{3}}\rangle) \\
|\mathrm{W}\rangle &= |E^{12\bar{3}}\rangle + |E^{1\bar{2}3}\rangle + |E^{\bar{1}23}\rangle \\
|\mathrm{GHZ}\rangle &= |E^{123}\rangle + |E^{\bar{1}\bar{2}\bar{3}}\rangle \\
|\mathrm{Sympl}_3\rangle &= (|E^{1\bar{1}}\rangle + |E^{2\bar{2}}\rangle + |E^{3\bar{3}}\rangle) \wedge |E^7\rangle \\
|\mathrm{Sympl}_3/\mathrm{SEP}\rangle &= (|E^{1\bar{1}}\rangle + |E^{2\bar{2}}\rangle + |E^{3\bar{3}}\rangle) \wedge |E^7\rangle + |E^{123}\rangle \\
|\mathrm{Sympl}_3/\mathrm{BISEP}\rangle &= (|E^{1\bar{1}}\rangle + |E^{2\bar{2}}\rangle + |E^{3\bar{3}}\rangle) \wedge |E^7\rangle + |E^{1}\rangle \wedge (|E^{23}\rangle + |E^{\bar{2}\bar{3}}\rangle) \\
|\mathrm{Sympl}_3/\mathrm{W}\rangle &= (|E^{1\bar{1}}\rangle + |E^{2\bar{2}}\rangle + |E^{3\bar{3}}\rangle) \wedge |E^7\rangle + |E^{12\bar{3}}\rangle + |E^{1\bar{2}3}\rangle + |E^{\bar{1}23}\rangle \\
|\mathrm{Sympl}_3/\mathrm{GHZ}\rangle &= (|E^{1\bar{1}}\rangle + |E^{2\bar{2}}\rangle + |E^{3\bar{3}}\rangle) \wedge |E^7\rangle + |E^{123}\rangle + |E^{\bar{1}\bar{2}\bar{3}}\rangle.
\end{aligned} \tag{3.87}$$

The origin of the name $|\mathrm{Sympl}_3\rangle$ is that a two particle state $\frac{1}{2}\omega_{ij}q^i q^j|0\rangle$ is given by an antisymmetric matrix $\omega_{ij}$ which in turn defines a symplectic form in six dimensions ($i, j$ runs from 1 to 6 now). In $|\mathrm{Sympl}_3\rangle$ the $\omega_{ij}$ is in the normal form for a rank 3 symplectic matrix. In table 3.4 we summarize the SLOCC orbits and label them by the above representatives. We also give their resolution in terms of the above introduced covariants. There is a single orbit, labeled by the state $|\mathrm{Sympl}_3/\mathrm{GHZ}\rangle$, where the invariant $\mathcal{J}_7$ is

[8]For now, this is just a fancy way of changing the indices $4, 5, 6$ to $\bar{1}, \bar{2}, \bar{3}$. The relations will be important to relate to the second type of presentation.

nonzero. This is the dense orbit.

| Representative | rank $\mathcal{N}_{\mathcal{P}}$ | rank $\mathcal{M}_{\mathcal{P}}$ | Extended SLOCC class |
|---|---|---|---|
| $\|\mathrm{SEP}\rangle$ | 0 | 0 | I |
| $\|\mathrm{BISEP}\rangle$ | 0 | 1 | II |
| $\|\mathrm{W}\rangle$ | 0 | 3 | III |
| $\|\mathrm{GHZ}\rangle$ | 0 | 6 | V |
| $\|\mathrm{Sympl}_3\rangle$ | 1 | 1 | III |
| $\|\mathrm{Sympl}_3/\mathrm{SEP}\rangle$ | 1 | 4 | VI |
| $\|\mathrm{Sympl}_3/\mathrm{BISEP}\rangle$ | 2 | 6 | VI |
| $\|\mathrm{Sympl}_3/\mathrm{W}\rangle$ | 4 | 7 | VII |
| $\|\mathrm{Sympl}_3/\mathrm{GHZ}\rangle$ | 7 | 7 | VIII |

TABLE 3.4: Entanglement classes of three fermions with seven single particle states, the ranks resolving them and their extended SLOCC class, when considered as states in $\mathcal{F}^-$ (see table 3.3). Notice that no orbits are inside the dense extended SLOCC orbit IX. Also, there are states (like $|\mathrm{W}\rangle$ and $|\mathrm{Sympl}_3\rangle$) which can be converted into each other using extended SLOCC but not ordinary SLOCC.

The other way of writing representatives for these classes is to reexpress them in terms of the $p^a$ modes through (3.85). This is clearly just a SLOCC transformation, hence the class of the states does not change. For example, for the representative of the dense orbit, we have

$$|\mathrm{Sympl}_3/\mathrm{GHZ}\rangle = 2(p^{123} - p^{156} + p^{246} - p^{345} + p^{147} + p^{257} + p^{367})|0\rangle. \quad (3.88)$$

The structure of this state is encoded in the incidence structure of the lines of the oriented Fano plane which is also encoding the multiplication table of the octonions. Motivated by this, the way we associate a graphical representative to a state is the following. We draw seven nodes corresponding to the seven single particle states and represent a Slater determinant by drawing an oriented line through three points, e.g. for $p^{123}|0\rangle$, we draw a line going through nodes 1,2 and 3, with orientation pointing towards 3. A reverse orientation means a minus sign and multiple lines mean that we add up the Slater determinants. With this dictionary in mind, we can represent the states in table 3.4 as parts of the Fano plane, as depicted on figure 3.3.

Notice that the number of points which are part of at least two lines (i.e. more than one lines intersect there) is just given by rank$\mathcal{M}_{\mathcal{P}}$. Similarly, the number of points which are part of at least three lines is given by rank$\mathcal{N}_{\mathcal{P}}$. Although an analysis to determine what kind of correlations the covariants $\mathcal{M}_{\mathcal{P}}$ and $\mathcal{N}_{\mathcal{P}}$ and the invariant $\mathcal{J}_7$ measure, similar to the one performed for three fermions with six modes in section 3.4.4, has not yet been done, it is tempting to think about $\mathcal{M}_{\mathcal{P}}$ as being connected to two body entanglement and about $\mathcal{N}_{\mathcal{P}}$ as being connected to three body entanglement. Indeed, in all the previously discussed cases, the bilinear invariant was directly connected to two body separability, while for six single particle modes, the quartic invariant measured three body entanglement. For seven single particle states, the natural quartic object is not a scalar but the matrix $\mathcal{L}_{\mathcal{P}}$. The previous observation about the ranks of these covariants seems to be very

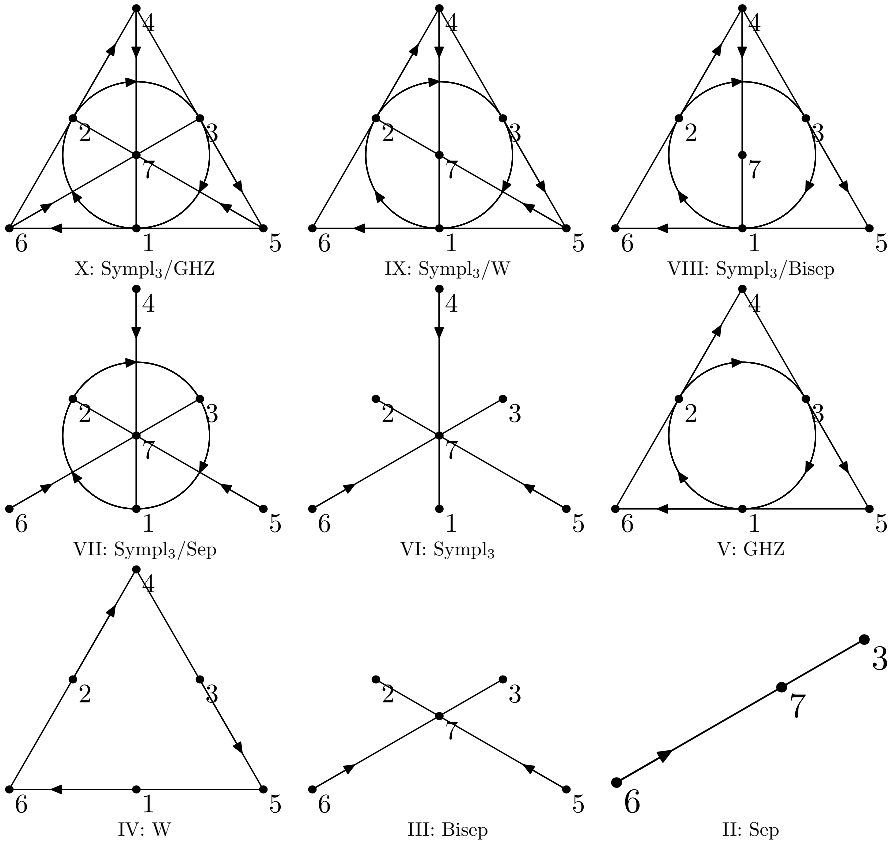


FIGURE 3.3: Graphical representation of the SLOCC classes of three fermions with seven single particle states.

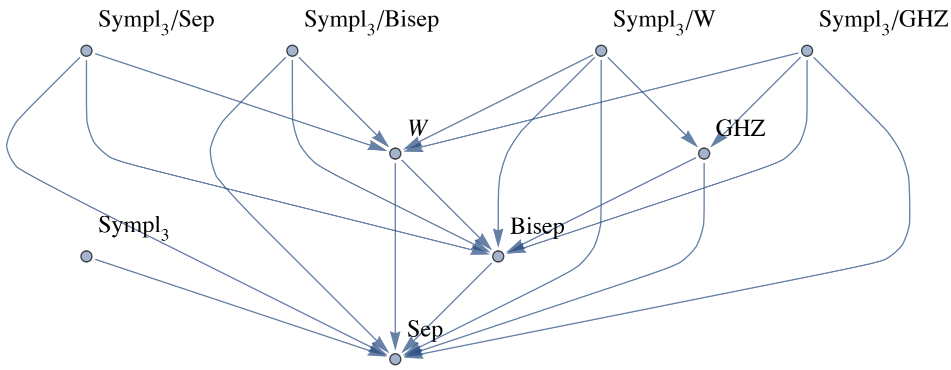


FIGURE 3.4: SLOCC classes for three fermions with seven single particle modes. An arrow points from class $A$ to class $B$ if all the ranks of class $B$ are smaller than those of class $A$.

well in line with this intuition. The hierarchy of the classes under noninvertibe SLOCC can be seen on figure 3.4.

## 3.6 Eight modes

### 3.6.1 Extended SLOCC classification

In this case we have $\mathcal{H} = \mathbb{C}^8$ and the fixed parity Fock spaces $\mathcal{F}^\pm$ are 128 dimensional. The relevant bilinear is the one with $d - 4 = 4$ indices i.e.

$$K(\psi)_{IJKL} = (\psi, e_I e_J e_K e_L \psi). \tag{3.89}$$

We can equivalently write

$$K(\psi)_{IJKL} = \frac{1}{4}(\psi, [e_I, e_J][e_K, e_L]\psi) + \frac{1}{4} g_{IJ} g_{KL}(\psi, \psi). \tag{3.90}$$

The second term gives rise to a nonzero quadratic invariant $(\psi, \psi)$. We can think about the first term as a linear map

$$\begin{aligned} R_\psi &: \mathfrak{so}(16) \to \mathfrak{so}(16) \\ T = T^{IJ}\frac{1}{4}[e_I, e_J] &\mapsto \big(\frac{1}{4}(\psi, [e^I, e^J][e_K, e_L]\psi)T^{KL}\big)\frac{1}{4}[e_I, e_J], \end{aligned} \tag{3.91}$$

where indices, as usual, are raised with $g^{IJ}$. The map $R_\psi$ has a $\Gamma_0$ invariant rank, moreover, we have the relative invariants

$$\mathcal{I}_{2p} = \mathrm{Tr} R_\psi^p, \tag{3.92}$$

with homogeneous degree $2p$. Now in contrary to the cases discussed so far, the pair $(\mathbb{C}^\times \times Spin_0(16), \wedge^{odd}\mathbb{C}^8)$ is not a prehomogeneous vector space. In fact, it can be shown that the algebra of polynomial invariants under $Spin_0(16)$ is eight dimensional[52]. An algebraically independent set of invariants can be given as[44]

$$\{\mathcal{I}_2, \mathcal{I}_8, \mathcal{I}_{12}, \mathcal{I}_{14}, \mathcal{I}_{18}, \mathcal{I}_{20}, \mathcal{I}_{24}, \mathcal{I}_{30}\}. \tag{3.93}$$

The basic observation behind finding the orbit decomposition is to notice that one can write the Lie algebra of $\mathfrak{e}_8$ as a semisimple complex symmetric space decomposition

$$\mathfrak{e}_8 = \mathfrak{so}_{16} \oplus \wedge^{even}\mathbb{C}^8, \tag{3.94}$$

where $[\mathfrak{so}_{16}, \mathfrak{so}_{16}] \subseteq \mathfrak{so}_{16}$, $[\wedge^{even}\mathbb{C}^8, \wedge^{even}\mathbb{C}^8] \subseteq \mathfrak{so}_{16}$ and $[\mathfrak{so}_{16}, \wedge^{even}\mathbb{C}^8] \subseteq \wedge^{even}\mathbb{C}^8$. This latter is just the action of the group on the even Fock space that we are after. We refrain from listing the orbit structure explicitly, which was identified [52]. Instead, we note some general features of the classification which turn out to be important when we consider the subcases of four fermions and four qubits. First, notice that as we may think of any state $|\psi\rangle$ as a Lie algebra element, $|\psi\rangle$ admits a *unique* Jordan decomposition into semisimple and nilpotent elements

$$|\psi\rangle = [|\psi\rangle]_s + [|\psi\rangle]_n, \tag{3.95}$$

such that $[|\psi_s\rangle, |\psi_n\rangle] = 0$. It is clear that this decomposition is invariant under the adjoint action, hence, after exponentiation, we have $g[|\psi\rangle]_s = [g|\psi\rangle]_s$ and $g[|\psi\rangle]_n = [g|\psi\rangle]_n$ for every $g \in Spin_0(16)$ i.e. the semisimple and nilpotent parts of a state transform *independently*. The entanglement classes can be grouped into *families* according to the type of their nilpotent parts[9]. There is a single family with vanishing nilpotent part, the family of semisimple states. It can be shown that this is also the collection of all Zariski-closed orbits. It is very easy to resolve the extended SLOCC classes within this family. The claim is that [52] the $Spin_0(16)$ orbits of semisimple states are in one-to-one correspondence with the points of the affine variety $\wedge^{odd}\mathbb{C}^8/Spin_0(16) \cong \mathbb{C}^8$. Practically this means that two semisimple states $|\psi_1\rangle$ and $|\psi_2\rangle$ are in the same $Spin_0(16)$ orbit iff all the invariants (3.93) take the same values on them. It follows that they are in the same extended SLOCC orbit if there exists a $\lambda$ nonzero complex number such that $\mathcal{I}_{2p}(\psi_1) = \lambda^{2p}\mathcal{I}_{2p}(\psi_2)$. A convenient representative of the family of semisimple orbits was given in [44] and reads as

$$|G\rangle = \sum_{\alpha=1}^{7} y_\alpha|E_\alpha\rangle + y_8(1+p^{12345678})|0\rangle, \tag{3.96}$$

where

$$\begin{aligned} |E_1\rangle &= (p^{1234}+p^{5678})|0\rangle, & |E_2\rangle &= (p^{1357}+p^{6824})|0\rangle, \\ |E_3\rangle &= (p^{1562}+p^{8437})|0\rangle, & |E_4\rangle &= (p^{1683}+p^{4752})|0\rangle, \\ |E_5\rangle &= (p^{1845}+p^{7263})|0\rangle, & |E_6\rangle &= (p^{1467}+p^{2358})|0\rangle, \\ |E_7\rangle &= (p^{1278}+p^{3456})|0\rangle. \end{aligned} \tag{3.97}$$

This way one can trade in the eight invariants of (3.93) to the eight parameters $y_\alpha$ to parametrize the semisimple orbits.

This simple criterion is not true if there is a nilpotent part to the state. It is not difficult to see that nilpotent states have the zero vector in the closure of their $Spin_0(16)$ orbit. From this, it follows that the value of any continous invariant is actually independent of the nilpotent part of a state.

Now we proceed with a short review of the ordinary SLOCC classification and the systems connection to four qubits.

### 3.6.2 SLOCC classification

There are three different fixed particle subspace where nontrivial entanglement can occur. The easiest one, as usual, is the one of two particles. In this case the SLOCC classes are identified by the rank of the antisymmetric coefficient matrix, as before. This can be 0,2,4,6 and 8. The other two cases are the one of three fermions (dual to five) and the one of four fermions. The former case corresponds to the pair $(GL(8,\mathbb{C}), \wedge^3\mathbb{C}^8)$ which is actually a prehomogeneous vector space, the last one for three fermions. As a result, there is a single relative invariant and the orbit structure is qualitatively similar to the previously discussed $d \leq 7$ cases. The latter case, four fermions with eight single particle states, is the one which is directly related to four qubits through single occupancy embedding.

[9]The classification of nilpotent orbits, and hence possible nilpotent parts, is done via the Kostant-Sekiguchi correspondence, see e.g. [53].

### Three fermions

We consider the three fermion state

$$|P\rangle = \frac{1}{3!} P_{abc} p^{abc} |0\rangle. \tag{3.98}$$

We can extract two covariants for the ordinary SLOCC group $GL(8,\mathbb{C})$ from $K(\psi)_{IJKL}$. The first just comes from $(P, p^{a_1} p^{a_2} p^{a_3} n_b P)$:

$$\left(E^{a_1 a_2 a_3}\right)_b = \frac{1}{12} \epsilon^{a_1 a_2 a_3 i_1 i_2 i_3 i_4 i_5} P_{b i_1 i_2} P_{i_3 i_4 i_5}. \tag{3.99}$$

For the second one, as in the case of seven modes, we need to be a little trickier. The way to go is to add the seven particle state $\frac{1}{7!} v^{a_1} \epsilon_{a_1 a_2 \ldots a_8} p^{a_2 \ldots a_8} |0\rangle$ to (3.98) and consider the quartic covariant $Q_{IJ} = K(\psi)_{IKLM} K(\psi)_J{}^{KLM}$. It will have non-vanishing elements for

$$Q^a{}_b \sim v^c \left(F^a\right)_{bc}, \tag{3.100}$$

where we have introduced the degree three covariant of $|P\rangle$

$$\left(F^a\right)_{b_1 b_2} = \frac{1}{24} \epsilon^{a i_1 i_2 i_3 i_4 i_5 i_6 i_7} P_{b_1 i_1 i_2} P_{b_2 i_3 i_4} P_{i_5 i_6 i_7}. \tag{3.101}$$

These two turn out to keep in hand the SLOCC orbits of this system, but in order to separate the classes we need to introduce some compositons. Define first the $8 \times 8$ symmetric matrix

$$G_{ab} \equiv \left(F^c\right)_{ad} \left(F^d\right)_{bc}, \tag{3.102}$$

which is of homogeneous degree six. Alternatively, we can define a degree ten $8 \times 8$ symmetric matrix

$$H^{ab} = \left(F^a\right)_{ci} \left(E^{ckl}\right)_j \left(F^b\right)_{dk} \left(E^{dij}\right)_l. \tag{3.103}$$

As mentioned previously, we have a single relative invariant which we can now write as

$$\mathcal{I}_{16}(P) = \mathrm{Tr}(GH), \tag{3.104}$$

and it has homogeneous degree 16. Finally, let us introduce the degree five combination of $E$ and $F$

$$(F \bullet E)^{akl}_{ij} \equiv \left(F^a\right)_{ci} \left(E^{ckl}\right)_j, \tag{3.105}$$

which we need in addition to resolve all the SLOCC classes.

The orbit structure of $\wedge^3 \mathbb{C}^8$ is available in the mathematical literature[51]. It turns out that in addition to the 9 classes of three fermions with seven single particle states we have 13 more classes.

The entanglement classes and the corresponding ranks are shown in TABLE 3.5. The representative states are encoded as

$$\Lambda = \alpha|E^{123}\rangle + \beta|E^{567}\rangle + \gamma|E^{154}\rangle + \delta|E^{264}\rangle + \epsilon|E^{374}\rangle + \lambda|E^{278}\rangle + \mu|E^{368}\rangle, \tag{3.106}$$

| Name | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ | $\epsilon$ | $\lambda$ | $\mu$ | Rank $G$ | Rank $F$ | Rank $E$ | Rank $F \bullet E$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| XI | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 3 | 6 | 0 |
| XII | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 4 | 7 | 0 |
| XIII | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 4 | 8 | 0 |
| XIV | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 5 | 8 | 1 |
| XV | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 6 | 8 | 2 |
| XVI | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 8 | 8 | 1 |
| XVII | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 8 | 8 | 2 |
| XVIII | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 8 | 8 | 4 |
| XIX | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 2 | 8 | 8 | 2 |
| XX | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 8 | 8 | 5 |
| XXI | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 3 | 8 | 8 | 7 |
| XXII | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 5 | 8 | 8 | 8 |
| XXIII | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 8 | 8 | 8 | 8 |

TABLE 3.5: Entanglement classes of three fermions with eight single particle states. The classes of table 3.4. are omitted here.

where as ususal $|E^{ijk}\rangle = p^i p^j p^k|0\rangle$. The continuous invariant of Eq. (3.104) is only non-zero for the class XXIII which is the single Zariski-open orbit of the prehomogeneous vector space[16] $(GL(8,\mathbb{C}), \wedge^3\mathbb{C}^8)$.

**Four fermions**

We may write a four fermion state as

$$|Z\rangle = \frac{1}{4!} Z_{ijkl} p^{ijkl}|0\rangle. \tag{3.107}$$

Now there is a lone SLOCC covariant that we can extract from the four index extended SLOCC bilinear (3.89). It reads as[44]

$$\mathcal{R}^{ij}{}_{kl} = (\psi, p^i p^j n_k n_l \psi) = \frac{1}{2!4!}\epsilon^{ijabcdef} Z_{lkab} Z_{cdef}, \tag{3.108}$$

while all the other components of $K(\psi)_{IJKL}$ are zero or can be expressed with $\mathcal{R}^{ij}{}_{kl}$. This covariant was first introduced by Katanova[54].

It turns out that a description in terms of a semisimple complex symmetric space is possible in this case as well and is very similar to the extended SLOCC classification of the total four mode system. It is based on the decomposition

$$\mathfrak{e}_7 = \mathfrak{sl}_8 \oplus \wedge^4\mathbb{C}^8, \tag{3.109}$$

and the orbit structure was identified in [55] and first considered in a physical setting in [31]. From this decomposition it follows that it again makes sense to talk about semisimple and nilpotent states and the orbits are conveniently grouped into families according to their nilpotent part. The family of semisimple orbits is easy to resolve again: it is just isomorphic to the affine variety $\wedge^4\mathbb{C}^8/SL(8,\mathbb{C}) \cong \mathbb{C}^7$ and the orbits are resolved by the values of the seven independent $SL(8,\mathbb{C})$ invariants

$$\{I_2, I_6, I_8, I_{10}, I_{12}, I_{14}, I_{18}\}, \tag{3.110}$$

where

$$I_{2p} = \mathcal{R}^{i_1 j_1}{}_{i_2 j_2}\mathcal{R}^{i_2 j_2}{}_{i_3 j_3}...\mathcal{R}^{i_p j_p}{}_{i_1 j_1}. \tag{3.111}$$

Representative for the semisimple orbits can be constructed by setting $y_8 = 0$ in (3.96):

$$|\mathcal{G}\rangle = \sum_{\alpha=1}^{7} y_\alpha |E_\alpha\rangle, \tag{3.112}$$

which is in a different $SL(8, \mathbb{C})$ class four each values of the parameters $y_1, ..., y_7$ for which the invariants of (3.110) can be traded in.

### 3.6.3 Four qubits

Now consider a four qubit state

$$|\chi\rangle = \sum_{ijkl=0}^{1} \chi_{ijkl}|ijkl\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2. \tag{3.113}$$

It turns out that the four qubit SLOCC problem can also be treated as a semisimple complex symmetric space as one has

$$\mathfrak{so}_8 = \left(\bigoplus_{i=1}^{4} \mathfrak{sl}_2\right) \oplus \left(\bigotimes_{i=1}^{4} \mathbb{C}^2\right). \tag{3.114}$$

The classification problem was solved partially in [30] and completed in [56]. We list the families obtained in these references, along with representatives, in table 3.6. From the nine families of this table the first one corresponds to the family of semisimple states. The last three families do not have a semisimple part. The nilpotent states characterizing families 2-6 may be obtained by setting all the continuous parameters to zero.

Now let us move on to the fermionic description of four qubits. As before, a single occupancy fermionic state can be associated to this through (2.92):

$$|\psi_\chi\rangle = \sum_{ijkl=0}^{1} \chi_{ijkl} p^{i+1} p^{j+3} p^{k+5} p^{l+7}|0\rangle. \tag{3.115}$$

These states span $\mathcal{F}^{0000} \subset \mathcal{F}$, the subspace of single occupancy states. The spaces with mixed occupancy are otained via acting with the first four Majorana operators of (3.42).

$$\mathcal{F}^{\mu_1\mu_2\mu_3\mu_4} = \gamma_1^{\mu_1}\gamma_2^{\mu_2}\gamma_3^{\mu_3}\gamma_4^{\mu_4}\mathcal{F}^{0000}. \tag{3.116}$$

Then, the total Fock space can be written[44] as the direct sum of mixed occupancy four qubit state spaces $\mathcal{F} = \bigoplus_{\mu_1\mu_2\mu_3\mu_4=0}^{1} \mathcal{F}^{\mu_1\mu_2\mu_3\mu_4}$. It is clear that the single occupancy subspace $\mathcal{F}^{0000}$ is the one which is also inside the four particle subspace $\wedge^4\mathbb{C}^4$.

The invariant algebra of the action of $SL(2, \mathbb{C})^{\times 4}$ on four qubits is generated by four invariants of homogeneous degree 2,4,4 and 6 respectively. We omit the definition of these invariants here, they can be found in [56]. It is natural to identify orbits different under the permutation of the four qubits and hence look at the orbit structure under the action of $S_4 \rtimes SL(2, \mathbb{C})^{\times 4}$. In

| Family | Canonical form |
|---|---|
| $G_{abcd}$ | $\frac{a+d}{2}(\vert 0000\rangle+\vert 1111\rangle)+\frac{a-d}{2}(\vert 0011\rangle+\vert 0011\rangle)+$ $\frac{b+c}{2}(\vert 0101\rangle+\vert 1010\rangle)+\frac{b-c}{2}(\vert 0110\rangle+\vert 1001\rangle)$ |
| $L_{abc_2}$ | $\frac{a+b}{2}(\vert 0000\rangle+\vert 1111\rangle)+\frac{a-b}{2}(\vert 0011\rangle+\vert 0011\rangle)+$ $c(\vert 0101\rangle+\vert 1010\rangle)+\vert 0110\rangle$ |
| $L_{a_2b_2}$ | $a(\vert 0000\rangle+\vert 1111\rangle)+b(\vert 0101\rangle+\vert 1010\rangle)+\vert 0110\rangle+$ $\vert 0011\rangle$ |
| $L_{ab_3}$ | $a(\vert 0000\rangle+\vert 1111\rangle)+\frac{a+b}{2}(\vert 0101\rangle+\vert 1010\rangle)+$ $\frac{a-b}{2}(\vert 0110\rangle+\vert 1001\rangle)+$ $\frac{i}{\sqrt{2}}(\vert 0001\rangle+\vert 0010\rangle+\vert 0111\rangle+\vert 1011\rangle)$ |
| $L_{a_4}$ | $a(\vert 0000\rangle+\vert 0101\rangle+\vert 1010\rangle+\vert 1111\rangle)+i\vert 0001\rangle+$ $\vert 0110\rangle-i\vert 1011\rangle$ |
| $L_{a_2 0_{3\oplus\bar{1}}}$ | $a(\vert 0000\rangle+\vert 1111\rangle)+\vert 0011\rangle+\vert 0101\rangle+\vert 0110\rangle$ |
| $L_{0_{5\oplus\bar{3}}}$ | $\vert 0000\rangle+\vert 0101\rangle+\vert 1000\rangle+\vert 1110\rangle$ |
| $L_{0_{7\oplus\bar{1}}}$ | $\vert 0000\rangle+\vert 1011\rangle+\vert 1101\rangle+\vert 1110\rangle$ |
| $L_{0_{3\oplus\bar{1}}0_{3\oplus\bar{1}}}$ | $\vert 0000\rangle+\vert 0111\rangle$ |

TABLE 3.6: Representatives for the nine families of SLOCC orbits of four qubits, see [30] and [56]. The first line is the family of generic semisimple states where each different value of the parameters $a, b, c, d$ correspond to a different orbit. The last three orbits are nilpotent and cannot have a semisimple part. There are eight nilpotent orbits under $S_4 \ltimes SL(2,\mathbb{C})^{\times 4}$ which are obtained by sending all the parameters in these representatives to zero.

this case the invariant algebra is generated by four invariants of degree 2, 6, 8 and 12 respectively. A generating set can be easily written in terms of the four fermion invariants (3.110) evaluated on the image of the embedding (3.115). It is just

$$\{I_2, I_6, I_8, I_{12}\}, \tag{3.117}$$

while in this case we can express all the other invariants of (3.110) with these four. For expressions relating these invariants to the ones of [56], see section 6.7 of [44].

The embedding (3.115) of four qubits into the single occupancy subspace of the fermionic Fock space has very nice properties, similarly to the embedding of three qubits into the single occupancy subspace of three fermions with six single particle states, as shown in [31]. In particular the embedding is injective, i.e. whenever the fermionic SLOCC class of two single occupancy states is different, their four qubit SLOCC class is different as well. However, contrary to the case of three qubits, the correspondence between fermionic and four qubit SLOCC classes is not one-to-one. There are fermionic SLOCC orbits which are not intersected by single occupancy states. Another nice result of [31] is that two states in a given family are $SL(2,\mathbb{C})^{\times 4} \rtimes S_4$ equivalent iff all the invariants $I_2$,...,$I_{12}$ take the same values on them.

In summary, the entanglement classification of four qubits, four fermions with eight modes and the extended SLOCC classification of fermions with eight modes is based on the semisimple symmetric space decomposition of the following sequence of Lie algebras

$$\begin{array}{rl} \text{Four qubits} & \mathfrak{so}_8 = \left(\bigoplus_{i=1}^{4} \mathfrak{sl}_2\right) \oplus \left(\bigotimes_{i=1}^{4} \mathbb{C}^2\right), \\ & \cap \\ \text{Four fermions} & \mathfrak{e}_7 = \mathfrak{sl}_8 \oplus \wedge^4 \mathbb{C}^8, \\ & \cap \\ \text{Fock space with eight modes} & \mathfrak{e}_8 = \mathfrak{so}_{16} \oplus \wedge^{even} \mathbb{C}^8. \end{array} \tag{3.118}$$

When we restrict the map (3.115) to semisimple states it gives rise to the sequence of embeddings of the corresponding affine varieties

$$\begin{array}{r} \left(\bigotimes_{i=1}^{4} \mathbb{C}^2\right) / SL(2,\mathbb{C})^{\times 4} \cong \mathbb{C}^4 \\ \downarrow \\ \wedge^4 \mathbb{C}^8 / SL(8,\mathbb{C}) \cong \mathbb{C}^7 \\ \downarrow \\ \wedge^{even} \mathbb{C}^8 / Spin_0(16) \cong \mathbb{C}^8, \end{array} \tag{3.119}$$

and these embeddings are injective. This property is clearly required for the map between SLOCC classes to be injective. We know that this is the case between the SLOCC classes of four qubits and four fermions [31] but the question is not yet settled for the extended SLOCC classes. This provides an interesting line of future research.

A somewhat more physical question, which is still open, is the precise quantification of tripartite and fourpartite entanglement for these systems. This is not yet settled even for the case of four qubits, appart from a couple of remarks in [30] about the type of multipartite entanglement each family posesses. It is likely, that the bilinear (3.108) plays a central role in this question and a similar analysis as in section 3.4.4 can be conducted. Hopefully, this question will be a subject of future investigations.

## 3.7 Nine modes

Let us consider the single particle Hilbert space $\mathcal{H} = \mathbb{C}^9$. The fixed chirality spaces $\mathcal{F}^{\pm}$ are of dimension 256 and they are dual under particle-hole symmetry, as always when we have an odd dimensional single particle space.

There are currently no results available about the extended SLOCC classification for the nine mode case. Following the scheme so far, it is likely that a central role is played by the bilinear with $d-4=5$ indices i.e. $K(\psi)_{I_1\dots I_5}$.

On the other hand, there are extensive results available for the SLOCC classification of the three particle subspace which turn out to be in a deep connection with the system of three qutrits[10] [P.3], in a very similar manner as the three fermions with six modes are connected to three qubits and four fermions with eight modes are connected to four qubits. Let us consider a three fermion state with nine modes

$$|P\rangle = \frac{1}{3!}P_{i_1 i_2 i_3}p^{i_1}p^{i_2}p^{i_3}|0\rangle \in \wedge^3\mathbb{C}^9, \tag{3.120}$$

and review the classification under the SLOCC group $GL(V)=GL(9,\mathbb{C})$.

The orbit structure is available due to the work of Vinberg and Élashvili [57]. It is obtained via a similar method to the semisimple symmetric space decomposition of four fermions with eight modes. We can decompose the Lie algebra $\mathfrak{e}_8$ as

$$\mathfrak{e}_8 = \mathfrak{sl}_9 \oplus \wedge^3\mathbb{C}^9 \oplus \left(\wedge^3\mathbb{C}^9\right)^*, \tag{3.121}$$

where $*$ denotes the dual vector space. Notice that, contrary to the case of four fermions, there are two copies of the state space inside $\mathfrak{e}_8$. This split behaves under the Lie bracket as

$$\begin{aligned}
[\mathfrak{sl}_9,\mathfrak{sl}_9] &\subseteq \mathfrak{sl}_9, & [\mathfrak{sl}_9,\wedge^3\mathbb{C}^9] &\subseteq \wedge^3\mathbb{C}^9,\\
[\mathfrak{sl}_9,\left(\wedge^3\mathbb{C}^9\right)^*] &\subseteq \left(\wedge^3\mathbb{C}^9\right)^*, & [\wedge^3\mathbb{C}^9,\wedge^3\mathbb{C}^9] &\subseteq \left(\wedge^3\mathbb{C}^9\right)^*,\\
[\left(\wedge^3\mathbb{C}^9\right)^*,\left(\wedge^3\mathbb{C}^9\right)^*] &\subseteq \wedge^3\mathbb{C}^9, & [\left(\wedge^3\mathbb{C}^9\right)^*,\wedge^3\mathbb{C}^9] &\subseteq \mathfrak{sl}_9,
\end{aligned} \tag{3.122}$$

and hence the action of the SLOCC group is again conveniently described as a special case of the adjoint action of $E_8$ on its Lie algebra. It again makes sense to define semisimple and nilpotent states. Recall that semisimple states have closed orbits under $SL(9,\mathbb{C})$ while the closure of orbits of nilpotent states contains the zero vector. The $GL(9,\mathbb{C})$ orbits can be grouped into seven families according to the type of their semisimple part.

Let us move on to discussing the invariants. We recall that continuous invariants are independent of the nilpotent part of a state. The family of semisimple orbits is isomoprhic to the affine variety $\wedge^3\mathbb{C}^9/SL(9,\mathbb{C}) \cong \mathbb{C}^4$ so that we have four algebraically independent invariants, or equivalently, we may represent all these orbits with a four parameter family of states. A generating set for the invariants were first found by Egorov[58], but a construction based on covariants were first used by Katanova[54]. Let us define the covariant

$$T^{abc}{}_{def} = \frac{1}{2!3!}\epsilon^{abcpqrstu}P_{dep}P_{fqr}P_{stu}. \tag{3.123}$$

It is clear that $T$ has homogeneous degree 3. Although an explicit computation has not yet been performed, it is reasonable to think that $T$ indeed emerges from the five index extended SLOCC bilinear $K(\psi)_{I_1\dots I_5} = (\psi, e_{I_1}\dots e_{I_2}\psi)$. To motivate this we add the top state $\xi p^{1\dots 9}|0\rangle$ to the state (3.120) thus turning it into an element of $\mathcal{F}^-$. We calculate the quantity

[10] We call a three state system with Hilber space $\mathbb{C}^3$ a qutrit.

$Q_{IJKLMN} = K(\psi)_{IJK}{}^{I'J'K'}K(\psi)_{I'J'K'LMN}$ of degree 4, for which the component with three upper little and lower little indices has to take the form

$$Q^{abc}{}_{def} \sim \xi T^{abc}{}_{def}. \tag{3.124}$$

Now because $T$ has 3 upper and 3 lower indices one can take its powers wich will have the same index structure:

$$\begin{aligned} (T^2)^{a_1b_1c_1}{}_{a_3b_3c_3} &= T^{a_1b_1c_1}{}_{a_2b_2c_2}T^{a_2b_2c_2}{}_{a_3b_3c_3}, \\ &\vdots \\ (T^m)^{a_1b_1c_1}{}_{a_{m+1}b_{m+1}c_{m+1}} &= T^{a_1b_1c_1}{}_{a_2b_2c_2}...T^{a_mb_mc_m}{}_{a_{m+1}b_{m+1}c_{m+1}}. \end{aligned} \tag{3.125}$$

More strictly speaking by antisymmetrization of its lower indices, $T$ can be regarded as a linear map $T : \wedge^3\mathbb{C}^9 \to \wedge^3\mathbb{C}^9$ hence one can compose $T$ with itself and form $T^2 = T \circ T$, ...,$T^m = T \circ ... \circ T$. Now define a set of relative invariants by

$$\phi_{3n} = \mathrm{Tr}T^n = (T^n)^{abc}_{\ \ abc}. \tag{3.126}$$

The subscript $3n$ denotes the homogeneous degree of $\phi_{3n}$ in the amplitudes $P_{ijk}$. Note that we have $\phi_{3n} = 0$ for $n$ odd and $n = 2$. Using these, a generating set for the four dimensional algebra of $SL(9, \mathbb{C})$ invariants can be conveniently given as

$$\{\phi_{12}, \phi_{18}, \phi_{24}, \phi_{20}\}. \tag{3.127}$$

For later convenience, we introduce the following rescaled versions of these invariants

$$\begin{aligned} J_{12} &= \frac{1}{2^7 3^3 7}\phi_{12}, \qquad & J_{18} &= -\frac{1}{2^{10}3^3 7 \cdot 13}\phi_{18}, \\ J_{24} &= \frac{1}{2^{11}3^2 7 \cdot 19}\phi_{24}, \qquad & J_{30} &= -\frac{1}{2^{12}3^3 5 \cdot 7 \cdot 13}\phi_{30}. \end{aligned} \tag{3.128}$$

Now any semisimple state $|P_s\rangle$ can be brougth by an $SL(V)$ transformation to the following form[57]

$$|P_s\rangle = a|q_1\rangle + b|q_2\rangle + c|q_3\rangle + d|q_4\rangle, \tag{3.129}$$

where for simplicity we have introduced

$$\begin{aligned} |q_1\rangle &= (p^{123} + p^{456} + p^{789})|0\rangle, \quad & |q_2\rangle &= (p^{147} + p^{258} + p^{369})|0\rangle, \\ |q_3\rangle &= (p^{159} + p^{267} + p^{348})|0\rangle, \quad & |q_4\rangle &= (p^{168} + p^{249} + p^{357})|0\rangle. \end{aligned} \tag{3.130}$$

We may group these semisimple orbits into families in the following way. For certain subvarieties of the space $\mathbb{C}^4$ of semisimple $SL(9, \mathbb{C})$ orbits it may happen that the invariants (3.128) become dependent. The number of independent invariants is measured by the rank of the matrix

$$M = \begin{pmatrix} \partial_a J_{12} & \partial_a J_{18} & \partial_a J_{24} & \partial_a J_{30} \\ \partial_b J_{12} & \partial_b J_{18} & \partial_b J_{24} & \partial_b J_{30} \\ \partial_c J_{12} & \partial_c J_{18} & \partial_c J_{24} & \partial_c J_{30} \\ \partial_d J_{12} & \partial_d J_{18} & \partial_d J_{24} & \partial_d J_{30} \end{pmatrix}, \tag{3.131}$$

which is nothing else but the Jacobian between coordinates $\{a, b, c, d\}$ and

$\{J_{12}, J_{18}, J_{24}, J_{30}\}$ on the variety $\wedge^3\mathbb{C}^9/SL(9,\mathbb{C})$ of semisimple orbits. Note that the determinant of $M$ is a not identically zero degree 80 polynomial expression in the coefficients $a, b, c, d$:

$$\frac{1}{2^{14}3^4 5^7 11^2 \cdot 6 \cdot 199} \det M = a^2b^2c^2d^2 \\ ((a^3+b^3-c^3)^3+(3abc)^3)^2 \\ ((a^3-b^3+d^3)^3+(3abd)^3)^2 \\ ((c^3+b^3+d^3)^3-(3cbd)^3)^2 \\ ((c^3+a^3-d^3)^3+(3cad)^3)^2. \tag{3.132}$$

Whenever $M$ is not full rank, we may find one or more combination of the invariants (3.128) that vanishes. This gives a convenient way to determine which family a generic state is in. We now review the seven families of semisimple orbits as described in [57]. Along the way, we find the invariants which vanish on each family [P.3].

**First family**
This family contains only semisimple states with no possible nilpotent part. This family is specified by the condition[57]

$$\det M \neq 0, \tag{3.133}$$

on the canonical states and it forms a Zariski-open subset in the space of all states. It is dense in the variety of semisimple states.

**Second family**
The semisimple part has the canonical form

$$a|q_1\rangle - b|q_2\rangle + d|q_4\rangle. \tag{3.134}$$

Formaly, we can obtain this by putting $c = 0$ and $b \to -b$ in (3.129). The canonical amplitudes in this family satisfy[57]

$$abd(a^3-b^3)(a^3-d^3)(b^3-d^3)((a^3+b^3+d^3)^3-(3abd)^3) \neq 0. \tag{3.135}$$

Now one can check that this is equivalent to dropping the third row of $M$, putting $c = 0$ and $b \to -b$ in it and requiring any of the $3 \times 3$ subdeterminant of the resulting $3 \times 4$ matrix to be non zero. This is equivalent with

$$\text{rank} M = 3 \tag{3.136}$$

for these semisimple states. The vanishing of $\det M$ means that it is possible that a function of the four invariants exists which equals zero. Indeed, we have found an invariant of degree 132 which vanishes for this family[11]:

$$
\begin{aligned}
\Delta_{132} = {} & J_{12}^{11} - \frac{44940218765172270463}{2232199994248855116} J_{12}^{8} J_{18}^{2} + \frac{113325967730636958495085217}{10091809656998987712226274} J_{12}^{5} J_{18}^{4} \\
& - \frac{11518845901768651039}{329340982758027804} J_{12}^{2} J_{18}^{6} - \frac{188875}{1526823} J_{12}^{9} J_{24} + \frac{20955843759677134000}{1506734996117977203 3} J_{12}^{6} J_{18}^{2} J_{24} \\
& - \frac{4809875789927509262 5}{15067349961179772033} J_{12}^{3} J_{18}^{4} J_{24} + \frac{156259946875}{27974261679948} J_{12}^{7} J_{24}^{2} \\
& - \frac{43381098724294271875}{244091069371112306934 6} J_{12}^{4} J_{18}^{2} J_{24}^{2} - \frac{32778366465625}{48591292538069676} J_{12} J_{18}^{4} J_{24}^{2} \\
& - \frac{37339826093750}{327991224631970313} J_{12}^{5} J_{24}^{3} - \frac{198339133437500}{741017211205562559} J_{12}^{2} J_{18}^{2} J_{24}^{3} \\
& + \frac{351718750000}{327991224631970313} J_{12}^{3} J_{24}^{4} - \frac{1250000000}{327991224631970313} J_{12} J_{24}^{5} \\
& + \frac{522717082571600510}{5022449987059924011} J_{12}^{7} J_{18} J_{30} - \frac{46317981762782284329748 60}{454131434564954447051823 3} J_{12}^{4} J_{18}^{3} J_{30} \\
& + \frac{45691574382263590}{741017211205562559} J_{12} J_{18}^{5} J_{30} - \frac{951594557840795000}{1356061496506179482 97} J_{12}^{5} J_{18} J_{24} J_{30} \\
& + \frac{2133816827644645000}{135606149650617948297} J_{12}^{2} J_{18}^{3} J_{24} J_{30} + \frac{140973248590625000}{12204553468555615346 73} J_{12}^{3} J_{18} J_{24}^{2} J_{30} \\
& + \frac{10890275000000}{200074647025501890 93} J_{12} J_{18} J_{24}^{3} J_{30} - \frac{8007699664851700}{45202049883539316099} J_{12}^{6} J_{30}^{2} \\
& + \frac{668635746252714792530 0}{151377144854984815683941 1} J_{12}^{3} J_{18}^{2} J_{30}^{2} + \frac{1392403335812500}{135606149650617948297} J_{12}^{4} J_{24} J_{30}^{2} \\
& - \frac{2371961791512500}{135606149650617948297} J_{12} J_{18}^{2} J_{24} J_{30}^{2} - \frac{216716472500000}{1220455346855561534673} J_{12}^{2} J_{24}^{2} J_{30}^{2} \\
& - \frac{1444554057104171200 0}{1513771448549848156839411} J_{12}^{2} J_{18} J_{30}^{3} + \frac{34328756109890000}{4541314345649544470518233} J_{12} J_{30}^{4}.
\end{aligned}
\tag{3.137}
$$

We have $\Delta_{132} = 0$ for any state in this family. It follows that the first family is *defined* by the equation $\Delta_{132} \neq 0$ which confirms that it is Zariski-open. There are three types of possible nilpotent parts in this family. These can be found in [57].

**Third family**

The canonical form of the semisimple part is

$$a|q_1\rangle + d|q_4\rangle. \tag{3.138}$$

We can obtain this by putting $b = 0$ in the canonical form of the second family. The coefficients satisfy $ad(a^6 - d^6) \neq 0$ which forces rank$M = 2$. There are nine types of possible nilpotent parts[57]. One can check that $\Delta_{132} = 0$ in this family. As the rank of the matrix $M$ is 2, one expects that there exists one more combination of the invariants which is identically zero in this family. Indeed, we have found that

[11] This expression is found by considering multinomials of the four invariants with a fixed homogeneous degree $k$ like $\Delta_k = \sum_{n_1 n_2 n_3 n_4 | 12n_1 + 18n_2 + 24n_3 + 30n_4 = k} \alpha_{n_1 n_2 n_3 n_4} J_{12}^{n_1} J_{18}^{n_2} J_{24}^{n_3} J_{30}^{n_4}$ and numerically generating a bunch of equations $\Delta_k = 0$ for some randomly selected states from the second family. The result is an overcomplete system of linear equations for the coefficients $\alpha_{n_1 n_2 n_3 n_4}$. The first value with a solution to this system is $k = 132$. The found coefficients were rationalized with a computer algebra system and the resulting expression was tested analytically to vanish in the second family.

the invariant of homogeneous degree 48 given by

$$\begin{aligned}\Delta_{48} = J_{24}^2 + \frac{13 \cdot 23^2 \cdot 293}{2^2 5^4} J_{12}^4 + \frac{3^2 \cdot 11 \cdot 127 \cdot 199^2}{2^3 5^4 \cdot 61} J_{12} J_{18}^2 \\ - \frac{257 \cdot 3^2}{5 \cdot 2^3} J_{12}^2 J_{24} - \frac{11 \cdot 199^2}{2^2 5^3 \cdot 61} J_{18} J_{30},\end{aligned} \tag{3.139}$$

vanishes in this family but $\Delta_{48} \neq 0$ for the first and the second family.

**Fourth family**
The canonical form of the semisimple part is

$$a|q_1\rangle + b|q_2\rangle - b|q_3\rangle. \tag{3.140}$$

Formaly, one obtains this by putting $d = 0$ and $c = -b$ in (3.129). The coefficients must satisfy $ab(a^3 - b^3)(a^3 + 8b^3) \neq 0$ which sets the rank of the matrix $M$ to be 2. We have

$$\Delta_{48} = 2^2 \cdot 5 \cdot 11^2 \cdot 199^2 \cdot b^9 \left(a^3 - b^3\right)^9 \left(a^4 + 8ab^3\right)^3 \tag{3.141}$$

for this family. The condition $\Delta_{48} \neq 0$ is obviously equivalent with the previous one. Of course we have $\Delta_{132} = 0$ but we can find another invariant of degree 48 wich vanishes here:

$$\begin{aligned}\Delta'_{48} = 113 \cdot 193 J_{12}^4 - \frac{11 \cdot 199^2 \cdot 21347}{3^5 61} J_{12} J_{18}^2 + \frac{2 \cdot 5^3 \cdot 257}{3^4} J_{12}^2 J_{24} \\ - \frac{2^4 5^4}{3^6} J_{24}^2 + \frac{2^3 \cdot 5 \cdot 11 \cdot 199^2}{3^5 \cdot 61} J_{18} J_{30}.\end{aligned} \tag{3.142}$$

One can check that $\Delta'_{48} \neq 0$ for the first, second and third families. There are six types of possible nilpotent parts, as listed in [57].

**Fifth family**
The canonical form of the semisimple part is

$$-c|q_2\rangle + c|q_3\rangle. \tag{3.143}$$

This is just the canonical form of the fourth family with $a = 0$. We require $c \neq 0$. The matrix $M$ has rank 1. We have $\Delta_{132} = \Delta_{48} = \Delta'_{48} = 0$. There are 18 different types of possible nilpotent parts.

**Sixth family**
The canonical form of the semisimple part is

$$a|q_1\rangle, \tag{3.144}$$

with $a \neq 0$. This is just the state (3.129) with $b = c = d = 0$. The matrix $M$ has rank 1 and $\Delta_{132} = \Delta_{48} = \Delta'_{48} = 0$. Moreover it is fairly easy to see that the degree 24 invariant

$$\Delta_{24} = J_{12}^2 - \frac{1}{111} J_{24} \tag{3.145}$$

is zero for this family while it is non-zero for families 1-5. There are 25 different types of possible nilpotent parts.

**Seventh family**
The semisimple part is zero here thus this family is the family of nilpotent states. Clearly, all continuous invariants vanish here. There are 102 different types of nilpotent states listed in the work of Vinberg and Élashvili[57]. All the previously discussed three fermion states (with 6, 7 and 8 single particle states) when considered as states of the nine dimensional system are in this family.

| Family | $\Delta_{132}$ | $\Delta_{48}$ | $\Delta'_{48}$ | $\Delta_{24}$ | Rank $T$ |
|---|---|---|---|---|---|
| First | $\neq 0$ | $\neq 0$ | $\neq 0$ | $\neq 0$ | 80 |
| Second | 0 | $\neq 0$ | $\neq 0$ | $\neq 0$ | 78 |
| Third | 0 | 0 | $\neq 0$ | $\neq 0$ | 76 |
| Fourth | 0 | $\neq 0$ | 0 | $\neq 0$ | 72 |
| Fifth | 0 | 0 | 0 | $\neq 0$ | 70 |
| Sixth | 0 | 0 | 0 | 0 | 56 |

TABLE 3.7: Values of the new continuous invariants on families of three fermions with nine single particle states. The last column is the rank of the map $T$ on semisimple states of the given family.

A summary of the families and their resolution with the new invariants $\Delta_{132}, \Delta_{48}, \Delta'_{48}, \Delta_{24}$ can be found in table 3.7. We have calculated in this table the rank of $T$ for semisimple states in each family. Comparing to ref. [57] we observe that the rank of the linear map $T : \wedge^3\mathbb{C}^9 \to \wedge^3\mathbb{C}^9$ defined in (3.123) satisfies the relation

$$\text{rank}T = 80 - \dim \text{stab}(|P_s\rangle), \tag{3.146}$$

where $\text{stab}(|P_s\rangle) \subseteq SL(9, \mathbb{C})$ is the stabilizer of the semisimple part $|P_s\rangle$.

### 3.7.1 Entanglement of three qutrits

A qutrit is a three state quantum system with Hilbert space $\mathcal{H} = \mathbb{C}^3$. The Hilbert space of three distinguishable qutrits is just $\mathcal{H}^{\otimes 3} \cong \mathbb{C}^9$. With $\{|1\rangle, |2\rangle, |3\rangle\}$ being a basis of $\mathcal{H}$ a general 3 qutrit state can be written as

$$|\chi\rangle = \sum_{\mu_1,\mu_2\mu_3=1}^{3} \chi_{\mu_1\mu_2\mu_3} |\mu_1\mu_2\mu_3\rangle. \tag{3.147}$$

The SLOCC group is $GL(3, \mathbb{C})^{\times 3}$ and it acts on the 9 complex amplitudes as

$$\chi_{\mu_1\mu_2\mu_3} \mapsto (S_1)_{\mu_1}^{\nu_1}(S_2)_{\mu_2}^{\nu_2}(S_3)_{\mu_3}^{\nu_3}\chi_{\nu_1\nu_2\nu_3}, \quad S_1 \otimes S_2 \otimes S_3 \in GL(3, \mathbb{C})^{\times 3}. \tag{3.148}$$

The mathematical problem of finding the SLOCC classes was solved by Nurmiev[59, 60]. Explicit expressions for the three continuous invariants generating the invariant algebra of this system was found by Briand et al.[61] where the problem was also recognized as the problem of SLOCC classification of three qutrits. Later Bremner and Hu managed to express the hyperdeterminant of a $3 \times 3 \times 3$ array with these three invariants[32, 62]. In the following we identify the problem of three qutrit entanglement as a special case of entanglement of three fermions with nine single particle states. We relate the invariants $I_6, I_9, I_{12}$ and the hyperdeterminant $\Delta_{333}$ of Bremner and Hu with the invariants of eq. (3.128).

According to Nurmiev[59, 60] any $3\times3\times3$ array can be uniquely written as the sum of a semisimple and a nilpotent part. Just like in the case of three fermions a semisimple state is defined to have a closed $SL(3,\mathbb{C})^{\times 3}$ orbit while a nilpotent state has the zero vector in the closure of its orbit. Now any semisimple state can be brought to a so called normal form:

$$|\chi_0\rangle = a|X_1\rangle - b|X_2\rangle + c|X_3\rangle, \tag{3.149}$$

where

$$\begin{aligned} |X_1\rangle &= |111\rangle + |222\rangle + |333\rangle, \quad |X_2\rangle = |123\rangle + |231\rangle + |312\rangle, \\ |X_3\rangle &= |132\rangle + |213\rangle + |321\rangle. \end{aligned} \tag{3.150}$$

There are a total of 43 orbits under the action of $GL(3,\mathbb{C})^{\times 3}$ and these can be grouped into five families according to the type of their semisimple part. The three fundamental invariants evaluated at the normal form $|\psi_0\rangle$ are[62]:

$$\begin{aligned} I_6 &= a^6 + 10a^3b^3 + b^6 - 10a^3c^3 + 10b^3c^3 + c^6, \\ I_9 &= (a+b)(a-c)(b+c)\left(a^2 - ab + b^2\right)\left(a^2 + ac + c^2\right)\left(b^2 - bc + c^2\right), \\ I_{12} &= -a^9b^3 - 4a^6b^6 - a^3b^9 + a^9c^3 - 2a^6b^3c^3 + 2a^3b^6c^3 - b^9c^3 - 4a^6c^6 \\ &\quad - 2a^3b^3c^6 - 4b^6c^6 + a^3c^9 - b^3c^9. \end{aligned} \tag{3.151}$$

The hyperdeterminant for $3\times3\times3$ arrays has degree 36 and expressed with these invariants as[62]:

$$\Delta_{333} = I_6^3 I_9^2 - I_{12}^2 I_6^2 - 32 I_{12}^3 + 36 I_{12} I_6 I_9^2 + 108 I_9^4. \tag{3.152}$$

It has the propierty that it is zero for all families except the first one. Now as $3\times 3 = 9$ it is clear that we can identify this system via (2.92) with the single occupancy states of the system of three fermions with nine single particle states. Let us denote this embedding by $\mathcal{E} : \mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3 \to \wedge^3\mathbb{C}^9$:

$$\mathcal{E} : |\chi\rangle \mapsto |P_\chi\rangle = \sum_{\mu_1,\mu_2,\mu_3=1}^{3} \chi_{\mu_1\mu_2\,\mu_3} p^{\mu_1} p^{3+\mu_2} p^{6+\mu_3} |0\rangle. \tag{3.153}$$

Now it is very easy to check that

$$\mathcal{E}(|X_1\rangle) = |P_{X_1}\rangle = |q_2\rangle, \quad \mathcal{E}(|X_2\rangle) = |P_{X_2}\rangle = |q_3\rangle, \quad \mathcal{E}(|X_3\rangle) = |P_{X_3}\rangle = |q_4\rangle, \tag{3.154}$$

where $|q_1\rangle, ..., |q_4\rangle$ are defined in e.q. (3.130).

We found that on $\mathrm{Im}\mathcal{E} \subset \wedge^3\mathbb{C}^9$ the invariants of (3.128) can be expressed with the fundamental invariants of three qutrits as

$$\begin{aligned}
J_{12} &= I_6^2 + 20I_{12}, \\
J_{18} &= I_6^3 + 30I_{12}I_6 + 100I_9^2, \\
J_{24} &= 111I_6^4 + 4440I_6^2I_{12} + 2\cdot 3^4\cdot 193I_{12}^2 + 2^2\cdot 11\cdot 199I_6I_9^2, \\
J_{30} &= 2\cdot 3^2\cdot 5^2\cdot 2521I_9^2I_{12} + 3^3\cdot 5\cdot 2521I_6I_{12}^2 \\
&\quad + 2\cdot 5\cdot 17\cdot 383I_6^2I_9^2 + 2^4\cdot 5^2\cdot 73I_6^3I_{12} + 2^3\cdot 73I_6^5.
\end{aligned} \tag{3.155}$$

Moreover the invariant $\Delta_{48}$ is expressed with the hyperdeterminant as

$$\Delta_{48} = -\frac{5\cdot 11^2\cdot 199^2}{2}\Delta_{333}I_{12}. \tag{3.156}$$

For the other $\Delta$ invariants we have:

$$\begin{aligned}
\Delta_{138} &= 0, \\
\Delta'_{48} &= \frac{2^4 5^5 11^2 199^2}{3^5}\left(2^3I_{12} + \frac{1}{3}I_6^2\right)I_9^4, \\
\Delta_{24} &= \frac{2\cdot 11\cdot 199}{37}\left(I_{12}^2 - \frac{2}{3}I_6I_9^2\right).
\end{aligned} \tag{3.157}$$

Define

$$D_{36} = \Delta_{333}, \quad D_{24} \ = I_{12}^2 - \frac{2}{3}I_6I_9^2, \quad D_{21} = (2^3I_{12} + \frac{1}{3}I_6^2)I_9. \tag{3.158}$$

Clearly, the invariants $D_{36}, D_{24}, D_{21}$ completely separate the five families of three qutrits. One can find representatives of these five families e.g. in the work of Bremner et. al.[62]. We followed the enumeration of the families used there. The first family has $D_{36} \neq 0$, the second family has $D_{36} = 0, D_{24} \neq 0, D_{21} \neq 0$, the third family has $D_{36} = D_{24} = D_{21} = 0$ and finally the fourth family has $D_{36} = D_{21} = 0, D_{24} \neq 0$. For the nilpotent orbits of the fifth family every fundamental invariant vanishes. On figure 3.5. we sketched how the embedding $\mathcal{E}$ works. The mapping between the families is injective.

It remains an open question whether $\mathcal{E}$ is injective between the SLOCC classes, as is the case between four qubits and four fermions with eight single particle states. Another intriguing question is whether an analysis, similar to the one performed in section 3.4.4, can quantify the amount of genuine bipartite and tripartite entanglement for both three qutrits and three fermions.

## 3.8 Outlook

### 3.8.1 Pinning of occupation numbers

As a possibly relevant physical application, we now comment on the connection of the ordinary SLOCC classification problem for fermionic quantum states with the constraints of Klyachko [63] on the eigenvalues of the one particle reduced density matrix (or one-matrix). This section is based in part on [P.3].

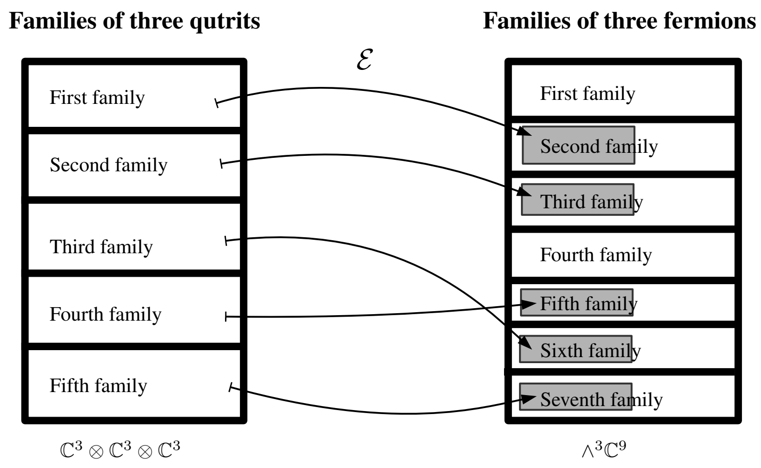


FIGURE 3.5: A sketch showing how the embedding $\mathcal{E}$ : $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \to \wedge^3\mathbb{C}^9$ defined in (3.153) works. The grey rectangles on the right side represent the image of the three qutrit families under $\mathcal{E}$. Different families are mapped into different families.

The constraints of Klyachko define a polytope in the space of possible eigenvalues of the one-matrix. An important concept is the so called pinning of occupation numbers which means the saturation of these constraints[64, 65]. It is widely believed that energy minima of many fermion systems usually do not lie inside the Klyachko polytope thus the ground state will be on the boundary and hence it will be pinned. Indeed there are both analytical[65] and numerical[66] results that such a pinning occurs in ground states of realistic systems. As shown by Klyachko[64] pinning of a state imposes selection rules on it reducing the number of separable states or Slater determinants that it contains. This is particularly useful in molecular physics since it simplifies the form of the ansatz one must use in variational methods to find the ground state.

### Selection rules from pinning

Now we review the selection rules imposed by saturation of these inequalities. Let us concentrate on three fermion states with an arbitrary single particle Hilbert space $\mathcal{H}$, i.e.

$$|P\rangle = \frac{1}{3!}P_{ijk}p^i p^j p^k|0\rangle. \tag{3.159}$$

Recall, that the one particle reduced density matrix is given by

$$\rho_i{}^j = \frac{1}{2}P_{inm}\bar{P}^{jnm}, \tag{3.160}$$

and that we use Löwdin normalization $\text{Tr}\rho = 3\langle P|P\rangle$. Let us also assume that the state is normalized

$$\langle P|P\rangle = \frac{1}{3!}\bar{P}^{i_1 i_2 i_3}P_{i_1 i_2 i_3} = 1. \tag{3.161}$$

Now every Klyachko inequality can be brought to the form

$$\sum_{i=1}^{\dim \mathcal{H}} y_i \lambda_i \geq 0, \tag{3.162}$$

where $\lambda_i$ are eigenvalues of the one particle reduced density matrix and $y_i$ are real numbers. Indeed, if there is a constant term in an inequality it can always be multiplied by $1 = \frac{1}{3}\sum_i \lambda_i$ which homogenizes it.

Now we show that if a Klyachko constraint $\sum_{i=1}^{\dim \mathcal{H}} y_i \lambda_i \geq 0$ is saturated (i.e. $\sum_{i=1}^{\dim \mathcal{H}} y_i \lambda_i = 0$) for the eigenvalues of the one particle reduced density matrix of $|P_0\rangle$ then $|P_0\rangle$ satisfies

$$\left(\sum_{i=1}^{\dim \mathcal{H}} y_i a_i^\dagger a_i\right)|P_0\rangle = 0, \tag{3.163}$$

where $a_i^\dagger$ creates[12] a *natural orbital* i.e. $\langle 0|a_i \rho a_j^\dagger|0\rangle = \lambda_i \delta_{ij}$. To see this, define the function $f : \wedge^3\mathcal{H} \to \mathbb{R}_{\geq 0}$ as

$$f(|P\rangle) = \sum_{i=1}^{\dim \mathcal{H}} y_i \lambda_i. \tag{3.164}$$

We can express this as

$$\begin{aligned} f(|P\rangle) &= \sum_{i=1}^{\dim \mathcal{H}} y_i \lambda_i \\ &= \sum_{i=1}^{\dim \mathcal{H}} y_i \rho_{ii} \\ &= \sum_{ikl} y_i |P_{ikl}|^2, \end{aligned} \tag{3.165}$$

where $P_{ijk}$ are the amplitudes of $|P\rangle$ *when expanded on its natural orbitals*. Using antisymmetry of $P_{ijk}$ we can write

$$\sum_{ikl} y_i |P_{ikl}|^2 = \frac{1}{3}\sum_{ikl}(y_i|P_{ikl}|^2 + y_i|P_{kil}|^2 + y_i|P_{kli}|^2). \tag{3.166}$$

Now in the second term relabel indices as $i \leftrightarrow k$ and in the third term as $i \leftrightarrow l$. We obtain

$$f(|P\rangle) = \frac{1}{3}\sum_{ikl}|P_{ikl}|^2(y_i + y_k + y_l) \tag{3.167}$$

Recall that $f(|P\rangle) \geq 0$ for all states. Then, by the ususal Ritz variational reasoning we have $\delta f(|P_0\rangle) = 0$ for minima of $f$. Hence a pinned state $|P_0\rangle$ should satisfy

$$\frac{\partial f}{\partial P_{ikl}^*}(|P_0\rangle) = \frac{1}{3}(P_0)_{ikl}(y_i + y_k + y_l) = 0, \quad \forall i,k,l, \tag{3.168}$$

[12] Here we are using the Hermitian inner product $\langle .|.\rangle$ so that we have an adjoint operation. Natural orbitals are orthonormal with respect to this inner product so $\{a_i, a_j^\dagger\} = \delta_{ij}$.

hence for a given set of indices $i, k, l$ either $(P_0)_{ikl} = 0$ or $y_i + y_k + y_l = 0$. Now it is easy to see that for a Slater determinant made up from three natural orbitals we have

$$\left(\sum_i y_i a_i^\dagger a_i\right) a_m^\dagger a_n^\dagger a_o^\dagger |0\rangle = (y_m + y_n + y_o) a_m^\dagger a_n^\dagger a_o^\dagger |0\rangle. \qquad (3.169)$$

Using that by (3.168) any Slater determinant in $|P_0\rangle$ satisfies $y_i + y_k + y_l = 0$ we arrive at

$$\left(\sum_i y_i a_i^\dagger a_i\right) |P_0\rangle = 0. \qquad (3.170)$$

As a byproduct, we are now in position to understand why every three fermion state with six single particle states (discussed in section 3.4.2) can be transformed into a three qubit state with a local unitary transformation. Recall that for this system the Borland-Dennis relations (3.50) are true for the eigenvalues of any one particle RDM. Applying (3.163) to the first three equations of (3.50) tells us that in the basis of natural orbitals, the only possible nonzero amplitudes are

$$\begin{array}{cccc} P_{123}, & P_{124}, & P_{135}, & P_{145}, \\ P_{236}, & P_{246}, & P_{356}, & P_{456}, \end{array} \qquad (3.171)$$

which is indeed a three qubit subspace.

#### Constrainst on SLOCC classes from pinning

Consider first the case of three fermions with six single particle states discussed in section 3.4.2. The classical Borland-Dennis result[48] is that if one orders the eigenvalues of the one-matrix as $\lambda_{i+1} \geq \lambda_i$ then one has a nontrivial inequality

$$\lambda_5 + \lambda_6 \geq \lambda_4. \qquad (3.172)$$

Note that this inequality is independent of the normalization of the original pure state. Now if (3.172) is saturated for a state $|P\rangle$ then according to (3.163) it must have the form[65, 64]

$$|P\rangle = \alpha a_1^\dagger a_2^\dagger a_3^\dagger |0\rangle + \beta a_1^\dagger a_4^\dagger a_5^\dagger |0\rangle + \gamma a_2^\dagger a_4^\dagger a_6^\dagger |0\rangle, \qquad (3.173)$$

in the basis of natural orbitals. It is clear that transforming an arbitary state to its natural orbital form amounts to a local unitary transformation hence it does not change the SLOCC class of it. Now if we calculate the matrix $K_P$ of (3.26) for the state (3.173) we get

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\beta\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\alpha\gamma & 0 & 0 & 0 & 0 \\ -2\alpha\beta & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \qquad (3.174)$$

This matrix has rank 3, 1 or 0 depending on the value of the coefficients. Looking at table 3.2 we already conclude that *pinning is impossible for states* *in the GHZ class* or otherwise stated pinning is impossible for states with $\mathcal{D}(P) = \frac{1}{6}\mathrm{Tr}K_P^2 \neq 0$. One might think that this means that all states with $\mathcal{D}(P) = 0$ are pinned but this is not the case since the spectrum of the one-matrix is not invariant under general SLOCC transformations thus pinning is not a SLOCC invariant concept. Indeed one can easily find both pinned and unpinned states in the W class. Note that these observations are in perfect agreement with the numerical work done by C. L. Benavides-Riveros et. al.[66] where pinning was studied in finite rank variational approximations of the ground state of lithium. It was observed there that pinning in the rank six approximation can only occur if the invariant $\mathcal{D}(P)$ is zero.

Consider now the case of seven single particle states of Section 3.5.2. Pinning for this system is investigated by Klyachko as it is important in studying the first excited state of beryllium[64]. Moreover it is used as the rank 7 approximation of lithium orbitals where pinning was also observed[66]. For seven single particle states we have four non-trivial Klyachko constraints:

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_4 + \lambda_7 \leq 2, \\ \lambda_1 + \lambda_2 + \lambda_5 + \lambda_6 \leq 2, \\ \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \leq 2, \\ \lambda_1 + \lambda_3 + \lambda_4 + \lambda_6 \leq 2. \end{aligned} \tag{3.175}$$

Suppose we saturate the first one: $\lambda_1 + \lambda_2 + \lambda_4 + \lambda_7 = 2$ for a normalized state $\mathcal{P}$. Then the selection rules arising from (3.163) imply[64] that $|\mathcal{P}\rangle \in \wedge^2\mathbb{C}^4 \otimes \mathbb{C}^3 \subset \wedge^3\mathbb{C}^7$. In particular in the basis of natural orbitals $|\mathcal{P}\rangle$ must be a linear combination of separable states with two indices from the set $\{1, 2, 4, 7\}$ and one index from the set $\{3, 5, 6\}$. One can easily calculate the covariants $\mathcal{N}_{\mathcal{P}ab}$ and $\mathcal{M}_{\mathcal{P}}{}^{ab}{}_c$ of eqs. (3.80) and (3.75) for such states and conclude that rank $\mathcal{N}_{\mathcal{P}} = 4$ and rank $\mathcal{M}_{\mathcal{P}} = 7$. Looking at table 3.4. we already deduce that there is no pinning for states in class of $|\mathrm{Sympl}_3/\mathrm{GHZ}\rangle$, or equivalently, for states with a non-vanishing $\mathcal{J}(\mathcal{P})$ invariant.

Now suppose we saturate three (the first two and the last one in e.q. (3.175)) of the constraints. In this case $|\mathcal{P}\rangle$ must have the form[64]

$$|\mathcal{P}\rangle = \alpha p^{123}|0\rangle + \beta p^{145}|0\rangle + \gamma p^{167}|0\rangle + \delta p^{246}|0\rangle, \tag{3.176}$$

when expanded on its natural orbitals. Calculating the relevant ranks for this state gives rank $\mathcal{N}_{\mathcal{P}} = 1$ and rank $\mathcal{M}_{\mathcal{P}} = 4$ which identifies the class of $|\mathrm{Sympl}_3/\mathrm{SEP}\rangle$ in table 3.4. Moreover one can check that one cannot increase the rank of $\mathcal{M}_{\mathcal{P}}$ by setting any of the coefficients to zero. This means that states of the form (3.176) cannot be in class $|\mathrm{GHZ}\rangle$. However, they do cross the remaining classes of table 3.4 with smaller ranks, so we deduce that pinning of three Klyachko constraints is only possible for states in these classes. If we require the saturation of all four constraints then we have to put $\gamma = 0$ in (3.176) and we get back to a state of the form (3.173). Thus pinning of all four constraints is only possible in a six single particle subspace and only in the classes $|\mathrm{SEP}\rangle$, $|\mathrm{BISEP}\rangle$ and $|\mathrm{W}\rangle$.

# Chapter 4

# Stringy black holes and entanglement

This chapter is mainly devoted to the construction of the entropy formula for nonextremal black holes of the STU model using the covariants introduced in section 3.4 in the fermionic context. Along the way we review the construction of these black hole solutions and, towards the end, we review some aspects of the black hole/qubit correspondence. In particular, in section 4.1.11, we comment on how general nonextremal STU black holes fit into the four qubit classification of STU black holes.

## 4.1 Single centered black holes and duality invariant entropy formulae

### 4.1.1 $\mathcal{N}=2$ supergravity

The term STU model refers to ungauged $\mathcal{N}=2$ supergravity in four dimensions coupled to 3 vector multiplets[67, 68]. The model is of central importance in string theory as it can be obtained from most string theories and is related to various other supergravity theories through dualities. Also, a suitable solution of the STU model can be used[69] to generate all single centered stationary black holes of maximal $\mathcal{N}=8$ supergravity[70, 71] which describes the low energy limit of M-theory compactified on $T^7$. The black hole solutions in the theory can be obtained from the bosonic part of the action without the hypermultiplets. There are several formulations of this action, here we simply pick one and refer to the literature on further details. The action we choose is

$$S=\frac{1}{16\pi}\int \mathrm{d}x^4\sqrt{|g|}\{-\frac{R}{2}+G_{i\bar{j}}\partial_\mu\tau^i\partial^\mu\bar{\tau}^{\bar{i}} + (\mathrm{Im}\mathcal{N}_{IJ}(\mathcal{F}^I)_{\mu\nu}(\mathcal{F}^J)^{\mu\nu}+\mathrm{Re}\mathcal{N}_{IJ}(\mathcal{F}^I)_{\mu\nu}(\star\mathcal{F}^J)^{\mu\nu})\}. \tag{4.1}$$

Here, $\mathcal{F}^I$, $I=0,1,2,3$ are four $U(1)$ gauge field strengths and $\star\mathcal{F}^I$ is the dual of $\mathcal{F}^I$. The complex scalars $\tau^i$, $i=1,2,3$ are coordinates of the projective special Kähler manifold $[SL(2,\mathbb{R})/SO(2)]^{\times 3}$ and $G_{i\bar{j}}$ is the Kähler metric of this space. The four dimensional Ricci scalar is $R$. We will usualy use the decomposition of $\tau^i$ into real and imaginary parts as

$$\tau^i=x_i+iy_i. \tag{4.2}$$

Then, the Kähler metric explicitly reads as $G_{i\bar{j}} = \frac{\delta_{i\bar{j}}}{(2y_i)^2}$. The period matrix $\mathcal{N}_{IJ}$ depends only on the scalars $\tau^i$ and is given as

$$\mathcal{N} = \begin{pmatrix} -2x_1x_2x_3 - iy_1y_2y_3\left(1+\sum_{i=1}^3 \frac{x_i^2}{y_i^2}\right) & x_2x_3+i\frac{x_1y_2y_3}{y_1} & x_1x_3+i\frac{x_2y_1y_3}{y_2} & x_1x_2+i\frac{x_3y_1y_2}{y_3} \\ x_2x_3+i\frac{x_1y_2y_3}{y_1} & -i\frac{y_2y_3}{y_1} & -x_3 & -x_2 \\ x_1x_3+i\frac{x_2y_1y_3}{y_2} & -x_3 & -i\frac{y_1y_3}{y_2} & -x_1 \\ x_1x_2+i\frac{x_3y_1y_2}{y_3} & -x_2 & -x_1 & -i\frac{y_1y_2}{y_3} \end{pmatrix}. \tag{4.3}$$

Note that the equations of motion coming from (4.1) are invariant under the $SL(2,\mathbb{R})_U^{3\times}$ subgroup of the full U-duality group $E_{7(7)}$ mapping the STU model to itself. An element

$$S_1 \otimes S_2 \otimes S_3 \in SL(2,\mathbb{R})_U^{3\times}, \tag{4.4}$$

is parametrized as

$$S_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, \quad a_id_i - b_ic_i = 1, \quad i = 1,2,3, \tag{4.5}$$

and it acts on the 3 dimensional fields the following way. The scalars $\tau^i$ are transformed as

$$\tau^i \mapsto \frac{a_i\tau^i + b_i}{c_i\tau^i + d_i}, \tag{4.6}$$

while the electromagnetic potentials $\zeta^I$, $\tilde{\zeta}_I$ transform in the fundamental $(2,2,2) = 8$ dimensional representation. Explicitely if one defines the three index tensor $\psi_{ijk}$ corresponding to the amplitudes of a 3 qubit state as

$$\begin{pmatrix} \psi_{000} & \psi_{001} & \psi_{010} & \psi_{011} \\ \psi_{100} & \psi_{101} & \psi_{110} & \psi_{111} \end{pmatrix} = \begin{pmatrix} \tilde{\zeta}_4 & \zeta^3 & \zeta^2 & -\tilde{\zeta}_1 \\ \zeta^1 & -\tilde{\zeta}_2 & -\tilde{\zeta}_3 & -\zeta^4 \end{pmatrix} \tag{4.7}$$

the transformation rule is

$$\psi_{ijk} \mapsto (S_1)_i{}^{i'}(S_2)_j{}^{j'}(S_3)_k{}^{k'}\psi_{i'j'k'}. \tag{4.8}$$

In the following when we write $SL(2)$ we are referring ot $SL(2,\mathbb{R})$ unless explicitly otherwise stated.

### 4.1.2 Timelike dimensional reduction

To describe stationary black hole solutions we proceed with the usual procedure of dimensional reduction along the time coordinate. For details of this procedure see e.g. [72, 22]. The ansatz for the metric and the gauge fields is

$$\begin{aligned} \mathrm{d}s^2 &= -e^{2U}(\mathrm{d}t+\omega)^2 + e^{-2U}\mathrm{d}s^2_{(3d)}, \\ \mathcal{F}^I &= \mathrm{d}\mathcal{A}^I = \mathrm{d}(\zeta^I(\mathrm{d}t+\omega) + A^I). \end{aligned} \tag{4.9}$$

Here, $\mathrm{d}s^2_{(3d)} = h_{ab}\mathrm{d}x^a\mathrm{d}x^b$ is the three dimensional line element with $a, b$ being 3d indices. The scalars $U$ and $\zeta^I$ and the one-forms $\omega$ and $A^I$ are considered to be three dimensional fields. One then dualizes $\omega$ and $A^I$ to scalars $\sigma$ and $\tilde{\zeta}_I$ as

$$\begin{aligned} \mathrm{d}\tilde{\zeta}_I &= \mathrm{Re}\mathcal{N}_{IJ}\mathrm{d}\zeta^J - e^{2U}\mathrm{Im}\mathcal{N}_{IJ} \star_{(3d)} (\mathrm{d}A^J + \zeta^J \mathrm{d}\omega), \\ \mathrm{d}\sigma &= e^{4U} \star_{(3d)} \mathrm{d}\omega + \zeta^I \mathrm{d}\tilde{\zeta}_I - \tilde{\zeta}_I \mathrm{d}\zeta^I. \end{aligned} \tag{4.10}$$

The resulting three dimensional theory is Euclidean gravity coupled to 16 real scalars $\{U, \sigma, \tau^i, \bar{\tau}^{\bar{i}}, \zeta^I, \tilde{\zeta}_I\}$ parametrizing the coset space $\frac{SO(4,4)}{SL(2,\mathbb{R})^{\times 4}}$. The Lagrangian can be written as

$$\mathcal{L} = -\frac{1}{2}\sqrt{h}R[h] + \sqrt{h}g_{mn}\partial_a \Phi^m \partial^a \Phi^n, \tag{4.11}$$

where $\Phi^m$ denotes the scalars with $m = 1, .., 16$. The metric $g_{mn}$ on $\frac{SO(4,4)}{SL(2,\mathbb{R})^{\times 4}}$ is expressed through the line element as

$$\begin{aligned} \frac{1}{4}g_{mn}\mathrm{d}\Phi^m \mathrm{d}\Phi^n = {} & G_{i\bar{j}}\mathrm{d}\tau^i \mathrm{d}\bar{\tau}^{\bar{j}} + \mathrm{d}U^2 + \frac{1}{4}e^{-4U}(\mathrm{d}\sigma + \tilde{\zeta}_I \mathrm{d}\zeta^I - \zeta^I \mathrm{d}\tilde{\zeta}_I)^2 \\ & + \frac{1}{2}e^{-2U}\big[\mathrm{Im}\mathcal{N}_{IJ}\mathrm{d}\zeta^I \mathrm{d}\zeta^J \\ & + (\mathrm{Im}\mathcal{N})^{-1}_{IJ}(\mathrm{d}\tilde{\zeta}_I - \mathrm{Re}\mathcal{N}_{IK}\mathrm{d}\zeta^K)(\mathrm{d}\tilde{\zeta}_J - \mathrm{Re}\mathcal{N}_{JL}\mathrm{d}\zeta^L)\big]. \end{aligned} \tag{4.12}$$

We can describe the second term in the 3d Lagrangian (4.11) as a sigma model with target space $SO(4,4)/SL(2)^{\times 4}$. The Lie algebra $\mathfrak{so}(4,4)$ has 28 generators. To describe this coset model we have to split this as $\mathfrak{so}(4,4) = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h}$ is an $\mathfrak{sl}_2^{\times 4}$ subalgebra and $\mathfrak{m}$ is its 16 dimensional fundamental representation which we denote as $(2,2,2,2)$. There are actually three ways to perform this split. We summarize bellow two of them which are relevant for our purposes.

$$\begin{array}{r} (2,2,2,2)\big\{ \\ \hline \mathfrak{sl}_2^{\times 4}\big\{ \end{array} \underbrace{\begin{array}{cc} H_\Lambda, & p_\Lambda = E_\Lambda + F_\Lambda \\ \hline & k_\Lambda = E_\Lambda - F_\Lambda \end{array}}_{(\mathfrak{sl}_2^{\times 4})_U} \Bigg| \underbrace{\begin{array}{cc} p^{Q_I} = E^{Q_I} - F^{Q_I}, & p^{P^I} = E^{P^I} - F^{P^I} \\ \hline k^{Q_I} = E^{Q_I} + F^{Q_I}, & k^{P^I} = E^{P^I} + F^{P^I} \end{array}}_{(2,2,2,2)_U} \tag{4.13}$$

Here, $H_\Lambda$, $E_\Lambda$, $F_\Lambda$, $\Lambda = 0, ..., 3$ and $E^{Q_I}$, $E^{P^I}$, $F^{Q_I}$, $F^{P^I}$, $I = 1, ..., 4$ are the 28 generators of $\mathfrak{so}(4,4)$. For an explicit definition of these generators and a detailed review of the splits, we refer to Appendix A.1. The subscript $U$ refers to the splitting suited to describe the action of the U-duality group $SL(2)^{\times 3}$. The extra $SL(2)$ factor in this split is the so called Ehlers symmetry, which always appears when an Einstein metric is dimensionally reduced along the time direction with a stationary ansatz. The subalgebras without the subscript answer the split suited to describe the 3d fields parametrizing the coset $SO(4,4)/SL(2)^{\times 4}$.

Let us represent an element of this coset by an $SO(4,4)$ matrix as[73, 22]

$$\mathcal{V} = e^{-UH_0} e^{-\frac{1}{2}\sum_i \log y_i H_i} e^{-\sum_i x_i E_i} e^{-\sum_I (\zeta^I E^{Q_I} + \tilde{\zeta}_I E^{P^I})} e^{-\frac{1}{2}\sigma E_0}, \tag{4.14}$$

Notice that there are no $F$-type generators in the above formula. This choice of gauge is called the Iwasawa gauge. The next step is to project the Mauer-Cartan 1-form to the 16 dimensional space spanned by the first line of (4.13) as

$$P_* = \frac{1}{2}(\mathrm{d}\mathcal{V}\mathcal{V}^{-1} + (\mathrm{d}\mathcal{V}\mathcal{V}^{-1})^{\#}). \tag{4.15}$$

Here, $^{\#}$ is the anti-involution defineing the horizontal split of (4.13) (see also e.q. (A.9)). One finds that the target space metric (4.12) can be written as the right invariant metric on $SO(4,4)/SL(2)^{\times 4}$:

$$g_{mn}\mathrm{d}\Phi^m \mathrm{d}\Phi^n = \mathrm{Tr}(P_*^2). \tag{4.16}$$

The other way of writing this line element is to define the matrix

$$\mathcal{M} = \mathcal{V}^{\#}\mathcal{V}. \tag{4.17}$$

Then we have

$$g_{mn}\mathrm{d}\Phi^m \mathrm{d}\Phi^n = \frac{1}{4}\mathrm{Tr}((\mathcal{M}^{-1}\mathrm{d}\mathcal{M})^2). \tag{4.18}$$

We note that on the contrary to $P_*$, in general $\mathcal{M}^{-1}\mathrm{d}\mathcal{M}$ is not required to sit inside the 16 dimensional subspace which is spanned by the first line of (4.13). The action of the 3d theory (4.11) can be conveniently written as

$$\begin{aligned} S &= \frac{1}{2}\int \star_{3\mathrm{d}} R[h] - \frac{1}{4}\int \mathrm{Tr}\left(\mathcal{M}^{-1}\mathrm{d}\mathcal{M} \wedge \star_{3\mathrm{d}}\mathcal{M}^{-1}\mathrm{d}\mathcal{M}\right) \\ &= \frac{1}{2}\int \star_{3\mathrm{d}} R[h] - \int \mathrm{Tr}\left(P_* \wedge \star_{3\mathrm{d}} P_*\right), \end{aligned} \tag{4.19}$$

so that the equations of motion are readily found to be

$$\begin{aligned} \mathrm{d}\left(\star_{3\mathrm{d}}\mathcal{M}^{-1}\mathrm{d}\mathcal{M}\right) &\equiv 2\cdot \mathrm{d}\left(\star_{3\mathrm{d}}\mathcal{V}^{-1}P_*\mathcal{V}\right) = 0, \\ R[h]_{ab} - \frac{1}{2}R[h]h_{ab} &= \mathrm{Tr}\big(-2(\mathcal{M}^{-1}\partial_a\mathcal{M})(\mathcal{M}^{-1}\partial_b\mathcal{M}) \\ &\quad + h_{ab}(\mathcal{M}^{-1}\partial_a\mathcal{M})(\mathcal{M}^{-1}\partial^a\mathcal{M})\big). \end{aligned} \tag{4.20}$$

The 16 three dimensional fields can be extracted directly from $\mathcal{M}$ (see [22] for details) so we may proceed describing the theory in terms of $\mathcal{M}$. A group element $h \in SO(4,4)$ acts naturaly on $\mathcal{V}$ as

$$\mathcal{V} \mapsto q\mathcal{V}h, \tag{4.21}$$

where $q \in SL(2)^{\times 4}$ is a (possibly field dependent) compensator which puts $\mathcal{V}$ back to the Iwasawa gauge. We will see an example of this in sec. 4.1.4 when we work out the action of the U-duality group on $\mathcal{V}$. The same action in terms of $\mathcal{M}$ reads as

$$\mathcal{M} \mapsto h^{\#}\mathcal{M}h. \tag{4.22}$$

It is then manifest that the line element and hence the 3d Lagrangian is invariat under this action of $SO(4,4)$.

### 4.1.3 Asymptotic charges

The general single centered, non-extremal, rotating, asymptotically Taub-NUT black hole solution of the STU model have 11 independent conserved charges[21, 22]. These are the mass, NUT charge, angular momentum and 8 independent dyonic charges. These charges can be extracted from the fields as follows. Define the inverse radial coordinate $\rho = \frac{1}{r}$ and expand the fields around asymptotic infinity. We assume the following expansion

$$\begin{aligned} e^{2U} &= 1 - 2M\rho + O(\rho^2), & \zeta^I &= Q_I\rho + O(\rho^2), \\ \sigma &= -4N\rho + (4J\cos\theta + c)\rho^2 + O(\rho^3), & \tilde{\zeta}_I &= P^I\rho + O(\rho^2), \\ y_i &= Y_i\left(1 - \Sigma_i\rho + O(\rho^2)\right), & x_i &= X_i + \Xi_i\rho + O(\rho^2). \end{aligned} \tag{4.23}$$

Here $M$ and $J$ are the ADM mass and angular momentum of the spacetime, $Q_I$ and $P^I$ are electric and magnetic charges associated to the original 4d $U(1)$ gauge fields and $\Xi_i$, $\Sigma_i$ denote 6 scalar charges. The constant $c$ is not playing any role in the following. We can conveniently pack these asymptotic charges into an element $Q$ of $\mathfrak{so}(4,4)$. In order to do this first expand the coset matrix $\mathcal{M}$ of (4.17) around asymptotic infinity as

$$\mathcal{M} = \mathcal{M}^{(0)} + \mathcal{M}^{(1)}\rho + O(\rho^2). \tag{4.24}$$

Note that the matrix $\mathcal{M}^{(1)}$ does not contain the angulat momentum $J$ which only enters in subleading order. Now notice that $\mathcal{M} = \mathcal{V}^{\#}\mathcal{V}$ is a proper element of $SO(4,4)$ for every value of the coordinates. It follows that $\mathcal{M}^{(0)}$ and $(\mathcal{M}^{(0)})^{-1}\mathcal{M}$ are all good $SO(4,4)$ elements. Expanding this latter yields

$$(\mathcal{M}^{(0)})^{-1}\mathcal{M} = I + Q\rho + O(\rho^2), \tag{4.25}$$

where we have defined the "dressed" charge matrix[69]

$$Q = (\mathcal{M}^{(0)})^{-1}\mathcal{M}^{(1)}. \tag{4.26}$$

As this is an expansion of an element of $SO(4,4)$ around the identity, $Q$ is indeed an element of the Lie algebra $\mathfrak{so}(4,4)$. We can also write this charge matrix in terms of the coset representative (4.14). In order to do this we assume the expansion

$$\mathcal{V} = \mathcal{V}^{(0)} + \mathcal{V}^{(1)}\rho + O(\rho^2). \tag{4.27}$$

Plugging this into the projected Mauer-Cartan form (4.15) and comparing the result with the definition of $\mathcal{M}$ (4.17) we see that $P_*$ has the expansion

$$P_* = \frac{1}{2}\mathcal{V}^{(0)}Q(\mathcal{V}^{(0)})^{-1}\mathrm{d}\rho + O(\rho), \tag{4.28}$$

which is the usual form in which the asymptotic charge matrix appears in the literature [74] for static extremal solutions[1] .

In most of what follows we set the moduli to $X_i = 0$ and $Y_i = 1$ so that $\mathcal{M}^{(0)}$ is just the identity matrix. However, we stress that $X_i$ and $Y_i$ do transform under U-duality. One should not worry about this too much as everything we do in the following relies only on the fact that $Q$ is an element of $\mathfrak{so}(4,4)$ which is independent of this choice and hence straightforwardly generalize to arbitrary $X_i$ and $Y_i$. We will treat the cases when $X_i \neq 0$ and $Y_i \neq 1$ separately in section 4.1.5. Setting $X_i = 0$ and $Y_i = 1$ results in the

[1] The equation of motion for $P_*$ can be found from (4.19), and it is just the conservation law $\mathrm{d}(\star_{3\mathrm{d}}\mathcal{V}^{-1}P_*\mathcal{V}) = 0$. Conservation here means conservation with respect to radial evolution in $\rho$. This only makes sense when we can slice the 3d metric into nice $S^2$s, i.e. for static solutions. In this case the corresponding Noether charge is precisely $2\pi Q$.

simple form of the charge vector $Q$

$$Q = 2MH_0 + \sum_i \Sigma_i H_i + 2Np_0 - \sum_i \Xi_i p_i - \sum_I (Q_I p^{Q_I} + P^I p^{P^I}), \quad (4.29)$$

which we then call the "undressed" charge matrix. We see that in this case $Q$ lives in the 16 dimensional subspace $(2,2,2,2)$ spanned by the first line of (4.13).

### 4.1.4 The action of the U-duality group

In this section we describe in detail the transformation properties of the asymptotic charges of the black hole under U-duality. This allows us to construct polynomial invariants of these in the next section.

#### Action on fields

Recall that the coset element is parametrized in Iwasawa gauge as

$$\mathcal{V} = e^{-UH_0} e^{-\frac{1}{2}\sum_i \log y_i H_i} e^{-\sum_i x_i E_i} e^{-\sum_I (\zeta^I E^{Q_I} + \tilde{\zeta}_I E^{P^I})} e^{-\frac{1}{2}\sigma E_0}. \quad (4.30)$$

Notice that the gauge is such that the scalar fields are in an "upper triangular" form in the sence that the $F$ type generators are not present. We claim that the U-duality group is generated by the generators $H_i$, $E_i$, $F_i$, $i = 1,2,3$ of (4.13) and the action is a simple right action on $\mathcal{V}$ followed by the left action of a local compensator $q \in SO(2)^{\times 3} \subset SL(2)_U^{\times 3}$ restoring the Iwasawa gauge:

$$\mathcal{V} \mapsto q\mathcal{V}g, \quad g \in SL(2)_U^{\times 3}. \quad (4.31)$$

Indeed, we may write this as

$$q\mathcal{V}g = e^{-UH_0}\left(qe^{-\frac{1}{2}\sum_i \log y_i H_i} e^{-\sum_i x_i E_i} g\right)\left(g^{-1} e^{-\sum_I (\zeta^I E^{Q_I} + \tilde{\zeta}_I E^{P^I})} g\right) e^{-\frac{1}{2}\sigma E_0}, \quad (4.32)$$

as the generators $E_0$ and $H_0$ commute with $SL(2)_U^{\times 3}$. We see that the part with the potentials transform simply with the adjoint action. It is convenient to describe the 16 dimensional representation $(2,2,2,2)_U$ and its element $\sum_I (\zeta^I E^{Q_I} + \tilde{\zeta}_I E^{P^I})$ with the amplitudes $\psi_{ijkl}$ of a four qubit state through (A.7). Since there are no $F^{P^I}$ and $F^{Q_I}$ generators present this four qubit state has $\psi_{1jkl} \equiv 0$ and therefore it is actually a three qubit state. We can think of $\psi_{1jkl} \equiv 0$ as the gauge fixing condition. Then, under the U-duality transformation

$$\sum_I (\zeta^I E^{Q_I} + \tilde{\zeta}_I E^{P^I}) \mapsto g^{-1} \sum_I (\zeta^I E^{Q_I} + \tilde{\zeta}_I E^{P^I}) g, \quad g \in SL(2)_U^{\times 3}, \quad (4.33)$$

this associated state transform as

$$\psi_{ijkl} \mapsto (S_1)_j{}^{j'} (S_2)_k{}^{k'} (S_3)_l{}^{l'} \psi_{ij'k'l'}, \quad S_1 \otimes S_2 \otimes S_3 \in SL(2)_U^{\times 3}, \quad (4.34)$$

where, according to (A.7), $S_1 \otimes S_2 \otimes S_3$ is the $SL(2)_U^{\times 3}$ element associated to $g^{-1}$. We see that the gauge condition $\psi_{1jkl} \equiv 0$ on the coset element does not change. Therefore, this gives the required action of the U-duality group on the potentials given in (4.7) and no compensator is needed. We only need to worry about the gauge condition on the scalars. Multiplying the scalar term from the right with $g$ spoils the gauge we chose and hence we need a local compensator. Using the fact that each scalar parametrizes the coset space $SL(2)/SO(2)$ we expect the local compensator to be from $SO(2)^{\times 3}$[75] and therefore to have the form

$$q = e^{\sum_{i=1}^{3}\alpha_i k_i} = e^{\sum_{i=1}^{3}\alpha_i(E_i - F_i)}. \tag{4.35}$$

Now it is easy to verify that if we let $g$ to be the $SL(2)_U^{\times 3}$ element corresponding to

$$\begin{pmatrix} d_1 & -b_1 \\ -c_1 & a_1 \end{pmatrix} \otimes \begin{pmatrix} d_2 & -b_2 \\ -c_2 & a_2 \end{pmatrix} \otimes \begin{pmatrix} d_3 & -b_3 \\ -c_3 & a_3 \end{pmatrix}, \tag{4.36}$$

in the standard representation, then in order to restore the Iwasawa gauge the $\alpha_i$ of the compensator have to be chosen as

$$\tan\alpha_i = \frac{c_i y_i}{c_i x_i + a_i}. \tag{4.37}$$

Then, we have

$$q e^{-\frac{1}{2}\sum_i \log y_i H_i} e^{-\sum_i x_i E_i} g = e^{-\frac{1}{2}\sum_i \log y_i' H_i} e^{-\sum_i x_i' E_i}, \tag{4.38}$$

with the primed scalars being

$$x_i' = \frac{(d_i + c_i x_i)(b_i + a_i x_i) + a_i c_i (x_i^2 + y_i^2)}{(d_i + c_i x_i)^2 + c_i^2 y_i^2}, \quad y_i' = \frac{y_i}{(d_i + c_i x_i)^2 + c_i^2 y_i^2}, \tag{4.39}$$

which just corresponds to the usual action of the U-duality group on the scalars

$$\tau_i' = \frac{a_i \tau_i + b_i}{c_i \tau_i + d_i}, \tag{4.40}$$

with $\tau_i = x_i + i y_i$.

### Action on asymptotic charges

Now recall that the asymptotic values of the fields are conveniently encoded in a charge matrix $Q$ (see (4.26)) defined from the series expasion of $\mathcal{M} = \mathcal{V}^{\#}\mathcal{V}$ around asymptotic infinity. From the previous subsection we conclude that $\mathcal{M}$ transforms under 4d U-duality as $\mathcal{M} \mapsto g^{\#}\mathcal{M}g$, and hence the charge matrix simply transforms with the adjoint action

$$Q \mapsto g^{-1} Q g, \quad g \in SL(2)_U^{\times 3}. \tag{4.41}$$

Now from (4.29) we easily see that under the decomposition (4.13) of $\mathfrak{so}(4,4)$ suitable for U-duality, $Q$ has components both in ${\mathfrak{sl}_2^{\times 4}}_U$ and $(2,2,2,2)_U$. As the splitting ensures that these components do not mix under the adjoint action of ${\mathfrak{sl}_2^{\times 4}}_U$ we may consider these parts separately

$$Q = Q_- + Q_+, \tag{4.42}$$

where

$$
\begin{aligned}
Q_- &= 2MH_0 + \sum_i \Sigma_i H_i + 2Np_0 - \sum_i \Xi_i p_i \in \mathfrak{sl}_2^{\times 4}{}_U, \\
Q_+ &= -\sum_I (Q_I p^{Q_I} + P^I p^{P^I}) \in (2,2,2,2)_U.
\end{aligned}
\tag{4.43}
$$

Let us first consider $Q_-$. Clearly, $M$ and $N$ are invariant under the adjoint action of $SL(2)_U^{\times 3}$ as expected. The six scalar charges $\Sigma_i$, $\Xi_i$ parametrize an element $\mathfrak{sl}_2^{\times 3}{}_U$ and hence they transform in the *adjoint representation* of $SL(2)_U^{\times 3}$. We define the matrices

$$
R_i = \begin{pmatrix} \Sigma_i & -\Xi_i \\ -\Xi_i & -\Sigma_i \end{pmatrix}, \tag{4.44}
$$

transforming under U-duality as

$$
R_i \mapsto \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} R_i \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}^{-1}, \tag{4.45}
$$

with $\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in SL(2)$. Note that the symmetricity of $R_i$ is spoiled by this transformation but this is simply a consequence of fixing the asymptotic values of the scalars which, again, are not invariant under U-duality (see section 4.1.5. for the general form of $R$).

The element $Q_+$ transforms as a four qubit state $\psi_{ijkl}$ under the full $SL(2)_U^{\times 4}$ and it decomposes into a pair of three qubit states $(\psi_1)_{jkl} \equiv \psi_{0jkl}$ and $(\psi_2)_{jkl} \equiv \psi_{1jkl}$ when just the U-duality group $SL(2)_U^{\times 3}$ is used. The explicit amplitudes can be read off using (A.7) and are given as

$$
\left(\begin{array}{cccc}
\psi_{0000} & \psi_{0001} & \psi_{0010} & \psi_{0011} \\
\psi_{0100} & \psi_{0101} & \psi_{0110} & \psi_{0111} \\ \hline
\psi_{1000} & \psi_{1001} & \psi_{1010} & \psi_{1011} \\
\psi_{1100} & \psi_{1101} & \psi_{1110} & \psi_{1111}
\end{array}\right)
=
\left(\begin{array}{cccc}
-P^4 & -Q_3 & -Q_2 & P^1 \\
-Q_1 & P^2 & P^3 & Q_4 \\ \hline
-Q_4 & P^3 & P^2 & Q_1 \\
P^1 & Q_2 & Q_3 & -P^4
\end{array}\right). \tag{4.46}
$$

Note that this pair is related through

$$
|\psi_2\rangle = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} |\psi_1\rangle. \tag{4.47}
$$

For the corresponding pair of three qubit states dressed with non-trivial scalar asymptotics, see again the next section 4.1.5. The index corresponding to the first qubit transforms as a doublet under the extra Ehlers $SL(2)$. Note that the scalar charges are singlets under the Ehlers symmetry: the adjoint 28 of $SO(4,4)$, where $Q$ lives in, decomposes under the maximal subgroup $SL(2)_U^{\times 4} = SL(2)_{Ehlers} \times (SL(2)_U^{\times 3})$ as $28 = (3,1) \oplus (1,9) \oplus (2,8)$.

### 4.1.5 Dealing with scalar asymptotics

Here we consider explicitly the case of general asymptotic values $X_i$ and $Y_i$ of the moduli. We expand the "dressed" charge matrix defined in (4.26) using (4.23) as

$$Q = q_{H_\Lambda} H_\Lambda + q_{E_\Lambda} E_\Lambda + q_{F_\Lambda} F_\Lambda + q_{E^{Q_I}} E^{Q_I} + q_{E^{P^I}} E^{P^I} + q_{F^{Q_I}} F^{Q_I} + q_{F^{P^I}} F^{P^I}, \tag{4.48}$$

Note that in general this matrix does not live in the 16 dimensional four qubit subspace corresponding to the first line of (4.13). However, we will soon see that there is always a 4d U-duality transformation which rotates it back into this subspace. The part in $\mathfrak{sl}_2^{\times 4}{}_U$ reads as

$$\begin{aligned} Q_- &= 2MH_0 + 2Np_0 \\ &+ \sum_{i=1}^{3} \left[ \left( \Sigma_i - \frac{\Xi_i X_i}{Y_i^2} \right) H_i + \left( -\Xi_i - 2\Sigma_i X_i + \frac{\Xi_i X_i^2}{Y_i^2} \right) E_i - \frac{\Xi_i}{Y_i^2} F_i \right], \end{aligned} \tag{4.49}$$

and hence the $R_i$ matrices of (4.44) obtain the following dressing

$$R_i = \begin{pmatrix} \Sigma_i - \frac{\Xi_i X_i}{Y_i^2} & -\Xi_i - 2\Sigma_i X_i + \frac{\Xi_i X_i^2}{Y_i^2} \\ -\frac{\Xi_i}{Y_i^2} & -\Sigma_i + \frac{\Xi_i X_i}{Y_i^2} \end{pmatrix}. \tag{4.50}$$

The four qubit state in $(2,2,2,2)_U$ is

$$Q_+ = q_{E^{Q_I}} E^{Q_I} + q_{E^{P^I}} E^{P^I} + q_{F^{Q_I}} F^{Q_I} + q_{F^{P^I}} F^{P^I}. \tag{4.51}$$

The amplitudes corresponding to the first three qubit state (see (A.7)) are unchanged

$$\begin{aligned} \begin{pmatrix} \psi_{0000} & \psi_{0001} & \psi_{0010} & \psi_{0011} \\ \psi_{0100} & \psi_{0101} & \psi_{0110} & \psi_{0111} \end{pmatrix} &\equiv \begin{pmatrix} q_{E^{P^4}} & q_{E^{Q_3}} & q_{E^{Q_2}} & -q_{E^{P^1}} \\ q_{E^{Q_1}} & -q_{E^{P^2}} & -q_{E^{P^3}} & -q_{E^{Q_4}} \end{pmatrix} \\ &= \begin{pmatrix} -P^4 & -Q_3 & -Q_2 & P^1 \\ -Q_1 & P^2 & P^3 & Q_4 \end{pmatrix}. \end{aligned} \tag{4.52}$$

On the other hand the second three qubit state

$$\begin{pmatrix} \psi_{1000} & \psi_{1001} & \psi_{1010} & \psi_{1011} \\ \psi_{1100} & \psi_{1101} & \psi_{1110} & \psi_{1111} \end{pmatrix} = \begin{pmatrix} -q_{F^{Q_4}} & q_{F^{P^3}} & q_{F^{P^2}} & q_{F^{Q_1}} \\ q_{F^{P^1}} & q_{F^{Q_2}} & q_{F^{Q_3}} & -q_{F^{P^4}} \end{pmatrix} \tag{4.53}$$

has the following dressing

$$\psi_{1ijk} = D^{(1)}_{ii'} D^{(2)}_{jj'} D^{(3)}_{kk'} \psi_{0i'j'k'}, \tag{4.54}$$

or equivalently,

$$|\psi_2\rangle = (D^{(1)} \otimes D^{(2)} \otimes D^{(3)}) |\psi_1\rangle, \tag{4.55}$$

with

$$D^{(i)} = \frac{1}{Y_i} \begin{pmatrix} X_i & -X_i^2 - Y_i^2 \\ 1 & -X_i \end{pmatrix}. \tag{4.56}$$

This is the generalization of the relation (4.47) and shows that the pair of three qubit states are related by a moduli dependent $SL(2)_U^{3\times}$ transformation.

Now we can also relate the charge matrix of an arbitrary solution through

U-duality to an auxiliary charge matrix with trivial scalar asymptotics. Indeed, looking at (4.14) it is manifest that we can remove the scalar hair from the coset element $\mathcal{V}$ by the U-duality transformation $\mathcal{V} \mapsto \mathcal{V}\mathcal{J}^{-1}$ where

$$\mathcal{J} = e^{-\frac{1}{2}\sum_i \log Y_i H_i} e^{-\sum_i X_i E_i} \equiv \mathcal{V}^{(0)}, \tag{4.57}$$

is the $SO(4,4)$ matrix corresponding to the $SL(2)_U^{\times 3}$ element

$$J_1 \otimes J_2 \otimes J_3 \in SL(2)_U^{\times 3}, \quad J_i = \frac{1}{\sqrt{Y_i}} \begin{pmatrix} 1 & -X_i \\ 0 & Y_i \end{pmatrix}. \tag{4.58}$$

Notice that due to (4.37) this transformation does not require a local compensator to go back to the Iwasawa gauge with $\mathcal{V}$. One may check with explicit computation that the asymptotic value of the coset element (4.24) indeed satisfies

$$\mathcal{J}^{\#}\mathcal{J} = \mathcal{M}^{(0)}. \tag{4.59}$$

As a consequence, for any general charge matrix $Q$ we may define a duality transformed one

$$\tilde{Q} = \mathcal{J} Q \mathcal{J}^{-1}, \tag{4.60}$$

which corresponds to a black hole with trivial moduli. From (4.28) it is immediate that the projected Mauer-Cartan form expands as

$$P_* = \frac{1}{2}\tilde{Q}\mathrm{d}\rho + O(\rho), \tag{4.61}$$

and hence $\tilde{Q}$ is inside the 16 dimensional subspace $(2,2,2,2)$ spanned by the first line of (4.13). To obtain the explicit expression for $\tilde{Q}$ we use the relations

$$D^{(i)} = J_i^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} J_i, \quad R_i = J_i^{-1} \begin{pmatrix} \Sigma_i & -\frac{\Xi_i}{Y_i} \\ -\frac{\Xi_i}{Y_i} & -\Sigma_i \end{pmatrix} J_i \tag{4.62}$$

to relate the auxiliary charges of $\tilde{Q}$ to the physical ones of $Q$. Following 4.1.4 we deduce that the dyonic charge vectors $|\tilde{\psi}_{1,2}\rangle$ of $\tilde{Q}$ are expressed with the physical charges $|\psi_1\rangle$ as

$$\begin{aligned} |\tilde{\psi}_1\rangle &= (J_1 \otimes J_2 \otimes J_3)|\psi_1\rangle, \\ |\tilde{\psi}_2\rangle &= (J_1 \otimes J_2 \otimes J_3)(D^{(1)} \otimes D^{(2)} \otimes D^{(3)})|\psi_1\rangle \\ &\equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (J_1 \otimes J_2 \otimes J_3)|\psi_1\rangle. \end{aligned} \tag{4.63}$$

We see that the vectors $|\tilde{\psi}_{1,2}\rangle$ are indeed related as in (4.47) which is valid only for the trivial moduli. For the scalar charges one has

$$\tilde{R}_i = J_i R_i J_i^{-1} \equiv \begin{pmatrix} \Sigma_i & -\frac{\Xi_i}{Y_i} \\ -\frac{\Xi_i}{Y_i} & -\Sigma_i \end{pmatrix}. \tag{4.64}$$

Since the entropy of black holes is blind to the 4d U-duality transformation (4.60) we obtained the result that for any asymptotics we may calculate the entropy by just using the formula for canonical moduli with replacing the dyonic charges by $(J_1 \otimes J_1 \otimes J_3)|\psi_1\rangle$ and scaling the scalar charges as

$\Xi_i \mapsto \frac{\Xi_i}{Y_i}$, i.e.

$$S(X_i, Y_i, |\psi_1\rangle, \Sigma_i, \Xi_i) = S\left(X_i = 0, Y_i = 1, (J_1 \otimes J_2 \otimes J_3)|\psi_1\rangle, \Sigma_i, \frac{\Xi_i}{Y_i}\right). \tag{4.65}$$

### 4.1.6 The most general non-extremal single center black hole solution

Here we give a short review of some of the results and the solution generating technique presented in [22]. We have seen that if $\mathcal{M}$ is a solution to the equations of motion then so is $h^{\#}\mathcal{M}h$. This fact can be used to generate new solutions from a known seed. This method is used by the authors of [22] to find the most general single centered stationary black hole solutions of the STU model. They chose the four dimensional Kerr-Taub-NUT metric as their seed with mass $m$, NUT charge $n$ and angular momentum $J = ma$ (here $a$ is a parameter). This solution can be given in terms of the 3d coset language as follows[76, 22]. The 3d metric $h_{ab}$ is

$$\mathrm{d}s^2_{3\mathrm{d}} = \frac{R\mathcal{U}}{a^2}\mathrm{d}\phi^2 + (R - \mathcal{U})\left(\frac{\mathrm{d}r^2}{R} + \frac{\mathrm{d}u^2}{\mathcal{U}^2}\right), \tag{4.66}$$

where $u = n + a\cos\theta$, $\phi$ and $\theta$ are the usual angular coordinates and

$$R = r^2 + a^2 - n^2 - 2mr, \quad \mathcal{U} = a^2 - n^2 - u^2 + 2nu. \tag{4.67}$$

The coset matrix is expressed as

$$\mathcal{M}_{KTN} = \frac{1}{R - \mathcal{U}}\times \\ \begin{pmatrix} R-\mathcal{U} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & R-\mathcal{U} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r^2+u^2 & 0 & 0 & 0 & 0 & 2(mu-nr) \\ 0 & 0 & 0 & r^2+u^2 & 0 & 0 & -2(mu-nr) & 0 \\ 0 & 0 & 0 & 0 & R-\mathcal{U} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & R-\mathcal{U} & 0 & 0 \\ 0 & 0 & 0 & -2(mu-nr) & 0 & 0 & (r-2m)^2+(u-2n)^2 & 0 \\ 0 & 0 & 2(mu-nr) & 0 & 0 & 0 & 0 & (r-2m)^2+(u-2n)^2 \end{pmatrix}. \tag{4.68}$$

This solution is then charged up with the group element[22]

$$h = e^{-\sum_I \gamma_I k^{P^I}} e^{-\sum_I \delta_I k^{Q_I}}, \tag{4.69}$$

as

$$\mathcal{M} = h^{\#}\mathcal{M}_{KTN}h. \tag{4.70}$$

The next thing we need to know is how the physical charges of the black hole are expressed with the 11 parameters $m, n, a, \delta_I, \gamma_I$. Without $J$, there are all together 16 asymptotic charges in $Q$, which can be expressed in terms of 10 seed & charge parameters, $m$, $n$, $\delta_I$, $\gamma_I$. Therefore, the scalar charges are not independent of $M$, $N$ and the dyonic charges, but we will keep them explicit until section 4.1.9. Notice though that when we turn on the asymptotic moduli $X_i$ and $Y_i$ this black hole is described by 16 independent parameters, which is the number of components in the auxilary charge matrix $\tilde{Q}$ of (4.60). Therefore, there is a sense in which we can think of the scalar charges as being independent variables: we exchange the moduli dependence for them.

We quote the formula for the mass and the NUT charge

$$M = \mu_1 m + \mu_1 n, \quad N = \nu_1 m + \nu_2 n, \tag{4.71}$$

where $\mu_1$, $\mu_2$, $\nu_1$, $\nu_2$ are functions only of $\delta_I$, $\gamma_I$ and are given explicitly as

$$\begin{aligned} \mu_1 &= 1 + \sum_I \left( \frac{s_{\delta I}^2 + s_{\gamma I}^2}{2} - s_{\delta I}^2 s_{\gamma I}^2 \right) + \frac{1}{2} \sum_{I,J} s_{\delta I}^2 s_{\gamma J}^2, \\ \mu_2 &= \sum_I s_{\delta I} c_{\delta I} \left( \frac{s_{\gamma I}}{c_{\gamma I}} c_{\gamma 1234} - \frac{c_{\gamma I}}{s_{\gamma I}} s_{\gamma 1234} \right), \end{aligned} \tag{4.72}$$

$$\nu_1 = \sum_I s_{\gamma I} c_{\gamma I} \left( \frac{c_{\delta I}}{s_{\delta I}} s_{\delta 1234} - \frac{s_{\delta I}}{c_{\delta I}} c_{\delta 1234} \right), \quad \nu_2 = \iota - D, \tag{4.73}$$

with

$$\begin{aligned} \iota &= c_{\delta 1234} c_{\gamma 1234} + s_{\delta 1234} s_{\gamma 1234} + \sum_{I<J} c_{\delta 1234} \frac{s_{\delta IJ}}{c_{\delta IJ}} \frac{c_{\gamma IJ}}{s_{\gamma IJ}} s_{\gamma 1234}, \\ D &= c_{\delta 1234} s_{\gamma 1234} + s_{\delta 1234} c_{\gamma 1234} + \sum_{I<J} c_{\delta 1234} \frac{s_{\delta IJ}}{c_{\delta IJ}} \frac{s_{\gamma IJ}}{c_{\gamma IJ}} c_{\gamma 1234}. \end{aligned} \tag{4.74}$$

Here the notation is resolved as follows: $c_{\delta_I} = \cosh \delta_I$, $s_{\delta_I} = \sinh \delta_I$ and the same for $\gamma_I$. Multiple indices denote that one should take the product of the hyperbolic functions e.g. $c_{\delta IJ} = \cosh \delta_I \cosh \delta_J$. The rest of the charges can be obtained through the formulae

$$\frac{\partial 2M}{\partial \delta_I} = Q_I, \quad \frac{\partial 2N}{\partial \delta_I} \quad = -P^I, \tag{4.75}$$

$$\frac{\partial Q_I}{\partial \delta_J} = \begin{pmatrix} 2M-\Sigma_1+\Sigma_2+\Sigma_3 & & & \\ & 2M+\Sigma_1-\Sigma_2+\Sigma_3 & & \\ & & 2M+\Sigma_1+\Sigma_2-\Sigma_3 & \\ & & & 2M-\Sigma_1-\Sigma_2-\Sigma_3 \end{pmatrix}, \tag{4.76}$$

$$\frac{\partial P^I}{\partial \delta_J} = \begin{pmatrix} -2N & \Xi_3 & \Xi_2 & -\Xi_1 \\ \Xi_3 & -2N & \Xi_1 & -\Xi_2 \\ \Xi_2 & -\Xi_1 & -2N & -\Xi_3 \\ -\Xi_1 & -\Xi_2 & -\Xi_3 & -2N \end{pmatrix}, \tag{4.77}$$

$$\begin{aligned} \frac{\partial \Sigma_i}{\partial \delta_J} &= \begin{pmatrix} -Q_1 & Q_2 & Q_3 & -Q_4 \\ Q_1 & -Q_2 & Q_3 & -Q_4 \\ Q_1 & Q_2 & -Q_3 & -Q_4 \end{pmatrix}, \\ \frac{\partial \Xi_i}{\partial \delta_J} &= \begin{pmatrix} -P^4 & P^3 & P^2 & -P^1 \\ P^3 & -P^4 & P^1 & -P^2 \\ P^2 & P^1 & -P^4 & -P^3 \end{pmatrix}. \end{aligned} \tag{4.78}$$

Note that these identities are simple consequences of the fact that we have defined the charging up element $h$ with the $\delta_I$ parameters being on the right. Therefore, for the charge matrix $Q = h^{\#} Q_{KTN} h$ one has

$$\frac{\partial Q}{\partial \delta_I} = [k^{Q_I}, Q], \tag{4.79}$$

and hence taking the $\delta_I$ derivative of a component of $Q$ in some basis just amounts to multiplying with the adjoint representation of $k^{Q_I}$ in the same basis. We have used here that $h$ is generated by the subalgebra in the second line of (4.13) and hence we have $h^{\#} = h^{-1}$.

After reconstructing the 4d solution the Bekenstein-Hawking entropy of the black hole can be calculated. It reads as[22]

$$S = 2\pi(\sqrt{\Delta + F} + \sqrt{-J^2 + F}), \tag{4.80}$$

where $\Delta = -\frac{1}{16}\mathrm{HDet}(\psi_{ijk})$ is the quartic invariant (hyperdeterminant) of (1.24) formed from the dyonic charges via (4.46) and $F$ is expressed with the seed and charge parameters as

$$F = (m^2 + n^2)(m\nu_2 - n\nu_1)^2. \tag{4.81}$$

To obtain asymptotically flat black holes one can cancel the NUT charge by setting $n = -m\frac{\nu_1}{\nu_2}$.

### 4.1.7 The $F$-invariant

The aim of this section is to construct an expression for the quantity $F$ appearing in the entropy formula (4.80) in terms of the asymptotic charges of the black hole. In general, we expect $F$ to be a very complicated function, but we know that it has to be invariant under the four dimensional U-dualtity group $SL(2)_U^{\times 3}$. This is because the quantized version of this symmetry is expected to be a symmetry of the full string theoy spectrum[77] and the black hole entropy only counts the number of states with a given set of charges. As all other quantities appearing in the entropy are U-duality invariants, the quantity $F$ has to be an U-duality invariant as well. Our strategy to obtain an expression for $F$ is then the following. We construct the polynomial duality invariants which can be formed from the asymptotic charge vector $Q$ of (4.29). It turns out that there are only a reasonable amount of these invariants with a homogeneous degree agreeing with that of $F$. Then we take linear combinations of these invariants and find the coefficients by requiring the expression to agree with the parametric expression of (4.81) for $F$.

In order to be able to write up invariants we first need to construct covariants with indices transforming the same way. We have seen that this transforms under four dimensional U-duality as a (not general) vector in $9 \oplus 8 \oplus 8$, where $9$ refers to the adjoint representation, while $8$ is the fundamental corresponding to a three qubit state. Luckily, there exists a construction, called the moment map, which allows one to associate an element transforming in $9$ to a *pair* of vectors in $8$. Unfortunately, this construction will result in an unnecessaryly large covariant algebra. We can significantly reduce this by incorporating "triality" symmetry of the STU model: the symmetry under permutation of the three $SL(2)$ factors. This leads us to consider the embedding $\mathfrak{sl}_2^{\times 3}{}_U \subset \mathfrak{sl}_6$ and to construct the moment map in the $SL(6)$ covariant language of three fermions with six single particle states, which is described in great detail in section 3.4

Now recall that an unnormalized three fermion state with six modes, as described in section 3.4, can be written as

$$|P\rangle = \frac{1}{3!} P_{abc} p^a p^b p^c |0\rangle \in \wedge^3(\mathbb{C}^6), \tag{4.82}$$

where $p^a$ are creation operators associated to some single particle basis and the antisymmetric tensor $P_{abc}$ have 20 independent components. The SLOCC group of this system contains $SL(6,\mathbb{C})$ which acts locally on the amplitudes as

$$P_{abc} \mapsto S_a{}^{a'} S_b{}^{b'} S_c{}^{c'} P_{a'b'c'}, \quad S \in SL(6,\mathbb{C}). \tag{4.83}$$

Now let us recall that we have a $6\times 6$ matrix $K_P$ defined in (3.26), bilinear in the amplitudes, that transforms nicely under this action

$$K_P \mapsto (S^T)^{-1} K_P S^T. \tag{4.84}$$

There is a single independent continous invariant that we can form for which is quartic in the amplitudes. We recall its formula here:

$$\mathcal{D}(P) = \frac{1}{6}\mathrm{Tr} K_P^2. \tag{4.85}$$

Note that this quantity is a measure of tripartite entanglement for the fermions, see section 3.4.4 for details. The situation is different if we have two states $|P\rangle$ and $|Q\rangle$ at our disposal. In this case we can define the following covariants

$$\begin{aligned}
(K_P)^a{}_b &= \frac{1}{2!3!}\epsilon^{ac_1c_2c_3c_4c_5} P_{bc_1c_2} P_{c_3c_4c_5}, \\
(K_Q)^a{}_b &= \frac{1}{2!3!}\epsilon^{ac_1c_2c_3c_4c_5} Q_{bc_1c_2} Q_{c_3c_4c_5}, \\
(K_{PQ})^a{}_b &= \frac{1}{2!3!}\epsilon^{ac_1c_2c_3c_4c_5} P_{bc_1c_2} Q_{c_3c_4c_5}, \\
(K_{QP})^a{}_b &= \frac{1}{2!3!}\epsilon^{ac_1c_2c_3c_4c_5} Q_{bc_1c_2} P_{c_3c_4c_5}.
\end{aligned} \tag{4.86}$$

This is the polarization of (3.26). Traces of products of these define invariants. In particular the fermionic Mukai pairing of section 2.4.1 can be expressed as

$$(P,Q) = \frac{1}{3}\mathrm{Tr}K_{PQ} = -\frac{1}{3}\mathrm{Tr}K_{QP} = \frac{1}{3!3!}\epsilon^{c_1c_2c_3c_4c_5c_6} P_{c_1c_2c_3} Q_{c_4c_5c_6}. \tag{4.87}$$

Notice that, in the Clifford algebra language of section 2.4.2 we can write these quantities as the bilinars

$$(K_{PQ})^a{}_b = -(P, p^a n_b Q). \tag{4.88}$$

Now recall that we can identify three qubit states with the single occupancy states of this fermionic system, see section 3.4.3 for details. The identification is given by

$$|\psi\rangle \mapsto |P_\psi\rangle = \sum_{i,j,k=0}^{1} \psi_{ijk} p^{i+1} p^{j+3} p^{k+5} |0\rangle. \tag{4.89}$$

Then, it is easy to see that a three qubit SLOCC transformation

$$\psi_{ijk} \mapsto (S_1)_i{}^{i'} (S_2)_j{}^{j'} (S_3)_k{}^{k'} \psi_{ij'k'}, \quad S_1 \otimes S_2 \otimes S_3 \in SL(2)_U^{\times 3}, \tag{4.90}$$

can be implemented in the language of of three fermions (4.83) by choosing

$$S = \begin{pmatrix} S_1 & & \\ & S_2 & \\ & & S_3 \end{pmatrix} \in SL(6, \mathbb{C}). \tag{4.91}$$

As permutations of these three $2 \times 2$ blocks is also in $SL(6)$, this way we also covariantize the triality symmetry of the STU model.

The 8 dyonic charges transform under U-duality as a three qubit state under SLOCC. We can therefore associate covariants transforming in the adjoint of $SL(2)_U^{\times 3}$ to the pair of three qubit states given in (4.46) using (4.86) and (4.89) as

$$\begin{aligned}
(K_{11})^a{}_b &= \frac{1}{2!3!} \epsilon^{ac_1c_2c_3c_4c_5} (P_{\psi_1})_{bc_1c_2} (P_{\psi_1})_{c_3c_4c_5}, \\
(K_{22})^a{}_b &= \frac{1}{2!3!} \epsilon^{ac_1c_2c_3c_4c_5} (P_{\psi_2})_{bc_1c_2} (P_{\psi_2})_{c_3c_4c_5}, \\
(K_{12})^a{}_b &= \frac{1}{2!3!} \epsilon^{ac_1c_2c_3c_4c_5} (P_{\psi_1})_{bc_1c_2} (P_{\psi_2})_{c_3c_4c_5}, \\
(K_{21})^a{}_b &= \frac{1}{2!3!} \epsilon^{ac_1c_2c_3c_4c_5} (P_{\psi_2})_{bc_1c_2} (P_{\psi_1})_{c_3c_4c_5}.
\end{aligned} \tag{4.92}$$

The main point of defining these bilinears borrowed from entanglement theory is that they transform exactly the same way under $SL(2)_U^{\times 3}$ as the scalar charge matrix

$$R = \begin{pmatrix} R_1^T & & \\ & R_2^T & \\ & & R_3^T \end{pmatrix} = \begin{pmatrix} \Sigma_1 & -\Xi_1 & 0 & 0 & 0 & 0 \\ -\Xi_1 & -\Sigma_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Sigma_2 & -\Xi_2 & 0 & 0 \\ 0 & 0 & -\Xi_2 & -\Sigma_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Sigma_3 & -\Xi_3 \\ 0 & 0 & 0 & 0 & -\Xi_3 & -\Sigma_3 \end{pmatrix}, \tag{4.93}$$

see e.q. (4.44).

The four matrices of (4.92) can be grouped into a $2 \times 2$ block matrix $K_{ab}$ transforming under the Ehlers $SL(2)$ as $2 \times 2 = 1 \oplus 3$ in its $ab$ indices. We note that the singlet part satisfies the relation

$$K_{21} - K_{12} = \Big( \sum_I (Q_I^2 + (P^I)^2) \Big) I, \tag{4.94}$$

and hence only three of the $K_{ab}$s give independent covariants.

Let us proceed by listing the independent primitive invariants of homogeneous degree less than or equal to four in the asymptotic charges, that can be formed by our covariants.

- Degree 1 invariants:
$$M, \quad N. \tag{4.95}$$
- Degree 2 invariants:
$$\text{Tr}(K_{12}), \quad \text{Tr}(R^2). \tag{4.96}$$
- Degree 3 invariants:
$$\text{Tr}(K_{12}R), \quad \text{Tr}(K_{11}R). \tag{4.97}$$
- Degree 4 invariants:
$$\text{Tr}K_{11}^2, \quad \text{Tr}K_{12}^2, \quad \text{Tr}(K_{11}K_{22}), \quad \text{Tr}(R^4). \tag{4.98}$$

To reduce the set of independent invariants we have used the identities

$$\begin{aligned} \text{Tr}(K_{11}R) &= -\text{Tr}(K_{22}R), & \text{Tr}(K_{11}^2) &= \text{Tr}(K_{22}^2), \\ \text{Tr}(K_{11}R^2) &= \text{Tr}(K_{22}R^2) = 0, & \text{Tr}(K_{12}K_{11}) &= -\text{Tr}(K_{12}K_{22}), \\ \text{Tr}(K_{12}R^2) &= \frac{1}{6}(\text{Tr}K_{12})(\text{Tr}R^2), & 3\text{Tr}(K_{11}K_{22}) + (\text{Tr}K_{12})^2 - 3\text{Tr}(K_{12}^2) &= 0. \end{aligned} \tag{4.99}$$

Note that we can write some well-known U-duality invariants in this language. First of all, Cayley's hyperdeterminant

$$\Delta = \frac{1}{16}\left(4(Q_1Q_2Q_3Q_4 + P^1P^2P^3P^4) + 2\sum_{J<K} Q_JQ_KP^JP^K - \sum_J (Q_J)^2(P^J)^2\right) \tag{4.100}$$

is just given as

$$\Delta = -\frac{1}{96}\text{Tr}K_{11}^2 = -\frac{1}{96}\text{Tr}K_{22}^2. \tag{4.101}$$

With the notations of section 3.4, we can write this as $\Delta = -\frac{1}{16}\mathcal{D}(P_\psi)$. Similarly, the relation to the hyperdeterminant HDet($\psi_{ijk}$) of (1.24) is just

$$\Delta = -\frac{1}{16}\text{HDet}(\psi_{ijk}), \tag{4.102}$$

where $\psi_{ijk}$ is given by (4.46). The asymptotic value of the quadratic symplectic invariant $I_2$ is

$$I_2^\infty = \frac{1}{4}\sum_I (Q_I^2 + (P^I)^2) = -\frac{1}{12}\text{Tr}K_{12}. \tag{4.103}$$

The quadratic invariant

$$S_2^\infty = \frac{1}{4}G_{i\bar{j}}\partial_r\tau^i\partial_r\bar{\tau}^{\bar{i}}|_{r\to\infty} = \frac{1}{4}\sum_i (\Xi_i^2 + \Sigma_i^2), \tag{4.104}$$

can be expressed as

$$S_2^\infty = \frac{1}{8}\text{Tr}(R^2). \tag{4.105}$$

Now recall the formula for the $F$ invariant in terms of the charge-up parameters $\delta_I$, $\gamma_I$ and seed variables $m$ and $n$:

$$F = (m^2 + n^2)(m\nu_2 - n\nu_1)^2, \tag{4.106}$$

where $\nu_1$ and $\nu_2$ are the functions given in e.q. (4.73) and they do not scale with the charges. We see that $F$ is of homogeneous degree 4 in the charges and we expect it to be U-duality invariant. From the 10 invariants that we have identified at the begining of this section we can form 22 monomials of homogeneous degree four. We can form a linear combination of these, equate it to $F$ and try to solve for the coefficients. The simplest way to do this is to generate various random sets of parameters $m$, $n$, $\delta_I$ and $\gamma_I$ and try to solve the resulting numerical, linear equations simultenaously. If this works for a considerably higher number of equations than the number of variables, which is 22, we can probably trust our coefficients. We did this procedure for 600 equations and we have found a single solution. The obtained numerical coefficients have been rationalized and the result was tested analyticaly with a computer algebra system. The result is [P.5]

$$\begin{aligned} F = M^4 + M^2N^2 + \frac{M^2}{12}\mathrm{Tr}K_{12} - \frac{M}{24}\mathrm{Tr}(K_{12}R) + \frac{N^2}{24}\mathrm{Tr}(R^2) \\ - \frac{N}{24}\mathrm{Tr}(K_{11}R) + \frac{1}{192}\left(\mathrm{Tr}(K_{11}^2) - \mathrm{Tr}(K_{11}K_{22}) - \frac{1}{2}(\mathrm{Tr}R^2)^2 + \mathrm{Tr}(R^4)\right). \end{aligned} \tag{4.107}$$

In the asymptotically flat case one sets $N = 0$ (or $n = -m\frac{\nu_1}{\nu_2}$). In this case the $F$ invariant reads as

$$\begin{aligned} F = m^4\frac{(\nu_1^2+\nu_2^2)^3}{\nu_2^4} = M^4 - M^2 I_2^\infty - \frac{M}{24}\mathrm{Tr}(K_{12}R) - \frac{1}{2}\Delta \\ - \frac{1}{192}\mathrm{Tr}(K_{11}K_{22}) - \frac{1}{6}(S_2^\infty)^2 + \frac{1}{192}\mathrm{Tr}(R^4), \end{aligned} \tag{4.108}$$

where we have reintroduced the familiar U-duality invariants where it is possible. We stress that for general scalar asymptotics one should use the $R$ and $K_{ab}$ matrices as given in section 4.1.5. It is useful to write the $F$ invariant without an explicit reference to the auxilary 6 dimensional representation that we have introduced. We may employ the invariant bilinear product of (4.87) to write

$$\begin{aligned} \mathrm{Tr}K_{12} &= 3(P_{\psi_1}, P_{\psi_2}), \\ \mathrm{Tr}(K_{12}R) &= -(P_{\psi_1}, R_* P_{\psi_2}), \\ \mathrm{Tr}(K_{11}K_{22}) &= -(P_{\psi_1}, (K_{22})_* P_{\psi_1}), \\ \mathrm{Tr}(K_{11}^2) &= -(P_{\psi_1}, (K_{11})_* P_{\psi_1}), \end{aligned} \tag{4.109}$$

where we have defined the action of a Lie algebra element $t \in \mathfrak{sl}(6)$ on $P \in \wedge^3\mathbb{C}$ as

$$(t_*P)_{abc} = t^d{}_a P_{dbc} + t^d{}_b P_{adc} + t^d{}_c P_{abd}. \tag{4.110}$$

Also, let us define the $8\times 8$ matrix $\hat{R}$ corresponding to $R$ in the fundamental representation of the U-duality group:

$$\hat{R} = R_1^T \otimes I \otimes I + I \otimes R_2^T \otimes I + I \otimes I \otimes R_3^T, \tag{4.111}$$

see (4.44) for the definition of $R_i$s. Then we have $\mathrm{Tr}\hat{R}^2 = 4\mathrm{Tr}R^2$ and

$$\mathrm{Tr}R^4 - \frac{1}{2}(\mathrm{Tr}R^2)^2 = -\frac{1}{8}\left(\mathrm{Tr}\hat{R}^4 - \frac{1}{8}(\mathrm{Tr}\hat{R}^2)^2\right). \tag{4.112}$$

We may then rewrite the $F$ invariant as

$$\begin{aligned} F = M^4 + M^2N^2 + \frac{M^2}{4}(P_{\psi_1}, P_{\psi_2}) + \frac{M}{24}(P_{\psi_1}, R_* P_{\psi_2}) + \frac{N^2}{96}\mathrm{Tr}(\tilde{R}^2) \\ + \frac{N}{24}(P_{\psi_1}, R_* P_{\psi_1}) - \frac{1}{192}\big(P_{\psi_1}, (K_{11} - K_{22})_* P_{\psi_1}\big) \\ + \frac{1}{1536}\left(\frac{1}{8}(\mathrm{Tr}\hat{R}^2)^2 - \mathrm{Tr}(\hat{R}^4)\right), \end{aligned} \tag{4.113}$$

which will be well suited for generalization to the $E_7$ invariant case.

As a final remark, we note that a single centered STU back hole parametrized by six moduli and eight dyonic charges is expected to have five independent U-duality invariants[78]. The reason that we have more than this in (4.96)-(4.98) is that we treat the scalar charges as independent variables. This allowed us to turn $F$ into a polynomial invariant.

### 4.1.8 $E_7$ invariant entropy for black holes in $\mathcal{N} = 8$ supergravity

As we have mentioned all 4d black holes of maximal $\mathcal{N} = 8$ supergravity can be obtained from STU black holes via U-dualities. Therefore, if we manage to write up (4.113) with the use of $E_7$ invariants, we automatically obtain the entropy formula for these black holes. This was partially done in [P.5] and then subsequently completed in [79]. In this section we review this generalization.

We have seen that in the STU case the charge matrix is an element of $\mathfrak{so}(4,4)$ and the U-duality group $SL(2)_U^{\times 3} \subset SO(4,4)$ acts on it through the adjoint representation of $SO(4,4)$. This representation decomposes as $28 = 1 \oplus 1 \oplus 1 \oplus 9 \oplus 8 \oplus 8$. There is a general way of constructing invariants on $9 \oplus 8 \oplus 8$ which allowed us to identify the $F$-invariant. In the $\mathcal{N} = 8$ case the 3d coset model is $E_{8(8)}/SO^*(16)$ and the U-duality group is $E_{7(7)}$. We expect that in this case the asymptotic charges of the black hole parametrize a Lie-algebra element $Q \in \mathfrak{e}_8$ and the U-duality group $E_{7(7)} \subset E_{8(8)}$ just acts by the adjoint action of $E_{8(8)}$. This representation decomposes as $248 = 1 \oplus 1 \oplus 1 \oplus 133 \oplus 56 \oplus 56$ and hence the relevant representation space is $133 \oplus 56 \oplus 56$ with 56 replacing three qubit states containing dyonic charges and 133 replacing 9 containing the 70 scalar charges [2]. The moment map from pairs of 56 to 133 can be formulated. The construction goes as follows. There is an $E_{7(7)}$ invariant antisymmetric bilinear form on 56, let us denote this by $\langle .,. \rangle$. We may define an $\mathfrak{e}_7$ element $T_{\Psi_1\Psi_2}$ associated to the pair $\Psi_1, \Psi_2 \in 56$ by demanding

$$\kappa(T, T_{\Psi_1\Psi_2}) = \langle \Psi_1, T\Psi_2 \rangle, \quad \forall T \in \mathfrak{e}_7. \tag{4.114}$$

Here, $\kappa$ is the Killing form on $\mathfrak{e}_7$. We note that the covariants (4.86) may be obtained in the same way from the Mukai pairing (4.87) so that this is

[2]Recall that in the STU case 9 contained 6 scalar charges, this is just an artifact of fixing the scalar asymptotics.

indeed a direct generalization of these to $E_7$. Using $56 \cong \wedge^2\mathbb{C}^8 \oplus \wedge^2(\mathbb{C}^8)^*$ we may parametrize $\Psi_a$, $a = 1, 2$ with a pair of antisymmetric $8 \times 8$ matrixes:

$$\Psi_a = \big((x^{(a)})^{ij}, y^{(a)}_{ij}\big), \quad (x^{(a)})^{ij} = -(x^{(a)})^{ji}, \quad y^{(a)}_{ij} = -y^{(a)}_{ji} \tag{4.115}$$

and using $\mathfrak{e}_7 \cong \mathfrak{sl}_8 \oplus \wedge^4\mathbb{C}^8$ we can parametrize the generators as

$$T = (\Lambda^i{}_j, \Sigma_{ijkl}), \quad \Lambda^i{}_i = 0, \tag{4.116}$$

and $\Sigma_{ijkl}$ totaly antisymmetric. We refer to appendix A.2 for the commutation relations, the action of $\mathfrak{e}_7$ on $56$ and the Killing form in terms of this parametrization. The invariant bilinear product reads as

$$\langle\Psi_1, \Psi_2\rangle = (x^{(1)})^{ij} y^{(2)}_{ij} - (x^{(2)})^{ij} y^{(1)}_{ij}. \tag{4.117}$$

Then, using the definition (4.114) of the moment map and equations (A.21), (A.22) and (A.23), a short excercise reveals the explicit form of the moment map to be

$$\begin{aligned} T_{\Psi_a\Psi_b} &= \big((\Lambda_{(ab)})^i{}_j, (\Sigma_{(ab)})_{ijkl}\big), \\ (\Lambda_{(ab)})^i{}_j &= -\frac{1}{6}\big((x^{(a)})^{in} y^{(b)}_{jn} + (x^{(b)})^{in} y^{(a)}_{jn}\big) + \frac{1}{48}\big((x^{(a)})^{nm} y^{(b)}_{nm} + (x^{(b)})^{nm} y^{(a)}_{nm}\big)\delta^i{}_j, \\ (\Sigma_{(ab)})_{ijkl} &= \frac{1}{48}\big(\epsilon_{ijklmnop}(x^{(a)})^{mn}(x^{(b)})^{op} - y^{(a)}_{[ij}\, y^{(b)}_{kl]}\big). \end{aligned} \tag{4.118}$$

Note that in this formalism we have the Cartan-Cremmer-Julia invariant expressed as

$$\begin{aligned} I_4 &\equiv \frac{1}{2}\langle\Psi, T_{\Psi\Psi}\Psi\rangle \\ &= x^{ij}x^{kl}y_{ik}y_{jl} - \frac{1}{4}(x^{ij}y_{ij})^2 \\ &+ \frac{1}{96}\big(\epsilon_{ijklmnop}x^{ij}x^{kl}x^{mn}x^{op} + \epsilon^{ijklmnop}y_{ij}y_{kl}y_{mn}y_{op}\big). \end{aligned} \tag{4.119}$$

The conventions of [22] are such that $I_4 = 4\diamondsuit$ and $\diamondsuit$ is the one that reduces to $\Delta$ of (4.100) for the STU duality frame. The dyonic charges parametrize $\Psi_1, \Psi_2 \in 56$ generalizing $\psi_1, \psi_2$. The STU charges (4.46) sit inside this $\Psi_1$ and $\Psi_2$ as

$$\left(\begin{array}{cccc} y^{(2)}_{12} & (x^{(2)})^{34} & (x^{(2)})^{56} & y^{(2)}_{78} \\ (x^{(2)})^{78} & y^{(2)}_{56} & y^{(2)}_{34} & (x^{(2)})^{12} \\ \hline y^{(1)}_{12} & (x^{(1)})^{34} & (x^{(1)})^{56} & y^{(1)}_{78} \\ (x^{(1)})^{78} & y^{(1)}_{56} & y^{(1)}_{34} & (x^{(1)})^{12} \end{array}\right) = \frac{1}{\sqrt{2}}\left(\begin{array}{cccc} \psi_{0000} & \psi_{0001} & \psi_{0010} & \psi_{0011} \\ \psi_{0100} & \psi_{0101} & \psi_{0110} & \psi_{0111} \\ \hline \psi_{1000} & \psi_{1001} & \psi_{1010} & \psi_{1011} \\ \psi_{1100} & \psi_{1101} & \psi_{1110} & \psi_{1111} \end{array}\right)$$
$$= \frac{1}{\sqrt{2}}\left(\begin{array}{cccc} -P^4 & -Q_3 & -Q_2 & P^1 \\ -Q_1 & P^2 & P^3 & Q_4 \\ \hline -Q_4 & P^3 & P^2 & Q_1 \\ P^1 & Q_2 & Q_3 & -P^4 \end{array}\right), \tag{4.120}$$

with all the remaining $(x^{(a)})^{ij}$ and $y^{(a)}_{ij}$ vanishing. See [80, 81] for details. One imediately verifies that the relation to the fermionic inner product of e.q. (4.87) is simply

$$\langle \Psi_1, \Psi_2 \rangle = (P_{\psi_1}, P_{\psi_2}), \tag{4.121}$$

and that $\diamondsuit$ reduces to $\Delta$.

As the next step, we have to parametrize an element $R \in \mathfrak{e}_7$ with the 70 scalar charges. Denote the corresponding $56 \times 56$ matrix in the adjoint representation with $\mathcal{R}$. In the STU duality frame $\mathcal{R}$ should reduce to $\hat{R}$ of (4.111) on the eight dimensional $SL(2)^{\times 4}$ invariant subspace of 56, where the pair of (4.120) lives, and to some apriory unknown action on its complement which completes it into an elment of $\mathfrak{e}_7$. Such a complement part is clearly required as one may check explicitly that for any generator $T$ of $E_7$ represented in $56$ the invariant $\mathrm{Tr}T^4$ satisfies

$$\mathrm{Tr}T^4 = \frac{1}{24}(\mathrm{Tr}T^2)^2, \tag{4.122}$$

and hence the last part of the F invariant (4.113) involving $\mathrm{Tr}\hat{R}^4$ cannot be written directly in this form using $\mathcal{R}$. Finding the scalar charge matrix $\mathcal{R}$ requires the explicit knowledge of how the scalars parametrize the 4d coset $E_{7(7)}/SU(8)$ and how the STU duality frame sits inside this parametrization. This program was done recently in [79]. The coset representative here is chosen to be

$$\mathcal{V} = \exp\left(-\frac{1}{2}\sum_{i=1}^{3}\log y_i \vec{v}_i \cdot \vec{H}\right)\exp\left(-x_1 E^{127} - x_2 E^{347} - x_3 E^{567}\right), \tag{4.123}$$

for the definitions of $\vec{H}$ and $E^{ijk}$ see appendix A.2. The vectors $\vec{v}_i$ are[3]

$$\begin{aligned}
\vec{v}_1 &= \left(-\frac{1}{2}, -\frac{3}{2\sqrt{7}}, \frac{1}{2\sqrt{21}}, \frac{1}{2\sqrt{15}}, \frac{1}{2\sqrt{10}}, \frac{1}{2\sqrt{6}}, -\frac{1}{\sqrt{3}}\right), \\
\vec{v}_2 &= \left(-\frac{1}{4}, -\frac{3}{4\sqrt{7}}, \frac{2}{\sqrt{21}}, \frac{2}{\sqrt{15}}, -\frac{1}{2\sqrt{10}}, -\frac{1}{2\sqrt{6}}, \frac{1}{\sqrt{3}}\right), \\
\vec{v}_3 &= \left(-\frac{1}{4}, -\frac{3}{4\sqrt{7}}, -\frac{\sqrt{\frac{3}{7}}}{2}, -\frac{\sqrt{\frac{3}{5}}}{2}, \frac{1}{\sqrt{10}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{3}}\right).
\end{aligned} \tag{4.124}$$

The scalars $x_i$ and $y_i$ are the scalar fields of the STU duality frame. The scalar charge matrix $\mathcal{R}$ is obtained via expanding $\mathcal{M} = \mathcal{V}^T\mathcal{V}$ around $\rho = \frac{1}{r} \to 0$

$$\mathcal{M} = \mathcal{M}^{(0)} + \mathcal{M}^{(1)}\rho + O(\rho^2), \tag{4.125}$$

$$\mathcal{R} = (\mathcal{M}^{(0)})^{-1}\mathcal{M}^{(1)}. \tag{4.126}$$

We have

$$\mathrm{Tr}\hat{R}^2 = \frac{1}{3}\mathrm{Tr}\mathcal{R}^2, \tag{4.127}$$

[3]Notice that our $\vec{v}_1$ is minus the one in [79]. This is to obtain a correct match of $\mathcal{R}$ with $\hat{R}$ on the STU subspace.

and we also have [79]

$$3(\mathrm{Tr}\hat{R}^2)^2 - 8\mathrm{Tr}\hat{R}^4 = \frac{2}{3^4 5}\big(5(\mathrm{Tr}\mathcal{R}^2)^2 + \sqrt{5}\sqrt{5^3(\mathrm{Tr}\mathcal{R}^2)^4 - 2^8 3^3 11 \mathrm{Tr}\mathcal{R}^2\mathrm{Tr}\mathcal{R}^6 + 2^9 3^6 \mathrm{Tr}\mathcal{R}^8}\big). \tag{4.128}$$

We can also confirm the following relation through explicit calculation:

$$(P_{\psi_1}, R_* P_{\psi_2}) = -\frac{1}{2}\langle \psi_1, \mathcal{R}\psi_2\rangle. \tag{4.129}$$

Now everything is in place to write the $F$ invariant (4.113) in a manifestly $E_7$ invariant way as:

$$\begin{aligned} F =\, & M^4 + M^2N^2 + \frac{M^2}{4}\langle\Psi_1, \Psi_2\rangle - \frac{M}{48}\langle\Psi_1, \mathcal{R}\Psi_2\rangle - \frac{N}{48}\langle\Psi_1, \mathcal{R}\Psi_1\rangle \\ & + \frac{N^2}{288}\mathrm{Tr}(\mathcal{R}^2) - \frac{1}{192}\langle\Psi_1, (T_{\Psi_1\Psi_1} - T_{\Psi_2\Psi_2})\Psi_1\rangle \\ & - \frac{1}{2^{10}3^4}\left((\mathrm{Tr}\mathcal{R}^2)^2 - \frac{5}{3^4 2}\sqrt{5^3(\mathrm{Tr}\mathcal{R}^2)^4 - 2^8 3^3 11 \mathrm{Tr}\mathcal{R}^2\mathrm{Tr}\mathcal{R}^6 + 2^9 3^6 \mathrm{Tr}\mathcal{R}^8}\right). \end{aligned} \tag{4.130}$$

### 4.1.9 Relations between scalar charges and physical charges

The formula (4.107) is entirely in terms of the asymptotic charges of the black hole and is manifestly invariant under U-duality and permutation of scalars. However, it does contain explicitly the scalar charges $\Sigma_i$ and $\Xi_i$ which are not independent of $M$, $N$, $Q_I$ and $P^I$. A formula entirely in terms of the physical charges would require solving for the functions $\Sigma_i(M,N,Q,P)$, $\Xi_i(M,N,Q,P)$. In this section, we provide constraints that these functions must satisfy and solve them for some special cases. We illustrate on the example of the four electric charge Cvetič-Youm black hole that in general it is not possible to give the $F$-invariant in terms of radicals of the physical charges.

We start by describing the constraint equations. Recall, that the charge matrix $Q$ of (4.26) tranforms as a four qubit state (see (A.6)) under the action of the $SL(2)^{4\times}$ spanned by the second line of (4.13). The charge-up matrix $h$ of e.q. (4.69) is an element of this $SL(2)^{4\times}$. We have reviewed in section 3.6.3 that this 16 dimensional representation admits four algebraically independent continous invariants. All of these can be checked to be proportional to the appropriate power of the combination $(m^2+n^2)$ of the seed parameters. It follows that they provide 3 independent polynomial equations among the 16 asymptotic charges in $Q$. We could in principle write up these equations in terms of the invariants of (3.117) but it is algebraically somewhat simpler (but equivalent) to proceed slightly differently. Consider the characteristic polynomial of the charge matrix $Q$:

$$p(\lambda) = \det(\lambda I - Q). \tag{4.131}$$

It is clear that the characteristic polynomial is invariant under the charge up operation and hence we have

$$p(\lambda) = p_0(\lambda), \tag{4.132}$$

where the characteristic polynomial for the seed solution is

$$\begin{aligned} p_0(\lambda) &= \det(\lambda I - Q_{KTN}) \\ &= \lambda^4 \left(\lambda^2 - 4m^2 - 4n^2\right)^2 \\ &= \lambda^4 \left(\lambda^2 - \frac{1}{4}\mathrm{Tr}Q^2\right)^2, \end{aligned} \tag{4.133}$$

where we have used the invariance of Tr$Q^2$ to express $m^2 + n^2$ in terms of asymptotic charges. The two polynomials agree iff all of their coefficients agree hence we have the following 9 polynomial equations

$$\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}(p(\lambda) - p_0(\lambda))|_{\lambda=0} = 0, \quad k = 0, ..., 8. \tag{4.134}$$

One can check that for $k = 1, 3, 5, 6, 7, 8$ these are trivially satisfied and hence we are left with 3 equations. These 3 equations have linearly independent gradients in the scalar charges which indicates a three dimensional solution set. However, this does not tell us anything about the set of real solutions. One can easily check that as one approaches the seed solution by taking $Q_I \to 0$ and $P^I \to 0$, there is only one real root satisfying the consistency requirement $\Sigma_i \to 0$. This shows that it is possible that the three equations are enough to determine the scalar charges uniquely. To say something more precise about this one would need to determine at least the real dimension of the semialgebraic set defined by these equations, but to our knowledge, there is no method to do this to date. Instead, in the next section we provide solutions to (4.134) for some special charge vectors, where the $F$-invariant is known explicilty, and hence we can compare our results with the existing literature.

Before doing so, we comment on what happens with these constraints when one considers black holes with non-trivial asymptotic moduli. In this case one can just replace $Q$ in (4.131) with the auxiliary charge vector $\tilde{Q}$, as readily seen from (4.60). This shows that whatever expressions $\Sigma_i = f_i(M, |\psi_1\rangle)$, $\Xi_i = g_i(M, |\psi_1\rangle)$ we find for trivial moduli by solving (4.134), we can safely use them for non-trivial moduli as $\Sigma_i = f_i(M, (J_1 \otimes J_2 \otimes J_3)|\psi_1\rangle)$ and $\Xi_i = Y_i g_i(M, (J_1 \otimes J_2 \otimes J_3)|\psi_1\rangle)$. Combine this with (4.65) to get the expected result

$$F(X_i, Y_i, |\psi_1\rangle) = F(X_i = 0, Y_i = 1, (J_1 \otimes J_2 \otimes J_3)|\psi_1\rangle), \tag{4.135}$$

which is then guaranteed to be valid as long as we can use (4.134) to solve for the scalar charges. This is the case for the first two of the following examples.

### 4.1.10 Non-extremal special cases

#### Klauza-Klein black hole

The Klauza-Klein black hole[82] is obtained by setting $Q_2 = Q_3 = Q_4 = 0$ and $P^2 = P^3 = P^4 = 0$ and $N = 0$. From the parametrization (4.75)-(4.78) we can deduce that $\Xi_i = 0$ and $\Sigma_2 = \Sigma_3 = -\Sigma_1$ in this case. We can put this into (4.134) as an ansatz and observe that it automaticaly solves two equations. The third equation reads as

$$\Sigma_1\left(8M^2+(P^1)^2+Q_1^2-2\Sigma_1^2\right)=2M\left((P^1)^2-Q_1^2\right). \tag{4.136}$$

As expected, there is only one root that vanishes for zero charges. For this root we have found numerical agreement between our formula

$$F=\left(M^2-\frac{1}{4}(P^1)^2\right)\left(M^2-\frac{1}{4}Q_1^2\right)+\frac{1}{8}M\Sigma_1((P^1)^2-Q_1^2)-\frac{1}{16}\Sigma_1^4, \tag{4.137}$$

for the F-invarant and the complicated expression presented in [22] for the Klauza-Klein black hole. We may further specialize by setting $P^1=0$. In this case the above equation factorizes as

$$(2M+\Sigma_1)\left(Q_1^2+4M\Sigma_1-2\Sigma_1^2\right)=0. \tag{4.138}$$

The physical root is $\Sigma_1=M-\sqrt{M^2+\frac{1}{2}Q_1^2}$. Upon substituing this into the formula for the $F$-invariant we get

$$F=\frac{1}{64}\left(32M^4-40M^2Q_1^2-Q_1^4+4M(4M^2+2Q_1^2)^{\frac{3}{2}}\right), \tag{4.139}$$

in complete agreement with [22].

### $-iX^0X^1$ supergravity black hole

Now let us consider the axion-dilaton black hole of [83]. We set the electric and magnetic charges pairwise equal $Q_1=Q_4$, $Q_2=Q_3$, $P^1=P^4$ and $P^2=P^3$. We also set the NUT charge to zero. From (4.75)-(4.78) we observe that in this case we have $\Sigma_2=\Sigma_3=\Xi_2=\Xi_3=0$. Using this as an ansatz in (4.134) we are left with a single equation

$$\begin{aligned}2(P^1)^2&\left(2M\Sigma_1+(P^2)^2-Q_1^2-Q_2^2\right)+2Q_2\left(4M\Xi_1(P^2)+2MQ_2\Sigma_1+Q_1^2Q_2\right)\\&=4M^2\Xi_1^2+4M^2\Sigma_1^2+(P^1)(8M\Xi_1Q_1-8(P^2)Q_1Q_2)\\&+2(P^2)^2\left(2M\Sigma_1+Q_1^2+Q_2^2\right)+4MQ_1^2\Sigma_1+(P^1)^4+(P^2)^4+Q_1^4+Q_2^4,\end{aligned} \tag{4.140}$$

which admits a single real solution

$$\begin{aligned}\Sigma_1&=\frac{(P^1)^2-(P^2)^2-Q_1^2+Q_2^2}{2M},\\ \Xi_1&=\frac{P^2Q_2-P^1Q_1}{M}.\end{aligned} \tag{4.141}$$

This agrees with the axion-dilaton charge obtained in [83]

$$\Upsilon=i(\Xi_1-i\Sigma_1)=-\frac{(Q_1+iP^1)^2+(-P^2+iQ_2)^2}{2M}. \tag{4.142}$$

Upon inserting this into our formula (4.107) for the $F$-invariant we obtain

$$F=\frac{1}{16}\left(4M^2-(P^1-P^2)^2-(Q_1-Q_2)^2\right)\left(4M^2-(P^1+P^2)^2-(Q_1+Q_2)^2\right), \tag{4.143}$$

in complete agreement with [22, 83]. We note here that when the single modulus of this model is turned on we have to replace the charges according to (4.135). Explicitly, this leads to the following replacement rule

$$\begin{pmatrix} P^1 \\ Q_1 \end{pmatrix} \mapsto \frac{1}{\sqrt{Y_1}} \begin{pmatrix} 1 & -X_1 \\ 0 & Y_1 \end{pmatrix} \begin{pmatrix} P^1 \\ Q_1 \end{pmatrix}, \qquad \begin{pmatrix} Q_2 \\ -P^2 \end{pmatrix} \mapsto \frac{1}{\sqrt{Y_1}} \begin{pmatrix} 1 & -X_1 \\ 0 & Y_1 \end{pmatrix} \begin{pmatrix} Q_2 \\ -P^2 \end{pmatrix}, \tag{4.144}$$

which, again, agrees with [83].

### Dilute gas limit

We can recover the dilute gas limit of [84] as well. In this limit, we have the following constraint among the magnetic charges

$$P^1 + P^2 + P^3 + P^4 = 0. \tag{4.145}$$

Provided that this is true, all three equations of (4.134) can be solved *exactly* by setting

$$\begin{aligned} \Sigma_1 &= -2M + Q_2 + Q_3, & \Sigma_2 &= \ \ 2M - Q_2 - Q_4, & \Sigma_3 &= 2M - Q_3 - Q_4, \\ \Xi_1 &= P^2 + P^3, & \Xi_2 &= \quad -P^2 - P^4, & \Xi_3 &= -P^3 - P^4. \end{aligned} \tag{4.146}$$

These roots cannot be physical for all values of the charges as for vanishing charges we must have $\Sigma_i = 0$ . However, in the dilute gas limit, the charges are large and hence this requirement is outside of the region of validity. Define the excitation energy as $\delta M = M - M_{BPS} = M - \frac{1}{4}(Q_1 + Q_2 + Q_3 + Q_4)$. We obtain the dilute gas limit of F by substituing (4.146) into (4.107) and scaling $Q_i \to \mu^2 Q_i$, $i = 2, 3, 4$ and $P^I \to \mu P^I$, while keeping $\delta M$ fixed. Upon $\mu \to \infty$ the leading order $\mu^8$ in F vanishes. The next to leading order contribution is the coefficient of $\mu^6$ which is

$$F_0 = \frac{1}{2}\delta M Q_2 Q_3 Q_4, \tag{4.147}$$

which agrees with the result of [84].

### Four charge Cvetič-Youm black hole

Here we set all the magnetic charges and the NUT charge to zero but allow for arbitrary electric charges[85]. We have all $\Xi_i = 0$ but we still need to solve for all the $\Sigma_i$. Instead of trying to solve the constraint equations we can simply use that as all $\gamma_I = 0$, the equations for the $\delta_I$ derivatives (4.75)-(4.78) give the full Jacobian for the change of variables from boost parameters to asymptotic charges. The best thing to do is to write differential equations for the functions $Q_I(2M, \Sigma_i)$ because these are remarkably easily solved. The solution is such that

$$\sqrt{4Q_I^2 + C_I} = \frac{\partial Q_I}{\partial \delta_I}, \tag{4.148}$$

where the right hand side is understood to be given through (4.76). The $C_I$ are constants of integration. Comparing with the actual parametrization reveals that $C_I = 4m$, $I = 1, ..., 4$. The solution for the scalar charges is then easily obtained to be

$$\begin{aligned}
\Sigma_1 &= -\frac{1}{2}\sqrt{m^2+Q_1^2} + \frac{1}{2}\sqrt{m^2+Q_2^2} + \frac{1}{2}\sqrt{m^2+Q_3^2} - \frac{1}{2}\sqrt{m^2+Q_4^2},\\
\Sigma_2 &= \frac{1}{2}\sqrt{m^2+Q_1^2} - \frac{1}{2}\sqrt{m^2+Q_2^2} + \frac{1}{2}\sqrt{m^2+Q_3^2} - \frac{1}{2}\sqrt{m^2+Q_4^2},\\
\Sigma_3 &= \frac{1}{2}\sqrt{m^2+Q_1^2} + \frac{1}{2}\sqrt{m^2+Q_2^2} - \frac{1}{2}\sqrt{m^2+Q_3^2} - \frac{1}{2}\sqrt{m^2+Q_4^2}.
\end{aligned} \tag{4.149}$$

Plugging these expressions into the formula (4.107) for the $F$-invariant we recover the expression given in [22]. This expression still depends on the seed parameter $m$. It is determined in terms of the physical charges through the equation

$$M = \frac{1}{4}\left(\sqrt{m^2+Q_1^2} + \sqrt{m^2+Q_2^2} + \sqrt{m^2+Q_3^2} + \sqrt{m^2+Q_4^2}\right). \tag{4.150}$$

We happily acknowledge that the right hand side is greater than $\frac{1}{4}\big(Q_1 + Q_2+Q_3+Q_4\big)$, and hence the requirement of solvability is $M \geq M_{BPS}$. Note that this example illustrates that it is in general not possible to express the $F$-invariant in terms of radicals of the physical charges. Indeed, one may rewrite (4.150) as a system of five polynomial equations as $M = \frac{1}{4}\sum_{I=1}^4 x_I$ and $x_I^2 = m^2 + Q_I^2$ for the five variables $m^2$ and $x_I$. Then one may use some algorithm to cast this system into regular chains. The first element of the chain can be chosen to depend only on $m^2$ and then to aquire $m$ we need to consider only this equation and forget about the others[4]. We do not present this equation here due to its length but it is a general, fifth order polynomial equation for $m^2$. Then, due to the Abel-Ruffini theorem, one cannot have an expression for $m$ in terms of radicals of the coefficients.

### 4.1.11 Extremal limits and the black hole/qubit correspondence

As we have already mentioned, the auxilary charge vector $\tilde{Q}$ of (4.60) sits always inside the 16 dimensional four qubit subspace spanned by the first line of (4.13). It makes sense then to compare the classification of single centered STU black holes under 3d dualities to classification of four qubits, briefly reviewed in 3.6.3. This program was carried out for *extremal* solutions in [23].

Let us first consider the general nonextremal solutions reviewed in section 4.1.6. We have mentioned before that the four independent four qubit invariants which are also permutation invariants are all proportional to

[4]Note that not all of the roots of this equation are solutions to (4.150).

$m^2 + n^2$. Explicitly, the invariants of (3.117) read as

$$\begin{aligned} I_2 &= -6(m^2+n^2) = -\frac{3}{8}\mathrm{Tr}Q^2, \\ I_6 &= -\frac{3}{2}(m^2+n^2)^3 = -\frac{3}{2\cdot 4^6}(\mathrm{Tr}Q^2)^3, \\ I_8 &= \frac{3}{4}(m^2+n^2)^4 = \frac{3}{4^9}(\mathrm{Tr}Q^2)^4, \\ I_{12} &= \frac{3}{16}(m^2+n^2)^6 = \frac{3}{4^{14}}(\mathrm{Tr}Q^2)^6. \end{aligned} \tag{4.151}$$

We see something very interesting. Though this state is in the generic semisimple orbit $G_{abcd}$, *there is only a single four qubit SLOCC class that it intersects*. Indeed, recall from section 3.6, that two semisimple states $|\psi_1\rangle$ and $|\psi_2\rangle$ are in the same $GL(2)^{\times 4}$ class if there is a nonzero number $\lambda$ such that $I_p(|\psi_1\rangle) = \lambda^p I_p(|\psi_2\rangle)$ for the invariants [31]. In this case this nonzero number is just $\sqrt{\mathrm{Tr}Q^2}$. We can write the canonical form of this state in a very convenient way.

$$\begin{aligned} |Q_{KTN}\rangle = \frac{i}{4}\sqrt{m^2+n^2}\big(&|0000\rangle - |0011\rangle + |0101\rangle - |0110\rangle \\ &- |1001\rangle + |1010\rangle - |1100\rangle + |1111\rangle\big), \end{aligned} \tag{4.152}$$

which is just the canonical form of the semisimple family $G_{abcd}$ of table 3.6 with parameters $a = b = 0$ and $d = c = \frac{i}{2}\sqrt{m^2+n^2} = \frac{i}{8}\sqrt{\mathrm{Tr}Q^2}$. One can easily check that this choice reproduces (4.151) which, according to the discussions in 3.6.3, ensures that the two states are on the same $SL(2,\mathbb{C})^{\times 4}$ orbit[5],[6]. It is in fact not unexpected that not all charge vectors correspond to single centered black hole solutions, we will comment more on this soon. However, it is a very interesting question what kind of other stationary solutions of the STU model we have for charge vectors which are not on this particular semisimple orbit. A strongly related question is the following. The 3d coset model in maximal $\mathcal{N} = 8$ supergravity is $E_{8(8)}/SO^*(16)$ so that the charge vector in this case can be regarded as an even or odd fermionic state with eight single particle modes (see sec. 3.6), with $SO^*(16)$ being a real form of the extended SLOCC group of this system, replacing the $SL(2,\mathbb{R})^{\times 4}$ of four qubits. We have seen in sec. 3.6 that the semisimple orbits are parametrized by eight numbers instead of four and that the mapping between qubit classes and fermion classes is only injective but not every fermionic class can be represented as four qubits. This suggests that there could be some nonextremal stationary solutions in $\mathcal{N} = 8$ supergravity which are invisible from the $\mathcal{N} = 2$ point of view.

Now let us move on to discuss the extremal limits. A black hole is called extremal if it has vanishing Hawking temperature. This condition is equivalent with the requirement that the outer and inner horizons are at the same location, thus forming an infinite throat for the black hole. The inner and

[5] Of course, this is only true if $Q_{KTN}$ doesn't have a nilpotent part. It can be easily checked that this is the case.

[6] Notice that we can elliminate the complex coefficients by acting with $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ on any one of the four qubits.

outer horizons are located at[22]

$$r_\pm = m \pm \sqrt{m^2 + n^2 - a^2}, \tag{4.153}$$

where $m$, $n$ and $a$ are parameters of the seed solution. Their difference is given by

$$r_+ - r_- = 2\sqrt{m^2 + n^2 - a^2} = \frac{1}{2}\sqrt{\mathrm{Tr}Q^2\left(1 - \frac{J^2}{F}\right)}, \tag{4.154}$$

We see that there are two ways to get an extremal solution: we either set $J^2 = F$ or $\mathrm{Tr}Q^2 = 0$. The first case corresponds to fast rotating extremal solutions which are important e.g. in Kerr/CFT [86]. We will not discuss these solutions here, the interested reader should consult [22]. We just note that their entropy (4.80) is clearly given by

$$S = 2\pi\sqrt{\Delta + J^2}. \tag{4.155}$$

The latter case corresponds to *nilpotent charge vectors*. Indeed, it is clear from (4.151) that all the invariants vanish and hence the four qubit state corresponding to $\tilde{Q}$ is nilpotent, therefore, it is in one of the eight[7] nilpotent classes of table 3.6. The identifiaction of these classes with explicit extremal supergravity solutions was done in [23], see table II of that reference. Here it is claimed that the nilpotent orbits $L_{a_4=0}$, $L_{0_{5\oplus\bar{3}}}$ and $L_{0_{7\oplus\bar{1}}}$ of table 3.6 cannot be obtained as a limit of the general nonextremal black hole. It was later revealed in [87] that these three orbits in fact correspond to extremal *multicenter* solutions, see also [88]. This shows that some charge vectors corresponding to stationary solitions may not describe single centered black holes. Along these lines, it would be interesting to learn more about what nonextremal configurations the charge vectors in different semisimple orbits than the one defined by (4.151) may correspond to. The remaining five nilpotent orbits corresponding to single centered black holes are summarized in table 4.1.

| | |
|---|---|
| $L_{abc_2}$ | doubly-critical $\frac{1}{2}$BPS |
| $L_{a_2b_2}$ | critical $\frac{1}{2}$BPS and non-BPS |
| $L_{a_20_{3\oplus\bar{1}}}$ | lightlike $\frac{1}{2}$BPS and non-BPS |
| $L_{ab_2}$ | large non-BPS |
| $L_{0_{3\oplus\bar{1}}0_{3\oplus\bar{1}}}$ | large $\frac{1}{2}$BPS and small non-BPS |

TABLE 4.1: Nilpotent orbits corresponding to single centered extremal black holes. For representative states, take the corresponding representative of table 3.6 and set all the parameters to zero.

Let us examine closer the entropy of extremal black holes with nilpotent charge vectors. The inner and outer horizon entropies are [22][8]

$$S_\pm = 2\pi(\sqrt{\Delta + F} \pm \sqrt{F - J^2}), \tag{4.156}$$

[7] There are eight nilpotent classes under the action of $S_4 \ltimes SL_2^{\times 4}$ which split into 16 classes when only the action of $SL_2^{\times 4}$ is considered.

[8] These are just $\frac{A_\pm}{4G}$.

hence naively, in the an extremal limit we must have $S_+ = S_-$, i.e.

$$\sqrt{F - J^2} = 0, \tag{4.157}$$

so that $F = J^2$ or $F = J = 0$. We have already met the first case. The second case corresponds to static large $\frac{1}{2}BPS$ black holes for which $\tilde{Q} \in L_{0_{3\oplus\bar{1}}0_{3\oplus\bar{1}}}$. For these black holes

$$S = 2\pi\sqrt{\Delta}, \tag{4.158}$$

thus they are only possible with charges satisfying $\Delta > 0$. We may ask what about extremal black holes where the charges give $\Delta < 0$? The answer is that we may only conclude $S_+ = S_-$ at extremality when $r_+ = r_- \neq 0$. We may instead formally take a limit where we first set $m = 0$ and then take a nilpotent limit of $\tilde{Q}$ [78]. This way, in the end we have $r_+ = r_- = 0$ but during the limiting procedure $r_+ = -r_-$ which implies[9] $S_+ = -S_-$, i.e.

$$F = -\Delta. \tag{4.159}$$

These are the so called slow rotating non-BPS black holes with entropy

$$S = 2\pi\sqrt{-\Delta - J^2}. \tag{4.160}$$

For this branch, we clearly need to have $\Delta < 0$ and $J^2 \leq |\Delta|$. These solutions comprise the nilpotent orbit $L_{ab_2}$. The remaining three orbits of table 4.1 correspond to small black holes, which by definition means that they have vanishing horizon area.

### 4.1.12 BPS attractors and the black hole/qubit correspondence

Notice that the extremal entropy formulae (4.155), (4.158) and (4.160) depend only on $\Delta$ and $J$, both of which are invariant under 4d duality. The dependence on the scalar hair can be taken into account via a duality transformation as we have described in e.q. (4.65). But all scalar charge dependence have explicitly dropped out from all the extremal entropies and the quartic invariant $\Delta$ of the charges is clearly invairant under the action of $J_1 \otimes J_2 \otimes J_3$. We can conclude that the entropy of extremal black holes is *independent of the asymptotic values of the moduli*. Behind this phenomenon is the *attractor mechanism* of extremal black holes[89, 90] which ensures that the moduli stabilize on the horizon to values depending only on the dyonic charges and not on their values at infinity[91, 92]For static extremal black holes, the entropy can be universally written as $S = 2\pi\sqrt{|\Delta|}$. Using equation (4.102) and the definition (1.23) of the three tangle measuring tripartite entanglement for three qubits, the static extremal entropy reads as

$$S = \frac{\pi}{2}\sqrt{|\mathrm{HDet}(\psi_{ijk})|} \equiv \frac{\pi}{4}\sqrt{\tau_{ABC}}. \tag{4.161}$$

[9]Notice that the entropy formula (4.156) has a Cardy form $\sqrt{N_L}+\sqrt{N_R}$. We see that there are two qualitatively district extremal limits: setting the right moving sector to its vaccum $N_R = 0$ gives the $\frac{1}{2}$BPS and the overspining extremal limits while when the left movers are in the vacuum $N_L = 0$ we end up with the non-BPS extremal branch.

This is the original observation [14] which led to the study of the relationship between simple entangled systems and stringy black holes. The attractor mechanism was later related to an entanglement distillation procedure in this language[93, 94].

Let us elaborate on this a little more. To describe the *attractor equations* of the STU model, which give the values of the scalar moduli $\tau^i$, $i = 1, 2, 3$ on the horizon, it is convenient to think about the STU model (4.1) as the bosonic sector of type IIB supergravity compactified on the Calabi-Yau threefold $T^2 \times T^2 \times T^2$. Calabi-Yau three-folds naturaly arise in superstring theory as the six dimensional manifolds preserving upon compactification 1/4 of the 32 supercharges of the ten dimensional parent theory. We will have a little more to say about Calabi-Yau manifolds in the next section, for now, it is enough for us that they are complex (Kähler) manifolds with a nowhere vanishing holomorphic 3-form $\Omega$. For the case of our interest, $T^2 \times T^2 \times T^2$, this 3-form can be explicitly written as

$$\Omega = e^{K/2} \mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3, \tag{4.162}$$

where $z^i = u^i + \tau^i v^i$ are the holomorphic coordinates[10] of the tori, $i = 1, 2, 3$. The numbers $\tau^i$ parametrize the space of complex structure deformations of $T^2 \times T^2 \times T^2$ which is $H^{(2,1)}$ i.e. the space of twice holomorphic, once antiholomorhic harmonic 3-forms. This space is Kähler with potential

$$K(\tau, \bar{\tau}) \equiv -\log \left( i \int_{T^2 \times T^2 \times T^2} \Omega \wedge \bar{\Omega} \right) = -\log(8 y_1 y_2 y_3), \tag{4.163}$$

where $y_i = \mathrm{Im}\tau^i$. Upon compactification of IIB supergravity to $T^2 \times T^2 \times T^2$ the complex structure moduli $\tau^i$ become 4d fields which are identified with the fields (4.2) of the STU model. Now we are in position to write down the BPS attractor equations:

$$2\mathrm{Im}\left(\bar{Z}(\Gamma)\Omega\right)|_{\text{horizon}} = \Gamma, \tag{4.164}$$

where $\Gamma$ is a moduli independent closed 3-form encoding the dyonic charges[11] of the black hole, $Q_I$ and $P^I$. The function

$$Z(\Gamma) = \int_{T^2 \times T^2 \times T^2} \Gamma \wedge \Omega, \tag{4.165}$$

boils down to the $\mathcal{N} = 2$ central charge when evaluated at asymptotic infinity. However, we need to plug its value at the horizon into (4.164) which is different. It satisfies

$$|Z(\Gamma)|^2|_{\text{horizon}} = e^{-2U}|_{\text{horizon}} \equiv \frac{S}{\pi} = 2\sqrt{\Delta}. \tag{4.166}$$

Notice that (4.164) can be explicitly written as

$$\Gamma = -i(\bar{Z}(\Gamma) e^{K/2} \mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3 - Z(\Gamma) e^{K/2} \mathrm{d}\bar{z}^1 \wedge \mathrm{d}\bar{z}^2 \wedge \mathrm{d}\bar{z}^3), \tag{4.167}$$

[10] The antiholomorphic coordinates are obviously $\bar{z}^i = u^i + \bar{\tau}^i v^i$ and the complex structure $J$ is defined implicitly via $J(\mathrm{d}z^i) = i\mathrm{d}z^i$ and $J(\mathrm{d}\bar{z}^i) = -i\mathrm{d}\bar{z}^i$

[11] The 3-form $\Gamma$ corresponds to the Poincaré dual of the 3-cycle that a bunch of D3 brane wraps. Indeed, extremal BPS STU black holes may be interpreted as supersymmetric bound states of D3 branes in the IIB duality frame [95].

which looks like a 3-qubit GHZ state provided we identify

$$|000\rangle = e^{K/2}\mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3, \quad |111\rangle \quad = e^{K/2}\mathrm{d}\bar{z}^1 \wedge \mathrm{d}\bar{z}^2 \wedge \mathrm{d}\bar{z}^3. \tag{4.168}$$

There is a precise sense in which this identification is true: the space of harmonic 3-forms on $T^2 \times T^2 \times T^2$ is 8 dimensional and has a natural Hermitian inner product under which the above vectors are orthonormal, see [15] for further details on this.

The attractor equation (4.164) may be solved in the following way. We have a dual 3-form $\hat{\Gamma}$ defined as the derivative of $|Z(\Gamma)|^2$ via

$$|Z(\Gamma + \epsilon\gamma)|^2 = |Z(\Gamma)|^2 + \epsilon \int \hat{\Gamma} \wedge \gamma + O(\epsilon^2). \tag{4.169}$$

Clearly,

$$\hat{\Gamma} = -2\mathrm{Re}(\bar{Z}(\Gamma)\Omega), \tag{4.170}$$

so that

$$2\bar{Z}(\Gamma)\Omega = -\hat{\Gamma} + i\Gamma. \tag{4.171}$$

This form will be very useful in the next section where we explain the relation to Hitchin functionals. For now, it is enough to see that it indeed fixes the moduli in terms of the charges. Indeed, picking a basis of 3-cocycles $\theta_A$, $A = 1, ..., 8$ in $H^3(T^2 \times T^2 \times T^2)$ with intersection matrix $I_{AB} = \int \theta_A \wedge \theta_B$ we may expand $\Omega = X^A \theta_A$ and $\Gamma = \Gamma^A \theta_A$. Then, the projective[12] coordinates on the complex structure moduli space can be extracted as

$$t^A = \frac{X^A}{X^0} = \frac{2I^{AB}\partial_{\Gamma^B}\sqrt{\Delta} + i\Gamma^A}{2I^{0B}\partial_{\Gamma^B}\sqrt{\Delta} + i\Gamma^0}. \tag{4.172}$$

With an appropriate choice of $\theta^A$ we can have $t^{i+1} = \tau^i$ for $i = 1, 2, 3$ and $\Gamma^A = (Q_I, P^J)$.

## 4.2 Hitchin's functionals and measures of entanglement

In this section we briefly discuss Hitchin functionals, which define form theories on certain manifolds. As we will see, they have a rich and fruitful connection to physics as they are related to topological strings, BPS black holes[96] and fermionic entanglement theory[P.1]. In fact, much of the work in chapter 3 was done via borrowing some formalism from Hitchin's original papers[97, 98]. Also, the introduction of the extended SLOCC group of chapter 2 was largely inspired by the existence of generlized Hitchin functionals[99]. Despite of this, we will take a reversed route in this section and rely heavily on notions introduced in chapters 2 and 3 and introduce these functionals via the discussed *fermionic entanglement measures*.

At the very heart of the possibility of defining Hitchin's functionals is the already familiar concept of prehomogeneous vector spaces. Here, the pair $(G, V)$ which needs to be prehomogeneous is the local diffeomorphism group[13] $G = GL(d, \mathbb{R})$ and the space of $q$-forms $V = \wedge^q T_p X^*$ at a point $p$ of the manifold $X$. Indeed, when this space is prehomogeneous, there is a

[12] The change in the overal normalization of $\Omega$ is not a complex structure deformation.

[13] Or generalized local diffeomorphism group, see later.

*single* scalar function that we can associate to our $q$-form which is a natural candidate to define an action functional. The fact that $q$-forms for which this invariant is nonzero lie in a dense, open orbit ensures that the variational problem is well-posed. This property of $q$-forms is called *stability* in the corresponding literature.

Another unifying theme of the manifolds with Hitchin functionals is the notion of special holonomy, which has its roots in the requirement that the manifold is capable of preserving supersymmetries. Special holonomy means that the holonomy group of the oriented Riemannian manifold $X$ is restricted

$$\mathrm{Hol}(X) \subsetneq SO(d). \tag{4.173}$$

This requirement can be characterised via $p$-forms which are invariant under $\mathrm{Hol}(X)$. For manifolds which admit a covariantly constant spinor $\nabla\xi = 0$ this is automatically satisfied by the $p$-forms

$$(\xi^\dagger \gamma_{i_1} ... \gamma_{i_p} \xi) e^{i_1} \wedge ... \wedge e^{i_p}, \tag{4.174}$$

where $e^i$ are the veilbein. This turns out to be nontrivial for a very restricted set of cases. For example, in the case of Calabi-Yau three-folds, the invariant $p$-form is just the holomorphic 3-form $\Omega$ and the holonomy group is $SU(3) \subsetneq SO(6)$.

### 4.2.1 Six dimensions

#### Three forms

Let us consider the three fermion states with six single particle modes of section 3.4.2. These are elements of the vector space $\wedge^3\mathbb{C}^6$. Now consider a six dimensional orientable manifold $X$ and its cotangent space $T_pX^*$ at a point $p$ which is a six dimensional *real* vector space. We may consider differential 3-forms

$$\varrho = \frac{1}{3!}\varrho_{abc} \mathrm{d}x^a \wedge \mathrm{d}x^b \wedge \mathrm{d}x^c \in \wedge^3 T_pX^*. \tag{4.175}$$

We may loosely think about these as three fermion states (3.32) with *real* and position dependent amplitudes. The local diffeomorphism group $GL(6,\mathbb{R}) \subset GL(6,\mathbb{C})$ acts on the amplitudes of $\varrho$ as a real SLOCC transformation, see (2.42,4.83). Recall, that the quartic invariant of (3.34), measuring the tripartite entanglement, transforms under this action as

$$\mathcal{D}(\varrho) \mapsto (\det G)^2 \mathcal{D}(\varrho), \quad G \in GL(6,\mathbb{C}), \tag{4.176}$$

and hence $\sqrt{\mathcal{D}(\varrho)}$ transforms under diffeomorphisms the same way as a volume form. This can be made more precise by considering the 3-form $\hat{\varrho}$ corresponding to the dual three fermion state of e.q. (3.30). We clearly have

$$\frac{1}{2}\hat{\varrho} \wedge \varrho = \sqrt{-\mathcal{D}(\varrho)} \mathrm{d}x^1 \wedge ... \wedge \mathrm{d}x^6, \tag{4.177}$$

which can be integrated over $X$ to define the *Hitchin functional*

$$V_H(\varrho) = \frac{1}{2}\int_X \hat{\varrho} \wedge \varrho. \tag{4.178}$$

Now under the action of $GL(6,\mathbb{R})$ the stable GHZ class of table 3.2 splits into two classes, one with $\mathcal{D}(\varrho) > 0$ and one with $\mathcal{D}(\varrho) < 0$. Recall, that for static extremal STU black holes, the former corresponds to non-BPS and the latter to BPS charge configurations, see section 4.1.11. This is the reason why we have inserted an extra minus sign under the square root in (4.177): we want $\varrho$ to be in the cohomology class of a BPS wrapping configuration. Indeed, as we shall see, there is a sense in which we can think about $X$ as the extra dimensions of our type IIB compactification over the horizon of a BPS black hole and a critical point $\varrho$ of (4.178) as a closed 3-form encoding the warping configuration of the corresponding D3 branes. Now we quote one of the central results of [97].

**Theorem:** Let $X$ be an oriented closed six dimensional manifold and $\varrho$ a closed ($\mathrm{d}\varrho = 0$) real 3-form. Then the following two statements are equivalent.

- $\mathcal{D}(\varrho) < 0$ at every point of $X$ and $\varrho$ is a critical point of the functional (4.178) in a fixed $[\varrho]$ cohomology class.
- The almost complex structure $J_\varrho = \frac{K_\varrho}{\sqrt{-\mathcal{D}(\varrho)}}$, $J_\varrho^2 = -I$ is integrable and $\Omega = \varrho + i\hat{\varrho}$ is a nowhere vanishing holomorphic (with respect to $J_\varrho$) 3-form, hence $X$ is a Calabi-Yau manifold.

For the definition of the $6 \times 6$ matrix $K_\varrho$, we again refer to section 3.4.2. We omit the full proof which can be found in [97]. The basic ingedients are the following. Since $V_H(\varrho)$ is homogeneous of degree 2 in $\varrho$ (see (4.177)), its derivative satisfies[14]

$$V_H(\varrho) = \frac{1}{2}\int \frac{\delta V_H(\varrho)}{\delta \varrho} \wedge \rho, \tag{4.179}$$

so that by (4.178) we have $\frac{\delta V_H(\varrho)}{\delta \varrho} = \hat{\varrho}$. Now varying (4.178) as $\varrho \mapsto \varrho + \mathrm{d}\alpha$ and using the previous equation along with the fact that $X$ is closed leads to the Euler-Lagrange equation

$$\mathrm{d}\hat{\varrho} = 0, \tag{4.180}$$

which is equivalent with $\mathrm{d}\Omega = 0$. Equation (3.38) and the discussion around it shows that $\Omega$ is a $(3,0)$ form with respect to $J_\varrho$. Then $\bar{\partial}\Omega = \mathrm{d}\Omega = 0$ shows that it is indeed holomorphic, while the condition $\mathcal{D}(\varrho) < 0$ ensures that it is nowhere vanishing.

Note that while it is in general a very hard task to explicitly extract the Calabi-Yau metric of $X$, the existence of the nowhere vanishing holomorphic three-form $\Omega$ guarantees that a unique such metric exists due to Yau's theorem [100, 101]. This way we see that critical points of (4.178) associate Calabi-Yau metrics to cohomology classes $[\varrho] \in H^3(X)$. This is the natural generalization of what the attractor mechanism does: there, we have fixed $X$ to be Calabi-Yau but we let the metric vary in the moduli space of all Calabi-Yau metrics on $X$. The moduli is then fixed by the eight charges of the black hole which encode the cohomology class of $\Gamma$, dual to the 3-cycles that the D3 branes wrap. We can in fact see explicitly that the formula $\Omega = \varrho + i\hat{\varrho}$ gives the solution to the attractor equations: it is just the same as equation (4.171)! There, $\hat{\Gamma}$ is the derivative of

[14] To derive this, write $V_H(\lambda\varrho) = \lambda^2 V_H(\varrho)$, take the $\lambda$ derivative and set $\lambda = 1$.

$|Z(\Gamma)|^2|_{\text{horizon}} = 2\sqrt{\Delta} \equiv \frac{1}{2}\sqrt{-\mathcal{D}(\Gamma)}$ so that we really have $\hat{\varrho} = \hat{\Gamma}$ provided that $\varrho = \Gamma$. Note that an overall rescaling of $\Omega$ in (4.171) by the coordinate independent factor $-i2\bar{Z}(\Gamma)$ does not change the complex structure of $X$. Actually, deriving the attractor equation (4.164) from a variational problem is an old idea: they can be obtained via the extremalisation of the *black hole potential* [102]

$$V_{BH} = \int_{T^2\times T^2\times T^2} \Gamma \wedge \star\Gamma, \tag{4.181}$$

with respect to the moduli $\tau^i$. Here $\star$ denotes the Hodge dualization of the tori $T^2 \times T^2 \times T^2$ with moduli $\tau^i$. At attractor values one has $\star\Gamma = \hat{\Gamma}$ and $V_{BH}$ agrees with twice the Hitchin functional (4.178). As at the attractor point we also have $V_{BH} = 2|Z(\Gamma)|^2|_{\text{horizon}} = 4\sqrt{\Delta}$, we see that the critical value of the Hitchi functional gives the BPS black hole entropy

$$S = 2\pi\sqrt{\Delta} = \pi V_H(\rho_c). \tag{4.182}$$

**Four forms**

There is an analoguos functional in six dimensions for closed 4-forms $\sigma$ [96]. The stability condition in this case is that the single invariant

$$\mathcal{C}(\sigma) = \frac{1}{384}\sigma_{a_1a_2b_1b_2}\sigma_{a_3a_4b_3b_4}\sigma_{a_5a_6b_5b_6}\epsilon^{a_1a_2a_3a_4a_5a_6}\epsilon^{b_1b_2b_3b_4b_5b_6}, \tag{4.183}$$

is nowhere vanishing. The square root of this invariant transforms as a volume form. In fact, the stability condition can be recast as the requirement that there is a 2-form $k$ such that $\sigma = \frac{1}{2}k \wedge k$ and then we have

$$\frac{1}{6}k \wedge k \wedge k = \sqrt{\mathcal{C}(\sigma)}\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^6. \tag{4.184}$$

The corresponding Hitchin functional is

$$V_S(\sigma) = \frac{1}{6}\int_X k \wedge k \wedge k, \tag{4.185}$$

which is just the *symplectic volume* of $X$ when $k$ is considered as a candidate symplectic 2-form. Upon fixing the cohomology class $[\sigma] \in H^4(X)$, variation of $V_S$ gives the equation of motion

$$\mathrm{d}k = 0, \tag{4.186}$$

thus $k$ indeed defines a good symplectic structure. There is a story about relating critical points of this functional to BPS black holes, similar to the one described in the previous section. However, the black holes in question are now the BPS black holes of the uplift of the STU model to five dimensions[103]. These can be regarded as solutions of 11 dimensional supergravity compactified on the six dimensional manifold in question. The BPS black holes are interpreted as supersymmetric bound states of M2 branes and thus the wraping configuration is encoded by 2-cycles which are Poincaré dual to 4-cocycles. The claim is that these 4-cocycles are critical points of $V_S(\sigma)$, which turns out to be proportional to the entropy of the black hole[96].

Now as both $V_H$ and $V_S$ are defined for six dimensional manifolds, a natural question is that when they are compatible in the sense that $k$ can be regarded as a Kähler form of the Calabi-Yau corresponding to $\Omega$. There are two neccessary conditions which are immediate. First we must have

$$k \wedge \varrho = 0, \tag{4.187}$$

which just states that $k$ is of type (1,1) under the complex structure[15] $J_\varrho$. Second, the volumes of $X$ must agree when determined by using either $k$ or $\Omega$ which leads to the constraint

$$2V_S(\sigma) = V_H(\varrho). \tag{4.188}$$

Taking a look at the Sato-Kimura classification [16] reveals that the stabilizer of a stable 3-form $\varrho$ is $SL(3,\mathbb{C}) \times SL(3,\mathbb{C}) \subset GL(6,\mathbb{R})$, which via $\Omega$ reduces the structure group to a single $SL(3,\mathbb{C})$. On the other hand, the stabilizer of the 2-form $k$ is obviously $Sp(6,\mathbb{R}) \subset GL(6,\mathbb{R})$. Given that $k \wedge \varrho = 0$ these two stable forms define an $SU(3)$ structure which is the intersection of their stabilizer groups. Now recall that the Kähler form is related to the Hermitian metric $h$ via

$$k(u,v) = h(u, Jv), \tag{4.189}$$

so given that we know compatible critical points $k = \frac{1}{2}k_{ab}\mathrm{d}x^a \wedge \mathrm{d}x^b$ and $\varrho$ we can reconstruct the Calabi-Yau metric as

$$h_{ab} = k_{ac}{(J_\varrho)^c}_b = \frac{1}{\sqrt{-\mathcal{D}(\varrho)}} k_{ac}{(K_\varrho)^c}_b. \tag{4.190}$$

The next section describes another way of describing such pairs.

### 4.2.2 Seven dimensions

Now let us move on to describe the Hitchin functional corresponding to the $\mathcal{J}_7$ invariant of (3.82) for three fermions with seven single particle states. As in the previous section, we now regard the three fermion states of (3.81) as closed differential 3-forms on a seven dimensional closed manifold $M$:

$$\Phi = \frac{1}{3!}\Phi_{abc}\mathrm{d}x^a \wedge \mathrm{d}x^b \wedge \mathrm{d}x^c \in \wedge^3 T_p M^* \tag{4.191}$$

The 35 coefficients $\Phi_{abc}$ are analoguos to $\mathcal{P}_{abc}$ in (3.81) but can depend on the coordinates $x^1, ..., x^7$. An element of the dense orbit has $\mathcal{J}_7 \neq 0$ and is called stable. Its stabilizer can be read off from [16] to be the exeptional group $G_2 \subset GL(7,\mathbb{R})$. This can also be seen from the discussion we had about the canonical form (3.88): this state encodes the multiplication table of the octonions which is well known to be stabilized by $G_2$. Therefore, the symmetric $7 \times 7$ matrix $\mathcal{N}_\Phi$ of (3.80) is a natural candidate to be a metric of

[15] One can actually derive from this that $k \wedge \hat{\varrho} = 0$ also holds (see e.g. the appendix of [P.3]), so that this constraint is indeed equivalent with $k \wedge \Omega = k \wedge \bar{\Omega} = 0$.

$G_2$ holonomy[16] on $M$. Let us define a rescaled version of this matrix

$$(\mathcal{B}_\Phi)_{ab} = -\frac{1}{6}(\mathcal{N}_\Phi)_{ab} \equiv -\frac{1}{144}\epsilon^{i_1i_2i_3i_4i_5i_6i_7}\Phi_{ai_1i_2}\Phi_{bi_3i_4}\Phi_{i_5i_6i_7}, \tag{4.192}$$

to be in accordance with the literature on Hitchin's functionals. Now recall that $\mathcal{N}_\Phi$ and thus $\mathcal{B}_\Phi$ picks up a determinant factor under the action of $GL(7,\mathbb{R})$, which we need to cancel by a suitable power of the relative invariant $\mathcal{J}_7$ in order to obtain a quantity transforming as a metric:

$$(g_\Phi)_{ab} = \frac{1}{(\mathcal{J}_7)^{\frac{1}{3}}}(\mathcal{B}_\Phi)_{ab}. \tag{4.193}$$

Clearly, our invariant functional will be the volume of $M$ determined by $g_\Phi$:

$$V_7(\Phi) = \int_M \sqrt{\det g_\Phi}\mathrm{d}^7x \equiv \int_M (\mathcal{J}_7)^{\frac{1}{3}}\mathrm{d}^7x. \tag{4.194}$$

There is a Hodge operator $\star_\Phi$ associated to the metric $g_\Phi$. One may check that the functional can equivalently be written as

$$V_7(\Phi) = \int_M \Phi \wedge \star_\Phi \Phi, \tag{4.195}$$

so that by the usual homogenity argument, its derivative is just $\frac{7}{3}\star_\Phi \Phi$. Critical points in a fixed cohomology class $[\Phi] \in H^3(M)$ are found by varying $\Phi \mapsto \Phi + \mathrm{d}B$ and setting the variation to zero, thus the equation for the critical points is

$$\mathrm{d}\star_\Phi \Phi = 0. \tag{4.196}$$

Another result of [97] is the following.

**Theorem:** Let $M$ be an oriented closed seven dimensional manifold and $\Phi$ a closed ($\mathrm{d}\Phi = 0$) real 3-form. Then the following two statements are equivalent.

- The matrix $\mathcal{B}_\Phi$ is nondegenerate and positive definite at every point of $M$ and $\Phi$ is a critical point of the functional (4.195) in a fixed $[\Phi]$ cohomology class.
- $g_\Phi = \frac{1}{(\mathcal{J}_7)^{\frac{1}{3}}}\mathcal{B}_\Phi$ is a metric with $G_2$ holonomy on $M$.

We can obviously have an equivalent treatment of this geometry in terms of a four-form.

Now let us briefly discuss a construction relating the functional $V_7$ to the previously discussed functionals $V_H$ and $V_S$ in six dimension. This gives a

[16] As $M$ is a real manifold, we need to choose a real form of $G_2^{\mathbb{C}}$ which corresponds to fixing the signature of this candidate metric. This is the same redundancy as in the six dimensional case: under $GL(7,\mathbb{R})$, the stable orbit splits into two, one with $\mathcal{J}_7 > 0$ and one with $\mathcal{J}_7 < 0$. The choice relevant in string theory is the one where the metric has signature $(+++++++)$. This corresponds to the compact real form of $G_2$ as the stabilizer subgroup. The other possible choice is the noncompact real form of $G_2^{\mathbb{C}}$ which is the automorphism group of the split octonions. It corresponds to a metric with signature $(+++----)$.

method of obtaining $G_2$ manifolds from manifolds with weak $SU(3)$ holonomy[17]. To this end, suppose that the seven dimensional manifold is locally a direct product of a six dimensional manifold $X$ with an interval $(a, b)$ of $\mathbb{R}$, i.e. $M = X \times (a, b)$. Now write the 3-form $\Phi$ in a decomposed way

$$\Phi = \varrho(t) + k(t) \wedge \mathrm{d}t, \tag{4.197}$$

where $t$ is a coordinate on the interval $(a, b)$. Now suppose that the compatibility conditions (4.187) and (4.188) are satisfied for the stable forms $\varrho$ and $k$ for all $t$ in the interval. As before, we assume $\varrho$ and $\sigma = \frac{1}{2}k \wedge k$ to be closed as six dimensional forms. In this case we have

$$\star_\Phi \Phi = \sigma + \hat{\varrho} \wedge \mathrm{d}t, \tag{4.198}$$

so that the equations of motion are easily written in the form

$$\begin{aligned} \mathrm{d}\Phi = 0 &\to \partial_t \rho = \mathrm{d}_6 k, \\ \mathrm{d} \star_\Phi \Phi = 0 &\to \partial_t \sigma = -\mathrm{d}_6 \hat{\rho}. \end{aligned} \tag{4.199}$$

Here, $\mathrm{d}_6$ denotes the exterior derivative on $X$ only, $\mathrm{d} = \mathrm{d}_6 + \mathrm{d}t \wedge \partial_t$. These two equations can be interpreted as Hamiltonian flow equations. The phase space is the space of variations of $\varrho$ and $\sigma$, i.e. $(\wedge^3 T_p X^*)_{\text{exact}} \times (\wedge^4 T_p X^*)_{\text{exact}}$. Writing $\delta\varrho = \mathrm{d}\alpha$ and $\delta\sigma = \mathrm{d}\beta$ the symplectic form is given by

$$\langle \delta\sigma, \delta\varrho \rangle = \int_X \alpha \wedge \mathrm{d}\beta. \tag{4.200}$$

The Hamiltonian is just

$$H = 2V_S(\sigma) - V_H(\varrho), \tag{4.201}$$

which vanishes onshell due to (4.188). Now insert the ansatz $\varrho(t) = f(t)\varrho_0$ and $k(t) = \lambda \partial_t f(t) k_0$ to (4.199). The resulting equations are

$$\begin{aligned} \varrho_0 &= \lambda \mathrm{d}_6 k_0, \\ \sigma_0 &= -\frac{2}{3}\lambda \mathrm{d}_6 \hat{\rho}_0, \end{aligned} \tag{4.202}$$

provided that $f(t)$ is a solution to

$$(\partial_t f(t))(\partial_t^2 f(t)) = \frac{3}{4\lambda} f(t). \tag{4.203}$$

General solutions to this equation can be given in terms of inverse hypergeometric functions, but a particularly simple solution is $f(t) = \frac{1}{24\lambda}t^3$. Equations (4.202) guarantee that the metric $(k_0)_{ac} J^c_{\varrho_0\, b}$ on $X$ has weak $SU(3)$ holonomy. In fact, they are the Euler-Lagrange equations giving the critical points of the functional $3V_H(\varrho) + 8V_S(\sigma)$ subject to the *constraint* $\langle \delta\varrho, \delta\sigma \rangle =$

[17] A weak holonomy manifold is defined to be a Riemannian manifold with a Killing spinor $\xi$ satisfying $\nabla_X \psi = \lambda X \psi$ for all vector fields $X$, where $\lambda \in \mathbb{C}$ is the Killing constant. By $X\psi$ we mean the action of $X$ on the spinor $\psi$ as an element of the Clifford algebra generated from the vector space $T_p M$ via the Riemannian metric at $p$, see (2.2). The subcase $\lambda = 0$ corresponds to special holonomy manifolds.

$const$. These constrained critical points indeed define weak $SU(3)$ holonomy metrics, see [98] for further details.

Note that the $\mathcal{B}_\Phi$ matrix of the decomposed form $\Phi = \varrho + k \wedge \mathrm{d}t$ can be written in the simple block matrix form [P.3]

$$\mathcal{B}_\Phi = \frac{1}{(\mathcal{J}_7)^{\frac{1}{3}}} \left( \begin{array}{c|c} \frac{1}{2}kK_\varrho & 0 \\ \hline 0 & -\mathrm{Pf}(k), \end{array} \right), \tag{4.204}$$

provided that the compatibility condition $k \wedge \varrho = 0$ holds. The Pfaffian of the $6 \times 6$ matrix $k_{ab}$ was defined in (3.31). Clearly, the invariant $\mathcal{J}_7$ also factorizes in this case:

$$\mathcal{J}_7 = \frac{1}{4}\mathrm{Pf}(k)\mathcal{D}(\varrho). \tag{4.205}$$

The compatibility condition (4.188) requires $\mathrm{Pf}(k) = \frac{1}{2}\sqrt{-\mathcal{D}(\varrho)}$, from which

$$\mathcal{J}_7 = \frac{1}{8}(-\mathcal{D}(\varrho))^{\frac{3}{2}}. \tag{4.206}$$

### 4.2.3 Generalized geometry

The fundamental concept underlying the construction of Hitchin is stability, which is the requirement that the structure group locally acts on a differential form in a prehomogeneous way. We have more or less exhausted[18] the examples of prehomogeneous vector spaces $(G, V)$ where the group $G$ can correspond to the general linear group of the tangent space of a point. But what about other prehomogeneous vector spaces? It turns out that the cases when $(G, V)$ is the spin group acting on polyforms (or in the terminology of chapter 2, the extended SLOCC group acting on the fermionic Fock space) emerge naturally in the field of *generalized geometry* [99, 104, 105, 106].

The basic idea of generalized geometry is to replace the tangent bundle $T_pX$ of a manifold $X$ with the generalized tangent bundle $E$ which is *locally* the direct sum $T_pX \oplus (T_pX)^*$. For sections of this bundle, we locally have $\mathcal{X} = X + \xi$, where $X$ is a vector field and $\xi$ is a 1-form. To glue together this bundle between coordinate patches $U_\alpha$ and $U_\beta$, we can introduce in addition to the usual Jacobian matrices $a_{(\alpha\beta)}$ a closed 2-form $\omega_{(\alpha\beta)}$:

$$x_{(\alpha)} + \xi_{(\alpha)} = a_{(\alpha\beta)}x_{(\beta)} + \left[(a^{-1}_{(\alpha\beta)})^T\xi_{(\beta)} - \iota_{a_{(\alpha\beta)}x_{(\beta)}}\omega_{(\alpha\beta)}\right], \tag{4.207}$$

satisfying $\omega_{(\alpha\beta)} + \omega_{(\beta\gamma)} + \omega_{(\gamma\alpha)} = 0$ on triple intersections $U_\alpha \cap U_\beta \cap U_\gamma$. The interior product $\iota$ is the same as the action of annihilation operators on the exterior algebra, see footnote 3. In fact, joining $T_pX$ and $T_pX^*$ is moraly the same as joining fermionic creation and annihilation operators as discussed in chapter 2 in great detail. In particular, the symmetric bilinear product introduced in (2.2) becomes a metric on $E$

$$(X + \xi, Y + \eta) = \frac{1}{2}(\eta(X) + \xi(Y)), \tag{4.208}$$

[18] In fact there are a couple of other examples where Hitchin functionals can be defined, e.g. for three forms in eight dimension for which we have discussed the invariant in sec. 3.6.2. See [98] for details on the geometric construction.

which has real signature $(d, d)$. This introduces an $O(d, d)$ structure on $E$. The differentiable structure of $E$ is encoded in the *Courant bracket* as a replacement of the standard Lie bracket $[X, Y]_{\text{Lie}}$ of vector fields $X$ and $Y$:

$$\begin{aligned}[X+\xi, Y+\eta] &= [X,Y]_{\text{Lie}} + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}\mathrm{d}(\eta(X)-\xi(Y)),\\ X, Y &\in T_pX, \quad \xi,\eta \in T_pX^*,\end{aligned} \tag{4.209}$$

where $\mathcal{L}_X$ is the Lie-derivative with respect to $X$. This bracket is invariant under the off-diagonal action

$$X+\xi \mapsto X + (\xi - \iota_X B), \quad B \in \wedge^2 T_pX^*, \tag{4.210}$$

when $\mathrm{d}B = 0$. This is indeed realized inside $O(d, d)$ and is the same B-transformation as the one appearing in (2.48). In the fermionic picture this is just a special Bogoliubov transformation. The Courant bracket is the antisymmetrization of the generalized Lie-derivative

$$\mathbb{L}_{\mathcal{V}}\mathcal{X} = [V, X]_{\text{Lie}} + (\mathcal{L}_V\xi - \iota_X \mathrm{d}\zeta), \tag{4.211}$$

where $\mathcal{X} = X+\xi$ and $\mathcal{V} = V+\zeta$. This action defines infinitesimally the generalized diffeomorphisms. There are various structures that can be defined on $E$ such as a *generalized complex structure*, which is a linear automorphism of $E$ which squares to $-1$, or a *generalized metric* which one may understand as reduction of the structure group to $O(d)\times O(d) \subset O(d,d)$, see [105, 106]. This latter can always be written as

$$\mathcal{H} = \left(\begin{array}{c|c} g - Bg^{-1}B & Bg^{-1} \\ \hline -g^{-1}B & g^{-1}\end{array}\right), \tag{4.212}$$

where $g$ is a Riemaninan metric on $T_pX$, or equivalently, a nondegenerate map $T_pX \to T_pX^*$, while $B$ is a 2-form gauge field. Under the generalized Lie-derivative (4.211) the fields $g$ and $B$ transforms as $g \mapsto g + \mathcal{L}_V g$ and $B \mapsto B + \mathcal{L}_V B - \mathrm{d}\zeta$, and hence generalized diffeomorphisms contain the gauge transformations of $B$. In fact, the generalized metric on $E$ is a natural way of encoding closed string backgrounds in a way that makes T-duality manifest[19] [106]. In this picture, $B$ is just the Neveu-Schwartz 2-form $B$-field. There are far reaching applications of generalized geometry in supergravity, such us double field theory [107, 108].

A generalized almost complex structure is defined to be an endomorphism $\mathcal{J}$ of $E$ which satisfies $\mathcal{J}^2 = -1$ and it is orthogonal, i.e. $(\mathcal{J}\mathcal{X}, \mathcal{J}\mathcal{Y}) = (\mathcal{X}, \mathcal{Y})$. If it satisfies an additional Courant integrability condition [99, 105], which is a certain compatibility condition with the bracket (4.209), then we say that its a generalized complex structure, and an (even dimensional) manifold equipped with such a structure is a generalized complex manifold. In a similar way as a generalized metric unifies two different objects,

[19] In this picture, T-dualities are just the intertwiners between the even and odd particle Fock spaces, discussed in section 2.4.4. In particular, any non-isotropic element of $E$ defines a T-duality via (2.87). We may think of a general element of $E$ as being a $B$-transformed version of a section locally of the form $\mathcal{X}_0 = \partial_t + \mathrm{d}t$ for some local coordinate $t$. Then T-dualization happens along the coordinate $t$. The generalized metric (or complex structure) transforms as $\mathcal{H} \mapsto (\mathcal{X}, \mathcal{X})\mathcal{V}_{\mathcal{X}}{}^T\mathcal{H}\mathcal{V}_{\mathcal{X}}$ as described in (2.89).

the metric and a 2-form field, the generalized complex structure unifies ordinary complex structures and symplectic structures: a $\mathcal{J}$ with no mixing among $T_pX$ and $T_pX^*$ defines an almost complex structure while a $\mathcal{J}$ which takes every element of $T_pX$ to $T_pX^*$ and vice-versa, defines a symplectic structure. The Courant integrability condition can also shown [105] to interpolate between the integrability of a complex structure and a symplectic structure.

### 4.2.4 Generalized Hitchin functionals

With this short, and far from complete introduction to the idea of generalized geometry, now we are in position to briefly discuss the Hitchin functionals which are defined by the invariants (3.22), (3.27) and (3.72) which, in the fermionic context, were candidate entanglement measures for fermions with six and seven modes respectively.

Let us start with the quartic invariants (3.22), (3.27) defined in six dimensions. Let again $X$ be a compact oriented six dimensional manifold. We take the differential polyforms $\phi \in \wedge^{\text{even}} T_pX^*$ and $\psi \in \wedge^{\text{odd}} T_pM^*$ which can be expanded in an analoguos way as in equations (3.19) and (3.24). We can define the functionals

$$\begin{aligned} V_{GH}^{\text{even}}(\phi) &= \int_X \sqrt{-q_{\text{even}}(\phi)}\mathrm{d}^6x, \\ V_{GH}^{\text{odd}}(\psi) &= \int_X \sqrt{-q_{\text{odd}}(\psi)}\mathrm{d}^6x. \end{aligned} \tag{4.213}$$

Note that these functionals are T-dual to each other: for any $\mathcal{X} \in E$ with $(\mathcal{X}, \mathcal{X}) \equiv 1$ we have

$$V_{GH}^{\text{even}}(\phi) = V_{GH}^{\text{odd}}(\mathcal{X}\phi), \tag{4.214}$$

where $\mathcal{X}\phi$ denotes the same Clifford action as in (2.86) and is now interpreted as the T-dual of $\phi$ along $\mathcal{X}$.

It is clear that the matrices $K(\phi)^I{}_J$ and $K(\psi)^I{}_J$ of (3.20) with either (3.21) or (3.25) inserted can be used to define generalized almost complex structures:

$$\begin{aligned} \mathcal{J}_{\phi}{}^I{}_J &= \frac{1}{\sqrt{-q_{\text{even}}(\phi)}} K(\phi)^I{}_J, \\ \mathcal{J}_{\psi}{}^I{}_J &= \frac{1}{\sqrt{-q_{\text{odd}}(\psi)}} K(\psi)^I{}_J. \end{aligned} \tag{4.215}$$

These satisfy the Courant integrability conditions, when the stable forms $\phi$ and $\psi$ are critical points of the functionals $V_{GH}^{\text{even}}(\phi)$ and $V_{GH}^{\text{odd}}(\psi)$ in a fixed cohomology class respectively, in a very similar way as critical points of $V_H$ give complex and critical points of $V_S$ give symplectic manifolds. In addition to this, due to (3.28), the polyforms

$$\begin{aligned} \phi = \phi + i\hat{\phi}, &\quad \bar{\phi} = \phi - i\hat{\phi}, \\ \psi = \psi + i\hat{\psi}, &\quad \bar{\psi} = \psi - i\hat{\psi}, \end{aligned} \tag{4.216}$$

are complex *pure spinors* which are the analogues of the *holomorphic* 3-form $\Omega$. This motivates the notion of *generalized Calabi-Yau manifolds* [99, 105]. These are generalized complex manifolds $X$ for which there is a closed complex poly-form $\varphi \in \wedge^{even/odd} T_pX^* \otimes \mathbb{C}$ such that it is a pure spinor and $(\varphi, \bar{\varphi}) = 0$ for the Mukai pairing. Here, conjugation is with respect to the generalized complex structure. Note that Calabi-Yau and symplectic manifolds are both generalized Calabi-Yau manifolds, but there are more exotic examples as well, see for example [105].

Now that we see that the invariants (3.22) and (3.27) of the prehomogeneous vector spaces $(\mathbb{C}^\times \times Spin_0(12,\mathbb{C}), \wedge^{even/odd}\mathbb{C}^6)$ define action functionals which give meaningful geometrical structures in the context of generalized geometry, we move on to briefly mention their possible physical significance. The fact that these functionals are T-dual to each other hints that they have some connection to string theory. This is indeed the case, it turns out that they are related to *topological strings*.

### 4.2.5 Topological strings

Let us give a lightning introduction into the idea of topological strings. For an excellent introduction, we refer the reader to [109]. In string theory one considers maps $\Phi$ from a two-dimensional world-sheet $\Sigma$ into a target space $X$. One weights these maps by the Polyakov action and integrates over all the maps and all the metrics on the world-sheets $\Sigma$ of different genera to obtain the perturbative series for string scattering amplitudes or the string partition function. This two dimensional gravity theory contains a large amount of gauge symmetries, so in the end the integral over metrics on $\Sigma$ boils down to a finite dimensional integral for the complex structure moduli of the Riemann surface $\Sigma$. Along the way, to ensure that our gauge symmetries are not anomalous, we are enforced to choose our world sheet theory (including ghosts coming from the Faddeev-Popov method in addition to the maps $\Phi$ and their superpartners) to be a superconformal field theory[20] with vanishing central charge $c = 0$. Now taking the Polyakov action seriously tells us that we need to choose the target space dimension to be 10. However, there is another way to change the central charge of a superconformal algebra: it is called twisting. The world sheet theory has a so called $\mathcal{N} = (2,2)$ superconformal symmetry which is generated by the left moving world sheet currents

$$J, G^+, G^-, T, \tag{4.217}$$

and their right moving counterparts which are denoted the same way with an additional bar on top. Here, $T$ is the stress energy tensor, $G^\pm$ are supertranslations and $J$ is the U(1) current generating left moving $R$-symmetry. The zero modes of these currents are analogous to the degree, $\bar{\partial}$, $\bar{\partial}^\dagger$ and $\Delta$ operators on target space differential forms. The zero mode part of the affine algebra of the modes of the currents corresponds to the algebra of these differential operators, extended by the central charge $c$. Now the redefined stress tensor

$$T' = T \pm \frac{1}{2}\partial J, \tag{4.218}$$

[20] Of course, the Polyakov action does not give automatically a conformal field theory for any target space. The "good" target spaces are precisely the solutions of the corresponding classical supergravity.

has no central extension in its operator product expansion with itself. With a similar redefinition in the right moving sector, we can eliminate the conformal anomaly by coupling the world sheet metric to $(T', \bar{T}')$ instead of $(T, \bar{T})$. This procedure is called twisting. The two different ways of twistings correspond to the so called $A$ and $B$ models of topological strings: when we twist $T$ and $\bar{T}$ with the same sign we obtain the $A$ and $\bar{A}$ models while when we twist them with a different sign we obtain the $B$ and $\bar{B}$ models[21]. Note that under $T'$, the conformal weights of the states change, but this is not the reason why the obtained world sheet theories are different from the ones we have started with. The source of the difference is that the theory becomes a *topological field theory*[22], moreover a very special kind of it, a *cohomological field theory*. This means that to obtain physical states we need to form the cohomology of a nilpotent operator $Q$. For cohomological field theories, the stress tensor is $Q$ exact which implies that correlation functions of physical operators are independent of any continuous changes of the metric. The nilpotent operator $Q$ in question is some combination of the zero modes of the currents $G^{\mp}$. The twisting also changes the spin of the corresponding current from $\frac{3}{2}$ to 1. This makes the analogy with target space differential operators even tighter. Forming the cohomology of these operators is then analogous to passing from the de Rahm complex to its cohomology. This analogy turns out to be sharp: the physical observables of the $A$ model are elements of $H^{(1,1)}(X)$, which are in fact in one-to-one correspondence with the moduli space of the Kähler structure of the target space $X$. The observables of the $B$ model on the other hand are deformations of the complex structure of the target space manifold $X$. They are represented with $O_i$, $i = 1, ..., h_{2,1} = \dim H^{(2,1)}(X)$ marginal operators.

Let us focus on the $B$-model. To obtain a topological string partition function, one must integrate some expectation value of the topological field theory over the world sheet metrics. This integral turns out to reduce to an integral over the moduli space $\mathcal{M}_g$ of the genus $g$ Riemann surfaces $\Sigma$. The tangent space of this space is spanned by *Beltrami differentials* $\mu_i = (\mu_i)^z_{\bar{z}} \mathrm{d}\bar{z} \otimes \partial_z$, $i = 3g - 3$, (0,1)-forms taking values in the holomorphic tangent bundle. They correspond to infinitesimal deformations of the complex structure of $\Sigma$ of the form $\mathrm{d}z \mapsto \mathrm{d}z + \mu_i(\mathrm{d}z)$. One needs to choose a measure on $\mathcal{M}_g$ which has the right invariance properties under changes of basis. The only nontrivial possibility for the $B$ model turns out to be

$$\prod_{i=1}^{3g-3} \mathrm{d}m^i \mathrm{d}\bar{m}^i \int_\Sigma G^-(\mu_i) \int_\Sigma \bar{G}^-(\bar{\mu}_i), \tag{4.219}$$

where $m^i$ are coordinates on $\mathcal{M}_g$ corresponding to $\mu_i$ while $G^-(\mu_i) = G^-_{zz}(\mu_i)^z_{\bar{z}} \mathrm{d}z \wedge \mathrm{d}\bar{z}$ is indeed a top-form. The genus $g$ free energy (log of the partition function) of the topological string is defined to be

$$F_g = \int_{\mathcal{M}_g} \langle \prod_{i=1}^{3g-3} \mathrm{d}m^i \mathrm{d}\bar{m}^i \int_\Sigma G^-(\mu_i) \int_\Sigma \bar{G}^-(\bar{\mu}_i) \rangle, \tag{4.220}$$

[21] Barred and undbarred models differ only by an overall complex conjugation.

[22] We roughly call a theory a topological field theory if it does not depend on continuous changes of the background metric.

where $\langle . \rangle$ is the topological field theory expectation value[23]. The full topological string free energy is defined perturbatively by the asymptotic series

$$F = \sum_{g=0}^{\infty} \lambda^{2g-2} F_g, \tag{4.221}$$

where $\lambda$ is the string coupling. The topological string partition function is

$$Z_B = e^F. \tag{4.222}$$

Now a generating functional can be defined by deforming the action of the topological theory with the marginal operator $\sum_{i=1}^{h_{2,1}} t^i \int_\Sigma O_i + c.c.$. The partition function now depends on a set of $t^i$ complex structure moduli in addition to the string coupling: $Z_B = Z_B(\lambda, t^i, \bar{t}^i, \Omega_0)$. Derivatives with respect to $t^i$ give multipoint functions of nonlocal operators. The $\Omega_0$ stands for the background holomorphic 3-form of the Calabi Yau target space. We note that the dependence of $Z_B$ on $\bar{t}^i$ is determined by the so called *holomorphic anomaly* equations [110]. The term anomaly appears because the operators $\bar{O}_i$ can be shown to be $Q$-exact so that one naively expects that $Z_B$ is independent of $\bar{t}^i$. However, the moduli space measure (4.219) is not invariant under the action of $Q$. Since it is a remnant of a path integral measure, the $\bar{t}^i$ dependence it introduces is indeed a Fujikawa-type quantum anomaly. The anomaly equations determine the $\bar{t}^i$ dependence completely, so that $Z_B$ is effectively the function of $h_{2,1} + 1$ parameters. In fact, this is just half the real dimension of the space $H^3(X, \mathbb{C})$ of complex harmonic 3-forms. This vector space is a symplectic space, where the symplectic product is evaluated as the integral wedge product of two 3-forms. It can be shown [111], that the holomorphic anomaly equations are equivalent with the statement that $Z_B$ is a wavefunction depending on half the coordinates of the classical phase space $H^3(X, \mathbb{C})$. This way we may think about the $B$-model as a quantization of $H^3(X, \mathbb{R})$.

### 4.2.6 Topological strings, black holes and generalized Hitchin functionals

There is a remarkable relation between the number of 4d BPS black hole microstates and the topological B-model, conjectured by Ooguri, Strominger and Vafa [112]. The relation is based on two relatively well known facts. The first is that the genus zero free energy $F_0$ of the $B$ model is just $i$ times the prepotential $\mathcal{F}$ of the Calabi-Yau target space. The second is that the classical BPS black hole entropy (4.158) can be obtained as the *Legendre transform* of the imaginary part of the same prepotential with respect to the imaginary parts of a set of particular *homogeneous* coordinates on the complex structure moduli space, called holomorphic sections. One then

[23] We note the subtlety that this definition is valid as it stands only for $g > 1$. One has to be careful that physical correlators must be neutral under the U(1) $R$-charge. It turns out that the topological string, similarly to the ordinary string, has something like a critical dimension: $R$-neutral correlators are possible for all $g$ only in the case of six dimensional target spaces. Moreover, these correlators must have 3,1, or 0 additional *local* insertions to the $G^-$ insertions for $g = 0, 1$ and $g > 1$ respectively. For $g > 1$ thus the expression for $F_g$ is $R$-neutral but for $g = 0, 1$ one has to extract $F_g$ from the 3 and 1-point function respectively.

has to substitute the attractor values of these sections into this Legendre transformed function to obtain the black hole entropy.

Now one can remove the factor of $\lambda^{-2}$ in the genus zero part of (4.221) by rescaling the holomorphic sections by $\lambda$. This does not change the complex structure moduli. Therefore, one may think about $Z_B$ as a function of the $h_{2,1}+1$ holomorphic sections. Then it is tempting to think that the above Legendre transform property of the BPS entropy is a *saddle point approximation*[24] of a relation like

$$\Omega(P^I, Q_I) = \int \mathrm{d}\Phi_I e^{-Q^I \Phi_I} |Z_B(P_I + i\Phi_I)|^2, \tag{4.223}$$

where $\Omega(P^I, Q_I) = e^S$ is the microcanonical partition function for the BPS black hole, $P^I$ and $Q_I$, $I = 1, ..., h_{2,1}+1$ are its dyonic charges and on the right hand side, $Z_B$ is evaluated at the *attractor values* of the moduli[25], shifted by a potential $\Phi_I$. Notice that on the right hand side we have the *Wigner transform* of the density matrix formed from $Z_B$ regarded as a wave function. At tree level we have $Z_B = e^{i\mathcal{F}}$ and for large charges this relation is equivalent with the statement about the black hole entropy being a Legendre transform. Beyond tree level, this relation is what is known as the OSV conjecture [112].

We have seen in section 4.2.1 that the Hitchin functional $V_H$ of (4.178) at its critical point in a fixed $[\varrho] \in H^3(X, \mathbb{R})$ cohomology class agrees with the BPS black hole entropy corresponding to the charges coming from $[\varrho]$. Now if we define the partition function

$$Z_H([\varrho]) = \int \mathcal{D}\beta e^{V_H(\varrho + \mathrm{d}\beta)}, \tag{4.224}$$

which is a quantization of the Hitchin functional regarded as an action, we see that the relation

$$Z_H([\varrho]) = \Omega(P^I, Q_I), \tag{4.225}$$

is true in the saddle point approximation when the charges of $[\varrho]$ are large. This was originaly observed in [96] where it was proposed that $Z_H$ calculates the partition function of both the $B \times \bar{B}$ model and the BPS black hole exactly. A one loop calculation was performed subsequently in [113] by Pestun and Witten. They found that the formula does not hold at one-loop. Instead, at one-loop the Wigner transform of the $B \times \bar{B}$ partition function agrees with the partition function defined from the generalized Hitchin functional (4.213)

$$Z_{GH}^{\mathrm{odd}}([\psi]) = \int \mathcal{D}\beta e^{V_{GH}^{\mathrm{odd}}(\psi + \mathrm{d}\beta)}. \tag{4.226}$$

Now we have seen that at tree level the two functionals *can* in fact have the same Calabi-Yau critical points. This amounts to choosing $[\psi]$ to be an element of $H^3(X, \mathbb{R})$ so that (3.27) agrees with $\mathcal{D}(\psi)$. However, the fluctuations $\mathrm{d}\beta$ can now have 1-form and 5-form parts which changes the result beyond tree level. The fact that it is the generalized Hitchin functional that can be related to topological strings is even more natural if we recall that

[24] Valid for large charges.

[25] With an appropriate choice of polarization on $H^3(X, \mathbb{C})$.

$V_{GH}^{\rm even}$ and $V_{GH}^{\rm odd}$ are T-dual to each other. Note that in [96] it was also put forward that the A-model is equivalent to the theory defined by $V_S$ of (4.185). This was checked at the semiclassical level but no proposal was made for a relation between a partition function defined from $V_S$ and the $A$-model partition function. If such a relation exists, it might be more natural to expect it to involve the generalized Hitchin functional $V_{GH}^{\rm even}$.

As a final remark, let us briefly mention the role of the seven dimensional functional (4.195) and its generalized version, which can be defined from the invariant[26] (3.72). It was put forward in [96] that a theory based on $V_7(\Phi)$ could be regarded as a topological $M$-theory. This was supported by several arguments, among them is the fact that this theory naturally incorporates and couples the two types of six dimensional Hitchin functionals, as reviewed in section 4.2.1, and that it seems to be the unique form theory in one dimension higher. This latter is not quite true, we can define a generalized Hitchin functional in seven dimensions based on the invariant (3.72). This was studied in [114] and connected to so called generalized $G_2$ manifolds, which are manifolds with a generalized metric reducing the generalized structure group all the way to $G_2 \times G_2 \subset O(7,7)$. In light of the fact that for the $B$-model one should use generalized Hitchin's functionals, one is tempted to think that the generalized $G_2$ functional could be related to topological $M$-theory. However, there are other interesting world sheet theories with $G_2$ target spaces such as the $G_2$ topological string of [115]. The quantization of both seven dimensional Hitchin functionals was performed at one loop in [116]. Here it was found that the $G_2$ string result is related to the generalized Hitchin functional at one loop, but only up to a multiplicative factor depending on a certain invariant of the background $G_2$ metric.

Getting a grasp on what topological M theory is requires one to compare the predictions of these form theories to something known. For example the six dimensional topological string calculates quantum corrected $F$-term superpotentials for the dimensionally reduced physical string. One could expect a similar relation between topological $M$-theory and the physical one reduced to four dimensions. This latter requirement points towards something called exceptional generalized geometry[117, 118], where the generalized structure group $O(d,d)$ is replaced by the $E_{d(d)}$ series. This would lead one to consider a functional of a tensor transforming in the 912 of $E_{7(7)}$ replacing the differential forms we have been discussing so far. However, this pair is not prehomogeneous, so there is no notion of stability.

[26] Indeed, $(\mathbb{C}^\times \times Spin_0(14,\mathbb{C}), \wedge^{even/odd}\mathbb{C}^7)$ is the last prehomogeneous vector space where the group is a spin group so that a functional can be defined in the context of generalized geometry.

# Appendix A

# Conventions on some Lie algebras

## A.1 Different ways of splitting $\mathfrak{so}(4,4)$ as $\mathfrak{sl}_2^{\times 4}\oplus(2,2,2,2)$

We define the group $SO(4,4)$ as the set of $8 \times 8$ matrices $O$ keeping the bilinear form

$$G = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{A.1}$$

i.e. $OGO^T = G$. Here, $I$ is the $4 \times 4$ identity matrix. The Lie algebra $\mathfrak{so}(4,4)$ is spanned by matrices $T$ satisfying $TG + GT^T = 0$. We use the same parametrization of these matrices as in [22]. We denote by $E_{ij}$ the $8 \times 8$ matrix with 1 in the $(i,j)$ component and zeros everywhere else. The Cartan generators are given by

$$\begin{aligned} H_0 &= E_{33} + E_{44} - E_{77} - E_{88}, \quad H_1 = E_{33} - E_{44} - E_{77} + E_{88}, \\ H_2 &= E_{11} + E_{22} - E_{55} - E_{66}, \quad H_3 = E_{11} - E_{22} - E_{55} + E_{66}, \end{aligned} \tag{A.2}$$

while the roots are parametrized as

$$\begin{aligned}
E_0 &= E_{47} - E_{38}, & E_1 &= E_{87} - E_{34}, & E_2 &= E_{25} - E_{16}, & E_3 &= E_{65} - E_{12}, \\
E^{Q_1} &= E_{45} - E_{18}, & E^{Q_2} &= E_{32} - E_{67}, & E^{Q_3} &= E_{36} - E_{27}, & E^{Q_4} &= E_{41} - E_{58}, \\
E^{P^1} &= E_{57} - E_{31}, & E^{P^2} &= E_{46} - E_{28}, & E^{P^3} &= E_{42} - E_{68}, & E^{P^4} &= E_{17} - E_{35}, \\
F_0 &= E_{74} - E_{83}, & F_1 &= E_{78} - E_{43}, & F_2 &= E_{52} - E_{61}, & F_3 &= E_{56} - E_{21}, \\
F^{Q_1} &= E_{54} - E_{81}, & F^{Q_2} &= E_{23} - E_{76}, & F^{Q_3} &= E_{63} - E_{72}, & F^{Q_4} &= E_{14} - E_{85}, \\
F^{P^1} &= E_{75} - E_{13}, & F^{P^2} &= E_{64} - E_{82}, & F^{P^3} &= E_{24} - E_{86}, & F^{P^4} &= E_{71} - E_{53}.
\end{aligned} \tag{A.3}$$

### A.1.1 U-duality split

It is easy to see that the generators $H_\Lambda$, $E_\Lambda$, $F_\Lambda$, $\Lambda = 0,1,2,3$ form four commuting $\mathfrak{sl}_2$ algebras:

$$[H_\Lambda, E_\Lambda] = 2E_\Lambda, \quad [H_\Lambda, F_\Lambda] = -2F_\Lambda, \quad [E_\Lambda, F_\Lambda] = H_\Lambda. \tag{A.4}$$

The remaining 16 generators $E^{P^I}$, $E^{Q_I}$, $F^{P^I}$, $F^{Q_I}$ form the fundamental $(2,2,2,2)_U$ representation of this $(\mathfrak{sl}_2^{\times 4})_U$ algebra under the adjoint action.

A vector of this representation can nicely be described by the 16 amplitudes $\psi_{ijkl}$ of a four qubit state

$$|\psi\rangle = \sum_{i,j,k,l=0}^{1} \psi_{ijkl}|ijkl\rangle, \tag{A.5}$$

transforming as

$$\psi_{ijkl} \mapsto (S_0)_i{}^{i'}(S_1)_j{}^{j'}(S_2)_k{}^{k'}(S_3)_l{}^{l'}\psi_{i'j'k'l'}, \quad S_0 \otimes S_1 \otimes S_2 \otimes S_3 \in SL(2)^{\times 4}, \tag{A.6}$$

under the action of the group $SL(2)^{\times 4}$ generated by the algebra (A.4). In terms of the Lie algebra generators one writes this vector as

$$\begin{aligned}\Psi = &\psi_{0000}E^{P^4} + \psi_{0001}E^{Q_3} + \psi_{0010}E^{Q_2} - \psi_{0011}E^{P^1} \\ &+ \psi_{0100}E^{Q_1} - \psi_{0101}E^{P^2} - \psi_{0110}E^{P^3} - \psi_{0111}E^{Q_4} \\ &- \psi_{1000}F^{Q_1} + \psi_{1001}F^{P^3} + \psi_{1010}F^{P^2} + \psi_{1011}F^{Q_1} \\ &+ \psi_{1100}F^{P^1} + \psi_{1101}F^{Q_2} + \psi_{1110}F^{Q_3} - \psi_{1111}F^{P_4},\end{aligned} \tag{A.7}$$

transforming as (A.6) under $\Psi \mapsto g\Psi g^{-1}$.

### A.1.2 Sigma model split

We can realize the split $\mathfrak{sl}_2^{\times 4} \oplus (2,2,2,2)$ in a different way suited to writing the timelike reduced STU action (4.11) as a sigma model on $SO(4,4)/SL(2)^{\times 4}$. We may introduce the symmetric bilinear form

$$\eta = \text{diag}(-1,-1,1,1,-1,-1,1,1), \tag{A.8}$$

and look for the subgroup $SO(2,2)\times SO(2,2) \cong SL(2)^{\times 4}$ keeping this fixed. This subgroup is generated by the $-1$ eigenspace of the involution

$$T^{\#} = \eta T^T \eta. \tag{A.9}$$

This eigenspace is spanned by the 12 generators

$$k_\Lambda = E_\Lambda - F_\Lambda, \quad k^{Q_I} = E^{Q_I} + F^{Q_I}, \quad k^{P^I} = E^{P^I} + F^{P^I}, \tag{A.10}$$

while the $+1$ eigenspace is spanned by

$$H_\Lambda, \quad p_\Lambda = E_\Lambda + F_\Lambda, \quad p^{Q_I} = E^{Q_I} - F^{Q_I}, \quad p^{P^I} = E^{P^I} - F^{P^I}. \tag{A.11}$$

To see that this indeed realizes a $\mathfrak{sl}_2^{\times 4} \oplus (2,2,2,2)$ split define[119, 120]

$$
\begin{aligned}
\tilde{H}_1 &= 1/2(-k^{Q_4} - k^{Q_1} - k^{Q_2} - k^{Q_3}),\\
\tilde{H}_2 &= 1/2(k^{Q_4} + k^{Q_1} - k^{Q_2} - k^{Q_3}),\\
\tilde{H}_3 &= 1/2(k^{Q_4} - k^{Q_1} + k^{Q_2} - k^{Q_3}),\\
\tilde{H}_4 &= 1/2(k^{Q_4} - k^{Q_1} - k^{Q_2} + k^{Q_3}),\\
\tilde{E}_1 &= 1/4(-k_0 + k_1 + k_2 + k_3 + k^{P^1} + k^{P^2} + k^{P^3} + k^{P^4}),\\
\tilde{E}_2 &= 1/4(k_0 - k_1 + k_2 + k_3 + k^{P^1} - k^{P^2} - k^{P^3} + k^{P^4}),\\
\tilde{E}_3 &= 1/4(k_0 + k_1 - k_2 + k_3 - k^{P^1} + k^{P^2} - k^{P^3} + k^{P^4}),\\
\tilde{E}_4 &= 1/4(k_0 + k_1 + k_2 - k_3 - k^{P^1} - k^{P^2} + k^{P^3} + k^{P^4}),\\
\tilde{F}_1 &= 1/4(k_0 - k_1 - k_2 - k_3 + k^{P^1} + k^{P^2} + k^{P^3} + k^{P^4}),\\
\tilde{F}_2 &= 1/4(-k_0 + k_1 - k_2 - k_3 + k^{P^1} - k^{P^2} - k^{P^3} + k^{P^4}),\\
\tilde{F}_3 &= 1/4(-k_0 - k_1 + k_2 - k_3 - k^{P^1} + k^{P^2} - k^{P^3} + k^{P^4}),\\
\tilde{F}_4 &= 1/4(-k_0 - k_1 - k_2 + k_3 - k^{P^1} - k^{P^2} + k^{P^3} + k^{P^4}).
\end{aligned}
\tag{A.12}
$$

One easily verifies that

$$
[\tilde{H}_J, \tilde{E}_J] = 2\tilde{E}_J, \quad [\tilde{H}_J, \tilde{F}_J] = -2\tilde{F}_J, \quad [\tilde{E}_J, \tilde{F}_J] = \tilde{H}_J, \tag{A.13}
$$

with $J = 1,..,4$ and all other commutators vanishing. We can write an element of the $+1$ eigenspace of $^{\#}$ in terms of four qubit amplitudes $\psi_{ijkl}$ transforming as in (A.6) under this new $SL(2)^{\times 4}$. It reads explicitly as

$$
\tilde{\Psi} = \tilde{\Psi}_{H_\Lambda} H_\Lambda + \tilde{\Psi}_{p_\Lambda} p_\Lambda + \tilde{\Psi}_{Q_I} p^{Q_I} + \tilde{\Psi}_{P^I} p^{P^I}, \tag{A.14}
$$

where

$$
\begin{aligned}
\tilde{\Psi}_{H_0} &= \psi_{0001} + \psi_{0010} + \psi_{0100} - \psi_{0111} - \psi_{1000} + \psi_{1011} + \psi_{1101} + \psi_{1110},\\
\tilde{\Psi}_{H_1} &= \psi_{0001} + \psi_{0010} - \psi_{0100} + \psi_{0111} + \psi_{1000} - \psi_{1011} + \psi_{1101} + \psi_{1110},\\
\tilde{\Psi}_{H_2} &= \psi_{0001} - \psi_{0010} + \psi_{0100} + \psi_{0111} + \psi_{1000} + \psi_{1011} - \psi_{1101} + \psi_{1110},\\
\tilde{\Psi}_{H_3} &= -\psi_{0001} + \psi_{0010} + \psi_{0100} + \psi_{0111} + \psi_{1000} + \psi_{1011} + \psi_{1101} - \psi_{1110},\\
\tilde{\Psi}_{p_0} &= -\psi_{0000} + \psi_{0011} + \psi_{0101} + \psi_{0110} - \psi_{1001} - \psi_{1010} - \psi_{1100} + \psi_{1111},\\
\tilde{\Psi}_{p_1} &= -\psi_{0000} + \psi_{0011} - \psi_{0101} - \psi_{0110} + \psi_{1001} + \psi_{1010} - \psi_{1100} + \psi_{1111},\\
\tilde{\Psi}_{p_2} &= -\psi_{0000} - \psi_{0011} + \psi_{0101} - \psi_{0110} + \psi_{1001} - \psi_{1010} + \psi_{1100} + \psi_{1111},\\
\tilde{\Psi}_{p_3} &= -\psi_{0000} - \psi_{0011} - \psi_{0101} + \psi_{0110} - \psi_{1001} + \psi_{1010} + \psi_{1100} + \psi_{1111},\\
\tilde{\Psi}_{Q_1} &= 2\left(\psi_{0100} - \psi_{1011}\right),\\
\tilde{\Psi}_{Q_2} &= 2\left(\psi_{0010} - \psi_{1101}\right),\\
\tilde{\Psi}_{Q_3} &= 2\left(\psi_{0001} - \psi_{1110}\right),\\
\tilde{\Psi}_{Q_4} &= 2(\psi_{1000} - \psi_{0111}),\\
\tilde{\Psi}_{P^1} &= \psi_{0000} + \psi_{0011} - \psi_{0101} - \psi_{0110} - \psi_{1001} - \psi_{1010} + \psi_{1100} + \psi_{1111},\\
\tilde{\Psi}_{P^2} &= \psi_{0000} - \psi_{0011} + \psi_{0101} - \psi_{0110} - \psi_{1001} + \psi_{1010} - \psi_{1100} + \psi_{1111},\\
\tilde{\Psi}_{P^3} &= \psi_{0000} - \psi_{0011} - \psi_{0101} + \psi_{0110} + \psi_{1001} - \psi_{1010} - \psi_{1100} + \psi_{1111},\\
\tilde{\Psi}_{P^4} &= -\psi_{0000} - \psi_{0011} - \psi_{0101} - \psi_{0110} - \psi_{1001} - \psi_{1010} - \psi_{1100} - \psi_{1111}.
\end{aligned}
\tag{A.15}
$$

Note that the $SL(2)^{\times 4}$ invariant $\mathrm{Tr}\tilde{\Psi}^2$ is a quadratic measure of four qubit entanglement[30].

### A.1.3 The third split

The previous two splits are related by triality of $\mathfrak{so}(4,4)$ and hence there must be one more inequivalent splitting of the algebra. Indeed this split is given by the $\pm 1$ eigenspaces of the involution

$$T \mapsto \eta_2 T^T \eta_2, \tag{A.16}$$

where

$$\eta_2 = \begin{pmatrix} \epsilon\otimes\epsilon & 0 \\ 0 & \epsilon\otimes\epsilon \end{pmatrix}, \tag{A.17}$$

where $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The $-1$ eigenspace is 12 dimensional and spans $\mathfrak{sl}_2^{\times 4}$ algebra. The $+1$ eigenspace forms the representation $(2,2,2,2)$ under this. We do not need this split in the following hence we omit the explicit form of the generators.

## A.2 Lie algebra of $E_7$

We will use the symmetric space splitting

$$\mathfrak{e}_7 \cong \mathfrak{sl}_8 \oplus \wedge^4\mathbb{C}^8, \tag{A.18}$$

to describe the Lie algebra of $E_7$, following [70]. Recall that this splitting plays a key role in finding the SLOCC classification of four fermions with four single particle states, see section 3.6. According to this splitting, we parametrize an element $T \in \mathfrak{e}_7$ by the pair

$$T = ({\Lambda^i}_j, \Sigma_{ijkl}), \quad {\Lambda^i}_i = 0, \tag{A.19}$$

where $\Sigma_{ijkl}$ totaly antisymmetric and the indices can take eight values. The commutation relations of the algbra are given explicitly as

$$T_3 = [T_1, T_2], \tag{A.20}$$

where

$$\begin{aligned} {\Lambda_3}^i{}_j &= [\Lambda_1, \Lambda_2]^i{}_j - \frac{1}{3}({}^*\Sigma_1^{iklm}\Sigma_{2klmj} - {}^*\Sigma_2^{iklm}\Sigma_{1klmj}) \\ \Sigma_{3ijkl} &= 4({\Lambda_1}^m{}_{[i}\Sigma_{2jkl]m} - {\Lambda_2}^m{}_{[i}\Sigma_{1jkl]m}). \end{aligned} \tag{A.21}$$

Here, the brackets denote antisymmetrization and ${}^*\Sigma_1^{ijkl} = \frac{1}{24}\epsilon^{ijklmnop}\Sigma_{mnop}$.

### A.2.1 The 56 of $E_7$

The fundamental 56 dimensional representation of $E_7$ is conveniently described by a pair of antisymmetric $8 \times 8$ matrixes $(x^{ij}, y_{ij})$. The generator $T$ acts in this representation via

$$T \begin{pmatrix} x^{ij} \\ y_{ij} \end{pmatrix} = \begin{pmatrix} 2\Lambda^{[i}{}_{[k}\delta^{j]}_{l]} & {}^*\Sigma^{ijkl} \\ \Sigma_{ijkl} & -2\Lambda^{[k}{}_{[i}\delta^{l]}_{j]} \end{pmatrix} \begin{pmatrix} x^{kl} \\ y_{kl} \end{pmatrix} \tag{A.22}$$

The Killing form of $\mathfrak{e}_7$ agrees with the trace of the product of generators in its fundamental. It reads as

$$\kappa(T_1, T_2) = \mathrm{Tr}(T_1T_2) = 12{\Lambda_1}^i{}_k{\Lambda_2}^k{}_i + 2\Sigma_{1ijkl}{}^*{\Sigma_2}^{ijkl}. \tag{A.23}$$

To express the representative of the coset $E_{7(7)}/SU(8)$ we borrow the basis used by [79]. We set

$$\begin{aligned} {G_a}^b &= ({\Lambda_a}^b, 0), & ({\Lambda_a}^b)^i{}_j &= \delta^i_a\delta^b_j - \frac{1}{8}\delta^a_b\delta^i_j, \\ G^{abcd} &= (0, \Sigma^{abcd}), & (\Sigma^{abcd})_{ijkl} &= \delta_i^{[a}\delta_j^b\delta_k^c\delta_l^{d]}, \end{aligned} \tag{A.24}$$

and then define

$$E_i{}^j = G_i{}^j, \quad E^{ijk} = -12G^{ijk8}, \quad D_i = G_i{}^8, \quad \vec{H} = \sum_{j=1}^{7}\left(-\vec{f}_j + \vec{g}\right)G_j{}^j, \tag{A.25}$$

where the indices $i, j$ only take the first 7 values and

$$\vec{f}_j = (\underbrace{0, ..., 0}_{j-1}, (10-j)s_j, s_{j+1}, ..., s_7),$$

$$
\begin{aligned}
\vec{g} &= 3(s_1, s_2, ..., s_7), \\
s_i &= \sqrt{\frac{2}{(10-i)(9-i)}}.
\end{aligned}
\tag{A.26}
$$

This is the basis used in e.q. (4.123)

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